Algebraic Geometry 9 (4) (2022) 502–512
doi:10.14231/AG-2022-015

Smoothing semi-smooth stable Godeaux surfaces

Barbara Fantechi, Marco Franciosi and Rita Pardini

Abstract

We show that all the semi-smooth stable complex Godeaux surfaces, classified in [M. Franciosi, R. Pardini and S. Rollenske, Ark. Mat. **56** (2018), no. 2, 299–317], are smoothable and that the moduli stack is smooth of the expected dimension 8 at the corresponding points.

1. Introduction

A *Godeaux surface* is (the canonical model of) a minimal complex surface of general type with $K^2 = 1$ and $h^1(\mathcal{O}) = h^2(\mathcal{O}) = 0$. A *stable Godeaux surface* is a stable surface with the same numerical invariants. It is *semi-smooth* if it has only double crossings and pinch points as singularities; see Section 3.1.

The algebraic fundamental group of a Godeaux surface is cyclic of order $m \leqslant 5$ (see [Miy75]). Almost fifty years have passed since Reid's seminal paper [Rei78] classifying Godeaux surfaces when $m \geqslant 3$, but a classification in the simply connected case is still lacking, in spite of much work on the subject (see Section 1.1 for a recap of known facts on Godeaux surfaces and their moduli). In particular, the question of irreducibility has not been decided yet.

An approach to investigate the dimension and singularities of the moduli is to construct non-canonical (that is, having worse than canonical singularities) stable surfaces X and show that they admit a smoothing. The first such construction can be found in [LP07, § 7], but many more examples are known nowadays (see, for instance, [SU16, § 5]). When the singularities are non-isolated, this is technically more difficult and sometimes no smoothing exists, even for hypersurface singularities and surfaces with $H^2(T_X) = 0$ (see [Rol16]).

In [FFP20], we obtained deformation-theoretical results that allow us to treat here the case of non-normal semi-smooth Godeaux surfaces; these are described explicitly in the classification of non-canonical stable Gorenstein surfaces in [FPR18b].

We verify for such surfaces the assumptions of Tziolas's formal smoothability criterion [Tzi10, Theorem 12.5].

Received 24 May 2021, accepted in final form 9 December 2021.
2020 Mathematics Subject Classification 14J10, 14D15, 14J29.
Keywords: stable surface, semi-smooth surface, Godeaux surface, deformation, smoothing.

This article is based upon work supported by the National Science Foundation under Grant No. 1440140, while the first and third author were in residence at the Mathematical Sciences Research Institute in Berkeley, California, during the spring semester of 2019. This collaboration started during the workshops "Derived Algebraic Geometry and Birational Geometry and Moduli Spaces" and "Connections for Women: Derived Algebraic Geometry", January 2019, MSRI Berkeley. This project was partially supported by the projects PRIN 2015EYPTSB_010 "Geometry of Algebraic Varieties" and PRIN 2017SSNZAW_004 "Moduli Theory and Birational Classification" of Italian MIUR. All authors are members of GNSAGA of INDAM.

Theorem 1.1. *Let X be a stable non-normal semi-smooth Godeaux surface. Then*

(A) *$\mathcal{T}^1_X$ is generated by global sections;*
(B) *$H^1(X, \mathcal{T}^1_X) = 0$;*
(C) *$H^2(X, T_X) = 0$.*

We prove Theorem 1.1 in Section 4. The proof combines the explicit classification of the relevant surfaces as push-outs of their normalizations from [FPR18b] (cf. Section 3) and the computation of T_X and $\mathcal{T}^1_X$ for a semi-smooth variety X, again in terms of its construction as a push-out, carried out in [FFP20]. We find the proof of part (C) (see Section 4.2) particularly interesting as it exploits the interplay between maps in cohomology and their geometrical interpretations.

For stable surfaces, formal smoothability is equivalent to geometric smoothability (see Section 2 for details); thus Theorem 1.1 has consequences on moduli.

Theorem 1.2. *Let X be a non-normal stable semi-smooth Godeaux surface.*

(i) *The moduli stack of stable surfaces is non-singular at $[X]$.*
(ii) *The general point of the unique irreducible component $\overline{M}_X$ containing $[X]$ corresponds to a non-singular surface.*
(iii) *The irreducible component $\overline{M}_X$ has (the expected) dimension 8.*

Remark 1.3. The fundamental groups of semi-smooth non-normal Godeaux surfaces range over all the groups $\mathbb{Z}_m$ for $m \leqslant 5$. By the semi-continuity of the fundamental group in families (cf. [FPR18b, Proposition 4.5]), it follows that for $m \geqslant 3$, the semi-smooth Godeaux surfaces with $\pi_1 = \mathbb{Z}_m$ can be smoothed to Godeaux surfaces with the same fundamental group. For $m = 1, 2$, it is possible that the general surface in the same component of the moduli space has larger fundamental group. However, by Theorem 1.2, each semi-smooth non-normal Godeaux surface lies in exactly one component, and this component has the expected dimension 8.

We expect that the techniques developed in this paper can be extended to other singular stable surfaces, in particular the surfaces mentioned in Remark 3.1.

1.1. Godeaux surfaces and their moduli. We give here some context on Godeaux surfaces, with the purpose of better framing our results and methods.

Godeaux surfaces have been an object of intense study over the last decades, but a complete classification and understanding of their moduli have not yet been achieved. As we recalled above, the algebraic fundamental group π_1^{alg} of a Godeaux surface is cyclic of order $m \leqslant 5$. It is a folklore conjecture that for each value of $m \leqslant 5$, the connected component of the moduli of Godeaux surfaces with $\pi_1^{\mathrm{alg}} = \mathbb{Z}_m$ is irreducible and rational of dimension 8. This conjecture is known to be true for $m \geqslant 3$ by [Rei78] (see also [CU18]) and for $m = 2$ by the recent preprint [DR20].

