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Moduli of elliptic $K3$ surfaces: Monodromy and Shimada root lattice strata

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With an appendix by Markus Kirschmer

Abstract

In this paper, we investigate two stratifications of the moduli space of elliptically fibred $K3$ surfaces. The first comes from Shimada's classification of connected components of the moduli of elliptically fibred $K3$ surfaces and is closely related to the root lattices of the fibration. The second is the monodromy stratification defined by Bogomolov, Petrov and Tschinkel. The main result of the paper is a classification of all positive-dimensional ambi-typical strata, that is, strata which are both Shimada root strata and monodromy strata. We also discuss the relationship with moduli spaces of lattice-polarised $K3$ surfaces. The appendix by M. Kirschmer contains computational results about the 1-dimensional ambi-typical strata.

1. Introduction

Elliptically fibred $K3$ surfaces have been studied over a long period from many different angles. We refer the reader in particular to the book [SS19], where $K3$ surfaces are especially treated in Sections 11 and 12. Due to the work of Miranda [Mir90], it is well known that the moduli space $\mathcal{F}$ of elliptically fibred $K3$ surfaces with a section, also known as Jacobian fibrations, can be described as a geometric invariant theory (GIT) quotient of an open subset V in the weighted projective space $\mathbb{P}_{8,12}(9,13)$ by the group $\mathrm{SL}(2,\mathbb{C})$. Alternatively, the moduli space $\mathcal{F}$ can also be constructed as the moduli space of lattice-polarised $K3$ surfaces, more precisely U-quasi-polarised $K3$ surfaces (where U is the hyperbolic plane spanned by the classes of the section and a general fibre).

The moduli space $\mathcal{F}$ itself is a rational variety by a result of Lejarraga [Lej 93]. Geometric properties of elliptically fibred $K3$ surfaces provide this space with many interesting geometric features.

The starting point of our work is two stratifications of the moduli space $\mathcal{F}$. The first of these was introduced by Bogomolov, Petrov and Tschinkel in [BPT02]. The strata of this decomposition are defined by the property that they are maximal locally closed irreducible subvarieties with

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constant monodromy group Γ, where Γ is a fixed subgroup of $\mathrm{SL}(2,\mathbb{Z})$ modulo conjugation. Bogomolov, Petrov and Tschinkel prove the remarkable fact that all of these *monodromy strata* are themselves rational varieties.

The other stratification is due to the work of Shimada [Shi00, Shi16a, Shi18] in which he classifies all connected components of the moduli of elliptic $K3$ surfaces with fixed combinatorial type. This means the following: Given an elliptically fibred $K3$ surface $f\colon S \to \mathbb{P}^1$, the components of the singular fibres not meeting the 0-section define a root lattice R which is the direct sum of some ADE-lattices. This need not be saturated in the Néron–Severi group $\mathrm{NS}(S)$. Its saturation L corresponds, by lattice theory, to an isotropic subgroup G of the discriminant group of R. Shimada's work provides a complete classification of all connected families of elliptically fibred $K3$ surfaces with given (R, G). This leads to a total of 3932 families, and in this way, one obtains a second stratification of the moduli space $\mathcal{F}$. We refer to the strata in this stratification as *Shimada root strata* or, shorter, simply as *Shimada strata*. A priori, the stratifications given by monodromy strata and Shimada strata are not related, and none is a refinement of the other. However, both stratifications are refined by a third stratification, namely the one given by a fixed configuration of the singular fibres. We call these the *configuration strata*. Their properties were investigated by Kloosterman in [Klo07].

The starting point of our work is the observation that there are some strata which appear in both the monodromy stratification and Shimada's stratification. We call these *ambi-typical* strata, and the main purpose of this paper is to understand these special strata. Our main result (Theorem 4.2) is the following.

Main Theorem. *There are exactly* 50 *positive-dimensional ambi-typical strata. These are listed in Table* 11.

We then prove in Theorem 4.3 that the ambi-typical Shimada strata are, with the exception of two strata, already completely determined by the root lattice R itself. In Theorem 4.4 and Table 12, we further characterise the ambi-typical strata in terms of local monodromy around the singular fibres and the branching behaviour of the j-invariant.

Clearly, there is a connection with moduli spaces of lattice-polarised $K3$ surfaces. Indeed, every ambi-typical stratum gives rise to a priori several moduli spaces of lattice-polarised $K3$ surfaces. It turns out that the total number of possible components of moduli spaces associated with a given ambi-typical stratum is either 1, 2 or 4, as follows from Corollary 11.4 and Proposition 11.7. For each such component $\mathcal{N}$ associated with an ambi-typical stratum $\mathcal{M}$, there is a natural finite dominant map $\mathcal{N} \to \mathcal{M}$; we compute its degree. This is the content of Proposition 12.1, Theorem 12.8 and Corollary 12.9.

We will now discuss the contents of the paper in some more detail: In Section 2, we recall basic facts about elliptically fibred $K3$ surfaces and Miranda's construction of the moduli space $\mathcal{F}$. The monodromy and Shimada stratifications are introduced in Section 3, where we also recall the principal results of Shimada's theory. In Section 4, we formulate the main results of our paper.

Sections 5 through 10 are dedicated to the proof of the main theorem. We start in Section 5 by recalling the configuration strata studied by Kloosterman [Klo07]. It is sufficient to look for ambi-typical configuration strata; this will lead us to the severe restriction that the generic element of an ambi-typical stratum cannot have singular fibres of type II, III, IV, II^* or III^* (Proposition 5.7). In Section 6, we recall a natural factorisation of the j-invariant $j(\mathcal{E}) = j_{\bar{\Gamma}} \circ j_{\mathcal{E}}$ via the modular curve $X(\bar{\Gamma})$ defined by the modular monodromy group $\bar{\Gamma} \subset \mathrm{PSL}(2,\mathbb{Z})$. The restrictions on fibre configurations then translate into branching properties of $j_{\bar{\Gamma}}$ and $j_{\mathcal{E}}$ over the points 0, 1, ∞. In Section 7, we use an Euler number consideration to find a list of possible candidates for the modular monodromy group $\bar{\Gamma}$. In Proposition 7.1, we prove that the index of the modular group $\bar{\Gamma}$ in $\mathrm{PSL}(2,\mathbb{Z})$ is bounded by 18. For the rest of the proof, we shall distinguish between low index (at most 6) and high index (greater than 6). The next step in the proof is that we provide Weierstraß data for all possible ambi-typical strata with modular monodromy group $\bar{\Gamma}$ of low index in Proposition 8.1. We also obtain some results about the Mordell–Weil groups and the monodromy of the six families listed in this proposition. In Sections 9 and 10, we finally complete the proof of the classification in the low- and high-index case, respectively.

Sections 11 and 12 are devoted to relating the ambi-typical strata which we have found to moduli spaces of lattice-polarised $K3$ surfaces.

The appendix by M. Kirschmer contains explicit calculations concerning the 1-dimensional ambi-typical strata, in particular the genus of the moduli spaces of lattice-polarised $K3$ surfaces covering these ambi-typical strata and the degree of the covering map. We find it remarkable that although the ambi-typical 1-dimensional strata all have genus 0, in accordance with the rationality result of Bogomolov, Petrov and Tschinkel, the genus of the moduli space of lattice-polarised $K3$ surfaces can be as high as 13.

2. Elliptically fibred $K3$ surfaces

A $K3$ surface S is *elliptically fibred* if there exists a surjective morphism $f\colon S \to \mathbb{P}^1$ whose generic geometric fibre is a genus 1 curve. We say that $f\colon S \to \mathbb{P}^1$ is a *Jacobian fibration* if it also has a section s. We denote the *Néron–Severi group* by $\mathrm{NS}(S)$ and its rank, also called the *Picard rank*, by $\rho(S)$. A general fibre of f and the 0-section s define a hyperbolic sublattice $U \subset \mathrm{NS}(S)$. Conversely, the existence of a hyperbolic plane U in $\mathrm{NS}(S)$ guarantees the existence of a Jacobian fibration on S. But it should be noted that the class of the fibre and the section need not be contained in U. This is only the case if U contains a nef or, equivalently, base-point-free, isotropic class and an irreducible -2-curve.

We will typically denote $K3$ surfaces with a Jacobian fibration by $f\colon \mathcal{E} \to \mathbb{P}^1$ or simply by $\mathcal{E}$, where we think of $\mathcal{E}$ as an elliptic curve over the function field $\mathbb{C}(\mathbb{P}^1) \cong \mathbb{C}(t)$. The sections of $\mathcal{E}$ form a finitely generated abelian group, the *Mordell–Weil group* of $\mathcal{E}$, which we denote by $\mathrm{MW}(\mathcal{E})$.

The components of all singular fibres of $\mathcal{E}$ which do not meet the section generate a sublattice $R(\mathcal{E})$ of $\mathrm{NS}(\mathcal{E})$ which is a root lattice, more precisely an orthogonal sum of lattices of ADE type. The *trivial part* of the Néron–Severi group $\mathrm{NS}(\mathcal{E})$ is

$$\mathrm{NS}_{\mathrm{tr}}(\mathcal{E}) = U + R(\mathcal{E})\,,$$

where U is the hyperbolic plane defined by the Jacobian fibration. To simplify notation, from here on we simply write + for a direct orthogonal sum. We will denote the rank of the trivial part by $\rho_{\mathrm{tr}}(\mathcal{E})$. In general, $R(\mathcal{E})$ is not saturated in $\mathrm{NS}(\mathcal{E})$, and we will denote its saturation by $L(\mathcal{E})$. It is well known, see [SS19, Corollary 6.20], that

$$L(\mathcal{E})/R(\mathcal{E}) \cong \mathrm{MW}_{\mathrm{tors}}(\mathcal{E})\,.$$

We recall that a Jacobian fibration $\mathcal{E}$ is called *extremal* if $\rho_{\mathrm{tr}}(\mathcal{E}) = 20$. Such surfaces correspond to isolated points in $\mathcal{F}$ and will not be considered here; see, however, [BM08] for the classification of those with a semi-stable fibration.

Jacobian fibrations can be classified via their Weierstraß models; this is the approach taken by Miranda. For the basic theory of Weierstraß equations, we refer the reader to Miranda's paper

[Mir81]. Any Jacobian fibration $f\colon \mathcal{E} \to \mathbb{P}^1$, where $\mathcal{E}$ is a $K3$ surface, is birational to a minimal Weierstraß model

$$y^2 z = 4x^3 - g_2 x z^2 - g_3 z^3 , \quad \text{where } g_2 \in H^0\big(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(8)\big) , \ g_3 \in H^0\big(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(12)\big) , \tag{2.1}$$

and where the following (open) conditions hold:

(1) $\Delta = g_2^3 - 27 g_3^2 \not\equiv 0$.

(2) For every point $q \in \mathbb{P}^1$, the inequality $\min\{3\nu_q(g_2), 2\nu_q(g_3)\} < 12$ holds, where $\nu_q(g)$ is the vanishing order of a polynomial g at q.

The latter condition ensures that the Weierstraß equation is *minimal* in the sense that we cannot write $g_2 = h^4 \overline{g}_2$ and $g_3 = h^6 \overline{g}_3$ with $\overline{g}_2 \in H^0\big(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(4)\big)$ and $\overline{g}_3 \in H^0\big(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(6)\big)$.

Conversely, an equation given as above defines a surface $\mathcal{E}'$ in the projective bundle $\mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(4) \oplus \mathcal{O}_{\mathbb{P}^1}(6) \oplus \mathcal{O}_{\mathbb{P}^1})$ with at most rational double points. A minimal resolution $\mathcal{E}$ of $\mathcal{E}'$ is then a $K3$ surface, and the projection onto the base of the projective bundle gives a Jacobian fibration $f\colon \mathcal{E} \to \mathbb{P}^1$. Weierstraß equations with coefficients g_2, g_3 and g_2', g_3', respectively, define isomorphic Jacobian fibrations if and only if there exists a coordinate transformation $\mathrm{GL}(2, \mathbb{C})$ which maps the pair (g_2, g_3) to (g_2', g_3'). This allows us to describe the moduli of Jacobian fibrations in terms of a GIT quotient.

In order to do this, we consider the weighted projective space $\mathbb{P}_{8,12}(9, 13)$ associated with $H^0\big(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(8)\big) \oplus H^0\big(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(12)\big)$. The open conditions described above define an open subset $V \subset \mathbb{P}_{8,12}(9, 13)$, and the group $\mathrm{SL}(2, \mathbb{C})$ acts on $\mathbb{P}_{8,12}(9, 13)$ as well as on the open subset V. By [Mir81, Proposition 5.1], all points of V are stable with respect to the action of $\mathrm{SL}(2, \mathbb{C})$. Hence we can form the quotient

$$\mathcal{F} := V /\!/ \mathrm{SL}(2, \mathbb{C}) ,$$

which is a quasi-projective variety. Its geometric points correspond bijectively to isomorphism classes of Jacobian fibrations $f\colon \mathcal{E} \to \mathbb{P}^1$, where $\mathcal{E}$ is a $K3$ surface.

In Section 11, we will discuss the theory of moduli spaces of lattice (quasi-)polarised $K3$ surfaces. Taking the hyperbolic plane U (which one should think of as being spanned by the classes of the section and of a general fibre), one obtains an alternative construction of the moduli space $\mathcal{F}$ as a quotient of a homogeneous domain of type IV by an arithmetic group. As we shall see, both points of view are relevant for our purposes. An interesting result of Odaka and Oshima [OO21, Theorem 7.9] even says that the GIT compactification and the Baily–Borel compactification of this space coincide; that is, $\mathcal{F}^{\mathrm{GIT}} \cong \mathcal{F}^{\mathrm{BB}}$.

3. Monodromy and Shimada strata

In this section, we will introduce the main protagonists of this paper, namely *monodromy strata* and *Shimada strata*.

Monodromy strata

Monodromy strata for Jacobian fibrations were first introduced by Bogomolov, Petrov and Tschinkel in [BPT02]. Here we recall some of their results, restricting ourselves to the case in hand, namely Jacobian fibrations of $K3$ surfaces.

The first step is to consider the open subset $V' \subset V$ corresponding to non-isotrivial fibrations. It is naturally obtained by replacing condition (1) above by the slightly stronger condition

(1′) $\Delta = g_2^3 - 27g_3^2$ and g_2^3 are not proportional (which also excludes the case $g_2 \equiv 0$ and the case $\Delta \equiv 0$).

Since the j-invariant is the quotient of these two expressions, this translates directly into several equivalent, more geometric, characterisations of points $v \in V'$:

(1) The j-invariant of the Jacobian elliptic fibration $\mathcal{E}_v$ is non-constant.

(2) The Jacobian elliptic fibration $\mathcal{E}_v$ is not isotrivial.

(3) The number of singular fibres of multiplicative type I_ν or additive type I_ν^*, for $\nu > 0$, is positive.

Clearly V' is invariant (as a set) under the action of $\mathrm{SL}(2,\mathbb{C})$; we denote the quotient by $\mathcal{F}' \subset \mathcal{F}$. This is the open subset parameterising all non-isotrivial Jacobian fibrations of $K3$ surfaces.

To decompose V' (and thus $\mathcal{F}'$) in a geometrically meaningful way, [BPT02] exploit the *monodromy group* of elliptic fibrations: the complement $\mathcal{E}'$ of the union of singular fibres of a Jacobian fibration $\mathcal{E}$ is topologically equivalent to a torus bundle over the base punctured at the critical values of the fibration. The image of the associated monodromy representation in the automorphisms of the first homology of a fibre is orientation preserving. Upon the choice of a basis, it becomes the *monodromy group* $\Gamma(\mathcal{E})$, a subgroup of $\mathrm{SL}(2,\mathbb{Z})$, well defined up to conjugacy; see [Mir89, Lecture VI.3]. Note that according to this definition, we will consider monodromy groups as subgroups of $\mathrm{SL}(2,\mathbb{Z})$, and keep in mind that it is the conjugacy class which is the invariant. We will use $\sim$ to denote conjugate subgroups. The group $\Gamma(\mathcal{E})$ determines a group $\bar{\Gamma}(\mathcal{E}) \subset \mathrm{PSL}(2,\mathbb{Z})$ which we will call the *modular monodromy group*.

The initial observation in [BPT02] is the following semi-continuity property with respect to the Zariski topology on the quasi-projective variety $\mathcal{F}'$: the monodromy group can only change if the configuration of singular fibres changes, that is, if some fibres split or come together. The latter case only occurs on closed subsets of the base of a family, and the monodromy group will be a subgroup after the process. Therefore, the property that the monodromy group is a subgroup of a fixed subgroup of $\mathrm{SL}(2,\mathbb{Z})$ (up to conjugacy) defines a closed subset.

This can be rephrased as follows. We consider the set P of all conjugacy classes of subgroups of $\mathrm{SL}(2,\mathbb{Z})$ together with the partial ordering induced by inclusion. On this we define the *Alexandroff topology* for which a set U is open if and only if for $p \in U$, the inequality $p \leqslant q$ implies $q \in U$. The semi-continuity property observed by Bogomolov, Petrov and Tschinkel then says that the map $V' \to P$ is continuous with respect to the Alexandroff topology on P and the Zariski topology on V'. It implies that the fibres V'_Γ of this map are locally closed for a fixed (conjugacy class of a) subgroup $\Gamma \subset \mathrm{SL}(2,\mathbb{Z})$. We can decompose each such set into its irreducible components $V'_\Gamma = \cup_i V'_{\Gamma,i}$. Bogomolov, Petrov and Tschinkel showed that the number of possible monodromy groups is finite and in this way obtained the following result.

Lemma 3.1 (cf. [BPT02, Lemma 3.1]). *The variety V' is a finite union of locally closed irreducible subvarieties $V'_{\Gamma,i}$, each preserved under the action of* $\mathrm{SL}(2,\mathbb{C})$ *such that for every $v \in V'_{\Gamma,i}$ one has $\Gamma(\mathcal{E}_v) \sim \Gamma$.*

We set $\mathcal{F}'_{\Gamma,i} = V'_{\Gamma,i} /\!\!/ \mathrm{SL}(2,\mathbb{C})$, and in this way one obtains a finite decomposition

$$\mathcal{F}' = \cup_i \mathcal{F}'_{\Gamma,i} \tag{3.1}$$

into irreducible locally closed subvarieties.

Definition 3.2. We refer to (3.1) as the *monodromy stratification* of the moduli space of $K3$ surfaces with a non-isotrivial Jacobian fibration and call the subvarieties $\mathcal{F}'_{\Gamma,i}$ the *monodromy strata* of $\mathcal{F}'$.

While the objective of [BPT02] is the rationality of the monodromy strata $\mathcal{F}'_{\Gamma,i}$ in the moduli space, for our arguments it is important to note the following maximality condition

$$v \in \overline{V'_{\Gamma,i}} \setminus V'_{\Gamma,i} \quad \Longrightarrow \quad \Gamma(\mathcal{E}_v) \not\sim \Gamma\,. \tag{3.2}$$

Shimada strata

We have already seen the root lattice $R(\mathcal{E})$ associated with a Jacobian fibration $f\colon \mathcal{E} \to \mathbb{P}^1$ of a $K3$ surface. In general, $R(\mathcal{E})$ is not saturated in $\mathrm{NS}(\mathcal{E})$, and we denoted its saturation by $L(\mathcal{E})$. The overlattice $L(\mathcal{E})$ of $R(\mathcal{E})$ defines, and can be reconstructed from, the finite group $G(\mathcal{E}) = L(\mathcal{E})/R(\mathcal{E}) \subset D(R(\mathcal{E}))$. Here we use standard notation and standard facts from lattice theory. If L is an even lattice, we denote by $L^\vee$ the *dual lattice* which can be defined as $L^\vee = \{x \in L_{\mathbb{Q}} \mid (x,y) \in \mathbb{Z} \text{ for all } y \in L\}$. The *discriminant* of L is the quotient $D(L) = L^\vee/L$. This is a finite group, and it is equipped with a quadratic form with values in $\mathbb{Q}/2\mathbb{Z}$. The importance of the discriminant group in lattice theory was fully developed in Nikulin's seminal paper [Nik79]. Overlattices of L correspond to isotropic subgroups $G \subset D(L)$; see [Nik79, Proposition 1.4.1].

In his paper [Shi00], Shimada investigated the question which pairs (R,G), where R is an ADE root lattice and $G \subset D(R)$ is a finite (isotropic) subgroup of the discriminant group, can occur as $(R(\mathcal{E}), G(\mathcal{E}))$, and how many irreducible families these Jacobian fibrations form. At this point it is important to note that, in general, a pair (R,G) does not necessarily determine a unique family. There are several reasons for this. Firstly, the abstract group G does not necessarily define a unique overlattice L of R; more precisely, one must choose a specific isotropic subgroup G in $D(R)$, and it can happen that G can be embedded in several ways as an isotropic subgroup in $D(R)$. Secondly, given L one must also specify a primitive embedding of $M = U \oplus L$ into the $K3$ lattice

$$L_{K3} = 3\,U + 2E_8\,,$$

where E_8 is the unique even unimodular negative definite lattice of rank 8, and the lattice M may have several such embeddings (modulo the orthogonal group $\mathrm{O}(L_{K3})$). Given such an embedding, one can consider the moduli space of *lattice-polarised* $K3$ surfaces with lattice polarisation M. Note, however, that a lattice polarisation possibly contains more information than the pair (R,G) as different lattice polarisations can give rise to the same elliptic $K3$ surface with a given configuration of singular fibres. We shall discuss this relationship in detail in Sections 11 and 12. The Shimada strata correspond to finite quotients of (open subsets) of moduli spaces of lattice-polarised $K3$ surfaces. We also note that the moduli spaces can have one or two components and this, thirdly, can also increase the number of strata. If there are two such components, then they are complex conjugate to each other; that is, surfaces of one component are complex conjugate to those of the other component; cf. [FM94, Section 2.7.4, Definition 7.13].

Shimada's classification [Shi18] gives the following result.

Theorem 3.3 (Shimada). (i) *There are* 3278 *different root lattices which occur as* $R(\mathcal{E})$ *for a Jacobian fibration* $\mathcal{E}$ *of a* $K3$ *surface. Of these,* 2953 *belong to non-extremal fibrations.*

(ii) *This decomposes the set of all Jacobian fibrations of* $K3$ *surfaces into* 3932 *connected families, of which* 3469 *belong to non-extremal fibrations.*

Shimada's result defines a stratification of the moduli space $\mathcal{F}'$ into maximal (with respect to inclusion) locally closed irreducible subvarieties $\mathcal{M}'_{R,G,j}$, and this defines a finite decomposition

$$\mathcal{F}' = \cup_j \mathcal{M}'_{R,G,j} \,. \tag{3.3}$$

DEFINITION 3.4. We call the decomposition (3.3) the *Shimada stratification* of $\mathcal{F}'$ and the subvarieties $\mathcal{M}'_{R,G,j}$ the *Shimada strata* of $\mathcal{F}'$.

For future use we also note the following remark, which is simply obtained by counting the conditions imposed by the rank of the Néron–Severi group.

Remark 3.5. The dimension of a component $\mathcal{M}'_{R,G,j}$ is given by

$$\dim \mathcal{M}'_{R,G,j} = 18 - \operatorname{rank}(R) \,. \tag{3.4}$$

We observe that this depends only on R and not on G or the specific component we are considering. Note that all components of $\mathcal{M}'_R$ have the same dimension, which we will also refer to as the dimension of $\mathcal{M}'_R$.

We note that the decomposition in Shimada strata (3.3) is a topological poset stratification in the sense of [YY19].

4. The main classification result

The primary goal of this paper is to compare the two stratifications of the moduli space $\mathcal{F}'$ of non-isotrivial Jacobian fibrations on $K3$ surfaces which we have defined above, namely the monodromy stratification and the Shimada stratification. More precisely, we want to determine all positive-dimensional monodromy and Shimada strata whose generic points coincide. In other words, we want to determine all pairs $\mathcal{F}'_{\tilde{\Gamma},i}$ and $\mathcal{M}'_{R,G,j}$ such there intersection is non-empty and open (and hence dense) in both $\mathcal{F}'_{\Gamma,i}$ and $\mathcal{M}'_{R,G,j}$. This leads us to the following definition.

DEFINITION 4.1. A positive-dimensional irreducible closed subset $\mathcal{A} \subset \mathcal{F}'$ is called an *ambi-typical stratum* if there exist a monodromy stratum $\mathcal{F}'_{\Gamma,i}$ and a Shimada stratum $\mathcal{M}'_{R,G,j}$ such that

$$\mathcal{A} = \overline{\mathcal{F}'_{\Gamma,i}} = \overline{\mathcal{M}'_{R,G,j}} \,. \tag{4.1}$$

(The monodromy stratum and the Shimada stratum are then uniquely defined.)

Our main result is the following.

THEOREM 4.2. *There are* 50 *ambi-typical strata in* $\mathcal{F}'$.

As we have discussed, a Shimada stratum determines a pair (R, G), but not necessarily the other way round. There are, however, cases where the root lattice defines a unique Shimada stratum. Indeed, this is the case for most of the ambi-typical strata. The data (R, G) for the ambi-typical strata which are referred to in the following theorem are listed in Table 11.

THEOREM 4.3. *In all but* 3 *cases, the root lattice* R *of an ambi-typical stratum* $\mathcal{A}$ *determines a unique Shimada stratum. The exceptions are as follows:*

(i) *The root lattice* $D_4 + 2A_6 + A_1$ *in* 38/39 *determines two Shimada strata; these are complex conjugate to each other.*

(ii) *The root lattice $2A_3 + 8A_1$ in 6 determines two non-conjugate Shimada strata. Only one of these occurs as an ambi-typical stratum.*[1]

By definition, a monodromy stratum determines a monodromy group Γ, but the converse will in general not be true. In fact, as we will see in the rest of the paper, monodromy strata are determined by some fairly involved data. This includes the modular monodromy group $\bar{\Gamma} \subset \mathrm{PSL}(2,\mathbb{Z})$ but also information about the j-invariant $j(\mathcal{E})\colon \mathbb{P}^1 \to \mathbb{P}^1$. We shall see in Section 6 that this can be factored in a unique way as $j(\mathcal{E}) = j_{\bar{\Gamma}} \circ j_{\mathcal{E}}$, where $j_{\bar{\Gamma}}$ is a Belyi map. The additional data which determine a monodromy stratum are given by a space of rational maps $j_{\mathcal{E}}\colon \mathbb{P}^1 \to \mathbb{P}^1$ and a rational family of divisors on $\mathbb{P}^1$ corresponding to the $*$-fibres. However, in the case of ambi-typical strata these data are fully determined by much more accessible ones.

Theorem 4.4. *The 50 ambi-typical monodromy strata determine the list of invariants given in Table 12, consisting of*

(i) *the local monodromies of $j_{\bar{\Gamma}}$ at 0, 1, ∞,*

(ii) *the branching of $j_{\mathcal{E}}$ for a generic element $\mathcal{E}$ at the non-critical pre-images of 0, 1 and at poles of $j_{\bar{\Gamma}}$,*

(iii) *the number of $*$-fibres.*

Conversely, the data are pairwise distinct, and each determines a unique monodromy stratum with the following exception: the strata 38/39 correspond to two distinct maps $j_{\bar{\Gamma}}$ having the same combinatorial type but with monodromy factorisations in distinct conjugation classes.

The proof of these theorems is quite involved and takes up most of the remainder of this paper. Below is a rough outline of our strategy.

In Section 5, we study configurations of singular fibres where we use the work of Kloosterman [Klo07, Section 4]. This leads to a semi-continuous invariant which induces a stratification which refines both the stratifications by monodromy and by root lattice. Hence it suffices to look for ambi-typical fibre configuration strata. We find that to be ambi-typical poses severe restrictions on the possible singular fibres.

In Section 6, we recall the factorisation $j(\mathcal{E}) = j_{\bar{\Gamma}} \circ j_{\mathcal{E}}$ of the j-invariant. The restrictions on fibre configurations then translate into branching properties of $j_{\bar{\Gamma}}$ and $j_{\mathcal{E}}$ over the points 0, 1, ∞.

