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Punctual Hilbert schemes for Kleinian singularities as quiver varieties

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Abstract

For a finite subgroup $\Gamma \subset \mathrm{SL}(2,\mathbb{C})$ and $n \geqslant 1$, we construct the (reduced scheme underlying the) Hilbert scheme of n points on the Kleinian singularity $\mathbb{C}^2/\Gamma$ as a Nakajima quiver variety for the framed McKay quiver of Γ, taken at a specific non-generic stability parameter. We deduce that this Hilbert scheme is irreducible (a result previously due to Zheng), normal and admits a unique symplectic resolution. More generally, we introduce a class of algebras obtained from the preprojective algebra of the framed McKay quiver by removing an arrow and then 'cornering', and we show that fine moduli spaces of cyclic modules over these new algebras are isomorphic to quiver varieties for the framed McKay quiver and certain non-generic choices of the stability parameter.

1. Introduction

Let $\Gamma \subset \mathrm{SL}(2,\mathbb{C})$ be a finite subgroup. One can associate various Hilbert schemes to the action of Γ on the affine plane $\mathbb{C}^2$ and the Kleinian singularity $\mathbb{C}^2/\Gamma$. For $N := |\Gamma|$ and any natural number n, the action of Γ on $\mathbb{C}^2$ induces an action of Γ on the Hilbert scheme $\mathrm{Hilb}^{[nN]}(\mathbb{C}^2)$ of nN points on the affine plane. The scheme $n\Gamma\text{-}\mathrm{Hilb}(\mathbb{C}^2)$, parametrising Γ-invariant ideals I in $\mathbb{C}[x,y]$ such that the quotient $\mathbb{C}[x,y]/I$ is isomorphic to the direct sum of n copies of the regular representation of Γ, is a union of components of the fixed point set of the Γ-action on $\mathrm{Hilb}^{[nN]}(\mathbb{C}^2)$. It is thus non-singular and quasi-projective. One may also consider the Hilbert scheme of n points $\mathrm{Hilb}^{[n]}(\mathbb{C}^2/\Gamma)$ on the singular surface $\mathbb{C}^2/\Gamma$, parametrising ideals in the invariant ring $\mathbb{C}[x,y]^\Gamma$ that have codimension n. This Hilbert scheme is quasi-projective, and in this introduction we endow it with the *reduced* scheme structure.

These two kinds of Hilbert schemes are related by the morphism

$$n\Gamma\text{-}\mathrm{Hilb}(\mathbb{C}^2) \longrightarrow \mathrm{Hilb}^{[n]}(\mathbb{C}^2/\Gamma) \tag{1.1}$$

sending a Γ-invariant ideal I in $\mathbb{C}[x,y]$ to the ideal $I \cap \mathbb{C}[x,y]^\Gamma$; this set-theoretic map is indeed a morphism of schemes by Brion [Bri13, Section 3.4]. By composing with the Hilbert–Chow morphism of the surface $\mathbb{C}^2/\Gamma$, we see that (1.1) is in fact a morphism of schemes over the affine scheme $\mathrm{Sym}^n(\mathbb{C}^2/\Gamma)$.

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Until recently, not much was known about the schemes $\mathrm{Hilb}^{[n]}\left(\mathbb{C}^2/\Gamma\right)$ for $n > 1$. Gyenge, Némethi and Szendrői [GNS18] computed the generating function of their Euler characteristics for Γ of type A and D (the cyclic and dihedral cases), giving an answer with modular properties. They also conjectured an analogous formula for type E. Zheng [Zhe17] proved that $\mathrm{Hilb}^{[n]}\left(\mathbb{C}^2/\Gamma\right)$ is always irreducible and gave a homological characterisation of its smooth points through a detailed analysis of Cohen–Macaulay modules over $\mathbb{C}^2/\Gamma$. Yamagishi [Yam17] studied symplectic resolutions of the Hilbert squares $\mathrm{Hilb}^{[2]}\left(\mathbb{C}^2/\Gamma\right)$ and described completely the central fibres of these resolutions, from which he deduced that $\mathrm{Hilb}^{[2]}\left(\mathbb{C}^2/\Gamma\right)$ admits a unique symplectic resolution.

