



On the local étale fundamental group of KLT threefold singularities

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With an appendix by János Kollár

ABSTRACT

Let S be a KLT threefold singularity over an algebraically closed field of positive characteristic $p > 5$. We prove that its local étale fundamental group is tame and finite. Further, we show that every finite unipotent torsor over a big open subset of S is realized as the restriction of a finite unipotent torsor over S .

1. Introduction

Consider the following setup.

SETUP 1.1. Let (X, Δ) be a Kawamata log terminal (KLT for short) log variety of dimension $d \geq 2$ defined over a field \mathbb{k} of characteristic $p \geq 0$. Let $\bar{x} \rightarrow X$ be a geometric closed point. Set $R := \mathcal{O}_{X, \bar{x}}^{\text{sh}}$ and denote its field of fractions by K . Let $Z \subset S := \text{Spec } R$ be a closed subscheme of codimension at least 2, and set $S^\circ := S \setminus Z$. In other words, S° is an arbitrary big open subset of S . We choose and fix a geometric generic point $\bar{\eta} \rightarrow S^\circ$ (that is, a separable closure of K) and let $\pi_1^{\text{ét}}(S^\circ) = \pi_1^{\text{ét}}(S^\circ, \bar{\eta})$ denote the étale fundamental group of S° with base point $\bar{\eta}$. If Z cuts out the singular locus, we may refer to it as the (étale) fundamental group of the singularity.

In this paper, we are concerned with the problem of determining the finiteness of $\pi_1^{\text{ét}}(S^\circ)$. In [Xu14], Xu proved that if $\mathbb{k} = \mathbb{C}$, then $\pi_1^{\text{ét}}(S \setminus \{x\})$ is finite; cf. [GKP16, TX17, Bra21]. Inspired by this result, Schwede, Tucker, and the first-named author proved that if $p > 0$ and S is strongly F -regular, then $\pi_1^{\text{ét}}(S^\circ)$ is finite, and its order is at most $1/s(R)$ and prime-to- p ; see [CST18], cf. [Car22]. (Here, $s(R)$ denotes the F -signature of R .) It is worth recalling that Bhatt–Gabber–Olsson later introduced in [BGO17] a spreading-out technique to prove Xu’s results from that in [CST18]. We also recommend to see [JS22], where a characteristic-free approach is discussed in terms of differential signatures. However, to the best of the authors’ knowledge, little is known about the finiteness of $\pi_1^{\text{ét}}(S^\circ)$ if S is a non-strongly- F -regular KLT singularity in positive characteristic—not even in sufficiently large characteristic. In this work, we attempt

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to address this case in dimension 3 with $p > 5$. In positive characteristic, there is another aspect regarding $\pi_1^{\text{ét}}(S^\circ)$ that is worth investigating, namely, its *tameness*. For strongly F -regular S , we know that this group is very tame as p does not divide its order [CST18, Corollary 2.11]; cf. [Car22, Theorem F]. The same cannot be true for general KLT S by the (independent) examples of Yasuda and Totaro; see [Yas19, Tot19], respectively.

It is worth observing that in dimension 2 and characteristics $p > 5$, KLT singularities and strongly F -regular singularities coincide; see [HX15, Har98]. Therefore, if the characteristic is large enough ($p > 5$), the aforementioned problem is solved by [CST18]; cf. [Art77, Art75].¹ This is no longer the case in dimension 3 as there exist non-strongly- F -regular but KLT threefold singularities; see [CTW18].

In dimension 3 and characteristic $p > 5$, we recall that Xu and Zhang have demonstrated in [XZ19, Theorem 3.4] that a certain tame quotient of $\pi_1^{\text{ét}}(S \setminus \{x\})$ is finite. Their methods are for the most part identical to the ones in [Xu14], which can only handle the tame part. In this article, we complement Xu–Zhang’s work on the matter by proving that $\pi_1^{\text{ét}}(S^\circ) = \pi_1^{\text{ét}}(S \setminus Z)$ is finite and tame for arbitrary Z . If G is a profinite group, we denote by $G^{(p)}$ the prime-to- p part of G , which is the inverse limit over all finite quotients of G whose order is not divisible by p . Then, our main result is the following.

THEOREM A (Corollary 4.5, Theorem 4.3). *Working in Setup 1.1, assume $d = 3$ and \mathbb{k} to be algebraically closed of characteristic $p > 5$. Then, the canonical quotient $\pi_1^{\text{ét}}(S^\circ) \twoheadrightarrow \pi_1^{\text{ét}}(S^\circ)^{(p)}$ is an isomorphism between finite groups.*

Remark 1.2. By the purity of the branch locus theorem, we may always assume that Z cuts out the singular locus of S . Further, since S° is a 2-dimensional KLT scheme (of characteristic $p > 5$), it has finitely many singular points which all have finite fundamental group. Hence, we may find a Galois quasi-étale cover $S' = \text{Spec } R' \rightarrow S$ such that every quasi-étale cover over the punctured spectrum of R' is étale. In other words, we may also assume $Z = \{x\}$. However, we do not do this here as we want our methods to generalize to higher dimensions.

Just as in [BCG⁺19], we obtain the following application.

THEOREM B (Corollaries 4.7 and 4.8). *Working in Setup 1.1, assume $d = 3$ and \mathbb{k} to be algebraically closed of characteristic $p > 5$. Then, every quasi-étale cover over X whose degree is a power of p is étale. Moreover, there exists a quasi-étale Galois cover $\tilde{X} \rightarrow X$ with prime-to- p degree such that every quasi-étale cover over \tilde{X} is étale. Moreover, if (X, Δ) is log Fano and $U \subset X$ is a big open, then $\pi_1^{\text{ét}}(U)$ is finite and its order is prime-to- p .*

Given a scheme X , we denote by X_{fl} the (small) fppf site of X , that is, the flat topology on X ; see [Mil80]. In particular, $H^1(X_{\text{fl}}, G)$ represents the isomorphism classes of (fppf) G -torsors over X , where G is a group scheme over X . See [Mil80, § III.4] for further details. If $\pi_1^{\text{ét}}(S^\circ) \twoheadrightarrow \pi_1^{\text{ét}}(S^\circ)^{(p)}$ is an isomorphism, the restriction map of torsors $H^1(S_{\text{fl}}, G) \rightarrow H^1(S_{\text{fl}}^\circ, G)$ is surjective for all finite étale unipotent group schemes G/\mathbb{k} (that is, for the p -groups). Inspired by [Car22, Theorem F], we push this further by using the rationality hypothesis.

¹To the best of the authors’ knowledge, this is still open if $p \in \{2, 3, 5\}$ and only known for canonical singularities using Artin–Lipman’s explicit classification of rational double points [Art77, Lip69]. The problem is that we do not know whether canonical covers of log terminal surface singularities remain log terminal (when the characteristic does divide the Gorenstein index).

THEOREM C (Corollary 2.19). *Working in Setup 1.1, assume $d = 3$ and \mathcal{K} to be algebraically closed of characteristic $p > 5$. If S is rational, then the restriction map of torsors*

$$\varrho_S^1(G) : H^1(S_{\text{fl}}, G) \rightarrow H^1(S_{\text{fl}}^\circ, G)$$

is surjective for all finite unipotent group schemes G/\mathcal{K} .

The problem of the rationality of KLT singularities in positive characteristics has been of great interest in recent years, where several examples of non-rational KLT singularities have been found.² There are essentially two known ways to construct such counterexamples. On one hand, we have the examples obtained from counterexamples of Kodaira or Kawamata–Viehweg vanishing theorems (by taking affine cones over them); see, for example, [GNT19, CT19, Kov18, Ber21, Tot19]. On the other hand, we have examples constructed by taking quasi-étale $\mathbb{Z}/p\mathbb{Z}$ -quotients of regular germs; see [Yas19, Yas14, Tot19]. In all these examples, either the dimension is large, or the characteristic is small, which is predicted by [HW19], where it is shown that KLT threefold singularities are rational in sufficiently large characteristics. Further, in Yasuda’s construction of certain quasi-étale $\mathbb{Z}/p\mathbb{Z}$ -quotients $\hat{\mathbb{A}}_{\mathcal{K}}^d \rightarrow S$, we see that S cannot be simultaneously rational (that is, Cohen–Macaulay) and KLT.³ In those examples, $\pi_1^{\text{ét}}(S) = \mathbb{Z}/p\mathbb{Z}$ as $\hat{\mathbb{A}}_{\mathcal{K}}^d \rightarrow S$ is a universal Galois cover. Our theorem goes further by establishing that KLT threefold singularities in characteristic $p > 5$ cannot be realized as wild étale quotients of regular germs. We also make the following observation.

THEOREM D (Corollary 2.17). *Working in Setup 1.1, assume $d = 3$ and \mathcal{K} to be algebraically closed of characteristic $p > 5$. If S is F -injective, then it is rational.*

Remark 1.3. When a first version of this work was made public, [ABL22] was not available. In that paper, rationality of KLT threefold singularities is established in characteristic $p > 5$. This makes, for example, Theorem D obsolete and together with Theorem C answers Question 2.22; see [ABL22]. However, we have decided to preserve our original paper as much as possible in this second version. In particular, we keep emphasizing what type of results need rationality and which need not. On the other hand, by using Appendix B, kindly provided by János Kollár, we are able to remove the use of rationality from our original arguments regarding the finiteness of the fundamental group.

1.1. Outline of the paper. Throughout this work, we follow the ideas in [HW19] to study KLT threefold singularities in characteristic $p > 5$ by strongly F -regular ones via the so-called generalized pure log terminal (PLT for short) blowups. Note that all theorems we have claimed in this introduction are true for strongly F -regular singularities. What we do for the most part of this paper is to prove that the same properties can be transferred across generalized PLT blowups which, loosely speaking, is what Hacon–Witaszek do in [HW19] for rationality.

In Section 2, we investigate the problem of (unipotent) tameness. This problem has to do with the action of the Frobenius morphism on (local) cohomology. Therefore, what we do in Section 2 is to compare the action of the Frobenius morphism on cohomology with supports along a generalized PLT blowup.

With tameness in place, we prove the finiteness of the fundamental group in Section 4. Our proof starts identically to Xu’s argument in [Xu14] (which was later mimicked in [XZ19, § 3]).

²The examples are, in fact, not Cohen–Macaulay. However, for KLT singularities, Cohen–Macaulayness and rationality are equivalent notions; see [Kov17].

³In fact, in dimension 3, these are never KLT.

That is, we start off by considering a PLT blowup $(T, E) \rightarrow (S, x)$ at our KLT point x to extract a Kollár component E . Then, the tame fundamental group of T with respect to E , say $\pi_1^{t,E}(T)$, can be used to study tame quasi-étale covers over S . More precisely, we use Grothendieck–Murre’s result [GM71, Corollary 9.8], see Proposition 4.1, to establish an isomorphism between $\pi_1^{t,E}(T)^{(p)}$ and $\pi_1^{\text{ét}}(S)^{(p)}$. From here, both approaches diverge significantly. On one hand, Xu–Zhang use the minimal model program in positive characteristics to study the finiteness of $\pi_1^{t,E}(T)^{(p)}$. Our approach, however, follows the philosophy in [HW19] in the sense that we rather exploit that (T, E) has purely F -regular singularities. More precisely, we use that (T, E) has purely F -regular singularities to prove that $\pi_1^{t,E}(T)^{(p)}$ is finite via a local-to-global argument and Galois-theoretic considerations. The local part is the content of the authors’ former preprint [CS23], whereas we explain the local-to-global methods in Section 3. In the end, just as in [Xu14] and [XZ19], we are left with studying the structure of totally ramified tame covers over (T, E) . We do this by examining the local structure of these covers via our singular Abhyankar’s lemma in [CS23]. This reduces the problem to studying the finiteness of the prime-to- p torsion of the divisor class group of S . This may be easily obtained by using [Bou78] under the rationality hypothesis. For the reader’s convenience, we provide details in Appendix A. However, this has been substantially improved by János Kollár in Appendix B, where the rationality hypothesis is not employed directly.

CONVENTION 1.4. In this paper, all schemes and rings are defined over \mathbb{F}_p . We denote the e th iterate of the Frobenius endomorphism by $F^e: X \rightarrow X$. We assume all our schemes and rings to be F -finite, noetherian, and so excellent.

2. Generalized PLT blowups and tameness

We begin by recalling the definition of Kollár components and PLT blowups; see [LX20, § 1.1], [Xu14, Lemma 1]. In dimension 3 and positive characteristic $p > 5$, we also recall the refined notion of generalized PLT blowups, which, roughly speaking, can be used to approximate KLT threefold singularities by strongly F -regular ones. After this, we explain why this approximation is good enough to impose tameness conditions on those KLT singularities.

2.1. Generalized PLT blowups. Consider the following definition.

DEFINITION 2.1 (PLT blowups and Kollár components). Let X be a variety, and let $x \in X$ be a closed point. One says that a proper birational morphism $f: Y \rightarrow X$ from a log pair (Y, Δ_Y) is a *PLT blowup at x* if the following conditions hold:

- (a) The morphism f induces an isomorphism over $X \setminus \{x\}$.
- (b) The divisor $E := f^{-1}(x)$ on Y is normal and prime, and $(Y, E + \Delta_Y)$ is a PLT log pair.
- (c) The divisor $-(K_Y + E + \Delta_Y)$ is \mathbb{Q} -Cartier and f -ample.
- (d) The divisor $-E$ is \mathbb{Q} -Cartier and f -nef.

The exceptional divisor E of a PLT blowup $f: Y \rightarrow X$ is called a *Kollár component* of X at the closed point x .

Remark 2.2. Let X be variety, $x \in X$ be a closed point, and $f: Y \rightarrow X$ be a PLT blowup extracting a Kollár component E . Further, if we set an equality

$$(K_Y + E + \Delta_Y)|_E = K_E + \Delta_E$$

given by adjunction [Kol13, Chapter 4], then one observes that (E, Δ_E) is a KLT pair ([Kol13, Lemma 4.8] using property (b)) and $-(K_E + \Delta_E)$ is ample by property (c). In other words, (E, Δ_E) is a log Fano pair. We may often refer to this pair as a Kollár component of X at x as well. Furthermore, since $-(K_Y + E + \Delta_Y)$ is ample over X and $(Y, E + \Delta_Y)$ is PLT, we have that $(X, f_*\Delta_Y)$ is KLT; see for instance [KM98, Corollary 3.44].

Regarding the existence of PLT blowups and Kollár components, we may say the following. For further details, see [Pro00, Xu14, GNT19, HW22b, HW22a].

THEOREM 2.3. *Let $x \in (X, \Delta)$ be a KLT closed point of a log pair of dimension $d \geq 2$ and characteristic p . Then, a Kollár component at x exists if either $p = 0$ or $d \leq 3$.*

The following result is crucial in this work. It establishes that for KLT threefolds in characteristic $p > 5$, PLT blowups can even be arranged to be “purely F -regular blowups.” This result is extracted from [HW19, Propositions 2.8 and 2.11].

THEOREM 2.4 (Generalized PLT blowups). *Let (X, Δ) be a KLT threefold over an algebraically closed field \mathbb{k} of characteristic $p > 5$ and $x \in X$ a closed point. Then, for a suitable open neighborhood U of x , there exists a \mathbb{Q} -factorial PLT blowup $f: (Y, \Delta_Y) \rightarrow U$ centered at x extracting a Kollár component $(E, \Delta_E) \subset Y$ such that (Y, E) is a purely F -regular log pair. Moreover, if Δ has standard coefficients, then so does (E, Δ_E) , and so $(Y, E + \Delta_Y)$ is purely F -regular.*

Proof. Recall that the non- \mathbb{Q} -factorial points of X are isolated. To wit, they form a closed subset by [BGS11, Theorem 6.1]. By [GNT19, Theorem 2.14], there exists a *small*⁴ birational morphism $Y \rightarrow X$, where Y is \mathbb{Q} -factorial. We conclude that the non- \mathbb{Q} -factorial locus must have codimension at least 3. Hence, we find an open neighborhood U of x such that each point in U except possibly x is \mathbb{Q} -factorial. Now we can apply [HW19, Theorem 2.11] (where we possibly shrink U again).

For the final statement, simply use that KLT and strong F -regularity coincide for 2-dimensional pairs with standard coefficients [Har95, HX15, CGS16]. Hence, one may apply inversion of F -adjunction to conclude that $(Y, E + \Delta_Y)$ is purely F -regular [Das15, Tay19]. \square

As a consequence, Hacon–Witaszek proved the following.