Many "sporadic" examples of Godeaux surfaces with trivial π_1^{alg} are known [Bar85, CG94, LP07, SU16], and an irreducible component of dimension 8 of the moduli space has very recently been constructed in [SS20] by homological algebra methods, but the geometry of the moduli space is still mysterious. For instance, it is known [CP00, Theorem 0.31] that the local moduli space of the surface in [CG94] is smooth of dimension 8 and that the same is true for the Barlow surface [CL97], but in general one has no clue which among the various examples belong to the same component of the moduli space.

The most classical approach to the construction (and eventually the classification) of Godeaux surfaces with $\pi_1^{\mathrm{alg}} = \mathbb{Z}_m$ goes back to [Rei78] and consists in writing down the canonical ring of the universal cover of the surface, keeping track of the $\mathbb{Z}_m$-action. Clearly this method is ineffective when $m = 1$, and different techniques have been used in order to produce examples with trivial π_1^{alg}. One method (cf. [LP07, § 7] and also [SU16]) consists in constructing a normal surface with rational singularities and showing that it admits a $\mathbb{Q}$-Gorenstein smoothing to a simply connected Godeaux surface. Namely, instead of constructing the Godeaux surface directly, one produces a surface in the boundary of the moduli space of stable surfaces with $K^2 = 1$ and $h^1(\mathcal{O}) = h^2(\mathcal{O}) = 0$ and then proves its smoothability by deformation-theoretical arguments. In this paper, we apply this approach to non-normal surfaces.

Our starting point is the systematic analysis of a part of the boundary of the moduli space of stable Godeaux surfaces carried out in [FPR18b], where all non-canonical Gorenstein stable Godeaux surfaces have been classified explicitly. The question of whether these surfaces actually belong to the closure of the moduli space of smooth Godeaux surfaces is partially answered in [FPR18b] and [FR18, Rol16], but the smoothability of some of the non-normal examples is still to be decided.

In Theorem 1.2, we answer this question in the affirmative for semi-smooth surfaces; furthermore, we show that the moduli stack is smooth of dimension 8 at the corresponding points, as predicted by the folklore conjecture mentioned above. In particular, the moduli stack is locally irreducible near these points; so while it is possible that the closures of connected components of the moduli space of canonical Godeaux with different fundamental groups meet at the boundary, this does not happen at the points corresponding to semi-smooth stable surfaces.

2. Smoothability conditions

The local analysis of the moduli stack of stable surfaces relies on the study of deformations of a stable surface. We recall here the smoothability criterion we will use.

Assumption 2.1. In this section, X is a proper, pure-dimensional complex variety with complete intersection singularities (cf. [FFP20, Definition 2.2]). In particular, X is reduced.

We are interested in the existence of a geometric smoothing.

Definition 2.2. A *geometric smoothing* of X is a Cartesian diagram

$$\begin{array}{ccc} X & \xrightarrow{\ \iota\ } & \mathcal{X} \\ \downarrow & & \downarrow{\scriptstyle \pi} \\ c & \longrightarrow & C\,, \end{array}$$

where C is a smooth irreducible curve, $c \in C$ is a closed point and π is a flat and proper morphism that is generically smooth. We say that X is *geometrically smoothable* if it has a geometric smoothing.

We denote by $T_X := \mathcal{H}om(\Omega_X, \mathcal{O}_X)$ the tangent sheaf of X and by $\mathcal{T}^1_X$ the sheaf $\mathcal{E}xt^1(\Omega_X, \mathcal{O}_X)$. The key theorem we use is the following result of Tziolas guaranteeing the existence of a *formal smoothing* [Tzi10, Definition 11.6].

Theorem 2.3 ([Tzi10, Theorem 12.5]). *If the conditions*

(A) $\mathcal{T}^1_X$ *is generated by global sections,*
(B) $H^1\big(X, \mathcal{T}^1_X\big) = 0$,
(C) $H^2\big(X, T_X\big) = 0$

hold, then X is formally smoothable; that is, it admits a formal smoothing.

Every geometric smoothing induces a formal smoothing, but the converse is in general not true. However, in our case the existence of the formal smoothing is sufficient, in view of the following result.

Theorem 2.4 ([Nob22]). *If one of the conditions*

(i) $H^2\big(X, \mathcal{O}_X\big) = 0$,
(ii) *either the dualizing sheaf ω_X or its dual $\omega_X^\vee$ is ample*

holds, then X is formally smoothable if and only if it is geometrically smoothable.

Remark 2.5. In general, the assumptions (B) and (C) of Theorem 2.3 imply that X has unobstructed deformations. By [Ill71, Proposition 2.1.2.3], an obstruction space for deformations is given by $\operatorname{Ext}^2\left(\mathbb{L}_X, \mathcal{O}_X\right)$. Since X has complete intersection singularities and is reduced, the cotangent complex $\mathbb{L}_X$ of X is equivalent to Ω_X in the derived category (cf. also [FFP20, § 2]). Thus $\operatorname{Ext}^2\left(\Omega_X, \mathcal{O}_X\right)$ is an obstruction space for X. Since $\mathcal{E}xt^2\big(\Omega_X, \mathcal{O}_X\big) = 0$ because X is reduced and has complete intersection singularities, the result follows by the local-to-global spectral sequence of Ext.

Recall that a proper surface X is *stable* if it has semi-log canonical singularities (see [KS88, § 4 and in particular Definition 4.17]) and K_X is ample as a $\mathbb{Q}$-Cartier divisor.

Remark 2.6. By Remark 2.5, if X is a stable surface that satisfies conditions (B) and (C) of Theorem 2.3, then the moduli stack of stable surfaces is smooth at $[X]$ of dimension equal to $\dim \operatorname{Ext}^1\left(\Omega_X, \mathcal{O}_X\right)$.

