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On global generation of vector bundles on the moduli space of curves from representations of vertex operator algebras

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Abstract

We consider global generation of sheaves of coinvariants on moduli of curves given by simple modules over certain vertex operator algebras, extending results for affine vertex operator algebras at integrable levels on stable pointed rational curves. A number of examples illustrate the subtlety of the problem.

1. Introduction

Given an object in any category, a natural objective is to find the maps admitted by it. On $\overline{\mathcal{M}}_{g,n}$, the moduli stack parametrizing families of stable n-pointed curves of genus g, globally generated coherent sheaves define rational maps, which are regular on the locus where they are free.

Sheaves of coinvariants, determined by n simple admissible modules W^i over a vertex operator algebra V (a VOA), are defined on $\overline{\mathcal{J}}_{g,n}$, the moduli stack parametrizing families of stable pointed curves with first-order tangent data. Under mild assumptions, they descend to sheaves $\mathbb{V}_g(V;W^\bullet)$ on $\overline{\mathcal{M}}_{g,n}$; see [DGT21, DGT19]. If V is C_2-cofinite, these sheaves are coherent [DGK22]; if V is also rational, they are vector bundles [DGT19]; and if V is strongly rational, their Chern classes are tautological [DGT22] (see § 2 for definitions). Examples include those given by affine VOAs, certain W-algebras, even lattice VOAs, and holomorphic VOAs (like the moonshine module), and others obtained as tensor products, orbifold algebras, and through coset constructions.

Affine VOAs are derived from (quotients of) the affinization of a Lie algebra $\mathfrak{g}$, and $\ell \in \mathbb{C}$, with $-\ell$ not equal to the dual Coxeter number. The simple affine VOA $L_\ell(\mathfrak{g})$, generated by its degree 1 component $\mathfrak{g}$, is strongly rational if and only if $\ell \in \mathbb{Z}_{>0}$. For $\mathfrak{g}$ reductive, $\mathbb{V}_g(L_\ell(\mathfrak{g});W^\bullet)$ was shown to be a vector bundle on $\overline{\mathcal{M}}_{g,n}$ in [TUY89] and globally generated on $\overline{\mathcal{M}}_{0,n}$ in [Fak12].

In this work, we investigate global generation in a more general context. Our main result is the following.

Theorem 1. *Sheaves of coinvariants defined by simple admissible modules over a vertex operator algebra, strongly generated in degree* 1*, are globally generated on* $\overline{\mathcal{J}}_{0,n}$*, and on* $\overline{\mathcal{M}}_{0,n}$ *if defined.*

Here we assume that all VOAs are of CFT-type. By [Lia94], the VOAs in Theorem 1 are

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quotients of the affinization of a not-necessarily reductive Lie algebra structure on their degree 1 component (see Remark 2.4.1). By [DM06], if V is strongly rational (rational, C_2-cofinite, simple, and self-contragredient), then we have $V \cong \bigotimes_{i=1}^{r} L_{\ell_i}(\mathfrak{g}_i)$, where the $\mathfrak{g}_i$ are simple Lie algebras, $\ell_i \in \mathbb{Z}_{>0}$, and $V_1 \cong \bigoplus_{i=1}^{r} \mathfrak{g}_i$. In Theorem 1, the VOA V need not be simple, C_2-cofinite, or rational, and may for instance be applied to $L_\ell(\mathfrak{g})$, for $\mathfrak{g}$ simple and such that ℓ is admissible but not in $\mathbb{Z}_{>0}$. Such VOAs $L_\ell(\mathfrak{g})$ are not C_2-cofinite but are quasi-lisse, a natural generalization of C_2-cofiniteness, introduced in [AK18]. It follows from [Ara16, Main Theorem] that simple admissible highest-weight modules over $L_\ell(\mathfrak{g})$ have rational conformal weights, as do more general V-modules (see Remark 2.7.3). In particular, their associated sheaves of coinvariants are defined on $\overline{\mathcal{M}}_{0,n}$ (see Remark 3.2.2). As in [Lia94], there are many other examples to which Theorem 1 applies.

By Corollary A, sheaves described in Theorem 1 are coherent. This improves [AN03], giving coherence on $\mathcal{M}_{0,n}$ for C_2-cofinite and self-contragredient V, and [DGK22], where coherence was proved on $\overline{\mathcal{M}}_{0,n}$ for C_2-cofinite V. By Corollary B, such sheaves are vector bundles on $\mathcal{M}_{0,n}$. If V is C_2-cofinite and rational, by [DGT19], these are vector bundles on $\overline{\mathcal{M}}_{g,n}$, and by Corollary C, these vector bundles are globally generated on $\overline{\mathcal{M}}_{0,n}$, extending [TUY89, Fak12].

While we have not found conditions to guarantee global generation for $g > 0$, or for VOAs which are not strongly generated in degree 1, to illustrate the subtlety of this problem, we give several representative examples, including

- globally generated and positive bundles of coinvariants (see §§ 7–9),
- sheaves of coinvariants that are not globally generated (see §§ 8 and 9).

Influenced by these, we ask questions and pose potential extensions of Theorem 1 (see § 10).

We next describe our methods, and our findings in more detail.

Given n simple admissible modules W^i over a vertex operator algebra V of CFT-type, we define a constant sheaf of finite rank and a morphism of sheaves (3.2) from it to the sheaf of coinvariants (Lemma 3.2.1). Each simple admissible V-module W^i is a direct sum of vector spaces W^i_d, graded by the natural numbers, and the fibers of the constant sheaf are isomorphic to the tensor product of the lowest-weight spaces $\bigotimes_i W^i_0$.

To prove that the sheaf of coinvariants is globally generated, we show that the map (3.2) from Lemma 3.2.1 is surjective. For this, it suffices to prove that the induced map on fibers is so, and to achieve this, we use a filtration induced by the grading on $W^\bullet$ by degree. In particular, we show that all positive-degree elements come from the degree zero part of the filtration, which is naturally a quotient of the constant sheaf. Crucial to our argument is Zhu's result that any simple admissible V-module is generated in degree zero over V; see [Zhu96, Theorem 2.2.2].

The proof of the surjectivity of (3.2) restricted to fibers at smooth pointed curves is a reinterpretation of the core of the argument of Tsuchiya, Ueno, and Yamada [TUY89, Proposition 2.3.1], that implies the global generation of bundles defined by affine Lie algebras at integrable levels $\ell \in \mathbb{Z}_{>0}$ on $\mathcal{M}_{0,n}$, as their analysis has essential features in common with ours. However, at pointed curves with singularities, due to differences in the definitions of the Lie algebras used to define the coinvariants in these two settings, our proof of surjectivity is considerably more involved than Fakhruddin's proof [Fak12] of the global generation of $\mathbb{V}_0(L_\ell(\mathfrak{g}); W^\bullet)$ on $\overline{\mathcal{M}}_{0,n}$ for $\ell \in \mathbb{Z}_{>0}$.

The terms we refer to are defined in § 2, and in § 3 we construct the constant bundle, proving Lemma 3.2.1. Theorem 1 is proved in § 5, after preparation is given in § 4. Proofs of the corollaries are given in § 6. We remark that Corollary A follows primarily from the proof of Theorem 1, while Corollary B is obtained by combining Corollary A with results from [DGT21]. Corollary C follows from Theorem 1 and results from [DGT19].

If $\mathbb{V}_g(V;W^\bullet)$ is a globally generated bundle on $\overline{\mathcal{M}}_{g,n}$ for $g>0$, as we next explain, it is necessary that V has non-negative central charge. Globally generated sheaves will have non-negative rank, and first Chern classes will be nef (a divisor is nef if it non-negatively intersects all curves). Suppose that V is C_2-cofinite, and rational so $\mathbb{V}_g(V;W^\bullet)$ is known to be a vector bundle [DGT19]. If V is also self-contragredient, then by [DGT22, Corollary 2], the first Chern classes of $\mathbb{V}_g(V;W^\bullet)$ are linear combinations of tautological classes, including λ (the first Chern class of the Hodge bundle). By [GKM02, Theorem 2.1], for $g\geqslant 2$, the coefficient of λ is a rational expression involving the rank of the bundle and the central charge of V. In particular, the central charge must be non-negative for $\mathbb{V}_g(V;W^\bullet)$ to be nef. While not necessary, strong unitarity of V is sufficient to guarantee that the central charge is positive, and such examples are therefore of interest.

In § 10, we discuss questions inspired by a number of examples we have studied, represented in simple cases here. The leitmotif is that sheaves of coinvariants derived from vertex operator algebras related to affine VOAs seem to be geometric in nature. For instance, in Question 1 we ask whether $\mathbb{V}_g\big(\bigotimes_{j=1}^r L_{\ell_j}(\mathfrak{g}_j);\bigotimes_{j=1}^r W_j^\bullet\big)$ is isomorphic to $\bigotimes_{j=1}^r \mathbb{V}_g\big(L_{\ell_j}(\mathfrak{g}_j);W_j^\bullet\big)$. In this case, dual sheaves of coinvariants could be identified with generalized theta functions and would be subject to strange dualities (see Remark 10.0.1).

Our remaining questions are about positivity. As illustrated in Example 8.0.5, in cases where the rank of the constant bundle $\mathcal{W}_0^\bullet$ was at least as large as the rank of the sheaf of coinvariants on $\overline{\mathcal{M}}_{0,n}$, the latter had positivity properties whenever an integral degree condition was satisfied (see Definition 2.3.1). In particular, since Chern classes of globally generated bundles are base-point-free, we were able to establish non–global generation by checking that if $n=4$, the degree of the sheaf was negative.

In § 7, we consider a class of line bundles defined by holomorphic VOAs and the affine VOA given by E_8. The associated sheaves $\mathcal{W}_0^\bullet$ also have rank 1. While Theorem 1 does not apply in positive genus, we can see that these bundles are globally generated on $\overline{\mathcal{M}}_{g,n}$ for $g>0$.

Tools from VOA theory like factorization, Zhu's algebra, and Zhu's character formula can often help one compute Chern classes and the ranks of the sheaves of coinvariants and of $\mathcal{W}_0^\bullet$, the latter of which is determined by the dimensions of the degree zero components of the module.

In [Zhu96], Zhu introduced an associative algebra $A(V)$ and established functors between categories of $A(V)$-modules and V-modules. By [Zhu96, Theorem 2.2.2], any simple admissible V-module $W=\bigoplus_{d\in\mathbb{N}} W_d$ corresponds to the simple $A(V)$-module W_0. In particular, if $A(V)$ is commutative, then W_0 is 1-dimensional, and the constant sheaf has rank 1. Minimal series principal W-algebras $\mathcal{W}^\ell(\mathfrak{g})$ and their simple quotients $\mathcal{W}_\ell(\mathfrak{g})$ have commutative Zhu algebras (see Example 3.2.3). These rational, C_2-cofinite, self-contragredient VOAs have appeared prominently in the literature and are related to other important families of VOAs. For instance, given $\ell\in\mathbb{Z}_{\geqslant 2}$ and $r=-\ell+(\ell+1)/(\ell+2)$, there is a well-known isomorphism between $\mathcal{W}_r(\mathfrak{sl}_2,\ell)$ and the Parafermion algebras $\mathrm{K}(\mathfrak{sl}_2,\ell)$, first proven in [ALY19]. A complete list of such isomorphisms between $\mathcal{W}_r(\mathfrak{sl}_n,\ell)$ and $\mathrm{K}(\mathfrak{sl}_2,\ell)$, for any r and n, is given in [Lin21, Theorem 10.3]. Both $\mathcal{W}^\ell(\mathfrak{g})$ and $\mathcal{W}_\ell(\mathfrak{g})$ can be realized as cosets of tensor products of affine vertex operator algebras (proved for simply laced $\mathfrak{g}$ in [ALY19, ACL19] and for non-simply laced $\mathfrak{g}$ in [CL22]). In § 8, we consider bundles of coinvariants defined by modules over $\mathrm{K}(\mathfrak{sl}_2,2)$ equal to the Virasoro VOA $\mathrm{Vir}_{\frac{1}{2}}$.

Many VOAs do not have a commutative Zhu algebra. For instance, generally speaking, affine Lie algebras and even lattice VOAs do not. One can often compute the rank of the constant bundle $\mathcal{W}_0^\bullet$ using Zhu's chararacter formula, which involves modular forms, as we explain in § 9. There we consider sheaves on $\overline{\mathcal{M}}_{0,n}$ generated by modules over even lattice vertex algebras, which have rank 1 by [DGT22]. Except in special cases, even lattice VOAs do not coincide with, nor are they constructed from, affine VOAs. We see that depending on how the modules are chosen, first Chern classes may be negative (so sheaves are not globally generated), zero (so sheaves are constant), or positive (so sheaves are possibly globally generated).

2. Background

Following [FLM88, TUY89, FB04, DGT19], we briefly give notation and results used here. We also recommend [DGT22, DGK22], which were written primarily for algebraic geometers.

2.1 Virasoro (Lie) algebra

The Virasoro (Lie) algebra Vir is a 1-dimensional central extension of the Witt (Lie) algebra Der $\mathcal{K}$. The Witt algebra represents the functor which assigns to a $\mathbb{C}$-algebra R the Lie algebra Der $\mathcal{K}(R) := R(\!(z)\!)\partial_z$ generated over R by the derivations $L_p := -z^{p+1}\partial_z$, for $p \in \mathbb{Z}$, with relations $[L_p, L_q] = (p-q)L_{p+q}$. In particular, we can view the Lie algebra Vir as representing the functor which assigns to R the Lie algebra generated over R by a formal vector K and the elements L_p, for $p \in \mathbb{Z}$, with Lie bracket given by

$$[K, L_p] = 0 \quad \text{and} \quad [L_p, L_q] = (p-q)L_{p+q} + \tfrac{1}{12}K\big(p^3 - p\big)\delta_{p+q,0}\,.$$

2.2 Vertex operator algebras

By a *vertex operator algebra*, we mean a four-tuple $(V, \mathbf{1}, \omega, Y(\cdot, z))$, where

(i) $V = \bigoplus_{i\in\mathbb{N}} V_i$ is a non-negatively graded $\mathbb{C}$–vector space with $\dim V_i < \infty$;

(ii) $\mathbf{1}$ is an element in V_0, called the *vacuum vector*;

(iii) ω is an element in V_2, called the *conformal vector*;

(iv) $Y(\cdot, z)\colon V \to \operatorname{End}(V)[\![z, z^{-1}]\!]$ is a linear map taking $A \in V$ to $Y(A,z) := \sum_{i\in\mathbb{Z}} A_{(i)} z^{-i-1}$. The series $Y(A,z)$ is called the *vertex operator* assigned to A.

The datum $(V, \mathbf{1}, \omega, Y(\cdot, z))$ is required to satisfy four axioms which we state in [DGT19], including axioms that vertex operators satisfy a weak version of commutativity and a weak version of associativity. One may regard this as framing a VOA as generalizing a commutative associative algebra. We highlight here the main properties that we will use in this paper, and we refer to [DGT21, DGT19, FLM88] for more details.