The aim of our paper is to study the spaces appearing in (1.1) and all possible ways in which the morphism from (1.1) can be decomposed, using quiver-theoretic techniques in a uniform way. The starting point is the McKay correspondence, which associates a quiver (oriented graph) to the subgroup $\Gamma \subset \mathrm{SL}(2,\mathbb{C})$. Representation spaces of a framed variant of the McKay quiver, each depending on a stability parameter, were introduced in Kronheimer and Nakajima [KN90] and studied further by Nakajima [Nak94]. Subsequently, for any $n \geqslant 1$ and for a special choice of framing, Kuznetsov [Kuz07] determined a pair of cones $C_\pm$ in the space of stability parameters for which the corresponding representation space $\mathfrak{M}_\theta$ is isomorphic to the punctual Hilbert scheme $\mathrm{Hilb}^{[n]}(S)$ of the minimal resolution S of $\mathbb{C}^2/\Gamma$ for $\theta \in C_-$ and to the scheme $n\Gamma\text{-}\mathrm{Hilb}\left(\mathbb{C}^2\right)$ from (1.1) for $\theta \in C_+$. Much more recently, Bellamy and Craw [BC20] gave a complete description of the wall-and-chamber structure on the space of stability parameters in this situation and identified a simplicial cone F containing $C_\pm$ that is isomorphic as a fan to the movable cone of $n\Gamma\text{-}\mathrm{Hilb}\left(\mathbb{C}^2\right)$ for $n > 1$; in particular, chambers in this simplicial cone correspond one-to-one with projective, symplectic resolutions of $\mathrm{Sym}^n\left(\mathbb{C}^2/\Gamma\right)$ (see Figure 1 below for an example).

The main result of our paper reconstructs the morphism from (1.1) by variation of geometric invariant theory (GIT) quotient. Explicitly, we vary a generic stability parameter $\theta \in C_+$ to a parameter θ_0 in a particular extremal ray of the closure of C_+; the induced morphism $\mathfrak{M}_\theta \to \mathfrak{M}_{\theta_0}$ coincides with the morphism (1.1). As a corollary, we obtain the following result.

Theorem 1.1. *Let $\Gamma \subset \mathrm{SL}(2,\mathbb{C})$ be a finite subgroup, and let $n \geqslant 1$. The (reduced) Hilbert scheme $\mathrm{Hilb}^{[n]}\left(\mathbb{C}^2/\Gamma\right)_{\mathrm{red}}$ is an irreducible, normal scheme with symplectic, hence rational Gorenstein, singularities. Furthermore, it admits a unique projective, symplectic resolution given by (1.1).*

We reiterate that irreducibility is due originally to Zheng [Zhe17]. The existence of a nowhere-vanishing $2n$-form in the type A case, which follows from having symplectic singularities, was shown in the same paper [Zhe17, Theorem D], while the existence and uniqueness of the symplectic resolution for $n = 2$ is due to Yamagishi [Yam17].

Our main tool is to furnish $\mathrm{Hilb}^{[n]}\left(\mathbb{C}^2/\Gamma\right)$ with a quiver-theoretic interpretation as a fine quiver moduli space by the process of cornering [CIK18]. More generally, we provide a fine moduli space description of the quiver varieties $\mathcal{M}_\theta$ for all non-generic stability parameters that lie in the closure of the cone C_+. Our methods give conceptual proofs of the geometric properties of $\mathrm{Hilb}^{[n]}\left(\mathbb{C}^2/\Gamma\right)$ listed in Theorem 1.1 and allow us to obtain all possible projective factorisations of the morphism (1.1) by universal properties of the resulting fine moduli spaces. Our proofs for all statements concerning $\mathrm{Hilb}^{[n]}\left(\mathbb{C}^2/\Gamma\right)$ avoid case-by-case analysis with respect to the ADE classification of Γ. We use only one such case-by-case argument for our more general quiver varieties $\mathcal{M}_\theta$ to establish a bound on the dimension vector for quiver representations that are stable with respect to a non-generic stability condition; see the appendix, Sections A.2–A.4.