3. Semi-smooth stable Godeaux surfaces

3.1. Semi-smooth surfaces. We recall that a surface is *semi-smooth* if it is locally étale isomorphic to $\left\{u^2 - v^2 w = 0\right\}$ ("double crossings points"). The singular points corresponding to the origin are called *pinch points*, and the remaining singular points are *double crossings points* (see, for example, [KS88, Definition 4.1]).

Semi-smooth surfaces have a smooth normalization, and the preimage of the singular locus via the normalization map is a smooth curve. More precisely, by [FFP20, Proposition 3.11], any quasi-projective semi-smooth surface X can be obtained as follows. Let $\bar{X}$ be a smooth surface, $\bar{Y} \subset \bar{X}$ a smooth curve and $g\colon \bar{Y} \to Y$ a double cover with Y smooth. Then X fits in the following push-out diagram:

$$\begin{array}{ccc} \bar{Y} & \xrightarrow{\;g\;} & Y \\ {\scriptstyle \bar{\jmath}}\downarrow & & \downarrow{\scriptstyle j} \\ \bar{X} & \xrightarrow{\;f\;} & X\,. \end{array} \tag{3.1}$$

The maps $\bar{j}$ and j are closed embeddings, and f is finite and birational (and so f is the normalization map). The singular locus of X is Y, and the pinch points are the images of the branch points of g. One sometimes says that X is obtained from $\bar{X}$ by gluing/pinching along $\bar{Y}$ via g and writes $X := \bar{X} \sqcup_{\bar{Y}} Y$.

3.2. Semi-smooth Godeaux surfaces. We call *stable Godeaux surface* a stable surface with $K^2 = 1$ and $h^1(\mathcal{O}) = h^2(\mathcal{O}) = 0$. The stable non-canonical (that is, with worse than canonical singularities) Gorenstein Godeaux surfaces have been completely classified in [FPR18b]. In particular, the semi-smooth ones are of type (E_+); namely, their normalization is the symmetric product of an elliptic curve. Here we recall briefly the construction of these surfaces and set the notation.

Fix an elliptic curve E, let $P \in E$ be the origin, and let $\bar{X} = S^2E$ be the second symmetric product of E. The addition map $E \times E \to E$ induces the Albanese map $\pi\colon \bar{X} \to E$. The map π gives $\bar{X}$ the structure of a $\mathbb{P}^1$-bundle over E. In fact, we have $\bar{X} = \mathbb{P}_E(\mathcal{E})$, where $\mathcal{E}$ is the only non-trivial extension:

$$0 \to \mathcal{O}_E \to \mathcal{E} \to \mathcal{O}_E(P) \to 0\,. \tag{3.2}$$

We denote the numerical equivalence class of $\mathcal{O}_{\mathbb{P}_E(\mathcal{E})}(1)$ by h and that of a fiber of π by F. The images in $\bar{X}$ of the "coordinate curves" $\{Q\} \times E$ are smooth curves of genus 1 representing h. One has $h^2 = hF = 1$.

We let $\bar{Y} \subset \bar{X}$ be a smooth curve of class $3h - F$. Since $K_{\bar{X}}$ is numerically equivalent to $-2h + F$, we have $\bar{Y}^2 = 3$ and $K_{\bar{X}}\bar{Y} = -1$, so $\bar{Y}$ has genus 2. We assume in addition that $\bar{Y}$ admits an involution ι with quotient a smooth curve Y of genus 1, and we denote the quotient map by $g\colon \bar{Y} \to Y$. The existence and classification of such $(\bar{Y}, \iota)$ has been established in [FPR18b, FPR18a]. So we can define $X := \bar{X} \sqcup_{\bar{Y}} Y$ as the semi-smooth surface obtained by pinching $\bar{X}$ along $\bar{Y}$ via g. By the Hurwitz formula, the branch locus of g consists of two points, so X has two pinch points.

The line bundle $\omega_X = \mathcal{O}_X(K_X)$ is ample, $K_X^2 = 1$, and $h^i(\mathcal{O}_X) = 0$ for $i > 0$; namely, X is a stable Godeaux surface. By [FPR18b], all semi-smooth non-normal stable Godeaux surfaces arise in this way.

Remark 3.1. In fact, the construction of type (E_+) Godeaux surfaces in [FPR18b] also includes non–semi-smooth surfaces, for special choices of the curve E. Assume that the curve E admits an endomorphism of degree 2. In this case, $\bar{X}$ contains a curve $\bar{Y}$ of class $3h - F$ that decomposes as $Y_1 \cup Y_2$, where Y_1 has class h, Y_2 has class $2h - F$, and Y_1 and Y_2 meet transversely at one point R and are both isomorphic to E. One can also take ι to be an involution of $\bar{Y}$ that exchanges Y_1 and Y_2 leaving R fixed and set $X := \bar{X} \sqcup_{\bar{Y}} Y$ in this case. The surface X is again a Gorenstein stable Godeaux surface, but it has worse singularities since it has a degenerate cusp at the image point of R. In this case, our methods do not allow us to prove directly the smoothability of X. However, X can be obtained as a limit of non-normal semi-smooth Godeaux surfaces, and so it is smoothable too, but we do not know whether the moduli space is irreducible at $[X]$.

4. Proofs of Theorems 1.1 and 1.2

Notation 4.1. We keep the notation of Section 3.2. In addition, we denote the fixed points of ι on $\bar{Y}$ by $\bar{y}_i$, for $i = 1, 2$, and we set $y_i = g(\bar{y}_i) \in Y$. The action of ι induces a decomposition into eigenspaces $g_*\mathcal{O}_{\bar{Y}} = \mathcal{O}_Y \oplus L^{-1}$, where L is a line bundle. The multiplication map of $g_*\mathcal{O}_{\bar{Y}}$

induces an isomorphism $L^{\otimes 2} \cong \mathcal{O}_Y(B)$, where $B := y_1 + y_2$ is the branch locus of g.