(i) *Conformal structure:* The Virasoro algebra acts on V through the identifications $L_p = \omega_{(p+1)}$ and $K = c_V \operatorname{Id}_V$ for some complex number c_V called central charge of V.

(ii) *Vacuum axiom:* $Y(\mathbf{1}, z) = \operatorname{Id}_V$.

(iii) *Graded action:* If $A \in V_k$, then $A_{(j)}V_\ell \subseteq V_{\ell+k-j-1}$.

(iv) *Commutator formula:* $[A_{(i)}, B_{(j)}] = \sum_{k\geqslant 0} \binom{i}{k} \big(A_{(k)}(B)\big)_{(i+j-k)}$.

(v) *Associator formula:* $\big(A_{(i)}(B)\big)_{(j)} = \sum_{k\geqslant 0} (-1)^k \binom{i}{k} \big(A_{(i-k)}B_{(j+k)} - (-1)^i B_{(i+j-k)}A_{(k)}\big)$.

A VOA V is *strongly rational*, or *of CohFT-type*, if V is simple, self-contragredient, and if V is

(I) *of CFT-type:* $V = \bigoplus_{i\in\mathbb{N}} V_i$ with $V_0 \cong \mathbb{C}\mathbf{1}$;

(II) *rational:* there are finitely many simple V-modules, and every finitely generated module is a direct sum of simple modules;

(III) *C_2-cofinite:* the space $C_2(V) := \mathrm{span}_{\mathbb{C}}\{A_{(-2)}B\colon A, B \in V\}$ has finite codimension in V.

2.3 V-Modules

By a V-module W, we mean what in the literature is known as an *admissible V-module*, that is a pair $(W, Y^W(-,z))$ consisting of

(i) an $\mathbb{N}$-graded vector space $W = \bigoplus_{i\geqslant 0} W_i$ with $\dim(W_i) < \infty$ and $W_0 \neq 0$,

(ii) a linear map $Y^W(-,z)\colon V \to \mathrm{End}(W)[\![z, z^{-1}]\!]$, $A \mapsto Y^W(A,z) = \sum_{i\in\mathbb{Z}} A^W_{(i)} z^{-i-1}$.

In order for this pair to define an admissible V-module, certain axioms need to hold [DGT19, FHL93, DL93]. Instead of reporting all the properties that $(W, Y^W(z,-))$ must satisfy, we list here only those that will be used in this paper (see [DGT19] for more details).

(i) *Conformal structure:* The Virasoro algebra acts on W through the identification $L_p \cong \omega^W_{(p+1)}$.

(ii) *Vacuum axiom:* $Y^W(\mathbf{1}, z) = \mathrm{Id}_W$.

(iii) *Graded action:* If $A \in V_k$, then $A^W_{(j)} W_\ell \subseteq W_{\ell+k-j-1}$, and we write $\deg\left(A^W_{(j)}\right) = \deg(A) - j - 1$.

(iv) *Commutator formula:* $\left[A^W_{(i)}, B^W_{(j)}\right] = \sum_{k\geqslant 0} \binom{i}{k} \left(A_{(k)}(B)\right)^W_{(i+j-k)}$.

(v) *Associator formula:* $\left(A_{(i)}(B)\right)^W_{(j)} = \sum_{k\geqslant 0} (-1)^k \binom{i}{k} \left(A^W_{(i-k)} B^W_{(j+k)} - (-1)^i B^W_{(i+j-k)} A^W_{(k)}\right)$.

In what follows, the endomorphism $A^W_{(j)}$ will simply be denoted by $A_{(j)}$. It is important to observe that V is a V-module and that the commutator and associator formulas for V and for V-modules both arise from the *Jacobi identities* for V and for V-modules. Moreover, when W is a simple V-module, there exists an element $\alpha \in \mathbb{C}$, called the *conformal dimension of* W, such that $L_0(w) = (\alpha + \deg(w))w$ for every homogeneous element $w \in W$.

Definition 2.3.1. An n-tuple $\left(W^1, \ldots, W^n\right)$ of admissible V-modules W^i of conformal dimension α_i is said to satisfy the integrality condition, or integrality property, if the sum of conformal dimensions $\sum_{i=1}^n \alpha_i$ is an integer (which can be zero).

2.4 Strong finite generation

A vertex algebra V is called *finitely strongly generated* if there exist finitely many elements $A^1, \ldots, A^r \in V$ such that V is spanned by the elements of the form

$$A^{i_1}_{(-n_1)} \cdots A^{i_r}_{(-n_r)} \mathbf{1},$$

with $r \geqslant 0$ and $n_i \geqslant 1$ (see [Ara12]). We say that $V = \bigoplus_{i\in\mathbb{N}} V_i$ is *strongly generated in degree d* if it is possible to choose the generators A^{i_j} to be in V_m for $m \leqslant d$.

By [Liu22, Proposition 2.5], one has that V is finitely strongly generated if and only if V is C_1-cofinite. If V is C_2-cofinite, then it is C_1-cofinite. However, the quasi-lisse but not C_2-cofinite affine VOAs defined by simple Lie algebras and admissible, non-integral levels are strongly finitely generated in degree 1.

Remark 2.4.1. VOAs of CFT-type, strongly generated in degree 1, were classified in [Lia94]. More generally, for any so-called preVOA V of CFT-type, by [Lia94, Theorem 3.7], the degree 1 component V_1 has the structure of a Lie algebra, with bracket $[A, B] = A_{(0)}(B)$. This Lie algebra $(V_1, [\ ,\ ])$, which need not be simple, or reductive, is equipped with a symmetric invariant bilinear form $\langle A, B\rangle = A_{(1)}(B)$. Roughly speaking, in the terminology of [Lia94], a preVOA satisfies many of the properties of a VOA except those involving a conformal vector. Given any pair consisting of a Lie algebra $\mathfrak{g}$ and symmetric invariant bilinear form $\langle\ ,\ \rangle$, Lian defines the affinization, and proves in [Lia94, Theorem 4.11] that for any preVOA V of CFT-type, if V is strongly generated in degree 1, then it is isomorphic to a quotient of the affinization of $(V_1, \langle\ ,\ \rangle)$ by some ideal. The last step in the classification is to determine which preVOAs admit a Virasoro vector and have the structure of a VOA. He classifies such Virasoro vectors (see [Lia94, Corollary 6.15]). As pointed out to us by a referee of our paper (who told us of this work), the most interesting aspect of [Lia94] is that this class of examples is much richer than the affinizations of reductive Lie algebras and their quotients. New examples are given in [Lia94, § 6.4].

2.5 Coordinatized curves

As the sheaf of coinvariants on $\overline{\mathcal{M}}_{g,n}$, the constant sheaf constructed in Lemma 3.2.1 is defined first on a covering $\widehat{\mathcal{M}}_{g,n}$ of $\overline{\mathcal{M}}_{g,n}$ and then descended to $\overline{\mathcal{M}}_{g,n}$ along two maps which we recall here. At the second step, one applies Tsuchimoto's method, as in [DGT21, DGT19]. By $\widehat{\mathcal{M}}_{g,n}$, we mean the moduli space of triples $(C, P_\bullet, t_\bullet)$, where $(C, P_\bullet) \in \overline{\mathcal{M}}_{g,n}$ and $t_\bullet$ is an n-tuple of formal coordinates t_i at each of the marked points P_i. This space is described in detail in [DGT19, § 2.2.2]. To understand the maps along which the two sheaves are to descend, we next describe the group scheme $\operatorname{Aut}\mathcal{O}$ and the varieties $\mathcal{A}ut_{C/S}$ on which it acts.

Consider the functor which assigns to a $\mathbb{C}$-algebra R the group

$$\operatorname{Aut}\mathcal{O}(R) = \left\{ z \mapsto \rho(z) = a_1 z + a_2 z^2 + \cdots \mid a_i \in R,\ a_1 \text{ a unit} \right\}$$

of continuous automorphisms of the algebra $R[\![z]\!]$ preserving the ideal $zR[\![z]\!]$. The group law is given by composition of series: $\rho_1 \cdot \rho_2 := \rho_2 \circ \rho_1$. This functor is represented by a group scheme, denoted by $\operatorname{Aut}\mathcal{O}$.

First suppose that C is a smooth curve, and let $\mathcal{A}ut_C$ be the smooth variety whose set of points are pairs (P, t), with $P \in C$ and $t \in \widehat{\mathcal{O}}_P$ such that $t \in \mathfrak{m}_P \setminus \mathfrak{m}_P^2$ a formal coordinate at P. Here $\mathfrak{m}_P$ is the maximal ideal of $\widehat{\mathcal{O}}_P$, the completed local ring at the point P. There is a simply transitive right action of $\operatorname{Aut}\mathcal{O}$ on $\mathcal{A}ut_C \to C$, given by changing coordinates:

$$\mathcal{A}ut_C \times \operatorname{Aut}\mathcal{O} \longrightarrow \mathcal{A}ut_C\,, \quad ((P,t),\rho) \longmapsto (P, t\cdot\rho := \rho(t))\,,$$

making $\mathcal{A}ut_C$ a principal $(\operatorname{Aut}\mathcal{O})$-bundle on C. A choice of formal coordinate at P gives a trivialization

$$\operatorname{Aut}\mathcal{O} \xrightarrow{\simeq_t} \mathcal{A}ut_P\,, \quad \rho \longmapsto \rho(t)\,.$$

If C is a *nodal* curve, then to define a principal $(\operatorname{Aut}\mathcal{O})$-bundle on C, one may give a principal $(\operatorname{Aut}\mathcal{O})$-bundle on its normalization, together with a gluing isomorphism between the fibers over the preimages of each node. For simplicity, suppose that C has a single node Q, and let $\widetilde{C} \to C$ denote its normalization, with Q_+ and Q_- the two preimages of Q in $\widetilde{C}$. A choice of formal coordinates $s_\pm$ at $Q_\pm$, respectively, determines a smoothing of the nodal curve C over $\operatorname{Spec}(\mathbb{C}[\![q]\!])$ such that, locally around the point Q in C, the family is defined by $s_+ s_- = q$. One may identify the fibers at $Q_\pm$ by the gluing isomorphism induced from the identification $s_+ = \gamma(s_-)$:

$$\mathcal{A}ut_{Q_+} \simeq_{s_+} \operatorname{Aut}\mathcal{O} \xrightarrow{\cong} \operatorname{Aut}\mathcal{O} \simeq_{s_-} \mathcal{A}ut_{Q_-}, \quad \rho(s_+) \longmapsto \rho \circ \gamma(s_-),$$

where $\gamma \in \operatorname{Aut}\mathcal{O}$ is the involution defined as

$$\gamma(z) := \frac{1}{1+z} - 1 = -z + z^2 - z^3 + \cdots .$$

This may be carried out in families to define $\mathcal{A}ut_{C/S} \to C/S$. The identification of the universal curve $\overline{\mathcal{C}}_g \cong \overline{\mathcal{M}}_{g,1} \to \overline{\mathcal{M}}_g$ leads to the principal $(\operatorname{Aut}\mathcal{O})$-bundle $\widehat{\mathcal{M}}_{g,1} \to \overline{\mathcal{M}}_{g,1}$.

The group scheme $\operatorname{Aut}_+\mathcal{O}$ represents the functor assigning to a $\mathbb{C}$-algebra R the group:

$$\operatorname{Aut}_+\mathcal{O}(R) = \left\{ z \mapsto \rho(z) = z + a_2 z^2 + \cdots \mid a_i \in R \right\},$$

and one has $\operatorname{Aut}\mathcal{O} = \mathbb{G}_m \ltimes \operatorname{Aut}_+\mathcal{O}$. In particular, by [DGT21], the projection $\widehat{\mathcal{M}}_{g,n} \to \overline{\mathcal{M}}_{g,n}$ is an $(\operatorname{Aut}\mathcal{O})^{\oplus n}$-torsor and factors as the composition of an $(\operatorname{Aut}_+\mathcal{O})^{\oplus n}$-torsor and a $\mathbb{G}_m^{\oplus n}$-torsor:

$$\begin{tikzcd} \widehat{\mathcal{M}}_{g,n} \arrow[rr, "(\operatorname{Aut}\mathcal{O})^{\oplus n}"] \arrow[dr, "(\operatorname{Aut}_+\mathcal{O})^{\oplus n}"'] & & \overline{\mathcal{M}}_{g,n} \\ & \overline{\mathcal{J}}_{g,n}, \arrow[ur, "\mathbb{G}_m^{\oplus n}"'] & \end{tikzcd}$$

where $\overline{\mathcal{J}}_{g,n}$ parametrizes objects of type $(C, P_\bullet, \tau_\bullet)$, where $(C, P_\bullet)$ is a stable n-pointed genus g curve and $\tau_\bullet = (\tau_1, \dots, \tau_n)$ with τ_i a non-zero 1-jet of a formal coordinate at P_i, for each i.

2.6 Lie algebras that act

To define the sheaf of coinvariants $\mathbb{V}(V; W^\bullet)$ on $\widehat{\mathcal{M}}_{g,n}$, one uses the sheaf of ancillary Lie algebras $\mathfrak{L}(V)^n$ and the sheaf of chiral Lie algebras $\mathcal{L}_{\widehat{\mathcal{C}}_{g,n} \setminus P_\bullet}(V)$, each of which acts on the tensor product $W^\bullet = \bigotimes_i W^i$. The fiber of $\mathfrak{L}(V)^n$ over $(C, P_\bullet, t_\bullet)$ is given by the direct sum $\bigoplus_{i=1}^n \mathfrak{L}_{P_i}(V)$ of the ancillary Lie algebras

$$\mathfrak{L}_{P_i}(V) := \frac{V \otimes \mathbb{C}(\!(t_i)\!)}{\operatorname{Im}\nabla} \quad \text{with } \nabla = L_{-1} \oplus \partial_{t_i},$$

with Lie bracket given by $\left[A_{[j]}, B_{[k]}\right] = \sum_{\ell \geqslant 0} \binom{j}{\ell} \left(A_{(\ell)}(B)\right)_{[j+k-\ell]}$, where $A_{[j]}$ is the class of the element $A \otimes t_i^j$ in $\mathfrak{L}_{P_i}(V)$. In particular, the diagonal action of $\bigoplus_{i=1}^n \mathfrak{L}_{P_i}(V)$ on $W^\bullet \otimes \mathcal{O}_{\widehat{\mathcal{M}}_{g,n}}$ is induced by the map $\mathfrak{L}_{P_i}(V) \to \operatorname{End}(W^i)$ that takes $A_{[j]}$ to the endomorphism $A_{(j)}^{W^i}$. The degree of every element of $\mathfrak{L}_{P_i}(V)$ is identified with the degree of the associated endomorphism; that is, for homogeneous elements $A \in V$ and $j \in \mathbb{Z}$, we have $\deg(A_{[j]}) = \deg(A) - j - 1$.

