Algebraic Geometry 10 (6) (2023) 754–778
doi:10.14231/AG-2023-027

The diagonal of quartic fivefolds

Nebojsa Pavic and Stefan Schreieder

Abstract

We show that a very general quartic hypersurface in $\mathbb{P}^6_k$ over a field of characteristic different from 2 does not admit a decomposition of the diagonal, hence is not retract rational. This generalizes a result of Nicaise–Ottem, who showed stable irrationality over fields of characteristic zero. To prove our result, we introduce a new cycle-theoretic obstruction that may be seen as an analogue of the motivic obstruction for rationality in characteristic zero, introduced by Nicaise–Shinder and Kontsevich–Tschinkel.

1. Introduction

A variety X over a field k is rational if it is isomorphic to $\mathbb{P}^{\dim X}_k$ after removing proper closed subvarieties from both sides; it is stably rational if $X \times \mathbb{P}^n_k$ is rational for some $n \geqslant 0$. Moreover, X is retract rational if there exist an integer N and rational maps $f : X \dashrightarrow \mathbb{P}^N$ and $g\colon \mathbb{P}^N \dashrightarrow X$ such that $g \circ f$ is defined and coincides with the identity: $g \circ f = \mathrm{id}_X$. Finally, X is unirational if it receives a dominant rational map from some projective space. We have the following well-known implications:

$$\text{rational} \implies \text{stably rational} \implies \text{retract rational} \implies \text{unirational}. \tag{1.1}$$

By [BCSS85] and [AM72], the first and last implications are both strict over algebraically closed fields; it is an open problem whether this holds true for the second implication as well.

Retract rational varieties admit a decomposition of the diagonal, which means that the point of the diagonal $\delta_X \in X_{k(X)}$ is rationally equivalent to $z_{k(X)}$ for some k-rational point $z \in X$. In [Voi15], with improvements in [CP16, Sch19a], Voisin used this implication to initiate a cycle-theoretic degeneration technique which, roughly speaking, allows one to disprove retract rationality for varieties X that admit a degeneration to a mildly singular variety Y with a cohomological obstruction for the existence of a decomposition of the diagonal, such as global differential forms [Tot16] or unramified cohomology [CP16, HPT18, Sch19b]. This method can be extended to cases where the special fibre of the degeneration breaks up into several pieces (see for example [Tot16, Lemma 2.4], [Sch21a, Proposition 6.1] and [Sch21b, Theorem 8.5]). However, at least over algebraically closed fields, it is crucial that at least one component Y_{i_0} of the special fibre Y is irrational with some cohomological obstruction, while that obstruction class must vanish on

Received 15 June 2021, accepted in final form 10 October 2022.
2020 Mathematics Subject Classification 14J70, 14C25 (primary), 14M20, 14E08 (secondary).
Keywords: hypersurfaces, quartics, algebraic cycles, rationality, retract rationality.

This project has received funding from the European Research Council (ERC) under the European Union's Horizon 2020 research and innovation programme under grant agreement No 948066.

the intersection of Y_{i_0} with any other component Y_j (for example if each component of $Y_{i_0} \cap Y_j$ is rational).

Using the weak factorization theorem, Nicaise–Shinder [NS19] and Kontsevich–Tschinkel [KT19] introduced powerful motivic obstructions for rationality and stable rationality in characteristic zero. Their approach proves (stable) irrationality for varieties that admit degenerations to simple normal crossing varieties $Y = \bigcup_{i\in I} Y_i$ such that

$$\left[\mathbb{P}_k^{\dim Y}\right] + \sum_{\emptyset \neq J \subset I} (-1)^{|J|} \left[Y_J \times \mathbb{P}_k^{|J|-1}\right]$$

is non-zero in the free abelian group generated by (stable) birational equivalence classes of smooth projective k-varieties, where $Y_J = \bigcap_{j\in J} Y_j$. For instance, it follows from this formula that a variety that specializes to a union of two smooth rational varieties that meet along a stably irrational variety is stably irrational. This approach has been implemented successfully by Nicaise–Ottem [NO22], who used it to prove that a very general quartic fivefold over an uncountable algebraically closed field of characteristic zero is not stably rational, by writing down a degeneration into two components (that may be chosen to be rational) such that their intersection is stably irrational by the work of Hassett–Pirutka–Tschinkel [HPT18].

The motivic method of Nicaise–Shinder and Kontsevich–Tschinkel does not seem to generalize to detect retract rationality. In particular, one may speculate that the motivic obstruction from [NS19, KT19] could be a suitable tool to distinguish retract rational varieties from stably rational varieties, and quartic fivefolds treated in [NO22] yield the first potential candidates regarding the strictness of the second implication in (1.1).

In this paper, we prove that quartic fivefolds are in fact not retract rational and hence do not give counterexamples to strictness of the second implication in (1.1).

THEOREM 1.1. *Let k be an uncountable field of characteristic different from 2. A very general quartic $X \subset \mathbb{P}_k^6$ does not admit a decomposition of the diagonal, hence is not retract rational.*

By a theorem of Morin (see [CM98]), a general quartic fivefold $X \subset \mathbb{P}_k^6$ as in Theorem 1.1 is known to be unirational. On the other hand, over fields of positive characteristic, even rationality of quartic fivefolds was previously open.

The rationality problem for Fano hypersurfaces $X \subset \mathbb{P}_k^{n+1}$ is a classical problem in algebraic geometry; see for example [CG72, IM71, Kol95, CP16, Tot16, Sch19b, Sch21a]. In arbitrary dimension, the best-known bound is due to [Sch19b, Sch21a], where it is shown that over uncountable fields k, very general hypersurfaces of dimension $n \geqslant 3$ and degree $d \geqslant \log_2 n + 2$ (or $d \geqslant \log_2 n + 3$ if $\mathrm{char}(k) = 2$) do not admit a decomposition of the diagonal and hence are neither retract rational nor stably rational. The case of cubic threefolds [CG72, Mur73], the aforementioned result of Nicaise–Ottem [NO22] and Theorem 1.1 above are the only cases where this bound has been improved.

In order to use a similar degeneration in Theorem 1.1 as Nicaise–Ottem used in [NO22], we, roughly speaking, face the problem of disproving the existence of a decomposition of the diagonal for the geometric generic fibre of a family that degenerates into a union of two rational components $Y = Y_1 \cup Y_2$ such that $Y_{12} = Y_1 \cap Y_2$ is integral and does not admit a decomposition of the diagonal. The naive idea is to perform a $2:1$ base change and to blow up Y_{12}, to arrive at a semi-stable model whose special fibre has three components: two rational end components and a component in the middle that is a $\mathbb{P}^1$-bundle over Y_{12} and hence does not admit a decomposition of the diagonal.

At this point, the following problem arises: we could have started with the trivial family $\mathbb{P}^n_R \to \operatorname{Spec} R$, blown up a subvariety Z of the special fibre and then blown up a general point of the exceptional divisor above Z. If Z does not admit a decomposition of the diagonal, then we arrive at a degeneration of $\mathbb{P}^n$ into a chain of three components where again the two end components are rational, while the component in the middle does not admit a decomposition of the diagonal. So how can we tell apart this degeneration from the one discussed above?

1.1 Method

We describe our solution to the above-mentioned problem briefly in this section. For this, let R be a discrete valuation ring, and let $\mathcal{X} \to \operatorname{Spec} R$ be a projective strictly semi-stable R-scheme with special fibre $Y = \bigcup_{i\in I} Y_i$. There is a canonical map

$$\Phi_{\mathcal{X}} \colon \operatorname{CH}_1(Y) \longrightarrow \ker\left(\deg\colon \bigoplus_{i\in I}\operatorname{CH}_0(Y_i) \to \mathbb{Z}\right), \quad \gamma \longmapsto \sum_{i\in I} \iota_i^*(\iota_*\gamma)\,,$$

where $\iota\colon Y \to \mathcal{X}$ and $\iota_i\colon Y_i \to \mathcal{X}$ denote the respective closed immersions, cf. Section 3.1 below. It is easy to see that $\Phi_{\mathcal{X}}$ does not depend on $\mathcal{X}$ but in fact only on the special fibre Y. Indeed, if $\gamma \in \operatorname{CH}_1(Y)$ is supported on Y_{i_0}, then the contribution $\Phi_{\mathcal{X},Y_i}(\gamma) \in \operatorname{CH}_0(Y_i)$ is given by

$$\Phi_{\mathcal{X},Y_i}(\gamma) = \begin{cases} \gamma|_{Y_i\cap Y_{i_0}} & \text{if } i \neq i_0\,,\\ -\sum_{j\neq i_0} \gamma|_{Y_i\cap Y_j} & \text{if } i = i_0\,, \end{cases}$$

where $\gamma|_{Y_i\cap Y_j}$ is the zero-cycle on Y_i given by the intersection of γ (viewed as a 1-cycle on Y_i) with $Y_i \cap Y_j$ (viewed as a divisor on Y_i). This description allows one to compute $\Phi_{\mathcal{X}}$ effectively.

If A/R is an unramified extension of discrete valuation rings (dvr's), then the base change $\mathcal{X}_A$ is again strictly semi-stable, and we get a map $\Phi_{\mathcal{X}_A}$ as above. Even if the residue field κ of R is algebraically closed, the residue field of A may be a function field over κ (for example $\kappa(Y_i)$ for some i), in which case $\Phi_{\mathcal{X}_A}$ is a map between Chow groups of varieties over interesting non-closed fields. In particular, even if $\Phi_{\mathcal{X}}$ is surjective, this may very well fail for $\Phi_{\mathcal{X}_A}$.

Theorem 1.2. *Let R be a discrete valuation ring with algebraically closed residue field, and let $\pi\colon \mathcal{X} \to \operatorname{Spec} R$ be a projective strictly semi-stable R-scheme.*

(i) *If the generic fibre of π admits a decomposition of the diagonal, then for any unramified extension A/R of dvr's, the map $\Phi_{\mathcal{X}_A}$ is surjective.*
(ii) *If the geometric generic fibre of π admits a decomposition of the diagonal and the dual graph of the special fibre is a straight line (that is, the components form a chain), then for any unramified extension A/R of dvr's, the map $\Phi_{\mathcal{X}_A}$ is surjective modulo 2.*

If the geometric generic fibre of π admits a decomposition of the diagonal, then up to a base change and a modification of the total space, we arrive at a situation where the generic fibre admits such a decomposition, and so item (i) may be applied. While this route could be taken to prove Theorem 1.1, it is significantly easier to work with the enhanced version in item (ii), where no additional base change and blow-ups are necessary, and so it allows us to work with a special fibre that has only few components.

We like to think about Theorem 1.2 as a cycle-theoretic analogue of the motivic method from [NS19, KT19] exploited in [NO22]. Note however that there is no direct relation among the two methods, and it may be possible to find situations where one applies but not the other.

To give an idea of how to apply Theorem 1.2, we state the following consequence as an example. The used notion of torsion order is introduced in Section 2.3.

Corollary 1.3. *Let R be a dvr with algebraically closed residue field k, and let $\pi\colon \mathcal{X} \to \operatorname{Spec} R$ be a projective strictly semi-stable R-scheme whose special fibre $Y = Y_1 \cup Y_2$ has two components. Assume that*

- *Y is universally CH_1-trivial in the sense that for any field extension L/k, the natural map $\mathrm{CH}_1(Y) \to \mathrm{CH}_1(Y_L)$ is surjective;*
- *$Y_{12} := Y_1 \cap Y_2$ is integral, and its torsion order is even.*

Then the geometric generic fibre of π does not admit a decomposition of the diagonal.

Theorem 1.1 will be deduced from Theorem 1.2 and a degeneration that is inspired by [NO22]. Note however that checking that the condition in item (ii) of Theorem 1.2 is violated is subtle and requires several additional degenerations. In particular, even though Corollary 1.3 illustrates well the basic idea, the technical details are more complicated, and we are not able to literally apply the statement of Corollary 1.3. The key input of course remains the striking example of Hassett–Pirutka–Tschinkel in [HPT18].

The problem of finding cycle-theoretic obstructions for rationality that are sensitive to semi-stable degenerations into unions of rational varieties has also been considered in [BGvG22a, BGvG22b]. Their approach relies on explicit identities in the prelog Chow ring and is different from ours.

Remark 1.4. In [NO22, Theorem 7.1], Nicaise–Ottem showed that a very general complete intersection $X_{23} \subset \mathbb{P}^6_k$ of type $(2, 3)$ is not stably rational over any uncountable algebraically closed field k of characteristic zero. Similarly to the case of quartic fivefolds, this result generalizes as follows: for any uncountable field of characteristic different from 2, a very general complete intersection $X_{23} \subset \mathbb{P}^6_k$ of type $(2, 3)$ does not admit a decomposition of the diagonal and hence is not retract rational. While this can also be deduced from Theorem 1.2, the case of complete intersections of type $(2, 3)$ in $\mathbb{P}^6$ is in fact much easier than the case of quartic fivefolds treated in this paper. Indeed, by [NO22, proof of Theorem 7.1], such complete intersections admit a degeneration into a union of two varieties meeting along a rational variety, such that one component does not admit a decomposition of the diagonal, and so [Tot16, Lemma 2.4] applies to give the result. In [Ska23], Skauli proves an enhancement of this result by actually constructing examples over $\mathbb{Q}$, where due to additional singularities in the special fibre, slightly more careful arguments are necessary. Note that these lines of arguments do not work for quartic fivefolds, where the 'obstruction for rationality' lies really in the intersection of the components of the degeneration and not in the components themselves.

2. Notation and conventions

2.1 Conventions

All schemes are separated. A variety is an integral scheme of finite type over a field. If X is a scheme over a ring R and A/R is ring extension, then we write $X_A := X \times_R A$. A very general point of an irreducible scheme is a closed point outside a countable union of proper closed subsets. For an integral scheme X over a field k, we write $k(X)$ or $\kappa(X)$ for the function field of X; we use the latter whenever we prefer to make the ground field of X not explicit in our notation.

For an abelian group G, we write $G/2$ instead of $G \otimes \mathbb{Z}/2\mathbb{Z}$. If $G_1 \to G_2$ is a group homomorphism of abelian groups, then we write by abuse of notation $G_2/G_1 := \operatorname{coker}(G_1 \to G_2)$.