THEOREM 2.5. *With notation as in Theorem 2.4, if Kawamata–Viehweg vanishing holds for (E, Δ_E) , then $R^k f_* \mathcal{F} = 0$ for all $k > 0$ if \mathcal{F} is either \mathcal{O}_Y or ω_Y , and so $\mathcal{O}_{X,x}$ is Cohen–Macaulay and further rational. Conversely, if $\mathcal{O}_{X,x}$ is rational, then $R^k f_* \mathcal{F} = 0$ for all $k > 0$ if \mathcal{F} is either \mathcal{O}_Y or ω_Y .*

Proof. The first part is the content of [HW19, §4]. To see the converse, note that Y is strongly F -regular, so F -rational, and further rational. Additionally, we may find a common resolution

$$\begin{array}{ccc} Y & \xleftarrow{g} & Y' \\ f \downarrow & & \swarrow h \\ & & U, \end{array}$$

where it is noted that U has rational singularities if $x \in U$ is rational. Hence,

$$\mathbf{R}f_* \mathcal{O}_Y = \mathbf{R}f_* (\mathbf{R}g_* \mathcal{O}_{Y'}) = \mathbf{R}h_* \mathcal{O}_{Y'} = \mathcal{O}_U,$$

⁴That is, it is an isomorphism in codimension-1.

where we used the rationality of T and U in the first and last equality, respectively. The same argument works replacing \mathcal{O} by ω using [Kov17, Lemma 8.2]; cf. [HW19, Proposition 2.3]. \square

2.2. Tameness. We apply the above results to obtain tameness conditions for KLT threefold singularities in characteristic $p > 5$. We need, however, some standard results on fundamental groups for log del Pezzo surfaces.

2.2.1 *Étale fundamental groups of big opens of log del Pezzo surfaces.* We remind the reader that a *weak log del Pezzo surface* is a KLT surface (X, Δ) for which $-(K_X + \Delta)$ is nef and big. We assume that the following result is well known to experts; see [Tak00, Zha06, GZ94, Xu09]. However, we include a proof for the sake of completeness. It will play a fundamental role in our study of tameness.

THEOREM 2.6. *Let (X, Δ) be a weak log del Pezzo surface over an algebraically closed field \mathbb{k} of positive characteristic p (note that X is in particular projective over \mathbb{k}). Then, $\pi_1^{\text{ét}}(X) = 1$, and further, if $p > 5$, then $\pi_1^{\text{ét}}(U)$ is finite for all big opens $U \subset X$.*

Proof. The first statement is just as in [Tak00, Theorem 1.1] if we know $\chi(X, \mathcal{O}_X) = 1$ for every weak log del Pezzo surface (X, Δ) . Indeed, granted this, if $f: Y \rightarrow X$ is a (connected) finite étale morphism of generic degree n , we have that $(Y, f^*\Delta)$ is a weak log del Pezzo surface and so by [Ful84, Example 18.3.9],

$$1 = \chi(Y, \mathcal{O}_Y) = n \cdot \chi(X, \mathcal{O}_X) = n,$$

which means that f is trivial and so is $\pi_1^{\text{ét}}(X)$.

We believe the following claim is well known to experts, yet we sketch a proof here for lack of an adequate reference. We extracted the following argument from unpublished work by Z. Patakfalvi; we are thankful to him for letting us include (a sketch of) his argument in this work.

CLAIM 2.7. *We have $\chi(X, \mathcal{O}_X) = 1$.*

Proof of the claim. Of course, this could be thought of as a direct consequence of Kawamata–Viehweg vanishing for weak log del Pezzo surfaces, which is now known to hold for $p > 5$ by [ABL22, Theorem 1.1]. In the general case, we may proceed as follows.

The key (vanishing) fact is that Grauert–Riemenschneider vanishing holds for log surfaces; see [KK94].⁵ Thus, we may let $f: Y \rightarrow X$ be a minimal resolution and $g: Y \rightarrow Z$ be a minimal model. Since both Y and Z are smooth, they have rational singularities. Hence, the Leray spectral sequence shows that the vanishing of $H^i(X, \mathcal{O}_X)$ is equivalent to the vanishing of $H^i(Z, \mathcal{O}_Z)$. Now, define Γ on Y by the formula $K_Y + \Gamma = f^*(K_X + \Delta)$. Then, by the negativity lemma, Γ is effective, and further $-(K_Y + \Gamma)$ is big and nef. Moreover, the coefficients of Γ are at most 1 as (X, Δ) has KLT singularities. On the other hand, one sees that $-(K_Z + g_*\Gamma)$ is big and nef, and so $-K_Z$ is big. In particular, Z has negative Kodaira dimension, and so Z is either a ruled surface or the projective plane. Finally, one rules out the possibility that Z is ruled over a non-rational curve by using that $K_Z + g_*\Gamma$ is big and nef.⁶ That is, Z is either the projective plane or a Hirzebruch

⁵In other words, if $f: S' \rightarrow S$ is a proper birational morphism between normal surfaces (over an algebraically closed field) and S' is smooth, then $R^1 f_* \omega_{S'} = 0$.

⁶To see this, consider a ruling $Z \rightarrow C$ with $C \not\cong \mathbb{P}^1$. Consider the cases $Z \cong \mathbb{P}^1 \times C$ and $Z \not\cong \mathbb{P}^1 \times C$ separately. In the former case, intersect $-(K_Z + g_*\Gamma)$ with general fiber of f to contradict that $-(K_Z + g_*\Gamma)$ is big. In the latter case, intersect $-(K_Z + g_*\Gamma)$ with the exceptional section of f to contradict that $-(K_Z + g_*\Gamma)$ is nef.

surface, and so $H^1(Z, \mathcal{O}_Z) = 0$.

To see that $H^2(Z, \mathcal{O}_Z) = 0$, simply notice that $H^2(Z, \mathcal{O}_Z) \cong H^0(Z, K_Z)$ and use that $-K_Z$ is big to obtain the desired vanishing. \square

Finally, let $U \subset X$ be a big open. Since (X, Δ) is a KLT *surface*, it is \mathbb{Q} -Gorenstein [Tan18, Corollary 4.11] and X with no boundary is also KLT (see [KM98, Corollary 2.35]). Hence, by [Har98], we deduce that X is strongly F -regular as long as $p > 5$. Then, we may use [BCG⁺19] to conclude the existence of a finite Galois cover $f: Y \rightarrow X$ that is étale over U (in particular quasi-étale) such that for $V = f^{-1}U$ we have $\pi_1^{\text{ét}}(V) = \pi_1^{\text{ét}}(Y)$. Nonetheless, we must have that $(Y, f^*\Delta)$ is weak log del Pezzo as f is quasi-étale, and so $\pi_1^{\text{ét}}(Y) = 1$. Hence, $\pi_1^{\text{ét}}(U) = \text{Gal}(Y/X)$ is finite. \square

2.2.2 τ -modules. We shall see that tameness can be understood in terms of the action of the Frobenius morphism on local cohomology modules so that we need to study modules equipped with a Frobenius action. Following [BP09], one has the following definition.

DEFINITION 2.8. Let X be an \mathbb{F}_p -scheme. A τ -module on X is a quasi-coherent \mathcal{O}_X -module \mathcal{F} equipped with an \mathcal{O}_X -linear map $\phi: \mathcal{F} \rightarrow F_*\mathcal{F}$. Equivalently, letting $\mathcal{O}_X[F]$ denote the sheaf of non-commutative rings given by $\mathcal{O}_X\{F\}/(F \cdot a - a^p \cdot F)$, where $\mathcal{O}_X\{F\}$ denotes the \mathcal{O}_X -algebra obtained by adding a free non-commutative variable, a τ -module is nothing but a left $\mathcal{O}_X[F]$ -module (that is quasi-coherent as a sheaf of \mathcal{O}_X -modules).

As the reader may expect, the category of coherent τ -modules over a field \mathbb{k} (of positive characteristic) should resemble the category of finitely generated modules over the principal ideal domain $\mathbb{k}[t]$. Indeed, if \mathbb{k} is perfect, a right division algorithm holds for $\mathbb{k}[F]$. Consequently, every left ideal of $\mathbb{k}[F]$ is principal, and every submodule of a finite rank free module is free. In particular, a τ -module over \mathbb{k} is a finite direct sum of cyclic modules $\mathbb{k}[F]/\mathbb{k}[F]f$ with $f = 0$ or a power of an irreducible polynomial in $\mathbb{k}[F]$. See [Mil17, § 14.e] for details. In particular, we have the following well-known result (see for instance [Mil80, Lemma III.4.13] and the references therein).

THEOREM 2.9. *Let \mathbb{k} be a perfect field of characteristic $p > 0$. Let V be a coherent τ -module and $\phi: V \rightarrow F_*V$ be the \mathbb{k} -linear map realizing the action of F on V (that is, $\phi(v) = F \cdot v$). Then, V admits a decomposition $V = V^{\text{nil}} \oplus V^{\text{ss}}$ of τ -modules such that ϕ is nilpotent on V^{nil} and ϕ is an isomorphism on V^{ss} . Moreover, if \mathbb{k} is algebraically closed, then V^{ss} admits a \mathbb{k} -basis $\{v_1, \dots, v_n\}$ such that $\phi(v_i) = v_i$ and further $V^F = \langle v_1, \dots, v_n \rangle_{\mathbb{F}_p}$.*

In this work, our main examples of τ -modules are going to be cohomology modules. More precisely, let $f: X \rightarrow Y$ be a morphism of \mathbb{F}_p -schemes and \mathcal{F} be a τ -module with structural map $\phi: \mathcal{F} \rightarrow F_*\mathcal{F}$ on X . Since Frobenius is functorial and affine (in particular $\mathbf{R}F_* = F_*$), we see that $\mathbf{R}f_*\phi$ defines a left $\mathcal{O}_Y[F]$ -module structure on $\mathbf{R}f_*\mathcal{F}$ and so a τ -module structure on its cohomology $R^k f_*\mathcal{F}$, for

$$\mathbf{R}f_*(F_*\mathcal{F}) = \mathbf{R}f_*(\mathbf{R}F_*\mathcal{F}) = \mathbf{R}F_*(\mathbf{R}f_*\mathcal{F}) = F_*\mathbf{R}f_*\mathcal{F}.$$

If $g: Y \rightarrow Z$ is a further morphism of \mathbb{F}_p -schemes, then the left $\mathcal{O}_Z[F]$ -module structure of $\mathbf{R}(g \circ f)_*\mathcal{F}$ is obtained by mapping (pushing forward) the left $\mathcal{O}_Y[F]$ -module structure of $\mathbf{R}f_*\mathcal{F}$ by $\mathbf{R}g_*$.

We shall focus our attention on two subsheaves of \mathcal{F} , namely $\mathcal{F}^{\perp F} := \ker \phi$, which is an \mathcal{O}_X -submodule of \mathcal{F} , and also $\mathcal{F}^F := \ker(\phi - \text{id})$, which is a subsheaf of \mathcal{F} as sheaves of \mathbb{F}_p -

modules, where $\text{id}: \mathcal{F} \rightarrow F_*\mathcal{F}$ is (locally defined) as the additive map $m \mapsto F_*m$. We observe that $(-)^{\perp F}$ defines a left exact functor from the category of τ -modules on X to the category of \mathcal{O}_X -modules, whereas $(-)^F$ defines a left exact functor from τ -modules to \mathbb{F}_p -sheaves on X (sheaves of \mathbb{F}_p -modules). Observe that all the previous remarks apply to sheaves on the étale (or flat) topology of X .

THEOREM 2.10 ([BP09, Proposition 10.1.7, Lemma 10.3.1]). *Let X be a noetherian \mathbb{F}_p -scheme. The functor of taking Frobenius fixed elements $(-)^F$ induces an exact functor from the category of étale τ -modules to the category of étale \mathbb{F}_p -sheaves. In particular, the canonical morphism*

$$R^k f_* (\mathcal{F}^F) \rightarrow (R^k f_* \mathcal{F})^F$$

is an isomorphism for all étale τ -modules \mathcal{F} on X .

Proof. For the last statement, notice that by applying $\mathbf{R}f_*$ to the exact sequence

$$0 \rightarrow \mathcal{F}^F \rightarrow \mathcal{F} \xrightarrow{F-\text{id}} F_*\mathcal{F},$$

we obtain a quasi-isomorphism $\mathbf{R}f_*(\mathcal{F}^F) \rightarrow (\mathbf{R}f_*\mathcal{F})^F$. The statement then follows by taking cohomology and using the exactness of $(-)^F$ to say that the k th cohomology of $(\mathbf{R}\mathcal{F})^F$ is $(\mathbf{R}^k f_* \mathcal{F})^F$. \square

Remark 2.11. As pointed out to us by Jakub Witaszek, the use of Theorem 2.10 can also be replaced by an argument using perfections of schemes. This is formally related to the above theorem due to [BL19, §1.2], where it is shown that the category of τ -crystals embeds into the category of perfect Frobenius modules in a way compatible with the Frobenius fixed point functor.

2.2.3 Main results concerning tameness. We are ready to apply the previous results in this section after recalling the following result from [Car22], which is well known to experts.

LEMMA 2.12. *Let X be an integral \mathbb{F}_p -scheme and $Z \subset X$ a proper closed subscheme. If X is strongly F -regular, then $\mathcal{H}_Z^k(X, \mathcal{O}_X)^F = 0$ for all k . If X is F -pure, then $\mathcal{H}_Z^k(X, \mathcal{O}_X)^{\perp F} = 0$ for all k .*

Proof. It suffices to consider the affine case. Write $X = \text{Spec } R$ with R an integral domain and $Z = V(\mathfrak{a})$ such that $\mathfrak{a} = \sqrt{\mathfrak{a}} \neq 0$. If R is F -pure, meaning that $F^\#: R \rightarrow F_*R$ is split, then $\phi := H_{\mathfrak{a}}^k(F^\#): H_{\mathfrak{a}}^k(R) \rightarrow F_*H_{\mathfrak{a}}^k(R)$ is split and so injective; that is, $0 = \ker \phi = H_{\mathfrak{a}}^k(R)^{\perp F}$.

Let $a \in H_{\mathfrak{a}}^k(R)$, and choose $r \neq 0$ such that $r \cdot a = 0$. If R is strongly F -regular, we find $e \gg 0$ and an R -linear map $\vartheta: F_*^e R \rightarrow R$ such that the following composition of R -linear maps is the identity:

$$R \xrightarrow{F^{e,\#}} F_*^e R \xrightarrow{F_*^e r} F_*^e R \xrightarrow{\vartheta} R.$$

Applying the functor $H_{\mathfrak{a}}^k(-)$, we have that the composition

$$H_{\mathfrak{a}}^k(R) \xrightarrow{\phi^e} F_*^e H_{\mathfrak{a}}^k(R) \xrightarrow{F_*^e r} F_*^e H_{\mathfrak{a}}^k(R) \xrightarrow{\vartheta} H_{\mathfrak{a}}^k(R)$$

is the identity as well. However, if $a \in H_{\mathfrak{a}}^k(R)^F$, that is, $\phi(a) = F_*a$, then a is mapped to zero along that composition, and so it must be zero. \square

THEOREM 2.13. *Work in Setup 1.1 and suppose that \mathbb{k} is algebraically closed of characteristic $p > 5$ and $d = 3$. Then, $H^1(S^\circ, \mathcal{O}_{S^\circ})^F = H^2(S, \mathcal{O}_S)^F = 0$.*

Proof. We consider the diagram

$$\begin{array}{ccc} S^\circ & \xrightarrow{j} & T \\ & \searrow i & \downarrow \pi \\ & & S, \end{array}$$

where i and j are the open immersions, whereas π is a generalized PLT blowup. Strictly speaking, π is the pullback of a PLT blowup $f: (Y, \Delta_Y) \rightarrow U$ (as in Theorem 2.4) along the canonical morphism $S \rightarrow U$. Denote by (E, Δ_E) the Kollár component of π , which is a log del Pezzo surface.