By [DGT19], there is a coordinate-independent version of $\mathfrak{L}_{P_i}(V)$, which we briefly recount. One can define a coordinate-independent version of the ancillary Lie algebras as well as give a description of the sheaf of chiral Lie algebras, and their actions on $W^\bullet$, using the sheaf of vertex algebras $\mathcal{V}_C$ on the curve C. The sheaf $\mathcal{V}_C$ was originally defined on smooth curves in [FB04]. To define the sheaf $\mathcal{V}_C$ on a nodal curve C, it is enough to define it on open subsets which do not include nodes and then for each node Q, define the sheaf on the normalization of the curve and specify isomorphisms of fibers over the preimages of each node. This is explained in detail in [DGT19, § 2.5], where it is shown that $\mathcal{V}_C$ is a sheaf of $\mathcal{O}_C$-modules.

If t_i is a local coordinate at P_i, then the ancillary Lie algebra of V at P_i is isomorphic to

$$\mathrm{H}^0\left(D_{P_i}^\times, \mathcal{V}_C \otimes \omega_C / \operatorname{Im}\nabla\right) \xrightarrow{\simeq_{t_i}} \mathfrak{L}_{P_i}(V),$$

the chiral Lie algebra for $(C, P_\bullet)$ is defined to be

$$\mathcal{L}_{C\setminus P_\bullet}(V) := \mathrm{H}^0\big(C \setminus P_\bullet, \mathcal{V}_C \otimes \omega_C/\mathrm{Im}\nabla\big)\,,$$

and one has a map of sheaves of Lie algebras given by restriction:

$$\mathcal{L}_{C\setminus P_\bullet}(V) \longrightarrow \bigoplus_{i=1}^{n} \mathrm{H}^0\big(D_{P_i}^\times, \mathcal{V}_{\widetilde{C}} \otimes \omega_{\widetilde{C}}/\,\mathrm{Im}\,\nabla\big) \xrightarrow{\simeq_{t_i}} \bigoplus_{i=1}^{n} \mathfrak{L}_{P_i}(V)\,. \tag{2.1}$$

In what follows, the image of σ in $\bigoplus_{i=1}^n \mathfrak{L}_{P_i}(V)$ will be denoted by $(\sigma_{P_1}, \dots, \sigma_{P_n})$.

2.7 Coinvariants in two steps

We describe how the sheaf of coinvariants, first defined over $\widehat{\mathcal{M}}_{g,n}$, descends to a sheaf over $\overline{\mathcal{M}}_{g,n}$. Details of this construction can be found in [DGT19].

2.7.1 The sheaf of Lie algebras $\mathcal{L}_{\widehat{C}_{g,n}\setminus P_\bullet}(V)$ on $\widehat{\mathcal{M}}_{g,n}$ acts on the constant vector bundle $W^\bullet \otimes \mathcal{O}_{\widehat{\mathcal{M}}_{g,n}}$ so that $\mathcal{L}_{\widehat{C}_{g,n}\setminus P_\bullet}(V) \cdot \big(W^\bullet \otimes \mathcal{O}_{\widehat{\mathcal{M}}_{g,n}}\big)$ is a submodule of $W^\bullet \otimes \mathcal{O}_{\widehat{\mathcal{M}}_{g,n}}$. The sheaf of coinvariants is the quotient

$$\widehat{\mathbb{V}}_g(V; W^\bullet) = \frac{W^\bullet \otimes \mathcal{O}_{\widehat{\mathcal{M}}_{g,n}}}{\mathcal{L}_{\widehat{C}_{g,n}\setminus P_\bullet}(V)\big(W^\bullet \otimes \mathcal{O}_{\widehat{\mathcal{M}}_{g,n}}\big)}\,. \tag{2.2}$$

We recall that the transitive action given by changing coordinates gives $\widehat{\mathcal{M}}_{g,n}$ the structure of a principal $(\mathrm{Aut}\,\mathcal{O})^n$-bundle over $\overline{\mathcal{M}}_{g,n}$, where $\pi\colon \widehat{\mathcal{M}}_{g,n} \to \overline{\mathcal{M}}_{g,n}$, is the forgetful map (see § 2.5). The actions of $(\mathrm{Aut}\,\mathcal{O})^n$ and of $\mathcal{L}_{\widehat{C}_{g,n}\setminus P_\bullet}(V)$ on $W^\bullet \otimes \mathcal{O}_{\widehat{\mathcal{M}}_{g,n}}$ are compatible [DGT19], and the action of $(\mathrm{Aut}\,\mathcal{O})^n$ on $W^\bullet \otimes \mathcal{O}_{\widehat{\mathcal{M}}_{g,n}}$ preserves the submodule $\mathcal{L}_{\widehat{C}_{g,n}\setminus P_\bullet}(V)\big(W^\bullet \otimes \mathcal{O}_{\widehat{\mathcal{M}}_{g,n}}\big)$, inducing an action of $(\mathrm{Aut}\,\mathcal{O})^n$ on $\widehat{\mathbb{V}}_g(V; W^\bullet)$. The sheaf of coinvariants on $\overline{\mathcal{M}}_{g,n}$ is then defined to be

$$\mathbb{V}_g(V; W^\bullet) := \big(\pi_* \widehat{\mathbb{V}}_g(V; W^\bullet)\big)^{\mathrm{Aut}\,\mathcal{O}^n}\,.$$

2.7.2 More explicitly, the descent of coinvariants is carried out in two steps:

$$\widehat{\mathcal{M}}_{g,n} \xrightarrow{\text{step 1}} \overline{\mathcal{J}}_{g,n} \xrightarrow{\text{step 2}} \overline{\mathcal{M}}_{g,n}\,.$$

In the first step, the group scheme $(\mathrm{Aut}_+\mathcal{O})^n$ acts equivariantly on $W^\bullet \otimes \mathcal{O}_{\widehat{\mathcal{M}}_{g,n}}$, and the quotient by this action descends to a vector bundle $\mathbb{V}^{\mathcal{J}}(V; W^\bullet)$ on $\mathcal{J} = \overline{\mathcal{J}}_{g,n}$, with fibers

$$\mathbb{V}^{\mathcal{J}}(V; W^\bullet)_{(C,P_\bullet,\tau_\bullet)} = \bigotimes_{i=1}^{n} W^i_{P_i,\tau_i} \Big/ \mathcal{L}_{C\setminus P_\bullet}(V) \cdot \Big(\bigotimes_{i=1}^{n} W^i_{P_i,\tau_i}\Big).$$

Here, $W^i_{P_i,\tau_i}$ is the coordinate-independent realization of the V-module W^i assigned at (P_i, τ_i) as defined in [DGT21]. In the second step, we then descend $\mathbb{V}^{\mathcal{J}}(V; W^\bullet)$ to $\overline{\mathcal{M}}_{g,n}$. The action of $\mathbb{G}_m^n \cong (\mathbb{C}^\times)^n$ is induced by the $\mathbb{Z}$-gradation of each W^i; it is described as follows: for $(z_1, \dots, z_n) \in (\mathbb{C}^\times)^n$ and $w^\bullet = w^1 \otimes \cdots \otimes w^n \in \bigotimes_j W^j$ given by homogeneous $w^i \in W$, we set

$$(z_1, \dots, z_n) \cdot w^\bullet := {z_1}^{-\deg(w^1)} w^1 \otimes {z_2}^{-\deg(w^2)} w^2 \otimes \cdots \otimes {z_n}^{-\deg(w^n)} w^n\,.$$

When descending along the $\mathbb{G}_m^{\oplus n}$-torsor to $\overline{\mathcal{M}}_{g,n}$, one applies the following method, inspired by Tsuchimoto in [Tsu93] and used in [DGT21, DGT19]. This is explained in detail using a root stack in case the conformal dimensions of modules are rational in [DGT19, § 8.7], while a more general procedure without rationality assumption is described in [DGT19, Remark 8.7.3(ii)].

Remark 2.7.3. There are a number of sufficient conditions for V that ensure that simple V-modules will have rational conformal dimension. For $L_k(\mathfrak{g})$, where $\mathfrak{g}$ is a simple Lie algebra and k is an admissible level that is not a positive integer, as mentioned in the introduction, by [Ara16, Main Theorem], the conformal weights of modules in category $\mathcal{O}$ are rational, as they are from (slightly) larger categories (that allow for dense modules, spectral flow twists, and finite length extensions as studied in for instance [CRW14]) where the weights are determined by those in Arakawa's classification by formulas that preserve rationality. Rationality has also been shown using various types of modularity of characters in different contexts [AM88, Zhu96, Miy04]. For instance, this is proved for C_2-cofinite VOAs in [Miy04, Corollary 5.10] using that the span over $\mathbb{C}$ of ordinary and pseudo trace functions is $\mathrm{SL}(2,\mathbb{Z})$-invariant. Admissible affine VOAs have the modular invariance property [KW88].

2.8 Formulas for the rank

An important feature of the sheaves of coinvariants $\mathbb{V}_g(V;W^\bullet)$ from a vertex algebra V of CohFT-type is that their rank can be computed by induction on the genus g. This is a consequence of the factorization theorem [DGT19] and can be seen via the equality

$$\operatorname{rank}\mathbb{V}_g(V;W^\bullet) = \sum_{W\in\mathcal{W}} \operatorname{rank}\mathbb{V}_{g-1}(V;W^\bullet\otimes W\otimes W'),$$

where the sum is over the finite set of simple admissible V-modules $\mathcal{W}$. That $\mathcal{W}$ is finite follows from the assumption that V is rational as well as from the assumption that V is C_2-cofinite. Moreover, for every non-negative integer i smaller than or equal to g and for every partition $I\sqcup I^c$ of $\{1,\ldots,n\}$, the following equality also holds:

$$\operatorname{rank}\mathbb{V}_g(V;W^\bullet) = \sum_{W\in\mathcal{W}} \operatorname{rank}\mathbb{V}_{g-i}\big(V;W^I\otimes W\big)\operatorname{rank}\mathbb{V}_i\big(V;W^{I^c}\otimes W'\big). \tag{2.3}$$

2.9 Formulas for first Chern classes in case $V\cong V'$

One can use first Chern classes to test the positivity of a given vector bundle. First Chern classes of globally generated vector bundles are base-point-free, and they define nef divisors. A divisor D on a projective variety X is nef if it non-negatively intersects every curve on X. We recall the formula derived in [DGT22, Corollary 2] for first Chern class of a vector bundle of coinvariants defined by a rational, C_2-cofinite self-contragredient vertex operator algebra V of CFT-type, with central charge c, and n simple V-modules W^i of conformal dimension a_i:

$$c_1\big(\mathbb{V}_g(V;W^\bullet)\big) = \operatorname{rank}\mathbb{V}_g(V;W^\bullet)\Big(\frac{c}{2}\lambda + \sum_{i=1}^n a_i\psi_i\Big) - b_{\mathrm{irr}}\delta_{\mathrm{irr}} - \sum_{i,I} b_{i:I}\delta_{i:I}\,, \tag{2.4}$$

where

$$b_{\mathrm{irr}} = \sum_{W\in\mathcal{W}} a_W \operatorname{rank}\mathbb{V}_{g-1}(V;W^\bullet\otimes W\otimes W')$$

and

$$b_{i:I} = \sum_{W\in\mathcal{W}} a_W \operatorname{rank}\mathbb{V}_i\big(V;W^I\otimes W\big)\operatorname{rank}\mathbb{V}_{g-i}\big(V;W^{I^c}\otimes W'\big).$$

In the coefficients b_{irr} of the boundary divisor δ_{irr} and $b_{i:I}$ of $\delta_{i:I}$ in (2.4), we sum over the finite set of simple admissible V-modules $\mathcal{W}$.

2.10 Test curves

F-Curves, which span $A_1(\overline{\mathcal{M}}_{g,n})$, are defined to be the numerical equivalence classes of the image of prescribed clutching maps from $\overline{\mathcal{M}}_{1,1}$ and $\overline{\mathcal{M}}_{0,4}$. A picture of all possible such maps and formulas for the intersections of F-curves with divisors is given in [GKM02]. The F-curves on $\overline{\mathcal{M}}_{0,n}$ are given by a partition $\{1,\dots,n\} = N_1 \cup N_2 \cup N_3 \cup N_4$ into four non-empty sets, which determines a map from $\overline{\mathcal{M}}_{0,4}$ to $\overline{\mathcal{M}}_{0,n}$, where given $(C, q_\bullet) \in \overline{\mathcal{M}}_{0,4}$, we obtain a point in $\overline{\mathcal{M}}_{0,n}$ by attaching to q_i, for $i \in \{1,\dots,4\}$, any stable $|N_i|+1$ pointed curve of genus zero by gluing q_i to the $+1$ point. The F-curve, denoted by F_{N_1,N_2,N_3,N_4}, is defined to be the numerical equivalence class of the image of this map.

3. Constant sheaves associated with coinvariants

Here we show how to associate with any n-tuple $\left(W^1,\dots,W^n\right)$ of admissible V-modules a sheaf $\mathcal{W}_k^\bullet$ based on the degree k part of the standard filtration $\mathcal{F}_k(\bigotimes_j W^j)$, defined in § 3.1. Each of these sheaves, considered in § 3.2, descends from a constant sheaf on $\widehat{\mathcal{M}}_{g,n}$ and in case $k=0$ remains constant. In Example 3.2.3, we discuss the rank 1 constant sheaves associated with any n-tuple of representations over the minimal series principal W-algebra related in some cases to the parafermions and the discrete series Virasoro VOAs, considered in § 8.

3.1 Filtration

The standard filtration on the sheaf of coinvariants $\widehat{\widehat{\mathbb{V}}}_g(V;W^\bullet)$ on $\widehat{\mathcal{M}}_{g,n}$ defined in (2.2) is induced from a filtration on $\mathcal{L}_{C\setminus P_\bullet}(V)$, given for $k \in \mathbb{N}$ by

$$\mathcal{F}_k\,\mathcal{L}_{C\setminus P_\bullet}(V) := \left\{\sigma \in \mathcal{L}_{C\setminus P_\bullet}(V) \mid \deg \sigma_{P_i} \leqslant k, \text{ for all } i\right\};$$

so $\mathcal{L}_{C\setminus P_\bullet}(V)$ is a filtered Lie algebra. There is also a filtration on $W^\bullet$ defined for $k\in\mathbb{N}$ by

$$\mathcal{F}_k W^\bullet = \bigoplus_{0\leqslant d\leqslant k} W_d^\bullet\,, \quad \text{where } W_d^\bullet := \sum_{d_1+\cdots+d_n=d} W^1_{d_1}\otimes\cdots\otimes W^n_{d_n}\,.$$

Since $\mathcal{F}_k\,\mathcal{L}_{C\setminus P_\bullet}(V)\cdot\mathcal{F}_\ell W^\bullet \subset \mathcal{F}_{k+\ell}W^\bullet$, it follows that $W^\bullet$ is a filtered $\mathcal{L}_{C\setminus P_\bullet}(V)$-module, and we set

$$\mathcal{F}_k\big(W^\bullet_{\mathcal{L}_{C\setminus P_\bullet}(V)}\big) := \big(\mathcal{F}_k W^\bullet + \mathcal{L}_{C\setminus P_\bullet}(V)\cdot W^\bullet\big)/\mathcal{L}_{C\setminus P_\bullet}(V)\cdot W^\bullet\,.$$

3.2 Sheaves $\mathcal{W}_k^\bullet$

In this section, we consider an n-tuple $\left(W^1,\dots,W^n\right)$ of simple admissible V-modules.