4.1. The sheaf $\mathcal{T}^1_X$. The points y_1 and y_2 are the pinch points of X. The singular scheme X_{sing} of X is supported on Y, but it is non-reduced since it has embedded points at y_1 and y_2. Indeed, locally in the étale topology, X is defined by the equation $h(u,v,w) := u^2 - v^2 w = 0$, and X_{sing} is the scheme defined by the vanishing of h and its derivatives, namely by $u = vw = v^2 = 0$ (see [FFP20, § 2.1]). The sheaf $\mathcal{T}^1_X$ is a line bundle on X_{sing} whose restriction to Y we denote by $\mathcal{L}$. So we have a short exact sequence

$$0 \to \mathcal{K} \to \mathcal{T}^1_X \to \mathcal{L} \to 0\,, \tag{4.1}$$

where $\mathcal{K}$ is isomorphic to $\mathbb{C}_{y_1} \oplus \mathbb{C}_{y_2}$. We now show how in this case claims (A) and (B) of Theorem 1.1 follow easily from [FFP20, Theorem 5.5].

Lemma 4.2. (i) *We have $h^0(\mathcal{T}^1_X) = 7$, and $\mathcal{T}^1_X$ is generated by global sections.*
(ii) *We have $h^1(\mathcal{T}^1_X) = 0$.*

Proof. One has $H^i(\mathcal{K}) = 0$ for $i > 0$ because $\mathcal{K}$ has support of dimension zero, so taking global sections in (4.1) gives an exact sequence

$$0 \to H^0(\mathcal{K}) \to H^0(\mathcal{T}^1_X) \to H^0(\mathcal{L}) \to 0$$

and an isomorphism $H^1(\mathcal{T}^1_X) \cong H^1(\mathcal{L})$. It follows that $\mathcal{T}^1_X$ is generated by global sections if and only if $\mathcal{L}$ is. By [FFP20, Theorem 5.5], there is an isomorphism $g^*\mathcal{L} \cong g^*L^{\otimes 2} \otimes N_{\bar{Y}|\bar{X}} \otimes \iota^* N_{\bar{Y}|\bar{X}}$; since $\deg N_{\bar{Y}|\bar{X}} = 3$, we have $\deg g^*\mathcal{L} = 10$ and therefore $\deg \mathcal{L} = 5$. Since Y is an elliptic curve, we have $h^1(\mathcal{L}) = 0$, the line bundle $\mathcal{L}$ is generated by global sections and $h^0(\mathcal{L}) = 5$. Hence $\mathcal{T}^1_X$ is generated by global section, and $h^0(\mathcal{T}^1_X) = 5 + 2 = 7$, proving assertions (i) and (ii). □

4.2. The sheaf T_X. The proof of claim (C) of Theorem 1.1 also relies on the results of [FFP20] but is far more involved than the proofs of claims (A) and (B).

We start with some standard computations.

Lemma 4.3. (i) *We have $h^0(\bar{X}, T_{\bar{X}}) = h^1(\bar{X}, T_{\bar{X}}) = 1$ and $h^2(\bar{X}, T_{\bar{X}}) = 0$,*
(ii) *We have $h^0(T_{\bar{X}}|_{\bar{Y}}) = 1$ and $h^1(T_{\bar{X}}|_{\bar{Y}}) = 2$.*

Proof. (i) Taking the dual of the relative differentials sequence for the Albanese morphism $\pi\colon \bar{X} \to E$

$$0 \to \pi^*\omega_E = \mathcal{O}_{\bar{X}} \to \Omega^1_{\bar{X}} \to \omega_{\bar{X}|E} = \omega_{\bar{X}} \to 0\,,$$

one gets

$$0 \to \mathcal{O}_{\bar{X}}(-K_{\bar{X}}) \to T_{\bar{X}} \to \mathcal{O}_{\bar{X}} \to 0\,. \tag{4.2}$$

We have $h^i(-K_{\bar{X}}) = 0$ for all i (see [CC93, § 2, Equation (5)]); hence the long cohomology sequence associated with (4.2) gives isomorphisms $H^i(\bar{X}, T_{\bar{X}}) \cong H^i(\bar{X}, \mathcal{O}_{\bar{X}})$ for every i. The claim follows.

(ii) Twisting (4.2) by $\mathcal{O}_{\bar{X}}(-\bar{Y})$, we get

$$0 \to \mathcal{O}_{\bar{X}}(-C) \to T_{\bar{X}}(-\bar{Y}) \to \mathcal{O}_{\bar{X}}(-\bar{Y}) \to 0\,, \tag{4.3}$$

where C is a divisor in the numerical class h (recall that $K_{\bar{X}}$ is numerically equivalent to $-2h+F$). Since both C and $\bar{Y}$ are ample by [Har77, Proposition V.2.21], by the Kodaira vanishing theorem, the long exact sequence associated with (4.3) gives $H^0\big(T_{\bar X}(-\bar Y)\big) = H^1\big(T_{\bar X}(-\bar Y)\big) = 0$ and a short exact sequence

$$0 \to H^2\big(\bar X, \mathcal{O}_{\bar X}(-C)\big) \to H^2\big(\bar X, T_{\bar X}(-\bar Y)\big) \to H^2\big(\bar X, \mathcal{O}_{\bar X}(-\bar Y)\big) \to 0\,.$$

By the Riemann–Roch theorem and Kodaira vanishing theorem, we have $h^2\big(\bar X, \mathcal{O}_{\bar X}(-C)\big) = \chi\big(\mathcal{O}_{\bar X}(-C)\big) = 0$ and $h^2\big(\bar X, \mathcal{O}_{\bar X}(-\bar Y)\big) = \chi\big(\mathcal{O}_{\bar X}(-\bar Y)\big) = 1$, and so $h^2\big(\bar X, T_{\bar X}(-\bar Y)\big) = 1$. Now consider the sequence

$$0 \to T_{\bar X}(-\bar Y) \to T_{\bar X} \to T_{\bar X}|_{\bar Y} \to 0\,. \tag{4.4}$$