Recall that $H^1(S^\circ, \mathcal{O}_{S^\circ})$ is the R -module given by the global sections of $R^1i_*\mathcal{O}_{S^\circ}$ (as $S = \text{Spec } R$ is affine). On the other hand, by [Gro05, Corollaire I.2.11(29)], there are canonical isomorphisms of τ -modules on S

$$R^1i_*\mathcal{O}_{S^\circ} \cong \mathcal{H}_Z^2(S, \mathcal{O}_S), \quad R^1j_*\mathcal{O}_{S^\circ} \cong \mathcal{H}_{\pi^{-1}Z}^2(T, \mathcal{O}_T).$$

In particular, $(R^1j_*\mathcal{O}_{S^\circ})^F = 0$ by Lemma 2.12 (and using Theorem 2.4 to ensure F -regularity of T). On the other hand, $\mathbf{R}i_* = \mathbf{R}\pi_* \circ \mathbf{R}j_*$, so that $\mathbf{R}i_*\mathcal{O}_{S^\circ} = \mathbf{R}\pi_*(\mathbf{R}j_*\mathcal{O}_{S^\circ})$. From the corresponding Grothendieck–Leray spectral sequence, one extracts the following exact sequence of τ -modules on S :

$$0 \rightarrow R^1\pi_*(j_*\mathcal{O}_{S^\circ}) \rightarrow R^1i_*\mathcal{O}_{S^\circ} \rightarrow \pi_*(R^1j_*\mathcal{O}_{S^\circ}).$$

See [Mil80, Appendix B, Theorem 1]. Applying the left-exact functor $(-)^F$, we have

$$0 \rightarrow (R^1\pi_*(j_*\mathcal{O}_{S^\circ}))^F \rightarrow (R^1i_*\mathcal{O}_{S^\circ})^F \rightarrow \pi_*(R^1j_*\mathcal{O}_{S^\circ})^F = 0.$$

Therefore, it suffices to prove the vanishing $(R^1\pi_*(j_*\mathcal{O}_{S^\circ}))^F = 0$. Observe that the cohomology sheaf in question is defined on S , which is the spectrum of a strictly local ring. In particular, étale cohomology on S coincides with Zariski cohomology. Carrying out computations on the étale site, we have

$$(R^1\pi_*(j_*\mathcal{O}_{S^\circ}))^F = (R^1\pi_*(j_*\mathbb{G}_a))^F = R^1\pi_*(j_*\mathbb{G}_a^F) = R^1\pi_*(j_*\mathbb{Z}/p\mathbb{Z}) = R^1\pi_*(\mathbb{Z}/p\mathbb{Z})$$

by the Artin–Schreier short exact sequence

$$0 \rightarrow \mathbb{Z}/p\mathbb{Z} \rightarrow \mathbb{G}_a \xrightarrow{F-\text{id}} \mathbb{G}_a \rightarrow 0$$

and Theorem 2.10. However, $R^1\pi_*(\mathbb{Z}/p\mathbb{Z})_{\bar{x}} = H^1(T_{\text{ét}}, \mathbb{Z}/p\mathbb{Z})$ by [Mil80, Theorem III.1.15]. On the other hand, pulling back along the closed embedding $E \rightarrow T$ induces an isomorphism

$$H^1(T_{\text{ét}}, \mathbb{Z}/p\mathbb{Z}) \xrightarrow{\cong} H^1(E_{\text{ét}}, \mathbb{Z}/p\mathbb{Z})$$

by the proper base change theorem [Mil80, Corollary VI.2.7]. However,

$$H^1(E_{\text{ét}}, \mathbb{Z}/p\mathbb{Z}) \cong \text{Hom}_{\text{cont}}(\pi_1^{\text{ét}}(E), \mathbb{Z}/p\mathbb{Z})$$

as pointed sets; see [Mil80, § III.4]. The result then follows from Theorem 2.6. \square

Remark 2.14. With notation as in the proof of Theorem 2.13, we have more generally that pulling back along $E \rightarrow T$ induces an isomorphism $\pi_1^{\text{ét}}(E) \xrightarrow{\cong} \pi_1^{\text{ét}}(T)$, see [DB77, Proposition IV.2.2], cf. [Art69], and so $\pi_1^{\text{ét}}(T) = 1$ by Theorem 2.6. Consider the short exact sequence

$$0 \rightarrow H^0(T, \mathcal{O}_T)/(F - \text{id})H^0(T, \mathcal{O}_T) \rightarrow H^1(T_{\text{ét}}, \mathbb{Z}/p\mathbb{Z}) \rightarrow H^1(T, \mathcal{O}_T)^F \rightarrow 0;$$

see [Mil80, Chapter III, Proposition 4.12]. Notice that $R \rightarrow H^0(T, \mathcal{O}_T)$ is an isomorphism as any global section of T restricts to a global section $T \setminus E \cong S \setminus \{x\}$ which is an element of R (for R is normal and so \mathbf{S}_2). Moreover, since R is strictly local, we see that $H^0(T, \mathcal{O}_T)/(F - \text{id})H^0(T, \mathcal{O}_T) = 0$.⁷ Thus, we obtain the vanishing

$$H^1(T, \mathcal{O}_T)^F = 0,$$

which would have been a consequence of the rationality of S .

COROLLARY 2.15. *In the setup of Theorem 2.13, we have $H^1(S_{\mathfrak{fl}}^\circ, G) = *$ for all p -groups G . In particular, for all p -groups G , every G -torsor over S° is trivial.*

Proof. Just as in [Car22, § 4.3], this is a formal consequence of Theorem 2.13. Indeed, say the order of G is p^e . We may proceed by induction on e . The case $e = 1$ follows from [Mil80, Proposition III.4.12], which establishes a short exact sequence

$$0 \rightarrow R/(F - \text{id})R \rightarrow H^1(S_{\text{ét}}^\circ, \mathbb{Z}/p\mathbb{Z}) \rightarrow H^1(S^\circ, \mathcal{O}_{S^\circ})^F \rightarrow 0.$$

However, $R/(F - \text{id})R = 0$ as R is strictly local (see footnote 7 on page 703). By Theorem 2.13, we have $H^1(S^\circ, \mathcal{O}_{S^\circ})^F = 0$. Then, since $\mathbb{Z}/p\mathbb{Z}$ is abelian and étale, $H^1(S_{\mathfrak{fl}}^\circ, \mathbb{Z}/p\mathbb{Z}) = H^1(S_{\text{ét}}^\circ, \mathbb{Z}/p\mathbb{Z}) = 0$. For arbitrary $e > 1$, use [Lan02, Corollary I.6.6]. That is, there is a short exact sequence

$$1 \rightarrow \mathbb{Z}/p\mathbb{Z} \rightarrow G \rightarrow H \rightarrow 1,$$

where H is a p -group of order p^{e-1} . Then, we conclude by taking (non-abelian) cohomology [Mil80, Proposition III.4.5] and using the inductive hypothesis. \square

For the next corollary, we use the following lemma.

LEMMA 2.16. *Let (R, \mathfrak{m}) be a quasi-excellent noetherian normal local ring of dimension $d \geq 3$; then $H_{\mathfrak{m}}^2(R)$ is a finitely generated \mathcal{K} -module.*

Proof. Quite generally, $H_{\mathfrak{m}}^2(R)$ is an artinian R -module [BH98, Proposition 3.5.4(a)]. It then suffices to show that $H_{\mathfrak{m}}^2(R)$ is noetherian. To that end, note that \hat{R} is a quotient of a regular ring by the Cohen structure theorem. However, $H_{\mathfrak{m}}^2(\hat{R}) = \hat{R} \otimes_R H_{\mathfrak{m}}^2(R)$, and so $H_{\mathfrak{m}}^2(R)$ is finitely generated as an R -module if and only if so is $H_{\mathfrak{m}}^2(\hat{R})$ as an \hat{R} -module; see [Sta22, Tag 03C4]. Now observe that \hat{R} is normal [Sta22, Tag 0C23]. Then one uses [Gro05, Exposé VIII, Corollaire 2.3], which establishes that $H_{\mathfrak{m}}^2(\hat{R})$ is finitely generated if $H_{\mathfrak{p}R_{\mathfrak{p}}}^1(\hat{R}_{\mathfrak{p}}) = 0$ for all prime ideals $\mathfrak{p} \subset \hat{R}$ of height $d - 1 \geq 2$, which follows from normality. \square

COROLLARY 2.17. *In the setup of Theorem 2.13, if S is F -injective, then it is rational.*

Proof. For a KLT singularity, rationality is the same as Cohen–Macaulayness [Kov17]. Hence, the statement amounts to the vanishing $H_{\mathfrak{m}}^2(R) = 0$. However, putting together Lemma 2.16, Theorem 2.9, and Theorem 2.13, it suffices to show $H_{\mathfrak{m}}^2(R)^{\perp F} = 0$. This, however, is trivially the case for F -injective singularities. \square

Next, we study the Frobenius annihilated elements on $H^1(S^\circ, \mathcal{O}_{S^\circ})$ assuming rationality.

THEOREM 2.18. *Work in the setup of Theorem 2.13. If S is rational, then $H^1(S^\circ, \mathcal{O}_{S^\circ})^{\perp F} = H_{\mathbb{Z}}^2(S, \mathcal{O}_S)^{\perp F} = 0$.*

⁷Indeed, non-zero elements of $R/(F - \text{id})R$ correspond to non-trivial $\mathbb{Z}/p\mathbb{Z}$ -torsors over S via $r \in R/(F - \text{id})R \mapsto \text{Spec}(R[t]/(t^p - t + r)) \rightarrow S$, which are trivial if R is strictly local.

Proof. The proof commences identically to the one for Theorem 2.13 up to the point where we need to prove the vanishing $(R^1\pi_*(j_*\mathcal{O}_{S^\circ}))^{\perp F} = 0$. Unfortunately, we do not know of any result analogous to Theorem 2.10 for the functor $(-)^{\perp F}$. Hence, we need to prove such a vanishing by different methods, which in our case shall require rationality. However, the following proof is valid for both functors $(-)^F$ and $(-)^{\perp F}$. We proceed as follows. By [Gro05, Corollaire I.2.11(28)], there is a natural exact sequence

$$0 \rightarrow \mathcal{H}_{\pi^{-1}Z}^0(T, \mathcal{O}_T) \rightarrow \mathcal{O}_T \rightarrow j_*\mathcal{O}_{S^\circ} \rightarrow \mathcal{H}_{\pi^{-1}Z}^1(T, \mathcal{O}_T) \rightarrow 0,$$

where we see straightaway that $\mathcal{H}_{\pi^{-1}Z}^0(T, \mathcal{O}_T) = 0$ as T is integral. Hence, using the vanishing $R^k\pi_*\mathcal{O}_T$ for all $k > 0$ (see Theorem 2.5), we have that

$$R^1\pi_*(j_*\mathcal{O}_{S^\circ}) \cong R^1\pi_*\mathcal{H}_{\pi^{-1}Z}^1(T, \mathcal{O}_T) = H^1(T, \mathcal{H}_{\pi^{-1}Z}^1(T, \mathcal{O}_T))^\sim,$$

where the actions of the Frobenius morphism are compatible by the naturality of the short exact sequence (that is, these are isomorphisms of τ -modules). On the other hand, we have a spectral sequence

$$H^k(T, \mathcal{H}_{\pi^{-1}Z}^l(T, \mathcal{O}_T)) \Rightarrow H_{\pi^{-1}Z}^{k+l}(T, \mathcal{O}_T),$$

compatible with Frobenius actions by functoriality, and so it is a spectral sequences of τ -modules. From this spectral sequence, we extract the following exact sequence (of τ -modules) of low degrees (starting with $E_2^{2,0} = 0$, cf. [Mil80, Appendix B, paragraph before Theorem 1]):

$$E_2^{2,0} = 0 \rightarrow E_1^2 \rightarrow E_2^{1,1} \rightarrow E_2^{3,0} = H^3(T, \mathcal{H}_{\pi^{-1}Z}^0(T, \mathcal{O}_T)) = 0,$$

where

$$E_1^2 = \ker(H_{\pi^{-1}Z}^2(T, \mathcal{O}_T) \rightarrow H^0(T, \mathcal{H}_{\pi^{-1}Z}^2(T, \mathcal{O}_T))).$$

Since $H^1(T, \mathcal{H}_{\pi^{-1}Z}^1(T, \mathcal{O}_T)) = E_2^{1,1} \cong E_1^2$, it therefore suffices to show that $H_{\pi^{-1}Z}^2(T, \mathcal{O}_T)^{\perp F} = 0$. Write $\pi^{-1}Z = Z' \cup E$, where Z' is the strict transform of Z along π (that is, Z' is the Zariski closure of $Z \cap (S \setminus \{x\})$ in T). We then have the Mayer–Vietoris sequence

$$\cdots \rightarrow H_{Z' \cap E}^2(T, \mathcal{O}_T) \rightarrow H_{Z'}^2(T, \mathcal{O}_T) \oplus H_E^2(T, \mathcal{O}_T) \rightarrow H_{\pi^{-1}Z}^2(T, \mathcal{O}_T) \rightarrow H_{Z' \cap E}^3(T, \mathcal{O}_T) \rightarrow \cdots,$$

which, by its naturality, is a sequence of τ -modules. Notice that $(Z' \cap E)_{\text{red}}$ is either empty or a finite set of closed points of T ; say $x_1, \dots, x_n \in T$.⁸ In particular,

$$H_{Z' \cap E}^2(T, \mathcal{O}_T) = \bigoplus_{i=1}^n H_{x_i}^2(T, \mathcal{O}_T) = \bigoplus_{i=1}^n H_{\mathfrak{m}_{x_i}}^2(\mathcal{O}_{T, x_i}) = 0,$$

which is zero as T is Cohen–Macaulay and has dimension 3. Hence, we have an exact sequence

$$0 \rightarrow H_{Z'}^2(T, \mathcal{O}_T) \oplus H_E^2(T, \mathcal{O}_T) \rightarrow H_{\pi^{-1}Z}^2(T, \mathcal{O}_T) \rightarrow \bigoplus_{i=1}^n H_{\mathfrak{m}_{x_i}}^3(\mathcal{O}_{T, x_i}).$$

Since $(-)^{\perp F}$ is left exact and $H_{\mathfrak{m}_{x_i}}^3(\mathcal{O}_{T, x_i})^{\perp F} = 0$ by Lemma 2.12, we obtain

$$H_{\pi^{-1}Z}^2(T, \mathcal{O}_T)^{\perp F} = H_{Z'}^2(T, \mathcal{O}_T)^{\perp F} \oplus H_E^2(T, \mathcal{O}_T)^{\perp F}.$$

Next, observe that $Z' \subset T$ is a closed subscheme that passes through only finitely many closed points, namely x_1, \dots, x_n . Hence, if we denote by $\mathfrak{a}_i \subset \mathcal{O}_{T, x_i}$ the ideal defining the pullback of Z'

⁸Indeed, $Z \cap (S \setminus \{x\}) = Z \setminus \{x\}$ is either empty or a finite set of codimension 2 points (which correspond to the generic points of the irreducible components of Z). Hence, in the latter case, Z' consists of all the specializations of these points in T , which must be all closed points in T . In other words, Z' meets E only at closed points.

along $\text{Spec } \mathcal{O}_{T,x_i} \rightarrow T$, the canonical homomorphism

$$\rho: H_{Z'}^2(T, \mathcal{O}_T) \rightarrow \bigoplus_{i=1}^n H_{\mathfrak{a}_i}^2(\mathcal{O}_{T,x_i})$$

is injective. To see this, first note that $\mathcal{H}_{Z'}^i(T, \mathcal{O}_T) = 0$ for $i = 0, 1$ as the complement of Z' in T is a big open (and T is normal). Therefore, $E_2^{2,0} = 0$ and $E_2^{1,1} = 0$ in the spectral sequence

$$H^k(T, \mathcal{H}_{Z'}^l(T, \mathcal{O}_T)) \Rightarrow H_{Z'}^{k+l}(T, \mathcal{O}_T).$$

From this, we deduce that $E_1^2 = \ker(H_{Z'}^2(T, \mathcal{O}_T) \rightarrow H^0(T, \mathcal{H}_{Z'}^2(T, \mathcal{O}_T))) = 0$; that is, the canonical map

$$H_{Z'}^2(T, \mathcal{O}_T) \rightarrow H^0(T, \mathcal{H}_{Z'}^2(T, \mathcal{O}_T))$$

is injective. Of course, ρ is none other than the composition

$$H_{Z'}^2(T, \mathcal{O}_T) \rightarrow H^0(T, \mathcal{H}_{Z'}^2(T, \mathcal{O}_T)) \rightarrow \bigoplus_{i=1}^n H_{\mathfrak{a}_i}^2(\mathcal{O}_{T,x_i}),$$

where the right arrow is defined via the restriction-to-stalks map and the fact that $\mathcal{H}_{Z'}^2(T, \mathcal{O}_T)_{x_i} = H_{\mathfrak{a}_i}^2(\mathcal{O}_{T,x_i})$ (which holds as local cohomology commutes with localizations [ILL⁺07, Proposition 7.15]). Since x_1, \dots, x_n is the list of closed points supporting the sheaf $\mathcal{H}_{Z'}^2(T, \mathcal{O}_T)$, that map is injective and so is ρ .