Lemma 3.2.1. *Given n admissible V-modules W^j, there is a natural map*

$$\phi^{\mathcal{J}}\colon (\mathcal{W}_k^\bullet)^{\mathcal{J}} = \big(\mathcal{F}_k(W^\bullet)\otimes\pi_*\mathcal{O}_{\widehat{\mathcal{M}}_{g,n}}\big)^{\mathrm{Aut}_+\mathcal{O}^n} \longrightarrow \big(\pi_*\widehat{\widehat{\mathbb{V}}}_g(V;W^\bullet)\big)^{\mathrm{Aut}_+\mathcal{O}^n} = \mathbb{V}^{\mathcal{J}}(V;W^\bullet) \tag{3.1}$$

from the sheaf $(\mathcal{W}_k^\bullet)^{\mathcal{J}}$ on $\mathcal{J}=\overline{\mathcal{J}}_{g,n}$ with fibers at closed points given by $(\mathcal{W}_k)_{(C,P_\bullet)}\cong\mathcal{F}_k(W^\bullet)$. If the sheaf of coinvariants descends to $\overline{\mathcal{M}}_{g,n}$, then $\phi^{\mathcal{J}}$ descends to a map

$$\phi\colon \mathcal{W}_k^\bullet = \big(\mathcal{F}_k(W^\bullet)\otimes\pi_*\mathcal{O}_{\widehat{\mathcal{M}}_{g,n}}\big)^{\mathrm{Aut}\,\mathcal{O}^n} \longrightarrow \big(\pi_*\widehat{\widehat{\mathbb{V}}}_g(V;W^\bullet)\big)^{\mathrm{Aut}\,\mathcal{O}^n} = \mathbb{V}_g(V;W^\bullet) \tag{3.2}$$

of sheaves over $\overline{\mathcal{M}}_{g,n}$. In case $k=0$, the sheaf $\mathcal{W}_0^\bullet$ (respectively, $(\mathcal{W}_0^\bullet)^{\mathcal{J}}$) on $\overline{\mathcal{M}}_{g,n}$ (respectively, on $\mathcal{J}$) is constant, with fibers $(\mathcal{W}_0)_{(C,P_\bullet)} = \bigotimes_j W_0^j$.

Remark 3.2.2. If the conformal dimensions of the modules are rational, then the sheaf of coinvariants $\mathbb{V}^{\mathcal{J}}(V;W^\bullet)$ on $\mathcal{J}=\overline{\mathcal{J}}_{g,n}$ descends to the sheaf $\mathbb{V}_g(V;W^\bullet)$ on $\overline{\mathcal{M}}_{g,n}$ (see [DGT19, § 8]).

Proof of Lemma 3.2.1. Consider the constant bundle $\mathcal{F}_k(\bigotimes_j W^j)\otimes \mathcal{O}_{\widehat{\mathcal{M}}_{g,n}}$ on $\widehat{\mathcal{M}}_{g,n}$, where $\mathcal{F}_k(\bigotimes_j W^j)$ is the degree k part of the filtration defined in § 3.1. Then $\mathcal{F}_k(\bigotimes_j W^j)\otimes \mathcal{O}_{\widehat{\mathcal{M}}_{g,n}}$ is a subbundle of $W^\bullet\otimes\mathcal{O}_{\widehat{\mathcal{M}}_{g,n}}$, and there is a natural composition

$$\mathcal{F}_k(\textstyle\bigotimes_j W^j)\otimes \mathcal{O}_{\widehat{\mathcal{M}}_{g,n}} \hookrightarrow W^\bullet\otimes\mathcal{O}_{\widehat{\mathcal{M}}_{g,n}} \longrightarrow \dfrac{W^\bullet\otimes\mathcal{O}_{\widehat{\mathcal{M}}_{g,n}}}{\mathcal{L}_{\widehat{\mathcal{C}}_{g,n}\setminus P_\bullet}\left(W^\bullet\otimes\mathcal{O}_{\widehat{\mathcal{M}}_{g,n}}\right)}=\widehat{\mathbb{V}}_g(V;W^\bullet)\,. \tag{3.3}$$

To show that this induces the maps (3.1) and (3.2), it is enough to show that $\mathcal{F}_k(\bigotimes_j W^j)$ is an $\operatorname{Aut}\mathcal{O}^n$-equivariant subset of $W^\bullet$. For this purpose, recall that $\operatorname{Aut}\mathcal{O}=\mathbb{G}_m\ltimes\operatorname{Aut}_+\mathcal{O}$. The action of $\operatorname{Aut}_+\mathcal{O}^n$ is given by exponentiating the action of L_i for $i\geqslant 1$, which shifts the degree in the negative direction, hence preserving $\mathcal{F}_k(W^\bullet)$. By definition, an element $z\in\mathbb{G}_m$ sends a homogeneous element $w\in W^j$ to $z^{-\deg w}w$; hence it preserves the degree of the element. It follows that $\mathcal{W}_k^\bullet := \left(\mathcal{F}_k(W^\bullet)\otimes\pi_*\mathcal{O}_{\widehat{\mathcal{M}}_{g,n}}\right)^{\operatorname{Aut}\mathcal{O}^n}$ is well defined, and the morphisms (3.1) and (3.2) are induced from (3.3).

For the last claim, it is enough to show that every element of $\operatorname{Aut}\mathcal{O}^n$ acts on $W_0^\bullet=\bigotimes_j W_0^j$ as the identity. From what we have just observed, every element of $\operatorname{Aut}^+\mathcal{O}$ acts on W_0^j as the identity because the modules are positively graded. From the description of the action of $\mathbb{G}_m$ on W^j given above, we have that the action of $z_\bullet=(z_1,\dots,z_n)\in\mathbb{G}_m^n$ on the element $w^\bullet=w_0^1\otimes\cdots\otimes w_0^n\in W_0^\bullet$ is given by

$$z_\bullet\cdot w^\bullet = z_1^{\deg(w_0^1)}w_0^1\otimes z_2^{\deg(w_0^2)}w_0^2\otimes\cdots\otimes z_n^{\deg(w_0^n)}w_0^n = w^\bullet\,,$$

so the restriction of this action to $W_0^\bullet$ is the identity, as wanted. It follows that the action of $(\operatorname{Aut}\mathcal{O})^n$ on $W_0^\bullet\otimes\mathcal{O}_{\widehat{\mathcal{M}}_{g,n}}$ is only given by the action of $(\operatorname{Aut}\mathcal{O})^n$ on $\mathcal{O}_{\widehat{\mathcal{M}}_{g,n}}$, and hence

$$\pi_*\left(W_0^\bullet\otimes\mathcal{O}_{\widehat{\mathcal{M}}_{g,n}}\right)^{(\operatorname{Aut}\mathcal{O})^n} = W_0^\bullet\otimes\left(\pi_*\mathcal{O}_{\widehat{\mathcal{M}}_{g,n}}\right)^{(\operatorname{Aut}\mathcal{O})^n} = W_0^\bullet\otimes\mathcal{O}_{\overline{\mathcal{M}}_{g,n}}\,,$$

which concludes the proof. □

Example 3.2.3. When $\mathfrak{g}$ is simply laced, the minimal series principal W-algebras $\mathcal{W}_\ell(\mathfrak{g})$ are simple, rational, C_2-cofinite, and of CFT-type for any (non-degenerate) admissible level ℓ; see [Ara15b, Ara15a]. In case $\ell+h^\vee=(k+h^\vee)/(k+h^\vee+1)$, for any positive integer k, these algebras are unitary. The W-algebra $\mathcal{W}_\ell(\mathfrak{g})$ is the simple quotient of the universal W-algebra $\mathcal{W}^\ell(\mathfrak{g})$. Zhu's algebra $A\big(\mathcal{W}^\ell(\mathfrak{g})\big)$ is isomorphic to the center $Z(U(\mathfrak{g}))$ of the universal enveloping algebra of $\mathfrak{g}$, and $A(\mathcal{W}_\ell(\mathfrak{g}))$ is a quotient of $Z(U(\mathfrak{g}))$ (see, for example, [DK06]). In particular, these algebras are commutative. Since the irreducible representations of a commutative algebra are 1-dimensional, any constant sheaf $\mathcal{W}_0^\bullet$ made from simple modules over $\mathcal{W}^\ell(\mathfrak{g})$ or $\mathcal{W}_\ell(\mathfrak{g})$ on $\overline{\mathcal{M}}_{0,n}$ is a line bundle.

As mentioned in the introduction, by [ACL19, Main Theorem 2] in types A, D, and E, and by [CL22, Corollary 4.1 and Theorem 7.1]) in types B and C, both $\mathcal{W}^\ell(\mathfrak{g})$ and $\mathcal{W}_\ell(\mathfrak{g})$ can be realized as cosets of tensor products of affine vertex operator algebras. The result in type A with $k=1$ was also proved in [ALY19]. In particular, except possibly when $k+h^\vee\in\mathbb{Q}_{\leqslant 0}$,

$$\mathcal{W}^\ell(\mathfrak{g})\cong\operatorname{Com}(V_{k+1}(\mathfrak{g}),V_k(\mathfrak{g})\otimes L_1(\mathfrak{g}))\quad\text{and}\quad\mathcal{W}_\ell(\mathfrak{g})\cong\operatorname{Com}(L_{k+1}(\mathfrak{g}),L_k(\mathfrak{g})\otimes L_1(\mathfrak{g}))\,.$$

Remark 3.2.4. Although by [DGT22], bundles of coinvariants for even lattice VOAs have rank 1 on $\overline{\mathcal{M}}_{0,n}$, it is not true that Zhu's algebra $A(V_L)$ will be commutative, even for an even lattice of rank 1.

4. Preparations for the proof of Theorem 1

Here we develop some tools that will be useful for proving Theorem 1. Although the theorem only discusses vertex algebras which are strongly generated in degree 1, we first analyze properties of vertex algebras strongly generated in degree d for $d \geqslant 1$ and then restrict to $d = 1$.

Lemma 4.0.1. *Assume that V is strongly finitely generated in degree d and W is an admissible V-module. Then every element $w \in W$ can be written as a linear combination of elements of the form*

$$A^1_{-j_1} A^2_{-j_2} \cdots A^m_{-j_m} w_0 \tag{4.1}$$

for some $m \geqslant 0$, with $A^i \in \mathcal{F}_d(V) \setminus V_0$ such that

(i) *if $\deg(w) = 0$, then $\deg\left(A^m_{-j_m}\right) \geqslant 0$, and*

(ii) *if $\deg(w) > 0$, then $\deg A^m_{-j_m} \geqslant 1$.*

Proof. We begin by observing that we can exclude the case where $A^i \in V_0 = \mathbb{C}\mathbf{1}$ since $\mathbf{1}_{-j}$ acts on W either by zero or by the identity.

By the proof of [Zhu96, Theorem 2.1.2], every element $w \in W$ can be written as a combination of $B^1_{-j_1} B^2_{-j_2} \cdots B^\ell_{-j_\ell} u_0$ for some B^i in V and $u_0 \in W_0$. Since V is strongly generated in degree d, we know that every element B^i of V can be written as a combination of elements of the type $B^{i,1}_{-k^i_1} \cdots B^{i,n_i}_{-k^i_{n_i}} \mathbf{1}$, where $\deg\left(B^{i,s}\right) \leqslant d$ for all $s \in \{1, \dots, n_i\}$.

Using this notation, we are then left to prove that every element written as

$$w = \left(B^{1,1}_{-k^1_1} \cdots B^{1,n_1}_{-k^1_{n_1}} \mathbf{1}\right)_{-j_1} \cdot \left(B^{2,1}_{-k^2_1} \cdots B^{2,n_2}_{-k^2_{n_2}} \mathbf{1}\right)_{-j_2} \cdots \left(B^{\ell,1}_{-k^\ell_1} \cdots B^{\ell,n_\ell}_{-k^\ell_{n_\ell}} \mathbf{1}\right)_{-j_\ell} \cdot u_0$$

can be rewritten as a linear combination of elements as in equation (4.1). This result is true by repeated use of the associator formula applied from left to right. That is, we first write

$$w_1 := \left(B^{2,1}_{-k^2_1} \cdots B^{2,n_2}_{-k^2_{n_2}} \mathbf{1}\right)_{-j_2} \cdots \left(B^{\ell,1}_{-k^\ell_1} \cdots B^{\ell,n_\ell}_{-k^\ell_{n_\ell}} \mathbf{1}\right)_{-j_\ell} \cdot u_0 \,,$$

so that

$$w = \left(B^{1,1}_{-k^1_1} \cdot B^{1,2}_{-k^1_2} \cdots B^{1,n_1}_{-k^1_{n_1}} \mathbf{1}\right)_{-j_1} w_1 = \Big(B^{1,1}_{-k^1_1} \underbrace{\left(B^{1,2}_{-k^1_2} \cdots B^{1,n_1}_{-k^1_{n_1}} \mathbf{1}\right)}_{=:\, D^{1,2}}\Big)_{-j_1} w_1 = \left(B^{1,1}_{-k^1_1}\left(D^{1,2}\right)\right)_{-j_1} w_1 \,.$$

We can then expand $\left(B^{1,1}_{-k^1_1}\left(D^{1,2}\right)\right)_{-j_1}$ using the associator formula. Following the expansion, we rewrite $D^{1,2}$ as $B^{1,2}_{-k^1_2}\left(D^{1,3}\right)$, where $D^{1,3} = \left(B^{1,3}_{-k^1_3} \cdots B^{1,n_1}_{-k^1_{n_1}}\right)$, and again expand using the associator formula. Repeating this for all $D^{1,i} = \left(B^{1,i}_{-k^1_i} \cdots B^{1,n_1}_{-k^1_{n_1}}\right)$, expanding using the associator formula, and then carrying out the same procedure for the factors of w_1, we arrive at a linear combination of terms of the form described in (4.1).

We note that once in the form given in (4.1), if the term $A^m_{-j_m}$ adjacent to w_0 has negative degree, then $A^m_{-j_m} w_0 = 0$ since W is graded by $\mathbb{N}$. If $A^m_{-j_m}$ has degree zero, then $A^m_{-j_m} w_0 = u_0 \in W_0$, and we may as well replace it. ☐

Given Lemma 4.0.1, we make the following definition for the length of an element in W, which will be used to argue by induction in the proof of Lemma 4.0.3.