By the previous computations, taking cohomology, one gets an isomorphism $H^0\big(\bar X, T_{\bar X}\big) \cong H^0\big(\bar X, T_{\bar X}|_{\bar Y}\big)$ and an exact sequence

$$0 \to H^1\big(\bar X, T_{\bar X}\big) \to H^1\big(\bar X, T_{\bar X}|_{\bar Y}\big) \to H^2\big(\bar X, T_{\bar X}(-\bar Y)\big) \to 0\,.$$

Therefore, we have $h^0\big(\bar X, T_{\bar X}|_{\bar Y}\big) = 1$ and $h^1\big(\bar X, T_{\bar X}|_{\bar Y}\big) = h^1\big(\bar X, T_{\bar X}\big) + h^2\big(\bar X, T_{\bar X}(-\bar Y)\big) = 1+1 = 2$ by assertion (i). □

The next step is an analysis of $H^0\big(N_{\bar Y|\bar X}\big)$. Since $\bar Y^2 = 3$ and $\bar Y$ has genus 2, by the Riemann–Roch theorem, this is a 2-dimensional vector space. Consider the cohomology sequences

$$0 \to \mathcal{O}_{\bar X} \to \mathcal{O}_{\bar X}\big(\bar Y\big) \to N_{\bar Y|\bar X} \to 0 \tag{4.5}$$

and

$$0 \to T_{\bar Y} \to T_{\bar X}|_{\bar Y} \to N_{\bar Y|\bar X} \to 0\,, \tag{4.6}$$

and let $\gamma\colon H^0\big(N_{\bar Y|\bar X}\big) \to H^1\big(\mathcal{O}_{\bar X}\big)$ and $\delta\colon H^0\big(N_{\bar Y|\bar X}\big) \to H^1\big(T_{\bar Y}\big)$ be the coboundary maps induced by (4.5) and (4.6), respectively.

Lemma 4.4. (i) *The kernels* $\ker\gamma$ *and* $\ker\delta$ *have dimension* 1.
(ii) *We have* $H^0\big(N_{\bar Y|\bar X}\big) = \ker\gamma \oplus \ker\delta$.

Proof. By [Mum66, Lecture 15], there is a scheme $\mathcal{P}$ parametrizing the curves of $\bar X$ algebraically equivalent to $\bar Y$, and $H^0\big(N_{\bar Y|\bar X}\big)$ is canonically isomorphic to the tangent space to $\mathcal{P}$ at the point $\big[\bar Y\big]$. Denote by $\mathrm{Pic}^{[\bar Y]}\big(\bar X\big)$ the connected component of $\mathrm{Pic}\big(\bar X\big)$ containing the class of $\mathcal{O}_{\bar X}\big(\bar Y\big)$, and let $c\colon \mathcal{P} \to \mathrm{Pic}^{[\bar Y]}\big(\bar X\big)$ be the characteristic map, which sends $[C] \in \mathcal{P}$ to the class of $\mathcal{O}_{\bar X}(C)$. Since the numerical class of $\bar Y$ is $3h - F = h + K_{\bar X}$ and h is ample, by the Riemann–Roch theorem and Kodaira vanishing theorem, we have $h^0\big(\mathcal{O}_{\bar X}(C)\big) = 2$ for any curve algebraically equivalent to $\bar Y$, so the map c gives $\mathcal{P}$ the structure of a $\mathbb{P}^1$-bundle over the genus 1 curve $\mathrm{Pic}^{[\bar Y]}\big(\bar X\big)$. The differential of c at $\big[\bar Y\big]$ is γ, and the long cohomology sequence coming from (4.5) shows that $\ker\gamma$ is the image of $H^0\big(\mathcal{O}_{\bar X}(\bar Y)\big) \to H^0\big(N_{\bar Y|\bar X}\big)$ and has dimension 1.

The diagonal action of E on $E \times E$ by translation descends to an action on $\bar X = S^2E$. We denote the automorphism of $\bar X$ induced by translation by a point $P \in E$ by g_P; it acts on the curves in the numerical class of h as twisting by P, where we regard P as an element of $E = \mathrm{Pic}^0\big(\bar X\big)$. Since $\bar Y$ is numerically equivalent to $h + K_{\bar X}$, we have that $g_P^*\bar Y$ is in the linear system $\big|\bar Y - P\big|$. So $S := \big\{\big[g_P^*\bar Y\big] \mid P \in E\big\} \subset \mathcal{P}$ is a section of the $\mathbb{P}^1$-bundle $\mathcal{P}$, and its tangent space W at $\big[\bar Y\big]$ is a 1-dimensional subspace of $H^0\big(N_{\bar Y|\bar X}\big)$ that is mapped isomorphically to $H^1\big(\mathcal{O}_{\bar X}\big)$. We claim that $W = \ker\delta$.

Indeed, the elements of $H^0\big(N_{\bar Y|\bar X}\big)$ are the first-order deformations of $\bar Y \subset \bar X$, and they are mapped by δ to the corresponding deformation of $\bar Y$. Since S gives a trivial deformation of $\bar Y$, it is clear that W is contained in $\ker\delta$. To finish the proof, it is enough to observe that $\ker\delta$ has dimension 1. This follows from the long exact sequence associated with (4.6) since $h^1(N_{\bar{Y}|\bar{X}}) = 0$ because $\bar{Y}^2 = 3$ and $\bar{Y}$ has genus 2, $h^1(T_{\bar{Y}}) = 3$ and, by Lemma 4.3, we have $h^1(T_{\bar{X}|\bar{Y}}) = 2$. □

The next step is an analysis of $H^0(K_{\bar{Y}})$. We start with a couple of general remarks.