In Section 7, we use an Euler number restriction to find a list of candidates for the modular monodromy group $\bar{\Gamma}$ and give an upper bound for the dimension of the corresponding strata.

In Section 8, we parameterise closed subsets of V' which give Weierstraß data for all possible ambi-typical strata with modular monodromy group $\bar{\Gamma}$ of low index, that is, index at most 6.

In Section 9, we determine the ambi-typical strata among these families of Weierstraß data and their invariants.

In Section 10, we address the classification of ambi-typical strata with modular monodromy group $\bar{\Gamma}$ of high index, that is, index at least 7. In the first step, we determine the possible corresponding root lattices and the topological types of the $j_{\mathcal{E}}$ factor of the j-function. In the second step, we determine the topology of the maps $j_{\bar{\Gamma}}$ and the corresponding monodromy groups.

This programme will ultimately allow us to complete the proofs of our theorems.

[1] We will say more on this in Remark 9.6.

5. Singular fibre configurations and generic invariants

In this section, we consider the configuration of singular fibres of elliptic surfaces. The possible singular fibre types have been classified by Kodaira and are listed in Table 1 with some of their invariants; see [BHP+04, Table V.6]. The relation between singular fibre types and Weierstraß data is given in Table 2. This contains all information necessary to apply the *Tate algorithm* over the complex numbers, that is, to determine the fibre types from the vanishing orders of $\nu_2 = \nu(g_2)$, $\nu_3 = \nu(g_3)$ and $\nu_\Delta = \nu(\Delta)$ of g_2, g_3 and Δ, respectively.

Fibre type	ADE-type	Euler number	Local monodromy	Local j-expansion
I_0	–	0	$\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$	$j = s^{3k}$, $j = 1 + s^{2k}$ or $j \neq 0, 1$
I_1	–	1	$\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$	pole of order 1
I_b $(b \geqslant 2)$	A_{b-1}	b	$\left(\begin{smallmatrix} 1 & b \\ 0 & 1 \end{smallmatrix}\right)$	pole of order b
I_0^*	D_4	6	$\left(\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$	same as in first case
I_b^* $(b \geqslant 1)$	D_{4+b}	$6+b$	$\left(\begin{smallmatrix} -1 & -b \\ 0 & -1 \end{smallmatrix}\right)$	pole of order b
II	–	2	$\left(\begin{smallmatrix} 1 & 1 \\ -1 & 0 \end{smallmatrix}\right)$	$j = s^{3k+1}$
III	A_1	3	$\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$	$j = 1 + s^{2k+1}$
IV	A_2	4	$\left(\begin{smallmatrix} 0 & 1 \\ -1 & -1 \end{smallmatrix}\right)$	$j = s^{3k+2}$
IV^*	E_6	8	$\left(\begin{smallmatrix} -1 & -1 \\ 1 & 0 \end{smallmatrix}\right)$	$j = s^{3k+1}$
III^*	E_7	9	$\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$	$j = 1 + s^{2k+1}$
II^*	E_8	10	$\left(\begin{smallmatrix} 0 & -1 \\ 1 & 1 \end{smallmatrix}\right)$	$j = s^{3k+2}$

Table 1. Fibre types of elliptic surfaces

In the ensuing discussion, we will use notation as introduced in Kloosterman's paper [Klo07]. For a Jacobian fibration $f\colon S \to \mathbb{P}^1$, let $C(f)$ denote the configuration of its singular fibres. Since we are restricting ourselves to non-isotrivial fibrations, we can assume that $C(f)$ contains at least one fibre of type I_ν or I_ν^* with $\nu > 0$. For an abstract configuration C of singular fibres we define

$$L(C) := \left\{ \left[f\colon S \to \mathbb{P}^1 \right] \in \mathcal{F}' \mid C(f) = C \right\}. \tag{5.1}$$

Remark 5.1. We recall from [Klo07, Lemma 4.6] that $L(C) \subset \mathcal{F}'$ is constructible. Fibre configurations are partially ordered by degeneration of several singular fibres into fewer more complicated ones. Degeneration occurs on closed subsets; thus semi-continuity holds, and $\mathcal{F}'$ is topologically stratified by the components of the sets $L(C)$.

Type	I_0			$I_k, k>0$	I_0^*			$I_k^*, k>0$	II	III	IV	IV^*	III^*	II^*
j	0	1	*gen.*	∞	0	1	*gen.*	∞	0	1	0	0	1	0
ν_2	>0	0	0	0	>2	2	2	2	>0	1	>1	>2	3	>3
ν_3	0	>0	0	0	3	>3	3	3	1	>1	2	4	>4	5
ν_Δ	0	0	0	k	6	6	6	$k+6$	2	3	4	8	9	10

Table 2. Tate algorithm (cf. [Mir89, Table IV.3.1])

Definition 5.2. We call $L(C)$ a *configuration locus* and the components of the sets $L(C)$ the *configuration strata*.

As we will need the dimension of the components of $L(C)$, we recall the following.

Lemma 5.3 ([Klo07, Lemma 4.6, Assumption 4.3]). *Assume that C is a configuration of singular fibres arising from a non-isotrivial Jacobian fibration. Then all components of $L(C)$ have dimension*

$$\dim L(C) = \#\{\textit{singular fibres}\} + \#\{\textit{fibres of type } II^*, III^*, IV^*, I_\nu^*, \nu \geqslant 0\} - 6\,.$$

By construction, each irreducible component of $L(C)$ corresponds to Jacobian elliptic fibrations with topologically equivalent complements of singular fibres. In particular, all elements in an irreducible component of $L(C)$ have the same monodromy group, and each irreducible component $\mathcal{F}'_{\Gamma,i}$ is a finite union of irreducible components of certain $L(C)$. The only one of these which is open thus corresponds to the configuration of a generic member of the monodromy stratum. We call this configuration $C(\mathcal{F}'_{\Gamma,i})$, and all components of $L(C(\mathcal{F}'_{\Gamma,i}))$ have the same dimension as $\mathcal{F}'_{\Gamma,i}$.

By definition, all Jacobian fibrations $\mathcal{E} \in L(C)$ have the same root configuration and hence the same rank $\rho_{\mathrm{tr}}(\mathcal{E})$ of the trivial lattice $\mathrm{NS}_{\mathrm{tr}}(\mathcal{E})$.

From this one easily obtains the following.

Lemma 5.4 ([Klo07, Proposition 4.7]). *Let C be a configuration of singular fibres containing at least one I_ν- or I_ν^*-fibre ($\nu > 0$) with $L(C) \neq \emptyset$, and let $\mathcal{E}$ be any element in $L(C)$. Then*

$$\dim L(C) = 20 - \rho_{\mathrm{tr}}(\mathcal{E}) - \#\{\textit{fibres of type } II, III \textit{ or } IV\}\,.$$

Proof. We use Lemma 5.3 and apply the fact that the rank of the trivial lattice $\rho_{\mathrm{tr}}(\mathcal{E})$ is given by

$$\begin{aligned}\rho_{\mathrm{tr}}(\mathcal{E}) &= 2 + \sum_{F \text{ multiplicative}} (e(F) - 1) + \sum_{F \text{ additive}} (e(F) - 2)\\ &= 26 - \#\{\text{multiplicative fibres}\} - 2\#\{\text{additive fibres}\}\,.\end{aligned}$$

Here $e(F)$ denotes the Euler number of a fibre F. □

Any component of a stratum $L(C)$ defines an embedding $U + R \hookrightarrow L_{K3}$ (up to isometries in $\mathrm{O}(L_{K3})$) and thus an embedding into a root stratum. Hence the stratification given by the configuration strata refines both the monodromy stratification and the root lattice stratification. In particular, every Shimada stratum $\mathcal{M}'_{R,G,j}$ and every monodromy stratum $\mathcal{F}'_{\Gamma,i}$ contains a unique configuration stratum which is open and dense in this stratum.

This also allows us to talk about the properties of a *generic* element of a Shimada stratum or a monodromy stratum. In particular, the following data, in addition to the configuration $C(\mathcal{E})$, are invariant for all members $\mathcal{E}$ of a configuration stratum: the monodromy group $\Gamma(\mathcal{E})$ (as we have already pointed out), the root lattice $R(\mathcal{E})$, the trivial lattice $\mathrm{NS}_{\mathrm{tr}}(\mathcal{E})$, the saturation $L(\mathcal{E})$ of $R(\mathcal{E})$ in $\mathrm{NS}(\mathcal{E})$ and the torsion of the Mordell–Weil group, $\mathrm{MW}_{\mathrm{tors}}(\mathcal{E}) \cong L(\mathcal{E})/R(\mathcal{E})$.

We are now ready to start the classification of ambi-typical strata. For this, we first collect necessary conditions which a generic element of such a stratum must satisfy. Indeed, the first restrictions on the possible configurations can be derived easily.

Proposition 5.5. *Suppose that Γ is a proper subgroup of $\mathrm{SL}(2,\mathbb{Z})$ of finite index and $\mathcal{E} \in \mathcal{F}'_{\Gamma,i}$ a generic element of a monodromy stratum with monodromy group Γ. Then the following properties are equivalent:*

(i) *The Jacobian fibration $\mathcal{E}$ has no fibres of type II, III or IV.*

(ii) *The dimensions of the Shimada stratum $\mathcal{M}'_{R(\mathcal{E}),G(\mathcal{E}),j}$ which contains $\mathcal{E}$ and of $\mathcal{F}'_{\Gamma,i}$ coincide.*

In particular, the generic element of an ambi-typical stratum does not have any fibres of type II, III or IV.

Proof. Since $\mathcal{E}$ is a generic element of $\mathcal{F}'_{\Gamma,i}$, it lies in a unique open and dense configuration stratum, namely a component of $L(C(\mathcal{E}))$. Hence

$$\dim \mathcal{F}'_{\Gamma,i} = \dim L(C(\mathcal{E})) = 20 - \rho_{\mathrm{tr}}(\mathcal{E}) - \#\{\text{fibres of type } II, III \text{ or } IV\}\,,$$

where the last equality follows from Lemma 5.4. Since the Shimada stratum has dimension equal to $20 - \rho_{\mathrm{tr}}(\mathcal{E})$, the claim follows immediately. □

Remark 5.6. Lemma 5.4 shows that if the generic element of a monodromy stratum $\mathcal{F}'_{\Gamma,i}$ has fibres of type II, III or IV, then this monodromy stratum is contained in a Shimada stratum of strictly bigger dimension.

We can find further severe restrictions on ambi-typical strata by studying how the monodromy behaves under certain fibre degenerations.

Proposition 5.7. *Assume that $\mathcal{E}$ is a generic element of an ambi-typical stratum. Then $\mathcal{E}$ has no singular fibres of type II, III, IV, II^* or III^*. If $-\mathrm{id} \in \Gamma(\mathcal{E})$, then $\mathcal{E}$ also has no singular fibres of type $I^*_{>0}$ or IV^*.*

Proof. We have already seen in Proposition 5.5 that fibres of type II, III or IV cannot exist.

Our strategy is the following: Suppose that $\mathcal{E}$ has a singular fibre of type II^*, III^*, $I^*_{>0}$ or IV^*, where in the last two cases we assume $-\mathrm{id} \in \Gamma(\mathcal{E})$. We will then construct a family containing $\mathcal{E}$ where the monodromy remains constant, but where the rank of the trivial Néron–Severi group drops. This contradicts the assumption that $\mathcal{E}$ is a generic element of a Shimada stratum.

To do this, pick a Weierstraß datum g_2, g_3 for $\mathcal{E}$. By inspection of the Tate Table 2, we see that the presence of a $*$-fibre implies that g_2, g_3 have a common zero at this fibre. More precisely, we can factor $g_2 = \breve{g}_2 x^2$ and $g_3 = \breve{g}_3 x^3$, where x is a linear form vanishing at this fibre. We then consider the family of Weierstraß data $\breve{g}_2(x-t)^2$, $\breve{g}_3(x-t)^3$ where t varies. It has the same j-invariant as $\mathcal{E}$, hence the projective monodromy group $\bar{\Gamma}(\mathcal{E}_t)$ is constant; see [BHP+04, Section V.11, p. 211]. If in addition $-\mathrm{id} \in \Gamma(\mathcal{E})$, then the monodromy group $\Gamma(\mathcal{E}_t)$ is also constant since it can only get smaller on a closed subset. By Table 1, for fibres of type II^* or III^* a power of the local monodromy is $-\mathrm{id}$, so this assumption is fulfilled. For fibres of type $I^*_{>0}$ or IV^*, this is part of our assumptions. Hence the monodromy group remains constant if we vary t; however, for $t \neq 0$, the fibre of $*$-type is replaced by an I_0^*-fibre and the fibre without $*$ that has the same local monodromy up to $-\mathrm{id}$; see Table 1. But then, again by Table 1, this implies that the rank of the trivial Néron–Severi group $\mathrm{NS}_{\mathrm{tr}}(\mathcal{E}_t)$ group drops as $t \neq 0$, giving the desired contradiction. □

Remark 5.8. A typical example for the situation discussed is when a type II^* fibre splits into a type IV fibre and a type I_0^* fibre. Here the rank of the trivial Néron–Severi drops by 2. This gives examples where a Shimada stratum is contained as a proper subset in a bigger monodromy stratum.

6. Factorisations of the j-invariant

An essential tool for our classification result is the j-function associated with a Jacobian fibration $f\colon \mathcal{E} \to \mathbb{P}^1$. As usual, we denote the upper half-plane by $\mathbb{H}_1$ and set

$$\overline{\mathbb{H}}_1 = \mathbb{H}_1 \cup \mathbb{Q} \cup \{\infty\}\,.$$

The quotient

$$X(1) := \overline{\mathbb{H}}_1 / \operatorname{PSL}(2,\mathbb{Z}) \cong \mathbb{P}^1$$

is the modular curve of level 1, namely the compactification of the j-line. If the Jacobian fibration $f\colon \mathcal{E} \to \mathbb{P}^1$ has monodromy group $\Gamma \subset \operatorname{SL}(2,\mathbb{Z})$, then we denote its image in $\operatorname{PSL}(2,\mathbb{Z})$ by $\bar\Gamma$. This defines the modular curve $X(\bar\Gamma) := \overline{\mathbb{H}}_1/\bar\Gamma$, which in our case is again isomorphic to $\mathbb{P}^1$. The j-function

$$j(\mathcal{E})\colon \mathbb{P}^1 \to X(1) \cong \mathbb{P}^1$$

has a unique factorisation

$$j(\mathcal{E}) = j_{\bar\Gamma} \circ j_{\mathcal{E}} \tag{6.1}$$

up to deck transformations of $j_{\bar\Gamma}$, where

$$j_{\bar\Gamma}\colon X(\bar\Gamma) \cong \mathbb{P}^1 \to X(1) \cong \mathbb{P}^1$$

is the natural quotient map. We also refer the reader to [BPT02, Section 0, p. 1107]. Another characterisation of $\bar\Gamma$, using the factorisation of $j(\mathcal{E})$, is provided by the following result.

Lemma 6.1. *The group $\bar\Gamma$ has the following minimality property:*

$$\textit{if } j(\mathcal{E}) \textit{ factors through } j_{\bar\Gamma'}, \textit{ then } \bar\Gamma \subset \bar\Gamma' \textit{ up to conjugation}. \tag{6.2}$$

Proof. Given a holomorphic surjective map $j\colon \mathbb{P}^1 \to X(1)$, by [BT03, Lemma 2.3] there exists a unique, up to conjugation, subgroup $\bar\Gamma_j \subseteq \operatorname{PSL}(2,\mathbb{Z})$ of finite index such that

$$j \text{ factors through } j_{\bar\Gamma_j}\,, \tag{6.3}$$

$$\text{if } j \text{ factors through } j_{\bar\Gamma'}, \text{ then } \bar\Gamma_j \subseteq \bar\Gamma' \text{ up to conjugation}. \tag{6.4}$$

We apply this result to $j(\mathcal{E})$ and thus have to show that $\bar\Gamma_{j(\mathcal{E})} = \bar\Gamma$. The inclusion $\bar\Gamma_{j(\mathcal{E})} \subseteq \bar\Gamma$ (up to conjugacy) follows from (6.4).

For the other direction, we use the discussion in [BHP+04, Section V.11, p. 211]. Every j-map gives rise to a representation class $r = r(j)$ with values in $\operatorname{PSL}(2,\mathbb{Z})$ and, in particular, $r(j_{\bar\Gamma'})$ has image $\bar\Gamma'$. By Kodaira's theory of elliptic fibrations, $r(j(\mathcal{E}))$ is the composition of the monodromy homomorphism with $\operatorname{SL}(2,\mathbb{Z}) \to \operatorname{PSL}(2,\mathbb{Z})$ and hence has image $\bar\Gamma$. By property (6.3), the map $j(\mathcal{E})$ factors through $j_{\bar\Gamma(j_{\mathcal{E}})}$ and hence $\operatorname{Im} r(j(\mathcal{E})) \subseteq \operatorname{Im} r(j_{\bar\Gamma(j_{\mathcal{E}})})$, which implies that $\bar\Gamma \subseteq \bar\Gamma(j_{\mathcal{E}})$ (up to conjugation). $\square$

Remark 6.2. From here on, it will be very helpful to investigate subgroups of $\operatorname{PSL}(2,\mathbb{Z})$ by means of the following three sets, where S_d denotes the symmetric group of d elements:

(1) subgroups $\bar\Gamma$ in $\operatorname{PSL}(2,\mathbb{Z})$ of index d up to conjugacy in $\operatorname{PSL}(2,\mathbb{Z})$;

(2) (connected) branched covers $j\colon C \to X(1)$ of degree d up to equivalence of covers, branched only over 0, 1, ∞, with multiplicities 1, 3 over 0 and 1, 2 over 1;

(3) homomorphisms $\mu\colon \pi_1(\mathbb{C} \setminus \{0,1\}) \to S_d$ up to conjugacy in S_d such that simple loops around 0, respectively 1, map to elements of order 1 or 3, respectively, 1 or 2, and the image acts transitively.

Since $\pi_1(\mathbb{C}\setminus\{0,1\})$ is freely generated by simple loops around 0 and 1, while $\mathrm{PSL}(2,\mathbb{Z})$ as the free product $\mathbb{Z}/2 * \mathbb{Z}/3$ is generated by $\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 1 & 1 \\ -1 & 0 \end{smallmatrix}\right)$, these sets are in bijective correspondence with each other via the following maps:

(1)→(2) Send $\bar\Gamma$ to the branched cover $j_{\bar\Gamma}\colon X(\bar\Gamma) \to X(1)$.

(1)→(3) Send $\bar\Gamma$ to left multiplication $\mathrm{PSL}(2,\mathbb{Z}) \to S(\mathrm{PSL}(2,\mathbb{Z})/\bar\Gamma) \cong S_d$ on cosets and then compose with $\pi_1(\mathbb{C}\setminus\{0,1\}) \to \mathrm{PSL}(2,\mathbb{Z})$ to get μ.

(2)→(3) Given $j\colon C \to \mathbb{P}^1$, restrict to $\mathbb{C}\setminus\{0,1\}$, which is a topological cover of degree d, and send j to the representation $\mu\colon \pi_1(\mathbb{C}\setminus\{0,1\}) \to S_d$ of the restriction by permutations of a fibre.

(3)→(1) Given μ, note that by our assumptions this factors through a homomorphism $\bar\mu\colon \mathrm{PSL}(2,\mathbb{Z}) \to S_d$ and send μ to the stabiliser subgroup $\bar\Gamma \subset \mathrm{PSL}(2,\mathbb{Z})$ of 1.

(3)→(2) Send μ to the connected topological cover over $\mathbb{C}\setminus\{0,1\}$ of degree d that extends to a branched cover $j\colon C \to \mathbb{P}^1$ by Riemann existence.

Using Remark 6.2, we obtain a characterisation of the factorisation (6.1) from Lemma 6.1.

Lemma 6.3. *Given the holomorphic map $j(\mathcal{E})\colon \mathbb{P}^1 \to X(1)$, a factorisation $j_2 \circ j_1$ is equivalent to $j_{\bar\Gamma} \circ j_{\mathcal{E}}$ if and only if*

$$j_2 \text{ is branched only over } 0,\ 1,\ \infty \text{ with multiplicities } 1,\ 3 \text{ over } 0 \text{ and } 1,\ 2 \text{ over } 1\,, \tag{6.5}$$

$$j_1 \text{ has no proper left factor } j',\ \text{so that } j_2 \circ j' \text{ has the property above}\,. \tag{6.6}$$

In the topological analysis of more arbitrary (connected) branched covers of $\mathbb{P}^1$, we first fix a base point not in the branch locus. We then use the monodromy homomorphism taking homotopy classes of closed paths inside the complement of the branch locus around this base point to the permutation group of the fibre over the base point. This defines a local monodromy which associates with a branch point the conjugacy class of the monodromy of the positively oriented boundary of a sufficiently small disc centred at the branch point. While the monodromy depends on the choice of a fibre, the conjugacy class does not, and it is thus an invariant, which we may as well give by the cycle type of the permutation or the corresponding partition of the fibre cardinality. The sizes of the parts of that partition are exactly the multiplicities of the pre-images of the branch point.

Accordingly, we can associate with $j_{\bar\Gamma}$ the three conjugacy classes C_0, C_1, C_∞ corresponding to the branch points 0, 1, ∞. In practice, we will give the three conjugacy classes as a 3-tuple of partitions of $\deg j_{\bar\Gamma}$. The degree of $j_{\bar\Gamma}$ and the numbers e_3 and e_2 of fixed points of elements in C_0 and C_1, respectively, are related by the following congruences:

$$\deg j_{\bar\Gamma} \equiv_2 e_2\,, \quad \deg j_{\bar\Gamma} \equiv_3 e_3\,. \tag{6.7}$$

This follows since the difference of the two numbers in question is the sum of cycle lengths of transpositions and 3-cycles, respectively. Note that for the subgroup $\bar\Gamma$ of $\mathrm{PSL}(2,\mathbb{Z})$ corresponding to $j_{\bar\Gamma}$, the numbers e_2 and e_3 count the non-ramified pre-image of 1 and 0, respectively, under $j_{\bar\Gamma}$, which we call 2-*torsion* points and 3-*torsion* points, respectively.

For the next lemma, we only require a small part of the branching datum of $j_{\mathcal{E}}$, namely the partitions corresponding to the local monodromy of $j_{\mathcal{E}}$ around the 2-torsion and 3-torsion points. In particular, if $f^{-1}(p)$, with $p \in \mathbb{P}^1$, is a fibre of type IV^*, then the map $j_{\mathcal{E}}$ is locally unbranched and maps the point p to a 3-torsion point.

Lemma 6.4. *The following properties hold for the invariants associated with the factors $j_{\bar{\Gamma}}$ and $j_{\mathcal{E}}$ of a generic element $\mathcal{E}$ in an ambi-typical monodromy stratum:*

(i) *Partitions associated with $j_{\mathcal{E}}$ at 2-torsion points have only even size parts.*

(ii) *The parts of the partitions associated with $j_{\mathcal{E}}$ at 3-torsion points all have size divisible by 3 except for a total of $\#IV^*$ parts of size equal to $1 \pmod 3$.*

(iii) $e_2 < 2$.

(iv) $e_3 < 2$ *if* $\#IV^* = 0$.

(v) $e_3 < 3$ *if* $\#IV^* \leqslant 2$.

(vi) $\deg j_{\mathcal{E}} \equiv_3 1$ *and* $\#IV^* = 2$ *if* $e_3 = 2$ *and* $\#IV^* < 3$.

Proof. (i) Let s be a local coordinate at a point corresponding to an odd part. Then the j-invariant takes value 1 and has odd multiplicity at $s = 0$. Hence the local expansion is $j = 1 + s^{2k+1}$. But then according to Table 1, the corresponding fibre of $\mathcal{E}$ would be of type III or III^*. We have already excluded this in Proposition 5.7.

(ii) If s is a local coordinate at a point corresponding to a part of size ℓ, then the j-invariant has value 0 and multiplicity equal to ℓ. From the local expansion $j = s^\ell$, we can again determine the corresponding fibre type from Table 1. It is IV or II^* in case $\ell \equiv_3 2$, but these are excluded by Proposition 5.7. It is II or IV^* in case $\ell \equiv_3 1$, but the former is excluded again by the same reason and hence the number of IV^*-fibres is equal to the number of parts of size $\ell \equiv_3 1$.

(iii) Suppose $e_2 \geqslant 2$. Then there are at least two partitions with only even size parts. By Proposition 6.6 below, $j_{\mathcal{E}}$ then has a proper factor j' which violates condition (6.6) on the factorisation of the j-invariant.

(iv) Suppose $e_3 \geqslant 2$ and $\#IV^* = 0$. Then there are at least two partitions with parts of size divisible by 3. Again by Proposition 6.6 below, $j_{\mathcal{E}}$ then has a proper factor j' violating condition (6.6).

(v) The claim follows since the number of parts of size $\ell \equiv_3 1$ is bounded below by e_3 if $\deg j_{\mathcal{E}}$ is not divisible by 3; otherwise, the number of parts of size $\ell \equiv_3 1$ is at least 3 or part (iv) applies.

(vi) By part (iv), it is not possible to have $\#IV^* = 0$. In case $\deg j_{\mathcal{E}} \equiv_3 0$ or 2, the number of parts of size $\ell \equiv_3 1$ is therefore at least 3, respectively 4. So $\deg j_{\mathcal{E}} \equiv_3 1$, and the number of parts of size $\ell \equiv_3 1$ is two, and hence $\#IV^* = 2$. □

In order to prove the factorisation result used above, we now study branched coverings of the Riemann sphere more systematically from a topological point of view. Consider a finite branched covering $\mathbb{P}^1 \to \mathbb{P}^1$ of degree d, and let r denote the number of branch points. The fundamental group of the complement with respect to a base point is generated by elements associated with a *geometric* sequence of r simple loops around these points, that is, chosen carefully such that they meet only at the base point and their product loop is homotopic to the boundary of a disc containing all base points. Of course, the product loop represents the trivial element and is known to suffice as the only relation.

These elements act by permutations on the set $I = \{1, \dots, d\}$ in bijection with the elements of the fibre over the base point, giving rise to the monodromy elements $\sigma_1, \dots, \sigma_r \in S(I)$, where σ_i is given by monodromy along the simple loop around the ith branch point. Of course, the choices of paths form an orbit under the action of the Hurwitz braid group [Hur91]. The induced *Hurwitz* *action* on r-tuples of monodromy elements is generated by transformations on adjacent pairs

$$(\sigma_1, \dots, \sigma_i, \sigma_{i+1}, \dots, \sigma_r) \mapsto \left(\sigma_1, \dots, \sigma_{i+1}, \sigma_{i+1}^{-1}\sigma_i\sigma_{i+1}, \dots, \sigma_r\right).$$

LEMMA 6.5. *Suppose that the covering $g\colon \mathbb{P}^1 \to \mathbb{P}^1$ has degree kh with $k > 1$ and I has a partition into parts $I_1, \dots, I_k$ each of cardinality h, such that*

(a) *all σ_i for $i > 2$ preserve all parts, and*
(b) *σ_1 and σ_2 permute the parts.*

Then

(i) *the covering map g is the composition of two factors $g = g_2 \circ g_1$, where*
(ii) *the second factor g_2 is a cyclic branched cover of degree k branched at the two branch points corresponding to $\sigma_1$and σ_2.*

Note that the claim and conclusion are insensitive to Hurwitz transformations, except that the indices may be relabelled.

Proof. Let us consider the topological cover over the complement of the branch points and an element in I_1. Its stabiliser in the fundamental group determines the covering. The setwise stabiliser of I_1 determines an intermediate cover g_2 which is of degree $k > 1$ over the base. By assumption, σ_i for $i > 2$ stabilises I_1, so g_2 is cyclically branched of degree k over the two points corresponding to σ_1 and σ_2. □

We now prove the factorisation of $j_{\mathcal{E}}$ into two factors, which we used in the proof of Lemma 6.4 in a more abstract setting.