Definition 4.0.2. Suppose that V is strongly generated in degree d and that W is a simple V-module. For $\ell \in \mathbb{N}$, set

$$G^\ell(W) := \operatorname{Span}\left\{A^1_{-j_1} \cdots A^\ell_{-j_\ell} w_0 \mid A^j \in \mathcal{F}_d(V) \setminus V_0,\ w_0 \in W_0\right\}.$$

We say that w has *d-length ℓ* if $w \in L^\ell(W) := G^\ell W \setminus G^{\ell-1}(W)$.

When there is no ambiguity on d, we will simply use length in place of d-length. Given two elements w_1 and $w_2 \in W$, we say that w_1 is shorter than w_2 if the length of w_1 is smaller than the length of w_2. For the proof of Theorem 1, we will need a refined version of Lemma 4.0.1.

Lemma 4.0.3. *Assume that V is strongly finitely generated in degree d. Then every element $w \in W$ such that $\deg(w) > 0$ can be written as a combination of elements of the form in (4.1) with the additional properties:*

(i) *If $\deg\left(A^1\right) = 1$, then $j_1 \geqslant 1$.*

(ii) *In case $d = 1$, every element $A^i_{-j_i}$ has positive degree or, equivalently, $j_i \geqslant 1$ for every i.*

Proof. We start by proving part (ii) and observe that by induction on the length of the elements, it is enough to consider only the case $m = 2$. Using the commutator formula, we have that

$$A^1_{-j_1} A^2_{-j_2} w_0 = A^2_{-j_2} A^1_{-j_1} w_0 + \sum_{k \geqslant 0} \left(A^1_k A^2\right)_{-j_1 - j_2 - k} w_0 \,.$$

We now show that each non-zero term on the right-hand side of the equality is a term which either is of degree zero or satisfies the wanted property. The first term of the right-hand side is zero if $\deg\left(A^1_{-j_1}\right)$ is negative and $A^2_{-j_2} u_0$ for some $u_0 \in W^0$ if $\deg\left(A^1_{-j_1}\right) = 0$. Else, $\deg\left(A^1_{-j_1}\right)$ is positive. We now look at the other terms. Since both A^1 and A^2 have degree 1, the only non-zero summands are those where $k = 0$ or $k = 1$. When $k = 0$, we have that $B = A^1_0 A^2$ is an element of degree 1; hence we reduce the statement to the case $m = 1$. When $k = 1$, the element $A^1_1 A^2$ has degree zero, which implies that it is a multiple of the vacuum vector, and so it can act on w_0 only by a scalar.

We are left to prove part (i). In this case too, the proof follows from the commutator formula and induction on the length of elements. With more details, let $w = A^1_{-j_1} \cdots A^m_{-j_m} u_0$ as in Lemma 4.0.1, and define the K-value of w, denoted by $K(w)$, as the smallest integer in $\{0, 1, \ldots, m\}$ such that $\deg\left(A^{K(w)}\right) \geqslant 2$, with $K(w) = 0$ if all the elements have degree 1. It is enough to show that every element with $K(w) \geqslant 2$ can be written as a sum of shorter elements and elements with smaller K-value. By repeating the argument, we reduce to the case where either $K = 0$ or $K = 1$. Observe that if $K = 0$, then we are done by part (ii) above, while if $K = 1$, then we are in the situation $\deg\left(A^1\right) \geqslant 2$. When $K(w) = K \geqslant 2$, using the commutator formula, we can write w as

$$\begin{aligned} & A^1_{-j_1} \cdots A^K_{-j_K} \cdot A^{K-1}_{-j_{K-1}} \cdot A^{K+1}_{-j_{K+1}} \cdots A^m_{-j_m} u_0 \\ & \quad + \sum_{k \geqslant 0} A^1_{-j_1} \cdots A^{K-2}_{-j_{K-2}} \cdot \left(A^{K-1}_k\left(A^K\right)\right)_{-j_K - j_{K-1} - k} \cdot A^{K+1}_{-j_{K+1}} \cdots A^m_{-j_m} u_0 \,. \end{aligned}$$

The first term is an element with K-value less than K. Moreover, since $\deg\left(A^{K-1}\right) = 1$, for every $k \geqslant 0$, we have that $\deg A^{K-1}_k\left(A^K\right) \geqslant d$, which shows that the terms in the second line are shorter than w. ☐

5. Proof of Theorem 1

Here we prove Theorem 1. We recall that in § 2.7.1 we describe the descent of coinvariants from $\widehat{\mathcal{M}}_{g,n}$ to $\overline{\mathcal{M}}_{g,n}$, which is carried out in two steps, first to a vector bundle $\mathbb{V}^{\mathcal{J}}(V; W^\bullet)$ on $\overline{\mathcal{J}}_{g,n}$ and then, if possible, to $\overline{\mathcal{M}}_{g,n}$. To show that the sheaf of coinvariants is globally generated, we show that each of the maps $\phi^{\mathcal{J}}$ and ϕ (when it exists) defined in Lemma 3.2.1 from the constant bundle $\mathcal{W}_0^\bullet$ to $\mathbb{V}_0(V;W^\bullet)$ is surjective. For this, it is sufficient to show that the map under consideration is surjective on fibers.

Since fibers of either sheaf are isomorphic, and the arguments are the same, without loss of generality we make the argument for the map ϕ and assume $(C,P_\bullet)\in\overline{\mathcal{M}}_{0,n}$. Then the restriction $\phi|_{(C,P_\bullet)}$ of ϕ to the fiber is induced by the composition

$$W_0^\bullet \hookrightarrow W^\bullet \twoheadrightarrow W^\bullet/\mathcal{L}_{C\setminus P_\bullet}(V)W^\bullet \cong W^\bullet_{\mathcal{L}_{C\setminus P_\bullet}(V)}\,.$$

It is enough to show that every element of $W^\bullet$ can be written as a combination of elements of $W_0^\bullet$ and elements in $\mathcal{L}_{C\setminus P_\bullet}(V)\cdot W^\bullet$.

For this purpose, it suffices to show that for every $d\geqslant 1$, we can write any element $w^\bullet$ of $W_d^\bullet=\mathcal{F}_d(W^\bullet)\setminus\mathcal{F}_{d-1}(W^\bullet)$ as a linear combinations of element in $\mathcal{L}_{C\setminus P_\bullet}(V)\cdot W^\bullet\cup\mathcal{F}_{d-1}(W^\bullet)$. In other words, we will prove that there exist elements

$$\sigma\in\mathcal{L}_{C\setminus P_\bullet}(V)\quad\text{and}\quad v^\bullet\in W^\bullet$$

such that

$$\sigma(v^\bullet)-w^\bullet\in\mathcal{F}_{\deg(w^\bullet)-1}(W^\bullet)\,. \tag{5.1}$$

In view of part (ii) of Lemma 4.0.3, we can further assume that $w^\bullet$ is of the form $w^1\otimes\cdots\otimes w^n$, where

- every w^k is a homogeneous element of W^k of degree d_k;
- for one $i\in\{1,\dots,n\}$, we can write $w^i=A_{-j}u^i$, with u^i a homogeneous elements of W^i, $A\in V_1$, and $j\geqslant 1$.

We start by showing that there exists an element $\sigma=A\otimes\mu\in\mathcal{L}_{C\setminus P_\bullet}(V)$ that has a pole of order j at P_i and is regular at all the other points. Since the description of $\mathcal{L}_{C\setminus P_\bullet}(V)$ over nodal curves requires some extra work, the proof continues treating separately the cases of C being a smooth or a nodal curve.

5.1 The smooth case: $C\cong\mathbb{P}^1$

Since $C\setminus P_\bullet$ is affine, we can deduce that

$$\mathcal{L}_{C\setminus P_\bullet}(V)\cong \mathrm{H}^0(C\setminus P_\bullet,\mathcal{V}_C\otimes\omega_C/\mathrm{Im}\nabla)\cong \mathrm{H}^0(C\setminus P_\bullet,\mathcal{V}_C\otimes\omega_C)/\nabla\mathrm{H}^0(C\setminus P_\bullet,\mathcal{V}_C)\,.$$

By [DGT19], one has

$$\mathrm{H}^0(C\setminus P_\bullet,\mathcal{V}_C\otimes\omega_C)\cong\bigoplus_{m\geqslant 0}\mathrm{H}^0\big(C\setminus P_\bullet,(\omega_C^{1-m})^{\dim V_m}\big)\cong\bigoplus_{m\geqslant 0}V_m\otimes\mathrm{H}^0\big(C\setminus P_\bullet,\omega_C^{1-m}\big)\,.$$

Using the Riemann–Roch theorem, if $D=\mathcal{O}_C(jP_i)$, the subset

$$\bigoplus_{m\geqslant 0}V_m\otimes\mathrm{H}^0\big(C,\omega_C^{1-m}(D)\big)\subseteq\bigoplus_{m\geqslant 0}V_m\otimes\mathrm{H}^0\big(C\setminus P_\bullet,\omega_C^{\otimes 1-m}\big)$$

is non-zero. In particular, it contains an element $\sigma=A\otimes\mu$, where $\mu\in\mathrm{H}^0(C,\mathcal{O}(jP_i))$ has a pole of order j at P_i and is regular elsewhere.

We next show that the element $v^\bullet=w^1\otimes\cdots\otimes w^{i-1}\otimes u^i\otimes w^{i+1}\otimes\cdots w^n$ satisfies (5.1). The action of $\mathcal{L}_{C\setminus P_\bullet}(V)$ on $W^\bullet$ is given by the diagonal action of σ_{P_k} on W^k (this is the image of σ

along the map $\mathcal{L}_{C\setminus P_\bullet} \to \mathfrak{L}_{P_k}(V)$ arising from (2.1)). From the definition of σ, we have that

$$\sigma_{P_i} = A_{[-j]} + \sum_{m\geqslant 1} \alpha_{-j+m} A_{[-j+m]}\,,$$

and since σ is regular at the other points, we have that $\sigma_{P_k} = \sum_{q\geqslant 0} b_q A_{[q]}$ with $b_q \in \mathbb{C}$, hence

$$\sigma_{P_k} \cdot W^j_{d_k} \subset W^j_{d_k}\,. \tag{5.2}$$

Summarizing, we conclude that

$$\begin{aligned}\sigma(v^\bullet) - w^\bullet = &\sum_{m\geqslant 1} \alpha_{-j+m}\big(w^1 \otimes \cdots \otimes w^{i-1} \otimes A_{(-j+m)}u^i \otimes w^{i+1} \otimes \cdots \otimes w^n\big)\\ &+ \sum_{k\neq i} w^1 \otimes \cdots \otimes u^i \otimes \cdots \otimes w^{k-1} \otimes \sigma_{P_k}\big(w^k\big) \otimes w^{k+1} \otimes \cdots w^n\,.\end{aligned} \tag{5.3}$$

Terms in the first line of the right-hand side of (5.3) are in $\mathcal{F}_{\deg(w^\bullet)-m}(W^\bullet)$ for $m \geqslant 1$. Each summand in the second line of (5.3) is in $\mathcal{F}_{\deg(w^\bullet)-j}(W^\bullet)$ by (5.2). Since $j \geqslant 1$ by assumption, we can conclude that (5.1) holds, concluding the proof in the smooth case.

5.2 C is nodal

The stability condition ensures that $C\setminus P_\bullet$ is an affine curve. Moreover, without loss of generality, we can assume that C has two components C_+ and C_- which meet at only one node Q as in Figure 1. The marked points on C_+ will be indexed by $P_{\bullet,+}$, and the marked points on C_- will be indexed by $P_{\bullet,-}$. Assume that the point $P_i = P$ lies in the component C_+. The preimages of Q via the normalization morphism $\eta\colon \widetilde{C} = C_+ \sqcup C_- \to C$ are the points Q_+ and Q_- with local coordinates t_+ and t_-.

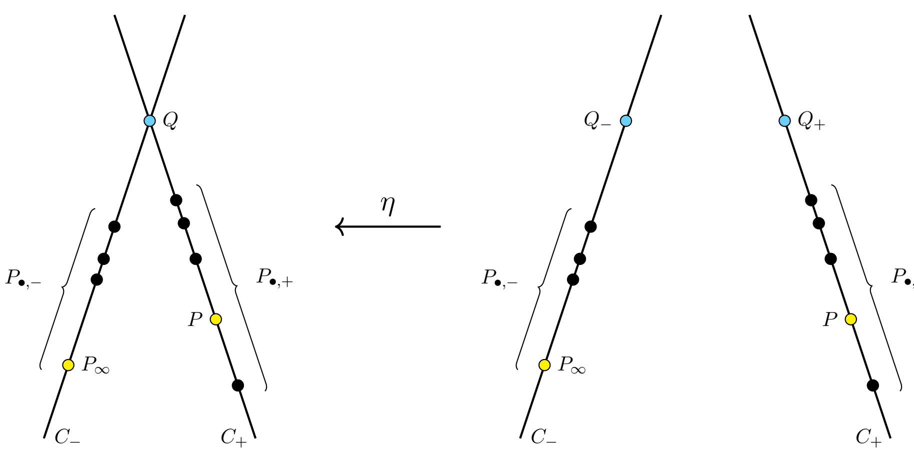

Figure 1. Normalization map

As in the smooth case, the goal is to construct an element of $\mathcal{L}_{C\setminus P_\bullet}(V)$ such that (5.1) holds. To do so, we view $\mathcal{L}_{C\setminus P^\bullet}(V)$ as consisting of elements of $\mathcal{L}_{\widetilde{C},P_\bullet\sqcup Q_\bullet}(V) = \mathcal{L}_{C_+\setminus P_{\bullet,+}\sqcup Q_+}(V) \oplus \mathcal{L}_{C_-\setminus P^\bullet_-\sqcup Q_-}(V)$ satisfying conditions described in [DGT19, Proposition 3.3.1] and stated here for convenience. For this purpose, recall that $\mathfrak{L}_{Q_\pm}(V) \cong \mathfrak{L}(V)$ is filtered, so that it admits a triangular decomposition $\mathfrak{L}(V) = \mathfrak{L}(V)_{<0} \oplus \mathfrak{L}(V)_0 \oplus \mathfrak{L}(V)_{>0}$. Let $\sigma_{Q_\pm} \in \mathfrak{L}_{Q_\pm}(V)$ be the image of $\sigma \in \mathcal{L}_{\widetilde{C}\setminus P_\bullet\sqcup Q_\bullet}(V)$, and let $\big[\sigma_{Q_\pm}\big]_0$ be the image of $\sigma_{Q_\pm}$ under the projection $\mathfrak{L}_{Q_\pm}(V) \cong$

$\mathfrak{L}(V) \to \mathfrak{L}(V)_0$. The involution ϑ of $\mathfrak{L}(V)$, which restricts to an involution on $\mathfrak{L}(V)_0$, is given for a homogeneous element $B \in V$ of degree b by

$$\vartheta\left(B_{[b-1]}\right) = (-1)^{b-1} \sum_{i \geqslant 0} \frac{1}{i!} L_1^i B_{[b-i-1]} \,.$$

With this notation, [DGT19, Proposition 3.3.1] says that

$$\mathcal{L}_{C \setminus P_\bullet}(V) = \left\{ \sigma \in \mathcal{L}_{\widetilde{C} \setminus P_\bullet \sqcup Q_\bullet}(V) \mid \sigma_{Q_+}, \sigma_{Q_-} \in \mathfrak{L}(V)_{\leqslant 0}, \text{ and } [\sigma_{Q_-}]_0 = \vartheta([\sigma_{Q_+}]_0) \right\}.$$

Following the argument of § 5.1, we can show that there exists an element $\sigma_+ = A \otimes \mu$ of $\mathcal{L}_{C_+ \setminus P_{\bullet,+}}(V) \subseteq \mathcal{L}_{C_+ \setminus P_{\bullet,+} \sqcup Q_+}(V)$, where μ is a form which has a pole of order j at P and is regular at other points. The expansion of σ_+ at the other points $P_{k,+} \neq P$ can be seen as an endomorphism of W^k of degree less than or equal to zero as in (5.2). The expansion of σ_+ at the point Q_+ can be written as the element

$$\sigma_{+Q_+} = \sum_{i \geqslant 0} a_i A_{[i]} \in \mathfrak{L}(V) \,. \tag{5.4}$$

Note that the components of σ_{+Q_+} have non-positive degree and $[\sigma_+]_0 := [\sigma_{+Q_+}]_0 = a_0 A_{[0]}$.