Remark 4.5. Let $h\colon C_1 \to C_2$ be a finite morphism of curves of positive genus, and write $J_i := \mathrm{Jac}(C_i)$ for $i = 1, 2$. Choosing base points $x_1 \in C_1$ and $x_2 := h(x_1) \in C_2$, we have a commutative diagram

$$\begin{array}{ccc} C_1 & \xrightarrow{a_1} & J_1 \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle h_*} \\ C_2 & \xrightarrow[a_2]{} & J_2\,, \end{array}$$

where a_1 and a_2 are the Abel–Jacobi maps with base points x_1 and x_2, respectively, and h_* is the morphism of abelian varieties induced by h. The differential of h_* at the origin is the transpose of the pull-back map $h^*\colon H^0(K_{C_2}) \to H^0(K_{C_1})$, so the tangent space to $\ker h_*$ at the origin is $\left(h^*\left(H^0(K_{C_2})\right)\right)^{\perp}$.

Remark 4.6. In the situation of Remark 4.5, assume in addition that h does not factor through a non-trivial étale cover of C_2. This happens, for instance, if $\deg h$ is a prime and h is not étale. Then the morphism $h^*\colon J_2 = \mathrm{Pic}^0(C_2) \to J_1 = \mathrm{Pic}^0(C_1)$ is injective, and therefore the kernel A of the dual morphism $h_*\colon J_1 \to J_2$ is connected. In particular, if C_2 has genus 1, then A is a connected divisor with $A \cdot a_1(C_1) = \deg h$.

We now apply the previous remarks in our situation. Under the action of the involution ι induced by the double cover $g\colon \bar{Y} \to Y$, the vector space $V := H^0(K_{\bar{Y}})$ splits as the direct sum $V = V^+ \oplus V^-$ of 1-dimensional eigenspaces, with $V^+ = g^* H^0(K_Y)$. Denote by σ the hyperelliptic involution of $\bar{Y}$; then σ and ι generate a group isomorphic to $\mathbb{Z}_2^2$, and σ acts on V as multiplication by -1, so V^- is invariant under the action of $\iota' := \iota \circ \sigma$. So if $g'\colon \bar{Y} \to Y' := \bar{Y}/\iota'$ denotes the quotient map, the curve Y' has genus 1 and $V^- = g'^* H^0(K_{Y'})$.

Set $Z := V^\vee$; since ι acts trivially on $H^1(K_{\bar{Y}})$, we have $Z^- = (V^+)^{\perp}$ and $Z^+ = (V^-)^{\perp}$ by Serre duality. The space Z also contains the 1-dimensional subspace $W := \left(p^* H^0(K_E)\right)^{\perp}$, where $p\colon \bar{Y} \to E$ is the degree 3 morphism induced by the Albanese map of $\bar{X}$.

We have the following.

Lemma 4.7. *One has* $Z^+ \cap W = Z^- \cap W = 0$.

Proof. Set $J := \mathrm{Jac}(\bar{Y})$ and denote the image of $\bar{Y}$ via the Abel–Jacobi map by $\Theta \subset J$. By definition, Z is the tangent space to J at the origin. By Remark 4.5, the subspace Z^- is the tangent space at the origin to the kernel D of $g_*\colon J \to \mathrm{Jac}(Y)$, the subspace Z^+ is the tangent space at the origin to the kernel D' of $g'_*\colon J \to \mathrm{Jac}(Y')$, and W is the tangent space to the kernel E' of $p_*\colon J \to \mathrm{Jac}(E)$. By Remark 4.6, the curves D, D' and E' are connected and satisfy $\Theta \cdot D = \Theta \cdot D' = 2$ and $\Theta \cdot E' = 3$. Summing up, the three abelian subvarieties D, D' and E' are distinct; since an abelian subvariety is determined by its tangent space at the origin, Z^+, Z^- and W are pairwise distinct. □

The next result is the key ingredient of the proof of fact (C).

Lemma 4.8. *We have* $\mathrm{Im}\,\delta \cap H^1(T_{\bar{Y}})^+ = 0$.

Proof. The space $H^1(T_{\bar Y})$ is Serre dual to $H^0(2K_{\bar Y})$. Since $\bar Y$ has genus 2, the multiplication map $\mu\colon H^0(K_{\bar Y})\otimes H^0(K_{\bar Y})\to H^0(2K_{\bar Y})$ induces an isomorphism $\rho\colon S^2H^0(K_{\bar Y})\to H^0(2K_{\bar Y})$. Via these identifications, we have $H^1(T_{\bar Y})^+=(Z^+\otimes Z^+)\oplus(Z^-\otimes Z^-)$. Also, there is an isomorphism $H^1(T_J)\cong Z\otimes Z$, and the dual map ${}^t\mu\colon H^1(T_{\bar Y})\to Z\otimes Z$ is the differential at $[\bar Y]$ of the Torelli map, sending a curve of genus 2 to its Jacobian.

To simplify the notation in what follows, we set $\psi := p_*\colon J\to \mathrm{Jac}(E)\cong E$. The differentials sequence $0\to T_{J/E}\to T_J\to\psi^*T_E\to 0$ can be rewritten more explicitly as

$$0\to W\otimes\mathcal{O}_J\to Z\otimes\mathcal{O}_J\to W'\otimes\mathcal{O}_J\to 0\,, \tag{4.7}$$

where $W'=H^0(K_E)^\vee$ is the tangent space to E at the origin. Following the notation of [Ser06, § 3.4.2], we denote the deformations with fixed target of the map $\psi\colon J\to E$ by $\mathrm{Def}_{\psi/E}$. By [Ser06, Theorem 3.4.8 and Lemma 3.4.7(iv)], the tangent space to $\mathrm{Def}_{\psi/E}$ is $H^1(T_{J/E})=W\otimes Z$; moreover, the map $H^1(T_{J/E})\to H^1(T_J)$ is clearly an inclusion.