2.2 Decomposition of the diagonal

Let X be a variety over a field k. We denote by $\mathrm{CH}_i(X)$ the Chow group of algebraic cycles of X of dimension i. One says that X admits a decomposition of the diagonal if

$$[\Delta_X] = [z \times_k X] + [Z] \in \mathrm{CH}_{\dim(X)}(X \times_k X)\,, \tag{2.1}$$

where $\Delta_X \subset X \times_k X$ denotes the diagonal, z is a zero-cycle on X and Z is a cycle on $X \times_k X$ which does not dominate the second factor. Now let $k(X)$ denote the fraction field of X, and write $X_{k(X)} := X \times_k k(X)$. We denote by

$$\delta_X \in \mathrm{CH}_0(X_{k(X)})$$

the zero-cycle on $X_{k(X)}$ that is induced by pulling back Δ_X via $X_{k(X)} \to X \times_k X$. By the localization sequence [Ful98, Proposition 1.8], relation (2.1) is equivalent to $\delta_X = z_{k(X)} \in \mathrm{CH}_0(X_{k(X)})$.

A proper variety X over k is said to have universally trivial Chow group of zero-cycles if for any field extension $F \supset k$, the degree map $\deg\colon \mathrm{CH}_0(X_F) \to \mathbb{Z}$ is an isomorphism. If X is in addition geometrically integral and smooth (over k), then X has universally trivial Chow group of zero-cycles if and only if X admits a decomposition of the diagonal (see [CP16, Proposition 1.4]).

2.3 Torsion order

The torsion order $\mathrm{Tor}(X)$ of a proper k-variety is the smallest positive integer N such that $N \cdot \Delta_X$ admits a decomposition as in (2.1) or, equivalently, such that $N \cdot \delta_X = z_{k(X)} \in \mathrm{CH}_0(X_{k(X)})$ for some zero-cycle $z \in \mathrm{CH}_0(X)$. The torsion order is ∞ if no such integer exists; cf. [CL17, Sch21a].

Lemma 2.1. *Let X be a smooth projective variety over an algebraically closed field k. Assume that the torsion order of X is finite and divisible by $e \geqslant 2$. Then for any $z \in \mathrm{CH}_0(X)$, the following holds in $\mathrm{CH}_0(X_{k(X)})$:*

$$\delta_X - z_{k(X)} \not\equiv 0 \mod e\,.$$

Proof. For a contradiction, assume that $\delta_X \equiv z_{k(X)} \mod e$. Then there is a zero-cycle $\epsilon \in \mathrm{CH}_0(X_{k(X)})$ with $\delta_X = z_{k(X)} + e\epsilon$. By our assumptions, there is also a positive integer d such that $\mathrm{Tor}(X) = d \cdot e$. Hence, $ed\Delta_X = z' \times X + R$ for some zero-cycle z' on X and some cycle R on $X \times X$ that does not dominate the second factor. Base changing everything to $k(X)$, we get a similar decomposition of $ed \cdot \Delta_{X_{k(X)}}$. Since $\Delta_{X_{k(X)}}$ acts as the identity on $\mathrm{CH}_0(X_{k(X)})$, we find that

$$ed \cdot \epsilon = ed \cdot \Delta^*_{X_{k(X)}}(\epsilon) = \deg(\epsilon) \cdot z'_{k(X)}\,.$$

Hence,

$$d\delta_X = dz_{k(X)} + ed \cdot \epsilon = z''_{k(X)}$$

for some zero-cycle $z'' \in \mathrm{CH}_0(X)$. This implies that $\mathrm{Tor}(X)$ divides d and so $e = 1$, which contradicts our assumptions. This proves the lemma. □

2.4 Chains of divisors

We call a closed and reduced subscheme $D = \bigcup_{i=1}^n D_i$ of a scheme X with irreducible components D_i of pure codimension 1 a chain of divisors if the scheme-theoretic intersections $D_{i-1} \cap D_i$ and $D_i \cap D_{i+1}$ are disjoint from each other in D_i for all $1 < i < n$ and if all the other intersections

$D_i \cap D_j$ with $i \neq j$ are empty. We call $D = \bigcup D_i$ a chain of Cartier divisors if the D_i are in addition Cartier in X.

2.5 Semi-stable models

Let R be a discrete valuation ring with residue field k and fraction field K. A proper flat R-scheme $\mathcal{X} \to \operatorname{Spec} R$ is called strictly semi-stable if the special fibre $Y = \mathcal{X} \times_R k$ is a geometrically reduced simple normal crossing divisor on $\mathcal{X}$. In other words, the components of Y are smooth Cartier divisors, and the intersection of r different components is either empty or smooth of codimension r. The total space $\mathcal{X}$ of a strictly semi-stable R-scheme is automatically regular because the special fibre is contained in the regular locus by assumption and $\mathcal{X}$ is proper over R, so that any point of the generic fibre specializes to a point of the special fibre. The special fibre Y is called a chain of Cartier divisors if $Y \subset \mathcal{X}$ is a chain of Cartier divisors in the sense of Section 2.4.

3. Chow-theoretic obstruction: The generic fibre

3.1 The obstruction map

DEFINITION 3.1. Let R be a discrete valuation ring, and let $\mathcal{X} \to \operatorname{Spec} R$ be a strictly semi-stable R-scheme with special fibre Y. Let Y_i with $i \in I$ be the irreducible components of Y, and let $\iota\colon Y \to \mathcal{X}$ and $\iota_i\colon Y_i \to \mathcal{X}$ denote the natural embeddings. We define $\Phi_{\mathcal{X},Y_i}\colon \operatorname{CH}_1(Y) \to \operatorname{CH}_0(Y_i)$ to be the composition

$$\Phi_{\mathcal{X},Y_i}\colon \operatorname{CH}_1(Y) \xrightarrow{\iota_*} \operatorname{CH}_1(\mathcal{X}) \xrightarrow{\iota_i^*} \operatorname{CH}_0(Y_i)\,, \tag{3.1}$$

and we denote by $\Phi_{\mathcal{X}}$ the direct sum

$$\Phi_{\mathcal{X}} := \sum_{i\in I} \Phi_{\mathcal{X},Y_i}\colon \operatorname{CH}_1(Y) \longrightarrow \bigoplus_{i\in I} \operatorname{CH}_0(Y_i)\,. \tag{3.2}$$

We have the following simple but useful lemma, which shows that $\Phi_{\mathcal{X}}$ depends only on the special fibre Y of the R-scheme $\mathcal{X}$.

LEMMA 3.2. *In the notation of Definition 3.1, let $Y_{ij} := Y_i \cap Y_j$, denote by $\iota_{ij}\colon Y_{ij} \to Y_j$ and $\iota_i\colon Y_i \to Y$ the natural inclusions, and write $\gamma_i|_{Y_{ji}} := \iota_{ji}^*\gamma_i$ for $\gamma_i \in \operatorname{CH}_1(Y_i)$.*

(i) *For any $\gamma_i \in \operatorname{CH}_1(Y_i)$, we have*

$$\Phi_{\mathcal{X},Y_j}((\iota_i)_*\gamma_i) = \begin{cases} (\iota_{ij})_*(\gamma_i|_{Y_{ji}}) \in \operatorname{CH}_0(Y_j) & \text{for } j \neq i\,, \\ -\displaystyle\sum_{\substack{k\in I,\\ k\neq i}} (\iota_{ki})_*(\gamma_i|_{Y_{ki}}) \in \operatorname{CH}_0(Y_i) & \text{for } j = i\,. \end{cases}$$

(ii) *Let $\gamma = \sum_{i\in I} (\iota_i)_*\gamma_i \in \operatorname{CH}_1(Y)$. Then*

$$\Phi_{\mathcal{X},Y_i}(\gamma) = \sum_{j\in I\setminus\{i\}} (\iota_{ji})_*\gamma_j|_{Y_{ji}} - \sum_{j\in I\setminus\{i\}} (\iota_{ji})_*\gamma_i|_{Y_{ji}} \in \operatorname{CH}_0(Y_i)\,.$$

Proof. Note that the restrictions ι_{ji}^* are well defined because Y is a simple normal crossing divisor on $\mathcal{X}$. Then in the case $j \neq i$, the first item follows directly from the definition of the intersection product (see [Ful98, Theorem 6.2(a)]), while in the case $j = i$, it follows from the fact that $[Y_i] = -\sum_{k\neq i}[Y_k]$ in $\operatorname{Pic}(\mathcal{X})$. The second item is a direct consequence of the first. $\square$

3.2 Obstructing decompositions on the generic fibre

The homomorphism $\Phi_{\mathcal{X}}$ from Definition 3.1 will be our main tool to obstruct the existence of decompositions of the diagonal in this paper. This rests on two observations. Firstly, if $\gamma \in \mathrm{CH}_1(Y)$, then

$$\deg(\Phi_{\mathcal{X}}(\gamma)) = \deg\left(\sum_{i\in I} \iota_i^* \iota_* \gamma\right) = \deg(\iota^* \iota_* \gamma) = 0\,,$$

and so the image of $\Phi_{\mathcal{X}}$ is always contained in the kernel of the degree map

$$\deg\colon \bigoplus_{i\in I} \mathrm{CH}_0(Y_i) \longrightarrow \mathbb{Z}\,, \quad (z_i)_{i\in I} \longmapsto \sum_{i\in I} \deg(z_i)\,.$$

Secondly, whenever A/R is an unramified extension of dvr's, $\mathcal{X}_A := \mathcal{X} \times_R A$ is a strictly semi-stable A-scheme. In particular, if L denotes the residue field of A (which will be an extension of the residue field of R), then we get from Definition 3.1 a homomorphism

$$\Phi_{\mathcal{X}_A}\colon\ \mathrm{CH}_1(Y_L) \longrightarrow \ker\left(\deg\colon \bigoplus_{i\in I} \mathrm{CH}_0(Y_{i,L}) \to \mathbb{Z}\right). \tag{3.3}$$

Item (i) in Theorem 1.2 is a consequence of the following result.

Proposition 3.3. *Let R be a discrete valuation ring with residue field k and fraction field K. Let $\mathcal{X} \to \operatorname{Spec} R$ be a strictly semi-stable projective R-scheme whose generic fibre X admits a decomposition of the diagonal. Let Y_i with $i \in I$ be the components of the special fibre Y. Then for any unramified extension A/R of dvr's, the homomorphism in (3.3) is surjective.*

Proof. By Lemma 3.2, $\Phi_{\mathcal{X}_A}$ depends only on the special fibre Y_L, and hence it remains the same after replacing A by its completion. In particular, we may assume that A is complete. Let $(z_i)_{i\in I} \in \oplus_{i\in I} \mathrm{CH}_0(Y_{i,L})$ be a collection of zero-cycles with $\sum_i \deg(z_i) = 0$. By the moving lemma (see for example [Rob72] or [Lev05, Theorem 2.13]), we may assume that the support of each z_i is contained in the smooth locus of Y_L.

Combining inflation of local rings (see Lemma 4.2) with Hensel's lemma (see for example [Gro67, Theorem 18.5.17]), it follows that any zero-cycle supported on the smooth locus of Y_L lifts to a 1-cycle on $\mathcal{X}_A$. In particular, there is a 1-cycle $z \in \mathrm{CH}_1(\mathcal{X}_A)$ that is flat over A and such that $z \cap Y_{i,L} = z_i$ for all $i \in I$.

The restriction of z to the generic fibre of $\mathcal{X}_A \to \operatorname{Spec} A$ is then a zero-cycle of degree zero, and so it is rationally equivalent to zero because X has universally trivial Chow group of zero-cycles by assumption. The restriction map $\mathrm{CH}_1(\mathcal{X}_A) \to \mathrm{CH}_0(X_{\operatorname{Frac} A})$ fits into the following localization exact sequence (see [Ful75, Section 4.4]):

$$\mathrm{CH}_1(Y_L) \xrightarrow{\iota_*} \mathrm{CH}_1(\mathcal{X}_A) \longrightarrow \mathrm{CH}_0(X_{\operatorname{Frac} A}) \longrightarrow 0\,,$$

and so we find that there is a 1-cycle $\gamma \in \mathrm{CH}_1(Y_L)$ with

$$z = \iota_* \gamma \in \mathrm{CH}_1(\mathcal{X}_A)\,. \tag{3.4}$$

Since $z \cap Y_{i,L} = z_i$ for all $i \in I$, this implies that $\Phi_{\mathcal{X}_A}(\gamma) = (z_i)_{i\in I} \in \bigoplus_{i\in I} \mathrm{CH}_0(Y_{i,L})$. Hence, $\Phi_{\mathcal{X}_A}$ in (3.3) is surjective, as we want. ☐

4. Chow-theoretic obstruction: The geometric generic fibre

The purpose of this section is to prove item (ii) of Theorem 1.2. This yields an obstruction to decompositions of the diagonal of a smooth projective variety which specializes to a chain of smooth varieties such as the union $Y = Y_1 \cup Y_2$ of two smooth varieties meeting along a smooth irreducible divisor $Y_1 \cap Y_2$. We will apply this obstruction later to a degeneration of quartic fivefolds that is similar to the one in [NO22]. The precise statement of our obstruction is as follows.

THEOREM 4.1. *Let R be a discrete valuation ring with algebraically closed residue field, and let $\mathcal{X} \to \operatorname{Spec} R$ be a strictly semi-stable projective R-scheme whose special fibre $Y = \bigcup_{i\in I} Y_i$ is a chain of Cartier divisors. Assume that the geometric generic fibre of $\mathcal{X} \to \operatorname{Spec} R$ has a decomposition of the diagonal. Then for any unramified extension A/R of dvr's, with induced extension L/k of residue fields, the natural map*

$$\Phi_{\mathcal{X}_A}\colon \operatorname{CH}_1(Y_L)/2 \longrightarrow \ker\left(\deg\colon \bigoplus_{i\in I} \operatorname{CH}_0(Y_{i,L})/2 \to \mathbb{Z}/2\right)$$

given by reduction modulo 2 of (3.3) is surjective.

In applications it will be useful to note that by inflation of local rings, an unramified extension A/R of dvr's exists for any given extension L/k of residue fields.