PROPOSITION 6.6. *Suppose that $P_1, \dots, P_r$ are the partitions associated with the branch points of a branched covering $g\colon \mathbb{P}^1 \to \mathbb{P}^1$ of degree hk with $k > 1$. If all parts of P_1 and P_2 have length divisible by k, then*

(i) *the covering map g is the composition of two factors $g = g_2 \circ g_1$, and*
(ii) *the second factor g_2 is a cyclic branched cover of degree k branched at the two branch points corresponding to P_1 and P_2.*

Proof. We choose a base point and a geometric sequence of r simple loops around the r points making up the branch locus. Let $\sigma_1, \dots, \sigma_r$ be the permutations associated with these paths. We recall that the conjugacy class of σ_i is determined by P_i and does not depend on the chosen paths. The product of the σ_i is the identity. It suffices to show that there is a decomposition of the fibre I over the base point into k subsets of cardinality h, the *blocks*, such that the hypothesis of Lemma 6.5 is met.

We will rely on the following elementary observation. If σ is a permutation in S_n and τ the transposition of two elements a and b, then

either (1) a and b belong to the orbit of the same cycle c of σ of length ℓ, and
- (a) there exists a minimal $\ell_1 > 0$ such that $\sigma^{\ell_1}(a) = b$,
- (b) the permutations σ and $\sigma\tau$ have all cycles of σ except c in common,
- (c) the orbit of c is the union of orbits of two cycles of $\sigma\tau$ which are of lengths ℓ_1 and $\ell_2 = \ell - \ell_1$;

or (2) a and b belong to orbits of distinct cycles c_1 and c_2 of σ of lengths ℓ_1 and ℓ_2, and
- (a) the permutations σ and $\sigma\tau$ have all cycles of σ except c_1 and c_2 in common,

(b) the union of the orbits of c_1 and c_2 is the orbit of one cycle of $\sigma\tau$ which has length $\ell = \ell_1 + \ell_2$.

Without loss of generality, we may assume σ_1 and σ_2 to have h cycles of length k each, and all other σ_i for $i > 2$ to be transpositions. We call this the *generic case*.

In fact, in any other case, there are fewer monodromy elements. We can factor each of the first two elements into a permutation with cycles of length k only and a minimal number of transpositions. Any other element can be factored into a minimal number of transpositions. Then the sequence of factors – after a suitable reordering using Hurwitz transformations – is just a sequence of permutations as in the generic case. Thus is suffices to prove the claim in the generic case since the monodromy elements meet the hypotheses of Lemma 6.5 if the factors do.

Recall that the cycles of σ_i correspond bijectively to the points in the ith fibre. The Riemann Hurwitz formula, which relates the Euler number of the domain to the Euler number of the target, the degree and the fibre defects reads

$$2 = 2hk - 2(hk - h) - (r - 2) \qquad \Longrightarrow \qquad r = 2h\,.$$

Let us write our monodromy elements as $\sigma_1, \sigma_2, \tau_3, \ldots, \tau_{2h}$, where the τ_i are transpositions.

Next we exploit the transitivity of the generated group, which needs only the elements σ_2 and τ_i for $i > 2$ since the product equals the identity. The element σ_2 has h orbits, so it generates a transitive group only in case $h = 1$. For $h > 1$ there must be a sequence $\tau_{i_1}, \ldots, \tau_{i_{h-1}}$ with $i_1 < \cdots < i_{h-1}$ such that $\rho := \sigma_2\tau_{i_1}\cdots\tau_{i_{h-1}}$ is an hk-cycle. Using Hurwitz transformations, we may assume without loss of generality that $i_1 = 3, \ldots, i_{h-1} = h + 1$.

The element ρ^k has order h, hence I decomposes into k orbits of length h. It remains to show that this is the decomposition into blocks we need for Lemma 6.5.

Assume to the contrary that τ_i for some $2 < i \leqslant h + 1$ does not preserve the blocks. Using Hurwitz transformations, we may write

$$\sigma_2 = \rho\tau_{h+1}\cdots\tau_3 = \rho\tau_i\tau'_h\cdots\tau'_3\,. \tag{6.8}$$

The difference in the number of cycles for σ_2 and ρ is $h - 1$, so in each of the $h - 1$ compositions with a transposition on the right, we are in case (1), where the number of cycles goes up.

Then τ_i transposes two elements of the cycle of ρ. By observation (1.a), the permutation $\rho\tau_i$ has a cycle of length ℓ_1 which k does not divide since τ_i is assumed *not* to preserve the blocks. We remain in case (1) for all the following τ', so σ_2 also has a cycle of length not divisible by k, contrary to the hypothesis of the proposition.

Since blocks are permuted by the element ρ and preserved by the τ_i for $2 < i \leqslant h + 1$, the element σ_2 permutes the blocks.

Repeating the discussion for the remaining elements, we assume to the contrary that some τ_i with $i > h + 1$ does not preserve the blocks. We may then write

$$\sigma_1^{-1} = \sigma_2\tau_3\cdots\tau_{2h} = \rho\tau_{h+2}\cdots\tau_{2h} = \rho\tau_i\tau'_{h+3}\cdots\tau'_{2h}\,. \tag{6.9}$$

The argument applied above to (6.8) can now be applied here to conclude that σ_1^{-1} and thus σ_1 has a cycle of length not divisible by k, contrary to the hypothesis of the proposition. We conclude, too, that σ_1 permutes the blocks.

Therefore, we are in a position to apply Lemma 6.5, and this concludes the proof. □

7. Restrictions on possible modular monodromy groups

The results collected so far are sufficient to derive a first characterisation of the modular monodromy groups $\bar{\Gamma}$ which can occur for the elliptic surfaces $\mathcal{E}$ we consider.

Since the singular fibres of a generic element are all of the form I_k, I_k^* and IV^*, we can, using Table 1, write the Euler number formula in the following form:

$$24 = \sum_{I_k, I_k^*} k + \sum_{I_k^*} 6 + 8\#IV^* = \deg j_{\bar{\Gamma}} \cdot \deg j_{\mathcal{E}} + 6\#I^* + 8\#IV^* \,. \tag{7.1}$$

We will consider all numerically possible combinations of the four integers $\deg j_{\mathcal{E}} > 0$, $\#I^* \geqslant 0$, $\#IV^* \geqslant 0$ and $\deg j_{\bar{\Gamma}} > 0$, and we discard the trivial case $\deg j_{\bar{\Gamma}} = 1$ corresponding to $\bar{\Gamma} = \mathrm{PSL}(2, \mathbb{Z})$ and hence to $\Gamma = \mathrm{SL}(2, \mathbb{Z})$. We set up the corresponding table of combinations in two parts, namely *low* ($\leqslant 6$) and *high* (> 6) index $[\mathrm{PSL}(2, \mathbb{Z}) : \bar{\Gamma}]$, and we add two rows giving e_2 and e_3. Since $\#IV^* \leqslant 2$ in every column, they are determined by

(1) $e_2 < 2$ and $e_2 \equiv_2 \deg j_{\bar{\Gamma}}$, according to (6.7) and Lemma 6.4(iii);
(2) $e_3 < 3$ and $e_3 \equiv_3 \deg j_{\bar{\Gamma}}$, according to (6.7) and Lemma 6.4(v).

In a last row we mark columns which we *discard* from further consideration according to one of the following arguments:

(3) The equality $e_3 = 2$ implies $\#IV^* = 2$, according to Lemma 6.4(vi).
(4) If $\deg j_{\mathcal{E}} = 1$, then the j-invariant of $\mathcal{E}$ is rigid. Thus $\mathcal{E}$ is rigid except when there are I_0^*-fibres, which is obviously excluded for $\#I^* = 0$. We recall here that we are only concerned with positive-dimensional strata. But this is also excluded for $\#IV^* > 0$ since the presence of an I_0^*-fibre implies that $-\mathrm{id}$ is in the monodromy group, which in turn forbids the existence of a IV^*-fibre in $\mathcal{E}$ by Proposition 5.7.

$\deg j_{\bar{\Gamma}}$	2	2	2	2	2	2	2	2	2	3	3	3	3	4	4	4	4	4	5	6	6	6	6
$\deg j_{\mathcal{E}}$	1	2	3	4	5	6	8	9	12	2	4	6	8	1	2	3	4	6	2	1	2	3	4
$\#I^*$	1	2	3	0	1	2	0	1	0	3	2	1	0	2	0	2	0	0	1	3	2	1	0
$\#IV^*$	2	1	0	2	1	0	1	0	0	0	0	0	0	1	2	0	1	0	1	0	0	0	0
e_2	0	0	0	0	0	0	0	0	0	1	1	1	1	0	0	0	0	0	2	0	0	0	0
e_3	2	2	2	2	2	2	2	2	2	0	0	0	0	1	1	1	1	1	2	0	0	0	0
Discard	4	3	3		3	3	3	3	3					4					3				

TABLE 3. Combinations of numerical invariants (low index)

$\deg j_{\bar{\Gamma}}$	8	8	8	9	10	12	12	16	18	24
$\deg j_{\mathcal{E}}$	1	2	3	2	1	1	2	1	1	1
$\#I^*$	0	0	0	1	1	2	0	0	1	0
$\#IV^*$	2	1	0	0	1	0	0	1	0	0
e_2	0	0	0	1	0	0	0	0	0	0
e_3	2	2	2	0	1	0	0	1	0	0
Discard	4	3	3		4			4		4

TABLE 4. Combinations of numerical invariants (high index)

Note that $e_2 = e_3 = 0$ is equivalent to $\bar\Gamma$ being torsion-free and that $\deg j_{\bar\Gamma}$ is equal to the index of $\bar\Gamma$ in $\mathrm{PSL}(2,\mathbb{Z})$. Furthermore, subgroups of $\mathrm{PSL}(2,\mathbb{Z})$ of index at most 6 are congruence subgroups [Woh64, Theorem 5]. Accordingly, our groups of low index occur in [CP03, Table 2] with genus 0, the genus of the domain of $j_{\bar\Gamma}$, and with $(I = \deg j_{\bar\Gamma}, e_2, e_3)$ as in one of the columns in our Table 3. We find the entries $2A^0$, $2B^0$, $3B^0$, $2C^0$ and $4B^0$. By [CP03, Table 4], they usually go by the standard names $\bar\Gamma(1)^2$, $\bar\Gamma_1(2)$, $\bar\Gamma_1(3)$, $\bar\Gamma(2)$ and $\bar\Gamma_1(4)$ – which we recall after Proposition 7.1 – and we get the following result.

Proposition 7.1. *Let $\bar\Gamma$ be a subgroup of $\mathrm{PSL}(2,\mathbb{Z})$ which appears in one of the columns of Table 3 or 4 which have not been discarded. Then it belongs to one of the following, mutually exclusive cases:*

(i) *$\bar\Gamma$ has index 18 in $\mathrm{PSL}(2,\mathbb{Z})$ and is torsion-free.*

(ii) *$\bar\Gamma$ has index 12 in $\mathrm{PSL}(2,\mathbb{Z})$ and is torsion-free.*

(iii) *$\bar\Gamma$ has index 9 in $\mathrm{PSL}(2,\mathbb{Z})$ with $e_2 = 1$ and $e_3 = 0$.*

(iv) *$\bar\Gamma$ is either $\bar\Gamma(2)$ or $\bar\Gamma_1(4)$, has index 6 in $\mathrm{PSL}(2,\mathbb{Z})$ and is torsion-free.*

(v) *$\bar\Gamma$ is $\bar\Gamma_1(3)$, which has index 4 with $e_2 = 0$ and $e_3 = 1$.*

(vi) *$\bar\Gamma$ is $\bar\Gamma_1(2)$, which has index 3 with $e_2 = 1$ and $e_3 = 0$.*

(vii) *$\bar\Gamma$ is $\bar\Gamma(1)^2$, which has index 2 with $e_2 = 0$ and $e_3 = 2$.*

Here we recall the standard notation for certain congruence subgroups of $\mathrm{SL}(2,\mathbb{Z})$ which are of importance for us. The *principal congruence subgroup of level* n is denoted by $\Gamma(n)$ and defined as

$$\Gamma(n) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \,\middle|\, a \equiv d \equiv 1,\ b \equiv c \equiv 0 \mod n \right\}.$$

Its geometric relevance is the fact that the modular curve $X^0(n) = \mathbb{H}_1/\Gamma(n)$ parameterises elliptic curves with a level n structure, that is, a symplectic basis with respect to the Weil form, of the group $E[n]$ of n-torsion points. Next we recall the group

$$\Gamma_1(n) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \,\middle|\, a \equiv d \equiv 1,\ c \equiv 0 \mod n \right\},$$

whose meaning is that the modular curve $X_1^0(n) = \mathbb{H}_1/\Gamma_1(n)$ parameterises elliptic curves with a fixed point of order n. We will also use the group

$$\Gamma_0(n) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \,\middle|\, c \equiv 0 \mod n \right\}.$$

Its significance is that the modular curve $X_0^0(n) = \mathbb{H}_1/\Gamma_0(n)$ is the moduli space of elliptic curves with a distinguished subgroup $\mathbb{Z}/n\mathbb{Z} \subset E[n]$ of n-torsion points. By $\bar\Gamma(n)$, $\bar\Gamma_1(n)$ and $\bar\Gamma_0(n)$ we denote the images of these groups in $\mathrm{PSL}(2,\mathbb{Z})$. Finally, we recall that $\bar\Gamma(1)^2$ is the subgroup of $\mathrm{PSL}(2,\mathbb{Z})$ which is generated by all squares.

Remark 7.2. We note here that $\bar\Gamma_0(n) = \bar\Gamma_1(n)$ for $n \leqslant 4$. This will be relevant for Proposition 9.3.

The data for the groups $\bar\Gamma$ in Proposition 7.1 can be further exploited to obtain upper bounds for the dimension of any corresponding monodromy stratum. For this purpose, we make the following definition.

DEFINITION 7.3. Let Γ be a subgroup of finite index of $\mathrm{SL}(2,\mathbb{Z})$. We define the *maximal dimension of a monodromy stratum associated with* Γ by

$$m(\Gamma) = \begin{cases} \max\{\dim \mathcal{F}_{\Gamma,i} \mid \mathcal{F}_{\Gamma,i} \text{ is a monodromy stratum associated with } \Gamma\}, \\ -\infty \text{ if there exists no monodromy stratum with monodromy group } \Gamma\,. \end{cases}$$

Together with the explicit bounds in the upcoming Lemma 7.4, this will be used later in the following way:

$$\begin{aligned} &\text{The closure of a Shimada stratum of dimension } d \text{ is ambi-typical} \\ &\text{if } d \geqslant m(\Gamma) \text{ for its generic monodromy } \Gamma\,. \end{aligned} \tag{7.2}$$

LEMMA 7.4. *Let $\bar\Gamma$ be a subgroup of* $\mathrm{PSL}(2,\mathbb{Z})$ *which appears in one of the columns of Table 3 or 4 which have not been discarded, and let Γ be any lift of $\bar\Gamma$ in* $\mathrm{SL}(2,\mathbb{Z})$*. The invariants are listed in Table 5.*

$\deg j_{\bar\Gamma}$	2	3	4	9	6	12	18
e_2	0	1	0	1	0	0	0
e_3	2	0	1	0	0	0	0
# poles of $j_{\bar\Gamma}$	1	2	2	3	3	4	5
$m(\Gamma)$	$\leqslant 6$	$\leqslant 10$	$\leqslant 6$	$\leqslant 2$	$\leqslant 6$	$\leqslant 2$	$\leqslant 1$

TABLE 5. Maximal dimension of Γ-strata

Proof. The first row in the table simply lists the possible degrees for $j_{\bar\Gamma}$ which occur in Table 3 or 4. The second and third row are also copied from these tables. The number of poles of $j_{\bar\Gamma}$ can be computed via the Riemann–Hurwitz formula and equals $2 + \deg j_{\bar\Gamma}/6 - 2e_3/3 - e_2/2$. Thus it remains to prove the upper bound for the dimension of the monodromy strata.

The dimension of a stratum is equal to the open dense irreducible subset of elliptic surfaces sharing the generic configuration of singular fibres. In Lemma 5.3, the dimension augmented by 6 is given as the sum of cardinality of singular fibres and the cardinality of $*$-fibres. Thus

$$6 + \dim \leqslant s_\infty + s_0 + s_1 + s^* + s^*\,, \tag{7.3}$$

where s_∞, s_0, s_1 are the numbers of singular fibres with $j = \infty, 0, 1$, respectively, and s^* is the number of $*$-fibres which is also an upper bound for the number of singular fibres with $j \notin \{0, 1, \infty\}$.

On the other hand, the Euler sum gives the bound

$$24 \geqslant \deg j_{\mathcal{E}} \deg j_{\bar\Gamma} + 4s_0 + 3s_1 + 6s^*\,, \tag{7.4}$$

as follows immediately from Table 1.

Indeed, there are further restrictions on s_∞, s_0, s_1 due to the map $j_{\bar\Gamma}$:

(1) The integer s_∞ is the number of poles of the j-invariant and is thus bounded above by $\deg j_{\mathcal{E}}$ times the number of poles of $j_{\bar\Gamma}$.
(2) The integer s_0 is the number of points with local j-expansion s^k for k not a multiple of 3, so $s_0 = 0$ if $e_3 = 0$.
(3) The integer s_1 is the number of points with local j-expansion $1 + s^k$ for k odd, so $s_1 = 0$ if $e_2 = 0$.

We shall now give the proof exemplary for the case of index 9. The other cases can be argued similarly. From (7.4) and $s_0 \geqslant 0$, we find

$$24 \geqslant \deg j_{\bar\Gamma} \deg j_{\mathcal{E}} + 3s_1 + 6s^* .$$

Since $j_{\bar\Gamma}$ has degree 9 and 3 poles, this implies

$$24 \geqslant 9\frac{s_\infty}{3} + 3s_1 + 6s^*$$

or, equivalently,

$$8 \geqslant s_\infty + s_1 + 2s^* .$$

Since $e_3 = 0$ in this case, and hence $s_0 = 0$, it then follows immediately from (7.3) that $m(\Gamma) \leqslant 2$. □

In the case of *low index*, we have ample information on the two factors in the factorisation $j(\mathcal{E}) = j_{\bar\Gamma} \circ j_{\mathcal{E}}$. Firstly, since we know the groups $\bar\Gamma$ explicitly, we can in fact also write down the maps $j_{\bar\Gamma}$ explicitly, as listed in Table 6.

$$\begin{aligned}
\bar\Gamma(2):&\quad j_{\bar\Gamma} = \frac{\left(z^2+3\right)^3}{z^2\left(z^2-9\right)^2} = 1 + \frac{27\left(z^2-1\right)^2}{z^2\left(z^2-9\right)^2}\\
\bar\Gamma_1(4):&\quad j_{\bar\Gamma} = \frac{4\left(z^2-4z+1\right)^3}{27z(z-4)} = 1 + \frac{(z-2)^2(2z^2-8z-1)^2}{27z(z-4)}\\
\bar\Gamma_1(3):&\quad j_{\bar\Gamma} = \frac{z(z+8)^3}{64(z-1)^3} = 1 + \frac{\left(z^2-20z-8\right)^2}{64(z-1)^3}\\
\bar\Gamma_1(2):&\quad j_{\bar\Gamma} = \frac{(z+3)^3}{27(z-1)^2} = 1 + \frac{z(z-9)^2}{27(z-1)^2}\\
\bar\Gamma(1)^2:&\quad j_{\bar\Gamma} = \frac{4z}{(z+1)^2} = 1 - \frac{(z-1)^2}{(z+1)^2}
\end{aligned}$$

Table 6. j-functions

There are various different explicit formulae in the literature, for example [FK16, MS01], since coordinates on the domain can be chosen arbitrarily. Indeed, to check our formulae, it suffices to check their degrees and their branching over 0, 1 and ∞, which are determined by the multiplicity sequences of the two numerators and the denominator, respectively.

With our choice of z-coordinate, we note that

row 1: $z = 0, \pm 3$ are the poles of $j_{\bar\Gamma}$, of pole order 2 each; (7.5)

row 2: $z = 0, 4, \infty$ are the poles of $j_{\bar\Gamma}$, of pole order 1, 1, 4, respectively; (7.6)

row 3: $e_2 = 0$ and $z = 0$ is the only 3-torsion point (non-critical point of value 0); (7.7)

row 4: $e_3 = 0$ and $z = 0$ is the only 2-torsion point (non-critical point of value 1); (7.8)

row 5: $e_2 = 0$ and $z = 0, \infty$ are the 3-torsion points mapping to 0. (7.9)

Secondly, information from Table 3 also allows us to obtain information about the map $j_{\mathcal{E}}$. In particular, we can list the possible degrees of this map as well as the branching behaviour over *special* points, namely the 3-torsion and 2-torsion points; see Lemma 6.4. Using the notation of

$\bar\Gamma$	deg $j_{\mathcal{E}}$	Special point(s)	Branch data	$j_{\mathcal{E}}$
$\bar\Gamma(2), \bar\Gamma_1(4)$	4	–	–	$\alpha_4 : \beta_4$
	3	–	–	$\alpha_3 : \beta_3$
	2	–	–	$\alpha_2 : \beta_2$
	1	–	–	$\alpha_1 : \beta_1$
$\bar\Gamma_1(3)$	6	$A = 0$	$(3,3)_A$	$\alpha_2^3 : \beta_6$
	4	$A = 0$	$(3,1)_A$	$\alpha_1^3\gamma_1 : \beta_4$
	3	$A = 0$	$(3)_A$	$\alpha_1^3 : \beta_3$
	2	$A = 0$	$(1,1)_A$	$\gamma_2 : \beta_2$
$\bar\Gamma_1(2)$	8	$B = 0$	$(2,2,2,2)_B$	$\alpha_4^2 : \beta_8$
	6	$B = 0$	$(2,2,2)_B$	$\alpha_3^2 : \beta_6$
	4	$B = 0$	$(2,2)_B$	$\alpha_2^2 : \beta_4$
	2	$B = 0$	$(2)_B$	$\alpha_1^2 : \beta_2$
$\bar\Gamma(1)^2$	4	$A_1 = 0, A_2 = \infty$	$(3,1)_{A_1}, (3,1)_{A_2}$	$\alpha_1^3\gamma_1 : \beta_1^3\delta_1$

TABLE 7. $j_{\mathcal{E}}$-functions for low index

[BPT02, Section 1], we will denote these by A and B, respectively. So we can derive the number of pre-images of special points and their multiplicities. They are included in Table 7 as the tuple of multiplicities with index the point they map to. The last column of this table gives the most general polynomial expression for $j_{\mathcal{E}}$ fitting the branching data. Here α_i, β_i, γ_i, δ_i denote coprime homogeneous bivariate polynomials of degree i.

8. Families of Weierstraß data for low index

In this section, we will analyse the Weierstraß data of the Jacobian fibrations with modular monodromy of low index. We will also discuss the Mordell–Weil group in these cases.

We start with the following table of Weierstraß data, whose relevance is that this includes all the Weierstraß data needed to describe the families from Section 7 of low monodromy index.

Note that in this table we still assume the polynomials to have the degree given by their index, but *no* assumption of coprimality is imposed.

#	$\bar\Gamma$	g_2	g_3	$\Delta = g_2^3 - 27g_3^2$
i)	$\bar\Gamma(2)$	$\alpha_4^2 + 3\beta_4^2$	$\beta_4(\alpha_4^2 - \beta_4^2)$	$\alpha_4^2(\alpha_4^2 - 9\beta_4^2)^2$
ii)	$\bar\Gamma_1(4)$	$12(\alpha_4^2 - 4\alpha_4\beta_4 + \beta_4^2)$	$4(\alpha_4 - 2\beta_4)(2\alpha_4^2 - 8\alpha_4\beta_4 - \beta_4^2)$	$2^4 3^6 \alpha_4(\alpha_4 - 4\beta_4)\beta_4^4$
iii)	$\bar\Gamma_1(3)$	$3\alpha_2(\alpha_2^3 + 8\beta_6)$	$\alpha_2^6 - 20\alpha_2^3\beta_6 - 8\beta_6^2$	$2^6 3^3 (\alpha_2^3 - \beta_6)^3 \beta_6$
iv)	$\bar\Gamma_1(3)$	$3\alpha_1\gamma_2^2(\alpha_1^3 + 8\beta_3)$	$\gamma_2^3(\alpha_1^6 - 20\alpha_1^3\beta_3 - 8\beta_3^2)$	$2^6 3^3 (\alpha_1^3 - \beta_3)^3 \beta_3\gamma_2^6$
v)	$\bar\Gamma_1(2)$	$3\alpha_4^2 + 9\beta_8$	$\alpha_4(\alpha_4^2 - 9\beta_8)$	$3^6(\alpha_4^2 - \beta_8)^2\beta_8$
vi)	$\bar\Gamma(1)^2$	$-12\alpha_1\beta_1\gamma_1^3\delta_1^3$	$4\gamma_1^4\delta_1^4(\alpha_1^3\gamma_1 - \beta_1^3\delta_1)$	$-2^4 3^3 (\alpha_1^3\gamma_1 + \beta_1^3\delta_1)^2\gamma_1^8\delta_1^8$

TABLE 8. Weierstraß families

Proposition 8.1. *Every elliptic surface with data as in Table 7 has Weierstraß datum in one of the families in Table 8. The correspondence between the monodromy group and the Weierstraß data is given by*

$$\begin{aligned} &\text{i)}:\quad \bar\Gamma(2)\,, &&\text{iii)}:\quad \bar\Gamma_1(3)\,,\ \deg j_{\mathcal{E}} \equiv_2 0\,, &&\text{v)}:\quad \bar\Gamma_1(2)\,,\\ &\text{ii)}:\quad \bar\Gamma_1(4)\,, &&\text{iv)}:\quad \bar\Gamma_1(3)\,,\ \deg j_{\mathcal{E}} = 3\,, &&\text{vi)}:\quad \bar\Gamma(1)^2\,. \end{aligned}$$

Proof. We have to show that for each row of Table 7, any elliptic surface $\mathcal{E}$ with the data provided by the row can be obtained by some suitable choice of Weierstraß datum from Table 8. Here we shall give the proof for $\mathcal{E}$ with modular monodromy $\bar\Gamma_1(3)$, which is the most subtle case. The other groups can be treated in an analogous way. We begin with the first corresponding row, so $\deg j_{\mathcal{E}} = 6$.

Composing the expressions for $j_{\bar\Gamma}$ and $j_{\mathcal{E}}$ from Tables 6 and 7, respectively, we get

$$j(\mathcal{E}) = j_{\bar\Gamma}\circ j_{\mathcal{E}} = \frac{\alpha_2^3\big(\alpha_2^3+8\beta_6\big)^3}{64\big(\alpha_2^3-\beta_6\big)^3\beta_6} = 1 + \frac{\big(\alpha_2^6-20\alpha_2^3\beta_6-8\beta_6^2\big)^2}{64\big(\alpha_2^3-\beta_6\big)^3\beta_6}\,.$$

We look at the general expression of the j-function in terms of Weierstraß data:

$$j = \frac{g_2^3}{g_2^3-27g_3^2} = \frac{(g_2/3)^3}{(g_2/3)^3-g_3^2} = 1 + \frac{g_3^2}{(g_2/3)^3-g_3^2}\,.$$

If we plug in the same α_2 and β_6, we get the identical expression for the j-invariant as above. Moreover, the analysis of common factors of g_2 and g_3 and their multiplicities according to the Tate algorithm, see Table 2, gives no singular fibres except of type I_ν since α_2 and β_6 are coprime by the assumption $\deg j_{\mathcal{E}} = 6$. Therefore, $\mathcal{E}$ and the elliptic surface given by the Weierstraß datum share the functional and the homological invariant and hence are isomorphic.