Now, since ρ is compatible with the Frobenius actions, $H_{Z'}^2(T, \mathcal{O}_T)^{\perp F}$ injects into

$$\bigoplus_{i=1}^n H_{\mathfrak{a}_i}^2(\mathcal{O}_{T,x_i})^{\perp F} = 0,$$

and so $H_{Z'}^2(T, \mathcal{O}_T)^{\perp F} = 0$ (the above vanishing is obtained again by using Lemma 2.12 and the strong F -regularity of T). Thus, we are left with proving the vanishings $H_E^2(T, \mathcal{O}_T)^{\perp F} = 0$. In fact, we have that $H_E^2(T, \mathcal{O}_T) = 0$. To see this, consider the long exact sequence

$$\cdots \rightarrow H^1(T, \mathcal{O}_T) \rightarrow H^1(T \setminus E, \mathcal{O}_{T \setminus E}) \rightarrow H_E^2(T, \mathcal{O}_T) \rightarrow H^2(T, \mathcal{O}_T) \rightarrow \cdots,$$

noticing that $H^1(T, \mathcal{O}_T) = 0 = H^2(T, \mathcal{O}_T)$ (this is the vanishing $R^i \pi_* \mathcal{O}_T = 0$, $i = 1, 2$). Hence,

$$H_E^2(T, \mathcal{O}_T) \cong H^1(T \setminus E, \mathcal{O}_{T \setminus E}) \cong H^1(S \setminus \{x\}, \mathcal{O}_{S \setminus \{x\}}) \cong H_{\mathfrak{m}}^2(R) = 0,$$

where the last vanishing holds as $S = \text{Spec } R$ is Cohen–Macaulay. \square

COROLLARY 2.19. *Working in Setup 1.1, suppose that \mathbb{k} is algebraically closed of characteristic $p > 5$ and $d = 3$. If S is rational, then the restriction map*

$$\varrho_S^1(G): H^1(S_{\mathfrak{h}}, G) \rightarrow H^1(S_{\mathfrak{h}}^{\circ}, G)$$

is surjective for all finite unipotent group schemes G/\mathbb{k} . In particular, if $Y \rightarrow X$ is a G -quotient by a finite unipotent group scheme G/\mathbb{k} that is a G -torsor in codimension 1, then $Y \rightarrow X$ is a G -torsor.

Proof. As in [Car22, § 4.3], the first, local, statement is a formal consequence of Theorems 2.13 and 2.18. The second, global, statement follows because checking whether a given G -quotient is a torsor can be done locally in the étale topology. Also, recall that being a torsor is an open condition [Car22, Corollary 3.13] \square

2.2.4 *Questions of interest.* Corollary 2.19 raises the following three questions, which should be compared to the ones in [Car22, § 4.7].

Question 2.20. Work in Setup 1.1. Is the restriction map $H^1(S_{\text{fl}}, G) \rightarrow H^1(S_{\text{fl}}^{\circ}, G)$ surjective for all unipotent group schemes G/\mathbb{k} if p is large enough?

Question 2.21. Work in Setup 1.1. Is the restriction map $H^1(S_{\text{fl}}, G) \rightarrow H^1(S_{\text{fl}}^{\circ}, G)$ surjective for all unipotent group schemes G/\mathbb{k} if S is rational?

Question 2.22. Letting \mathbb{k} be an algebraically closed field of characteristic $p > 5$, is there an action of α_p on $\hat{\mathbb{A}}_{\mathbb{k}}^3$ whose quotient $\hat{\mathbb{A}}_{\mathbb{k}}^3 \rightarrow \hat{\mathbb{A}}_{\mathbb{k}}^3/\alpha_p$ is an α_p -torsor in codimension 1 and for which $\hat{\mathbb{A}}_{\mathbb{k}}^3/\alpha_p$ is KLT?

Remark 2.23. As mentioned in Remark 1.3, the work [ABL22] shows that KLT threefold singularities are rational in characteristics $p > 5$. In particular, Corollary 2.19 answers Question 2.22 affirmatively. Of course, it also answers Question 2.20 affirmatively in the 3-dimensional case by taking $p > 5$.

3. On the structure of tame covers via local tame fundamental groups

In this section, we establish an analog of the main results in [BCG⁺19] (cf. [GKP16, Sti17]) in the context of tame covers and tame fundamental groups. In Section 4, we will apply this result to the case where X is the source of a generalized PLT blowup and P is the corresponding Kollár component. We consider the following setup and notation.

SETUP 3.1. Let X be a normal integral scheme, and let $Z \subset X$ be a closed subscheme of codimension at least 2 with open complement $U = X \setminus Z$. Let P be a prime divisor on X —whose restriction to U we denote by P as well—that is itself a normal (integral) scheme. We will denote the ideal sheaf corresponding to P by \mathfrak{p} . Given a geometric point $\bar{x} \rightarrow U$, we denote by $P_{\bar{x}}$ the pullback of P to $U_{\bar{x}} := \text{Spec } \mathcal{O}_{U, \bar{x}}^{\text{sh}}$.

Remark 3.2. Working in Setup 3.1, observe that $P_{\bar{x}}$ is a prime divisor on $\mathcal{O}_{U, \bar{x}}^{\text{sh}}$. Indeed, note that $\mathcal{O}_{U, \bar{x}}^{\text{sh}}$ is normal (as it is the colimit of étale $\mathcal{O}_{U, x}$ -algebras [Sta22, Lemma 033C], [Sta22, Lemma 037D]) and $\mathcal{O}_{U, x}$ is normal). Similarly, $P_{\bar{x}}$ is given by the spectrum of the strict henselization of $\mathcal{O}_{U, x}/\mathfrak{p}_x$ (see [Sta22, Lemma 05WS]) and thus also normal since $\mathcal{O}_{U, x}/\mathfrak{p}_x$ is normal. Since $P_{\bar{x}}$ is the spectrum of a local ring, it is integral.

Étale and tame fundamental groups. For the reader’s convenience, we briefly recall the fundamental groups of interest in this paper. For further details, we recommend [Mur67, GM71, Cad13, CS23]. Let X be a normal integral scheme with field of functions K , and let P be a prime divisor on X . Denoting by $\eta \in X$ the generic point of X , we write $\bar{\eta} \rightarrow X$ for a chosen geometric generic point, which amounts to the choice of a separable closure K^{sep} of K . We denote by $\text{FEt}(X)$ the Galois category of finite étale covers over X . Of course, $\text{FEt}(K)$ is nothing but the Galois category (whose minimal/connected objects consist) of finite separable extensions L/K , and $\text{FEt}(X)$ can be realized as the full Galois subcategory of $\text{FEt}(K)$ given by those finite separable extensions L/K such that the normalization of X in L , say X^L , is étale over X . In particular, we have a canonical surjective homomorphism of topological groups

$$\pi_1^{\text{ét}}(K) = \text{Gal}(K^{\text{sep}}/K) := \varprojlim_{\substack{K^{\text{sep}}/L/K \\ L/K \text{ Galois}}} \text{Gal}(L/K) \twoheadrightarrow \pi_1^{\text{ét}}(X) = \varprojlim_{\substack{K^{\text{sep}}/L/K \\ L/K \text{ Galois} \\ X^L/X \in \text{FEt}(X)}} \text{Gal}(L/K),$$

where we have implicitly chosen $\bar{\eta} \rightarrow X$ as our common base point. However, we will always drop it from our notation.

Further, we denote by $\text{Rev}^P(X)$ the Galois category of tame covers over X with respect to P . More precisely, $\text{Rev}^P(X)$ is the full Galois subcategory of $\text{FEt}(K)$ (whose minimal/connected objects are) given by the finite separable extensions L/K such that the normalization of X in L , say $Y \rightarrow X$, is such that $Y_{X \setminus P} \rightarrow X \setminus P$ is étale and L/K is tamely ramified with respect to the discrete valuation ring (DVR) $\mathcal{O}_{X,P} \subset K$. Thus, we consider

$$\pi_1^{t,P}(X) := \varinjlim_{\substack{K^{\text{sep}}/L/K \\ L/K \text{ Galois} \\ X^L/X \in \text{Rev}^P(X)}} \text{Gal}(L/K),$$

where the limit traverses all finite Galois extensions L/K (inside K^{sep}) such that $X^L \rightarrow X$ is étale away from P but at worst tamely ramified over the generic point of P .

Summing up, we have canonical continuous and surjective homomorphisms

$$\text{Gal}(K^{\text{sep}}/K) \twoheadrightarrow \pi_1^{\text{ét}}(X) \twoheadrightarrow \pi_1^{t,P}(X).$$

We now recall some terminology from [CS23] which will be important in the following discussion. To do so, it is important to notice that given a Galois object $f: Y \rightarrow X$ in $\text{Rev}^P(X)$ of (generic) degree d_f , we have an equality

$$d_f = n_f \cdot e_f \cdot i_f,$$

where n_f is the number of irreducible components of $f^{-1}P$, e_f is the *ramification index*, and i_f is the *inertial degree*. More precisely, if Q_1, \dots, Q_{n_f} are the irreducible components of $(f^{-1}(P))_{\text{red}}$ and so prime divisors on Y , then e_f is the (common) ramification index of the extension of DVRs $\mathcal{O}_{X,P} \subset \mathcal{O}_{Y,Q_j}$ for all $j = 1, \dots, n_f$ and $i_f := [K(Q_j) : K(P)]$ for all $j = 1, \dots, n_f$.

TERMINOLOGY 3.3 ([CS23, Terminology 3.21]). Let us recall the following properties of the Galois category $\text{Rev}^P(X)$:

- (a) *P-irreducibility*: Every connected cover $f: Y \rightarrow X$ in $\text{Rev}^P(X)$ satisfies that $Q := (f^{-1}(P))_{\text{red}}$ is a prime divisor on Y . If f is Galois, this means $n_f = 1$. In this case, we shall often represent an object in $\text{Rev}^P(X)$ as a pair (Y, Q) .
- (b) *Inertial boundedness*: There exists an $N \in \mathbb{N}$ such that $i_f \leq N$ for all Galois objects $f: Y \rightarrow X$ in $\text{Rev}^P(X)$.
- (c) *Inertial tameness*: The inertial degree i_f is prime-to- p for all Galois objects $f: Y \rightarrow X$ in $\text{Rev}^P(X)$.
- (d) *Inertial decantation*: Assuming the P -irreducibility of $\text{Rev}^P(X)$, inertial decantation means that every Galois cover $f: Y \rightarrow X$ in $\text{Rev}^P(X)$ dominates a quasi-étale Galois cover $Y' \rightarrow X$ in $\text{Rev}^P(X)$ whose degree is i_f . When this degree is 1, we say that f is *totally ramified* (that is, $d_f = e_f$).

Remark 3.4. The P -irreducibility of $\text{Rev}^P(X)$ implies its inertial decantation; see [CS23, Lemma 3.23].

THEOREM 3.5. Work in Setup 3.1, and suppose the following:

- (1) Every connected cover $f: V \rightarrow U$ in $\text{Rev}^P(U)$ is such that $f^{-1}P$ is connected.
- (2) The Galois category $\text{Rev}^{P_{\bar{x}}}(U_{\bar{x}})$ is $P_{\bar{x}}$ -irreducible and inertially bounded for all geometric points $\bar{x} \rightarrow U$.

Then, the Galois category $\text{Rev}^P(U)$ has the three following properties:

- (A) It is P -irreducible.
- (B) It is inertially decanted.
- (C) There exists a quasi-étale Galois cover (\bar{U}, \bar{P}) in $\text{Rev}^P(U)$ such that every quasi-étale Galois cover in $\text{Rev}^{\bar{P}}(\bar{U})$ is étale.

Furthermore, suppose that \bar{P} is normal and $\pi_1^{\text{ét}}(\bar{P} \cap \bar{U})$ is finite. Then, there exists an étale cover (\tilde{U}, \tilde{P}) in $\text{Rev}^{\bar{P}}(\bar{U})$ (with \tilde{P} normal) such that for all Galois covers (V, Q) in $\text{Rev}^{\tilde{P}}(\tilde{U})$, the following hold:

- (i) The morphism $(V, Q)/(\tilde{U}, \tilde{P})$ is totally ramified.
- (ii) The morphism $Q \rightarrow \tilde{P}$ is an isomorphism, and so Q is normal.
- (iii) The morphism $(V, Q)/(\tilde{U}, \tilde{P})$ is étale-locally a Kummer cover over \tilde{U}_{reg} . That is, given a geometric point $\bar{x} \rightarrow \tilde{U}_{\text{reg}}$, there exists an étale neighborhood $W \rightarrow \tilde{U}$ of \bar{x} such that $V_W \rightarrow W$ is isomorphic in $\text{Rev}^{P_W}(W)$ to $W[T]/(T^e - w) \rightarrow W$, where $\text{div } w = P_W$ for some $0 \neq w \in \Gamma(W, \mathcal{O}_W)$ and $e = [K(V) : K(\tilde{U})]$ is the ramification index of $(V, Q)/(\tilde{U}, \tilde{P})$.

Remark 3.6. Property (A) in Theorem 3.5 is truly essential. In its presence, we can think of connected objects in $\text{Rev}^P(U)$ as pairs (V, Q) , where V/U is a tamely ramified cover with respect to P and Q is the only prime divisor on V lying over P . In this way, $V \setminus Q \rightarrow U \setminus P$ is étale, and $\mathcal{O}_{U,P} \rightarrow \mathcal{O}_{V,Q}$ is a tamely ramified extension of DVRs, so that $[K(V) : K(U)] = e \cdot [K(Q) : K(P)]$, where e is the ramification index. Consequently, the Galois category consisting of the objects of $\text{Rev}^P(U)$ that lie over a given (connected) object (V, Q) is the same as the Galois category $\text{Rev}^Q(V)$. In particular, given a Galois object (V, Q) in $\text{Rev}^P(U)$, we have a short exact sequence of topological groups

$$1 \rightarrow \pi_1^{\dagger, Q}(V) \rightarrow \pi_1^{\dagger, P}(U) \rightarrow \text{Gal}(V/U) \rightarrow 1.$$

Before getting into the proof of Theorem 3.5, we consider instructive to have the following preliminary discussion on stratifications and Abhyankar’s lemma.

3.1 Stratifications. Let X be a normal integral scheme. Suppose that we are given a tower of quasi-étale Galois covers

$$X = X_0 \xleftarrow{\gamma_0} X_1 \xleftarrow{\gamma_1} X_2 \xleftarrow{\gamma_2} \dots$$

such that $\gamma_{i, \bar{x}}$ is étale for sufficiently large i for every geometric point $\bar{x} \rightarrow X$. More precisely, suppose that for all geometric points $\bar{x} \rightarrow X$, there is an $N_{\bar{x}} \in \mathbb{N}$ such that $\gamma_{i, \bar{x}}$ is étale for all $i \geq N_{\bar{x}}$. It is natural to ask whether this implies the existence of a uniform constant $N \in \mathbb{N}$ (independent of the geometric point) such that γ_i is étale for all $i \geq N$. This question was considered in the works [GKP16, BCG⁺19], where positive answers were given over the complex numbers and in positive characteristic, respectively. In both cases, a type of stratification was required. In [GKP16], the authors employed the so-called Whitney’s stratification, whereas in [BCG⁺19], the authors used a result of Gabber [Dég14, Théorèmes 1.1 and 1.3] to construct an analogous stratification. It is worth noting that Stibitz [Sti17] has recently introduced a characteristic-free stratification based on De Jong’s alterations [dJo96], that is suitable to answer this question positively.

THEOREM 3.7. *Let X be a normal integral excellent scheme. Let*

$$X = X_0 \xleftarrow{\gamma_0} X_1 \xleftarrow{\gamma_1} X_2 \xleftarrow{\gamma_2} \dots$$

be a tower of quasi-étale Galois covers over X . Suppose that for every geometric point $\bar{x} \rightarrow X$, there is an $N_{\bar{x}} \in \mathbb{N}$ such that $\gamma_{i,\bar{x}}$ is étale (hence trivial) for all $i \geq N_{\bar{x}}$. Then, there is an $N \in \mathbb{N}$ such that γ_i is étale for all $i \geq N$.

Proof. This follows straightaway from [BCG⁺19]; cf. [BCG⁺19, Remark 4.2] to see why bounding the order of the local fundamental groups in [BCG⁺19, Proposition 3.2] is superfluous. Note that if X_i admits a regular alteration for all large enough i , another proof can be found in [Sti17, § 4, proof of (i) \Rightarrow (iii)]. \square

3.2 Abhyankar’s lemma. Next, we come to the étale-local description of normal totally ramified Galois covers in Theorem 3.5, which we refer to as Abhyankar’s lemma; cf. [Sta22, Tag 0EYG]. The following result is based on a generalization of [Sta22, Tag 0EYG] to singular divisors given in [CS23].