LEMMA 4.2. *Let R be a discrete valuation ring with residue field k. For any field extension L/k, there is an unramified extension of dvr's A/R which induces L/k on the residue fields.*

Proof. By inflation of local rings, see [Bou06, Chapter IX, Appendice, Corollaire du Théorème 1], there is a flat local R-algebra A such that A is a dvr with $\mathfrak{m}_A = \mathfrak{m}_R R$ and $L = A/\mathfrak{m}_A$. In particular, A/R is an unramified extension of dvr's which induces L/k on the residue fields, as we want. □

Theorem 4.1 will follow from Proposition 3.3 together with a careful analysis of the effect of ramified base changes $R \subset \tilde{R}$ of dvr's.

4.1 A preliminary lemma

The following simple lemma will be needed in the proof of Theorem 4.1.

LEMMA 4.3. *Let $\mathcal{X} \to \operatorname{Spec} R$ be a strictly semi-stable R-scheme such that the special fibre $Y = \bigcup_{i=1}^n Y_i$ of $\mathcal{X}$ is a chain of Cartier divisors. Let $Y_{i,i+1} := Y_i \cap Y_{i+1}$, and let A/R be an unramified extension of dvr's with induced extension L/k of residue fields. Then we have the following, where on the right-hand side of (4.1), (4.2), (4.3) and (4.4) below, we leave out the respective pushforward maps by the natural inclusions by slight abuse of notation.*

(i) *Let*

$$\gamma = \gamma_1 + \ldots + \gamma_n \in \operatorname{CH}_1(Y_L)\,, \tag{4.1}$$

where γ_i is in $\operatorname{CH}_1(Y_{i,L})$. Then for all i, we have

$$\Phi_{\mathcal{X}_A,Y_{i,L}}(\gamma) = \gamma_{i-1}|_{Y_{i-1,i,L}} - \gamma_i|_{Y_{i-1,i,L}} - \gamma_i|_{Y_{i,i+1,L}} + \gamma_{i+1}|_{Y_{i,i+1,L}} \in \operatorname{CH}_0(Y_{i,L})\,. \tag{4.2}$$

(ii) *If, additionally, for some $0 < i < n$, the intersections $Y_{i-1,i}$ and $Y_{i,i+1}$ are isomorphic and if Y_i is a $\mathbb{P}^1_k$-bundle over $Y_{i-1,i} \simeq Y_{i,i+1}$ with projection $q_i\colon Y_i \to Y_{i-1,i}$, then any $\gamma \in \operatorname{CH}_l(Y_L)$ is of the form*

$$\gamma = \gamma_1 + \cdots + \gamma_{i-1} + q_i^*\alpha_i + \gamma_{i+1} + \cdots + \gamma_n \in \operatorname{CH}_l(Y_L) \tag{4.3}$$

for some $\alpha_i \in \operatorname{CH}_0(Y_{i-1,i,L})$ and $\gamma_j \in \operatorname{CH}_1(Y_{j,L})$, and we have that

$$\Phi_{\mathcal{X}_A,Y_{i,L}}(\gamma) = \gamma_{i-1}|_{Y_{i-1,i,L}} - 2\alpha_i + \gamma_{i+1}|_{Y_{i,i+1,L}} \in \operatorname{CH}_0(Y_{i,L})\,. \tag{4.4}$$

Proof. Item (i) follows directly from Lemma 3.2. Moreover, (4.4) follows from (4.3) and item (i), and so it suffices to prove (4.3). That is, we need to show that $\mathrm{CH}_1(Y_L)$ is generated by

$$\mathrm{CH}_1(Y_{1,L}) \oplus \cdots \oplus \mathrm{CH}_1(Y_{i-1,L}) \oplus q_i^* \,\mathrm{CH}_0(Y_{i,i-1,L}) \oplus \mathrm{CH}_1(Y_{i+1,L}) \oplus \cdots \oplus \mathrm{CH}_1(Y_{n,L})\,. \tag{4.5}$$

As $Y_{i,L}$ is a $\mathbb{P}^1_k$-bundle over $Y_{i-1,i,L}$ and as $\iota_{i-1,i}\colon Y_{i,i-1,L} \hookrightarrow Y_{i,L}$ is a section of $q_{Y_i}\colon Y_{i,L} \to Y_{i-1,i,L}$, we can write any x in $\mathrm{CH}_1(Y_{i,L})$ as

$$x = q_i^*(x_0) + (\iota_{i-1,i})_*(y)$$

with x_0 in $\mathrm{CH}_0(Y_{i-1,i,L})$ and y in $\mathrm{CH}_1(Y_{i-1,i,L})$ (see for example [Ful98, Theorem 3.3(b)]). It is clear, however, that $(\iota_{i-1,i})_*(y)$ in $\mathrm{CH}_1\left(\tilde{Y}_L\right)$ is an element which comes from $\mathrm{CH}_1(Y_{i-1,L})$. This proves (4.5), which concludes the proof of the lemma. □

4.2 Analyses of base change

Let $\tilde{R}/R$ be a finite (possibly ramified) extension of discrete valuation rings, and let $\mathcal{X} \to \operatorname{Spec} R$ be a strictly semi-stable R-scheme. We consider the base change $\mathcal{X}_{\tilde{R}} := \mathcal{X} \times_R \tilde{R}$. Following Hartl [Har01, proof of Proposition 2.2], there is a finite sequence of blow-ups

$$\tilde{\mathcal{X}} := \mathcal{X}_r \longrightarrow \mathcal{X}_{r-1} \longrightarrow \dots \longrightarrow \mathcal{X}_1 \longrightarrow \mathcal{X}_{\tilde{R}}\,,$$

where each step $\mathcal{X}_{i+1} \to \mathcal{X}_i$ is the consecutive blow-up of (strict transforms of) all irreducible components of Y that are not Cartier, such that

$$\tilde{\mathcal{X}} \to \operatorname{Spec} \tilde{R} \tag{4.6}$$

is strictly semi-stable. In particular, the special fibre $\tilde{Y}$ of $\tilde{\mathcal{X}}$ is given by

$$\tilde{Y} = Y_1 \cup \bigcup_{j=1}^{r} R_{1,j} \cup Y_2 \cup \bigcup_{j=1}^{r} R_{2,j} \cup \cdots \cup Y_n\,. \tag{4.7}$$

Note that $\tilde{Y}$ is again a chain of non-singular Cartier divisors, and there is a $\mathbb{P}^1_k$-bundle structure of $R_{i,j}$ over $Y_{i,i+1}$ for all $j > 0$ and all i. We denote the corresponding projection by $q_{R_{i,j}}\colon R_{i,j} \to Y_{i,i+1}$. Moreover, the intersections $Y_i \cap R_{i,1}$ and $R_{i,j} \cap R_{i,j+1}$ are isomorphic to $Y_{i,i+1}$, for all $1 \leqslant i < n$ and all $1 \leqslant j \leqslant r$.

Proposition 4.4. *Let R be a discrete valuation ring with algebraically closed residue field k. Let $\mathcal{X} \to \operatorname{Spec} R$ be a strictly semi-stable R-scheme such that the special fibre $Y = \bigcup_{i=1}^n Y_i$ of $\mathcal{X}$ is a chain of Cartier divisors. Let $R \subset \tilde{R}$ be a finite extension of discrete valuation rings, and let $\tilde{\mathcal{X}} \to \operatorname{Spec} \tilde{R}$ be as in (4.6) with special fibre $\tilde{Y}$ as in (4.7). Let A/R be an unramified extension of dvr's with induced extension L/k of residue fields, and denote by $\tilde{\mathcal{X}}_{\tilde{A}}$ the base change of $\tilde{\mathcal{X}}$ to an unramified extension $\tilde{A}$ of $\tilde{R}$ which induces L/k on the residue fields ($\tilde{A}$ exists by Lemma 4.2). Then the following holds: if $\Phi_{\tilde{\mathcal{X}}_{\tilde{A}}}$ is surjective modulo 2, then $\Phi_{\mathcal{X}_A}$ is surjective modulo 2.*

Proof. We will split the proof of the proposition into two cases, depending on the parity of the integer r appearing in (4.7). Before we do so, let us first make some general observations, which will be used in both cases.

By (4.3), any $\gamma \in \mathrm{CH}_1(\tilde{Y}_L)$ can be written as

$$\gamma = \gamma_{Y_1} + \sum_{j=1}^{r} q^*_{R_{1,j}} \alpha_{R_{1,j}} + \gamma_{Y_2} + \sum_{j=1}^{r} q^*_{R_{2,j}} \alpha_{R_{2,j}} + \cdots + \gamma_{Y_n}$$

for some $\gamma_{Y_i} \in \mathrm{CH}_1(Y_{i,L})$ and $\alpha_{R_{i,j}} \in \mathrm{CH}_0(Y_{i,i+1,L})$, where $Y_{i,i+1,L} := Y_{i,L} \cap Y_{i+1,L}$. We then have that

$$\Phi_{\tilde{\mathcal{X}}_{\tilde{A}},R_{i,j,L}}(\gamma) \equiv \alpha_{R_{i,j-1}} - 2\alpha_{R_{i,j}} + \alpha_{R_{i,j+1}} \equiv \alpha_{R_{i,j-1}} + \alpha_{R_{i,j+1}} \bmod 2$$

in $\mathrm{CH}_0(R_{i,j,L})/2$ for all $1 < j < r$ and all i, where in the first equality we use (4.4). (Here and in what follows we neglect, by slight abuse of notation, the respective pushforwards to $R_{i,j,L}$ and to $Y_{i,L}$ in our notation whenever no confusion is likely to arise.) Moreover, we have that

$$\begin{aligned} \Phi_{\tilde{\mathcal{X}}_{\tilde{A}},R_{i,1,L}}(\gamma) &\equiv \gamma_{Y_i}|_{Y_{i,i+1,L}} + \alpha_{R_{i,2}} \quad \bmod 2\,, \\ \Phi_{\tilde{\mathcal{X}}_{\tilde{A}},R_{i,r,L}}(\gamma) &\equiv \alpha_{R_{i,r-1}} + \gamma_{Y_{i+1}}|_{Y_{i,i+1,L}} \quad \bmod 2 \end{aligned}$$

by Lemma 4.3(i).

Let us now assume that $\Phi_{\tilde{\mathcal{X}}_{\tilde{A}},R_{i,j,L}}(\gamma) \equiv 0 \bmod 2$ for all $j > 0$ and all i. Pushing the above equations forward via $q_{R_{i,j}} \colon R_{i,j,L} \to Y_{i,i+1,L}$ for $j > 0$, we then obtain

$$\begin{aligned} \gamma_{Y_i}|_{Y_{i,i+1,L}} \equiv \alpha_{R_{i,2}} \equiv \alpha_{R_{i,4}} \equiv \cdots \equiv \alpha_{R_{i,2\lfloor r/2 \rfloor}} &\quad \bmod 2\,, \\ \alpha_{R_{i,r-2\lfloor r/2\rfloor+1}} \equiv \cdots \equiv \alpha_{R_{i,r-3}} \equiv \alpha_{R_{i,r-1}} \equiv \gamma_{Y_{i+1}}|_{Y_{i,i+1,L}} &\quad \bmod 2\,, \\ \alpha_{R_{i,1}} \equiv \alpha_{R_{i,3}} \equiv \cdots \equiv \alpha_{R_{i,2\lceil r/2\rceil -1}} &\quad \bmod 2 \end{aligned} \tag{4.8}$$

in $\mathrm{CH}_0(Y_{i,i+1,L})/2$.

Case 1. The integer r appearing in (4.7) is even.

In this case we will prove that for any $\gamma \in \mathrm{CH}_1(\tilde{Y}_L)$ such that $\Phi_{\tilde{\mathcal{X}}_{\tilde{A}},R_{i,j,L}}(\gamma) \equiv 0 \bmod 2$ for all j and all i, we have that

$$\Phi_{\mathcal{X}_A}(q_*(\gamma)) = q_*\Phi_{\tilde{\mathcal{X}}_{\tilde{A}}}(\gamma) \ \in \bigoplus_i \mathrm{CH}_0(Y_{i,L})/2\,, \tag{4.9}$$

where $q : \tilde{Y} \to Y$ denotes the morphism induced by $\tilde{\mathcal{X}} \to \mathcal{X}$. This clearly implies that if $\Phi_{\tilde{\mathcal{X}}_{\tilde{A}}}$ is surjective modulo 2, then $\Phi_{\mathcal{X}_A}$ is surjective modulo 2.

To prove (4.9), first note that since r is even, (4.8) reads as

$$\begin{aligned} \gamma_{Y_i}|_{Y_{i,i+1,L}} \equiv \alpha_{R_{i,2}} \equiv \alpha_{R_{i,4}} \equiv \cdots \equiv \alpha_{R_{i,r}} &\quad \bmod 2, \\ \alpha_{R_{i,1}} \equiv \alpha_{R_{i,3}} \equiv \cdots \equiv \alpha_{R_{i,r-1}} \equiv \gamma_{Y_{i+1}}|_{Y_{i,i+1,L}} &\quad \bmod 2 \end{aligned} \tag{4.10}$$

in $\mathrm{CH}_0(Y_{i,i+1,L})/2$. Furthermore, we note that the morphism $q \colon \tilde{Y} \to Y$ is the identity morphism onto its image when restricted to the components Y_i and is the projection $q_{R_{i,j}} \colon R_{i,j} \to Y_{i,i+1}$ when restricted to the components $R_{i,j}$. This, together with the assumption $\Phi_{\tilde{\mathcal{X}}_{\tilde{A}},R_{i,j,L}}(\gamma) \equiv 0 \bmod 2$ for all j and all i, implies that

$$q_*\Phi_{\tilde{\mathcal{X}}_{\tilde{A}}}(\gamma) = \sum_{i=1}^{n} \alpha_{R_{i-1,r}} - \gamma_{Y_i}|_{Y_{i-1,i,L}} - \gamma_{Y_i}|_{Y_{i,i+1,L}} + \alpha_{R_{i,1}} \in \bigoplus_i \mathrm{CH}_0(Y_{i,L})/2\,,$$

where we used Lemma 4.3(i). (Here, $\alpha_{R_{0,r}}$, $\gamma_{Y_1}|_{Y_{0,1,L}}$, $\gamma_{Y_n}|_{Y_{n,n+1,L}}$ and $\alpha_{R_{n,1}}$ are defined to be 0.)