Quiver varieties with degenerate stability conditions identical to ours were considered before in [Nak09]. In very recent work, Nakajima [Nak20] uses the main result of our paper and some results from the representation theory of quantum affine algebras to prove the conjecture of [GNS18].

Notation. Let $\pi\colon X \to Y$ be a projective morphism of schemes over an affine base Y. For a globally generated line bundle L on X, write $|L| := \operatorname{Proj}_Y \bigoplus_{k\geqslant 0} H^0\big(X, L^k\big)$ for the (relative) linear series of L and $\varphi_{|L|}\colon X \to |L|$ for the induced morphism over Y.

2. Variation of GIT quotient for quiver varieties

Let $\Gamma \subset \mathrm{SL}(2,\mathbb{C})$ be a finite subgroup. Let V denote its given two-dimensional representation, defined by this inclusion. Write $\rho_0, \rho_1, \dots, \rho_r$ for the irreducible representations of Γ, with ρ_0 the trivial one. The *McKay graph* of Γ has vertex set $\{0, 1, \dots, r\}$, where vertex i corresponds to the representation ρ_i of Γ, and there are $\dim \operatorname{Hom}_\Gamma(\rho_j, \rho_i \otimes V)$ edges between vertices i and j. By the McKay correspondence [McK80], the McKay graph is an extended Dynkin diagram of ADE type. Add a framing vertex ∞, together with an edge between vertices ∞ and 0, and let Q_1 be the set of pairs consisting of an edge in this graph and an orientation of the edge. If a is an edge with orientation, we write a^* for the same edge with the opposite orientation. The *framed McKay quiver* Q has vertex set $Q_0 = \{\infty, 0, 1, \dots, r\}$ and arrow set Q_1, where for each oriented edge $a \in Q_1$, we write $\mathrm{t}(a)$ and $\mathrm{h}(a)$ for the tail and head of a, respectively.

Let $\mathbb{C}Q$ denote the path algebra of Q. Let $\epsilon\colon Q_1 \to \{\pm 1\}$ be any map such that $\epsilon(a) \neq \epsilon(a^*)$ for all $a \in Q_1$. The preprojective algebra Π is the quotient of $\mathbb{C}Q$ by the ideal generated by the relation

$$\sum_{a\in Q_1} \epsilon(a) a a^* .$$

Equivalently, multiplying both sides of this relation by the vertex idempotent at vertex i shows that Π can be presented as the quotient of $\mathbb{C}Q$ by the ideal

$$\left(\sum_{\mathrm{h}(a)=i} \epsilon(a) a a^* \mid i \in Q_0 \right). \tag{2.1}$$

The preprojective algebra Π does not depend on the choice of the map ϵ; see [CH98, Lemma 2.2]. Let $R(\Gamma)$ denote the representation ring of Γ. Introduce a formal symbol ρ_∞ so that $\{\rho_i \mid i \in Q_0\}$ provides a $\mathbb{Z}$-basis for $\mathbb{Z}^{Q_0} \cong \mathbb{Z} \oplus R(\Gamma)$ considered as $\mathbb{Z}$-modules.

For a natural number $n \geqslant 1$ that we fix for the rest of the paper, consider the dimension vector

$$v := (v_i)_{i\in Q_0} := \rho_\infty + \sum_{i\geqslant 0} n \dim(\rho_i) \rho_i \in \mathbb{Z}^{Q_0} .$$