THEOREM 3.8. *Work in the setup of Theorem 3.5. Let $f: (V, Q) \rightarrow (U, P)$ be a totally ramified Galois cover in $\text{Rev}^P(U)$ of degree e . Then, for a given geometric point $\bar{x} \rightarrow X$ with $x \in P \cap U_{\text{reg}}$, there exists an étale neighborhood $W \rightarrow U$ of \bar{x} such that $V_W \rightarrow W$ is isomorphic in $\text{Rev}^{P_W}(W)$ to $W[T]/(T^e - w) \rightarrow W$, where $\text{div } w = P_W$ for some $0 \neq w \in \Gamma(W, \mathcal{O}_W)$.*

Proof. We use the notation of Theorem 3.5 and Setup 3.1. Consider the cartesian diagram

$$\begin{array}{ccc} V & \longleftarrow & V_{\bar{x}} \\ f \downarrow & & \downarrow f_{\bar{x}} \\ U & \longleftarrow & U_{\bar{x}}, \end{array} \tag{3.1}$$

where $f_{\bar{x}}$ is an object in $\text{Rev}^{P_{\bar{x}}}(U_{\bar{x}})$; see [GM71, Lemma 2.2.8]. (Note that $V_{\bar{x}}$ is normal as $V_{\bar{x}} \rightarrow V$ is pro-étale and V is normal.) Since $f: Q \rightarrow P$ is a finite birational (as $f: V \rightarrow U$ is totally ramified) morphism and P is normal, we obtain that $f: Q \rightarrow P$ is an isomorphism (this is a simple case of Zariski’s main theorem). Therefore, since $x \in P$, there is only one point of $V_{\bar{x}}$ lying over \bar{x} . Hence, $f_{\bar{x}}$ is connected and totally ramified. In particular, $V_{\bar{x}}$ is a connected normal scheme, which has the following consequence.

CLAIM 3.9. *In (3.1), the morphism $f_{\bar{x}}$ is Galois.*

Proof of the claim. Let $W \rightarrow U$ be a connected étale neighborhood of $\bar{x} \rightarrow U$, and consider the diagram

$$\begin{array}{ccccc} V & \longleftarrow & V_W & \longleftarrow & V_{\bar{x}} \\ f \downarrow & & \downarrow f_W & & \downarrow f_{\bar{x}} \\ U & \longleftarrow & W & \longleftarrow & U_{\bar{x}} := \text{Spec } \mathcal{O}_{U,\bar{x}}^{\text{sh}}, \end{array}$$

where both inner squares are cartesian and the outer square is none other than (3.1). In particular, V_W is a normal integral scheme.

Now, by localizing the left cartesian diagram at the generic point of U , we conclude that $K(V_W) = K(V) \otimes_{K(U)} K(W)$ because all four arrows in this diagram are quasi-finite morphisms between irreducible schemes. In other words, $K(V)$ and $K(W)$ are subfields of $K(W)$ whose compositum is $K(V_W)$. In particular, $K(V_W)/K(W)$ is Galois by [Lan02, Theorem VI.1.12], which is to say, f_W is Galois as an object of $\text{Rev}^{P_W}(W)$.

To prove that $K(V_{\bar{x}})/K(U_{\bar{x}})$ is Galois, we observe that separability is clear as étaleness is preserved under base change. We must explain the normality of $K(V_{\bar{x}})/K(U_{\bar{x}})$. To this end, we consider $0 \neq b \in K(V_{\bar{x}})$ and let $\alpha(t) = t^n + a_{n-1}t^{n-1} + \dots + a_1t + a_0 \in K(U_{\bar{x}})[t]$ be its minimal polynomial. By clearing denominators, let $0 \neq a \in \mathcal{O}_{U_{\bar{x}}}^{\text{sh}}$ be such that $a \cdot \alpha(t)$ has coefficients in $\mathcal{O}_{U_{\bar{x}}}^{\text{sh}}$. Moreover, we write $b = b'/b''$, where $0 \neq b', b'' \in \Gamma(V_{\bar{x}}, \mathcal{O}_{V_{\bar{x}}}) = f_*\mathcal{O}_V \otimes_{\mathcal{O}_U} \mathcal{O}_{U_{\bar{x}}}^{\text{sh}}$ (this is an equality of $\mathcal{O}_{U_{\bar{x}}}^{\text{sh}}$ -algebras). Recall that $\mathcal{O}_{U_{\bar{x}}}^{\text{sh}} = \varinjlim_W \Gamma(W, \mathcal{O}_W)$, where the colimit traverses all étale neighborhoods $W \rightarrow U$ of $\bar{x} \rightarrow U$. In particular, we may write

$$\Gamma(V_{\bar{x}}, \mathcal{O}_{V_{\bar{x}}}) = f_*\mathcal{O}_V \otimes_{\mathcal{O}_U} \mathcal{O}_{U_{\bar{x}}}^{\text{sh}} = \varinjlim_W (f_*\mathcal{O}_V \otimes_{\mathcal{O}_U} \Gamma(W, \mathcal{O}_W)) = \varinjlim_W \Gamma(V_W, \mathcal{O}_{V_W}),$$

where $f_*\mathcal{O}_V \otimes_{\mathcal{O}_U} \Gamma(W, \mathcal{O}_W) = \Gamma(V_W, \mathcal{O}_{V_W})$ is an equality of $\Gamma(W, \mathcal{O}_W)$ -algebras. Let $W \rightarrow U$ be a sufficiently small connected étale neighborhood of $\bar{x} \rightarrow U$ such that $\Gamma(W, \mathcal{O}_W) \ni a, aa_{n-1}, \dots, aa_0$,⁹ and such that $b', b'' \in f_*\mathcal{O}_V \otimes_{\mathcal{O}_U} \Gamma(W, \mathcal{O}_W) = \Gamma(V_W, \mathcal{O}_{V_W})$. In particular, we have that $a_i \in K(W)$ for all $i = 1, \dots, n-1$, and further $b \in K(V_W)$. However, we proved above that $K(V_W)/K(W)$ is Galois (so normal). Hence, the polynomial $\alpha(t)$ must split in $K(V_W)$ and consequently in $K(V_{\bar{x}})$. \square

By assumption $x \in U_{\text{reg}}$, and so $U_{\bar{x}} := \text{Spec } \mathcal{O}_{U_{\bar{x}}}^{\text{sh}}$ is a regular strictly henselian scheme and in particular factorial. Hence, $P_{\bar{x}} = \text{div } w$ for some $0 \neq w \in U_{\bar{x}} := \text{Spec } \mathcal{O}_{U_{\bar{x}}}^{\text{sh}}$. Moreover, by the purity of the branch locus for regular schemes, it follows that every Galois quasi-étale cover over $U_{\bar{x}} := \text{Spec } \mathcal{O}_{U_{\bar{x}}}^{\text{sh}}$ is trivial. Combining everything, we conclude that $\pi_1^{t, P_{\bar{x}}}(U_{\bar{x}}) = \hat{\mathbb{Z}}^{(p)}$ by [CS23, Theorem 3.29]. Hence, $V_{\bar{x}} \rightarrow U_{\bar{x}}$ is further cyclic and so a Kummer cover with ramification index e ; cf. [CS23, Lemma 3.34]. This description remains valid on a sufficiently small étale neighborhood W of $\bar{x} \rightarrow U$ where $\text{div } w = P_W$. \square

The following result will be useful in Section 4.

COROLLARY 3.10. *With the same hypothesis as in Theorem 3.8, set $U' := U_{\text{reg}} \setminus P$. Additionally, suppose that P contains all closed points of U . Then, $V_{U'} \rightarrow U'$ is a connected μ_e -torsor in the sense of [Mil80, § III.4] (that is, a torsor in the fppf or flat topology).*

Proof. Since P contains all closed points of U , the intersection $P \cap U_{\text{reg}}$ contains all the closed points of U_{reg} .¹⁰ Applying Theorem 3.8, we find an étale covering $\{W_i \rightarrow U_{\text{reg}}\}_{i \in I}$ such that $f_{W_i}: V_{W_i} \rightarrow W_i$ is a Kummer cover of degree e . In particular, the pullback of f_{W_i} to U' is a μ_e -torsor. Since these torsors are all induced by a morphism $f: V \rightarrow U$, they clearly glue. Finally, we observe that connectedness follows from normality. \square

Remark 3.11. Let $Y \rightarrow X$ be a connected μ_e -torsor (in the flat topology) with X, Y normal integral schemes. We may use Kummer theory to describe $Y \rightarrow X$ in terms of cyclic covers; see [Mil80, § III.4]. Indeed, there exists an invertible sheaf \mathcal{L} on X of index e such that $Y \rightarrow X$ is isomorphic to a cyclic cover $\text{Spec}_X \bigoplus_{i=0}^{e-1} \mathcal{L}^i \rightarrow X$. In particular, if the Picard group of X is finitely generated, e cannot be arbitrarily large.

⁹To be precise, we choose W such that the image of $\Gamma(W, \mathcal{O}_W)$ under the canonical homomorphism $\Gamma(W, \mathcal{O}_W) \rightarrow \mathcal{O}_{U_{\bar{x}}}^{\text{sh}}$ contains a and aa_i for all i .

¹⁰Indeed, a closed point of U_{reg} is necessarily a closed point of U by [Sta22, Tag 07P8] (or directly by [Sta22, Tag 0541] and the fact that the Zariski closure of a point is irreducible).

3.3 Proof of Theorem 3.5. With the above in place, we are ready to proceed with the proof of the main theorem of this section.

Proof of Theorem 3.5. First of all, we recall that $P_{\bar{x}}$ is a prime divisor on $\mathcal{O}_{X,\bar{x}}^{\text{sh}}$ so that our local hypothesis (2) makes sense; see Remark 3.2.

Proof of property (A). Let $f: V \rightarrow U$ be a connected cover in $\text{Rev}^P(U)$. Since f is finite and surjective, proving property (A) amounts to showing that there is exactly one DVR of $K(V)$ lying over $\mathcal{O}_{U,P} \subset K(U)$ or, equivalently, that there is exactly one point of V lying over the generic point of P . Indeed, $(f^{-1}P)_{\text{red}}$ is the reduced closed subscheme of V whose irreducible components are given by the prime divisors associated with the (necessarily) codimension 1 points of V lying over the generic point of P . Since we assume $f^{-1}P$ to be connected by hypothesis (1), so is $(f^{-1}P)_{\text{red}}$. Hence, it suffices to prove that the prime divisors of V lying over P do not intersect. Suppose for the sake of contradiction that two such divisors, say Q and Q' , meet at a closed point $y \in V$, and let $x = f(y) \in Y$. Let $W \rightarrow U_{\bar{x}}$ be the connected component of $f_{\bar{x}}: V_{\bar{x}} \rightarrow U_{\bar{x}}$ that contains $\bar{y} \rightarrow V_{\bar{x}}$; see [CS23, Lemma 3.31]. Then, $W \rightarrow U_{\bar{x}}$ is a connected object in $\text{Rev}^{P_{\bar{x}}}(U_{\bar{x}})$ that violates $P_{\bar{x}}$ -irreducibility and so our hypothesis (2); this gives a contradiction. Indeed, the pullbacks of Q and Q' to W are two prime divisors on W lying over $P_{\bar{x}}$. \square

Proof of property (B). Property (B) is a formal consequence of property (A); see Remark 3.4. \square

Proof of property (C). The existence of the stated quasi-étale Galois cover in $\text{Rev}^P(U)$ follows from applying Theorem 3.7. Indeed, consider a tower

$$U = U_0 \xleftarrow{\gamma_0} U_1 \xleftarrow{\gamma_1} U_2 \xleftarrow{\gamma_2} \dots$$

of quasi-étale Galois covers in $\text{Rev}^P(U)$. Since $\text{Rev}^{P_{\bar{x}}}(U_{\bar{x}})$ is inertially bounded for all geometric points $\bar{x} \rightarrow U$, the maps $\gamma_{i,\bar{x}}$ are trivial for sufficiently large i . By applying Theorem 3.7, we conclude that the maps γ_i are étale for sufficiently large i . By abstract nonsense regarding Galois categories (keeping in mind Remark 3.6), we obtain the desired statement. \square

Proof of the final statement. Consider a tower

$$(\bar{U}, \bar{P}) = (\bar{U}_0, \bar{Q}_0) \xleftarrow{\beta_0} (\bar{U}_1, \bar{Q}_1) \xleftarrow{\beta_1} (\bar{U}_2, \bar{Q}_2) \xleftarrow{\beta_2} \dots$$

of étale Galois covers in $\text{Rev}^{\bar{P}}(\bar{U})$. When restricting to \bar{P} , we obtain a tower of étale covers

$$\bar{P} = \bar{Q}_0 \xleftarrow{\alpha_0} \bar{Q}_1 \xleftarrow{\alpha_1} \bar{Q}_2 \xleftarrow{\alpha_2} \dots$$

Here, we use that the maps β_i are étale to say that $\bar{Q}_{i+1} = (\beta_i^{-1}\bar{Q}_i)_{\text{red}} = \beta_i^{-1}\bar{Q}_i$. In particular, all the prime divisors \bar{Q}_i are normal. In this manner, the α_i must eventually all be isomorphisms as $\pi_1^{\text{ét}}(\bar{P})$ is assumed finite. Using this, Remark 3.6, and property (B), we obtain an object (\tilde{U}, \tilde{P}) that satisfies property (i). We obtain that (\tilde{U}, \tilde{P}) satisfies properties (ii) and (iii) as a direct application of Theorem 3.8. \square

This completes the proof of Theorem 3.5. \square

4. Finiteness and tameness of local étale fundamental groups

We now come to the finiteness of $\pi_1^{\text{ét}}(S^\circ)^{(p)}$ and eventually to the tameness and finiteness of $\pi_1^{\text{ét}}(S^\circ)$. Throughout this section, we work in Setup 1.1 assuming $d = 3$ and \mathbb{k} to be algebraically closed of characteristic $p > 5$.

First, use Theorem 2.4 to ensure the existence of a *generalized* PLT blowup $\pi: (T, \Delta_T) \rightarrow S$ extracting a Kollár component (E, Δ_E) ,¹¹ which is a log del Pezzo pair. Recall that $(T, E + \Delta_T)$ is a PLT pair, whereas (T, E) is a purely F -regular one. Define $\Delta_S := \pi_*\Delta_T$, so (S, Δ_S) is KLT (cf. Remark 2.2). We denote by $Z' \subset T$ the strict transform of Z along π ,¹² and we write $U := T \setminus Z'$. By abuse of notation, we denote by E the restriction of E to U . Further, we observe that $S^\circ = U \setminus E$. The following commutative diagram depicts the situation:

$$\begin{array}{ccccc}
 S^\circ & \xrightarrow{\iota} & U & \xrightarrow{o} & T \\
 & \searrow i & \downarrow \varpi & \swarrow \pi & \\
 & & S & &
 \end{array} \tag{4.1}$$

where i , ι , and o are the open immersions. Next, we consider the following observation of Grothendieck–Murre, whose proof we revisit for the reader’s convenience and for the sake of completeness (as we work in a slightly different setup).