Using the relations in (4.10), we can then further rewrite this equation as

$$q_*\Phi_{\widetilde{\mathcal{X}}_{\tilde{A}}}(\gamma) \equiv \sum_{i=1}^{n} \gamma_{Y_{i-1}}|_{Y_{i-1,i,L}} - \gamma_{Y_i}|_{Y_{i-1,i,L}} - \gamma_{Y_i}|_{Y_{i,i+1,L}} + \gamma_{Y_{i+1}}|_{Y_{i,i+1,L}} \mod 2\,,$$
$$\equiv \sum_{i=1}^{n} \Phi_{\mathcal{X}_A}(\gamma_{Y_i}) \equiv \Phi_{\mathcal{X}_A}(q_*\gamma) \mod 2\,,$$

where in the second equation we used Lemma 4.3(i) again ($\gamma_{Y_0}|_{Y_{0,1,L}}$ and $\gamma_{Y_{n+1}}|_{Y_{n,n+1,L}}$ are set to be 0 here). This proves (4.9) and hence concludes the proof of the proposition in the case where r is even.

Case 2. The integer r appearing in (4.7) is odd.

We assume that $\Phi_{\widetilde{\mathcal{X}}_{\tilde{A}}}$ is surjective modulo 2 and that r is odd. We then need to show that $\Phi_{\mathcal{X}_A}$ is surjective modulo 2. A simple formula analogous to (4.9) does not seem to hold in this case. Instead, our argument relies on the claim that since r is odd and $\Phi_{\widetilde{\mathcal{X}}_{\tilde{A}}}$ is surjective modulo 2, we have that

$$\Psi : \bigoplus_{i=1}^{n} \mathrm{CH}_1(Y_{i,L}) \longrightarrow \bigoplus_{i=1}^{n-1} \mathrm{CH}_0(Y_{i,i+1,L})\,, \quad \sum_i \gamma_i \longmapsto \sum_i \gamma_i|_{Y_{i,i+1,L}} + \gamma_{i+1}|_{Y_{i,i+1,L}} \tag{4.11}$$

is surjective modulo 2. Indeed, let $\alpha' = \alpha'_1 + \ldots + \alpha'_{n-1}$ be in $\bigoplus_i \mathrm{CH}_0(Y_{i,i+1,L})$ with α'_i in $\mathrm{CH}_0(Y_{i,i+1,L})$. Denote by $\iota_i : Y_{i,i+1,L} \to Y_{i,L}$ the natural embedding, and let $\iota_{i,j} : Y_{i,i+1,L} \to R_{i,j,L}$ denote the embeddings $Y_{i,L} \cap R_{i,1,L} \subset R_{i,1,L}$ for $j = 1$ and $R_{i,j-1,L} \cap R_{i,j,L} \subset R_{i,j,L}$ for $1 < j \leqslant r$. We then consider

$$z := \sum_{i=1}^{n-1} \iota_{i*}\alpha'_i - \iota_{i,1*}\alpha'_i \in \bigoplus_{i=1}^{n-1} \mathrm{CH}_0(Y_{i,L}) \oplus \mathrm{CH}_0(R_{i,1,L}),$$

which by (4.7) is an element in the direct sum of the Chow groups of zero-cycles of the components of $\tilde{Y}_L$. It is clear from the definition of z that $\deg(z) = 0$. Hence, by the surjectivity of $\Phi_{\widetilde{\mathcal{X}}_{\tilde{A}}}$ modulo 2, there is a class

$$\bar{\gamma} = \bar{\gamma}_{Y_1} + \sum_{j=1}^{r} q^*_{R_{1,j}}\bar{\alpha}_{R_{1,j}} + \bar{\gamma}_{Y_2} + \sum_{j=1}^{r} q^*_{R_{2,j}}\bar{\alpha}_{R_{2,j}} + \ldots + \bar{\gamma}_{Y_n} \in \mathrm{CH}_1(\tilde{Y}_L)/2$$

with $\bar{\gamma}_{Y_i} \in \mathrm{CH}_1(Y_{i,L})/2$ and $\bar{\alpha}_{R_{i,j}} \in \mathrm{CH}_0(Y_{i,i+1,L})/2$ such that $\Phi_{\widetilde{\mathcal{X}}_{\tilde{A}},R_{i,1}}(\bar{\gamma}) \equiv -\iota_{i,1*}\alpha'_i \mod 2$ and $\Phi_{\widetilde{\mathcal{X}}_{\tilde{A}},R_{i,j}}(\bar{\gamma}) \equiv 0 \mod 2$ for $j > 1$ an all i.

Using a similar computation as in (4.8) and Lemma 4.3(ii), we then see that

$$-\alpha'_i \equiv \bar{\gamma}_{Y_i}|_{Y_{i,i+1,L}} + \bar{\alpha}_{R_{i,2}} \mod 2\,,$$
$$\bar{\alpha}_{R_{i,2}} \equiv \bar{\alpha}_{R_{i,4}} \equiv \ldots \equiv \bar{\alpha}_{R_{i,r-1}} \mod 2\,,$$
$$-\bar{\alpha}_{R_{i,r-1}} \equiv \bar{\gamma}_{Y_{i+1}}|_{Y_{i,i+1,L}} \mod 2$$

in $\mathrm{CH}_0(Y_{i,i+1,L})/2$, where we used the assumption that r is odd.

Combining these equations, we obtain that $\Psi(\sum \bar{\gamma}_{Y_i}) = \alpha'$ modulo 2, and so we have showed that Ψ from (4.11) is surjective modulo 2, as we want.

Let us now show that $\Phi_{\mathcal{X}_A}$ is surjective modulo 2. Let $\beta_i \in \mathrm{CH}_0(Y_{i,L})/2$ for $i = 1, \ldots, n$ with

$\sum_i \deg(\beta_i) \equiv 0 \mod 2$. Since $\Phi_{\widetilde{\mathcal{X}}_{\tilde{A}}}$ is surjective modulo 2, there is a class $\gamma \in \mathrm{CH}_1(\tilde{Y}_L)$ with

$$\Phi_{\widetilde{\mathcal{X}}_{\tilde{A}}, Y_{i,L}}(\gamma) = \beta_i \in \mathrm{CH}_0(Y_{i,L})/2$$

for all $i = 1, \ldots, n$ and

$$\Phi_{\widetilde{\mathcal{X}}_{\tilde{A}}, R_{i,j,L}}(\gamma) = 0 \in \mathrm{CH}_0(R_{i,j,L})/2$$

for all $i = 1, \ldots, n-1$ and $j = 1, \ldots, r$.

As before, we can write

$$\gamma = \gamma_{Y_1} + \sum_{j=1}^{r} q^*_{R_{1,j}} \alpha_{R_{1,j}} + \gamma_{Y_2} + \sum_{j=1}^{r} q^*_{R_{2,j}} \alpha_{R_{2,j}} + \ldots + \gamma_{Y_n}$$

for some $\gamma_{Y_i} \in \mathrm{CH}_1(Y_{i,L})$ and $\alpha_{R_{i,j}} \in \mathrm{CH}_0(Y_{i,i+1,L})$. Then, as r is odd, (4.8) implies that

$$\gamma_{Y_i}|_{Y_{i,i+1,L}} \equiv \alpha_{R_{i,2}} \equiv \alpha_{R_{i,4}} \equiv \ldots \equiv \alpha_{R_{i,r-1}} \equiv \gamma_{Y_{i+1}}|_{Y_{i,i+1,L}} \mod 2\,, \tag{4.12}$$

$$\alpha_{R_{i,1}} \equiv \alpha_{R_{i,3}} \equiv \ldots \equiv \alpha_{R_{i,r-2}} \equiv \alpha_{R_{i,r}} \mod 2 \tag{4.13}$$

in $\mathrm{CH}_0(Y_{i,i+1,L})/2$ for $i = 1, \ldots, n-1$. Moreover, $\beta_i = \Phi_{\widetilde{\mathcal{X}}_{\tilde{A}}, Y_{i,L}}(\gamma)$ modulo 2 means that

$$\beta_1 = \gamma_{Y_1}|_{Y_{1,2,L}} + \alpha_{R_{1,1}} \in \mathrm{CH}_0(Y_{1,L})/2\,, \tag{4.14}$$

$$\begin{aligned} \beta_n &= \gamma_{Y_n}|_{Y_{n-1,n,L}} + \alpha_{R_{n-1,r}} \\ &= \gamma_{Y_n}|_{Y_{n-1,n,L}} + \alpha_{R_{n-1,1}} \in \mathrm{CH}_0(Y_{n,L})/2 \end{aligned} \tag{4.15}$$

and

$$\begin{aligned} \beta_i &= \gamma_{Y_i}|_{Y_{i-1,i,L}} + \gamma_{Y_i}|_{Y_{i,i+1,L}} + \alpha_{R_{i-1,r}} + \alpha_{R_{i,1}} \\ &= \gamma_{Y_i}|_{Y_{i-1,i,L}} + \gamma_{Y_i}|_{Y_{i,i+1,L}} + \alpha_{R_{i-1,1}} + \alpha_{R_{i,1}} \in \mathrm{CH}_0(Y_{i,L})/2 \end{aligned} \tag{4.16}$$

for all $2 \leqslant i \leqslant n-1$. Note that we used (4.13) in the second equation of (4.15) and (4.16).

By the surjectivity of Ψ in (4.11), we get classes $\gamma'_{Y_i} \in \mathrm{CH}_1(Y_{i,L})$ with

$$\alpha_{R_{i,1}} = \gamma'_{Y_i}|_{Y_{i,i+1,L}} + \gamma'_{Y_{i+1}}|_{Y_{i,i+1,L}} \in \mathrm{CH}_0(Y_{i,i+1,L})/2 \tag{4.17}$$

for all $i = 1, \ldots, n-1$. Combining this with (4.14)–(4.16), we find

$$\gamma_{Y_1}|_{Y_{1,2,L}} + \gamma'_{Y_1}|_{Y_{1,2,L}} + \gamma'_{Y_2}|_{Y_{1,2,L}} = \beta_1 \in \mathrm{CH}_0(Y_{1,L})/2\,,$$
$$\gamma_{Y_n}|_{Y_{n-1,n,L}} + \gamma'_{Y_{n-1}}|_{Y_{n-1,n,L}} + \gamma'_{Y_n}|_{Y_{n-1,n,L}} = \beta_n \in \mathrm{CH}_0(Y_{n,L})/2$$

and

$$\begin{aligned} \gamma_{Y_i}|_{Y_{i-1,i,L}} + \gamma_{Y_i}|_{Y_{i,i+1,L}} + \gamma'_{Y_{i-1}}|_{Y_{i-1,i,L}} + \gamma'_{Y_i}|_{Y_{i-1,i,L}} + \gamma'_{Y_i}|_{Y_{i,i+1,L}} + \gamma'_{Y_{i+1}}|_{Y_{i,i+1,L}} \\ = \beta_i \in \mathrm{CH}_0(Y_{i,L})/2 \end{aligned}$$

for $2 \leqslant i \leqslant n-1$. Now let

$$\gamma' = \sum_{i=1}^{n} \gamma'_{Y_i} \in \mathrm{CH}_1(Y_L)/2 \quad \text{and} \quad \gamma^{\mathrm{even}} := \sum_{j=1}^{\lfloor n/2 \rfloor} \gamma_{Y_{2j}} \in \mathrm{CH}_1(Y_L)/2\,.$$

Using the relation $\gamma_{Y_i}|_{Y_{i,i+1,L}} = \gamma_{Y_{i+1}}|_{Y_{i,i+1,L}}$ modulo 2 from (4.12), we then conclude from the above computation that

$$\Phi_{\mathcal{X}_A, Y_{i,L}}(\gamma' + \gamma^{\mathrm{even}}) = \beta_i \in \mathrm{CH}_0(Y_{i,L})/2$$

for all $i = 1, \dots, n$. Hence, $\Phi_{\mathcal{X}_A}$ is surjective modulo 2, as we want. This concludes the proof of the proposition. □

4.3 Proof of Theorem 4.1

Proof of Theorem 4.1. Since $\Phi_{\mathcal{X}}$ depends only on the special fibre (see Lemma 3.2) and because the base change of $\mathcal{X}$ to the completion of R remains a strictly semi-stable family, we may replace R with its completion and assume that R is complete. As the geometric generic fibre $X_{\bar{K}}$ of $\mathcal{X} \to \operatorname{Spec} R$ has a decomposition of the diagonal, it follows that there is a finite field extension $F \supset K$ such that X_F has a decomposition of the diagonal.

Now let R_F be the integral closure of R in F. Since R is complete, R_F is again a dvr. We consider the strictly semi-stable model $\tilde{\mathcal{X}} \to \operatorname{Spec} R_F$ constructed as in (4.6), with special fibre $\tilde{Y}$ as described in (4.7) and with the induced morphism $\tilde{Y} \to Y$.

The residue field k of R is algebraically closed by our assumptions. It follows that the residue field of R_F is given by k as well. Let A/R be any unramified extension of dvr's with induced extension L/k of residue fields. We denote by $\tilde{\mathcal{X}}_{\tilde{A}}$ the base change of $\tilde{\mathcal{X}}$ to an unramified extension $\tilde{A}$ of R_F that induces L/k on residue fields (see Lemma 4.2). By Proposition 3.3, the map $\Phi_{\tilde{\mathcal{X}}_{\tilde{A}}}$ is surjective, hence surjective modulo 2. It then follows from Proposition 4.4 that $\Phi_{\mathcal{X}_A}$ is surjective modulo 2 as well, as we want. This concludes the proof of the theorem. □

4.4 Proofs of Theorem 1.2 and Corollary 1.3

Proof of Theorem 1.2. As aforementioned, item (i) of Theorem 1.2 follows from Proposition 3.3, and item (ii) follows from Theorem 4.1. □

Proof of Corollary 1.3. The main strategy here is to argue by contradiction. Indeed, we are going to assume that the geometric generic fibre has a decomposition of the diagonal, and we want to conclude that Y_{12} has odd torsion order, which contradicts the assumptions of the corollary.