Still in case $\bar\Gamma_1(3)$ but with $\deg j_{\mathcal{E}} = 4$, we obtain from Tables 6 and 7 that

$$j(\mathcal{E}) = j_{\bar\Gamma}\circ j_{\mathcal{E}} = \frac{\alpha_1^3\gamma_1\big(\alpha_1^3\gamma_1+8\beta_4\big)^3}{64\big(\alpha_1^3\gamma_1-\beta_4\big)^3\beta_4} = 1 + \frac{\big(\alpha_1^6\gamma_1^2-20\alpha_1^3\gamma_1\beta_4-8\beta_4^2\big)^2}{64\big(\alpha_1^3\gamma_1-\beta_4\big)^3\beta_4}\,.$$

This expression cannot be obtained as easily. Indeed, we have to recall that we are allowed to plug in polynomials into the families which are *not* necessarily coprime. Doing this with $\beta_6 = \gamma_1^2\beta_4$ and $\alpha_2 = \alpha_1\gamma_1$ in family iii), we get

$$j = \frac{\alpha_1^3\gamma_1^3\big(\alpha_1^3\gamma_1^3+8\beta_4\gamma_1^2\big)^3}{64\big(\alpha_1^3\gamma_1^3-\beta_4\gamma_1^2\big)^3\beta_4\gamma_1^2} = 1 + \frac{\big(\alpha_1^6\gamma_1^6-20\alpha_1^3\gamma_1^5\beta_4-8\gamma_1^4\beta_4^2\big)^2}{64\big(\alpha_1^3\gamma_1^3-\beta_4\gamma_1^2\big)^3\beta_4\gamma_1^2}\,,$$

which is exactly the expression for $j(\mathcal{E})$ expanded by γ_1^8. The Weierstraß datum is thus

$$g_2 = 3\alpha_1\gamma_1^3\big(\alpha_1^3\gamma_1+8\beta_4\big)\,,\quad g_3 = \gamma_1^4\big(\alpha_1^6\gamma_1^2-20\alpha_1^3\gamma_1\beta_4-8\beta_4^2\big)\,.$$

The analysis with the Tate algorithm shows the existence of one IV^*-fibre and otherwise only singular fibres of type I_ν since $\alpha_1\gamma_1$ and β_4 are coprime, and we may conclude again that $\mathcal{E}$ is isomorphic to a surface given by Weierstraß datum from family iii).

The case with $\bar\Gamma_1(3)$ and $\deg j_{\mathcal{E}} = 2$ is very similar but with $\beta_6 = \gamma_2^2\beta_2$ and $\alpha_2 = \gamma_2$ sharing even a factor of degree 2. The Weierstraß datum

$$g_2 = 3\gamma_2^3(\gamma_2+8\beta_2)\,,\quad g_3 = \gamma_2^4\big(\gamma_2^2-20\gamma_2\beta_2-8\beta_2^2\big)$$

then defines an elliptic surface sharing the functional and homological invariant with $\mathcal{E}$ again.

This leaves us with $\bar\Gamma_1(3)$ and $\deg j_{\mathcal{E}} = 3$. From Tables 6 and 7, we get

$$j(\mathcal{E}) = j_{\bar\Gamma} \circ j_{\mathcal{E}} = \frac{\alpha_1^3\big(\alpha_1^3 + 8\beta_3\big)^3}{64\big(\alpha_1^3 - \beta_3\big)^3\beta_3} = 1 + \frac{\big(\alpha_1^6 - 20\alpha_1^3\beta_3 - 8\beta_3^2\big)^2}{64\big(\alpha_1^3 - \beta_3\big)^3\beta_3}\,.$$

This time, we choose to plug the coprime α_1 and β_3 from an expression for $j(\mathcal{E})$ together with a still to be determined γ_2 into the family iv) and find the j-function of this Weierstraß datum to be

$$j = \frac{\alpha_1^3\gamma_2^6\big(\alpha_1^3 + 8\beta_3\big)^3}{64\big(\alpha_1^3 - \beta_3\big)^3\gamma_2^6\beta_3} = 1 + \frac{\gamma_2^6\big(\alpha_1^6 - 20\alpha_1^3\beta_3 - 8\beta_3^2\big)^2}{64\big(\alpha_1^3 - \beta_3\big)^3\gamma_2^6\beta_3}\,,$$

which is exactly the expression for $j(\mathcal{E})$ expanded by γ_2^6. Thus we get the j-invariant $j(\mathcal{E})$ with the Weierstraß datum

$$g_2 = 3\alpha_1\gamma_2^2\big(\alpha_1^3 + 8\beta_3\big)\,, \quad g_3 = \gamma_2^3\big(\alpha_1^6 - 20\alpha_1^3\beta_3 - 8\beta_3^2\big)\,.$$

Finally, we choose γ_2 to vanish at the two I_0^*-fibres of $\mathcal{E}$. Then γ_2 is coprime to $\beta_3\big(\alpha_1^3 - \beta_3\big)$ since the j-invariant of an I_0^*-fibre is finite.

The analysis of this datum with the Tate algorithm shows the existence of I_0^*-fibres precisely at the zeros of γ_2. Again we may conclude since the functional and homological invariant are shown to coincide. □

We next determine for the generic members of each family whether $-\mathrm{id} \in \Gamma$ or not. The following lemma will be used as a tool to determine the Mordell–Weil torsion from the monodromy group.

In Section 7 we already introduced the principal congruence subgroup $\Gamma(n)$. As we said there, the modular curve $X^0(n) = \mathbb{H}_1/\Gamma(n)$ is the classifying space of elliptic curves with a level n structure. This carries a universal family if $n \geqslant 3$. If $n = 2$, we no longer have a universal family of elliptic curves, but a universal Kummer family still exists. We denote by $X(n)$ the compactification of $X^0(n)$ which is obtained by adding the cusps; that is, $X(n) = \overline{\mathbb{H}}_1/\Gamma(n)$. The universal family over $X^0(n)$ can be extended to $X(n)$; the extension is known as a Shioda modular surface. This has n^2 sections which restrict to the n-torsion points on the smooth fibres of the universal family.

We had also introduced the group $\Gamma_1(n)$ and the curve $X_1^0(n) = \mathbb{H}_1/\Gamma_1(n)$. This is the classifying space of elliptic curves with a fixed n-torsion point. As above, we can compactify the curve $X_1^0(n)$ by adding the cusps and obtain a curve $X_1(n)$.

If m divides n, we consider the group

$$\Gamma_m(n) := \Gamma(m) \cap \Gamma_1(n)\,.$$

Obviously, $X_m^0(n) := \mathbb{H}_1/\Gamma_m(n)$ parameterises elliptic curves with a level m structure and additionally an n-torsion point. Again by adding the cusps, we obtain the compactification $X_m(n)$. The universal family over $X_m^0(n)$ extends to $X_m(n)$, and in addition to the sections giving the m-torsion points, we have a distinguished section of order n which restricts to the distinguished n-torsion point on the smooth fibres.

Lemma 8.2. *Let $\mathcal{E}$ be a Jacobian fibration with monodromy group Γ, and assume that m and n are positive integers with $m \mid n$. Then the following are equivalent:*

(i) *Up to conjugation, Γ is contained in $\Gamma_m(n)$.*

(ii) *The Mordell–Weil group $\mathrm{MW}(\mathcal{E})$ contains a subgroup $\mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}/m\mathbb{Z}$.*

Proof. We shall first prove the implication from (i) to (ii). For this let U be the subset of the base $\mathbb{P}^1$ of $\mathcal{E}$ which is given by removing the points $j(\mathcal{E})^{-1}\{0,1,\infty\}$ and the base points of the singular fibres. We want to construct sections which form a subgroup $\mathbb{Z}/n\mathbb{Z}\times\mathbb{Z}/m\mathbb{Z}$ of $\mathrm{MW}(\mathcal{E})$. It is enough to do this over U as the sections can then be extended to $\mathbb{P}^1$ (since the base has dimension 1). We choose a base point $x_0\in U$. For each point $x\in U$ we can choose a small disc $U(x)$ such that $\mathcal{E}|_{U(x)}\cong\mathbb{C}\times U(x)/(\mathbb{Z}+\mathbb{Z}\tau_x)$, where $\tau_x\colon U(x)\to\mathbb{H}_1$ is a local lift of the j-function $j(\mathcal{E})$ and $\mathbb{Z}+\mathbb{Z}\tau_x$ acts fibrewise on $\mathbb{C}\times U(x)$ by translation on $\mathbb{C}\times\{t\}$ with the lattice $\mathbb{Z}+\mathbb{Z}\tau_x(t)$.

In particular, the fibre $\mathcal{E}_{x_0}$ is isomorphic to $\mathbb{C}/(\mathbb{Z}+\tau_{x_0}(x_0))$, where we have chosen a fixed lift τ_{x_0}. Let $a,b\in\mathbb{Z}$ and $z_0=(a+b\tau_{x_0}(x_0))/\ell$ be an ℓ-torsion point in $\mathcal{E}_{x_0}$ (where ℓ will become either m or n). Using the above local uniformisation of $\mathcal{E}$, we can extend this to a local ℓ-torsion section of $\mathcal{E}|_{U(x_0)}$. We want to extend such a section to U. Given any point $y\in U$ we can choose a path $s\colon[0,1]\to U$ connecting x_0 with y. We can cover this path with finitely many open sets $U(x_i)$ for $i=0,\dots,N$, with $x_N=y$ and choose local lifts τ_{x_i} with $\tau_{x_i}|U_i\cap U_{i+1}=\tau_{x_{i+1}}|U_i\cap U_{i+1}$. Then we can move the point z_0 along the path s to an ℓ-torsion point $z_0(y)\in\mathcal{E}_y$. Clearly, this will a priori depend on the chosen path s. The point $z_0(y)$ will be independent of this choice if all elements in the monodromy group $\Gamma(\mathcal{E})$ fix the point $(a+b)/\ell\in\mathbb{Z}/\ell+\mathbb{Z}/\ell$ (where we have chosen a fixed representation $\Gamma(\mathcal{E})\to\mathrm{SL}(2,\mathbb{Z})$ and hence consider $\Gamma(\mathcal{E})$ as a subgroup of $\mathrm{SL}(2,\mathbb{Z})$). Given this observation, the claim now follows immediately from the definition of the group $\Gamma_m(n)$. The converse implication from (ii) to (i) follows by the same argument. □

As a consequence of the above discussion, we can now obtain some first results on the Mordell–Weil torsion.

Lemma 8.3. *Let $\mathcal{E}$ be a member of family iii). Then the following hold:*

(i) $-\mathrm{id}\notin\Gamma(\mathcal{E})$.

(ii) $\mathbb{Z}/3\mathbb{Z}\subset\mathrm{MW}(\mathcal{E})$.

Moreover, for a generic element $\mathcal{E}$ the equality $\mathrm{MW}(\mathcal{E})=\mathbb{Z}/3\mathbb{Z}$ holds.

Proof. Consider the following Weierstraß datum of a rational elliptic surface $\bar{\mathcal{E}}$ in homogeneous coordinates z_1 and z_0:

$$\bar{g}_2=3z_1^3(z_1+8z_0)\,,\quad \bar{g}_3=z_1^4\big(z_1^2-20z_1z_0-8z_0^2\big)\,.$$

A generic member $\mathcal{E}$ of family iii) is given by coprime β_6 and α_2. To compute the datum of the pull-back of $\bar{\mathcal{E}}$ along $j_{\mathcal{E}}$, we plug α_2^3 and β_6 into z_1 and z_0, respectively, resulting in

$$g_2'=3\alpha_2^9\big(\alpha_2^3+8\beta_6\big)\,,\quad g_3'=\alpha_2^{12}\big(\alpha_2^6-20\alpha_2^3\beta_6-8\beta_6^2\big)\,.$$

The proper Weierstraß datum of the normalised and possibly blown-down pull-back is then

$$g_2=3\alpha_2\big(\alpha_2^3+8\beta_6\big)\,,\quad g_3=\alpha_2^6-20\alpha_2^3\beta_6-8\beta_6^2\,,$$

which is exactly that of $\mathcal{E}$. By the Tate algorithm, we find that the fibre configuration on the rational elliptic surface $\bar{\mathcal{E}}$ is IV^*, I_1 or I_3.

Due to the singular fibres, the trivial lattice is E_6+A_2, which implies that the surface is number 69 in the list of Oguiso–Shioda [OS91] and thus has Mordell–Weil torsion $\mathbb{Z}/3$. In particular, $-\mathrm{id}$ does not belong to the monodromy.

Since torsion sections are pulled back to torsion sections, the Mordell–Weil torsion of $\mathcal{E}$ also contains $\mathbb{Z}/3$. Moreover, the monodromy group of $\mathcal{E}$ is generated by the monodromy elements of the rational elliptic surface $\bar{\mathcal{E}}$ associated with loops liftable along $j_{\mathcal{E}}$. So the monodromy of $\mathcal{E}$ is contained in that of the rational elliptic surface $\bar{\mathcal{E}}$ and hence does not contain $-\operatorname{id}$.

Conversely, the generic $\mathcal{E}$ does not have Mordell–Weil torsion properly containing $\mathbb{Z}/3$. Otherwise $j(\mathcal{E})$ factors through $j_{\bar{\Gamma}}$ for some $\bar{\Gamma}$ properly contained in $\bar{\Gamma}_1(3)$. This contradicts $j_{\bar{\Gamma}} \circ j_{\mathcal{E}}$ being the canonical factorisation.

In the above arguments, we have used that $\mathcal{E}$ is a generic element of the family. It thus remains to prove that $-\operatorname{id}$ is never contained in the monodromy group and $\mathbb{Z}/3$ is always contained in the Mordell–Weil torsion. But both follow by semi-continuity since the monodromy group can only get smaller under specialisation and the Mordell–Weil torsion can only get bigger. □

LEMMA 8.4. *Let $\mathcal{E}$ be a member of family ii). Then the following hold:*

(i) *If α_4 or β_4 is a perfect square, then $-\operatorname{id} \notin \Gamma(\mathcal{E})$.*

(ii) *In the two cases, the generic element has $\operatorname{MW}(\mathcal{E}) = \mathbb{Z}/2\mathbb{Z}$ and $\operatorname{MW}(\mathcal{E}) = \mathbb{Z}/4\mathbb{Z}$, respectively.*

Proof. In both cases, the elliptic fibration is a pull-back along $j_{\mathcal{E}}$ of degree 4 of a rational elliptic fibration whose fibre configuration is one of

$$I_4^* + 2\ I_1\,, \quad I_1^* + I_4 + I_1$$

and with two simple ramification points over the $*$-fibre. Again, it suffices to investigate the rational elliptic fibrations. The monodromy is freely generated by two elements, so $-\operatorname{id}$ does not belong to the monodromy since it is torsion.

The surfaces occur as numbers 64 and 72 in the list of Oguiso–Shioda [OS91] and have Mordell–Weil torsion $\mathbb{Z}/2$ and $\mathbb{Z}/4$, respectively. So via the pull-back, these groups are contained in the Mordell–Weil torsion. Equality holds for all $j_{\mathcal{E}}$ such that $j_{\bar{\Gamma}} \circ j_{\mathcal{E}}$ is the canonical factorisation, hence generically. □

LEMMA 8.5. *If $\mathcal{E}$ is a generic member of either family i) or ii), then $-\operatorname{id} \in \Gamma(\mathcal{E})$.*

Proof. Here we use the semi-continuity argument for the monodromy in the other direction. Since the group can only get smaller under specialisation, the generic monodromy for the family contains $-\operatorname{id}$ if it does so for some member.

If $\alpha_4 = \alpha_3\gamma_1$ and $\beta_4 = \beta_3\gamma_1$ with α_3, β_3, γ_1 pairwise coprime, then the Tate algorithm shows that for either family we get an I_0^*-fibre corresponding to γ_1. Hence $-\operatorname{id}$ is in the monodromy of such a special member. □

9. Classification for $j_{\bar{\Gamma}}$ of low degree

We are now ready to classify the ambi-typical strata with $\bar{\Gamma}$ of index at most 6. At the same time we determine the corresponding root lattices and Mordell–Weil torsion.

PROPOSITION 9.1. *There is a unique ambi-typical stratum with modular monodromy $\bar{\Gamma}_1(2)$. Its monodromy group is $\Gamma_1(2)$. A generic element of this stratum has the following root lattice and Mordell–Weil torsion:*

$$8A_1\,,\ \mathbb{Z}/2\mathbb{Z}\,.$$

Proof. Since $-\operatorname{id}$ is the only 2-torsion element in $\operatorname{SL}(2,\mathbb{Z})$, there is no splitting map to $\operatorname{SL}(2,\mathbb{Z})$ from any subgroup of $\operatorname{PSL}(2,\mathbb{Z})$ containing 2-torsion such as $\bar{\Gamma}_1(2)$. In particular, every $\bar{\Gamma}_1(2)$ monodromy stratum is actually a $\Gamma_1(2)$-monodromy stratum.

With our explicit knowledge of $j_{\bar{\Gamma}}$ and $j_{\mathcal{E}}$ for the family v), we can use the Tate algorithm and find that the generic fibre configuration is $8I_1 + 8I_2$. By Lemma 5.3, we find that the dimension of the configuration locus is $\dim L(8I_1 + 8I_2) = 10$. The corresponding generic root lattice is $8A_1$. The Shimada stratum with this root lattice and saturation given by the generic element of family v) also has dimension 10. This stratum and family v) therefore determine the same irreducible closed subset of $\mathcal{F}'$. It is ambi-typical by the argument of (7.2) since $\Gamma_1(2)$ belongs to the second column in Table 5 and thus $m(\Gamma_1(2)) \leqslant 10$ by Lemma 7.4.

Any other Shimada stratum with $\bar{\Gamma}_1(2)$ monodromy corresponds by Proposition 8.1 to a stratum in family v) and must be of dimension less than 10. But its monodromy group must still be $\Gamma_1(2)$, so it belongs to the monodromy stratum above, see (3.2), and cannot be a monodromy stratum of its own.

By Lemma 8.2, the Mordell–Weil torsion always contains $\mathbb{Z}/2\mathbb{Z}$, and for a generic element this is equal to $\mathbb{Z}/2\mathbb{Z}$. The proof is analogous to that in the proof of Lemma 8.3. □

Proposition 9.2. *There is a unique ambi-typical stratum with modular monodromy $\bar{\Gamma}(2)$ and monodromy group $\Gamma(2)$. A generic element of this stratum has the following root lattice and Mordell–Weil torsion:*

$$12A_1\,,\ \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}\,.$$

Proof. The group $\Gamma(2)$ is the pre-image of $\bar{\Gamma}(2)$ in $\mathrm{SL}(2,\mathbb{Z})$ and contains $-\mathrm{id}$, thus by Lemma 8.5 it is the monodromy group of the generic surface in family i). We can now argue very much as in the proof of Proposition 9.1. The generic fibre configuration is $12I_2$, which gives a generic root lattice $12A_1$, so both the corresponding strata are of dimension 6.

On the other hand, $\Gamma(2)$ belongs to column 5 in Table 5. Hence $m(\Gamma(2)) \leqslant 6$ by Lemma 7.4, and the irreducible closed subset of $\mathcal{F}'$ corresponding to family i) is ambi-typical by (7.2).

Any other Shimada stratum with $\Gamma(2)$ monodromy corresponds to a stratum in family i) by Proposition 8.1 and must be of dimension less than 6. Due to (3.2), it belongs to the monodromy stratum above and cannot be a monodromy stratum of its own.

The claim about the Mordell–Weil torsion again follows as in the previous proof. □

Proposition 9.3. *There is a unique ambi-typical stratum with modular monodromy $\bar{\Gamma}_1(4)$ and monodromy group $\Gamma_0(4)$. A generic element of this stratum has the following root lattice and Mordell–Weil torsion:*

$$4A_3\,,\ \mathbb{Z}/2\mathbb{Z}\,.$$

Proof. The group $\Gamma_0(4)$ is the pre-image in $\mathrm{SL}(2,\mathbb{Z})$ of $\bar{\Gamma}_1(4) = \bar{\Gamma}_0(4)$ (see Remark 7.2), thus contains $-\mathrm{id}$ and – by Lemma 8.5 – is the monodromy group of the generic surface in family ii). The argument then is as above, only applied to the family ii). □

To complete the classification of monodromy strata for the two torsion-free subgroups of $\mathrm{PSL}(2,\mathbb{Z})$, we have to exploit the fact that for fibrations with monodromy Γ not containing $-\mathrm{id}$, the quotient map $\mathrm{SL}(2,\mathbb{Z}) \to \mathrm{PSL}(2,\mathbb{Z})$ defines an isomorphism $\Gamma \cong \bar{\Gamma}$.

Proposition 9.4. *There is a unique ambi-typical stratum with modular monodromy $\bar{\Gamma}(2)$ and monodromy group Γ such that $\Gamma \to \bar{\Gamma}(2)$ is an isomorphism. The generic root lattice and Mordell–Weil torsion are*

$$2A_3 + 8A_1\,,\ \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}\,.$$

Remark 9.5. The group Γ in the proposition can be described as

$$\Gamma = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \,\middle|\, b, c \equiv 0 \mod 2,\ a, d \equiv 1 \mod 4 \right\}.$$

Proof of Proposition 9.4. By Proposition 8.1, the surfaces of any Γ-monodromy stratum are contained in family i), where by (7.5) of Table 6, the factor $j_{\bar\Gamma}$ has three poles at 0 and ± 3. The splitting isomorphism gives a map from $\pi_1^{\mathrm{orb}}(\mathbb{H}_1/\bar\Gamma) \cong \bar\Gamma$ to $\Gamma \subset \mathrm{SL}(2,\mathbb{Z})$, and $\mathbb{H}_1/\bar\Gamma$ is identified with the complement in $\mathbb{P}^1$ of these poles. The positive loop at each pole is mapped by $(j_{\bar\Gamma})_*$ to the conjugacy class of $\pm\left(\begin{smallmatrix} 1 & 2 \\ 0 & 1 \end{smallmatrix}\right)$. Since 1 is chosen to represent the class of $\pm\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$, this is further mapped to $2 \in \mathbb{Z}/6\mathbb{Z} \cong \mathrm{PSL}(2,\mathbb{Z})_{\mathrm{ab}}$. Hence the splitting isomorphism associates with each loop the conjugacy class of $\left(\begin{smallmatrix} 1 & 2 \\ 0 & 1 \end{smallmatrix}\right)$ or $\left(\begin{smallmatrix} -1 & -2 \\ 0 & -1 \end{smallmatrix}\right)$ corresponding to 2 and 8 in $\mathbb{Z}/12\mathbb{Z} = \mathrm{SL}(2,\mathbb{Z})_{\mathrm{ab}}$, respectively. Now the sum of the three elements in $\mathrm{SL}(2,\mathbb{Z})_{\mathrm{ab}}$ must be zero since the sum in homology of the three loops is zero. Under the assumption of a splitting isomorphism, we therefore can assume that the conjugacy class associated with the loop around 0 contains $\left(\begin{smallmatrix} -1 & -2 \\ 0 & -1 \end{smallmatrix}\right)$ – possibly after applying a deck transformation in $\bar\Gamma(1)/\bar\Gamma(2)$ of $j_{\bar\Gamma(2)}$ acting by the full permutation group on $\{0, \pm 3\}$.

We now consider the map $j_{\mathcal{E}} = (\alpha_4 : \beta_4)$ and try to understand the possible fibres over pre-images of 0. Such a pre-image corresponds to a linear factor γ_1 of α_4 which may occur with multiplicity $1 \leqslant k \leqslant 4$. It may occur in β_4 with multiplicity $l = 0$ or $l = 1$ as long as $k > l$. Indeed, if $k \leqslant l$, the image would not be 0; if $k > l > 1$, the Weierstraß coefficients g_2, g_3 would share a common factor of γ_1^{12}, which is not allowed.

In all possible combinations we can compute the local fibre type at the zero of γ_1 from the local monodromy. Recall that we made an assumption on the matrix associated with the loop around 0 in the codomain of $j_{\mathcal{E}}$. Since the multiplicity of $j_{\mathcal{E}}$ is $k-l$ at the zero of γ_1 in the domain of $j_{\mathcal{E}}$, the matrix associated with the loop around this zero must be conjugate to $\left(\begin{smallmatrix} -1 & -2 \\ 0 & -1 \end{smallmatrix}\right)^{k-l}$. These monodromy matrices determine the local fibre type by Table 1 in the way recorded in the last row of Table 9.

The other row gives the local fibre types determined by the Tate algorithm: The pole order of $j(\mathcal{E})$ is the product of the multiplicities of $j_{\mathcal{E}}$ at the zero of γ_1 and of $j_{\bar\Gamma}$ at 0, hence $2(k-l)$. The vanishing orders of g_2 and g_3 at the zero of γ_1 are determined by row i) in Table 8, so for $l = 0$ they do not vanish simultaneously, and $\nu_2 = 2$ and $\nu_3 = 3$ for $l = 1$.

(k,l)	$1,0$	$2,0$	$3,0$	$4,0$	$2,1$	$3,1$	$4,1$
Tate	I_2	I_4	I_6	I_8	I_2^*	I_4^*	I_6^*
Cover	I_2^*	I_4	I_6^*	I_8	I_2^*	I_4	I_6^*

TABLE 9. Comparisons of fibre type calculations

Since the fibre types in both rows of Table 9 must coincide, our assumption on the splitting implies that k is even. Hence we may conclude that $\alpha_4 = \gamma_2^2$. This condition generically corresponds to a map $j_{\mathcal{E}}$ branched outside the other poles of $j_{\bar\Gamma}$. Therefore, $j_{\mathcal{E}}$ induces a surjection on orbifold fundamental groups, and thus the monodromy group is Γ.

Generically, there are I_4-fibres at the zeros of $j_{\mathcal{E}}$ and an I_2-fibre at each of the four unramified pre-images of each of the other two poles ± 3 of $j_{\bar\Gamma}$. This yields generic fibre type $2I_4 + 8I_2$ and generic root lattice $2A_3 + 8A_1$ with corresponding strata of dimension 4. So we have found the unique ambi-typical Γ stratum of the claim since every stratum of root lattice type with constant

Γ monodromy was shown to belong to the closed set in family i) defined by $\alpha_4 = \gamma_2^2$ and we can use conclusion (3.2) again.

The assertion about the Mordell–Weil torsion follows from Lemma 8.2 since the modular monodromy is $\bar\Gamma(2)$ and is only contained in $\bar\Gamma_m(n)$ if m and n divide 2. □

Remark 9.6. Note that the generic root lattice and the Mordell–Weil torsion do not necessarily determine a unique Shimada stratum. In fact, entry 95 in Table 3 of Shimada [Shi18] shows that there are two families with root lattice $2A_3 + 8A_1$ and Mordell–Weil torsion $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, which can be distinguished *algebraically*. That – in the terminology of Shimada – means that the saturations of the root lattices are non-isomorphic for the two families.

Shimada also gives a recipe for how to analyse such a situation in his Section 6.1, and his Example 6.4 corresponding to number 91 in his table is similar to our case in many aspects. By lattice theory, the two possible saturations in case 95 correspond to two isotropy subgroups of the discriminant group of $2A_3 + 8A_1$ up to isometry. Shimada computes these and provides his result on his home page [Shi16a]. The subgroup belonging to our family consists of the trivial element and the isotropic elements in $\mathrm{discr}(2A_3 + 8A_1) = (\mathbb{Z}/4\mathbb{Z})^2 \times (\mathbb{Z}/2\mathbb{Z})^8$ given by

$$(0,0,1,1,1,1,1,1,1,1)\,, \quad (2,2,0,0,0,0,1,1,1,1)\,, \quad (2,2,1,1,1,1,0,0,0,0)\,,$$

where a basis for the discriminant group is chosen in the obvious way. To prove that we are in that case requires a detailed analysis of the torsion sections and their intersections with fibre components. We shall not give the details here.