The group $G(v) := \mathbb{C}^\times \times \prod_{0\leqslant i\leqslant r} \mathrm{GL}\big(n \dim(\rho_i), \mathbb{C}\big)$ acts on the space

$$\operatorname{Rep}(Q, v) := \bigoplus_{a\in Q_1} \operatorname{Hom}\left(\mathbb{C}^{v_{\mathrm{t}(a)}}, \mathbb{C}^{v_{\mathrm{h}(a)}}\right)$$

of representations of the quiver Q of dimension vector v by conjugation. The diagonal scalar subgroup acts trivially, and the action of the quotient $G := G(v)/\mathbb{C}^\times$ induces a moment map $\mu\colon \operatorname{Rep}(Q, v) \to \mathfrak{g}^*$ such that a closed point lies in $\mu^{-1}(0)$ if and only if the corresponding $\mathbb{C}Q$-module satisfies the relations (2.1) of the preprojective algebra Π, see, for example, [Cra21, Remark 3.4(2)]. If we write

$$\Theta_v = \left\{\theta\colon \mathbb{Z}^{Q_0} \to \mathbb{Q} \mid \theta(v) = 0\right\},$$

then each character of G is $\chi_\theta\colon G \to \mathbb{C}^\times$ for some integer-valued $\theta \in \Theta_v$, where $\chi_\theta(g) = \prod_{i\in Q_0} \det(g_i)^{\theta_i}$ for $g \in G(v)$.

Given a stability parameter $\theta \in \Theta_v$, recall that a Π-module M is θ-stable (respectively, θ-semistable) if $\theta(\dim M) = 0$ and for every proper, non-zero submodule $N \subset M$, we have $\theta(\dim N) > 0$ (respectively, $\theta(\dim N) \geqslant 0$). Two θ-semistable Π-modules M and M' are said to be S-equivalent if they admit filtrations

$$0 = M_0 \subset M_1 \subset \cdots \subset M_{s_1} = M \quad \text{and} \quad 0 = M'_0 \subset M'_1 \subset \cdots \subset M'_{s_2} = M'$$

such that each M_i and each M'_j is θ-semistable and

$$\bigoplus_{i=1}^{s_1} M_i/M_{i-1} \cong \bigoplus_{i=1}^{s_2} M'_i/M'_{i-1}\,.$$

Every S-equivalence class has a representative unique up to isomorphism that is a direct sum of θ-stable modules, the so-called *polystable* module.

Given $\theta \in \Theta_v$, the quiver variety

$$\mathfrak{M}_\theta \coloneqq \left(\mu^{-1}(0)/\!\!/_\theta G\right)_{\mathrm{red}}$$

is the categorical quotient of the locus of χ_θ-semistable points of $\mu^{-1}(0)$ by the action of G. It is the coarse moduli space of S-equivalence classes of θ-semistable Π-modules of dimension vector v. As indicated, we consider these GIT quotients with their reduced scheme structure everywhere below.

Lemma 2.1. *For all $\theta \in \Theta_v$, the scheme $\mathfrak{M}_\theta$ is irreducible and normal, with symplectic singularities.*

Proof. See Bellamy and Schedler [BS21, Theorem 1.2, Proposition 3.21]. □

The set of stability conditions Θ_v admits a preorder $\geqslant$, where $\theta \geqslant \theta'$ if and only if every θ-semistable Π-module is θ'-semistable. It is well known [DH98, Tha96] that we obtain a wall-and-chamber structure on Θ_v, where $\theta, \theta' \in \Theta_v$ lie in the relative interior of the same cone if and only if both $\theta \geqslant \theta'$ and $\theta' \geqslant \theta$ hold in this preorder, in which case $\mathfrak{M}_\theta \cong \mathfrak{M}_{\theta'}$. The interiors of the top-dimensional cones in Θ_v are *GIT chambers*, while the codimension one faces of the closure of each GIT chamber are *GIT walls*. We say that $\theta \in \Theta_v$ is *generic* with respect to v if it lies in some GIT chamber; equivalently, θ is generic if every θ-semistable Π-module is θ-stable. Since v is indivisible, King [Kin94, Proposition 5.3] proves that for generic $\theta \in \Theta_v$, the quiver variety $\mathfrak{M}_\theta$ is the fine moduli space of isomorphism classes of θ-stable Π-modules of dimension vector v. In this case, the universal family on $\mathfrak{M}_\theta$ is a tautological locally free sheaf

$$\mathcal{R} \coloneqq \bigoplus_{i\in Q_0} \mathcal{R}_i$$

together with a $\mathbb{C}$-algebra homomorphism $\phi\colon \Pi \to \mathrm{End}(\mathcal{R})$, where $\mathcal{R}_\infty$ is the trivial bundle on $\mathfrak{M}_\theta$ and where $\mathrm{rank}(\mathcal{R}_i) = n\dim(\rho_i)$ for $i \geqslant 0$.