First note that the Chow group of Y_{12} does not appear in the target of $\Phi_{\mathcal{X}}$, and so we need to modify our strictly semi-stable family. More concretely, we perform a $2:1$ base change and blow up Y_{12} to arrive at a strictly semi-stable model $\mathcal{X}' \to \operatorname{Spec} R'$ with special fibre $Y' = Y_1 \cup P \cup Y_2$, where $q\colon P \to Y_{12}$ is a $\mathbb{P}^1$-bundle that meets Y_1 and Y_2 along disjoint sections. This implies $\mathrm{CH}_1(Y'_L) \simeq \mathrm{CH}_1(Y_L) \oplus q^* \mathrm{CH}_0((Y_{12})_L)$ for any field extension L/k. By the construction of the above map, $\Phi_{\mathcal{X}',P}$ is zero modulo 2 on $q^* \mathrm{CH}_0((Y_{12})_L)$. Since $\mathrm{CH}_1(Y) \to \mathrm{CH}_1(Y_L)$ is surjective by assumption, we conclude that for any unramified extension A/R' of dvr's,

$$\operatorname{im}(\Phi_{\mathcal{X}'_A,P_L}) \equiv \operatorname{im}(\Phi_{\mathcal{X}',P}) \mod 2\,. \tag{4.18}$$

If the geometric generic fibre of π admits a decomposition of the diagonal, then, by Theorem 1.2, the map $\Phi_{\mathcal{X}'_A,P_L}$ is surjective modulo 2 for A the local ring of $\mathcal{X}'$ at the generic point of P. By (4.18), this implies that up to some multiples of 2, the point δ_P is contained in $\operatorname{im}(\mathrm{CH}_0(P) \to \mathrm{CH}_0(P_{k(P)})$ and so P has odd torsion order (see Lemma 2.1). Since the torsion order is a stable birational invariant of smooth projective varieties, we find that Y_{12} has odd torsion order as well, which contradicts our assumptions. □

Remark 4.5. The above proof shows that Corollary 1.3 remains true if for any field extension L/k, the map $\mathrm{CH}_1(Y) \to \mathrm{CH}_1(Y_L)$ is surjective modulo 2.

5. Very general quartic fivefolds have no decomposition of the diagonal

5.1 Overview

We aim to write down a smooth quartic fivefold $X \subset \mathbb{P}^6$ which has no decomposition of the diagonal. We sketch our construction here, before we give the technical details below.

As in [NO22, Theorem 5.1], we start by degenerating a general quartic fivefold to a union $Y_1 \cup Y_2$ of general quartic double covers of $\mathbb{P}^5$ (see Section 5.2). The two components Y_1 and Y_2 are isomorphic to each other via an exchange of variables, and they intersect each other in a general quartic double cover $Y_1 \cap Y_2 = Z$ of $\mathbb{P}^4$, which degenerates to the double quartic fourfold $Z_0 \to \mathbb{P}^4$ studied by Hassett–Pirutka–Tschinkel [HPT19]. The total space $\mathcal{X}$ of this degeneration is singular along a codimension 1 subvariety S of Z.

The blow-up $\mathcal{X}' := \mathrm{Bl}_{Y_2} \mathcal{X}$ resolves the singularities of the total space, and its special fibre is the union $Y_1 \cup \tilde{Y}_2$, where $\tilde{Y}_2$ is the blow-up of Y_2 along S. Moreover, this family is strictly semi-stable. A further $2:1$ base change, followed by a blow-up along $Z \simeq Y_1 \cap \tilde{Y}_2$, gives a strictly semi-stable family $\tilde{\mathcal{X}}$ whose generic fibre is a general quartic fivefold and whose special fibre is a union $\tilde{X}_0 = Y_1 \cup P_Z \cup \tilde{Y}_2$, where P_Z is a $\mathbb{P}^1$-bundle over Z (see Lemma 5.5).

Let A be the local ring of $\tilde{\mathcal{X}}$ at the generic point of P_Z, with residue field $\kappa(P_Z)$, where we recall our convention from Section 2.1 that the function field of an integral scheme X over a field is denoted by $\kappa(X)$ whenever we prefer to make the ground field in our notation not explicit. Then $\tilde{\mathcal{X}}_A \to \operatorname{Spec} A$ is strictly semi-stable, and we get a map

$$\Phi_{\tilde{\mathcal{X}}_A, P_Z} \colon \mathrm{CH}_1\left(\tilde{X}_0 \times \kappa(P_Z)\right) \longrightarrow \mathrm{CH}_0(P_Z \times \kappa(P_Z))\,;$$

see Definition 3.1. The main technical result of this section is then the assertion that for a zero-cycle $z \in \mathrm{CH}_0(P_Z)$ of degree 1, the zero-cycle

$$\delta_{P_Z} - z_{\kappa(P_Z)} \in \mathrm{CH}_0(P_Z \times \kappa(P_Z)) \tag{5.1}$$

is not in the image of $\Phi_{\tilde{\mathcal{X}}_A}$ modulo 2; see Proposition 5.7. By Theorem 4.1, this implies that the geometric generic fibre of $\tilde{\mathcal{X}} \to \operatorname{Spec} R$ has no decomposition of the diagonal, as we want.

We assume that (5.1) is contained in the image of $\Phi_{\tilde{\mathcal{X}}_A}$ modulo 2 and aim to find a contradiction. The strategy is to repeatedly apply Fulton's specialization map on Chow groups (in the form of Lemma 5.8 below) to simplify the contribution from $\mathrm{CH}_1\left(\tilde{X}_0 \times \kappa(P_Z)\right)$. The goal will be to finally arrive at the conclusion that the diagonal point $\delta_{Z_0} \in \mathrm{CH}_0(Z_0 \times \kappa(Z_0))$ satisfies

$$\delta_{Z_0} \in \mathrm{im}(\mathrm{CH}_0(Z_0) \to \mathrm{CH}_0(Z_0 \times \kappa(Z_0))) \mod 2\,.$$

We will show (see Lemma 5.13) that this contradicts some properties of the non-trivial unramified cohomology class with $\mathbb{Z}/2$-coefficients on Z_0 from [HPT18, HPT19].

5.2 A strictly semi-stable family

Let k_0 be an algebraically closed field of characteristic different from 2, and let

$$k := \overline{k_0(\lambda, u, v, s)}$$

be the algebraic closure of purely transcendental extension of k_0 of degree 4.

Our construction relies on the following choices.

Definition 5.1. Let $f, g \in k_0[z_0, \ldots, z_4]$ be general homogeneous polynomials with $\deg(f) = 4$ and $\deg(g) = 3$.

(i) Let

$$f_s := sf + f_0 \in k[z_0, \dots, z_4],$$

where

$$f_0 := z_0z_1z_3^2 + z_0z_2z_4^2 + z_1z_2\big(z_0^2 + z_1^2 + z_2^2 - 2(z_0z_1 + z_0z_2 + z_1z_2)\big). \tag{5.2}$$

(ii) Let

$$g_u := ug + z_2^3 \in k[z_0, \dots, z_4]. \tag{5.3}$$

(iii) For f_s and g_u as above, we define

$$F := F_{v,u,s} := v\big(z_5^4 + z_6^4\big) + (z_5 + z_6)g_u + f_s \in k[z_0, \dots, z_6], \tag{5.4}$$

which is symmetric in z_5 and z_6.

LEMMA 5.2. *Let f_s and g_u be as in Definition 5.1. Then*

$$\{f_s = 0\} \subset \mathbb{P}^4_k, \quad \{g_u = 0\} \subset \mathbb{P}^4_k \quad \text{and} \quad \{f_s = g_u = 0\} \subset \mathbb{P}^4_k$$

are smooth complete intersections. Moreover,

$$D_1 := \big\{vz_5^4 + z_5g_u + f_s = 0\big\} \subset \mathbb{P}^5_k \tag{5.5}$$

is smooth, and

$$\bar{D}_1 := \{z_5g_u + f_s = 0\} \subset \mathbb{P}^5_k \tag{5.6}$$

has a singularity of multiplicity 3 at $p = [0 : \dots : 0 : 1]$ and is smooth away from p.

Proof. It suffices to prove the assertion after specializing $s \to \infty$ and $u \to \infty$; that is, we may replace f_s with f and g_u with g. Since f and g are general, we thus conclude from Bertini's theorem (which works over infinite fields) that $\{f_s = 0\} \subset \mathbb{P}^4_k$ and $\{g_u = 0\} \subset \mathbb{P}^4_k$ are smooth hypersurfaces that meet in a smooth complete intersection $\{f_s = g_u = 0\} \subset \mathbb{P}^4_k$. This proves the first assertion of the lemma; the rest follows easily from this. □

Let $R := k[[t]]$, and let $\mathbb{P}_R(1^7, 2) := \operatorname{Proj}(R[z_0, \dots, z_6, w])$ be the weighted projective space such that the variables z_i have weight 1 and the variable w has weight 2. Consider the R-scheme

$$\mathcal{X} := \big\{\big(\lambda w - z_2^2\big)t - z_5z_6 = 0,\ w^2 - F = 0\big\} \subset \mathbb{P}_R\big(1^7, 2\big), \tag{5.7}$$

where F is as in (5.4). The R-scheme $\mathcal{X}$ is flat over R, and the generic fibre X_K is given by the equation

$$X_K = \big\{\lambda^{-2}\big(t^{-1}z_5z_6 + z_2^2\big)^2 - v\big(z_5^4 + z_6^4\big) - (z_5 + z_6)g_u - f_s = 0\big\} \subset \mathbb{P}^6_K,$$

which is a smooth quartic fivefold. (We note that $\mathcal{X}$ is not regular; in fact, the Weil divisors Y_1 and Y_2 are not Cartier.)

The special fibre X_0 of $\mathcal{X}$ has two irreducible components, $X_0 = Y_1 \cup Y_2$, such that

$$Y_i = \big\{w^2 - F = 0, z_{7-i} = 0\big\} = \big\{w^2 - vz_{i+4}^4 - z_{i+4}g_u - f_s = 0\big\} \subset \mathbb{P}_k\big(1^6, 2\big) \tag{5.8}$$

for $i = 1, 2$ is a smooth double quartic fivefold $Y_i \to \mathbb{P}^5_{[z_0:\dots:z_4:z_{4+i}]}$. Using the symmetry of F in z_5 and z_6, we get a canonical isomorphism $Y_1 \simeq Y_2$. Moreover, the intersection $Z := Y_1 \cap Y_2 \subset \mathbb{P}\big(1^5, 2\big)$ is the double quartic fourfold $Z \to \mathbb{P}^4$ given by the equation

$$Z := \big\{w^2 - f_s = 0\big\} \subset \mathbb{P}_k\big(1^5, 2\big). \tag{5.9}$$

Since $\{f_s = 0\}$ is smooth, so is Z.

LEMMA 5.3. *The singular locus of $\mathcal{X}$ in (5.7) is given by*

$$S = \left\{t = z_5 = z_6 = \lambda w - z_2^2 = w^2 - f_s = 0\right\} \subset \mathcal{X}\,, \tag{5.10}$$

which is smooth over k. Moreover, $\mathcal{X}$ has ordinary quadratic singularities of codimension 3 along S.

Proof. As described above, the special fibre X_0 is given by the union $Y_1 \cup Y_2$ of two smooth projective varieties. It follows that $\mathcal{X}$ is regular outside of $Y_1 \cap Y_2$, and so the singular locus S is contained in $\{t = z_5 = z_6 = 0\}$. Considering the Jacobian matrix of $\mathcal{X}$, the Jacobian criterion then easily shows that S is given as claimed in (5.10). Since $\lambda \in k$ is non-zero, S is isomorphic to

$$\left\{\lambda^{-2} z_2^4 - f_s = 0\right\} \subset \mathbb{P}^4_k\,,$$

which is smooth over k because $f_s = sf + f_0$ with f general.

Let $p \in S$. Since $S \subset \{z_5 = z_6 = 0\}$, the z_5- and z_6-coordinates of p vanish, and so at least one of the z_i-coordinates with $i = 0, \ldots, 4$ is non-zero. Since $\{f_s = 0\} \subset \mathbb{P}^4$ is smooth, we further see that for $i = 0, \ldots, 4$, at least one partial $\partial_{z_i} F = \partial_{z_i} f_s$ does not vanish at p (if they all vanished simultaneously, then f_s would vanish at p by the Leibniz identity, contradicting the smoothness of $\{f_s = 0\} \subset \mathbb{P}^4$). It follows that the tangent space of $\left\{w^2 - F = 0\right\}$ at p intersects the tangent cone of $\left\{\left(\lambda w - z_2^2\right)t - z_5 z_6 = 0\right\}$ at p transversely. The latter is Zariski locally isomorphic to the tangent cone of the ordinary quadratic singularity $\{xt - yz = 0\}$, thus proving the claim in the lemma. □

LEMMA 5.4. *Let $\mathcal{X}$ be the $R = k[[t]]$-scheme as defined in (5.7), let $K := k((t))$, and let Y_1, Y_2 be the components of the special fibre X_0 of $\mathcal{X}$. Then $\mathcal{X}' := \mathrm{Bl}_{Y_2}\mathcal{X}$ is strictly semi-stable with special fibre $Y_1 \cup \tilde{Y}_2$, where $\tilde{Y}_2 = \mathrm{Bl}_S Y_2$, and $Y_1 \cap \tilde{Y}_2 = \mathrm{Bl}_S Z = Z$, where $Z = Y_1 \cap Y_2$.*

Proof. Note that $Y_1 \to \mathbb{P}^5$ is a double cover branched along D_1. By Lemma 5.2, the divisor D_1 is smooth, and so Y_1 is smooth. Since $Y_2 \simeq Y_1$, the same holds true for Y_2. We have seen above that the singular locus S of $\mathcal{X}$ is also smooth. Locally at a point of S, the scheme $\mathcal{X}$ has ordinary quadratic singularities of codimension 3 (see Lemma 5.3), and a local computation shows that the special fibre of $\mathcal{X}'$ is given by $Y_1 \cup \tilde{Y}_2$, where $\tilde{Y}_2 = \mathrm{Bl}_S Y_2$. Since Y_2 and S are smooth, so is $\tilde{Y}_2$. Moreover, $Y_1 \cap \tilde{Y}_2 = \mathrm{Bl}_S Z = Z$, where the second equality comes from the fact that $S \subset Z$ is a divisor and Z is smooth. By construction, $\tilde{Y}_2$ is a Cartier divisor of $\mathcal{X}$; since the components of X_0' are reduced and X_0 is Cartier, we find that Y_1 is Cartier as well. Since Y_1, $\tilde{Y}_2$ and $Y_1 \cap \tilde{Y}_2 = Z$ are smooth and the components of X_0' are Cartier, it follows that $\mathcal{X}'$ is strictly semi-stable, as we want. □

LEMMA 5.5. *In the notation of Lemma 5.4, let $\mathcal{X}'' = \mathcal{X}' \times_{\substack{R \to R \\ t \to t^2}} R$. Then*

$$\tilde{\mathcal{X}} := \mathrm{Bl}_Z \mathcal{X}'' \to \operatorname{Spec} R \tag{5.11}$$

is a strictly semi-stable R-scheme with special fibre $\tilde{X}_0 = Y_1 \cup P_Z \cup \tilde{Y}_2$, where $\tilde{Y}_2$ is the blow-up of Y_2 along S and P_Z is a $\mathbb{P}^1_\kappa$-bundle over Z. The intersections $Y_1 \cap P_Z$ and $P_Z \cap \tilde{Y}_2$ are disjoint sections of $P_Z \to Z$. The generic fibre

$$\tilde{X}_K = \left\{\lambda^{-2}\left(t^{-2} z_5 z_6 + z_2^2\right)^2 - v\left(z_5^4 + z_6^4\right) - (z_5 + z_6) g_u - f_s = 0\right\} \subset \mathbb{P}^6_K \tag{5.12}$$

of $\tilde{\mathcal{X}}$ is a smooth quartic fivefold.