Proposition 9.7. *There are two ambi-typical strata with modular monodromy $\bar\Gamma_1(4)$ and monodromy group Γ such that $\Gamma \to \bar\Gamma_1(4)$ is an isomorphism. Their generic root lattice and Mordell–Weil torsion are, respectively,*

(i) $4A_3 + 2A_1$ *and* $\mathbb{Z}/4\mathbb{Z}$,

(ii) $2A_7$ *and* $\mathbb{Z}/2\mathbb{Z}$.

The two corresponding subgroups in $\mathrm{SL}(2,\mathbb{Z})$ *are not conjugate.*

Remark 9.8. The group is $\Gamma_1(4)$ in case (i) and in case (ii) can be given by

$$\Gamma = \left\{ M \;\middle|\; M \equiv \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}^k \mod 4,\ k = 0,1,2,3 \right\}.$$

Proof of Proposition 9.7. We argue as in the proof of Proposition 9.4, but with family ii) and the associated map $j_{\bar\Gamma}$ which by (7.6) has poles at 0, 4 and ∞. The analysis of the splitting isomorphism again provides information about the monodromy at these poles. However, this time only the poles at 0 and 4 are equivalent under deck transformation, so we have to consider two cases instead of one:

(1) The conjugacy class associated with the pole at 0 contains $\left(\begin{smallmatrix} -1 & -1 \\ 0 & -1 \end{smallmatrix}\right)$.

(2) The conjugacy class associated with the pole at ∞ contains $\left(\begin{smallmatrix} -1 & -4 \\ 0 & -1 \end{smallmatrix}\right)$.

In the first case, we apply the Tate algorithm to $j_{\mathcal{E}} = (\alpha_4 : \beta_4)$ and compare local monodromies at its zeros. We infer $\alpha_4 = \alpha_2^2$, then compute the generic fibre configuration $2I_2 + 4I_1 + 4I_4$, the generic root lattice $4A_3 + 2A_1$ and the dimension 4 of the corresponding strata. In the second case, we look at the poles instead and get $2I_8 + 8I_1$ and $2A_7$ and again dimension 4.

We can argue as before that these yield the ambi-typical strata of the claim and that there are no more.

Lemma 8.4 provides the Mordell–Weil torsion, showing in particular that the two subgroups are not conjugate. □

Proposition 9.9. *There is a unique ambi-typical stratum with modular monodromy $\bar{\Gamma}_1(3)$. Its monodromy group is $\Gamma_1(3)$. Its generic root lattice and Mordell–Weil torsion are*

$$6A_2\,,\ \mathbb{Z}/3\,.$$

Proof. Any ambi-typical stratum with modular monodromy $\bar{\Gamma}_1(3)$ belongs to family iii) or iv).

If it belongs to family iii), then by Lemma 8.3, the Mordell–Weil torsion contains $\mathbb{Z}/3$ and $-\operatorname{id}\notin\Gamma$. By the former property and Lemma 8.2, family iii) has monodromy contained in $\Gamma_1(3)$. Since $-\operatorname{id}\notin\Gamma_1(3)$, no proper subgroup surjects onto $\bar{\Gamma}_1(3)$; thus $\Gamma_1(3)$ is the monodromy group for the generic surface in family iii).

We follow the proofs of Propositions 9.1 and 9.2 and get generic fibre configuration $6I_3+6I_1$, generic root lattice $6A_2$, corresponding strata dimension 6 and $m(\Gamma_1(3))\leqslant 6$ by Lemma 7.4. Using (7.2) and (3.2), we conclude that family iii) constitutes a unique ambi-typical stratum, with the invariants given in the claim.

To address possible strata in family iv), we look at the additional family with Weierstraß datum

$$g_2=3\alpha_1\alpha_2\big(\alpha_1^3\alpha_2+8\beta_5\big)\,,\quad g_3=\alpha_2\big(\alpha_1^6\alpha_2^2-20\alpha_1^3\alpha_2\beta_5-8\beta_5^2\big)$$

that has associated j-invariant

$$j=\frac{(g_2/3)^3}{(g_2/3)^3-g_3^2}=\frac{\alpha_1^3\alpha_2\big(\alpha_1^3\alpha_2+8\beta_5\big)^3}{64\big(\alpha_1^3\alpha_2-\beta_5\big)^3\beta_5^2}\,,$$

which is the composition of $j_{\bar{\Gamma}}$ for $\bar{\Gamma}_1(3)$ with $j_{\mathcal{E}}=\alpha_1^3\alpha_2/\beta_5$. It contains the family iv) as a proper closed subset determined by the specialisation to $\alpha_2=\gamma_2$, $\beta_5=\gamma_2\beta_3$. A zero of γ_2 generically corresponds to a fibre of type I_0^* so both have generic monodromy $\Gamma_0(3)$, the pre-image in $\mathrm{SL}(2,\mathbb{Z})$ of $\bar{\Gamma}_1(3)=\bar{\Gamma}_0(3)$.

Repeating the argument above, using in particular (3.2), we deduce that no stratum of monodromy $\Gamma_0(3)$ is ambi-typical, except possibly that corresponding to the generic fibre configuration $2II+5I_3+5I_1$ in the additional family. However, by Proposition 5.5, the fibre II may not occur in generic surfaces of an ambi-typical stratum.

We are left to look for an ambi-typical stratum in family iv) with $\Gamma\to\bar{\Gamma}_1(3)$ an isomorphism. The splitting isomorphism gives a map from $\pi_1^{\mathrm{orb}}(\mathbb{H}/\Gamma)\cong\bar{\Gamma}$ to $\Gamma\subset\mathrm{SL}(2,\mathbb{Z})$. The positive loops around the 3-torsion point and the poles of $j_{\bar{\Gamma}}$ of order 1 and 3 are mapped to 2, 1 and 3 in $\mathbb{Z}/6=\mathrm{PSL}(2,\mathbb{Z})_{\mathrm{ab}}$, so the splitting isomorphism associates with these loops the elements 2 or 8, 1 or 7, and 3 or 9, respectively, in $\mathbb{Z}/12=\mathrm{SL}(2,\mathbb{Z})_{\mathrm{ab}}$.

In fact, we can be more precise. On the one hand, 2 cannot be associated with a 3-torsion point since then the corresponding monodromy is conjugate to $\left(\begin{smallmatrix}1&1\\-1&0\end{smallmatrix}\right)$, contradicting $-\operatorname{id}\notin\Gamma$. On the other hand, 1 and 3 cannot be associated with the two poles. Otherwise the corresponding elements of Γ are conjugate to powers of $\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)$ and so are all monodromies at poles of $j(\mathcal{E})$. Then the I^*-fibres – present according to Table 3 – are I_0^*-fibres, again contradicting $-\operatorname{id}\notin\Gamma$.

We again use the fact that the sum of the three elements in $\mathrm{SL}(2,\mathbb{Z})_{\mathrm{ab}}$ must be zero since the sum in homology of the three loops is zero. Thus the elements must be 8, 7 and 9, and $\mathcal{E}$ is a pull-back (up to normalisation and blow down) of an elliptic surface with singular fibre configuration $I_1^*+I_3^*+IV^*$ along $j_{\mathcal{E}}$.

By Table 7, the map $j_{\mathcal{E}}$ has degree 3 and ramification datum (3) at the 3-torsion point. By Table 3, the surface $\mathcal{E}$ has two I^*-fibres, so the ramification datum of $j_{\mathcal{E}}$ at the two poles must be $(2, 1)$. But then $j_{\mathcal{E}}$ is unramified outside these points and therefore violates the maximality condition (6.6). □

Proposition 9.10. *There is no ambi-typical stratum with* $\bar{\Gamma} = \bar{\Gamma}(1)^2$.

Proof. We first assume that there is such a stratum with $\Gamma \to \bar{\Gamma}(1)^2$ an isomorphism. By Lemma 8.1, we know that there is a corresponding generic surface $\mathcal{E}$ in the family vi).

The splitting isomorphism gives a map from $\pi_1^{\mathrm{orb}}(\mathbb{H}/\Gamma) \cong \bar{\Gamma}$ to $\Gamma \subset \mathrm{SL}(2,\mathbb{Z})$. The positive loops around both 3-torsion points and the pole of $j_{\bar{\Gamma}}$ are mapped to $2 \in \mathbb{Z}/6 = \mathrm{PSL}(2,\mathbb{Z})_{\mathrm{ab}}$, so the splitting isomorphism associates with each loop either 2 or 8 in $\mathbb{Z}/12 = \mathrm{SL}(2,\mathbb{Z})_{\mathrm{ab}}$. However, 2 can not be associated with a 3-torsion point since then the corresponding monodromy is conjugate to $\left(\begin{smallmatrix} 1 & 1 \\ -1 & 0 \end{smallmatrix}\right)$, contradicting $-\mathrm{id} \notin \Gamma$.

Again we use the fact that the sum of the three elements in $\mathrm{SL}(2,\mathbb{Z})_{\mathrm{ab}}$ must be zero. Thus all elements must be 8, and $\mathcal{E}$ is a pull-back (up to normalisation and blow down) of an elliptic surface with singular fibre configuration $I_2^* + 2IV^*$ along $j_{\mathcal{E}}$.

By Table 7, the map $j_{\mathcal{E}}$ has degree 4 and ramification datum $(3, 1)$ at the two 3-torsion points. By Table 3, the surface $\mathcal{E}$ has no I^*-fibre, so the ramification datum of $j_{\mathcal{E}}$ at the pole must be $(2, 2)$. But then $j_{\mathcal{E}}$ is unramified outside these points and therefore violates the maximality condition (6.6).

To show that there is also no stratum with $-\mathrm{id} \in \Gamma$, we look at the new family with Weierstraß datum given by

$$g_2 = -12\alpha_1\alpha_3\beta_1\beta_3\,, \quad g_3 = 4\alpha_3\beta_3\big(\alpha_1^3\alpha_3 - \beta_1^3\beta_3\big)$$

that has associated j-invariant

$$j = \frac{(g_2/3)^3}{(g_2/3)^3 - g_3^2} = \frac{4\alpha_3\beta_3(\alpha_1\beta_1)^3}{\big(\alpha_1^3\alpha_3 + \beta_1^3\beta_3\big)^2}\,,$$

which is the composition of $j_{\bar{\Gamma}}$ for $\bar{\Gamma}(1)^2$ with $j_{\mathcal{E}} = \big(\alpha_1^3\alpha_3 : \beta_1^3\beta_3\big)$. It contains the family vi) as a proper closed subset determined by the specialisation to $\alpha_3 = \gamma_1^2\delta_1$, $\beta_3 = \gamma_1\delta_1^2$, so both have the same generic monodromy Γ.

Repeating the argument above, using in particular (3.2), we deduce that no stratum of that monodromy is ambi-typical, except possibly that corresponding to the generic fibre configuration $6II + 6I_2$ of the new family. However, again by Proposition 5.5, the fibre II may not occur in generic surfaces of an ambi-typical stratum. □

10. Classification in the high-index cases

In this section, we will complete the classification of ambi-typical strata of high index. More precisely, we will classify the subgroups $\bar{\Gamma}$ of $\mathrm{PSL}(2,\mathbb{Z})$ of index at least 9 for which an ambi-typical stratum exists and give a description of the possible monodromy groups Γ in $\mathrm{SL}(2,\mathbb{Z})$ which cover $\bar{\Gamma}$.

10.1 Uniqueness of strata

We shall proceed as follows. First, given a group $\bar{\Gamma}$, we shall determine the possible groups Γ and prove the uniqueness of the Γ-ambi-typical strata. Second, we will determine the root lattices,

$j_{\mathcal{E}}$ branch data and the number of $*$-fibres of their generic members in terms of the ramification data of $j_{\bar{\Gamma}}$. In fact, we use the notation as in [BPT02] for the three cases that $j_{\mathcal{E}}$ can belong to:

- $(2)_B, 2$: $j_{\mathcal{E}}$ is of degree 2 with branching in a 2-torsion point and a non-torsion point,
- 2^2: $j_{\mathcal{E}}$ is of degree 2 with branching in two non-torsion points,
- 1: $j_{\mathcal{E}}$ is of degree 1 with no branching.

The actual list of ramification data of $j_{\bar{\Gamma}}$, and thus the list of possible groups $\bar{\Gamma}$, is postponed to the next subsection, as is the computation of the corresponding Mordell–Weil torsion.

Proposition 10.1. *Suppose that $\bar{\Gamma}$ has index 9 in $\mathrm{PSL}(2,\mathbb{Z})$. Any such subgroup defines at most one ambi-typical stratum, and in this case the monodromy group Γ is the pre-image of $\bar{\Gamma}$ in $\mathrm{SL}(2,\mathbb{Z})$.*

The generic invariants root lattice, $j_{\mathcal{E}}$ ramification and number of $*$-fibres are

$$D_4 + 2A_{i_1} + \cdots + 2A_{i_k}\,, \quad (2)_B, 2\,, \quad 1\,,$$

with k the number of poles of $j_{\bar{\Gamma}}$ of order at least two, $i_1+1, \ldots, i_k+1$ their orders and root lattice rank $4 + 2i_1 + \cdots + 2i_k = 16$.

Proof. By column 4 of Table 4, the map $j_{\mathcal{E}}$ is a double cover, and we have $e_2 = 1$ and $e_3 = 0$. So $\bar{\Gamma}$ contains 2-torsion and has no splitting map to $\mathrm{SL}(2,\mathbb{Z})$ since $-\mathrm{id}$ is the only 2-torsion element in $\mathrm{SL}(2,\mathbb{Z})$. In particular, every $\bar{\Gamma}$ monodromy stratum is actually a Γ monodromy stratum with pre-image Γ.

The map $j_{\mathcal{E}}$ is branched over the unique 2-torsion point according to Lemma 6.4(i). The family of such coverings is 1-dimensional and irreducible.

Moreover, it follows from Proposition 5.7 that the I^*-fibre in the generic member of the stratum must be I_0^*. Therefore, we get a 2-dimensional irreducible family of elliptic surfaces with monodromy Γ, where the second parameter is the position of the I_0^*-fibre. If we have an ambi-typical stratum, the rank of the root lattice must be 16.

The generic fibre type consists of the I_0^*-fibre and the I_ν-fibres corresponding to pole orders of the generic j-map $j_{\bar{\Gamma}} \circ j_{\mathcal{E}}$, which coincide with the lengths of the ramification partition at infinity. Since $j_{\bar{\Gamma}} \circ j_{\mathcal{E}}$ is the canonical factorisation, the map $j_{\mathcal{E}}$ is a double cover not branched over $j_{\bar{\Gamma}}^{-1}(\infty)$, and the ramification partition is twice that of $j_{\bar{\Gamma}}$.

We can now use the argument in (7.2) to conclude that Γ defines a monodromy stratum. This is the case since our family has varying moduli and is of maximal dimension by column 4 of Table 5. □

Proposition 10.2. *Suppose that $\bar{\Gamma}$ is of index 12 in $\mathrm{PSL}(2,\mathbb{Z})$. Let k be the number of poles of $j_{\bar{\Gamma}}$ of order at least two and $i_1+1, \ldots, i_k+1$ their orders. If $\bar{\Gamma}$ gives rise to an ambi-typical stratum, then it is torsion-free and there are two possibilities:*

(i) *There is a unique ambi-typical Γ stratum with $\Gamma \to \bar{\Gamma}$ an isomorphism. The generic invariants root lattice of rank 16, $j_{\mathcal{E}}$ ramification and number of $*$-fibres are*

$$2A_{i_1} + \cdots + 2A_{i_k}\,, \quad 2^2\,, \quad 0\,,$$

(ii) *There is a unique ambi-typical Γ stratum with $\Gamma \subset \mathrm{SL}(2,\mathbb{Z})$ the pre-image of $\bar{\Gamma}$. The generic invariants root lattice of rank 16, $j_{\mathcal{E}}$ ramification and number of $*$-fibres are*

$$2D_4 + A_{i_1} + \cdots + A_{i_k}\,, \quad 1\,, \quad 2\,.$$

Proof. By the hypothesis on $\bar{\Gamma}$, we are in the case of either column 6 or column 7 of Table 4, and we already know that $\bar{\Gamma}$ is torsion-free by Proposition 7.1(ii).

In the case of column 6 of Table 4, the map $j_{\mathcal{E}}$ is an isomorphism. Moreover, the two I^*-fibres in the general member of the stratum must be of type I_0^* according to Proposition 5.7 since, otherwise, the corresponding surface would be rigid. Therefore we get a 2-dimensional irreducible family of elliptic surfaces with monodromy Γ containing $-\mathrm{id}$, where the parameters are the positions of the I^*-fibres, given by a reduced divisor of degree 2 on $\mathbb{P}^1$.

The generic fibre type consists of the I_0^*-fibres and the I_ν-fibres corresponding to the poles of the generic j-map $j_{\bar{\Gamma}} \circ j_{\mathcal{E}}$. Since $j_{\mathcal{E}}$ is a isomorphism, the pole orders are those of $j_{\bar{\Gamma}}$.

In the case of column 7 of Table 4, the map $j_{\mathcal{E}}$ is a double cover. It is branched generically at two points since Lemma 6.4 imposes no additional conditions. The family of such coverings is 2-dimensional and irreducible. The generic fibre type consists of the I_ν-fibres corresponding to the pole orders of the generic j-map $j_{\bar{\Gamma}} \circ j_{\mathcal{E}}$, which coincide with the lengths of the ramification partition at infinity. Since $j_{\bar{\Gamma}} \circ j_{\mathcal{E}}$ is the canonical factorisation, the map $j_{\mathcal{E}}$ is a double cover not branched over $j_{\bar{\Gamma}}^{-1}(\infty)$, and the ramification partition is twice that of $j_{\bar{\Gamma}}$.

By column 6 of Table 5, our family is not contained in a monodromy stratum of larger dimension.

Since the dimension of both families is 2, the rank of the root lattice must be 16. □

Proposition 10.3. *Suppose that $\bar{\Gamma}$ is of index 18 in $\mathrm{PSL}(2,\mathbb{Z})$. If $\bar{\Gamma}$ is associated with an ambi-typical stratum, then it is torsion-free and the monodromy group Γ is the pre-image of $\bar{\Gamma}$ in $\mathrm{SL}(2,\mathbb{Z})$.*

The generic invariants root lattice of rank 17, $j_{\mathcal{E}}$ ramification and number of $*$-fibres are

$$D_4 + A_{i_1} + \cdots + A_{i_k}\,, \quad 1, \quad 1\,,$$

with k the number of poles of $j_{\bar{\Gamma}}$ of order at least 2 and $i_1+1, \ldots, i_k+1$ their orders.

Proof. By Proposition 7.1(i), we know that $\bar{\Gamma}$ is torsion-free. We also note that the I^*-fibre in the general member of the stratum must be of type I_0^* since $j_{\mathcal{E}}$ is of degree 1 by column 9 of Table 4; thus the positive-dimensionality of the stratum must be due to moduli of a movable I_0^*-fibre. Accordingly, $-\mathrm{id} \in \Gamma$, and Γ is the pre-image in $\mathrm{SL}(2,\mathbb{Z})$ of $\bar{\Gamma}$.

Therefore, we get a 1-dimensional irreducible family of elliptic surfaces with monodromy Γ, where the parameter is the position of the I_0^*-fibre; consequently, the rank of the root lattice must be 17.

The generic fibre type consists of the I_0^*-fibre and the I_ν-fibres corresponding to the poles of the generic j-map $j_{\bar{\Gamma}} \circ j_{\mathcal{E}}$. Since $j_{\mathcal{E}}$ is an isomorphism, the pole orders are those of $j_{\bar{\Gamma}}$.

The property of being a monodromy stratum follows as before by (7.2) and Table 5. □

10.2 Classification of relevant subgroups in the high-index cases

In Subsection 10.1, we related the monodromy strata uniquely, respectively, in a two-to-one way, with subgroups of $\mathrm{PSL}(2,\mathbb{Z})$ in the high-index case. So we have to classify these subgroups next.

To this end, we note that subgroups $\bar{\Gamma}$ are in bijective correspondence with maps $j_{\bar{\Gamma}}$, and such maps are in turn determined by triples of permutations μ_0, μ_1, μ_∞ whose product is the identity and which generate a group acting transitively on the sets of 9, 12 and 18 elements, respectively. Since condition (6.5) and the value of e_2 and e_3 impose restrictions on μ_0 and μ_1, we deduce the following:

(1) In the index 9 case, μ_0 has only 3-cycles and μ_1 only 2-cycles except for one fixed point.

(2) In the torsion-free cases, where the index is 12 or 18, the permutation μ_0 has only 3-cycles and μ_1 only 2-cycles.

Moreover, concerning the fibres of type I_ν for $\nu > 0$,

(3) the number of parts of μ_∞ is the number of poles, and the pole orders are the lengths of the parts and determine the corresponding fibre.

Proposition 10.4. *In the index* 9 *subcase, there are four relevant subgroups* $\bar{\Gamma}$ *of* $\mathrm{PSL}(2,\mathbb{Z})$. *These are in bijection with the following four partitions of* μ_∞ *of the corresponding map* $j_{\bar{\Gamma}}$*:*

$$(7,1,1)\,,\ (6,2,1)\,,(5,3,1)\,,\ (4,3,2)\,.$$

This translates into the following root lattices:

$$D_4+2A_6\,,\ D_4+2A_5+2A_1\,,\ D_4+2A_4+2A_2\,,\ D_4+2A_3+2A_2+2A_1$$

with corresponding Mordell–Weil torsion

$$\textit{trivial}\,,\quad \mathbb{Z}/2\,,\quad \textit{trivial}\,,\quad \mathbb{Z}/2\,.$$

Proof. Since each relevant subgroup gives an irreducible stratum, their number is bounded above by the number of Shimada components which have root lattice as described in Proposition 10.1. Shimada's list [Shi00, arXiv:math/0505140] published in the arXiv version shows that there are four such root lattices, entries $2171, 2179, 2190$ and 2198, which uniquely determine the Mordell–Weil torsion as claimed. They do not appear in the list of root lattices corresponding to multiple components; see Shimada [Shi18, Section 7, Table II, p. 38]. Therefore, each of these contributes one component.

To see that these Shimada strata are indeed ambi-typical, it suffices to exhibit the four groups, which we do in terms of four tuples of permutations:

$$(123)(456)(789)\,,\ (14)(27)(56)(89)\,,\ (1643297)(5)(8)\,, \tag{10.1}$$

$$(123)(456)(789)\,,\ (14)(26)(57)(89)\,,\ (16)(259743)(8)\,, \tag{10.2}$$

$$(123)(456)(789)\,,\ (14)(25)(67)(89)\,,\ (16975)(243)(8)\,, \tag{10.3}$$

$$(123)(456)(789)\,,\ (14)(27)(59)(68)\,,\ (167)(2943)(58)\,, \tag{10.4}$$

where we use the correspondence from (3) to (1) in Remark 6.2. □

Remark 10.5. One can give an alternative proof which avoids the use of Shimada's list by performing an exhaustive search for all homomorphisms as in Remark 6.2(3). This is straightforward though cumbersome, and we omit the details.

In the index 12 subcase, by Proposition 10.2, we get two monodromy strata for each relevant subgroup of $\mathrm{PSL}(2,\mathbb{Z})$. As it turns out, Beauville [Bea82] has classified exactly these torsion-free subgroups in his study of semi-stable rational elliptic fibrations. Note that Beauville uses the notation $\Gamma_0^0(n)$ for our groups $\Gamma_1(n)$.

Proposition 10.6. *In the index* 12 *subcase, there are six relevant subgroups of* $\mathrm{PSL}(2,\mathbb{Z})$. *These are in bijection with the following partitions of* μ_∞ *of the corresponding map* $j_{\bar{\Gamma}}$*:*

$$(9,1,1,1)\,,\ (8,2,1,1)\,,\ (6,3,2,1)\,,\ (5,5,1,1)\,,\ (4,4,2,2)\,,\ (3,3,3,3)\,.$$

This claim is proved in [Bea82], which is also used in the proof of the following proposition addressing case (i) of Proposition 10.2.

Proposition 10.7. *If $j_{\mathcal{E}}$ is generic of degree 2, so that $\mathcal{E}$ is the pull-back along $j_{\mathcal{E}}$ of a rational modular elliptic surface without $*$-fibre, then the monodromy group Γ is one of*

$$\Gamma_0(9)\cap\Gamma_1(3)\,,\quad \Gamma_0(8)\cap\Gamma_1(4)\,,\quad \Gamma_1(6)\,,\quad \Gamma_1(5)\,,\quad \Gamma_1(4)\cap\Gamma(2)\,,\quad \Gamma(3)\,,$$

with corresponding Mordell–Weil torsion

$$\mathbb{Z}/3\,,\quad \mathbb{Z}/4\,,\quad \mathbb{Z}/6\,,\quad \mathbb{Z}/5\,,\quad \mathbb{Z}/4\times\mathbb{Z}/2\,,\quad \mathbb{Z}/3\times\mathbb{Z}/3\,.$$

Proof. The list for the monodromy groups of the rational modular elliptic surfaces is taken from [Bea82]. Indeed, they do not change under generic pull-back since $(j_{\mathcal{E}})_*$ induces a surjection on fundamental groups of the complements of singular values. The claim about the Mordell–Weil torsion is then obtained using Lemma 8.2. It can also be verified by a check of the corresponding lines 2242, 2262, 2322, 2345, 2368, 2373 in [Shi00]. □

The other families of surfaces with torsion-free modular monodromy of index 12 are obtained from the Beauville elliptic surfaces, which are rational and rigid for deformations preserving the monodromy group, by replacing two smooth fibres by fibres of type I_0^*. This *generic quadratic twisting* corresponds to the transformation of the Weierstraß datum (g_2', g_3') of a rational Beauville elliptic surface into Weierstraß data $\left(\gamma_2^2 g_2', \gamma_2^3 g_3'\right)$, where the zeros of γ_2 avoid the singular fibres. Geometrically, this means the following. We first take the double cover branched at the smooth fibres over the zeros of γ_2. This double cover is acted on by the deck transformation and the involution which is $-\operatorname{id}$ on each fibre. The quotient by the product of these two involutions gives the twisted surface after resolution of its singularities. The choice of γ_2 or, equivalently, the position of the I_0^*-fibres, gives the two moduli of these families.

Proposition 10.8. *If $\mathcal{E}$ is a rational modular elliptic surface without $*$-fibre* twisted *at two smooth fibres, then the monodromy group Γ is one of*

$$\Gamma_0(9)\,,\quad \Gamma_0(8)\,,\quad \Gamma_0(6)\,,\quad \Gamma_1(5)\{\pm\operatorname{id}\}\,,\quad \Gamma_0(4)\cap\Gamma(2)\,,\quad \Gamma(3)\{\pm\operatorname{id}\}\,,$$

with corresponding Mordell–Weil torsion

$$\textit{trivial}\,,\quad \mathbb{Z}/2\,,\quad \mathbb{Z}/2\,,\quad \textit{trivial}\,,\quad \mathbb{Z}/2\times\mathbb{Z}/2\,,\quad \textit{trivial}\,.$$

Proof. The monodromy groups are the groups $\Gamma\{\pm\operatorname{id}\}$ generated by the groups from [Bea82] and $-\operatorname{id}$ due to the presence of I_0^*-fibres. The claim about the Mordell–Weil torsion is then obtained with Lemma 8.2. It is also immediate by a check of the corresponding lines 2148–2153 in [Shi00]. □

Proposition 10.9. *In the index 18 subcase, there are 26 relevant subgroups of $\operatorname{PSL}(2,\mathbb{Z})$ corresponding to root lattices and Mordell–Weil torsion given in lines 2762–2786 of Shimada's list. The lattice appearing in line 2776 gives rise to two different ambi-typical strata.*

Remark 10.10. We remark without proof that the two components corresponding to line 2776 are complex conjugate, as are the maps $j_{\bar{\Gamma}}$ for the corresponding modular monodromy groups. More precisely, this means that modulo conjugation in S_{18}, the corresponding maps $\mu\colon \pi_1(\mathbb{C}\setminus\{0,1\})\to S_{18}$ differ only by the automorphism of $\pi_1(\mathbb{C}\setminus\{0,1\})$ induced by complex conjugation.