To produce an element in $\mathcal{L}_{C \setminus P_\bullet}(V)$, we need to construct an element $\sigma_- \in \mathcal{L}_{C_- \setminus P_{\bullet,-}}(V)$ which is compatible with σ_+. The compatibility condition requires that

$$[\sigma_-]_0 = \vartheta[\sigma_+]_0 = a_0 A_{[0]} + a_0 L_1(A)_{[-1]} = a_0 A_{[0]} + a_1 \mathbf{1}_{[-1]} \tag{5.5}$$

for some $a_1 \in \mathbb{C}$ since V is of CFT-type.

We are left to show that there is an element σ_- in $\mathcal{L}_{C_- \setminus P_{\bullet,-} \sqcup Q_-}(V)$ whose image in $\mathfrak{L}(V)_0$ is $a_0 A_{[0]} + a_1 \mathbf{1}_{[-1]}$. To do so, we consider the two components independently and use the fact that $\mathcal{L}_{C_- \setminus P_{\bullet,-} \sqcup Q_-}(V)$ is a quotient of $\bigoplus_{k \geqslant 0} \mathrm{H}^0\big(C_- \setminus P_{\bullet,-} \sqcup Q_-, V_k \otimes \omega^{1-k}\big)$.

We first observe that $V_1 = \mathrm{H}^0(C_-, V_1 \otimes \mathcal{O}) \subseteq \bigoplus_{k \geqslant 0} \mathrm{H}^0\big(C_- \setminus P_{\bullet,-} \sqcup Q_-, V_k \otimes \omega^{1-k}\big)$. Hence we can lift $a_0 A_{[0]}$ to an element $\beta \in \mathcal{L}_{C_- \setminus P_{\bullet,-} \sqcup Q_-}(V)$ such that

$$\beta|_{Q_-} = a_0 A \quad \text{and} \quad \beta|_{P_j^-} = a_0 A \,. \tag{5.6}$$

After observing that $\mathrm{H}^0(C_-, V_0 \otimes \omega(Q_-)) = 0$, let P_∞ be any point in $P_{\bullet,-}$, and note that $V_0 \cong \mathrm{H}^0(C_-, V_0 \otimes \omega(Q_- + P_\infty)) \subseteq \bigoplus_{k \geqslant 0} \mathrm{H}^0\big(C_- \setminus P_{\bullet,-} \sqcup Q_-, V_k \otimes \omega^{1-k}\big)$ is 1-dimensional. Hence there exists an element $\gamma \in \mathcal{L}_{C_- \setminus P_{\bullet,-} \sqcup Q_-}(V)$ satisfying

$$\gamma|_{Q_-} = a_1 \mathbf{1} \otimes t_-^{-1} + \mathbf{1} \otimes F(t_-) \,, \quad \gamma|_{P_\infty} = a_\infty \mathbf{1} \otimes {t_\infty}^{-1} + \mathbf{1} \otimes G(t_\infty) \,, \quad \gamma|_{P_{j,-} \neq P_\infty} = a_j \otimes H(t_j) \tag{5.7}$$

with $F(t_-) \in \mathbb{C}[\![t_-]\!]$, $G(t_\infty) \in \mathbb{C}[\![t_\infty]\!]$ and $H(t_j) \in \mathbb{C}[\![t_j]\!]$.

It follows that the pair $(\sigma_+, \beta + \gamma)$ defines an element σ of $\mathcal{L}_{C \setminus P_\bullet}(V)$. We are left to prove that under this choice, (5.1) holds, but this follows from (5.6) and (5.7) and the fact that σ_+ has poles only at P.

6. The corollaries

The proof of Theorem 1, and the definition of admissible modules, implies the following statement.

Corollary A. *The sheaf of coinvariants $\mathbb{V}_0(V; W^\bullet)$ on $\overline{\mathcal{M}}_{0,n}$, defined by n simple admissible modules over a vertex operator algebra V of CFT-type and strongly generated in degree 1, is coherent.*

Proof. By definition, for every $i \in \{1, \dots, n\}$, the admissible V-module W^i has finite-dimensional lowest-weight component W^i_0. This fact, together with Lemma 3.2.1, gives that $\mathcal{W}^\bullet_0$ is a vector bundle of finite rank on $\overline{\mathcal{M}}_{0,n}$. By the proof of Theorem 1, the map $\phi\colon \mathcal{W}^\bullet_0 \to \mathbb{V}_0(V; W^\bullet)$ defined in (3.2) is surjective. From this and the coherence of $\mathcal{W}^\bullet_0$, we therefore deduce the coherence of $\mathbb{V}_0(V; W^\bullet)$, giving the assertion. □

A further consequence, owing to both Corollary A and [DGT21, Theorem 7.1], is the following.

Corollary B. *The sheaf of coinvariants $\mathbb{V}_0(V; W^\bullet)$, defined by n simple admissible modules over a vertex operator algebra V of CFT-type and strongly generated in degree 1, is locally free of finite rank on $\mathcal{M}_{0,n}$.*

Proof. Since by Corollary A, we have that $\mathbb{V}_0(V; W^\bullet)$ is coherent, it is enough to show that the restriction of $\mathbb{V}_0(V; W^\bullet)$ to $\mathcal{M}_{0,n}$ is equipped with a projectively flat connection. This is guaranteed by [DGT21, Theorem 7.1], whose hypotheses are actually weaker than what is assumed for Corollary B. □

Finally, applying results from [DGT19], we obtain the following.

Corollary C. *For V a rational, C_2-cofinite vertex operator algebra of CFT-type and strongly generated in degree 1, the sheaf $\mathbb{V}_0(V; W^\bullet)$ defined by n simple admissible V modules is a globally generated vector bundle on $\overline{\mathcal{M}}_{0,n}$.*

Proof. This follows from [DGT19, VB Corollary] and Theorem 1. □

7. Higher-genus examples

In this section, we describe vector bundles of coinvariants on $\overline{\mathcal{M}}_{g,n}$ defined by holomorphic vertex algebras of CFT-type, which are globally generated for positive genus (Example 7.0.1). As explained in Remark 7.0.2, global generation is not given by Theorem 1.

Example 7.0.1. A VOA is holomorphic if it is self-contragredient and the only irreducible V-module is itself. By [DGT22, § 1.6.1], using factorization, it was shown that bundles of coinvariants defined by holomorphic VOAs V of CFT-type have rank 1, with Chern class $\frac{1}{2}c_V\lambda$, where c_V is the central charge of V. Line bundles are globally generated if their first Chern class is base-point-free. It is well known that λ, the first Chern class of the Hodge bundle, is base-point-free, and non-trivial if $g > 0$. As explained in [LS19a, § 3], by [DM04b, Theorems 1 and 2], any holomorphic VOA of CohFT-type has positive central charge (in fact, c_V is divisible by 8). In particular, sheaves of coinvariants defined by holomorphic vertex operator algebras are globally generated on $\overline{\mathcal{M}}_{g,n}$.

Moreover, if $c_V \leqslant 24$, the character of V and the degree 1 component V_1 are uniquely determined, and in particular there are many examples for which $V_1 \neq \emptyset$. For instance, if V is a C_2-cofinite, holomorphic vertex operator algebra of CFT-type (in the language of [DM04b], V is strongly rational and holomorphic), then for $c = 8$, $V = V_L$ is the lattice VOA given by the E_8 root lattice [DM04a, Theorem 1]. In particular, the affine VOA bundles $\mathbb{V}_g(L_1(\mathfrak{e}_8); W^\bullet)$ have first Chern classes which are multiples of λ, so are base-point-free (see also [Fak12, Corollary 6.3 and Remark 6.4]). If $c = 16$, then $V = V_L$, where L is one of the two unimodular rank 16 lattices [DM04a, Theorem 2], and if $c_V = 24$, then if V_1 is abelian of rank 24, V is isomorphic to the Leech lattice VOA [DLM00], and if V_1 is zero, then $V \cong V^\natural$. If on the other hand V_1 is semi-simple, then relations between the dual Coxeter number, the dimension, and the level of its affinization and other constraints led Shellekens, in [Sch93], to propose a list of 69 other Lie algebra structures for V_1, that he conjectured would determine these holomorphic VOAs of conformal dimension 24 (these 71 make up what is called Schellekens' list). As described by Lam and Shimakura in [LS19a], all such V have now been constructed [Bor86, FLM88, Don93, DGM96, Lam11, LS12, Miy13, SS16, vEMS20a, Möl21, LS16, LL20]. The last cases required substantial development of orbifold theory and were completed in [vEMS20a, vEMS20b]. The uniqueness of these VOAs was proven in [vEMS20a, vEMS20b, LS19b, LS20b, LS20a, LS20b].

Remark 7.0.2. Since holomorphic vertex operator algebras of CFT-type have 1-dimensional degree zero components, associated sheaves $\mathcal{W}_0^\bullet$ have rank 1. However, although the $\mathbb{V}_g(V;W^\bullet)$ are globally generated, we do not know if it is possible to prove that the map (3.2) from Lemma 3.2.1 from $\mathcal{W}_0^\bullet$ to $\mathbb{V}_g(V;W^\bullet)$ is surjective (see Question 3).

8. Discrete series bundles

Sheaves defined from the discrete series representations of the Virasoro vertex algebra Vir_c were introduced in [BFM91]. Such VOAs are the simplest case of a family referred to as the minimal series principal W-algebras $\mathcal{W}_k(\mathfrak{g})$, see [Ara11, ALY14], and in case $\mathfrak{g}=\mathfrak{sl}_2$, one obtains Vir_c (see Example 3.2.3). The minimal series principal W-algebras arise in many contexts (see [ALY14] and references therein). Unlike affine VOAs, the minimal series principal W-algebras are not strongly generated in degree 1 [ACL19]; however, as discussed in Example 3.2.3, they are related to affine VOAs through a coset construction. In § 8.0.2, we describe the discrete series representations of Vir_c and their modules, afterwards giving a formula for their ranks, and a specific example of the Chern classes of bundles they define for $n=4$. But first, in § 8.0.1, we give a brief summary of our findings about them.

8.0.1 *Summary.* Since the Zhu algebra $A(\mathrm{Vir}_c)$ is commutative, for any bundle of coinvariants $\mathbb{V}(\mathrm{Vir}_c;W^\bullet)$, the associated constant sheaf $\mathcal{W}_0^\bullet$ has rank 1. As we show, one can cook up bundles of coinvariants $\mathbb{V}(\mathrm{Vir}_c;W^\bullet)$ of ranks 0, 1, and larger than 1. In all the examples we considered, if $\mathbb{V}(\mathrm{Vir}_c;W^\bullet)$ had rank 1, then it was positive. If the rank was larger than 1, it was positive if and only if its modules satisfied an integral degree condition (see Definition 2.3.1 and Question 2).

8.0.2 *Description of the discrete series representations of* Vir_c *and their modules.* Let $\mathrm{Vir}_{\geqslant 0} := \mathbb{C}K \oplus z\mathbb{C}[\![z]\!]\partial_z$ be a Lie subalgebra of the Virasoro Lie algebra Vir, and let $M_{c,h} := U(\mathrm{Vir})\otimes_{U(\mathrm{Vir}_{\geqslant 0})}\mathbb{C}\mathbf{1}$ be the Verma module of highest weight $h\in\mathbb{C}$ and central charge $c\in\mathbb{C}$ (note that $M_{c,h}$ is a module over $M_{c,0}$). There is a unique maximal proper submodule $J_{c,h}\subset M_{c,h}$. Set $L_{c,h} := M_{c,h}/J_{c,h}$ and $\mathrm{Vir}_c := L_{c,0}$. By [Wan93, Theorem 4.2 and Corollary 4.1], one has that Vir_c is rational if and only if $c=c_{p,q}=1-6(p-q)^2/pq$, where p and $q\in\{2,3,\dots\}$ are relatively prime. By [DLM00, Lemma 12.3] (see also [Ara12, Proposition 3.4.1]), the VOA Vir_c is C_2-cofinite for $c=c_{p,q}$, and by [FZ92, Theorem 4.3], the VOA Vir_c is of CFT-type. By [Wan93, Theorem 4.2], the modules $L_{c,h}$ are irreducible if and only if

$$h=\frac{(np-mq)^2-(p-q)^2}{4pq}\,,\quad \text{with } 0<m<p\,,\ \text{and } 0<n<q\,.$$

Note that, by definition, h is the conformal dimension of $L_{c,h}$. These vertex operator algebras are unitary if $|q-p|=1$.