By Lemma 4.4, the image of δ is $\delta(\ker\gamma)$; namely, it is generated by the first-order deformation ξ of $\bar Y$ obtained by letting $\bar Y$ vary in the linear pencil $|\bar Y|$ of $\bar X$. The element ${}^t\mu(\xi)$ is the corresponding first-order deformation of J and, since ξ induces a first-order deformation of ψ with fixed target, by the above discussion it lies in $H^1(T_{J/E})=W\otimes Z$. Using Lemma 4.7, it is an easy linear algebra exercise to show that the subspaces $H^1(T_{\bar Y})^+=(Z^+\otimes Z^+)\oplus(Z^-\otimes Z^-)$ and $H^1(T_{J/E})=W\otimes Z$ of $H^1(T_J)=Z\otimes Z$ intersect only in 0. □

4.3. Conclusion. We are finally ready complete the proofs.

Proof of Theorem 1.1. Claims (A) and (B) are proven in Lemmas 4.3 and 4.2, so we only have to prove claim (C). We first recall some facts from [FFP20]. By [FFP20, Theorem 5.1], there is a natural injective map $\alpha\colon T_X\to f_*T_{\bar X}$ which is an isomorphism on the smooth locus of X. Let $\mathcal{G}$ be the sheaf defined by the short exact sequence

$$0\to T_X\xrightarrow{\alpha} f_*T_{\bar X}\to\mathcal{G}\to 0\,. \tag{4.8}$$

The map α is an isomorphism on the smooth locus of X, so the sheaf $\mathcal{G}$ is supported on Y. By the same theorem, there is an exact sequence

$$0\to(g_*T_{\bar Y})^+\to g_*T_{\bar X}|_{\bar Y}\to\mathcal{G}\to 0\,. \tag{4.9}$$

Since $H^2(f_*T_{\bar X})=H^2(\bar X,T_{\bar X})=0$ by Lemma 4.3, by (4.8) it is enough to show that $h^1(\mathcal{G})=0$ or, equivalently, that the map $j\colon H^1((g_*T_{\bar Y})^+)\to H^1(g_*T_{\bar X}|_{\bar Y})$ is surjective. Since g is a finite map, we can make identifications $H^1((g_*T_{\bar Y})^+)\cong H^1(T_{\bar Y})^+$ and $H^1(g_*T_{\bar X}|_{\bar Y})\cong H^1(T_{\bar X}|_{\bar Y})$ and work on $\bar Y$.

Taking cohomology in (4.6), we get

$$0\to H^0(T_{\bar X}|_{\bar Y})\to H^0(N_{\bar Y|\bar X})\xrightarrow{\delta} H^1(T_{\bar Y})\xrightarrow{j_0} H^1(T_{\bar X}|_{\bar Y})\to 0\,. \tag{4.10}$$

So the map j is just the restriction to $H^1(T_{\bar Y})^+$ of the map j_0 in (4.10); since both $H^1(T_{\bar Y})^+$ and $H^1(T_{\bar X}|_{\bar Y})$ have dimension 2, the map j is surjective if and only if it is an isomorphism if and only if the kernel of j_0, which is the image of δ, intersects $H^1(T_{\bar Y})^+$ only in zero. This last statement is precisely the content of Lemma 4.8, so fact (C) is proven. □

Proof of Theorem 1.2. By Theorem 1.1, the assumptions of Theorem 2.3 are satisfied, so X is formally smoothable. Since ω_X is ample, or since $H^2(\mathcal{O}_X)=0$, Theorem 2.4 applies and, therefore, X is geometrically smoothable and claim (ii) is proven. Furthermore, by Remark 2.6, the stack $\overline{M}_X$ is smooth at $[X]$ of dimension equal to $\dim \operatorname{Ext}^1(\Omega_X, \mathcal{O}_X)$.

To complete the proof, we need to show that $\operatorname{Ext}^1(\Omega_X, \mathcal{O}_X)$ has dimension 8. Since $H^2(T_X) = 0$, by the local-to-global exact sequence for Ext, we have

$$0 \to H^1(T_X) \to \operatorname{Ext}^1(\Omega_X, \mathcal{O}_X) \to H^0(\mathcal{T}_X^1) \to 0\,;$$

hence $\dim \operatorname{Ext}^1(\Omega_X, \mathcal{O}_X) = h^1(T_X) + h^0(\mathcal{T}_X^1) = h^1(T_X) + 7$, where the last equality follows by Lemma 4.2. So we have to prove $h^1(T_X) = 1$. Again by the vanishing of $H^2(T_X)$, we have $h^1(T_X) = h^0(T_X) - \chi(T_X) = -\chi(T_X)$ since $H^0(T_X)$ is the tangent space at the origin of $\operatorname{Aut}(X)$ and $\operatorname{Aut}(X)$ is finite because X is stable (cf. [BHPS13, Lemma 2.5]). Finally, sequences (4.9) and (4.8) give

$$\begin{aligned}\chi(T_X) &= \chi(T_{\bar{X}}) - \chi(\mathcal{G}) \\ &= \chi(T_{\bar{X}}) - \chi(T_{\bar{X}}|_{\bar{Y}}) + \chi((g_* T_{\bar{Y}})^+) = 0 - (-1) - 2 = -1\,,\end{aligned}$$

where the last equality follows by Lemma 4.3 and by observing that

$$\chi((g_* T_{\bar{Y}})^+) = -\dim H^1(T_{\bar{Y}})^+ = -2$$

(see the proof of Lemma 4.8). □

Acknowledgements

We wish to thank Alessandro Nobile, Gian Pietro Pirola and Nikolaos Tziolas for useful mathematical discussions.