Proof of Proposition 10.9. Since each relevant subgroup gives an irreducible stratum, their number is bounded above by the number of Shimada strata which have root lattice of rank 17 of the form $D_4 + A_{i_1} + \cdots + A_{i_k}$ as in Proposition 10.3. The list in [Shi00] shows that there are 25 such root lattices, and the component count in [Shi18] shows that each of these contributes one component except for that of line 2776, where two components exist.

To see that all these components are related to torsion-free subgroups, we provide the corresponding list of 26 triples of monodromy permutations together with a proof that the two triples corresponding to line 2776 are not equal under conjugation. We give representatives for all orbits, with $\mu_1 = (12)(34)(56)(78)(9\,10)(11\,12)(13\,14)(15\,16)(17\,18)$ and μ_0 and μ_∞ in Table 10 below.

1	(123)(567)(9 10 11)(13 14 15)(4 8 17)(12 16 18), (1)(5)(9)(13)(2 3 17 16 14 15 12 10 11 18 8 6 7 4)	141111
2	(123)(567)(9 10 11)(4 8 17)(12 15 17)(14 18 16), (1)(5)(9)(15 18)(2 3 13 16 12 10 11 17 14 8 6 7 4)	132111
3	(123)(567)(9 10 11)(4 13 18)(8 15 14)(12 17 16), (1)(5)(9)(13 15 17)(2 3 18 12 10 11 16 8 6 7 14 4)	123111
4	(123)(458)(697)(10 11 14)(12 15 13)(16 17 18), (1)(17)(57)(11 13)(2 3 8 9 14 15 18 16 12 10 6 4)	122211
5	(123)(567)(489)(10 11 13)(12 17 15)(14 16 18), (1)(5)(16 17)(11 15 14)(2 3 9 13 18 12 10 8 6 7 4)	113211
6	(123)(567)(9 10 11)(4 8 14)(13 15 17)(12 16 18), (1)(5)(9)(10 11 18 15 12)(2 3 14 17 16 13 8 6 7 4)	105111
7	(123)(567)(4 9 11)(8 13 10)(12 15 17)(14 18 16), (1)(5)(15 18)(9 13 16 12)(2 3 11 17 14 8 6 7 10 9)	104211
8	(123)(567)(4 9 11)(8 13 15)(10 16 17)(12 18 14), (1)(5)(9 17 12)(13 18 16)(2 3 11 14 8 6 7 15 10 4)	103311
9	(123)(457)(698)(10 11 13)(12 17 15)(14 16 18), (1)(58)(16 17)(11 15 14)(2 3 7 9 13 18 12 10 6 4)	103221
10	(123)(567)(9 10 11)(4 13 18)(8 14 15)(12 17 16), (1)(5)(9)(6 7 15 17 13 8)(2 3 18 12 10 11 16 14 4)	96111
11	(123)(567)(4 9 11)(10 13 12)(14 15 17)(8 16 18), (1)(5)(9 12)(6 7 18 15 8)(2 3 11 13 17 16 14 10 4)	95211
12	(123)(457)(6 9 11)(8 13 15)(10 17 12)(14 18 16), (1)(9 12)(13 16)(5 11 17 14 8)(2 3 7 15 18 10 6 4)	85221
13	(123)(457)(6 9 11)(8 12 13)(10 15 17)(16 17 18), (1)(15 18)(5 11 8)(9 17 14 12)(2 3 7 13 16 10 6 4)	84321
14	(135)(274)(6 8 10)(9 11 13)(14 15 17)(12 18 16), (14)(15 18)(376)(11 16 14)(2 5 10 13 17 12 9 8)	83322
15	(123)(567)(4 9 11)(8 10 14)(12 16 18)(13 17 15), (1)(5)(16 17)(2 3 11 18 13 10 4)(6 7 14 15 12 9 8)	77211
16	(123)(567)(4 11 9)(8 14 10)(12 16 18)(13 17 15), (1)(5)(16 17)(2 3 9 14 15 12 4)(6 7 10 11 18 13 8)	77211
17	(123)(567)(4 9 11)(8 13 15)(10 14 17)(12 18 16), (1)(5)(9 17 12)(6 7 15 18 14 8)(2 3 11 16 13 10 4)	76311
18	(123)(457)(689)(10 11 13)(12 15 17)(14 18 16), (1)(15 18)(11 17 14)(2 3 7 6 4)(5 9 13 16 12 10 8)	75321
19	(123)(457)(6 9 11)(8 13 15)(12 17 14)(10 16 18), (1)(9 18 12)(13 17 16)(5 11 14 8)(2 3 7 15 10 6 4)	74331
20	(123)(567)(4 9 11)(8 13 15)(10 17 16)(12 14 18), (1)(5)(9 16 13 12)(2 3 11 18 10 4)(6 7 15 17 14 8)	66411
21	(135)(486)(7 9 11)(10 14 12)(13 15 17)(2 18 16), (36)(9 12)(15 18)(1 16 13 10 7 4)(2 5 8 11 14 17)	66222
22	(123)(567)(4 9 11)(8 13 15)(10 12 17)(14 16 18), (1)(5)(2 3 11 10 4)(6 7 15 14 8)(9 17 16 13 18 12)	65511
23	(123)(457)(6 9 11)(8 13 10)(12 15 17)(14 18 16), (1)(15 18)(9 13 12 16)(2 3 7 10 6 4)(5 18 17 14 8)	65421
24	(135)(2 8 10)(7 11 9)(4 12 14)(13 15 17)(6 18 16), (89)(15 18)(1 10 11 4)(3 14 17 6)(2 5 16 13 12 7)	64422
25	(135)(274)(6 9 11)(8 13 15)(10 16 17)(12 18 14), (14)(9 17 12)(13 18 16)(2 5 11 14 8)(3 7 15 10 6)	55332
26	(135)(279)(4 11 13)(6 14 15)(8 16 17)(10 18 12), (3 13 6)(7 17 10)(1 9 12 4)(2 5 15 8)(11 18 16 14)	44433

TABLE 10. Monodromy data of $j_{\bar{\Gamma}}$

Supposing that the factorisations in lines 15 and 16 are conjugate, there is a permutation σ which commutes with μ_1 and which conjugates the μ_0 and μ_∞ of line 15 to those of line 16, which for this proof we denote by ρ_0 and ρ_∞. From this we deduce the conditions

$$\sigma\big(\mu_*^k(n)\big) = \rho_*^k(\sigma(n)) \quad \text{for all } n,\, k \text{ and for } * = 0, 1, \infty\,.$$

In particular, for $1 = \mu_\infty(1)$, we get that $\sigma(1) = \sigma(\mu_\infty(1))$ implies $\sigma(1) = \rho_\infty(\sigma(1))$. Thus $\sigma(1)$ is a fixed point of ρ_∞ and thus either 1 or 5. Then for $2 = \mu_1(1)$, we get $\sigma(2) = \rho_1(\sigma(1)) \in \{\rho_1(1), \rho_1(5)\} = \{2, 6\}$. On the other hand, the 2-cycle $(16\,17)$ is unique in both μ_∞ and ρ_∞, so $(\sigma(16), \sigma(17)) = (16, 17)$ implies $\sigma(17) \in \{16, 17\}$. Consequently, for $18 = \mu_1(17)$, we get $\sigma(18) = \rho_1(\sigma(17)) \in \{\rho_1(16), \rho_1(17)\} = \{15, 18\}$. But in this way we arrive at a contradiction since $\sigma\big(\mu_\infty^4(2)\big) = \sigma(18) \in \{15, 18\}$ while $\rho_\infty^4(\sigma(2)) \in \big\{\rho_\infty^4(2), \rho_\infty^4(6)\big\} = \{14, 11\}$. □

This finally concludes the proof of Theorems 4.2, 4.3 and 4.4.

#	Root lattice	MW	dim	ind	|ker|	[Shi] #	Remarks
0		[1]	18	1	2		$\Gamma = \mathrm{SL}(2,\mathbb{Z})$
1	$8A_1$	[2]	10	3	2	99	$\Gamma = \Gamma_1(2)$
2	$6A_2$	[3]	6	4	1	559	$\Gamma = \Gamma_1(3)$
3	$4\,A_3$	[2]	6	6	2	547	$\Gamma = \Gamma_0(4)$
4	$12\,A_1$	[2, 2]	6	6	2	565	$\Gamma = \Gamma(2)$
5	$2\,A_7$	[2]	4	6	1	1134	$\bar\Gamma = \bar\Gamma_0(4),\ -\mathrm{id} \notin \Gamma$
6	$2\,A_3\,+\,8\,A_1$	[2, 2]	4	6	1	1223	$\bar\Gamma = \bar\Gamma(2),\ -\mathrm{id} \notin \Gamma$
7	$4\,A_3\,+\,2\,A_1$	[4]	4	6	1	1215	$\Gamma = \Gamma_1(4)$
8	$D_4 + 2A_6$	[1]	2	9	2	2171	non-congruence
9	$D_4 + 2A_5 + 2A_1$	[2]	2	9	2	2179	"
10	$D_4 + 2A_4 + 2A_2$	[1]	2	9	2	2190	"
11	$D_4 + 2A_3 + 2A_2 + 2A_1$	[2]	2	9	2	2198	"
12	$2\,D_4\,+\,A_8$	[1]	2	12	2	2148	twisted Beauville
13	$2\,D_4\,+\,A_7\,+\,A_1$	[2]	2	12	2	2149	"
14	$2\,D_4\,+\,A_5\,+\,A_2\,+\,A_1$	[2]	2	12	2	2150	"
15	$2\,D_4\,+\,2\,A_4$	[1]	2	12	2	2151	"
16	$2\,D_4\,+\,2\,A_3\,+\,2\,A_1$	[2, 2]	2	12	2	2152	"
17	$2\,D_4\,+\,4\,A_2$	[1]	2	12	2	2153	"
18	$2\,A_8$	[3]	2	12	1	2242	2:1 Beauville
19	$2\,A_7\,+\,2\,A_1$	[4]	2	12	1	2262	"
20	$2\,A_5\,+\,2\,A_2\,+\,2\,A_1$	[6]	2	12	1	2322	"
21	$4\,A_4$	[5]	2	12	1	2345	"
22	$4\,A_3\,+\,4\,A_1$	[4, 2]	2	12	1	2368	"
23	$8\,A_2$	[3, 3]	2	12	1	2373	"
24	$D_4\,+\,A_{13}$	[1]	1	18	2	2762	non-congruence
25	$D_4\,+\,A_{12}\,+\,A_1$	[1]	1	18	2	2763	"
26	$D_4\,+\,A_{11}\,+\,A_2$	[2]	1	18	2	2764	"
27	$D_4\,+\,A_{11}\,+\,2\,A_1$	[2]	1	18	2	2765	"
28	$D_4\,+\,A_{10}\,+\,A_2\,+\,A_1$	[1]	1	18	2	2766	"
29	$D_4\,+\,A_9\,+\,A_4$	[1]	1	18	2	2767	"
30	$D_4\,+\,A_9\,+\,A_3\,+\,A_1$	[2]	1	18	2	2768	"
31	$D_4\,+\,A_9\,+\,2\,A_2$	[1]	1	18	2	2769	"
32	$D_4\,+\,A_9\,+\,A_2\,+\,2\,A_1$	[2]	1	18	2	2770	"
33	$D_4\,+\,A_8\,+\,A_5$	[1]	1	18	2	2771	"
34	$D_4\,+\,A_8\,+\,A_4\,+\,A_1$	[1]	1	18	2	2772	"
35	$D_4\,+\,A_7\,+\,A_4\,+\,2\,A_1$	[2]	1	18	2	2773	"
36	$D_4\,+\,A_7\,+\,A_3\,+\,A_2\,+\,A_1$	[2]	1	18	2	2774	"
37	$D_4\,+\,A_7\,+\,2\,A_2\,+\,2\,A_1$	[2]	1	18	2	2775	"
38	$D_4\,+\,2\,A_6\,+\,A_1$	[1]	1	18	2	2776	"
39	$D_4\,+\,2\,A_6\,+\,A_1$	[1]	1	18	2	2776	"
40	$D_4\,+\,A_6\,+\,A_5\,+\,A_2$	[1]	1	18	2	2777	"
41	$D_4\,+\,A_6\,+\,A_4\,+\,A_2\,+\,A_1$	[1]	1	18	2	2778	"
42	$D_4\,+\,A_6\,+\,A_3\,+\,2\,A_2$	[1]	1	18	2	2779	"
43	$D_4\,+\,2\,A_5\,+\,A_3$	[2]	1	18	2	2780	"
44	$D_4\,+\,2\,A_5\,+\,3\,A_1$	[2, 2]	1	18	2	2781	"
45	$D_4\,+\,A_5\,+\,2\,A_4$	[1]	1	18	2	2782	"
46	$D_4\,+\,A_5\,+\,A_4\,+\,A_3\,+\,A_1$	[2]	1	18	2	2783	"
47	$D_4\,+\,A_5\,+\,2\,A_3\,+\,2\,A_1$	[2, 2]	1	18	2	2784	"
48	$D_4\,+\,2\,A_4\,+\,2\,A_2\,+\,A_1$	[1]	1	18	2	2785	"
49	$D_4\,+\,3\,A_3\,+\,2\,A_2$	[2]	1	18	2	2786	"

Table 11. Data of ambi-typical strata

#	$j_{\bar\Gamma}$	$\bar\Gamma$	$j_{\mathcal{E}}$	I_0^*-fibres
0	1	$\mathrm{PSL}(2,\mathbb{Z})$	$(3^8)_A$, $(2^{12})_B$, 2^{18}	-
1	$(3),(2,1),(2,1)$	$\bar\Gamma_1(2)$	$(2,2,2,2)_B$, 2^{10}	-
2	$(3,1),(2,2),(3,1)$	$\bar\Gamma_1(3)$	$(3,3)_A$, 2^6	-
3	$(3,3),(2,2,2),(4,1,1)$	$\bar\Gamma_1(4)$	2^6	-
4	$(3,3),(2,2,2),(2,2,2)$	$\bar\Gamma(2)$	2^6	-
5	$(3,3),(2,2,2),(4,1,1)$	$\bar\Gamma_1(4)$	$(2,2)_{4_\infty}$, 2^4	-
6	$(3,3),(2,2,2),(2,2,2)$	$\bar\Gamma(2)$	$(2,2)_{2_\infty}$, 2^4	-
7	$(3,3),(2,2,2),(4,1,1)$	$\bar\Gamma_1(4)$	$(2,2)_{1_\infty}$, 2^4	-
8	$(3,3,3),(2,2,2,2,1),(7,1,1)$		$(2)_B$, 2	1
9	$(3,3,3),(2,2,2,2,1),(6,2,1)$		$(2)_B$, 2	1
10	$(3,3,3),(2,2,2,2,1),(5,3,1)$		$(2)_B$, 2	1
11	$(3,3,3),(2,2,2,2,1),(4,3,2)$		$(2)_B$, 2	1
12	$(3,3,3,3),(2,2,2,2,2,2),(9,1,1,1)$		1	2
13	$(3,3,3,3),(2,2,2,2,2,2),(8,2,1,1)$		1	2
14	$(3,3,3,3),(2,2,2,2,2,2),(6,3,2,1)$		1	2
15	$(3,3,3,3),(2,2,2,2,2,2),(5,5,1,1)$		1	2
16	$(3,3,3,3),(2,2,2,2,2,2),(4,4,2,2)$		1	2
17	$(3,3,3,3),(2,2,2,2,2,2),(3,3,3,3)$		1	2
18	$(3,3,3,3),(2,2,2,2,2,2),(9,1,1,1)$		2^2	-
19	$(3,3,3,3),(2,2,2,2,2,2),(8,2,1,1)$		2^2	-
20	$(3,3,3,3),(2,2,2,2,2,2),(6,3,2,1)$		2^2	-
21	$(3,3,3,3),(2,2,2,2,2,2),(5,5,1,1)$		2^2	-
22	$(3,3,3,3),(2,2,2,2,2,2),(4,4,2,2)$		2^2	-
23	$(3,3,3,3),(2,2,2,2,2,2),(3,3,3,3)$		2^2	-
24	$(3,3,3,3,3,3),(2,2,2,2,2,2,2,2,2),(14,1,1,1,1)$		1	1
25	$(3,3,3,3,3,3),(2,2,2,2,2,2,2,2,2),(13,2,1,1,1)$		1	1
26	$(3,3,3,3,3,3),(2,2,2,2,2,2,2,2,2),(12,3,1,1,1)$		1	1
27	$(3,3,3,3,3,3),(2,2,2,2,2,2,2,2,2),(12,2,2,1,1)$		1	1
28	$(3,3,3,3,3,3),(2,2,2,2,2,2,2,2,2),(11,3,2,1,1)$		1	1
29	$(3,3,3,3,3,3),(2,2,2,2,2,2,2,2,2),(10,5,1,1,1)$		1	1
30	$(3,3,3,3,3,3),(2,2,2,2,2,2,2,2,2),(10,4,2,1,1)$		1	1
31	$(3,3,3,3,3,3),(2,2,2,2,2,2,2,2,2),(10,3,3,1,1)$		1	1
32	$(3,3,3,3,3,3),(2,2,2,2,2,2,2,2,2),(10,3,2,2,1)$		1	1
33	$(3,3,3,3,3,3),(2,2,2,2,2,2,2,2,2),(9,6,1,1,1)$		1	1
34	$(3,3,3,3,3,3),(2,2,2,2,2,2,2,2,2),(9,5,2,1,1)$		1	1
35	$(3,3,3,3,3,3),(2,2,2,2,2,2,2,2,2),(8,5,2,2,1)$		1	1
36	$(3,3,3,3,3,3),(2,2,2,2,2,2,2,2,2),(8,4,3,2,1)$		1	1
37	$(3,3,3,3,3,3),(2,2,2,2,2,2,2,2,2),(8,3,3,2,2)$		1	1
38	$(3,3,3,3,3,3),(2,2,2,2,2,2,2,2,2),(7,7,2,1,1)$		1	1
39	$(3,3,3,3,3,3),(2,2,2,2,2,2,2,2,2),(7,7,2,1,1)$		1	1
40	$(3,3,3,3,3,3),(2,2,2,2,2,2,2,2,2),(7,6,3,1,1)$		1	1
41	$(3,3,3,3,3,3),(2,2,2,2,2,2,2,2,2),(7,5,3,2,1)$		1	1
42	$(3,3,3,3,3,3),(2,2,2,2,2,2,2,2,2),(7,4,3,3,1)$		1	1
43	$(3,3,3,3,3,3),(2,2,2,2,2,2,2,2,2),(6,6,4,1,1)$		1	1
44	$(3,3,3,3,3,3),(2,2,2,2,2,2,2,2,2),(6,6,2,2,2)$		1	1
45	$(3,3,3,3,3,3),(2,2,2,2,2,2,2,2,2),(6,5,5,1,1)$		1	1
46	$(3,3,3,3,3,3),(2,2,2,2,2,2,2,2,2),(6,5,4,2,1)$		1	1
47	$(3,3,3,3,3,3),(2,2,2,2,2,2,2,2,2),(6,4,4,2,2)$		1	1
48	$(3,3,3,3,3,3),(2,2,2,2,2,2,2,2,2),(5,5,3,3,2)$		1	1
49	$(3,3,3,3,3,3),(2,2,2,2,2,2,2,2,2),(4,4,4,3,3)$		1	1

TABLE 12. j-factorisation data of ambi-typical strata

10.3 Conclusion

We obtain a list of 48 root lattices with uniquely associated Mordell–Weil torsion. The latter appear in Table 11 using the notation of [Shi00] with $[n]$ for $\mathbb{Z}/n$ and $[n, m]$ for $\mathbb{Z}/n \times \mathbb{Z}/m$. In addition, Table 11 gives the dimensions of the 50 ambi-typical strata, the index of $\bar{\Gamma}$ in $\mathrm{PSL}(2, \mathbb{Z})$, the cardinality of the kernel of $\Gamma \to \bar{\Gamma}$ and the corresponding number in Shimada's list. Although it is not needed in this paper, for completeness we also state which of the monodromy groups are congruence subgroups. This turns out to be the case if and only if the index is not divisible by 9. For index at most 6, this follows from [Woh64, Theorem 5]. For index 9 this follows from [CP03, Table 2]; alternatively, one can give an independent argument using the amplitudes of the cusps. Finally, for indices 12 and 18, the claim can be deduced from Sebbar's classification [Seb01, Table 1].

In Table 12, we list the branch behaviour of the maps $j_{\bar{\Gamma}}$ and $j_{\mathcal{E}}$.

11. Moduli spaces of lattice-polarised $K3$ surfaces and Shimada strata

In this section, we want to start our discussion about the precise relationship between moduli spaces of lattice-polarised $K3$ surfaces and ambi-typical strata. For this, we first recall some basic facts about lattice-polarised $K3$ surfaces and their moduli spaces. Our aim is to understand how many moduli spaces of lattice-polarised $K3$ surfaces dominate a given ambi-typical stratum and to compute the degree of the finite map from a component of such a moduli space to the ambi-typical stratum it dominates.

In Section 2, we encountered the following situation: We started with an elliptically fibred $K3$ surface $f\colon \mathcal{E} \to \mathbb{P}^1$ and root lattice $R(\mathcal{E})$ whose saturation in the $K3$ lattice was denoted by $L(\mathcal{E})$. Then $M(\mathcal{E}) = U + L(\mathcal{E})$ is a hyperbolic lattice contained in the Néron–Severi group of $\mathcal{E}$. This gives rise to a lattice polarisation.

To explain this in more detail, let M be an even lattice of signature $(1, t)$ which admits a primitive embedding $\iota\colon M \to L_{K3}$ into the $K3$ lattice L_{K3}. We denote the orthogonal complement of the image of ι by $T = T(\iota) = \iota(M)^{\perp}_{L_{K3}} \subset L_{K3}$. The lattice T has signature $(2, 19 - t)$. For the rest of this section, we shall assume that the rank of T is at least 3 (which is the case in our situation). Then T defines a type IV homogeneous domain

$$\Omega_T = \{[x] \in \mathbb{P}(T \otimes \mathbb{C}) \mid (x, x) = 0,\, (x, \bar{x}) > 0\} = \mathcal{D}_T \cup \mathcal{D}'_T$$

of dimension $19 - t$ consisting of two connected components, namely $\mathcal{D}_T$ and $\mathcal{D}'_T$.

We also consider the real cone

$$C_M = \{x \in M_{\mathbb{R}} \mid (x, x) > 0\} = C_M^+ \cup C_M^-\,,$$

which again consists of two connected components, of which we choose one, say C_M^+. Removing from C_M^+ all hyperplanes orthogonal to roots $\Delta_M = \{d \in M \mid d^2 = -2\}$ subdivides C_M^+ into different connected components, the so-called *Weyl chambers*, of which we choose one that we call C_M^{pol}. An (M, ι)*-polarised* $K3$ surface is a pair $(S, \tilde{\iota})$, where $\tilde{\iota}\colon M \to \mathrm{NS}(S) \subset H^2(S, \mathbb{Z})$ is a primitive embedding which is isomorphic to ι with respect to a suitable marking $\varphi\colon H^2(S, \mathbb{Z}) \to L_{K3}$ and is such that $\tilde{\iota}(C_M^{\mathrm{pol}})$ contains an ample class. In this case, we call $\varphi\colon H^2(S, \mathbb{Z}) \to L_{K3}$ an M*-polarised marking.* We call two M-polarised $K3$ surfaces $(S_1, \tilde{\iota}_1)$ and $(S_2, \tilde{\iota}_2)$ isomorphic if there is an isomorphims $f\colon S_1 \to S_2$ with $f^* \circ \tilde{\iota}_2 = \tilde{\iota}_1$.

In order to describe the moduli space of M-polarised $K3$ surfaces, we consider the group

$$\mathrm{O}(L_{K3}, M, \iota) := \{g \in O(L_{K3}) \mid g|_{\iota(M)} = \mathrm{id}_{\iota(M)}\}$$

and recall the definition of the *stable* orthogonal group

$$\widetilde{\mathrm{O}}(T) := \{g \in O(T) \mid g|_{D(T)} = \mathrm{id}_{D(T)}\}$$

consisting of all orthogonal transformations of T which act trivially on the discriminant $D(T)$. It is well known, see [Nik79, Corollary 1.5.2], that there is an isomorphism

$$\mathrm{O}(L_{K3}, M, \iota) \cong \widetilde{\mathrm{O}}(T)\,.$$

The group $O(T)$, and hence also $\widetilde{\mathrm{O}}(T)$, acts properly discontinuously on Ω_T as well as on the open subset

$$\Omega_T^{\mathrm{pol}} = \Omega_T \setminus \bigcup_{d \in T,\, d^2=-2} (H_d \cap \Omega_T)\,,$$

where $H_d = \langle d \rangle^\perp \subset \mathbb{P}(T \otimes \mathbb{C})$ is the hyperplane orthogonal to d.

PROPOSITION 11.1. *The quotient*

$$\mathcal{N}^a_{M,\iota} = \Omega_T^{\mathrm{pol}} / \widetilde{\mathrm{O}}(T)$$

is the moduli space of (M,ι)-polarised $K3$ surfaces.

Proof. See [Dol96, Section 3] or [BHP$^+$04, Section VIII.22, p. 360]. □

It is sometimes also useful to consider a weakening of (M,ι)-polarisations. Recall that a line bundle $\mathcal{L}$ is said to be a *quasi-polarisation* if it is nef and big. We say that $(S,\tilde{\iota})$ is an (M,ι)-quasi-polarised $K3$ surface if $\tilde{\iota}\big(C_M^{\mathrm{pol}}\big)$ contains a big and nef class. By [Dol96, Section 3], the quotient

$$\mathcal{N}_{M,\iota} = \Omega_T / \widetilde{\mathrm{O}}(T)$$

is in $1:1$ correspondence with the set of isomorphism classes of M-quasi-polarised $K3$ surfaces. It can be viewed as the moduli space of M-polarised $K3$ surfaces with ADE singularities. This contains $\mathcal{N}^a_{M,\iota}$ as an open subset.

At this point, some remarks are in order. In the literature, it is often tacitly assumed that the lattice M has a unique primitive embedding into the $K3$ lattice. This assumption then justifies that one talks about *the* moduli space of M-polarised $K3$ surfaces. For us it will be important to also allow the possibility that M possesses different primitive embeddings into the $K3$ lattice (the number of such embeddings modulo $\mathrm{O}(L_{K3})$ is always finite). We will then consider the union

$$\mathcal{N}_M = \cup_\iota \mathcal{N}_{M,\iota}\,,$$

where ι runs over all classes of different primitive embeddings of M into L_{K3}. Moreover, Dolgachev has formulated arithmetic conditions which lead to the notion of *m-admissible* lattices. This means in particular that the transcendental lattice T splits off a summand $U(m)$, that is, a multiple of a hyperbolic plane. This is relevant for mirror symmetry and the discussion of the Yukawa-coupling, but it plays no role for our purposes since the construction of the moduli space of lattice (quasi-)polarised $K3$ surfaces does not require this condition. In fact, a number of the lattices which we consider, in particular when M has rank 19, do not fulfil this condition; see the appendix. In these cases, the lattice T does not split off a summand U over $\mathbb{Q}$ resulting in compact moduli spaces of M-polarised $K3$ surfaces. Finally, we recall that the case $M = U$ gives us another construction for the moduli space $\mathcal{F}$ of elliptically fibred $K3$ surfaces with a section or Jacobian fibrations.