PROPOSITION 4.1 ([GM71, Corollary 9.8]). *The restriction functor defined by ι*

$$\mathrm{Rev}^E(U) \rightarrow \mathrm{FEt}(S^\circ)$$

induces a surjective homomorphism of topological groups

$$\lambda: \pi_1^{\acute{e}t}(S^\circ) \rightarrow \pi_1^{\mathrm{t},E}(U)$$

that induces an isomorphism on prime-to- p parts:

$$\mu = \lambda^{(p)}: \pi_1^{\acute{e}t}(S^\circ)^{(p)} \xrightarrow{\cong} \pi_1^{\mathrm{t},E}(U)^{(p)}.$$

Thus, we have surjective homomorphisms of topological groups

$$\pi_1^{\acute{e}t}(S^\circ) \twoheadrightarrow \pi_1^{\mathrm{t},E}(U) \twoheadrightarrow \pi_1^{\acute{e}t}(S^\circ)^{(p)}.$$

Proof. By abstract nonsense regarding Galois categories [Mur67, §5.2.1], to show that λ is surjective, we must prove that any connected (Galois) cover $V \rightarrow U$ in $\mathrm{Rev}^E(U)$ remains connected when restricted to S° . However, this follows from normality as connectedness and irreducibility are equivalent notions under normality and because irreducibility is preserved under restriction to open subschemes.¹³

To see why $\mu = \lambda^{(p)}$ is injective, we proceed as follows. We must verify that any Galois cover $Y \rightarrow S^\circ$ in $\mathrm{FEt}(S^\circ)$ of order prime-to- p is obtained as the restriction of some Galois cover in $\mathrm{Rev}^E(U)$ (whose order is necessarily prime-to- p). To this end, we consider the Galois field extension $K \subset K(Y)$. Since π is birational, we have $K(U) = K$. Thus, we may let $V \rightarrow U$ be the normalization of U in $K(Y)$. Of course, the pullback of $V \rightarrow U$ along ι is $Y \rightarrow S^\circ$, and moreover $K(Y)/K$ is tame with respect to $\mathcal{O}_{U,E}$ as p does not divide $[K(Y) : K]$. \square

Remark 4.2. In the proof of Proposition 4.1, we mentioned the relationship between the Galois categories $\mathrm{Rev}^E(U)$ and $\mathrm{FEt}(S^\circ)$. We would like expand on this as it will facilitate and clarify the

¹¹To be precise, we construct the PLT blowup on a Zariski open neighborhood of $x \in X$, and then we pull it back to S . However, as pointed out to us by an anonymous referee, this may now be done directly by using the recent work [BMP⁺23].

¹²That is, Z' is the Zariski closure of $Z \cap (S \setminus \{x\})$ in T . Note that, by construction, $S \setminus \{x\}$ is an open subscheme of both S and T .

¹³In general, any non-empty open subset of an irreducible topological space is (dense and) irreducible [Har77, Exercise I.1.6].

upcoming discussion. Let K_1/K be a finite Galois extension. We may then take the normalization of the diagram (4.1), so that we have

$$\begin{array}{ccccc}
 & & S_1^\circ & \xrightarrow{\iota_1} & U_1 & \xrightarrow{o_1} & T_1, \\
 & & \swarrow & & \downarrow \varpi_1 & & \swarrow \pi_1 \\
 S^\circ & \xrightarrow{\iota} & U & \xrightarrow{o} & T & \xrightarrow{k} & S_1 \\
 & \searrow i & \downarrow \varpi & & \swarrow \pi & & \swarrow \pi \\
 & & S & & & &
 \end{array}$$

where the dotted, unlabeled arrows represent the normalization morphisms. Note that both top (slanted) rectangles are cartesian, as well as the one most to the left, for i , ι , and o are open immersions. Note that $S_1 = \text{Spec } R_1$, where R_1 —the integral closure of R in K_1 —is a strictly local ring as R is strictly local and S_1 is connected. We may say that the content of Proposition 4.1 is that S_1°/S° is étale if U_1/U is at worst tamely ramified over E and the converse (trivially) holds provided that $[K_1 : K]$ is prime-to- p . Next consider the commutative square (4.2) making the lateral right face of the diagram:

$$\begin{array}{ccc}
 T & \xleftarrow{g} & T_1 \\
 \pi \downarrow & & \downarrow \pi_1 \\
 S & \xleftarrow{f} & S_1.
 \end{array} \tag{4.2}$$

This is the diagram we will focus on. Note that $(S_1, f^* \Delta_S =: \Delta_{S_1})$ is a KLT singularity [Kol13, Corollary 2.43]. Similarly, we define $\Delta_{T_1} := g^* \Delta_T$. Since g and f are finite (and so proper), we have that π_1 is proper because π is proper. Thus, π_1 is a proper birational morphism with exceptional locus equal to $E_1 := g^{-1}E$ by the commutativity of the diagram. We conclude that E_1 is then connected by Zariski’s main theorem; see [Har77, Corollary III.11.4].

With the above in place, our next goal is to prove that $\pi_1^{t,E}(U)$ is finite, so that $\pi_1^{\text{ét}}(S^\circ)^{(p)}$ is finite as well in the rational case.

THEOREM 4.3 (cf. [XZ19, Theorem 3.4], [HW22b, Proposition 5.2]). *The group $\pi_1^{t,E}(U)$, and so $\pi_1^{\text{ét}}(S^\circ)^{(p)}$, is finite.*

Proof. We apply Theorem 3.5 to the case $X \leftrightarrow T$, $Z \leftrightarrow Z'$, $U \leftrightarrow U$, and $P \leftrightarrow E$. First, we verify that the hypotheses (1) and (2) of Theorem 3.5 hold in this context.

CLAIM 4.4. *The Galois category $\text{Rev}^E(U)$ satisfies hypotheses (1) and (2) in Theorem 3.5. Furthermore, for any connected object (U_1, E_1) in $\text{Rev}^E(U)$,*

- E_1 is a normal prime divisor on T_1 ;
- (T_1, E_1) is a purely F -regular pair, and so $\pi_1 : (T_1, \Delta_{T_1}) \rightarrow S_1$ is a generalized PLT blowup with respective Kollár component (E_1, Δ_{E_1}) ;
- $\pi_1^{\text{ét}}(E_1 \cap U_1)$ is finite.

Proof of the claim. Why $\text{Rev}^E(U)$ satisfies hypothesis (1) was explained in the last paragraph of Remark 4.2.

We proceed to verify (2). Recall that (T, E) has purely F -regular singularities. In particular, for every geometric point $\bar{x} \rightarrow U$, the local pair $(U_{\bar{x}}, E_{\bar{x}})$ is a purely F -regular local pair in the

sense of [CS23, Definition 2.3]. Thus, we may apply [CS23, Theorem 4.12], which establishes that $\text{Rev}^{E_{\bar{x}}}(U_{\bar{x}})$ is $E_{\bar{x}}$ -irreducible, inertially bounded, and inertially tame.

Let us explain next why (U_1, E_1) is purely F -regular, so that E_1 is strongly F -regular and so normal. Further, E_1 is a normal prime divisor as it is connected. The pure F -regularity of (U_1, E_1) is a local condition that can be checked at every geometric point. Moreover, it can be checked at every geometric point $\bar{x} \rightarrow U$. Since $(U_{\bar{x}}, E_{\bar{x}})$ is purely F -regular, we may apply [CS23, Theorem 4.7] to conclude that every connected component of $(E_{1,\bar{x}}, E_{1,\bar{x}})$ is a purely F -regular local pair, as desired.

Since g is a tame cover with respect to E , the pair $(T_1, E_1 + \Delta_{T_1})$ is PLT by [Kol13, Corollary 2.43], and in fact $g^*(K_T + E + \Delta_T) = K_{T_1} + E_1 + \Delta_{T_1}$. Further, we see that $-(K_{T_1} + E_1 + \Delta_{T_1})$ is ample over S and the contraction of E_1 is precisely π_1 and $\pi_{1,*}\Delta_{T_1} = \Delta_{S_1}$. This concludes the proof of the second bullet point.

For the finiteness of $\pi_1^{\text{ét}}(E_1 \cap U_1)$, we use Theorem 2.6, noting that $E_1 \cap U_1$ is a big open of E_1 ; see footnote 8 on page 704. \square

By Claim 4.4, we may apply Theorem 3.5 to obtain a Galois quasi-étale cover $f: \tilde{U} \rightarrow U$ satisfying properties (i) and (ii) in Theorem 3.5. In particular, it suffices to explain why $\pi_{\mathbb{1}}^{\text{t},\tilde{E}}(\tilde{U})$ is finite. To this end, we use Corollary 3.10 and Proposition B.1. First, we observe that \tilde{E} goes through every closed point of \tilde{U} . Indeed, using the construction and notation of Remark 4.2, we observe that $\tilde{\pi}: \tilde{T} \rightarrow \tilde{S}$ is a proper morphism over the spectrum of a local ring and \tilde{E} is the fiber over *the* closed point, and so it must contain every closed point of \tilde{T} . Thus, by Corollary 3.10 (cf. Remark 3.11), the degree of a Galois cover in $\text{Rev}^{\tilde{E}}(\tilde{U})$ is bounded by the degree of a connected prime-to- p cyclic cover over

$$\tilde{U}_{\text{reg}} \setminus \tilde{E} = \tilde{S}^{\circ} \cap \tilde{U}_{\text{reg}} = \tilde{S}_{\text{reg}}^{\circ} = \tilde{S}^{\circ} \cap \tilde{S}_{\text{reg}} =: O,$$

and so by the index of a prime-to- p torsion element of $\text{Pic } O$. However, O is a regular big open of \tilde{S} , and so $\text{Pic } O = \text{Cl } O = \text{Cl } \tilde{S}$, which has finitely generated prime-to- p part by Proposition B.1. Thus, the degree of a Galois object in $\text{Rev}^{\tilde{E}}(\tilde{U})$ cannot be arbitrarily large. This implies the existence of a universal Galois cover in $\text{Rev}^{\tilde{E}}(\tilde{U})$ and further the finiteness of $\pi_{\mathbb{1}}^{\text{t},\tilde{E}}(\tilde{U})$. \square

The following is the main result of this paper, which we will prove as a formal Galois-theoretic consequence of Corollary 2.15 and Theorem 4.3.

COROLLARY 4.5. *The canonical surjection $\pi_{\mathbb{1}}^{\text{ét}}(S^{\circ}) \twoheadrightarrow \pi_{\mathbb{1}}^{\text{t},E}(U) \twoheadrightarrow \pi_{\mathbb{1}}^{\text{ét}}(S^{\circ})^{(p)}$ is an isomorphism. Hence, $\pi_{\mathbb{1}}^{\text{ét}}(S^{\circ})$ is tame and finite.*

Proof. Let K_1/K be a Galois extension whose Galois group $G = \text{Gal}(K_1/K)$ has order divisible by p . Suppose, for the sake of contradiction, that K_1/K is an object in $\text{FEt}(S^{\circ})$. Formally, this is tantamount to the existence of a homomorphism $\pi_{\mathbb{1}}^{\text{ét}}(S^{\circ}) \twoheadrightarrow G$. Let $S_1 \rightarrow S$ be the corresponding quasi-étale cover; that is, $S_1 = \text{Spec } R_1$, where R_1 is the integral closure of R in K_1 , which is a strictly local ring. Let H be the p -Sylow group of G , which is non-trivial by assumption. Let $S_0 := S_1/H = \text{Spec } R_1^H$ be the corresponding quotient (R_1^H is also a strictly local ring). Consider the induced factorization $S_1 \rightarrow S_0 \rightarrow S$ of $S_1 \rightarrow S$, noting that $S_0 \rightarrow S$ is étale over S° (and so quasi-étale) and $S_1 \rightarrow S_0$ is an H -torsor over the inverse image of S° along $S_0 \rightarrow S$, say S_0° . In particular, S_0 is a KLT singularity by [Kol13, Corollary 2.43] just as S is. Then, applying Corollary 2.15 to S_0 yields the sought contradiction. Indeed, the pullback of $S_1 \rightarrow S_0$ to S_0° defines a non-trivial element of $H^1(S_{0,\text{fl}}^{\circ}, H)$, which is a singleton according to Corollary 2.15. \square

Remark 4.6. Note that Corollary 4.5 and so Theorem 2.13 do not hold in characteristics $p = 2, 3, 5$ by examples in [Yas19, Tot19].

We finalize with the following—by now standard—application [GKP16, BCG⁺19, Sti17]. Roughly speaking, it establishes that threefolds with at worst KLT singularities *almost* satisfy Zariski–Nagata–Auslander purity of the branch locus.

COROLLARY 4.7. *Work in Setup 1.1 assuming $d = 3$ and \mathbb{k} to be algebraically closed of characteristic $p > 5$. Then, every quasi-étale cover over X whose degree is a power of p is étale. Moreover, there exists a quasi-étale Galois cover $\tilde{X} \rightarrow X$ with prime-to- p degree such that every quasi-étale cover over \tilde{X} is étale. Moreover, $\tilde{X} \rightarrow X$ can be taken so that it is étale over any prescribed big open of X .*

Proof. The first statement was already observed in more generality in Corollary 2.19. If $Y \rightarrow X$ is a quasi-étale cover whose degree is a power of p , then the same is true for $Y \times_X \text{Spec } \mathcal{O}_{X,\bar{x}}^{\text{sh}} \rightarrow \text{Spec } \mathcal{O}_{X,\bar{x}}^{\text{sh}}$ at every geometric closed point $\bar{x} \rightarrow X$. Hence they are all étale by Corollary 2.15, and so is $Y \rightarrow X$.

The existence of a quasi-étale Galois cover $X' \rightarrow X$ such that quasi-étale covers over X' are étale follows from the methods in [BCG⁺19], as well as [Sti17, Theorem 1] because X is a variety. However, it also follows by the same methods that there exists a quasi-étale Galois cover $\tilde{X} \rightarrow X$ with prime-to- p degree such that every quasi-étale cover over \tilde{X} with prime-to- p degree is étale. We claim that all quasi-étale covers over \tilde{X} are étale. It suffices to prove this for the connected and further Galois ones. Thus, let $Y \rightarrow \tilde{X}$ be a quasi-étale Galois cover. Let $H \subset \text{Gal}(Y/\tilde{X})$ be the corresponding p -Sylow group, which we may assume non-trivial as otherwise there is nothing to prove. Let $Y \rightarrow Y/H \rightarrow \tilde{X}$ be the corresponding quotient. Thus, $Y \rightarrow Y/H$ is a quasi-étale Galois cover with Galois group H and $Y/H \rightarrow \tilde{X}$ is quasi-étale and so étale. In particular, Y/H is a KLT threefold just as X and \tilde{X} are. Hence, by the first part of the corollary, $Y \rightarrow Y/H$ must be étale and so is $Y \rightarrow \tilde{X}$. \square

COROLLARY 4.8. *Work in Setup 1.1, assuming $d = 3$, the field \mathbb{k} to be algebraically closed of characteristic $p > 5$, and (X, Δ) to be a log Fano pair. If $U \subset X$ is a big open subset, then $\pi_1^{\text{ét}}(U)$ is finite and tame.*

Proof. By Corollary 4.7, there is a tame quasi-étale Galois cover $f: \tilde{X} \rightarrow X$ that is étale over U and such that the induced homomorphism $\pi_1^{\text{ét}}(\tilde{X}_U) \rightarrow \pi_1^{\text{ét}}(\tilde{X})$ is an isomorphism. In particular, it suffices to show that $\pi_1^{\text{ét}}(\tilde{X})$ is finite and tame. To this end, we observe that $(\tilde{X}, f^*\Delta)$ is a log Fano pair and so rationally chain connected by [GNT19, Theorem 1.5]. Then, the result follows from [Cha03]. \square

Appendix A. Local Picard schemes and local divisor class groups

Inspired by [Mum61, § II(a)], Grothendieck proposed in [Gro05, Exposé VIII, § 5] the construction of a local Picard scheme; see [FGI⁺05, Remark 9.4.19] for an account. Such a construction would provide a connection between the local cohomology module $H_{\mathfrak{m}_{\bar{x}}}^2(\mathcal{O}_{X,\bar{x}}^{\text{sh}})$ and the Picard group of the punctured spectrum of $\mathcal{O}_{X,\bar{x}}^{\text{sh}}$. Also see [EV10, § 3], where the 2-dimensional case is worked out. The construction of a local Picard scheme was obtained by Lipman [Lip76] and subsequently by Boutot [Bou78]. For the reader’s convenience, we briefly recall Boutot’s approach and main results. For details, see [Bou78].

Throughout this section, we let $(R, \mathfrak{m}, \mathbb{k})$ be a noetherian strictly local \mathbb{k} -algebra and denote by S and S° its spectrum and punctured spectrum, respectively.

For a given \mathbb{k} -algebra A , one defines $R \otimes_{\mathbb{k}}^{\text{sh}} A$ to be the henselization of $R \otimes_{\mathbb{k}} A$ along the ideal $\mathfrak{m} \otimes_{\mathbb{k}} A$. There is a canonical morphism

$$\alpha_A: \text{Spec}(R \otimes_{\mathbb{k}}^{\text{sh}} A) \rightarrow S,$$

and one defines

$$\tilde{S}_A^\circ := \alpha_A^{-1}(S^\circ).$$

Next, one considers the presheaf of abelian groups on \mathbb{k} -algebras given by

$$A \mapsto \text{Pic } \tilde{S}_A^\circ.$$

One defines the *local Picard functor* $\mathcal{P}ic_{S/\mathbb{k}}$ to be the sheafification of this presheaf in the étale topology on \mathbb{k} -algebras.

The first main result we need from [Bou78] is the following.