References

Bar85 R. Barlow, *A simply connected surface of general type with $p_g = 0$*, Invent. Math. **79** (1985), no. 2, 293–301; doi:10.1007/BF01388974.

BHPS13 B. Bhatt, W. Ho, Z. Patakfalvi and C. Schnell, *Moduli of products of stable varieties*, Compos. Math. **149** (2013), no. 12, 2036–2070; doi:10.1112/S0010437X13007288.

CC93 F. Catanese and C. Ciliberto, *Symmetric products of elliptic curves and surfaces of general type with $p_g = q = 1$*, J. Algebraic Geom. **2** (1993), no. 3, 389–411.

CG94 P. C. Craighero and R. Gattazzo, *Quintic surfaces of $\mathbf{P}^3$ having a nonsingular model with $q = p_g = 0$, $P_2 \neq 0$*, Rend. Sem. Mat. Univ. Padova **91** (1994), 185–198.

CL97 F. Catanese and C. LeBrun, *On the scalar curvature of Einstein manifolds*, Math. Res. Lett., **4** (1997), no. 6, 843–854; doi:10.4310/MRL.1997.v4.n6.a5.

CP00 F. Catanese and R. Pignatelli, *On simply connected Godeaux surfaces*, in *Complex Analysis and Algebraic Geometry* (de Gruyter, Berlin, 2000), 117–153; doi:10.1515/9783110806090-007.

CU18 S. Coughlan and G. Urzúa, *On $\mathbb{Z}/3$-Godeaux surfaces*, Int. Math. Res. Not. **2018** (2018), no. 18, 5609–5637; doi:10.1093/imrn/rnx049.

DR20 E. Dias and C. Rito, *$\mathbb{Z}/2$-Godeaux surfaces*, 2020, arXiv:2009.12645.

FFP20 B. Fantechi, M. Franciosi and R. Pardini, *Deformations and smoothability for semi-smooth varieties*, 2020, arXiv:2010.2296.

FPR18a M. Franciosi, R. Pardini and S. Rollenske, *(d, d')-elliptic curves of genus two*, Ark. Mat. **56** (2018), no. 2, 299–317; doi:10.4310/ARKIV.2018.v56.n2.a6.

FPR18b ———, *Gorenstein stable Godeaux surfaces*, Selecta Math. (N.S.) **24** (2018), no. 4, 3349–3379; doi:10.1007/s00029-017-0342-6.

FR18 M. Franciosi and S. Rollenske, *Canonical rings of Gorenstein stable Godeaux surfaces*, Boll. Unione Mat. Ital. **11** (2018), no. 1, 75–91; doi:10.1007/s40574-016-0114-9.

Har77 R. Hartshorne, *Algebraic geometry*, Grad. Texts in Math., vol. 52 (Springer-Verlag, New York – Heidelberg, 1977); doi:10.1007/978-1-4757-3849-0.

Ill71 L. Illusie, *Complexe cotangent et déformations. I*, Lecture Notes in Math., vol. 239 (Springer-Verlag, Berlin – New York, 1971); doi:10.1007/BFb0059052.

KS88 J. Kollár and N. I. Shepherd-Barron, *Threefolds and deformations of surface singularities*, Invent. Math. **91** (1988), no. 2, 299–338; doi:10.1007/BF01389370.

LP07 Y. Lee and J. Park, *A simply connected surface of general type with $p_g = 0$ and $K^2 = 2$*, Invent. Math. **170** (2007), no. 3, 483–505; doi:10.1007/s00222-007-0069-7.

Miy75 Y. Miyaoka, *Tricanonical map of a certain class of surfaces*, Sūrikaisekikenkyūsho Kōkyūroku **240** (1975), 31–47.

Mum66 D. Mumford, *Lectures on curves on an algebraic surface*, Annals of Math. Studies, vol. 59 (Princeton Univ. Press, Princeton, NJ, 1966).

Nob22 A. Nobile, *On formal schemes and smoothings*, Ph.D. Thesis, SISSA, 2022; available at https://iris.sissa.it/handle/20.500.11767/127709.

Rei78 M. Reid, *Surfaces with $p_g = 0$, $K^2 = 1$*, J. Fac. Sci. Univ. Tokyo Sect. IA Math. **25** (1978), no. 1, 75–92.

Rol16 S. Rollenske, *A new irreducible component of the moduli space of stable Godeaux surfaces*, Manuscripta Math. **149** (2016), no. 1–2, 117–130; doi:10.1007/s00229-015-0776-0.

Ser06 E. Sernesi, *Deformations of algebraic schemes*, Grundlehren math. Wiss., vol. 334 (Springer-Verlag, Berlin, 2006); doi:10.1007/978-3-540-30615-3.

SS20 F.-O. Schreyer and I. Stenger, *Godeaux surfaces I*, 2020, arXiv:2009.05357.

SU16 A. Stern and G. Urzúa, *KSBA surfaces with elliptic quotient singularities, $\pi_1 = 1$, $p_g = 0$, and $K^2 = 1, 2$*, Israel J. Math. **214** (2016), no. 2, 651–673; doi:10.1007/s11856-016-1350-5.

Tzi10 N. Tziolas, *Smoothings of schemes with nonisolated singularities*, Michigan Math. J. **59** (2010), no. 1, 25–84; doi:10.1307/mmj/1272376026.

Barbara Fantechi fantechi@sissa.it
SISSA, Via Bonomea 265, I-34136 Trieste, Italy

Marco Franciosi marco.franciosi@unipi.it
Dipartimento di Matematica, Università di Pisa, Largo B. Pontecorvo 5, I-56127 Pisa, Italy

Rita Pardini rita.pardini@unipi.it
Dipartimento di Matematica, Università di Pisa, Largo B. Pontecorvo 5, I-56127 Pisa, Italy