Since $\mathrm{O}(T)$ acts properly discontinuously on Ω_T, the quotient $\mathcal{N}_{M,\iota}$ has at most finite quotient singularities. By a well-known result of Baily–Borel, this is a quasi-projective variety. We also note that for small rank of the transcendental lattice T, there is a relation with Siegel spaces: if $t = 18$, then $\mathcal{D}_T \cong \mathbb{H}_1$ is the upper half-plane; if $t = 17$, then $\mathcal{D}_T \cong \mathbb{H}_1 \times \mathbb{H}_1$; and if $t = 16$, then $\mathcal{D}_T \cong \mathbb{H}_2$ is the Siegel upper half-plane of genus 2. We will discuss this in more detail in the case of $t = 18$ in the appendix.

The quotient $\Omega_T/\widetilde{\mathrm{O}}(T)$ can have one or two components. If it has two components, then these are complex conjugate to each other. An element $g \in \mathrm{O}(T)$ can either fix the two components $\mathcal{D}_T$ and $\mathcal{D}'_T$ or interchange them, depending on its spinor norm. We recall that the *real spinor norm* is a homomorphism

$$\mathrm{sn}_{\mathbb{R}}\colon\ \mathrm{O}(T) \to \mathbb{R}^*/(\mathbb{R}^*)^2 = \{\pm 1\}\,.$$

For a precise definition, we refer the reader to [GHS09, Section 1]. Our normalisation of the spinor norm is such that in the case of signature $(2, n)$, in which we here are, the transformation g fixes the two components of Ω_T if and only if $\mathrm{sn}_{\mathbb{R}}(g) = 1$, and it interchanges them if and only if $\mathrm{sn}_{\mathbb{R}}(g) = -1$. We define the groups

$$\mathrm{O}^+(T) = \{g \in \mathrm{O}(T) \mid \mathrm{sn}_{\mathbb{R}}(g) = 1\}$$

and

$$\widetilde{\mathrm{O}}^+(T) = \mathrm{O}^+(T) \cap \widetilde{\mathrm{O}}(T)\,.$$

The quotient $\Omega_T/\widetilde{\mathrm{O}}(T)$ then has two components if any only if $\widetilde{\mathrm{O}}(T) = \widetilde{\mathrm{O}}^+(T)$.

We first want to discuss the number of moduli spaces of M-polarised $K3$ surfaces and the number of connected components. Clearly, there is only one such moduli space if there is a unique primitive embedding of $\iota\colon M \to L_{K3}$ (up to $\mathrm{O}(L_{K3})$). In general, however, such an embedding need not be unique. Assume that there is at least one primitive embedding $\iota\colon M \to L_{K3}$, and let $T_\iota = \iota(M)^{\perp}_{L_{K3}}$. Then T_ι may depend on ι, but its genus does not. It is determined by

$$\mathrm{sign}(T_\iota) = (2, 18 - \mathrm{rank}(M)) \quad \text{and} \quad (D(T_i), q_{T_i}) \cong (D(M), -q_M)\,.$$

We call this the *genus orthogonal to M* and denote it by $\mathcal{G}_T$.

Proposition 11.2. *Let $T \in \mathcal{G}_T$, and let $\overline{\mathrm{O}}(T)$ be the image of $\mathrm{O}(T)$ in $\mathrm{O}(D(T))$. Then the index $[\mathrm{O}(D(T)) : \overline{\mathrm{O}}(T)]$ depends only on the genus $\mathcal{G}_T$ of T and not on T itself.*

Proof. This follows from the theory of Miranda and Morrison, in particular [MM09, Chapter VIII, Proposition 6.1(2)]. Here we recall that we are always in the situation that T is indefinite since we assume in this section that it has rank at least 3. □

It follows from the theory of Miranda–Morrison [MM09, Theorem VIII.7.2], see also [Shi18, Section 1, p. 514], that there is an exact sequence, which is completely determined by M, of the form

$$0 \to \mathrm{coker}\big(\mathrm{O}(T) \to \mathrm{O}(D(T))\big) \to \mathcal{M}_T \to \mathcal{G}_T \to 0\,, \tag{11.1}$$

where $\mathcal{M}_T$ is a finite group which is in $1:1$ correspondence with the primitive embeddings $\iota\colon M \to L_{K3}$ and thus with the moduli spaces of lattice-polarised $K3$ surfaces with lattice polarisation M.

The following result is far less obvious. We will not need it for the lattices we are concerned with, as in our cases the genus always consists of one element only, but we state it to complete the picture. The proof of this was communicated to us by Simon Brandhorst; here we only give a sketch.

PROPOSITION 11.3. *The index* $\big[\,\widetilde{\mathrm{O}}(T) : \widetilde{\mathrm{O}}^{+}(T)\big]$ *depends only on the genus* $\mathcal{G}_T$ *of* T *and not on* T *itself.*

Sketch of a proof. This is a consequence of the strong approximation theorem. It can be deduced from [MM09, Proposition VIII.6.1(2)] with extra bookkeeping of the real spinor norm using the fact that (in the terminology of [MM09]) the objects Γ_S, $\Sigma(T)$ and $\Sigma^\sharp(T)$ all depend only on the genus $\mathcal{G}_T$ and not on the lattice T itself. □

As a corollary of the theory of Miranda and Morrison and in particular sequence (11.1) together with Proposition 11.3, we thus obtain the following.

COROLLARY 11.4. *Let* M *be a hyperbolic lattice which admits a primitive embedding into the* $K3$*-lattice* L_{K3}*, and let* $\mathcal{G}_T$ *be the genus of the orthogonal complement of one, and hence any, such embedding of* M *into* L_{K3}*.*

(i) *The number of moduli spaces of* M*-polarised* $K3$ *surfaces is given by*

$$|\mathcal{M}_T| = \big[\,\mathrm{O}(D(T)) : \overline{\mathrm{O}}(T)\big] \cdot |\mathcal{G}_T|\,.$$

(ii) *The number of connected components of these moduli spaces is given by*

$$|\mathcal{M}_T|^c = \big[\,\mathrm{O}(D(T)) : \overline{\mathrm{O}}(T)\big] \cdot |\mathcal{G}_T| \cdot \frac{2}{[\widetilde{\mathrm{O}}(T) : \widetilde{\mathrm{O}}^{+}(T)]}\,.$$

We shall now discuss which values these numbers can have in our cases. The relevant data for the ambi-typical strata are collected in Table 11. The lattice data consist first of a root lattice R and an isotropic subgroup $G \subset D(R)$ in the discriminant group of R. This determines an overlattice L of R and the hyperbolic lattice $M = U + L$.

We shall first prove that in our situation the genus $\mathcal{G}_T$ always consists of a single element.

PROPOSITION 11.5. *Let* M *be a hyperbolic lattice associated with one of the families listed in Table 11. Then the genus given by the signature* $(2, 20 - \mathrm{rank}(M))$ *and discriminant group* $(D(M), -q_M)$ *consists of one element only.*

Proof. The claim for the entries (1), (2) and (3) of Table 11 follows immediately from the numerical conditions of Nikulin's theorem [Nik79, Theorem 1.14.2].

For the other lattices we shall use [CS99, Chapter 15]. Since we have indefinite lattices, according to [CS99, Section 15.9.7], it suffices to show that there are no non-tractable primes (for a definition of non-tractable primes, see [CS99, Chapter 15.9.6]). Let d be the discriminant of M, and let $n = 22 - \mathrm{rank}(M)$ be the rank of the orthogonal complement. According to [CS99, Theorem 15.20], a necessary condition for an odd prime p to be non-tractable is

$$d \text{ is divisible by } p^{\binom{n}{2}}\,.$$

Note that $n = 8$ in the case of entry (4), that is, Shimada's case 565, and $n = 4$ or $n = 3$ in all other cases. One can now check by hand that this condition is never fulfilled for the lattices coming from Table 11. In most cases this follows already from the discriminants of the root lattices R in the second column of this table. However, in some cases one has to be more careful. An example is entry (23), which is Shimada's family 2373. Here $n = 4$, and we must not have divisors of d of the form p^6. Now the lattice $8A_2$ has discriminant 3^8. However, the lattice L is an overlattice of $8A_2(-1)+U$ with torsion group $[3,3]$. But this means that the order of the discriminant group of M is $3^8/3^4 = 3^4$, and hence 3 is not a non-tractable prime. The other cases can be treated in the same way.

Thus the only possible non-tractable prime is $p=2$. By [CS99, Theorem 15.20] this implies that

$$4^{[n/2]}d \text{ is divisible by } 8^{\binom{n}{2}}.$$

Again, this can be checked by hand. For example, in case 2784, one has

$$4^{[n/2]}d(D_4+A_5+2A_3+2A_1+U) = 2^{11}\cdot 3.$$

However, taking the torsion into account, we obtain that $4^{[n/2]}d = 2^7\cdot 3$, which is not divisible by $8^3 = 2^9$. The other cases are similar. □

Corollary 11.6. *Let M be a hyperbolic lattice associated with one of the families listed in Table 11. Then the orthogonal complement of a primitive embedding $\iota\colon M\to L_{K3}$ depends only on M and not on the chosen embedding ι.*

This corollary allows us to speak of *the* orthogonal complement T of the lattice M in the $K3$ lattice L_{K3}, even if the primitive embedding $\iota\colon M\to L_{K3}$ is not uniquely defined (which can occur).

Proposition 11.7. *Let M be a hyperbolic lattice associated with one of the families listed in Table 11, and let T be the unique element in the genus orthogonal to M. Then the map $\mathrm{O}(T)\to\mathrm{O}(D(T))$ is always surjective with the exception of the following five root lattices R:*

(17) $2D_4+4A_2$,

(21) $4A_4$,

(37) $D_4+A_7+2A_2+2A_1$,

(48) $D_4+2A_4+2A_2+A_1$,

(49) $D_4+3A_3+2A_2$.

In these cases, the index $[\mathrm{O}(D(T)):\overline{\mathrm{O}}(T)]$ is 2; in particular, there are exactly two non-isomorphic primitive embeddings of M into L_{K3}.

Proof. For the cases (1), (2) and (3) from Table 11, the surjectivity of $\mathrm{O}(T)\to\mathrm{O}(D(T))$ can be seen directly by Nikulin's criterion [Nik79, Theorem 1.16.10]. In general, this follows from computations of Shimada, which are available on his website; see [Shi16b]. For the rank 17 cases, this follows independently from Kirschmer's computations; see Table 14, column 5 in the appendix. □

It now follows that the number of connected components of M-polarised $K3$ surfaces, where M is a lattice corresponding to an ambi-typical stratum, is either one, two or four. It is at least two in the cases listed in Proposition 11.7. To determine the exact number, one has to compute the index $\big[\widetilde{\mathrm{O}}(T):\widetilde{\mathrm{O}}^+(T)\big]$. Indeed, all three possible cases occur, as the following example shows.

Example 11.8. Assume that the root lattice R has rank 17. Then the following holds: out of the 25 rank 17 lattices, we have $|\mathcal{M}_T|^c = 1$ in 18 cases. In the four cases $D_4+A_7+A_4+2A_1$, $D_4+2A_6+A_1$, $D_4+A_6+A_3+2A_2$ and $D_4+2A_5+3A_1$, we have $|\mathcal{M}_T|=1$ and $\big[\widetilde{\mathrm{O}}(T):\widetilde{\mathrm{O}}^+(T)\big]=1$ and thus $|\mathcal{M}_T|^c=2$. Finally, in the three cases $D_4+A_7+2A_2+2A_1$, $D_4+2A_4+2A_2+A_1$ and $D_4+3A_3+2A_2$, we have $|\mathcal{M}_T|=2$ and $\big[\widetilde{\mathrm{O}}(T):\widetilde{\mathrm{O}}^+(T)\big]=1$ and hence $|\mathcal{M}_T|^c=4$. This follows from Kirschmer's computations; see Table 14 in the appendix. Note that we have two components if either $\left[\widetilde{\mathrm{O}}(T) : \widetilde{\mathrm{O}}^+(T)\right] = 1$ or $\left[\mathrm{O}(D(T)) : \overline{\mathrm{O}(T)}\right] = 2$ and four components if both of these indices are 1 and 2, respectively. In all other cases, we have one component.

Remark 11.9. It is interesting to compare this with Shimada's list of non-connected moduli of elliptic $K3$ surfaces; see [Shi18, Corollary 1.5 and Table II]. We first observe that the lattices which appear in Proposition 11.7 do not appear in Shimada's list. The reason is that different moduli spaces of lattice-polarised $K3$ surfaces can lead to the same Shimada stratum. This is due to symmetries of the root lattice R; we shall discuss this in the next section. On the other hand, there are three root lattices which appear in both Table 11 of our paper and Table 3 in [Shi18, Section 7, pp. 555–557]. These are $D_4+2A_6+A_1$, which is our cases (38/39), as well as $2A_3+8A_1$ and $4A_3+2A_1$, which are our cases (6) and (7), respectively. In the case of $D_4+2A_6+A_1$, we have two complex-conjugate components. The lattice $4A_3+2A_1$ appears in [Shi18, Table 3] in connection with the Mordell–Weil torsion $\mathbb{Z}/2\mathbb{Z}$. In this case, one has two components. However, in our situation, we have Mordell–Weil torsion $\mathbb{Z}/4\mathbb{Z}$, and this leads to one component only. Finally, in the case of $2A_3+8A_1$ the two components in Shimada's list come from inequivalent isotropic subgroups $\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}$ which correspond to different intersection behaviour of the torsion sections. Only one of these cases leads to an ambi-typical stratum; see the discussion in Remark 9.6.

12. The moduli maps

The aim of this section is to investigate the precise relationship between moduli spaces of lattice-polarised $K3$ surfaces and ambi-typical strata. In particular, we show that there is a finite map from certain moduli spaces of lattice-polarised $K3$ surfaces to ambi-typical strata and compute the degree of this map.

For this we consider an ambi-typical stratum

$$\mathcal{A} = \overline{\mathcal{F}'_{\Gamma,i}} = \overline{\mathcal{M}'_{R,G,\iota}}\,. \tag{12.1}$$

As before we denote by L the lattice defined by the isotropic group $G \subset D(R)$ and set $M = U + L$. Our first goal is to associate with such a given ambi-typical stratum certain moduli spaces of lattice-polarised $K3$ surfaces. First of all, given any surface S in such a stratum, we identify the sublattice spanned by the section and the fibre with a fixed summand U of the $K3$ lattice L_{K3}. More precisely, we first choose the standard basis e, f of U with $e^2 = f^2 = 0$ and $e.f = 1$ and identify the fibre class with f and the section s with $e - f$. This can be done once and for all simultaneously for all surfaces in this stratum. Now choose a generic surface S in $\overline{\mathcal{M}'_{R,G,\iota}}$ and a marking $\varphi\colon H^2(S,\mathbb{Z}) \to L_{K3}$ where the sublattice spanned by the fibre and the section are identified with a given summand U as described above. The components of the singular fibres not meeting the section define a sublattice isomorphic to R whose saturation in the $K3$ lattice is isomorphic to L. We note that the isomorphism with R is not canonical but depends on an identification of these components with a set of simple roots of R. There are finitely many ways of choosing such an identification. Adding the sublattice spanned by the fibre and the section, one obtains an embedding

$$\iota\colon M \to \mathrm{NS}(S) \subset H^2(S,\mathbb{Z}) \cong L_{K3}$$

and thus an element (S,ι) in a connected component $\mathcal{N}_{M,\ell}$ of the moduli space $\mathcal{N}_{M,\iota}$ of lattice-polarised $K3$ surfaces. We shall refer to such a component, as well as to the moduli space $\mathcal{N}_{M,\iota}$

itself, as *associated* with $\mathcal{A}$. Note that we do not claim that $\mathcal{N}_{M,\ell}$ or $\mathcal{N}_{M,\iota}$ are uniquely determined by the ambi-typical stratum since there is no canonical way of identifying the fibre components with a root system. Nor do we claim that we obtain a morphism from (an open part of) the stratum $\mathcal{A}$ to such a moduli space $\mathcal{N}_{M,\iota}$. Indeed, we always have (at least) the ambiguity given by the symmetries S_R of the Dynkin diagram associated with R. Changing the isomorphism by such an element can have two effects. One is that it defines different points in the same moduli space of M-lattice-polarised $K3$ surfaces. The other is that it leads to different moduli spaces of lattice-polarised $K3$ surfaces. Note, however, that two such moduli spaces of M-polarised $K3$ surfaces define the same Shimada stratum as the combinatorial type of the singular fibres is the same. As we shall see, both cases occur. We shall now discuss this in detail. Before doing this, we establish the following convention. Given such a moduli space $\mathcal{N}_{M,\iota}$ of lattice-polarised $K3$ surfaces and a $K3$ surface S which appears in this moduli space, we always have a distinguished copy U. In each component $\mathcal{N}_{M,\ell}$, we consider the non-empty open part of the moduli space where this copy of U contains a nef isotropic class and an irreducible (-2)-curve. This determines a Jacobian fibration $S \to \mathbb{P}^1$ whose fibre and section are contained in U. Unless stated otherwise, we will always work with this elliptic fibration. By $\mathcal{N}^0_{M,\ell}$ we further denote the open subset where the configuration of the singular fibres is generic.

Proposition 12.1. (i) *Given an ambi-typical stratum $\mathcal{A}$, there are only finitely many components of moduli spaces of lattice-polarised $K3$ surfaces associated with $\mathcal{A}$.*

(ii) *Let $\mathcal{N}_{M,\ell}$ be a component of a moduli space $\mathcal{N}_{M,\iota}$ of lattice-polarised $K3$ surfaces associated with $\mathcal{A}$, and let $\mathcal{N}^0_{M,\ell}$ be the open subset where the configuration of the singular fibres is generic. Then there is a natural finite-to-one dominant morphism $\mathcal{N}^0_{M,\ell} \to \mathcal{A}$.*

Proof. Claim (1) follows from Nikulin's theory since there are only finitely many inequivalent primitive embeddings $\iota\colon L \to L_{K3}$.

To prove claim (2), we recall that we have for every element in $\mathcal{N}_{M,\iota}$ a well-defined Jacobian fibration $S \to \mathbb{P}^1$ which can be written in Weierstraß form. The fact that $\mathcal{F}$ is a coarse moduli space of $K3$ surfaces with a Jacobian fibration defines a morphism $\mathcal{N}_{M,\ell} \to \mathcal{F}$. On the open subset $\mathcal{N}^0_{M,\ell}$, the monodromy is constant with monodromy group Γ, and hence we obtain a morphism $\mathcal{N}^0_{M,\ell} \to \mathcal{A}$. This map has finite fibres since there are only finitely many ways of identifying the components of the singular fibres not intersection the 0-section with a basis of the root lattice R. The map is dominant by the definition of an ambi-typical stratum. $\square$

We now want to understand these maps better. The next two statements will be useful for this.

Lemma 12.2. *Let (R, G) be a pair consisting of a root lattice and an isotropic subgroup $G \subset D(R)$ with associated overlattice L, arising from one of the families listed in Table 11. Then the roots of R and L coincide.*

Proof. A proof can be found in [Shi18, Proposition 3.2]. Indeed, this is a simple geometric argument. We can use the fact that $M = U + L$ is isomorphic to the Néron–Severi group $\mathrm{NS}(S)$ of some (sufficiently general) $K3$ surface S and that R is the subgroup of $\mathrm{NS}(S)$ generated by all fibre components which do not meet the 0-section. Then the claim is geometrically clear: assume that r is a root of L which is not a root of R. Then $\pm r$ is effective and meets neither the 0-section nor a general fibre since it is orthogonal to the summand U which contains the classes of the 0-section and a general fibre. Hence $\pm r$ defines a union of rational curves on the associated elliptic $K3$ surfaces consisting of components of singular fibres, which do not intersect the 0-section. But these roots are already contained in R. □

Remark 12.3. Here we use the specific geometric situation. In general such a statement is false. Indeed, let $R = 4A_1$, and let H be the subgroup generated by the diagonal $(1,1,1,1)$ of $D(R) = 4\mathbb{Z}/2\mathbb{Z}$. Then the overlattice is D_4, and one obtains a new root, namely $r = \frac{1}{2}\sum_{i=1}^{4} r_i$.

Using the above lemma we can now prove the following.

PROPOSITION 12.4. *Let (R,G) be a pair of a root lattice and an isotropic subgroup arising from one of the families listed in Table 11 with associated overlattice L. Then*

$$\mathrm{O}(L) \cong \{g \in \mathrm{O}(R) \mid \overline{g}(G) = G\}\,.$$

Proof. Clearly, the elements of $\mathrm{O}(R)$ which leave G invariant (as a subgroup) define isometries of L, by the construction of this lattice. Conversely, let $g \in \mathrm{O}(L)$. Then g maps roots of L to roots of L. By Lemma 12.2, these are exactly the roots of R, and hence R is mapped to itself. Hence g is an isometry of the lattice inclusion $R \subset L$, and thus $\overline{g}$ maps the isotropic subgroup G to itself. □

Let R be a root lattice. We recall that the Weyl group $W_R \subset \mathrm{O}(R)$ is the group generated by the reflections with respect to the roots $r \in R$. We denote by S_R the subgroup of $\mathrm{O}(R)$ which is induced by symmetries of the Dynkin diagram. We recall from [Hum90, Theorem 12.2] that the isometry group of R is the semi-direct product of the Weyl group W_R and the group S_R:

$$\mathrm{O}(R) = W_R \rtimes S_R\,.$$

Elements in the Weyl group W_R act trivially on the discriminant $D(R)$ and hence, in our situation, lift to isometries of L. In particular, we can consider $W_R \subset \mathrm{O}(L)$. We further denote the subgroup of S_R which leaves G invariant (as a subgroup) by S_R^G. It now follows from Proposition 12.4 that $\mathrm{O}(L)$ is generated by the Weyl group W_R together with the group S_R^G of diagram isometries which fix the subgroup G; that is,

$$\mathrm{O}(L) = W_R \rtimes S_R^G\,. \tag{12.2}$$

We can extend all elements in $\mathrm{O}(L)$ to isometries of $M = U + L$ by taking the identity on the first factor U. By doing this, we can consider S_R^G as a subgroup of $\mathrm{O}(M)$, and to simplify the notation, we shall denote the image of S_R^G in $O(M)$ by S_M. This notation is unambiguous in our situation as there is no lattice R in Table 11 which comes with more than one subgroup $G \subset D(R)$.

The lattice M has signature $(1, \mathrm{rank}(M)-1) = (1, \mathrm{rank}(R)+1)$. As before we fix a connected component $C^+(M)$ of the positive cone in $M_{\mathbb{R}}$. We have already mentioned that the hyperplanes orthogonal to the roots $r \in M$ subdivide $C^+(M)$ into connected components, the Weyl chambers of M. It is well known, see [Huy16, Proposition 8.2.6], that the Weyl group W_M of M acts simply transitively on the Weyl chambers in $C^+(M)$. By Lemma 12.2, we have

$$W_R = W_L \subset W_M \subset \widetilde{\mathrm{O}}^+(M)\,. \tag{12.3}$$

In particular, W_R also acts faithfully on the set of Weyl chambers of M.

For our applications, it will be essential that the group S_M maps Weyl chambers to themselves.

PROPOSITION 12.5. *Let (R,G) be a pair of a root lattice and an isotropic subgroup arising from one of the families listed in Table 11. The elements of S_M, that is, the symmetries of the Dynkin diagram of R fixing G as a group, map all Weyl chambers of $C^+(M)$ to themselves.*

Proof. Here we again make use of the special situation, namely the fact that there exists an M-polarised $K3$ surface S with $\mathrm{NS}(S) \cong M$ and such that the components of the singular fibres which do not intersect the 0-section generate the root lattice R. In fact, these define a set of simple roots which gives rise to the Dynkin diagram associated with R, and S_R acts on these. More precisely, we can choose a primitive embedding $\iota\colon M \to L_{K3}$ and a marking $\varphi\colon H^2(S,\mathbb{Z}) \to L_{K3}$ such that $\varphi(\mathrm{NS}(S)) = \iota(M)$. Let Δ^+ be the set of positive roots (that is, effective (-2)-classes in $\mathrm{NS}(S)$). By [Huy16, Section 8.2.3], it is enough to show that every element of S_M maps positive roots to positive roots. Let $g \in S_M$, and let f be the class of a fibre. Then $g(f) = f$. If s is a positive root with $(f,s) \neq 0$, then $(f,s) > 0$. Since $(f, g(s)) = (g(f), g(s)) = (f,s) > 0$, it follows that $g(s)$ is again positive. By the definition of the group S_M, we have $g(s_0) = s_0$, where s_0 is the class of the 0-section. Then the same argument also shows that $g(s)$ is positive for every positive root s with $(s, s_0) \neq 0$. It remains to consider the positive roots which are orthogonal to f and s_0, but these are exactly the positive roots of R, which are given by non-negative combinations of components of singular fibres which do not intersect the 0-section. Since g permutes these, the claim follows. ◻

We shall now discuss the action of the group of symmetries of the Dynkin diagram on the moduli spaces of M-lattice-polarised $K3$ surfaces in more detail. Let $(S, \tilde\iota)$ be a general element in $\mathcal{N}_{M,\iota}$, more precisely an element in $\mathcal{N}^0_{M,\ell}$, where $\mathcal{N}_{M,\ell}$ is a connected component of $\mathcal{N}_{M,\iota}$ and $\mathcal{N}^0_{M,\ell}$ denotes the open set where the fibre configuration is constant. Then the embedding $\tilde\iota\colon M \to \mathrm{NS}(S) \subset H^2(S,\mathbb{Z})$ is primitive, and there exists a marking $\varphi\colon H^2(S,\mathbb{Z}) \to L_{K3}$ such that $\tilde\iota = \varphi^{-1} \circ \iota$ and such that $\tilde\iota\big(C_M^{\mathrm{pol}}\big)$ contains an ample class on S, where C_M^{pol} is the fixed Weyl chamber in $C^+(M)$ which we have chosen once and for all. By Proposition 12.5, every element $g_M \in S_M \subset \mathrm{O}(M)$ has the property that it fixes the Weyl chambers in $C^+(M)$. Hence $\tilde\iota \circ g_M\colon M \to \mathrm{NS}(S)$ again defines an M-polarisation on S. Now two cases can occur. The first is that $\tilde\iota$ and $\tilde\iota \circ g_M$ define isomorphic embeddings of M into the $K3$ lattice L_{K3}. In this case, $(S,\tilde\iota)$ and $(S, \tilde\iota \circ g_M)$ define elements in the same moduli space $\mathcal{N}_{M,\iota}$ and g_M induces a map from $\mathcal{N}_{M,\iota}$ to itself identifying $(S,\tilde\iota)$ and $(S,\tilde\iota \circ g_M)$. Note that the moduli space $\mathcal{N}_{M,\iota}$ can have one or two components. If it has two components, then $g_M\colon \mathcal{N}_{M,\iota} \to \mathcal{N}_{M,\iota}$ can either fix the components or interchange them. By the discussion in Remark 11.8, all of these possibilities actually occur in our situation. In the second case, ι and $\iota \circ g_M$ define different embeddings, and g induces an isomorphism of moduli spaces $\mathcal{N}_{M,\iota} \to \mathcal{N}_{M,\iota\circ g_M}$. The lattices where more than one embedding exists are listed in Proposition 11.7. In these cases, we have two different embeddings, but there exist symmetries of the Dynkin diagram which define isomorphisms $\mathcal{N}_{M,\iota} \cong \mathcal{N}_{M,\iota\circ g_M}$ (by Remark 11.9). Again, note that $\mathcal{N}_{M,\iota}$ can have one or two components, but the latter does not occur among our cases.