THEOREM A.1 ([Bou78]). *Suppose $\text{depth } R \geq 2$ and that $H_{\mathfrak{m}}^2(R) = H^1(S^\circ, \mathcal{O}_{S^\circ})$ is finite over \mathbb{k} . Then, $\mathcal{P}ic_{S/\mathbb{k}}$ is represented by a (commutative) group scheme locally of finite type over \mathbb{k} , say $\text{Pic}_{S/\mathbb{k}}$. Moreover, the tangent space of $\text{Pic}_{S/\mathbb{k}}$ at the identity is isomorphic to $H^1(S^\circ, \mathcal{O}_{S^\circ})$ as \mathbb{k} -modules.*

DEFINITION A.2. With notation as in Theorem A.1, one refers to $\text{Pic}_{S/\mathbb{k}}$ as the *local Picard scheme* of R (or S). One denotes by $\text{Pic}_{S/\mathbb{k}}^0$ the connected component of $\text{Pic}_{S/\mathbb{k}}$ at the identity, which by general principles is an algebraic group over \mathbb{k} ; see [FGI+05, Lemma 9.5.1] or [Bri17, Theorem 2.4.1]. The *local Néron–Severi group* is defined as the quotient

$$\text{NS}_{S/\mathbb{k}} := \text{Pic}_{S/\mathbb{k}} / \text{Pic}_{S/\mathbb{k}}^0.$$

Remark A.3. With notation as in Theorem A.1, suppose that R is quasi-excellent and normal of dimension $d \geq 3$. Then, $\text{Pic}_{S/\mathbb{k}}$ exists by Lemma 2.16. In that case, the \mathbb{k} -points of $\text{Pic}_{S/\mathbb{k}}$ correspond to the Picard group of S° . More precisely,

$$\text{Pic}_{S/\mathbb{k}}(\text{Spec } \mathbb{k}) = \mathcal{P}ic_{S/\mathbb{k}}(\mathbb{k}) = \text{Pic } S^\circ \subset \text{Cl } S^\circ = \text{Cl } S.$$

More generally, if A is a strictly local \mathbb{k} -algebra, then the A -points on $\text{Pic}_{S/\mathbb{k}}$ can be computed as follows:

$$\text{Pic}_{S/\mathbb{k}}(\text{Spec } A) = \mathcal{P}ic_{S/\mathbb{k}}(A) = \text{Pic } \tilde{S}_A^\circ.$$

Technically speaking, this says that the canonical homomorphism $\text{Pic } \tilde{S}_A^\circ \rightarrow \mathcal{P}ic_{S/\mathbb{k}}(A)$ is an isomorphism if A is a strictly local \mathbb{k} -algebra; see [Bou78, Proposition II.3.2]. For instance, recall that the tangent space of $\text{Pic}_{S/\mathbb{k}}$ at the identity is

$$\text{Lie}(\text{Pic}_{S/\mathbb{k}}) := \ker(\mathcal{P}ic_{S/\mathbb{k}}(\mathbb{k}[\varepsilon]/\varepsilon^2) \rightarrow \mathcal{P}ic_{S/\mathbb{k}}(\mathbb{k})).$$

Hence, since \mathbb{k} and $\mathbb{k}[\varepsilon]/\varepsilon^2$ are both strictly local \mathbb{k} -algebras, there is a canonical isomorphism

$$\ker(\text{Pic}(S^\circ[\varepsilon]/\varepsilon^2) \rightarrow \text{Pic}(S^\circ)) \xrightarrow{\cong} \text{Lie}(\text{Pic}_{S/\mathbb{k}})$$

and further

$$\ker(H^1(S^\circ[\varepsilon]/\varepsilon^2, \mathcal{O}_{S^\circ[\varepsilon]/\varepsilon^2}^\times) \rightarrow H^1(S^\circ, \mathcal{O}_{S^\circ}^\times)) \xrightarrow{\cong} \text{Lie}(\text{Pic}_{S/\mathbb{k}}).$$

Now, considering the short exact sequence of sheaves on $S^\circ[\varepsilon]/\varepsilon^2$

$$0 \rightarrow \mathcal{O}_{S^\circ} \xrightarrow{a \mapsto 1+a\varepsilon} \mathcal{O}_{S^\circ[\varepsilon]/\varepsilon^2}^\times \rightarrow \mathcal{O}_{S^\circ}^\times \rightarrow 1$$

and the corresponding long exact sequence on cohomology, we see why $\mathrm{Lie}(\mathrm{Pic}_{S/\mathbb{k}})$ is canonically isomorphic to $H^1(S^\circ, \mathcal{O}_{S^\circ}) = H_{\mathfrak{m}}^2(R)$ (cf. [Har77, Exercise III.4.6]). For more details, see [Bou78, Proposition II.4.2], whose proof specializes to the above argument by taking $A' \rightarrow A \rightarrow A_0$ in *ibid.* to be $\mathbb{k}[\varepsilon]/\varepsilon^2 \rightarrow \mathbb{k} \rightarrow \mathbb{k}$. In particular, if R is further Cohen–Macaulay, then $\mathrm{Pic}_{S/\mathbb{k}}^0 = 0$ as its tangent space at the identity $H_{\mathfrak{m}}^2(R)$ is zero. That is, $\mathrm{NS}_{S/\mathbb{k}} \cong \mathrm{Pic}_{S/\mathbb{k}}$.

THEOREM A.4 ([Bou78]). *Suppose that \mathbb{k} is algebraically closed and R is excellent, normal, and has dimension 3. Then, $\mathrm{NS}_{S/\mathbb{k}}(\mathbb{k})$ is a finitely generated abelian group.*

Remark A.5. Boutot proved Theorem A.4 to hold in higher dimensions as well if R is further strongly desingularizable; see [Bou78, §V.3]. However, as Boutot remarks, if \mathbb{k} has characteristic 0, this condition follows from [Hir64] or else by [Abh66, CP19] if $\dim R \leq 3$.

As a direct consequence of Theorem A.4, one obtains the following important corollaries; cf. [Bou78, Corollaire V.4.9].

COROLLARY A.6. *Work in the setup of Theorem A.4. If S is Cohen–Macaulay, then $\mathrm{Pic} S^\circ$ is a finitely generated abelian group.*

Proof. As explained in Remark A.3, we have $\mathrm{NS}_{S/\mathbb{k}} \cong \mathrm{Pic}_{S/\mathbb{k}}$ if R is Cohen–Macaulay. Then, by computing \mathbb{k} -points under this isomorphism, we obtain the claimed statement by using Theorem A.4. \square

We are thankful to Thomas Polstra for his help with the following corollary, which answers [Pol22, Question 1] in dimension 3.

COROLLARY A.7. *Work in the setup of Theorem A.4. If S is rational, then $\mathrm{Cl} S$ is a finitely generated abelian group.*

Proof. Since rational threefold singularities are Cohen–Macaulay, $\mathrm{Pic} S^\circ$ is finitely generated by Corollary A.6. Note that the punctured spectrum of a rational threefold singularity is \mathbb{Q} -factorial; see [Lip69, EV10]. Indeed, S° is a 2-dimensional normal scheme with rational singularities, and so it has isolated 2-dimensional rational singularities which all have finite (and so torsion) divisor class group [EV10, Lemma 3.1]. In particular, there is an $n \in \mathbb{N}$ with $n > 0$ such that $n \cdot \mathrm{Cl} S \subset \mathrm{Pic} S^\circ$ (note that n can be taken to be 1 if S has an isolated singularity, meaning that S° is regular). Thus, there is an exact sequence of abelian groups

$$0 \rightarrow \mathrm{Pic} S^\circ \rightarrow \mathrm{Cl} S \rightarrow T \rightarrow 0,$$

where T is n -torsion. This immediately implies that torsion in $\mathrm{Cl} S$ is bounded. However, under closer inspection, T is finite, and hence $\mathrm{Cl} S$ is finitely generated as it is the extension of finitely generated groups. To see why T is finite, consider the following. Let $s_1, \dots, s_k \in S^\circ$ be the finitely many singular (rational) points of S° . Set $S_i := \mathrm{Spec} \mathcal{O}_{S^\circ, s_i}$. As we mentioned before, we know that the groups $\mathrm{Cl} S_i = \mathrm{Pic}(S_i \setminus s_i)$ are finite by [EV10, Lemma 3.1]. Note that $\delta \in \mathrm{Cl} S = \mathrm{Cl} S^\circ$ belongs to $\mathrm{Pic} S^\circ$ if and only if δ belongs to the kernel of each homomorphism $\mathrm{Cl} S \rightarrow \mathrm{Cl} S_i$. In other words,

$$\mathrm{Pic} S^\circ = \ker \left(\mathrm{Cl} S \rightarrow \bigoplus_{i=1}^k \mathrm{Cl} S_i \right),$$

which implies that T is isomorphic to a subgroup of the finite group $\bigoplus_{i=1}^k \mathrm{Cl} S_i$. \square

Appendix B. On local class groups

János Kollár

The aim of this appendix is to prove two propositions about factoriality properties of log terminal schemes.

From now on, we work with Noetherian and excellent schemes. We use the acronym klt for Kawamata log terminal and dlt for divisorial log terminal.

PROPOSITION B.1. *Let (y, Y, Δ) be a 3-dimensional, dlt, local scheme (see Remark B.2) of residue characteristic $p > 0$. Then the prime-to- p parts of $\text{Cl}(Y)$, $\text{Cl}(Y^{\text{h}})$ and of $\text{Cl}(\hat{Y})$ are finitely generated.*

Here $\text{Cl}(\)$ denotes the class group, Y^{h} the Henselisation and \hat{Y} the completion in the topology induced by the maximal ideal.

The p^∞ -part of an Abelian group A is the subgroup $A[p^\infty] \subset A$ consisting of elements whose order is a power of p . The quotient $A/A[p^\infty]$ is the *prime-to- p part* of A . If $A/A[p^\infty]$ is finitely generated, then $A \cong A[p^\infty] \oplus A/A[p^\infty]$.

Remark B.2. The current reference [Kol21] and work in progress by J. Witaszek cover the cases when X is essentially of finite type over a perfect field. Arbitrary excellent 3-folds (including mixed characteristic) are treated in [BMP+23] and in work in progress by C. Hacon and J. Witaszek.

Remark B.3. Let $(y \in Y)$ be a normal, local domain with Henselisation Y^{h} and completion \hat{Y} . There are injections

$$\text{Cl}(Y) \hookrightarrow \text{Cl}(Y^{\text{h}}) \hookrightarrow \text{Cl}(\hat{Y});$$

cf. [Mum76, §1C]. Thus finite generation for $\text{Cl}(\hat{Y})$ implies finite generation for $\text{Cl}(Y)$ and $\text{Cl}(Y^{\text{h}})$.

Frequently, $\text{Cl}(Y^{\text{h}})$ is much bigger than $\text{Cl}(Y)$ (see Example B.4 in dimension 3 or [Kol16, Example 27] for a 2-dimensional log canonical example in characteristic 0), but $\text{Cl}(Y^{\text{h}}) = \text{Cl}(\hat{Y})$ if Y has an isolated singularity [Art69, Theorem 3.10].

Note that Proposition B.1 is new only when (y, Y) is not (or not known to be) rational, which happens only if $\text{char } k(y) \in \{2, 3, 5\}$; see [ABL22, Corollary 1.3].

Example B.4. Let S be a normal, del Pezzo surface over a field k and L an ample line bundle on S such that $H^1(S, L^m) = 0$ for $m \geq 2$. Set $Y := \text{Spec}_k \oplus_{m \geq 0} H^0(S, L^m)$, the affine cone over (S, L) . Then $\text{Cl}(Y) = \text{Cl}(S)/[L]$ and there is an exact sequence

$$0 \rightarrow H^1(S, L) \rightarrow \text{Cl}(\hat{Y}) \rightarrow \text{Cl}(S)/[L] \rightarrow 0.$$

If $H^1(S, L) = 0$, then $\text{Cl}(\hat{Y}) = \text{Cl}(S)/[L]$; this holds if the characteristic is 0 or at least 7. However, in characteristics 2, 3, 5 there are examples with $H^1(S, L) \neq 0$; see [CT19, Ber21, ABL22]. For these the p^∞ -part of $\text{Cl}(\hat{Y})$ contains $H^1(S, L)$, which is infinite if k is.

Remark B.5 (Local class group and local Picard group). Let (y, Y) be a 3-dimensional singularity. Let $\eta_i \in \text{Sing } Y \setminus \{y\}$ be the generic points. There is an exact sequence

$$0 \rightarrow \text{Pic}^{\text{loc}}(y, Y) \rightarrow \text{Cl}(Y) \rightarrow \bigoplus_i \text{Cl}(Y_{\eta_i}).$$

Here the Y_{η_i} are 2-dimensional. If (y, Y, Δ) is klt, then the Y_{η_i} are rational; hence the $\text{Cl}(Y_{\eta_i})$ are finite by [Lip69]. Thus for 3-dimensional, klt singularities, $\text{Pic}^{\text{loc}}(y, Y) \subset \text{Cl}(Y)$ is a finite index subgroup.

For equicharacteristic local rings, [Bou71, Bou78] define a *local Picard scheme* $\mathbf{Pic}^{\text{loc}}(y, Y)$. Proposition B.1 is equivalent to saying that its connected component is a unipotent algebraic group.

It is reasonable to expect that the following generalization of Proposition B.1 holds in all dimensions.

CONJECTURE B.6. *Let $(y \in Y, \Delta)$ be a complete, dlt, local scheme and $p := \text{char } k(y)$. Then*

- (1) $\text{Cl}(Y)/\text{Cl}(Y)[p^\infty]$ is finitely generated, and
- (2) $\text{Cl}(Y)[p^\infty]$ has bounded exponent.

As part of the proof of Proposition B.1, we establish the following result about the formal-local class groups. This seems to be a special case of more general finiteness phenomena, which will be considered elsewhere; see [BGS11] for another example.

PROPOSITION B.7. *Let X be a normal scheme. Assume that its codimension 2 singular points are rational. Then there is a closed subset $W \subset X$ of codimension at least 3 such that*

- (1) $\text{Cl}(X \setminus W)/\text{Pic}(X \setminus W)$ is finite and
- (2) $X \setminus W$ is formal-locally \mathbb{Q} -factorial.

Moreover, the orders of the formal-local class groups $\text{Cl}(\hat{X}_x)$ are uniformly bounded for $x \in X \setminus W$.

Proof of Proposition B.7

Let $x \in X$ be a codimension 2 singular point. After localizing at x , we obtain a rational surface singularity (s, S) . Then (s, S) is \mathbb{Q} -factorial by [Lip69]. Thus if $D \subset X$ is any Weil divisor, then D is \mathbb{Q} -Cartier at x , hence also in an open neighborhood of x . This shows that every Weil divisor D is \mathbb{Q} -Cartier outside a closed subset $W(D) \subset X$ of codimension at least 3. We need to show that $W(D)$ is essentially independent of D . This is not obvious. However, we claim that one can follow the steps of Lipman's proof to get Proposition B.7. To see this, let us first recall the surface case.

B.8 Summary of Lipman's proof. I mostly follow the notation in [Kol13, Definition 10.7, Lemma 10.8, Proposition 10.9, Corollary 10.10].

Let $f: T \rightarrow S$ be a resolution of singularities with exceptional curves $\{C_i: i \in I\}$. We need the following properties:

- (1) There are finite field extensions $F_i/k(s)$ such that C_i is either $\mathbb{P}_{F_i}^1$ (we call these lines) or a regular conic in $\mathbb{P}_{F_i}^2$.
- (2) There exist a sequence of effective, f -exceptional cycles $Z_1 \leq \dots \leq Z_m$ and a map $\tau: \{1, \dots, m\} \rightarrow I$ such that
 - (a) $\mathcal{O}_T(-Z_m)$ is f -ample,
 - (b) $Z_1 = C_{\tau(1)}$,
 - (c) $C_{\tau(i+1)}$ is a line for $i \geq 1$, and we have exact sequences

$$0 \rightarrow \mathcal{O}_{C_{\tau(i+1)}}(-1) \rightarrow \mathcal{O}_{Z_{i+1}} \rightarrow \mathcal{O}_{Z_i} \rightarrow 0. \tag{B.1}$$

The \mathbb{Q} -factoriality is now established in three steps. The most important is the following.

CLAIM B.8.3. *Let L be any line bundle on T that has degree 0 on each C_i . Then L is the pullback of a line bundle from S . Equivalently, there is an exact sequence*

$$\mathrm{Pic}(S) \xrightarrow{f^*} \mathrm{Pic}(T) \xrightarrow{c_1} \bigoplus_{i \in I} \mathbb{Z}[C_i],$$

where $c_1(M) := (\deg(M|_{C_i}) : i \in I)$.