8.0.3 *A particular example.* Let $V = L_{\frac{1}{2},0} = \mathrm{Vir}_{c_{3,4}}$ be the discrete series vertex operator algebra with central charge $\frac{1}{2}$. This vertex operator algebra has only two non-trivial simple modules, $W_1 = L_{\frac{1}{2},\frac{1}{2}}$ and $W_2 = L_{\frac{1}{2},\frac{1}{16}}$. Divisors associated with bundles of rank zero are trivial and hence are trivially nef. We show that this rank is determined by the parity of W_1 and W_2:

Proposition 8.0.4. *On $\overline{\mathcal{M}}_{0,i+j+k}$, for $i+j+k \geqslant 3$, one has*

$$\operatorname{rank}\big(\mathbb{V}_0\big(V; V^{\otimes i} \otimes W_1^{\otimes j} \otimes W_2^{\otimes k}\big)\big) = \begin{cases} 2^{\ell} & \text{if } k = 2\ell+2 \text{ with } \ell \geqslant 0\,, \\ 1 & \text{if } j \text{ is even and } k = 0\,, \\ 0 & \text{otherwise}. \end{cases}$$

Example 8.0.5. On $\overline{\mathcal{M}}_{0,4}$,

$$\deg\big(\mathbb{V}_0\big(V; W_1^{\otimes j} \otimes W_2^{\otimes k}\big)\big) = \begin{cases} 1 & \text{if } j = k = 2\,, \\ 2 & \text{if } j = 4\,, \\ -1 & \text{if } k = 4\,. \end{cases}$$

In particular, the line bundles $\mathbb{V}_0\big(V; W_1^{\otimes 2} \otimes W_2^{\otimes 2}\big)$ and $\mathbb{V}_0\big(V; W_1^{\otimes 4}\big)$ are globally generated on $\overline{\mathcal{M}}_{0,4}$, while the bundle $\mathbb{V}_0\big(V; W_2^{\otimes 4}\big)$, which has rank 2, is not globally generated on $\overline{\mathcal{M}}_{0,4}$.

Proposition 8.0.4 is proved by induction, using formulas from § 2.8, with base case dependent on the following.

Lemma 8.0.6 ([DMZ94]). *For $V = \mathrm{Vir}_{c_{3,4}}$, $W_1 = L_{\frac{1}{2},\frac{1}{2}}$, and $W_2 = L_{\frac{1}{2},\frac{1}{16}}$, the dimension of $\mathbb{V}_0(V; W^\bullet)$ on $\overline{\mathcal{M}}_{0,3}$ is 1 if $W^\bullet$ is (V,V,V), (V,W_1,W_1), (V,W_2,W_2), or (W_1,W_2,W_2); it is zero otherwise.*

Proof of Proposition 8.0.4. By propagation of vacua, if $j+k \geqslant 3$, then the rank of the vector bundle $\mathbb{V}_0\big(V; V^{\otimes i} \otimes W_1^{\otimes j} \otimes W_2^{\otimes k}\big)$ is the same as the rank of $\mathbb{V}_0\big(V; W_1^{\otimes j} \otimes W_2^{\otimes k}\big)$. We then need to prove the theorem only for bundles of the form $\mathbb{V}_0\big(V; W_1^{\otimes j} \otimes W_2^{\otimes k}\big)$ for $j+k \geqslant 3$.

We first show that $\mathbb{V}_0\big(V; W_1^{\otimes j} \otimes W_2^{\otimes 2\ell+2}\big)$ has rank 2^ℓ by double induction on ℓ and j, where the cases $j=0,1$ and $\ell=0$ follow from Lemma 8.0.6. Using (2.3), we obtain that

$$\begin{aligned} \operatorname{rank}\mathbb{V}_0\big(V; W_1^{\otimes j} \otimes W_2^{\otimes 2\ell+2}\big) = {} & \operatorname{rank}\mathbb{V}_0\big(V; W_1^{\otimes j} \otimes W_2^{\otimes 2\ell}\big) \operatorname{rank}\mathbb{V}_0\big(V; W_2^{\otimes 2}\big) \\ & + \operatorname{rank}\mathbb{V}_0\big(V; W_1^{\otimes j+1} \otimes W_2^{\otimes 2\ell}\big) \operatorname{rank}\mathbb{V}_0\big(V; W_1 \otimes W_2^{\otimes 2}\big) \\ & + \operatorname{rank}\mathbb{V}_0\big(V; W_1^{\otimes j} \otimes W_2^{\otimes 2\ell+1}\big) \operatorname{rank}\mathbb{V}_0\big(V; W_2^{\otimes 3}\big)\,. \end{aligned}$$

By induction on ℓ and Lemma 8.0.6, we deduce that

$$\operatorname{rank}\mathbb{V}_0\big(V; W_1^{\otimes j} \otimes W_2^{\otimes 2\ell+2}\big) = 2^{\ell-1} + \operatorname{rank}\mathbb{V}_0\big(V; W_1^{\otimes j+1} \otimes W_2^{\otimes 2\ell}\big)\,,$$

and using (2.3) again, Lemma 8.0.6, and induction on ℓ and j, we obtain that

$$\begin{aligned} \operatorname{rank}\mathbb{V}_0\big(V; W_1^{\otimes j+1} \otimes W_2^{\otimes 2\ell}\big) = {} & \operatorname{rank}\mathbb{V}_0\big(V; W_1^{\otimes j-1} \otimes W_2^{\otimes 2\ell}\big) \operatorname{rank}\mathbb{V}_0\big(V; W_1^{\otimes 2}\big) \\ & + \operatorname{rank}\mathbb{V}_0\big(V; W_1^{\otimes j} \otimes W_2^{\otimes 2\ell}\big) \operatorname{rank}\mathbb{V}_0\big(V; W_1^{\otimes 3}\big) \\ & + \operatorname{rank}\mathbb{V}_0\big(V; W_1^{\otimes j-1} \otimes W_2^{\otimes 2\ell+1}\big) \operatorname{rank}\mathbb{V}_0\big(V; W_1^{\otimes 2} \otimes W_2\big) \\ = {} & \operatorname{rank}\mathbb{V}_0\big(V; W_1^{\otimes j-1} \otimes W_2^{\otimes 2\ell}\big) = 2^{\ell-1}, \end{aligned}$$

so that $\operatorname{rank} \mathbb{V}_0\big(V; W_1^{\otimes j} \otimes W_2^{\otimes 2\ell+2}\big) = 2^{\ell-1} + 2^{\ell-1} = 2^\ell$, as claimed.

We then show that $\operatorname{rank} \mathbb{V}_0\big(V; W_1^{\otimes j}\big)$ is equal to 1 when j is even and is zero when j is odd. This is shown by induction on j, knowing the result for $0 \leqslant j \leqslant 3$. Assume $j \geqslant 4$. Using (2.3), we obtain that

$$\begin{aligned}\operatorname{rank} \mathbb{V}_0\big(V; W_1^{\otimes j}\big) = \; & \operatorname{rank} \mathbb{V}_0\big(V; W_1^{\otimes j-2}\big) \operatorname{rank} \mathbb{V}_0\big(V; W_1^{\otimes 2}\big) \\ & + \operatorname{rank} \mathbb{V}_0\big(V; W_1^{\otimes j-1}\big) \operatorname{rank} \mathbb{V}_0\big(V; W_1^{\otimes 3}\big) \\ & + \operatorname{rank} \mathbb{V}_0\big(V; W_1^{\otimes j-2} \otimes W_2\}\big) \operatorname{rank} \mathbb{V}_0\big(V; W_1^{\otimes 2} \otimes W_2\big)\,,\end{aligned}$$

which in view of Lemma 8.0.6 is equal to $\operatorname{rank} \mathbb{V}_0\big(V; W_1^{\otimes 2n}\big) = \operatorname{rank} \mathbb{V}_0\big(V; W_1^{\otimes j-2}\big)$. So the result holds by induction on j, as claimed.

We prove by induction on ℓ and j that $\operatorname{rank} \mathbb{V}_0\big(V; W_1^{\otimes j} \otimes W_2^{\otimes 2\ell+1}\big) = 0$, knowing the result for $j = 0, 1$ and $\ell = 0$ or $j = 0$ and $\ell = 1$. By (2.3), Lemma 8.0.6, and induction on ℓ, we have that

$$\begin{aligned}\operatorname{rank} \mathbb{V}_0\big(V; W_1^{\otimes j} \otimes W_2^{\otimes 2\ell+1}\big) = \; & \operatorname{rank} \mathbb{V}_0\big(V; W_1^{\otimes j} \otimes W_2^{\otimes 2\ell-1}\big) \operatorname{rank} \mathbb{V}_0\big(V; W_2^{\otimes 2}\big) \\ & + \operatorname{rank} \mathbb{V}_0\big(V; W_1^{\otimes j+1} \otimes W_2^{\otimes 2\ell-1}\big) \operatorname{rank} \mathbb{V}_0\big(V; W_1 \otimes W_2^{\otimes 2}\big) \\ = \; & \operatorname{rank} \mathbb{V}_0\big(V; W_1^{\otimes j} \otimes W_2^{\otimes 2\ell}\big) \operatorname{rank} \mathbb{V}_0\big(V; W_2^{\otimes 3}\big) \\ = \; & \operatorname{rank} \mathbb{V}_0\big(V; W_1^{\otimes j+1} \otimes W_2^{\otimes 2\ell-1}\big)\,.\end{aligned}$$

Using (2.3) again, we have that

$$\begin{aligned}\operatorname{rank} \mathbb{V}_0\big(V; W_1^{\otimes j+1} \otimes W_2^{\otimes 2\ell-1}\big) = \; & \operatorname{rank} \mathbb{V}_0\big(V; W_1^{\otimes j-1} \otimes W_2^{\otimes 2\ell-1}\big) \operatorname{rank} \mathbb{V}_0\big(V; W_1^{\otimes 2}\big) \\ & + \operatorname{rank} \mathbb{V}_0\big(V; W_1^{\otimes j} \otimes W_2^{\otimes 2\ell-1}\big) \operatorname{rank} \mathbb{V}_0\big(V; W_1^{\otimes 2}\big) \\ & + \operatorname{rank} \mathbb{V}_0\big(V; W_1^{\otimes j-1} \otimes W_2^{\otimes 2\ell}\big) \operatorname{rank} \mathbb{V}_0\big(V; W_1^{\otimes 2} \otimes W_2\big)\,,\end{aligned}$$

which is zero by induction on j and by Lemma 8.0.6. □

Example 8.0.5*, continued.* We use the results of [DGT22] summarized in § 2.9 together with the rank computations to prove the degree results stated in Example 8.0.5. Observe that the rank of the bundle is trivial except in three cases analyzed below:

Case 1: $j = k = 2$. Since the degree of ψ_P on $\overline{\mathcal{M}}_{0,4}$ is 1, as is $\delta_{\{P_1,P_2\}}$, and since the conformal dimension of V is zero, we have that $\deg\big(\mathbb{V}_0\big(V; W_1^{\otimes 2} \otimes W_2^{\otimes 2}\big)\big) = 2\frac{1}{2} + 2\frac{1}{16} - 2\frac{1}{16} = 1$.

Case 2: $k = 0$, and $j = 4$. We have that $\deg(\mathbb{V}_0(V; W_1^{\otimes 4})) = \frac{4}{2} - 0 = 2$.

Case 3: $k = 4$, $j = 0$. We have that $\deg\big(\mathbb{V}_0\big(V; W_2^{\otimes 4}\big)\big) = 2\frac{4}{16} - \frac{3}{2} = -1$.

9. Lattice divisor classes

In a special case, lattice VOAs coincide with affine Lie algebras at level 1. But generally, they are distinct. In § 9.0.2, we describe these VOAs and their modules, giving representatives of the examples of lattice VOA bundles we have considered. But first, in § 9.0.1, we give a brief summary of our findings about them.

9.0.1 *Summary.* We show here two series of lattice VOA bundles of rank 1, the first with trivial first Chern class (Example 9.1.1), and the second with negative first Chern class (Example 9.1.3). For the simplest example in each case, we show that the constant bundle $\mathcal{W}_0^\bullet$ also has rank 1, where the rank is computed using Zhu's character formula (Example 9.2.1). In the second case, the simplest example has the property that the modules satisfy the integral degree condition specified in Definition 2.3.1.

9.0.2 *Description of lattice VOAs and their modules.* Let $V = V_L$ be the lattice vertex operator algebra associated with the even lattice (L, q), where $L = \bigoplus_{i=1}^{d} \mathbb{Z}e_i$ is a rank d lattice and q is an even positive-definite form on $\mathbb{Z}$ such that $q(e_i, e_i) = 2 \cdot k_i$ for some $k_i \in \mathbb{Z}_{\geqslant 1}$ (see [Bor86, FLM88, Don93]). By [Don93, Theorem 3.1], the set of irreducible representations of V_L is in bijection with the cosets of L inside $L' = \{a \in L \otimes \mathbb{Q} \mid q(a, e) \in \mathbb{Z} \text{ for all } e \in L\}$. Note that L'/L further has the structure of an abelian group. Given an element $[\lambda] \in L'/L$, its conformal dimension is given by the rational number $\frac{1}{2} \min_{e \in L} q(\lambda + e, \lambda + e)$. Assume $L'/L \cong \mathbb{Z}/m\mathbb{Z}$, so that the simple representations of L are indexed by elements in $\{0, \ldots, m-1\}$. From [DGT22], we know that

$$\operatorname{rank} \mathbb{V}_g\big(V_L; W_1^{\otimes n_1} \otimes \cdots \otimes W_{m-1}^{\otimes n_{m-1}}\big) = m^g \delta_{\sum_{j=1}^{m-1} j n_j \equiv 0 \pmod{m}},$$

so that on $\overline{M}_{0,N}$, these sheaves of coinvariants are either trivial or line bundles.

9.1 Particular examples: Computing degrees

In what follows, L will be the lattice $L = \mathbb{Z}e$ with pairing $q(e, e) = 4k$ for some positive $k \in \mathbb{Z}$. It follows that L'/L is isomorphic to $\mathbb{Z}/4k\mathbb{Z} \cong \{0, \ldots, 4k-1\}$.

In what follows, the sheaf $\mathbb{V}_0(V_L; W_{i_1} \otimes \cdots \otimes W_{i_r})$, with $i_j \in \{0, \ldots, 4k-1\}$, will be denoted by $\mathbb{V}_0(i_1, \ldots, i_r)$.

Example 9.1.1. Consider on $\overline{\mathcal{M}}_{0,4}$ the space of coinvariants associated with the representations $(1, 1, 1, 4k-3)$. The degree of the line bundle $\mathbb{V}_0(1, 1, 1, 4k-3)$ is given by the degree of $c_1(\mathbb{V}_0(1, 1, 1, 4k-3))$, that is,

$$\left(\frac{1}{8k}\psi_1 + \frac{1}{8k}\psi_2 + \frac{1}{8k}\psi_3 + \frac{9}{8k}\psi_4\right) - \left(\frac{4}{8k}\delta_{[1,1][1,4k-3]} + \frac{4}{8k}\delta_{[1,1][1,4k-3]} + \frac{4}{8k}\delta_{[1,4k-3][1,1]}\right),$$

where boundary classes are indexed by partitions of the four points. This can be seen to have zero degree:

$$\deg(\mathbb{V}_0(1, 1, 1, 4k-3)) = \left(\frac{1}{8k} + \frac{1}{8k} + \frac{1}{8k} + \frac{9}{8k}\right) - \left(\frac{4}{8k} + \frac{4}{8k} + \frac{4}{8k}\right) = 0.$$

Remark 9.1.2. In the simplest case where $k = 1$, the line bundle $\mathbb{V}_0(1, 1, 1, 1)$ has degree zero, and we will see in § 9.2 how to use Zhu's character formula to prove that the constant bundle $\mathcal{W}_0^{\bullet}$ associated with $(1, 1, 1, 1)$ is also a line bundle.