Variation of GIT quotient for the quiver varieties $\mathfrak{M}_\theta$ was investigated recently by the first author with Bellamy [BC20]. The following result records a surjectivity statement that will be useful later on.

LEMMA 2.2. *Let $\theta, \theta' \in \Theta_v$ satisfy $\theta \geqslant \theta'$. Then the morphism $\pi\colon \mathfrak{M}_\theta \to \mathfrak{M}_{\theta'}$ obtained by variation of GIT quotient is a surjective, projective and birational morphism of varieties over* $\mathrm{Sym}^n\left(\mathbb{C}^2/\Gamma\right)$.

Proof. If θ is generic and $\theta' = 0$, then the morphism $\mathfrak{M}_\theta \to \mathfrak{M}_0 \cong \mathrm{Sym}^n\left(\mathbb{C}^2/\Gamma\right)$ is a projective symplectic resolution [BC20, Theorem 4.5] and the result holds. For the general case, combining [BS21, Lemma 3.22] and [BC20, Lemma 4.4], we get $\dim \mathfrak{M}_\theta = 2n$. This holds for any $\theta \in \Theta_v$, so $\dim \mathfrak{M}_{\theta'} = 2n$. The morphism $\pi\colon \mathfrak{M}_\theta \to \mathfrak{M}_{\theta'}$ is projective, so the image $Z := \pi(\mathfrak{M}_\theta)$ is closed in $\mathfrak{M}_{\theta'}$. Deform θ if necessary to a generic parameter η such that $\eta \geqslant \theta$. Then the resolution $\mathfrak{M}_\eta \to \mathfrak{M}_0 \cong \mathrm{Sym}^n\left(\mathbb{C}^2/\Gamma\right)$ factors through π by variation of GIT quotient, so $\dim(Z) = 2n$ and hence π is birational onto its image. It follows that Z is an irreducible component of $\mathfrak{M}_{\theta'}$. However, $\mathfrak{M}_{\theta'}$ is irreducible [BS21, Proposition 3.21]; so π is surjective. □

The GIT wall-and-chamber structure on Θ_v was computed explicitly in [BC20, Theorem 4.6]. In this paper, we focus on the distinguished GIT chamber

$$C_+ := \left\{\theta \in \Theta_v \mid \theta(\rho_i) > 0 \text{ for } i \geqslant 0\right\}. \tag{2.2}$$

It is well known that the quiver variety $\mathfrak{M}_\theta$ for $\theta \in C_+$ admits a description as an equivariant Hilbert scheme. Recall from the introduction that $n\Gamma\text{-}\mathrm{Hilb}\left(\mathbb{C}^2\right)$ is the scheme parametrising Γ-invariant ideals $I \lhd \mathbb{C}[x,y]$ with quotient isomorphic as a representation of Γ to the direct sum of n copies of the regular representation of Γ.

THEOREM 2.3 ([VV99, Wan99, Kuz07]). *Let $\Gamma_n := \Gamma^n \rtimes \mathfrak{S}_n \subset \mathrm{Sp}(2n,\mathbb{C})$ denote the wreath product of Γ with the symmetric group $\mathfrak{S}_n$. For $\theta \in C_+$, there is a commutative diagram*

$$\begin{array}{ccc} n\Gamma\text{-}\mathrm{Hilb}\left(\mathbb{C}^2\right) & \xrightarrow{\ \sim\ } & \mathfrak{M}_\theta \\ \downarrow & & \downarrow{\scriptstyle\pi} \\ \mathbb{C}^{2n}/\Gamma_n \cong \mathrm{Sym}^n\left(\mathbb{C}^2/\Gamma\right) & \xrightarrow{\ \sim\ } & \mathfrak{M}_0 \end{array}$$

in which the horizontal arrows are isomorphisms and the vertical arrows are symplectic resolutions.