We can also formulate the above discussion in more group-theoretic terms. The choice of a primitive embedding $\iota\colon M \to L_{K3}$ with orthogonal complement T (up to isomorphism of embeddings) is equivalent to the choice of an isomorphism $\alpha_\iota\colon (D(M), q_M) \cong (D(T), -q_T)$ (modulo $\overline{\mathrm{O}}(T)$). Recall the definitions of the groups S_R^G and S_M from (12.2) and the subsequent paragraph. An element $g_M \in S_M$ defines an isometry $\overline{g}_M \in \mathrm{O}(D(M))$ and, via α_ι, an isometry $\alpha_\iota(\overline{g}_M) \in \mathrm{O}(D(T))$. The morphism which maps $(S,\tilde\iota)$ to $(S,\tilde\iota \circ g_M)$ maps $\mathcal{N}_{M,\iota}$ to itself if and only if $\alpha_\iota(\overline{g}_M) \in \overline{\mathrm{O}}(T)$.

If g_M induces a morphism from $\mathcal{N}_{M,\iota}$ to itself, then we can describe this map explicitly. In this case, we have $\alpha_\iota(\overline{g}_M) \in \overline{\mathrm{O}}(T)$ and we can lift this to an element $g_T \in \mathrm{O}(T)$ such that the pair (g_M, g_T) extends to an isometry of L_{K3}. The lift g_T is uniquely determined up to $\widetilde{\mathrm{O}}(T)$. The action of g_T on $\mathcal{N}_{M,\iota} = \Omega_T / \widetilde{\mathrm{O}}(T)$ then induces the map on $\mathcal{N}_{M,\iota}$ in question. Now $\mathcal{N}_{N,\iota}$ has two components if and only if $\widetilde{\mathrm{O}}^+(T) = \widetilde{\mathrm{O}}(T)$, and in this case g_T interchanges the two components if and only if g_T has real spinor norm -1, that is, if and only if $g_T \notin \widetilde{\mathrm{O}}^+(T)$ (which in this case is independent of the chosen lift).

Let $\pi_M \colon \mathrm{O}(M) \to \mathrm{O}(D(M))$ and $\pi_T \colon \mathrm{O}(T) \to \mathrm{O}(D(T))$ be the canonical projections. We define

$$\overline{S}_M = \pi_M(S_M)\,, \tag{12.4}$$

which we will, via $\alpha_\iota \colon D(M) \to D(T)$, also consider as a subgroup $\overline{S}_M \subset \mathrm{O}(D(T))$. As a subgroup, this depends on the embedding ι as it is defined via the isomorphism α_ι. This becomes important when we define

$$\overline{S}_{M,\iota} = \overline{S}_M \cap \overline{\mathrm{O}}(T)\,, \quad \overline{S}^+_{M,\iota} = \overline{S}_M \cap \overline{\mathrm{O}}^+(T) \tag{12.5}$$

and their pre-images

$$\Gamma_{M,\iota} = \pi_T^{-1}\big(\overline{S}_{M,\iota}\big) \subset \mathrm{O}(T)\,, \quad \Gamma^+_{M,\iota} = \pi_T^{-1}\big(\overline{S}^+_{M,\iota}\big) \subset \mathrm{O}^+(T)\,. \tag{12.6}$$

The elements in $\Gamma_{M,\iota}$ are those isometries of T which can be extended to the overlattice L_{K3} of $T \oplus \iota(M)$ (where we do not ask that these isometries act trivially on $\iota(M)$). The group $\Gamma_{M,\iota}$ acts on the period domain Ω_T and induces an action on the moduli space $\mathcal{N}_{N,\iota}$. An element in $\Gamma_{M,\iota}$ fixes the components of $\mathcal{N}_{N,\iota}$ if and only if it is in $\Gamma^+_{M,\iota}$.

By the above discussion, the group $\overline{S}_M$ operates on $\mathrm{O}(D(T))/\overline{\mathrm{O}}(T)$ via $h \mapsto h \circ \alpha_\iota(\overline{g})$. It will be important for us to know whether this action is transitive. The following is essentially a reformulation of Remark 11.9.

Proposition 12.6. *If M is a hyperbolic lattice arising from one of the families listed in Table 11, then $\overline{S}_M$ acts transitively on $\mathrm{O}(D(T))/\overline{\mathrm{O}}(T)$ unless we are in the case where $R = 2A_3 + 8A_1$.*

Proof. As we have said before (see Remark 11.9), comparing the lists in [Shi18, Corollary 1.5] and [Shi18, Table 3] with our Table 11, we find three lattices, namely $4A_3+2A_1$, $2A_3+8A_1$ and $D_4+2A_6+A_1$. The first lattice is irrelevant for us as it appears with Mordell–Weil torsion $\mathbb{Z}/2\mathbb{Z}$ in Shimada's lists, whereas we have torsion $\mathbb{Z}/4\mathbb{Z}$. The reason that $D_4 + 2A_6 + A_1$ appears in Shimada's lists is that there are two connected component, but they belong to the same primitive lattice embedding. Finally, for $2A_3 + 8A_1$ there exist two combinatorially different components of the moduli space coming from different lattice embeddings (but only one of them appears in our classification). □

Our previous discussion can now be summarised as follows. Let $\mathcal{A}$ be an ambi-typical stratum; respectively, let $\mathcal{A}\cup\overline{\mathcal{A}}$ be the union of the two complex-conjugated components 38/39. Then there are finitely many components of moduli spaces $\mathcal{N}_{N,\iota}$ of lattice-polarised $K3$ surfaces which are associated with $\mathcal{A}$ or $\overline{\mathcal{A}}$. The group $\overline{S}_M$ acts transitively on the set of all moduli spaces $\mathcal{N}_{M,\iota}$ whose components are associated with $\mathcal{A}$ or $\mathcal{A} \cup \overline{\mathcal{A}}$, respectively. The groups $\Gamma_{M,\iota}$ act on the moduli spaces $\mathcal{N}_{M,\iota}$ (and their elements may interchange connected components of these moduli spaces).

Remark 12.7. By inspection one sees that for all our lattices, we have $-1 \in \overline{S}_{M,\iota}$ and hence also $-1 \in \Gamma_{M,\iota}$ and hence $\Gamma_{M,\iota}/\big(\pm\,\widetilde{\mathrm{O}}(T)\big) \cong \overline{S}_{M,\iota}/(\pm 1)$.

Theorem 12.8. *Let $\mathcal{A}$ be an ambi-typical stratum, and let $\mathcal{N}_{M,\iota}$ be a moduli space of lattice-polarised $K3$ surfaces associated with $\mathcal{A}$. Then the following hold:*

(i) *If $\mathcal{A}$ is one of the ambi-typical strata different from 38/39, then the dominant map $\mathcal{N}^0_{M,\iota} \to \mathcal{A}$ is given by the action of the finite group $\Gamma_{M,\iota}/\big(\pm\widetilde{\mathrm{O}}(T)\big)$, which acts faithfully.*

(ii) *Let $\mathcal{A} \cup \overline{\mathcal{A}}$ be the union of the two complex-conjugate strata 38/39. Then $\mathcal{N}_{M,\iota}$ has two connected components $\mathcal{N}_{M,\ell}$ and $\overline{\mathcal{N}}_{M,\ell}$, and the dominant map $\mathcal{N}^0_{M,\ell} \cup \overline{\mathcal{N}}^0_{M,\ell} \to \mathcal{A} \cup \overline{\mathcal{A}}$ is given by the group $\Gamma_{M,\iota}/\big(\pm\widetilde{\mathrm{O}}(T)\big)$, which acts faithfully.*

Proof. We start with a surface $S \in \mathcal{A}$ (or $S \in \overline{\mathcal{A}}$). As we have explained before, identifying the fibre components not meeting the 0-section of the singular fibres with simple roots of R and the lattice spanned by the fibre and section with a copy of U, we obtain a lattice polarisation $\tilde{\iota}\colon M \to \mathrm{NS}(S) \subset H^2(S,\mathbb{Z})$. Let $\varphi\colon H^2(S,\mathbb{Z}) \to L_{K3}$ be a marking, and set $\iota = \varphi \circ \tilde{\iota}$. We have to determine when two lattice-polarised $K3$ surfaces in $\mathcal{N}_{M,\iota}$ are mapped to the same point in $\mathcal{A}$ or $\overline{\mathcal{A}}$, respectively.

Assume that two surfaces $(S_1, \tilde{\iota}_1)$ and $(S_2, \tilde{\iota}_2)$ in $\mathcal{N}_{N,\iota}$ define the same point in $\mathcal{A}$ or $\mathcal{A} \cup \overline{\mathcal{A}}$, respectively. Then there is an isomorphism $f\colon S_2 \to S_1$ which respects the elliptic fibration. This induces a map $f^*\colon \mathrm{NS}(S_1) \to \mathrm{NS}(S_2)$. Let $\varphi_i\colon H^2(S_i;\mathbb{Z}) \to L_{K3}$ be markings with $\iota = \tilde{\iota}_i \circ \varphi_i$ for $i = 1, 2$. Then $\varphi_2 \circ f^* \circ \varphi_1$ defines an isometry of $M = U + L$. This is the identity on U as the 0-section and the general fibre are mapped to the 0-section and the general fibre, respectively. By restriction, this then defines an isometry of L. Recall from (12.2) that $\mathrm{O}(L) = W_R \rtimes S_R^G$ and from (12.3) that $W_R = W_L \subset W_M \subset \widetilde{\mathrm{O}}^+(L)$. Since the isometry $\varphi_2 \circ f^* \circ \varphi_1|_M$ maps C_M^{pol} to itself and since the Weyl group acts faithfully on the Weyl chambers, it follows that $\varphi_2 \circ f^* \circ \varphi_1|_M$ defines an element in $\Gamma_{M,\iota}$. Conversely, if two lattice-polarised $K3$ surfaces in $\mathcal{N}^0_{N,\iota}$ are conjugate under $\Gamma_{M,\iota}$, the underlying elliptic $K3$ surfaces are isomorphic and hence define the same point in $\mathcal{A}$ or $\mathcal{A} \cup \overline{\mathcal{A}}$, respectively. Finally, since $\pm\,\mathrm{id}_T$ are the only elements in $\mathrm{O}(T)$ which act trivially on Ω_M, it follows that the group $\Gamma_{M,\iota}/(\pm\widetilde{\mathrm{O}}(T))$ acts faithfully on $\mathcal{N}_{M,\iota}$. □

Corollary 12.9. (i) *If $\mathcal{A}$ is an ambi-typical stratum different from 38/39, then the degree of the dominant map $\mathcal{N}^0_{M,\iota} \to \mathcal{A}$ is given by*

$$\big|\overline{S}_M/(\pm 1)\big|/|\mathcal{M}_T|^c = \big|\overline{S}^+_{M,\iota}/(\pm 1)\big|\,.$$

(ii) *If $\mathcal{A}$ (or $\overline{\mathcal{A}}$) is one of the strata 38/39, then the degree of the dominant map $\mathcal{N}^0_{M,\ell} \cup \overline{\mathcal{N}}^0_{M,\ell} \to \mathcal{A} \cup \overline{\mathcal{A}}$ is given by*

$$2\big|\overline{S}_M/(\pm 1)\big|/|\mathcal{M}_T|^c = \big|\overline{S}^+_{M,\iota}/(\pm 1)\big| = 24\,.$$

Proof. The first equality follows from the fact that $\overline{S}_M$ acts transitively on the connected components of M-polarised $K3$ surfaces with the exception of the cases 38/39, where we have two orbits. The second equality follows from the definition of the group $\overline{S}^+_{M,\iota}$ and the construction of the covering map. □

In the case of 1-dimensional strata, one can compute all the data of the covering map from the moduli space of lattice-polarised $K3$ surfaces to the ambi-typical stratum. In Table 12, we give these data for all strata of dimension 1 with more than one connected component. We list the case number, the root lattice, the number of connected components, the order of the group $\overline{S}_M$ coming from the symmetries of the Dynkin diagram, the degree of the covering map, the genus of the modular curve parameterising the lattice-polarised $K3$ surfaces and finally the genus g_{BPT}

#	Root lattice	$\lvert\mathcal{M}_T\rvert^c$	$\lvert\overline{S}_M/(\pm 1)\rvert$	$d = \lvert\overline{S}^+_{M,\iota}/(\pm 1)\rvert$	g	g_{BPT}
35	$D_4 + A_7 + A_4 + 2A_1$	2	8	4	0	0
37	$D_4 + A_7 + 2A_2 + 2A_1$	4	32	8	0	0
38/39	$D_4 + 2A_6 + A_1$	2	24	24	1	0
42	$D_4 + A_6 + A_3 + 2A_2$	2	96	48	13	0
44	$D_4 + 2A_5 + 3A_1$	2	24	12	0	0
48	$D_4 + 2A_4 + 2A_2 + A_1$	4	192	48	1	0
49	$D_4 + 3A_3 + 2A_2$	4	384	96	5	0

Table 13. Covering data for all 1-dimensional ambi-typical strata with more than one component

of the ambi-typical stratum. The last is always 0 in accordance with [BPT02, Theorem], which says that all monodromy strata are rational. We find it remarkable that in contrast, the genus of the components of the associated moduli spaces of lattice-polarised $K3$ surfaces can be as high as 13. Note that the group $\overline{S}_M$ is a subgroup of the symmetry group of the Dynkin diagram of the root lattice. If the isotropic group G is trivial, then the two groups coincide. Otherwise, the condition that G must be fixed can impose non-trivial extra conditions. This is the case for numbers $37, 48, 49$, where $\overline{S}_{M,\iota}$ is a proper subgroup of index 2 of $\overline{S}_M$. Furthermore, in these cases $\overline{S}^+_{M,\iota}$ is also an index 2 subgroup of $\overline{S}_{M,\iota}$. Altogether, $\overline{S}^+_{M,\iota}$ can have index 1, 2 or 4 in $\overline{S}_M$. The relevant computations, in particular of the genera g, were performed by Markus Kirschmer and are presented in the appendix.

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Appendix. Numerical calculations

Markus Kirschmer

In this appendix, we summarise some explicit computations concerning the 25 rank 17 lattices in Table 11. All computations were done in Magma [BCP97], and the complete code is available from `www.math.rwth-aachen.de/~Markus.Kirschmer/magma/K3.html`.

Let L be one of the rank 17 lattices in Table 11, and set $M := U \oplus L$. Further, let $\iota\colon M \hookrightarrow L_{K3}$ be a primitive embedding.

Constructing T

We first need to construct an integral lattice T isometric to $\iota(M)^\perp$. This can be done without constructing an embedding ι as follows:

Let (V, q) be the ambient quadratic space of T. The fact that $M \oplus T$ and L_{K3} lie in isometric quadratic spaces shows that (V, q) has signature $(2, 1)$ and determinant $\det(M)$. It also uniquely determines the Hasse–Witt-invariants of (V,q). Using [Kir16, Algorithm 3.4.3], we can construct a rational quadratic space isometric to (V,q). Let X be a maximal even lattice in (V,q), that is, $q(X) \subseteq 2\mathbb{Z}$ and no lattice properly containing X has that property; cf. [Kir16, Algorithm 3.5.5]. By [O'M73, Theorem 91:2], the genus of X is unique; thus we may assume $T \subseteq X$.

For any prime p dividing $\#D(L)$, let S_p be the p-Sylow subgroup of $D(L)$. Then

$$\left\{Y \subseteq X \mid D(Y) \cong S_p \text{ and } \#q_Y^{-1}(\{a\}) = \#q_L^{-1}(\{-a\}) \text{ for all } a \in \mathbb{Q}/2\mathbb{Z}\right\}$$

consists of a single genus. Let $Y^{(p)}$ be any representative. By Proposition 11.5, the lattice

$$T := \cap_p Y^{(p)}$$

is isometric to $\iota(M)^{\perp}$.

Computing $\mathrm{O}(T)$ and its subgroups

A finite generating set of $\mathrm{O}(T)$ can be constructed using a variation of Voronoi's algorithm by Mertens [Mer14]. A slight modification of Mertens' algorithms also yields a finite presentation of $\mathrm{O}(T)$ using Bass–Serre theory [BCNS15]. This modification was provided to us by Sebastian Schönnenbeck.

The group $\widetilde{\mathrm{O}}(T)$ is the kernel of the homomorphism $\pi_T\colon \mathrm{O}(T) \to \mathrm{O}(D(T), q_T)$. Since the group $\mathrm{O}(D(T), q_T)$ is finite, we can construct a finite generating set of $\widetilde{\mathrm{O}}(T)$ using the standard orbit stabiliser algorithm. Similarly, the spinor norm map $\mathrm{sn}_{\mathbb{R}}\colon \mathrm{O}(T) \to \{\pm 1\}$ yields finite generating sets for $\mathrm{O}^+(T)$ and $\widetilde{\mathrm{O}}^+(T)$. Note that spinor norms can be computed using Zassenhaus' trick [Zas62]. But one has to keep in mind that our normalisation of the spinor norm on (T,q) corresponds to Zassenhaus' spinor norm on the lattice $(T,-q)$.

Next we want to compute generators for the group $\Gamma_{M,\iota}$; cf. equation (12.6). The group $D(M)$ is finite and so is its automorphism group $\mathrm{Aut}(D(M))$. Thus the subgroup

$$\mathrm{O}(D(M), q_M) = \{f \in \mathrm{Aut}(D(M)) \mid q_M(f(x)) = q_M(x) \text{ for all } x \in D(M)\}$$

can be enumerated by brute force. Similarly, we enumerate the subgroup $\overline{S}_M \subseteq \mathrm{O}(D(M), q_M)$ induced by the automorphism group of the Dynkin diagram of the root lattice R. If $\overline{S}_M = \mathrm{O}(D(M), q_M)$, then $\Gamma_{M,\iota} = \mathrm{O}(T)$. Now suppose $\overline{S}_M \subsetneq \mathrm{O}(D(M), q_M)$. In these cases, it just happens that $\pi_T\colon \mathrm{O}(T) \to \mathrm{O}(D(T), q_T)$ is onto. We start by computing any isometry $\alpha\colon (D(M), q_M) \to (D(T), -q_T)$ using a backtrack approach. This gives us an isomorphism $\mathrm{O}(D(T)) \cong \mathrm{O}(D(M))$. The fact that $\mathrm{O}(T) \to \mathrm{O}(D(T), q_T)$ is onto shows that there exists some $f \in \mathrm{O}(T)$ such that $\pi_T(f) \circ \alpha = \alpha_\iota$. In particular, the pre-image of $\overline{S}_{M,\iota}$ under $\mathrm{O}(T) \to \mathrm{O}(D(T)) \cong \mathrm{O}(D(M))$ must be conjugate to $\Gamma_{M,\iota}$, and we find generators for this group using the orbit stabiliser algorithm.

Fuchsian groups

Let $\mathrm{SO}^+(T) = \{\varphi \in \mathrm{O}^+(T) \mid \det(\varphi) = 1\}$. We denote by $\mathrm{SO}^+(2,1)$ the connected component of the special orthogonal group of a real quadratic space of signature $(2,1)$. The sporadic isomorphism between $\mathrm{SO}^+(2,1)$ and $\mathrm{PSL}(2,\mathbb{R})$ implies that the type IV homogeneous domain associated with T is isomorphic to the upper half-plane $\mathbb{H}_1$. Moreover, it induces an injection $\mathrm{SO}^+(T) \hookrightarrow \mathrm{PSL}(2,\mathbb{R})$ and thus an action of $\mathrm{SO}^+(T)$ on the upper half-plane $\mathbb{H}_1$. This action is properly discontinuous. Hence any finite-index subgroup G of $\mathrm{SO}^+(T)$ is a Fuchsian group; cf. [Kat92, Theorem 2.2.6].

Suppose that G is a finite-index subgroup of $\mathrm{SO}^+(T)$, and let $g := g(\mathbb{H}_1/G)$ be the genus of the (compactified) curve $\mathbb{H}_1/G$. By [Kat92, Section 4.3], the group G admits a presentation

$$G \cong \left\langle a_1, b_1, \ldots, a_g, b_g, x_1, \ldots, x_d, y_1, \ldots, y_t \,\middle|\, \begin{array}{l} x_1^{m_1} = \cdots = x_d^{m_d} = 1 \text{ and} \\ x_1 \cdots x_d y_1 \cdots y_t [a_1, b_1] \cdots [a_g, b_g] = 1 \end{array} \right\rangle \tag{A.1}$$

with $t = 0$ if and only if $\mathbb{H}_1/G$ is compact.

Since the index of G in $\mathrm{O}(T)$ is finite, we can obtain a finite presentation of G from the presentation of $\mathrm{O}(T)$ using the Reidemeister–Schreier method [MKS76, Section 2.3]. Even though this presentation might not be in the form of equation (A.1), it is good enough to determine the genus of $\mathbb{H}_1/G$.

Lemma A.1. *Let L be one of the rank 17 lattices of Table 11, and let $\iota\colon L \oplus U \to L_{K3}$ be a primitive embedding. Further, let $T := \iota(L \oplus U)^{\perp}$, and let G be a finite-index subgroup of $\mathrm{SO}^+(T)$.*

(i) *The space $\mathbb{H}_1/G$ is compact.*

(ii) *The torsion-free part of G/G' has rank $2g(\mathbb{H}_1/G)$.*

Proof. It suffices to prove the first statement for $G = \mathrm{SO}^+(T)$. An explicit computation shows that the abelian group $\mathrm{SO}^+(T)/\mathrm{SO}^+(T)' \cong (\mathbb{Z}/2\mathbb{Z})^r$ is an elementary abelian 2-group. Suppose that $\mathbb{H}_1/\mathrm{SO}^+(T)$ is not compact; that is, the parameter t in equation (A.1) is non-zero. The isomorphism type of $\mathrm{SO}^+(T)/\mathrm{SO}^+(T)'$ implies that $g(\mathbb{H}_1/\mathrm{SO}^+(T)) = 0$ and

$$\mathrm{SO}^+(T) \cong \mathbb{Z}/2\mathbb{Z} * \cdots * \mathbb{Z}/2\mathbb{Z}$$

is a free product of r copies of $\mathbb{Z}/2\mathbb{Z}$. An explicit computation shows that for all lattices T, the groups $\mathrm{SO}^+(T)$ and $\mathbb{Z}/2\mathbb{Z} * \cdots * \mathbb{Z}/2\mathbb{Z}$ have different numbers of subgroups of small index. This proves the first assertion.

The second assertion follows immediately from the fact that the parameter t in equation (A.1) is zero as $\mathbb{H}_1/G$ is compact. □

Now suppose that G is a subgroup of $\mathrm{O}^+(T)$. We denote by $\mathbb{H}_1/G$ the space $(\mathbb{H}_1/\pm G \cap \mathrm{SO}^+(T))$, where $\pm G$ is the subgroup of $\mathrm{O}(T)$ generated by G and $-I_3$.

Results

For each rank 17 lattice in Table 11, we constructed a lattice T as well as generating sets for $\mathrm{O}(T)$, $\mathrm{O}^+(T)$, $\widetilde{\mathrm{O}}(T)$ and $\widetilde{\mathrm{O}}^+(T)$. It turns out that in all cases, $[\mathrm{O}(T) : \mathrm{O}^+(T)] = 2$ and $g(\mathbb{H}_1/\mathrm{O}^+(T)) = 0$. Table 14 lists the indices $I_1 := \left[\mathrm{O}^+(T) : \widetilde{\mathrm{O}}^+(T)\right]$, $I_2 := \left[\widetilde{\mathrm{O}}(T) : \widetilde{\mathrm{O}}^+(T)\right]$ and $I_3 := \left[\mathrm{O}(D(T)) : \overline{\mathrm{O}(T)}\right]$, the order of $\overline{S}_{M,\iota}/(\pm 1)$ and the genus of the moduli space of lattice-polarised $K3$ surfaces. We note that in all but three cases $\overline{S}_M/(\pm 1) = \overline{S}_{M,\iota}/(\pm 1)$. The only cases where this is not so are 37, 48 and 49, where $\overline{S}_{M,\iota}/(\pm 1)$ has index 2 in $\overline{S}_M/(\pm 1)$. Furthermore, in these cases, as well as in cases 35, 42 and 44, the group $\overline{S}^+_{M,\iota}/(\pm 1)$ has index 2 in $\overline{S}_{M,\iota}/(\pm 1)$.

#	Root lattice	I_1	I_2	I_3	$\lvert\overline{S}_M/(\pm 1)\rvert$	$g\big(\mathbb{H}_1/\widetilde{\mathrm{O}}^+(T)\big)$
24	$D_4 + A_{13}$	12	2	1	6	1
25	$D_4 + A_{12} + A_1$	12	2	1	6	1
26	$D_4 + A_{11} + A_2$	16	2	1	4	0
27	$D_4 + A_{11} + 2A_1$	8	2	1	4	0
28	$D_4 + A_{10} + A_2 + A_1$	24	2	1	12	1
29	$D_4 + A_9 + A_4$	72	2	1	12	4

#	Root lattice	I_1	I_2	I_3	$\lvert\overline{S}_M/(\pm 1)\rvert$	$g\big(\mathbb{H}_1/\widetilde{\mathrm{O}}^+(T)\big)$
30	$D_4 + A_9 + A_3 + A_1$	8	2	1	4	1
31	$D_4 + A_9 + 2A_2$	96	2	1	48	7
32	$D_4 + A_9 + A_2 + 2A_1$	8	2	1	4	0
33	$D_4 + A_8 + A_5$	72	2	1	12	4
34	$D_4 + A_8 + A_4 + A_1$	24	2	1	12	1
35	$D_4 + A_7 + A_4 + 2A_1$	8	1	1	8	0
36	$D_4 + A_7 + A_3 + A_2 + A_1$	16	2	1	8	1
37	$D_4 + A_7 + 2A_2 + 2A_1$	32	1	2	32	0
38/39	$D_4 + 2A_6 + A_1$	48	1	1	24	1
40	$D_4 + A_6 + A_5 + A_2$	48	2	1	24	7
41	$D_4 + A_6 + A_4 + A_2 + A_1$	48	2	1	24	1
42	$D_4 + A_6 + A_3 + 2A_2$	96	1	1	96	13
43	$D_4 + 2A_5 + A_3$	32	2	1	16	3
44	$D_4 + 2A_5 + 3A_1$	24	1	1	24	0
45	$D_4 + A_5 + 2A_4$	96	2	1	48	7
46	$D_4 + A_5 + A_4 + A_3 + A_1$	16	2	1	8	1
47	$D_4 + A_5 + 2A_3 + 2A_1$	16	2	1	8	0
48	$D_4 + 2A_4 + 2A_2 + A_1$	192	1	2	192	1
49	$D_4 + 3A_3 + 2A_2$	384	1	2	384	5

Table 14. Indices $I_1 = \big[\mathrm{O}^+(T) : \widetilde{\mathrm{O}}^+(T)\big]$, $I_2 = \big[\widetilde{\mathrm{O}}(T) : \widetilde{\mathrm{O}}^+(T)\big]$, $I_3 = \big[\mathrm{O}(D(T)) : \overline{\mathrm{O}(T)}\big]$, order of $\overline{S}_{M,\iota}/(\pm 1)$ and genus of the moduli space of lattice-polarised $K3$ surfaces

There are only four lattices T for which $\Gamma_{M,\iota} \neq \mathrm{O}(T)$. In these cases, the genus of $\hat{C}_T := \mathbb{H}_1/\Gamma^+_{M,\iota}$, which is birational to the ambi-typical stratum, is again 0, once more in concordance with [BPT02, Theorem]. Table 15 lists the indices $\big[\mathrm{O}^+(T) : \Gamma^+_{M,\iota}\big]$ and $\big[\Gamma_{M,\iota} : \Gamma^+_{M,\iota}\big]$ in these cases.

#	Root lattice	$\big[\mathrm{O}^+(T) : \Gamma^+_{M,\iota}\big]$	$\big[\Gamma_{M,\iota} : \Gamma^+_{M,\iota}\big]$
26	$D_4 + A_{11} + A_2$	2	2
29	$D_4 + A_9 + A_4$	3	2
33	$D_4 + A_8 + A_5$	3	2
38/39	$D_4 + 2A_6 + A_1$	1	1

Table 15. Indices of $\Gamma^+_{M,\iota}$ in $\mathrm{O}^+(T)$ and $\Gamma_{M,\iota}$

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