Example 9.1.3. Consider on $\overline{\mathcal{M}}_{0,4}$ the sheaf of coinvariants associated with the representations (k, k, k, k). The conformal dimension of the representation represented by k equals $k/8$. Following § 2.9, the first Chern class of this line bundle is

$$c_1(\mathbb{V}_0(k, k, k, k)) = \left(\frac{k}{8}\psi_1 + \frac{k}{8}\psi_2 + \frac{k}{8}\psi_3 + \frac{k}{8}\psi_4\right) - \left(\frac{k}{2}\delta_{[2,2][2,2]} + \frac{k}{2}\delta_{[2,2][2,2]} + \frac{k}{2}\delta_{[2,2][2,2]}\right),$$

where the boundary classes are indexed by the partitions of the four points. It follows that

$$\deg(\mathbb{V}_0(k, k, k, k)) = \frac{k}{8} + \frac{k}{8} + \frac{k}{8} + \frac{k}{8} - \left(\frac{k}{2} + \frac{k}{2} + \frac{k}{2}\right) = -k,$$

and so this bundle is not globally generated.

Remark 9.1.4. We note that in this example, the conformal dimensions of each of the modules is $k/8$, so for $k=2$, the sum of these conformal dimensions is integral (see Definition 2.3.1), but the bundle still has negative degree. This example explains why we restrict Question 2 to vertex algebras that can be obtained from affine vertex algebras through tensor products, orbifold, and coset constructions.

We will see in § 9.2 how to use Zhu's character formula to prove that the constant bundle $\mathcal{W}_\bullet$ associated with $\mathbb{V}_0(2,2,2,2)$ is also a line bundle. We had not seen this behavior for the Virasoro bundles. The main distinction between them in this instance is that these lattice VOAs are not constructed from an affine VOA.

9.2 Computing ranks with Zhu's character formula

Here we illustrate how to use Zhu's character formula to compute the dimension for the lowest-weight spaces of modules over even lattice VOAs. Suppose that $V=V_L$ is a vertex operator algebra associated with an even lattice L. In particular, the lattice L is determined by its rank d and the quadratic form $Q=q(\ ,\ )/2$. Let $W=V_{L+\lambda}$ be a simple admissible module of conformal dimension a_λ. By Zhu's character formula [Zhu96, Introduction, p. 238] and [MT10], for V of central charge c,

$$\begin{aligned} q^{a_W-\frac{c}{24}}\sum_{n\geqslant 0}\dim W_{a_W+n}q^n &= \frac{1}{\eta(\tau)^d}\sum_{\alpha\in L}q^{Q(\alpha+\lambda)} \\ &= \left(q^{\frac{-1}{24}}\prod_{n=1}^{\infty}\left(\frac{1}{1-q^n}\right)\right)^d\sum_{j\in\mathbb{Q}_{\geqslant 0}}|L_j^\lambda|q^j = q^{\frac{-d}{24}}\left(\prod_{n=1}^{\infty}\left(\frac{1}{1-q^n}\right)\right)^d\sum_{j\in\mathbb{Q}_{\geqslant 0}}|L_j^\lambda|q^j\,, \end{aligned} \tag{9.1}$$

where

$$L_j^\lambda := \{\alpha\in L \mid Q(\alpha+\lambda)=j\}\,. \tag{9.2}$$

We note that

$$\prod_{n=1}^{\infty}\left(\frac{1}{1-q^n}\right) = \left(\sum_{n_1=0}^{\infty}q^{n_1}\right)\cdot\left(\sum_{n_2=0}^{\infty}q^{2n_2}\right)\cdot\left(\sum_{n_3=0}^{\infty}q^{3n_3}\right)\cdots = \sum_{n=0}^{\infty}P(n)q^n\,, \tag{9.3}$$

where $P(n)$ is the number of ways to write n as a sum of positive integers and $P(0)=1$. Since $V=V_L$ has central charge $c=d$, we obtain from (9.1) and (9.3) that

$$\sum_{n\geqslant 0}^{\infty}\dim W_{a_W+n}q^n = \sum_{n\in\mathbb{Q}_{\geqslant 0}}\left(\sum_{\substack{n_1,n_2,\ldots,n_d\in\mathbb{N}\\ j\in\mathbb{Q}_{\geqslant 0},\,\sum n_i+j=n}}|L_j^\lambda|\prod_i P(n_i)\right)q^{n-a_W}\,.$$

In summary, the coefficient of q^0 on the right-hand side is equal to the number of ways to write

$$a_W = n_1+n_2+\cdots+n_d+j \quad \text{with } n_i\in\mathbb{Z}_{\geqslant 0} \text{ and } j\in\mathbb{Q}_{\geqslant 0}\,,$$

and for each such way, the contribution is given by the product $\left|L_m^\lambda\right|\prod_i P(n_i)$. For instance, taking the trivial module $W=V_L$, represented by $\lambda=0$ with $h=0$, we have $\dim(W_0)=1$ since $\left|L_0^0\right|=1$, and $P(0)=1$.

Example 9.2.1. Consider the lattice VOA from Example 9.1, and take $k=2$ and $L=e\mathbb{Z}$, with pairing $q(e,e)=8$, so $Q(a)=a^2\cdot 4$ for every $a\in\mathbb{Q}$. Then V_L has central charge 1, and the module $W=W_{\frac{1}{4}}$ has conformal dimension $Q(\frac{1}{4})=\frac{1}{4}$. From the argument above, it follows that the dimension of W_0 is given by the following recipe:

$$\dim W_0 = \sum_{N=0}^{\infty} \left|L^{\frac{1}{4}}_{\frac{1}{4}-N}\right| P(N)\,. \tag{9.4}$$

In fact, there is only one way in which we can write $\frac{1}{4} = a_w = N + j$ with $N \in \mathbb{Z}_{\geqslant 0}$ and $j \in \mathbb{Q}_{\geqslant 0}$, that is, $N = 0$ and $j = \frac{1}{4}$. Otherwise said, in (9.4) we can see that $\frac{1}{4} - N$ is non-negative only for $N = 0$, which implies that the only non-zero contribution from $\left|L^{\frac{1}{4}}_{\frac{1}{4}-N}\right|$ is obtained when $N = 0$. It follows that $\dim W_0 = \left|L^{\frac{1}{4}}_{\frac{1}{4}}\right|$. In particular, one has that $\dim W_0 = 1$ since by the definition of L^j_λ, we have

$$L^{\frac{1}{4}}_{\frac{1}{4}} = \left\{\alpha \in \mathbb{Z} \mid Q\left(\alpha + \tfrac{1}{4}\right) = \tfrac{1}{4}\right\} = \{\alpha \in \mathbb{Z} \mid 2\alpha(2\alpha+1) = 0\} = \{0\}\,.$$

10. Questions

Given r simple Lie algebras $\mathfrak{g}_j$, positive integers ℓ_j, and for $j \in \{1, \dots, r\}$, an n-tuple of simple $L_{\ell_j}(\mathfrak{g}_j)$-modules $(W^1_j, \dots, W^n_j)$, we can ask the following.

Question 1. Are $\mathbb{V}_g(\bigotimes_{j=1}^r L_{\ell_j}(\mathfrak{g}_j); \bigotimes_{j=1}^r W_j^\bullet)$ and $\bigotimes_{j=1}^r \mathbb{V}_g(L_{\ell_j}(\mathfrak{g}_j); W^\bullet_j)$ isomorphic?

Remark 10.0.1. Much is known about the classes of bundles of coinvariants for simple affine VOAs $L_\ell(\mathfrak{g})$, which are C_2-cofinite and rational if and only if $\ell \in \mathbb{Z}_{>0}$. For instance, by [Ber93, Tha94, Fal94, KNR94, BL94, Pau96], in this case, there are canonical isomorphisms between generalized theta functions with (the dual spaces to) vector spaces of coinvariants at *smooth* curves. It has been shown that this extends to families of stable pointed curves with singularities [BF19]:

$$\mathbb{V}(L_\ell(\mathfrak{g}); W^\bullet)|^\vee_{(C,P_\bullet)} \cong \mathrm{H}^0\big(\mathcal{B}un_G^{\mathrm{Par}}(C, P_\bullet), \mathcal{L}^\ell\big)\,. \tag{10.1}$$

Here $\mathcal{L}$ is a canonical line bundle on the stack $\mathcal{B}un_G^{\mathrm{Par}}(C, P_\bullet)$ of parabolic G-bundles, and G is a simple, simply connected algebraic group with $\mathrm{Lie}(G) = \mathfrak{g}$. For $G = \mathrm{SL}(r)$ and $W^\bullet = V^\bullet$, there is a natural map SD:

$$\mathbb{V}(L_\ell(\mathfrak{sl}_r); W^\bullet)|_{(C)} \cong \mathrm{H}^0\big(\mathcal{M}_{\mathrm{SL}(r)}(C), \mathcal{L}^\ell\big)^\vee \xrightarrow[\mathrm{SD}]{\cong} \mathrm{H}^0\big(\mathcal{M}_{\mathrm{GL}(\ell)(C)}, \theta^r\big)\,,$$

where $\mathcal{M}_{\mathrm{SL}(r)}(C)$ is the moduli space of semi-stable vector bundles of rank r with trivial determinant on C, $\mathcal{M}_{\mathrm{GL}(\ell)}(C)$ is the moduli space of semi-stable vector bundles of rank ℓ and degree $\ell(g-1)$ on C, and where one has $\theta = \left\{\mathcal{E} \in \mathcal{M}_{\mathrm{GL}(\ell)} \colon \mathrm{H}^0(C, \mathcal{E}) \neq 0\right\}$. Donagi and Tu [DT94] showed that the dimensions of these vector spaces were the same and stated what became known as the strange duality conjecture. Various special cases had appeared earlier in the physics literature, for example in [NS90] (see also [NT92]). Pantev [Pan94] generalized the dimension statement to the case where R is reductive and $G = [R, R]$ is semi-simple. The conjecture was proved in type A by Belkale [Bel08] and Marian–Oprea [MO07]. Strange duality was proved by Abe in [Abe08] in the symplectic setting conjectured by Beauville [Bea06] (see also [Bel12]) and has been studied for other cases [Muk16a, Muk16b, BP10, MW19]. In [DGT22, Question 1], it was asked whether there are analogous geometric interpretations of dual spaces for vector spaces of conformal blocks defined by vertex operator algebras.

If the answer to Question 1 is yes, then by Theorem 1, [DT94], and [Pan94], an induced level-rank duality dimension statement will hold for vector spaces of conformal blocks given by any simple, rational, C_2-cofinite, self-contragredient, vertex operator algebra V of CFT-type, strongly generated in degree 1 since by [DM06], we have that $V \cong \otimes_j L_{\ell_j}(\mathfrak{g}_j)$. One also obtains a canonical identification between generalized theta functions with (the dual spaces to) vector spaces of coinvariants from (10.1) for these spaces. Moreover, if the $\mathfrak{g}_j$ are (combinations) in types A or C, then by [Bel08, MO07, Abe08], the vector spaces will be subject to strange dualities.

In § 7, examples of globally generated line bundles defined on moduli spaces of positive genus curves were given. For these, the constant sheaf also had rank 1. In § 8, we have given a representative example where if the rank of the constant bundle $\mathcal{W}_0^\bullet$ was at least as large as the rank of the coinvariants, then the vector bundle of coinvariants was positive whenever its modules satisfied an integral degree condition (see Definition 2.3.1).

Question 2. Let V be a VOA that can be obtained from affine vertex algebras through tensor product, orbifold, or coset construction. Suppose that both of the following properties hold:

(i) The rank of the constant bundle $\mathcal{W}_0^\bullet$ is at least as large as the rank of the coinvariants.

(ii) The conformal dimensions of the modules sum to an integer.

Is $\mathbb{V}_0(V; W^\bullet)$ globally generated on $\overline{\mathcal{M}}_{0,n}$?

Question 3. Is there another constant bundle that maps to the sheaf of coinvariants $\mathbb{V}_g(V; W^\bullet)$?

Remark 10.0.2. Tsuchiya, Ueno, and Yamada have observed that the map from the dth part of the filtration $\mathcal{F}_d(W^\bullet)$ to $\mathrm{gr}_d(\mathbb{V}_g(L_\ell(\mathfrak{g}); W^\bullet)|_{(C,P_\bullet)})$ is surjective at integrable levels, for $(C, P_\bullet)$ a smooth n-pointed curve [TUY89, Proposition 3.23]. Together with factorization, this is used to prove the coherence of $\mathbb{V}_g(L_\ell(\mathfrak{g}); W^\bullet)$ on $\overline{\mathcal{M}}_{g,n}$. Using the Weierstrass gap theorem, one can extend their observation to stable curves with singularities for $g > 0$ and $n \gg 0$. This defines a surjective map from the sheaf $\mathcal{W}_d^\bullet$ introduced in Lemma 3.2.1 to $\mathbb{V}_g(L_\ell(\mathfrak{g}); W^\bullet)$, analogous to the surjective map from $\mathcal{W}_0^\bullet$ to $\mathbb{V}_0(L_\ell(\mathfrak{g}); W^\bullet)$ shown in the proof of Theorem 1. Also in Lemma 3.2.1, the sheaf $\mathcal{W}_0^\bullet$ is shown to be independent of a change of coordinates, so descends to a constant sheaf on $\overline{\mathcal{M}}_{0,n}$. However, we know from examples of non-nef divisors $c_1(\mathbb{V}_g(L_\ell(\mathfrak{g}); W^\bullet))$ that for positive genus g, without further assumptions, $\mathcal{W}_d^\bullet$ is not independent of a change of coordinates and does not descend to a constant sheaf on $\overline{\mathcal{M}}_{g,n}$.

One could also try to base a constant bundle on the product $\bigotimes_i W^i/C_2(W^i)$, which maps surjectively onto coinvariants (the key step for proving finite generation [DGT19, Proposition 5.1.1]). However, again, without further assumptions, such a sheaf would not be independent of a change of coordinates.

Remark 10.0.3. We have been asked whether bundles of coinvariants from modules over general vertex operator algebras give new nef classes, apart from those given by bundles from affine Lie algebras. The ranks of the more general bundles are the same as the ranks of the bundles from affine Lie algebras, but in the formulas for the first Chern class given in [DGT22], which are valid if V is self-contragredient, rational, C_2-cofinite, and of CFT-type, the coefficients determined by the conformal dimensions of the modules can be very different from those for simple affine VOAs. One can therefore obtain new classes, although we have not done a careful study to see if the cones obtained with more general VOAs are larger than cones generated by the classical divisors.

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