Proof. We extend the sequence of effective, f -exceptional cycles $Z_1 \leq \dots \leq Z_m$ by setting $Z_{am+i} := aZ_m + Z_i$. Then the sequences (B.1) become

$$0 \rightarrow \mathcal{O}_{C_{\tau(i+1)}}(-1) \otimes \mathcal{O}_T(-aZ_m) \rightarrow \mathcal{O}_{Z_{am+i+1}} \rightarrow \mathcal{O}_{Z_{am+i}} \rightarrow 0.$$

Since Z_m is f -ample,

$$H^1(C_{\tau(i)}, \mathcal{O}_{C_{\tau(i+1)}}(-1) \otimes \mathcal{O}_T(-aZ_m) \otimes L) = 0;$$

hence the constant 1 section of $L|_{C_{\tau(1)}} \cong \mathcal{O}_{C_{\tau(1)}}$ lifts to any order. By the theorem on formal functions, we get that f_*L is a line bundle and $L \cong f^*f_*L$. \square

CLAIM B.8.4. *The subgroup $\langle \mathcal{O}_T(C_i) : i \in I \rangle \subset \mathrm{Pic}(T)$ has finite index $d := \det(C_i \cdot C_j)$. Denote the quotient by $H(T)$.*

Proof. In abstract terms, we have a subgroup of $\bigoplus_{i \in I} \mathbb{Z}[C_i]$ generated by $C_j^* := ((C_i \cdot C_j) : i \in I)$. This shows that $H(T)$ is determined by the matrix $(C_i \cdot C_j)$, which is negative definite by the Hodge index theorem. The rest is linear algebra. \square

Putting Claims B.8.3 and B.8.4 together gives the following.

CLAIM B.8.5. *There is an injection $\mathrm{Pic}^{\mathrm{loc}}(s, S) \hookrightarrow H(T)$. In particular, $|\mathrm{Pic}^{\mathrm{loc}}(s, S)|$ divides $\det(C_i \cdot C_j)$.* \square

Let us axiomatize the properties (1) and (2) of this section.

DEFINITION B.9. Let X be a normal scheme with singular locus $V \subset X$ and such that V is regular. A *Lipman-type resolution* is a resolution $f : Y \rightarrow X$ with exceptional divisors $\{E_i : i \in I\}$ and the following properties:

- (1) There are finite, flat morphisms $V_i \rightarrow V$ such that E_i is either $\mathbb{P}_{V_i}^1$ or a regular conic bundle with irreducible fibers in $\mathbb{P}_{V_i}^2$. Furthermore, every $V_i \rightarrow V$ is the composite of a purely inseparable morphism $V_i \rightarrow V_i^*$ with an étale morphism $V_i^* \rightarrow V$.
- (2) There exist a sequence of effective, f -exceptional cycles $Z_1 \leq \dots \leq Z_m$ and a map $\tau : \{1, \dots, m\} \rightarrow I$ such that
 - (a) $\mathcal{O}_Y(-Z_m)$ is f -ample,
 - (b) $Z_1 = E_{\tau(1)}$ and
 - (c) $E_{\tau(i+1)} \cong \mathbb{P}_{V_{\tau(i+1)}}^1$ for $i \geq 1$, and we have exact sequences

$$0 \rightarrow \mathcal{O}_{E_{\tau(i+1)}}(-1) \rightarrow \mathcal{O}_{Z_{i+1}} \rightarrow \mathcal{O}_{Z_i} \rightarrow 0.$$

Note that a Lipman-type resolution is preserved by étale morphisms $U \rightarrow X$ and by completions $\hat{X}_x \rightarrow X$. The only complication is that the E_i may become reducible if $V_i \rightarrow V$ is not purely inseparable. We claim that the latter can be achieved after an étale base change $X' \rightarrow X$. Indeed,

$V_i^* \rightarrow V$ is standard étale in a neighborhood of the generic point of V by [Sta22, Tag 02GT]; hence it extends to a finite étale cover $X'_i \rightarrow X_i^\circ \subset X$. Taking the fiber product of all of these now gives $X' \rightarrow X^\circ := \cap_i X_i^\circ$. For the resulting $f': Y' \rightarrow X'$ with exceptional divisors E'_i , all the $V'_i \rightarrow V'$ are purely inseparable.

We call such an $f': Y' \rightarrow X'$ a *stabilized Lipman-type resolution*. For such a resolution, the E'_i stay irreducible after completion.

Proposition B.7 is now a direct consequence of the following two claims.

LEMMA B.10. *Let X be a normal scheme. Assume that its codimension 2 singular points are rational. Then there is a closed subset $W \subset X$ of codimension at least 3 such that $X \setminus W$ has a Lipman-type resolution.*

LEMMA B.11. *Let X be a normal scheme that has a Lipman-type resolution. Then*

- (1) $\mathrm{Cl}(X)/\mathrm{Pic}(X)$ is finite, and
- (2) X is analytically \mathbb{Q} -factorial.

Moreover, the orders of the local class groups $\mathrm{Cl}(\hat{X}_x)$ are uniformly bounded for $x \in X$.

Proof of Lemma B.10. Let $\{x_i : i \in I\}$ be the codimension 2 singular points of X . The minimal resolution at x_i is obtained by blowing up an $m_{x_i, X}$ -primary ideal sheaf $I_{x_i, X} \subset \mathcal{O}_{x_i, X}$. We extend these to ideal sheaves $I_i \subset \mathcal{O}_X$ whose co-support is the closure V_i of x_i . By blowing up we get $Y_i \rightarrow X$. There are open neighborhoods $x_i \subset U_i \subset X$ such that $Y_i \times_X U_i \rightarrow U_i$ is a Lipman-type resolution. Set

$$W := (\cup_i (V_i \setminus U_i)) \cup (\cup_{i \neq j} (V_i \cap V_j)).$$

Let J be the restriction of $\cap_i I_i$ to $X \setminus W$. Note that the $V_i \cap (X \setminus W)$ are disjoint from each other; hence blowing up J is the same as blowing up the $I_i|_{X \setminus W}$ one at a time. Thus over each $V_i \cap (X \setminus W)$ we obtain the Lipman-type resolution $Y_i \times_X U_i \rightarrow U_i$. \square

Proof of Lemma B.11. The claims are Zariski local; we may thus assume that $V = \mathrm{Sing} X$ is irreducible.

The claims also descend from finite étale base changes. Thus, as we noted in Definition B.9, we may assume that the set of irreducible components of the exceptional divisor, and hence the group $H(T)$, is invariant under completions.

Note that $L|_{E_i}$ is f -trivial if and only if its degree on the generic fiber is 0. If $L|_{E_i}$ is f -trivial for every i , then, as in Claim B.8.4, we can use the sequences in Definition B.9(2)(c) to lift this trivialization to get that $f_* L$ is a line bundle and $L \cong f^* f_* L$.

As in Claim B.8.5, we get injections $\mathrm{Cl}(X)/\mathrm{Pic}(X) \hookrightarrow H(T)$ and $\mathrm{Cl}(\hat{X}_x) \hookrightarrow H(T)$ for every completion. \square

Proof of Proposition B.1. First we take a suitable log resolution $g: X \rightarrow (Y, \Delta)$. If (Y, Δ) is klt, then we can take any resolution obtained by a sequence of blowups whose centers lie over $\mathrm{Sing} Y \cup \mathrm{Sing}(\mathrm{Supp} \Delta)$, and such that $\mathrm{Ex}(g) \cup g_*^{-1} \Delta$ is a simple normal crossing divisor. (The most general resolution of 3-dimensional schemes is proved in [CP19]; see also [KW21].) In this case $\mathrm{Ex}(g)$ supports a g -ample divisor H , and the coefficients of H can be chosen to be linearly independent over any given finitely generated subfield of \mathbb{R}^+ .

The general dlt case can be reduced to the klt case using [KM98, Proposition 2.43].

Then we run a suitable minimal model program (MMP) to get from X to Y . On X the completions of the local rings are regular, hence factorial. Thus it is enough to show that the finite generation of the prime-to- p part is preserved by each MMP step.

First we use that, by [Kol21], the MMP steps are simpler than expected; see Proposition B.12. Then we discuss how the divisor class groups change under MMP steps.

For divisorial contractions we use Lemma B.13. Note that assumption (1) of Lemma B.13 is satisfied by Proposition B.7 and assumption (2) of Lemma B.13 holds by Claim 2.7.

For flips we use Lemma B.15. In applying it, a key issue is the following. Let $g: X \rightarrow Z$ be a 3-dimensional flipping contraction with exceptional curve $C = \text{Ex}(g)$. In characteristic 0 we know that $R^1g_*\mathcal{O}_X = 0$. Thus $H^1(X, \mathcal{O}_C) = 0$; hence $\text{Pic}^\circ(C) = 0$. In positive characteristic the vanishing of $R^1g_*\mathcal{O}_X$ is not known and may not be true.

We go around this problem as follows. Let $C^{\text{wn}} \rightarrow C$ denote the weak normalization. We prove in Lemma B.16 that $h^1(C^{\text{wn}}, \mathcal{O}_{C^{\text{wn}}}) = 0$. Therefore, C is homeomorphic to a tree of smooth rational curves, which implies that $\text{Pic}^\circ(C)$ is a unipotent group scheme (hence p^∞ -torsion). This is enough for our purposes. \square

PROPOSITION B.12 ([Kol21]). *Let (Y, Δ) be a 3-dimensional klt scheme (see Remark B.2) and $g: X \rightarrow Y$ a log resolution with exceptional divisors $E = \sum E_i$. Assume that E supports a g -ample divisor H ; we can then choose its coefficients to be sufficiently general.*

Then the MMP starting with $(X^0, E^0 + \Delta^0) := (X, E + g_^{-1}\Delta)$ with scaling of H runs and terminates with (Y, Δ) . Each step $X^i \dashrightarrow X^{i+1}$ of this MMP is*

- (1) *either a contraction $\phi_i: X^i \rightarrow X^{i+1}$, whose exceptional set is an irreducible component of E^i ,*
- (2) *or a flip $X^i \xrightarrow{\phi_i} Z^i \xleftarrow{\psi_i} (X^i)^+ = X^{i+1}$. Moreover, there are irreducible components E_1^i, E_2^i such that both E_1^i and $-E_2^i$ are ϕ_i -ample.*

Divisorial contractions are handled by the following.

LEMMA B.13. *Let (y, Y) be a normal, local scheme and $\pi: X \rightarrow Y$ a proper modification with reduced exceptional set E , where X is normal. Assume $p := \text{char } k(y) > 0$ and that the following hold:*

- (1) *There is a finite point set $P \subset E$ such that*
 - (a) *$\text{Pic}(X \setminus P) \subset \text{Cl}(X \setminus P)$ has finite index and*
 - (b) *the prime-to- p part of $\text{Pic}^{\text{loc}}(x, X)$ is finitely generated for all $x \in P$.*
- (2) *The prime-to- p part of $\text{Pic}(E)$ is finitely generated.*

Then the prime-to- p part of $\text{Cl}(Y)$ is finitely generated.

Proof. First note that $\text{Cl}(Y) \cong \text{Cl}(X)/\langle E_i: i \in I \rangle$, where $\{E_i \subset E: i \in I\}$ are the irreducible components that have codimension 1 in X . Thus our claim is equivalent to proving that the prime-to- p part of $\text{Cl}(X)$ is finitely generated. Note that $\text{Cl}(X \setminus P) = \text{Cl}(X)$ and, by assumption (1)(a), the subgroup $\text{Pic}(X \setminus P) \subset \text{Cl}(X \setminus P)$ has finite index. Extending divisors from $X \setminus P$ to X gives an exact sequence

$$\text{Pic}(X) \rightarrow \text{Pic}(X \setminus P) \rightarrow \bigoplus_{x \in P} \text{Pic}^{\text{loc}}(x, X).$$

It remains to show that the prime-to- p part of $\text{Pic}(X)$ is finitely generated. The prime-to- p part of $\text{Pic}(E)$ is finitely generated by assumption (2), and the kernel of $\text{Pic}(X) \rightarrow \text{Pic}(E)$ is p^∞ -torsion by [CT20]. \square

(Note that [CT20] deals with \mathbb{F}_p -schemes; this is all we need for Proposition B.1. We plan to return to the general case later; see also [Wit22]. Also, [CT20] states only that the kernel of $\text{Pic}(X) \rightarrow \text{Pic}(E)$ is torsion. If L is in the kernel and has order m not divisible by p , then it gives a degree m étale cover of X that is trivial on E , hence on E_y . Thus $m = 1$; cf. [Art69, Theorem 3.1] or [Sta22, Tag 0BQC].)

For flips we need a more precise version.

LEMMA B.14. *Let (y, Y) be a 3-dimensional, normal, complete, local scheme and $\pi: X \rightarrow Y$ a proper modification, where X is normal and its codimension 2 singular points are rational. Assume that $C := \text{red } \pi^{-1}(y)$ is 1-dimensional and $p := \text{char } k(y) > 0$. The following are equivalent:*

- (1) *The prime-to- p part of $\text{Cl}(\hat{X}_x)$ is finitely generated for all $x \in C$, and $h^1(C^{\text{wn}}, \mathcal{O}_{C^{\text{wn}}}) = 0$.*
- (2) *The prime-to- p part of $\text{Cl}(Y)$ is finitely generated.*

Proof. The proof of (1) \Rightarrow (2) goes as in the proof of Lemma B.13.

For the converse, the interesting part is to show that the prime-to- p part of $\text{Cl}(\hat{X}_x)$ is finitely generated for all $x \in C$. As we noted in Remark B.5, it is enough to prove that the prime-to- p part of $\text{Pic}^{\text{loc}}(x, \hat{X}_x)$ is finitely generated.

Let $D_x \subset \hat{X}_x$ be a divisor that is Cartier outside x . Then it is linearly equivalent to a divisor $D'_x \subset \hat{X}_x$ such that $C \cap D'_x = \{x\}$. Since X is complete in the topology induced by the ideal sheaf of C , we can view D'_x as a divisor on X . That is, there is a surjection

$$\text{Pic}(X \setminus \{x\}) \twoheadrightarrow \text{Pic}^{\text{loc}}(x, \hat{X}_x).$$

Thus if the prime-to- p part of $\text{Pic}^{\text{loc}}(y, Y)$ is finitely generated, then so is the prime-to- p part of $\text{Pic}^{\text{loc}}(x, \hat{X}_x)$. \square

Applying Lemma B.14 twice gives the following.

COROLLARY B.15. *Consider a 3-dimensional flip*

$$(C \subset X, \Delta) \xrightarrow{g} Z \xleftarrow{g^+} (C^+ \subset X^+, \Delta^+).$$

Assume that the prime-to- p part of $\text{Cl}(\hat{X}_x)$ is finitely generated for all $x \in C$ and that we have $h^1(C^{\text{wn}}, \mathcal{O}_{C^{\text{wn}}}) = 0$. Then the prime-to- p part of $\text{Cl}(\hat{X}_{x^+}^+)$ is finitely generated for all $x^+ \in C^+$.

It remains to show that the condition $h^1(C^{\text{wn}}, \mathcal{O}_{C^{\text{wn}}}) = 0$ holds for the flips in Proposition B.12(2).

LEMMA B.16. *Let $g: (X, \Delta) \rightarrow Z$ be a 3-dimensional flipping contraction with exceptional curve $C = \text{Ex}(g)$. Assume that there is an irreducible component $S \subset [\Delta]$ such that $C \subset S$. Then $h^1(C^{\text{wn}}, \mathcal{O}_{C^{\text{wn}}}) = 0$.*

Proof. The normalization $\bar{S} \rightarrow S$ is a homeomorphism by Lemma B.17. Let $\bar{C} \subset \bar{S}$ denote the preimage of C . Then \bar{C} is the exceptional set of an $(\bar{S}, \text{Diff}_{\bar{S}}(\Delta - S))$ -negative contraction; hence $h^1(\bar{C}, \mathcal{O}_{\bar{C}}) = 0$. Furthermore, \bar{C} is weakly normal, and $\bar{C} \rightarrow C$ is a homeomorphism. Thus $\bar{C} = C^{\text{wn}}$. \square

LEMMA B.17. *Let $(X, D + \Delta)$ be a dlt pair with D irreducible and \mathbb{Q} -Cartier. Then the normalization $\bar{D} \rightarrow D$ is a universal homeomorphism.*

Proof. By [Kol13, Theorem 2.31], the divisor D is regular outside a subset of codimension at least 2. The rest follows from [Gro05, Théorème XIII.2.1]. \square

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