Proof. By Lemma 5.4, the map $\mathcal{X}' \to \operatorname{Spec} R$ is strictly semi-stable. The $2:1$ base change $\mathcal{X}''$ is thus regular away from the singular locus Z of the central fibre, and it has ordinary double point singularities along Z (because étale locally at the non-smooth locus, $\mathcal{X}'$ is given by $t = xy$, and so $\mathcal{X}''$ is given by $t^2 = xy$; cf. [Har01, Proposition 1.3]). Those singularities are resolved by the blow-up of Z, and the corresponding exceptional divisor will be a reduced component of the special fibre; see for example [Har01, Proposition 2.2]. Hence, $\tilde{\mathcal{X}}$ is regular, and the special fibre is given by $Y_1 \cup P_Z \cup \tilde{Y}_2$, where $P_Z \to Z$ is a smooth conic bundle. Moreover, $Y_1 \cap P_Z$ (as well as $\tilde{Y}_2 \cap P_Z$) is a section of $P_Z \to Z$, and so $P_Z \to Z$ is a Zariski locally trivial $\mathbb{P}^1$-bundle, as claimed. This proves the lemma. □

5.3 The main result

Theorem 5.6. *Let $\overline{K}$ be the algebraic closure of the fraction field K of R. Then the smooth quartic fivefold $\tilde{X}_{\overline{K}} \subset \mathbb{P}^6_{\overline{K}}$ given by the base change of (5.12) to $\overline{K}$ does not admit a decomposition of the diagonal.*

Proof. We aim to deduce Theorem 5.6 from Theorem 4.1. To this end, let $A = \mathcal{O}_{\tilde{\mathcal{X}},P_Z}$ with residue field $\kappa(P_Z)$. Then $R \to A$ is an unramified extension of dvr's, and so it follows from Lemma 5.5 that $\tilde{\mathcal{X}}_A \to \operatorname{Spec} A$ is strictly semi-stable. By Definition 3.1, we get a map

$$\Phi_{\tilde{\mathcal{X}}_A,P_Z} \colon \operatorname{CH}_1\left(\tilde{X}_0 \times \kappa(P_Z)\right) \longrightarrow \operatorname{CH}_0(P_Z \times \kappa(P_Z)).$$

By Theorem 4.1, Theorem 5.6 follows from Proposition 5.7 below, which is the main technical result of this section. □

Proposition 5.7. *Let $\tilde{\mathcal{X}} \to \operatorname{Spec} R$ be as in (5.11), and let $A = \mathcal{O}_{\tilde{\mathcal{X}},P_Z}$ with residue field $\kappa(P_Z)$. Then for any zero-cycle $z \in \operatorname{CH}_0(P_Z)$, the element*

$$\delta_{P_Z} - z_{\kappa(P_Z)} \in \operatorname{CH}_0(P_Z \times \kappa(P_Z))/2 \tag{5.13}$$

is not in the image of $\Phi_{\tilde{\mathcal{X}}_A,P_Z}$ modulo 2, where δ_{P_Z} denotes the diagonal point of $P_Z \times \kappa(P_Z)$.

Proof. Recall from Lemma 5.5 that the special fibre of $\tilde{\mathcal{X}} \to \operatorname{Spec} R$ is given by $Y_1 \cup P_Z \cup \tilde{Y}_2$, where $\tilde{Y}_2 \to Y_2$ is the blow-up along the smooth subvariety $S \subset Z \subset Y_2$. Since blow-ups commute with extensions of the base field, the blow-up formula for Chow groups yields a canonical isomorphism

$$\operatorname{CH}_1(Y_2 \times \kappa(P_Z)) \oplus \operatorname{CH}_0(S \times \kappa(P_Z)) \simeq \operatorname{CH}_1\left(\tilde{Y}_2 \times \kappa(P_Z)\right).$$

Since P_Z is a $\mathbb{P}^1$-bundle over Z, we have $\operatorname{CH}_0(Z\times\kappa(P_Z))\oplus\operatorname{CH}_1(Z\times\kappa(P_Z)) \simeq \operatorname{CH}_1(P_Z\times\kappa(P_Z))$. Since $Y_1 \cap P_Z$ is a section of $P_Z \to Z$, the contribution of $\operatorname{CH}_1(Z \times \kappa(P_Z))$ to $\operatorname{CH}_1\left(\tilde{X}_0 \times \kappa(P_Z)\right)$ is absorbed by $\operatorname{CH}_1(Y_1 \times \kappa(P_Z))$, and we get a canonical surjection

$$\operatorname{CH}_1(Y_1\times\kappa(P_Z))\oplus\operatorname{CH}_0(Z\times\kappa(P_Z))\oplus\operatorname{CH}_1(Y_2\times\kappa(P_Z))\oplus\operatorname{CH}_0(S\times\kappa(P_Z)) \twoheadrightarrow \operatorname{CH}_1\left(\tilde{X}_0\times\kappa(P_Z)\right).$$

We aim to compute the image of $\Phi_{\tilde{\mathcal{X}}_A,P_Z}$ modulo 2. By Lemma 4.3(ii), the contribution of $\operatorname{CH}_0(Z \times \kappa(P_Z))$ via $\Phi_{\tilde{\mathcal{X}}_A,P_Z}$ is divisible by 2, and so we may neglect it in what follows. Using the symmetry $Y_1 \simeq Y_2$ together with the fact that Y_1 and $\tilde{Y}_2$ meet P_Z in (disjoint) sections of the $\mathbb{P}^1$-bundle $P_Z \to Z$, we thus conclude that

$$\operatorname{im}(\Phi_{\tilde{\mathcal{X}}_A,P_Z}) = \operatorname{im}\left(\operatorname{CH}_1(Y_1 \times \kappa(P_Z)) \oplus \operatorname{CH}_0(S \times \kappa(P_Z)) \longrightarrow \operatorname{CH}_0(P_Z \times \kappa(P_Z))\right) \mod 2. \tag{5.14}$$

Here the map $\operatorname{CH}_1(Y_1\times\kappa(P_Z)) \to \operatorname{CH}_0(P_Z\times\kappa(P_Z))$ is given by restricting to the Cartier divisor $(Y_1 \cap P_Z) \times \kappa(P_Z)$ and pushing forward the resulting zero-cycle to $P_Z \times \kappa(P_Z)$. Moreover, the map $\operatorname{CH}_0(S \times \kappa(P_Z)) \to \operatorname{CH}_0(P_Z \times \kappa(P_Z))$ is the natural pushforward map, where we use the section of the $\mathbb{P}^1$-bundle $P_Z \to Z$ given by $P_Z \cap \tilde{Y}_2$ to identify $S \subset Z$ with a subscheme of P_Z.

The idea is now to perform some specializations to Y_1, S and P_Z, to make their Chow groups more accessible, so that we can control the image in (5.14). The main technical tool which allows us to perform these specializations is the following result of Fulton.

LEMMA 5.8. *Let B be a discrete valuation ring with fraction field F and residue field L. Let $p\colon \mathcal{X} \to \operatorname{Spec} B$ and $q\colon \mathcal{Y} \to \operatorname{Spec} B$ be a flat proper B-schemes with connected fibres. Denote by X_η, Y_η and X_0, Y_0 the generic and special fibres of p, q, respectively. Assume that there is a component $Y_0' \subset Y_0$ such that $A = \mathcal{O}_{\mathcal{Y},Y_0'}$ is a discrete valuation ring (this holds if Y_0 is reduced along Y_0'), and consider the flat proper A-scheme $\mathcal{X}_A \to \operatorname{Spec} A$ given by base change of π. Then Fulton's specialization map induces a specialization map*

$$\mathrm{sp}\colon \mathrm{CH}_i\left(X_\eta \times_F \overline{F}(Y_\eta)\right) \longrightarrow \mathrm{CH}_i\left(X_0 \times_L \overline{L}(Y_0')\right),$$

where $\overline{F}$ and $\overline{L}$ denote the algebraic closures of F and L, respectively, such that the following holds:

(i) *The map* sp *commutes with pushforwards along proper maps and pullbacks along regular embeddings.*

(ii) *If $\mathcal{X} = \mathcal{Y}$ and X_0 is integral, then $\mathrm{sp}(\delta_{X_\eta}) = \delta_{X_0}$, where $\delta_{X_\eta} \in \mathrm{CH}_0\left(X_\eta \times_F \overline{F}(X_\eta)\right)$ and $\delta_{X_0} \in \mathrm{CH}_0\left(X_0 \times_L \overline{L}(X_0)\right)$ denote the diagonal points.*

Proof. Let n be the relative dimension of $q\colon \mathcal{Y} \to \operatorname{Spec} B$. Consider the flat proper B-scheme $p \times q\colon \mathcal{X} \times_B \mathcal{Y} \to \operatorname{Spec} B$. By [Ful98, § 20.3] (see also [Ful75, Theorem 3.3(b)]), there is a specialization map

$$\mathrm{sp}\colon \mathrm{CH}_{i+n}\left(X_\eta \times_F Y_\eta \times_F \overline{F}\right) \longrightarrow \mathrm{CH}_{i+n}\left(X_0 \times_L Y_0 \times_L \overline{L}\right). \tag{5.15}$$

This map is compatible with respect to pushforwards along proper maps and pullbacks along regular embeddings; see [Ful98, Proposition 20.3]. For a given cycle γ on $X_\eta \times_F Y_\eta \times_F \overline{F}$, the specialization $\mathrm{sp}(\gamma)$ is constructed by first performing a base change so that γ is defined on the generic fibre of $p \times q$. We may then take the closure $\overline{\gamma}$ of γ in the total space $\mathcal{X} \times_B \mathcal{Y}$ and restrict $\overline{\gamma}$ to the special fibre. The cycle $\mathrm{sp}(\gamma)$ is then given by the image of $\overline{\gamma}|_{X_0 \times_L Y_0}$ via the natural map

$$\mathrm{CH}_{i+n}(X_0 \times_L Y_0) \longrightarrow \mathrm{CH}_{i+n}\left(X_0 \times_L Y_0 \times_L \overline{L}\right).$$

(Taking the image via this map is necessary to make the construction well defined because the base change performed above may replace L by a finite extension.)

Pullback of cycles yields a canonical isomorphism

$$\varinjlim_{\emptyset \neq U \subset Y_0'} \mathrm{CH}_{i+n}\left(X_0 \times_L U \times_L \overline{L}\right) \xrightarrow{\simeq} \mathrm{CH}_i\left(X_0 \times_L \overline{L}(Y_0')\right)$$

whose inverse is given by taking closures of cycles. We may thus consider the induced pullback (respectively, localization) map

$$\mathrm{CH}_{i+n}\left(X_0 \times_L Y_0 \times_L \overline{L}\right) \longrightarrow \mathrm{CH}_i\left(X_0 \times_L \overline{L}(Y_0')\right). \tag{5.16}$$

By the localization exact sequence [Ful98, Proposition 1.8], the kernel of (5.16) is generated by cycles on $X_0 \times_L Y_0 \times_L \overline{L}$ that do not dominate $Y_0' \times_L \overline{L}$. Using this together with the above description of the specialization map in (5.15), we find that the composition of (5.15) with (5.16) factorizes through

$$\mathrm{CH}_{i+n}\left(X_\eta \times_F Y_\eta \times_F \overline{F}\right) \longrightarrow \mathrm{CH}_i\left(X_\eta \times_F \overline{F}(Y_\eta)\right)$$

as the kernel of the latter is again generated by cycles that do not dominate $Y_\eta \times_F \overline{F}$. We thus arrive at the specialization map as claimed in the lemma. The compatibility results with respect to proper pushforwards and pullbacks along regular embeddings follow from the respective properties for (5.15), and the claim concerning the image of the diagonal point is clear from the construction. □

Since the specialization map in Lemma 5.8 commutes with pushforwards along proper maps and pullbacks along regular embeddings, we may compute the specialization of (5.14) simply by specializing the involved varieties. The assumption that (5.1) lies in (5.14) modulo 2 then implies that the same holds true after specialization, and we aim to finally arrive at a contradiction after specializing the parameters λ, v, u and s to zero.

Note that Z depends on s but not on λ, v and u, and so we write from now on $Z = Z_s$. Similarly, S depends on s and λ but not on u and v, and so we write $S = S_{s,\lambda}$. Also note that Y_1 depends on v, u and s but not on λ, but we will not need to indicate this in our notation. The required degenerations are captured in the following diagram:

$$\begin{aligned} S_{s,\lambda} \subset Z_s \subset P_{Z_s} &\overset{\lambda \to 0}{\rightsquigarrow} S_s \subset Z_s \subset P_{Z_s}\,, \\ Y_1 \overset{2:1}{\longrightarrow} \mathbb{P}^5 &\overset{v \to 0}{\rightsquigarrow} \bar{Y}_1 \overset{2:1}{\longrightarrow} \mathbb{P}^5\,, \\ T_{s,u} \subset Z_s &\overset{u \to 0}{\rightsquigarrow} T_s \subset Z_s\,, \\ S_s \subset Z_s\,, \quad T_s \subset Z_s &\overset{s \to 0}{\rightsquigarrow} S_0^{\mathrm{red}} = T_0^{\mathrm{red}} \subset Z_0\,. \end{aligned}$$

Here $T_{s,u} \subset Z_s \subset P_{Z_s}$ is a hypersurface of Z_s that depends on s and u and is such that

$$\begin{aligned} &\operatorname{im}\left(\mathrm{CH}_1\left(\bar{Y}_1 \times \kappa(P_{Z_s})\right) \longrightarrow \mathrm{CH}_0(P_{Z_s} \times \kappa(P_{Z_s}))\right) \\ &\qquad\qquad \subset \operatorname{im}\left(\mathrm{CH}_0(T_{u,s} \times \kappa(P_{Z_s})) \longrightarrow \mathrm{CH}_0(P_{Z_s} \times \kappa(P_{Z_s}))\right). \end{aligned}$$

We explain our notation and construction in detail in the following four steps. In each step Lemma 5.8 will be applied to a situation where the special fibre Y_0 of $\mathcal{Y}$ is integral, so that $Y_0' = Y_0$. To simplify the notation, we will not write down the total spaces of our degenerations explicitly but only indicate which parameter (that is λ, v, u or s) is sent to zero. For the same reason, we will use the following convention: if the defining equation of X does not depend on a parameter μ, then we denote the specialization of X via $\mu \to 0$ with the same letter. When applying the above specializations, the ground field $k = \overline{k_0(\lambda, u, v, s)}$ will change in each step in the sense that one has to delete one transcendental parameter each time (note however that the resulting field remains algebraically closed by the construction in Lemma 5.8). To simplify the notation, we will not make this change of the ground field explicit in our notation.

5.3.1 *Step 1.* In the first step, we aim to simplify the contribution from $S = S_{s,\lambda}$ by specializing $\lambda \to 0$. The union $Y_1 \cup P_{Z_s}$ does not depend on λ. On the other hand, $S_{s,\lambda}$ specializes by (5.10) to the hypersurface

$$S_s = \left\{z_5 = z_6 = z_2^2 = w^2 - f_s = 0\right\} \tag{5.17}$$

given as the pullback of the non-reduced plane $z_2^2 = 0$ via the double covering $Z_s \to \mathbb{P}^4$. We thus find by Lemma 5.8 that the image of (5.14) via the specialization $\lambda \to 0$ is given by

$$\operatorname{im}\left(\mathrm{CH}_1(Y_1 \times \kappa(P_{Z_s})) \oplus \mathrm{CH}_0(S_s \times \kappa(P_{Z_s})) \to \mathrm{CH}_0(P_{Z_s} \times \kappa(P_{Z_s}))\right) \mod 2\,. \tag{5.18}$$

5.3.2 *Step 2.* In the second step, we aim to get a hand on $\mathrm{CH}_1(Y_1)$. For this, we degenerate Y_1 via $v \to 0$, so that the double cover $Y_1 \to \mathbb{P}^5$ from (5.8) specializes to a singular double cover

$\bar{Y}_1 \to \mathbb{P}^5$, branched along the quartic $\bar{D}_1 \subset \mathbb{P}^5$ from (5.6). By Lemma 5.2, the quartic $\bar{D}_1$ has a triple point at $p = [0 : \cdots : 0 : 1]$ as its unique singularity. The triple point makes $\bar{Y}_1$ rational, and we will use this to control its first Chow group. Since Z_s as well as S_s do not depend on the parameter v, they specialize smoothly in this step. Hence the image of (4.18) via the specialization $\lambda \to 0$ followed by $v \to 0$ is given by

$$\operatorname{im}\left(\mathrm{CH}_1\left(\overline{Y}_1 \times \kappa(P_{Z_s})\right) \oplus \mathrm{CH}_0(S_s \times \kappa(P_{Z_s})) \to \mathrm{CH}_0(P_{Z_s} \times \kappa(P_{Z_s}))\right) \mod 2\,. \tag{5.19}$$

Let $\hat{Y}_1$ be the blow-up of $\overline{Y}_1$ along p. The projection from p given by $[z_0 : \cdots : z_5] \mapsto [z_0 : \cdots : z_4]$ induces a morphism

$$f \colon \hat{Y}_1 \longrightarrow \mathbb{P}^4\,.$$

Let

$$C_u := \{g_u = 0\} \subset \mathbb{P}^4\,.$$

LEMMA 5.9. *The base change of $f \colon \hat{Y}_1 \to \mathbb{P}^4$ to the open subset $U = \mathbb{P}^4 \setminus C_u$ is a Zariski locally trivial $\mathbb{P}^1$-bundle.*

Proof. We identify $\mathbb{P}^4$ with the hyperplane $H = \{z_5 = 0\} \subset \mathbb{P}^5$. For a point $y \in H$ that is not contained in C_u, the line through p and y in $\mathbb{P}^5$ meets $\bar{D}_1$ in exactly two points: in p with multiplicity 3 and in another point $q(y)$ with multiplicity 1. The fibre $f^{-1}(y)$ then identifies to the double cover of $\mathbb{P}^1$ branched at p and $q(y)$. Hence, $f^{-1}(y)$ is a smooth conic, and the branch point p, which does not depend on y, yields a section of the base change of f to U. Hence, $f^{-1}(U) \to U$ is a smooth conic bundle with a section, and so it identifies to a Zariski locally trivial $\mathbb{P}^1$-bundle. This proves the lemma. ☐

Remark 5.10. In the above notation, one can check that for $y \in C_u$, the point $q(y)$ collides with p, and the fibre $f^{-1}(y)$ is given by two lines that meet in one point, corresponding to the point p. We did not include this description as it will be irrelevant for our argument.

COROLLARY 5.11. *The canonical map $\mathrm{CH}_1\left(f^{-1}(C_u)\right) \to \mathrm{CH}_1\left(\hat{Y}_1\right)$ is universally surjective; that is, it is surjective after any extension of the base field.*

Proof. Since $f^{-1}(U) \to U$ is a $\mathbb{P}^1$-bundle, by Lemma 5.9, the isomorphism $\mathrm{CH}_1\left(f^{-1}(U)\right) \simeq \mathrm{CH}_0(U) \oplus \mathrm{CH}_1(U)$ holds universally, that is, after any extension of the base field. By the localization exact sequence, we get an exact sequence $\mathrm{CH}_0(C_u) \to \mathrm{CH}_0\left(\mathbb{P}^4\right) \to \mathrm{CH}_0(U) \to 0$, which holds again after any extension of the base field. Since C_u contains a rational point (as it is defined over an algebraically closed field), the first arrow in the localization exact sequence is surjective, and so $\mathrm{CH}_0(U) = 0$ holds after any extension of the base field. Similarly, $\mathrm{CH}_1(C_u) \to \mathrm{CH}_1\left(\mathbb{P}^4\right) \to \mathrm{CH}_1(U) \to 0$ is exact after any extension of the base field. Since C_u is a cubic threefold over an algebraically closed field, it contains a line, and so we conclude that $\mathrm{CH}_1(U) = 0$ holds after any extension of the base field. Altogether, we find that $\mathrm{CH}_1\left(f^{-1}(U)\right) = 0$ after any extension of the base field. The result thus follows from the localization exact sequence $\mathrm{CH}_1\left(f^{-1}(C_u)\right) \to \mathrm{CH}_1\left(\hat{Y}_1\right) \to \mathrm{CH}_1\left(f^{-1}(U)\right) \to 0$. This proves the corollary. ☐

Consider (5.19), and recall that the map

$$\mathrm{CH}_1\left(\overline{Y}_1 \times \kappa(P_{Z_s})\right) \longrightarrow \mathrm{CH}_0(P_{Z_s} \times \kappa(P_{Z_s})) \tag{5.20}$$

is given by intersecting a 1-cycle on $\overline{Y}_1 \times \kappa(P_{Z_s})$ with

$$\big(\overline{Y}_1 \cap P_{Z_s}\big) \times \kappa(P_{Z_s}) \simeq Z_s \times \kappa(P_{Z_s})\,.$$

The singular point of $\overline{Y}_1$ does not meet the above intersection. To compute the image of (5.20), we may thus replace $\overline{Y}_1$ with the blow-up $\hat{Y}_1$. Corollary 5.11 then shows that the image of (5.20) is contained in the image of

$$\mathrm{CH}_0(T_{s,u} \times \kappa(P_{Z_s})) \longrightarrow \mathrm{CH}_0(P_{Z_s} \times \kappa(P_{Z_s}))\,,$$

where

$$T_{s,u} := f^{-1}(C_u) \cap Z_s = \big\{w^2 - f_s = g_u = 0\big\} \subset \mathbb{P}\big(1^5, 2\big)\,; \tag{5.21}$$

cf. (5.9). We thus conclude that (5.19) is contained in

$$\mathrm{im}\,\big(\,\mathrm{CH}_0(T_{s,u} \times \kappa(P_{Z_s})) \oplus \mathrm{CH}_0(S_s \times \kappa(P_{Z_s})) \to \mathrm{CH}_0(P_{Z_s} \times \kappa(P_{Z_s}))\big) \mod 2\,. \tag{5.22}$$

5.3.3 *Step 3.* In this step, we specialize $u \to 0$. By the construction of g_u in (5.3), this specializes g_u to $g_0 = z_2^3$, and hence $T_{s,u}$ specializes to

$$T_s := \big\{w^2 - f_s = z_2^3 = 0\big\} \subset \mathbb{P}\big(1^5, 2\big)\,.$$

Applying the specialization map from Lemma 5.8 that corresponds to $u \to 0$ to (5.22), we then get

$$\mathrm{im}\,\big(\,\mathrm{CH}_0(T_s \times \kappa(P_{Z_s})) \oplus \mathrm{CH}_0(S_s \times \kappa(P_{Z_s})) \to \mathrm{CH}_0(P_{Z_s} \times \kappa(P_{Z_s}))\big) \mod 2\,.$$

Comparing the above description of T_s with that of S_s in (5.17), we find that $T_s^{\mathrm{red}} = S_s^{\mathrm{red}}$. Since Chow groups depend only on the underlying reduced schemes, the above image simplifies further to

$$\mathrm{im}\,\big(\,\mathrm{CH}_0(T_s^{\mathrm{red}} \times \kappa(P_{Z_s})) \to \mathrm{CH}_0(P_{Z_s} \times \kappa(P_{Z_s}))\big) \mod 2\,. \tag{5.23}$$

5.3.4 *Step 4.* In this final step, we specialize $s \to 0$. This way Z_s specializes to the double cover $Z_0 \to \mathbb{P}^4$ branched along the quartic f_0 of Hassett–Pirutka–Tschinkel from (5.2). Moreover, T_s^{red} specializes to

$$T_0^{\mathrm{red}} = \big\{w^2 - f_0 = z_2 = 0\big\} \subset \mathbb{P}\big(1^5, 2\big)\,.$$

Using the explicit description of f_0 in (5.2), we find

$$T_0^{\mathrm{red}} = \big\{w^2 - z_0 z_1 z_3^2 = z_2 = 0\big\} \subset \mathbb{P}\big(1^5, 2\big)\,.$$

LEMMA 5.12. *The scheme T_0^{red} has universally trivial Chow group of zero-cycles.*

Proof. We write for simplicity $T := T_0^{\mathrm{red}}$. The non-normal locus of T is given by the plane $\{w = z_3 = 0\}$. The normalization $\tilde{T} \to T$ of T is locally given by the equation $v^2 - z_0 z_1 = 0$, where we substituted $v = w/z_3$. That is, $\tilde{T}$ is an integral quadric given by the equation

$$\tilde{T} = \big\{z_4^2 - z_0 z_1 = 0\big\} \subset \mathbb{P}^4_{[z_0:\cdots:z_4]}\,.$$

Note that $\tilde{T}$ is a cone over a smooth conic with a rational point, hence a cone over $\mathbb{P}^1$, and so it has universally trivial Chow group of zero-cycles. Since the non-normal locus of T is a plane, which of course has universally trivial Chow group of zero-cycles as well, we conclude that T has universally trivial Chow group of zero-cycles, as we want. □

By Lemma 5.12, the specialization of (5.23) under $s \to 0$ is contained in

$$\operatorname{im}\big(\mathrm{CH}_0(P_{Z_0}) \to \mathrm{CH}_0(P_{Z_0} \times \kappa(P_{Z_0}))\big) \mod 2\,. \tag{5.24}$$

Our initial assumption that (5.13) is contained in (5.14) then implies that $\delta_{P_{Z_0}}$ is contained in (5.24). Since P_{Z_0} is a $\mathbb{P}^1$-bundle over Z_0, the pushforward of cycles yields a universal isomorphism $\mathrm{CH}_0(P_{Z_0}) \simeq \mathrm{CH}_0(Z_0)$. Moreover, $\mathrm{CH}_0(Z_0 \times \kappa(Z_0)) \simeq \mathrm{CH}_0(Z_0 \times \kappa(P_{Z_0}))$ because Chow groups do not change under purely transcendental field extensions. Since $\delta_{P_{Z_0}}$ maps to δ_{Z_0} via the composition

$$\mathrm{CH}_0(P_{Z_0} \times \kappa(P_{Z_0})) \xrightarrow{\simeq} \mathrm{CH}_0(Z_0 \times \kappa(P_{Z_0})) \xrightarrow{\simeq} \mathrm{CH}_0(Z_0 \times \kappa(Z_0))\,,$$

we thus conclude that

$$\delta_{Z_0} \in \operatorname{im}(\mathrm{CH}_0(Z_0) \to \mathrm{CH}_0(Z_0 \times \kappa(Z_0))) \mod 2\,.$$

The proof of Proposition 5.7 is then completed by Lemma 5.13 below, where we note that Z_0 is defined over k_0, and so the function field $\kappa(Z_0)$ of Z_0 may also be written as $k_0(Z_0)$. □

LEMMA 5.13. *Let $Z_0 = \{w^2 - f_0 = 0\} \subset \mathbb{P}_{k_0}(1^5, 2)$, where f_0 is as in (5.2). Then the class $\delta_{Z_0} \in \mathrm{CH}_0(Z_{0,k_0(Z_0)})$ is non-zero in the quotient*

$$0 \neq \delta_{Z_0} \in \frac{\mathrm{CH}_0(Z_{0,k_0(Z_0)})/2}{\mathrm{CH}_0(Z_0)/2}\,. \tag{5.25}$$

Proof. As noted in [HPT19], the fourfold Z_0 is birational to the $(2,2)$ hypersurface in $\mathbb{P}^2 \times \mathbb{P}^3$ as described in [HPT18]. The lemma thus follows from [HPT18, Proposition 11] and [Sch19b, Theorem 9.2] by arguments similar to those in [Sch19b, Propositions 3.1 and 7.1]. We give some details for the convenience of the reader.

Recall that the double cover $Z_0 \to \mathbb{P}^4$ is given by the equation

$$w^2 = z_0z_1z_3^2 + z_0z_2z_4^2 + z_1z_2\big(z_0^2 + z_1^2 + z_2^2 - 2(z_0z_1 + z_0z_2 + z_1z_2)\big)\,.$$

The branch locus in $\mathbb{P}^4$ has multiplicity 2 along the line $\ell := \{z_0 = z_1 = z_2 = 0\}$. Let $Z_0' = \mathrm{Bl}_\ell Z_0$, and let $\pi\colon Z_0' \to \mathbb{P}^2$ denote the morphism induced by projection from ℓ. The generic fibre of π is the quadric surface from [HPT18], and so $\alpha = (z_1/z_0, z_2/z_0) \in H^2\big(k_0\big(\mathbb{P}^2\big), \mathbb{Z}/2\big)$ satisfies

$$0 \neq \pi^*\alpha \in H^2_{\mathrm{nr}}(k_0(Z_0')/k_0, \mathbb{Z}/2)\,;$$

see [HPT18, Proposition 11] (this is written over the field of complex numbers, but the same arguments work over any algebraically closed field of characteristic different from 2; cf. [Sch19b, Example 4.2]). The exceptional divisor E of $\mathrm{Bl}_\ell Z_0 \to Z_0$ is given by the equation

$$w^2 = z_0z_1z_3^2 + z_0z_2z_4^2\,.$$

This is a conic bundle over $\mathbb{P}^2$ whose generic fibre is the conic that corresponds to the symbol α, and so $\pi^*\alpha$ vanishes when restricted to the generic point of E. If $e \in E$ is a regular point, then the pullback map $H^2(\operatorname{Spec}\mathcal{O}_{E,e}, \mathbb{Z}/2) \to H^2(k(E), \mathbb{Z}/2)$ is injective (see for example [Sch21b, Theorem 3.6(a)]), and so the restriction of $\pi^*\alpha$ to any regular point of E vanishes. In particular, $\pi^*\alpha$ vanishes at any point of the generic fibre of $E \to \mathbb{P}^2$. Now let $\tau'\colon Z_0'' \to Z_0'$ be an alteration whose degree is odd, which exists by Gabber's theorem; see [IT14]. Since k_0 is perfect, Z_0'' is smooth.

Consider the composition $\tau\colon Z_0'' \to Z_0$ of τ' with the blow-down map $Z_0' \to Z_0$. Clearly, τ is an alteration of odd degree. The base change of τ to $k_0(Z_0)$ is an alteration of odd degree of $Z_{0,k_0(Z_0)}$ that we denote by the same symbol. For a contradiction, we assume that there exist a class $z_1 \in \mathrm{CH}_0(Z_{0,k_0(Z_0)})$ and a class $z' \in \mathrm{CH}_0(Z_0)$ such that

$$\delta_{Z_0} = 2z_1 + z'_{k_0(Z_0)} \in \mathrm{CH}_0(Z_{0,k_0(Z_0)}) .$$

We restrict this class to the smooth locus of $Z_{0,k_0(Z_0)}$ and pull that back via f. The localization exact sequence [Ful98, Proposition 1.8] then shows that

$$\delta_\tau = z_2 + 2z'_1 + z''_{k_0(Z_0)} \in \mathrm{CH}_0\left(Z''_{0,k_0(Z_0)}\right) ,$$

where δ_τ is the zero-cycle on $Z''_{0,k_0(Z_0)}$ induced by the graph of τ, z_2 is supported on $\tau^{-1}\big(Z^{\mathrm{sing}}_{0,k_0(Z_0)}\big)$ and $z'_1 \in \mathrm{CH}_0\left(Z''_{0,k_0(Z_0)}\right)$ and $z'' \in \mathrm{CH}_0(Z''_0)$ are some classes.

We aim to compute the Merkurjev pairing (see for example [Mer08, Section 2.4], [Sch21b, Section 5]) of the above zero-cycle with the unramified cohomology class $\tau^*\alpha \in H^2_{\mathrm{nr}}(k_0(Z''_0)/k_0, \mathbb{Z}/2)$. Here we find

$$\langle \delta_\tau, \tau^*\pi^*\alpha \rangle = \deg(\tau) \cdot \pi^*\alpha \in H^2(k_0(Z_0), \mathbb{Z}/2) .$$

This class is non-zero because $\deg(\tau)$ is odd and $\pi^*\alpha$ is non-zero.

On the other hand,

$$\langle z_2 + 2z'_1 + z''_{k_0(Z_0)}, \tau^*\pi^*\alpha \rangle = \langle z_2 + z''_{k_0(Z_0)}, \tau^*\pi^*\alpha \rangle = \langle z_2, \tau^*\pi^*\alpha \rangle ,$$

where we used that $\langle z''_{k_0(Z_0)}, \tau^*\pi^*\alpha \rangle = 0$ as $\tau^*\pi^*\alpha$ restricts to zero on any closed point of Z_0 because k_0 is algebraically closed. We claim $\langle z_2, \tau^*\pi^*\alpha \rangle = 0$, and it suffices to show our claim in the case where z_2 is a single point. If z_2 does not map to the generic point of $\mathbb{P}^2$ via the composition $f := \pi \circ \tau \colon Z''_0 \to \mathbb{P}^2$, then the claim follows from [Sch19b, Theorem 9.2]. Otherwise, since z_2 is supported on $\tau^{-1}\big(Z^{\mathrm{sing}}_{0,k_0(Z_0)}\big)$ and the generic fibre of $\pi \colon Z'_0 \to \mathbb{P}^2$ is smooth, $\tau(z_2)$ is a point on the generic fibre of $E \to \mathbb{P}^2$. We have shown above that α vanishes when restricted to any point on the generic fibre of $E \to \mathbb{P}^2$, and this implies that $\langle z_2, \tau^*\pi^*\alpha \rangle = 0$, as we want. Altogether we have thus shown that

$$0 \neq \deg(\tau) \cdot \pi^*\alpha = \langle \delta_\tau, \tau^*\pi^*\alpha \rangle = \langle z_2 + 2z'_1 + z''_{k_0(Z_0)}, \tau^*\pi^*\alpha \rangle = 0 \in H^2(k_0(Z_0), \mathbb{Z}/2) .$$

This gives a contradiction, which concludes the proof of the lemma. ∎

Acknowledgements

We thank John Ottem for conversations and comments. We are also grateful to a referee for comments that helped to improve the exposition.

References

AM72 M. Artin and D. Mumford, *Some elementary examples of unirational varieties which are not rational*, Proc. London Math. Soc. (3) **25** (1972), 75–95; doi:10.1112/plms/s3-25.1.75.

BCSS85 A. Beauville, J.-L. Colliot-Thélène, J.-J. Sansuc, and P. Swinnerton-Dyer, *Variétés stablement rationnelles non rationnelles*, Ann. of Math. (2) **121** (1985), no. 2, 283–318; doi:10.2307/1971174.

BGvG22a C. Böhning, H.-C. Graf von Bothmer, and M. van Garrel, *Prelog Chow groups of self-products of degenerations of cubic threefolds*, Eur. J. Math. **8** (2022), no. 1, 260–290; doi:10.1007/s40879-021-00510-8.

BGvG22b ———, *Prelog chow rings and degenerations*, Rend. Circ. Mat. Palermo. (2) (2022); doi:10.1007/s12215-022-00750-x.

Bou06 N. Bourbaki, *Éléments de mathématique. Algèbre commutative. Chapitres 8 et 9* (Springer, Berlin, 2006); doi:10.1007/3-540-33980-9.

CL17 A. Chatzistamatiou and M. Levine, *Torsion orders of complete intersections*, Algebra Number Theory **11** (2017), no. 8, 1779–1835; doi:10.2140/ant.2017.11.1779.

CG72 C. H. Clemens and P. A. Griffiths, *The intermediate Jacobian of the cubic threefold*, Ann. of Math. (2) **95** (1972), 281–356; doi:10.2307/1970801.

CP16 J.-L. Colliot-Thélène and A. Pirutka, *Hypersurfaces quartiques de dimension* 3*: non-rationalité stable*, Ann. Sci. Éc. Norm. Supér. (4) **49** (2016), no. 2, 371–397; doi:10.24033/asens.2285.

CM98 A. Conte and J. P. Murre, *On a theorem of Morin on the unirationality of the quartic fivefold*, Atti Accad. Sci. Torino Cl. Sci. Fis. Mat. Natur. **132** (1998), 49–59.

Ful75 W. Fulton, *Rational equivalence on singular varieties*, Inst. Hautes Études Sci. Publ. Math. **45** (1975), 147–167; doi:10.1007/BF02684300.

Ful98 ———, *Intersection theory*, 2nd ed., Ergeb. Math. Grenzgeb. (3), vol. 2 (Springer-Verlag, Berlin, 1998); doi:10.1007/978-1-4612-1700-8.

Gro67 A. Grothendieck, *Éléments de géométrie algébrique. IV. Étude locale des schémas et des morphismes de schémas IV*, Inst. Hautes Études Sci. Publ. Math. **32** (1967), 5–361; http://www.numdam.org/item/PMIHES_1967__32__5_0/.

Har01 U. T. Hartl, *Semi-stability and base change*, Arch. Math. (Basel) **77** (2001), no. 3, 215–221; doi:10.1007/PL00000484.

HPT18 B. Hassett, A. Pirutka, and Y. Tschinkel, *Stable rationality of quadric surface bundles over surfaces*, Acta Math. **220** (2018), no. 2, 341–365; doi:10.4310/ACTA.2018.v220.n2.a4.

HPT19 ———, *A very general quartic double fourfold is not stably rational*, Algebr. Geom. **6** (2019), no. 1, 64–75; doi:10.14231/ag-2019-004.

IT14 L. Illusie and M. Temkin, *Exposé X. Gabber's modification theorem (log smooth case)*, Astérisque **363**–**364** (2014), 167–212.

IM71 V. A. Iskovskih and Y. I. Manin, *Three-dimensional quartics and counterexamples to the Lüroth problem*, Math. USSR-Sb. **15** (1971), 141–166; doi:10.1070/SM1971v015n01ABEH001536.

Kol95 J. Kollár, *Nonrational hypersurfaces*, J. Amer. Math. Soc. **8** (1995), no. 1, 241–249; doi:10.2307/2152888.

KT19 M. Kontsevich and Y. Tschinkel, *Specialization of birational types*, Invent. Math. **217** (2019), no. 2, 415–432; doi:10.1007/s00222-019-00870-9.

Lev05 M. Levine, *Mixed motives*, in *Handbook of K-Theory, Vol. 1* (E. M. Friedlander and D. R. Grayson, eds) (Springer, Berlin, 2005), 429–521; doi:10.1007/3-540-27855-9_10.

Mer08 A. Merkurjev, *Unramified elements in cycle modules*, J. Lond. Math. Soc. (2) **78** (2008), no. 1, 51–64; doi:10.1112/jlms/jdn011.

Mur73 J. P. Murre, *Reduction of the proof of the non-rationality of a non-singular cubic threefold to a result of Mumford*, Compos. Math. **27** (1973), 63–82; http://www.numdam.org/item/?id=CM_1973__27_1_63_0.

NO22 J. Nicaise and J. C. Ottem, *Tropical degenerations and stable rationality*, Duke Math. J. **171** (2022), no. 15, 3023–3075; doi:10.1215/00127094-2022-0065.

NS19 J. Nicaise and E. Shinder, *The motivic nearby fiber and degeneration of stable rationality*, Invent. Math. **217** (2019), no. 2, 377–413; doi:10.1007/s00222-019-00869-2.

Rob72 J. Roberts, *Chow's moving lemma*, Algebraic Geometry (Oslo 1970) Proc. Fifth Nordic Summer School in Math. (Wolters-Noordhoff, Groningen, 1972), 89–96.

Sch19a S. Schreieder, *On the rationality problem for quadric bundles*, Duke Math. J. **168** (2019), no. 2, 187–223; doi:10.1215/00127094-2018-0041.

Sch19b ______, *Stably irrational hypersurfaces of small slopes*, J. Amer. Math. Soc. **32** (2019), no. 4, 1171–1199; doi:10.1090/jams/928.

Sch21a ______, *Torsion orders of Fano hypersurfaces*, Algebra Number Theory **15** (2021), no. 1, 241–270; doi:10.2140/ant.2021.15.241.

Sch21b ______, *Unramified cohomology, algebraic cycles and rationality*, Rationality of Varieties, Progr. Math., vol. 342 (Birkhäuser/Springer, Cham, 2021), 345–388; doi:10.1007/978-3-030-75421-1_13.

Ska23 B. Skauli, *A* $(2,3)$*-complete intersection fourfold with no decomposition of the diagonal*, Manuscripta Math. **171** (2023), 473–486; doi:10.1007/s00229-022-01386-y.

Tot16 B. Totaro, *Hypersurfaces that are not stably rational*, J. Amer. Math. Soc. **29** (2016), no. 3, 883–891; doi:10.1090/jams/840.

Voi15 C. Voisin, *Unirational threefolds with no universal codimension* 2 *cycle*, Invent. Math. **201** (2015), no. 1, 207–237; doi:10.1007/s00222-014-0551-y.

Nebojsa Pavic pavic@math.uni-hannover.de
Institute of Algebraic Geometry, Leibniz University Hannover, Welfengarten 1, 30167 Hannover, Germany

Stefan Schreieder schreieder@math.uni-hannover.de
Institute of Algebraic Geometry, Leibniz University Hannover, Welfengarten 1, 30167 Hannover, Germany