We now study partial resolutions of $\mathrm{Sym}^n\left(\mathbb{C}^2/\Gamma\right)$ through which the resolution from Theorem 2.3 factors. The result of [BC20, Proposition 6.1] implies that for $n > 1$, the nef cone of $n\Gamma\text{-}\mathrm{Hilb}\left(\mathbb{C}^2\right)$ over $\mathrm{Sym}^n\left(\mathbb{C}^2/\Gamma\right)$ is isomorphic to the closure $\overline{C_+}$ of the chamber from (2.2). For $n = 1$, the relation between these two cones is described in [BC20, Proposition 7.11] (see Remark 6.5 for more on the case $n = 1$). In any case, for $n \geqslant 1$, the partial resolutions of interest can all be obtained as follows: choose a face of $\overline{C_+}$ and any GIT parameter from the relative interior of that face; then perform variation of GIT quotient as the parameter moves to the origin in Θ_v.

Every face of $\overline{C_+}$ is of the form

$$\sigma_J := \left\{\theta \in \overline{C_+} \mid \theta(\rho_j) > 0 \text{ if and only if } j \in J\right\}$$

for some (possibly empty) subset $J \subseteq \{0, 1, \ldots, r\}$. The parameter $\theta_J \in \overline{C_+}$ defined by setting

$$\theta_J(\rho_i) = \begin{cases} -\sum_{j\in J} n\dim(\rho_j) & \text{for } i = \infty\,, \\ 1 & \text{if } i \in J\,, \\ 0 & \text{if } i \in \{0,1,\ldots,r\}\setminus J \end{cases}$$

lies in the relative interior of σ_J. To simplify notation, in the case $J = \{0\}$ we occasionally write $\theta_0 := \theta_{\{0\}}$.

Proposition 2.4. *The face poset of the cone $\overline{C_+}$ can be identified with the poset on the set of quiver varieties $\mathfrak{M}_{\theta_J}$ for subsets $J \subseteq \{0, 1, \dots, r\}$, where edges in the Hasse diagram of the poset are realised by the surjective, projective and birational morphisms $\pi_{J,J'} \colon \mathfrak{M}_{\theta_J} \to \mathfrak{M}_{\theta_{J'}}$.*

Proof. This is standard for variation of GIT quotient apart from the surjectivity and birationality of each $\pi_{J,J'}$. This was established in Lemma 2.2. □

Remark 2.5. When $J' = \varnothing$ and $J = \{0, \dots, r\}$, the morphism $\mathfrak{M}_{\theta_J} \to \mathfrak{M}_{\theta_{J'}}$ is the resolution $n\Gamma\text{-Hilb}(\mathbb{C}^2) \to \operatorname{Sym}^n(\mathbb{C}^2/\Gamma)$ from Theorem 2.3. The statement of Proposition 2.4 implies that the paths in the Hasse diagram of the face poset of $\overline{C_+}$ from the unique maximal element to the unique minimal element provide all possible ways in which this resolution can be decomposed via primitive morphisms [Wil92].

Example 2.6. Consider the case $\Gamma \cong \mu_3$, corresponding to Dynkin type A_2 and $n = 3$. Figure 1 shows a transverse slice of the GIT wall-and-chamber structure inside a specific closed cone F in the space Θ_v of stability parameters. According to [BC20, Theorem 1.2], this decomposition of the cone is isomorphic as a fan to the closure of the movable cone of this particular $n\Gamma\text{-Hilb}(\mathbb{C}^2)$, with its natural subdivision into nef cones of birational models. The open subcone C_+ corresponds to the ample cone of $n\Gamma\text{-Hilb}(\mathbb{C}^2)$ itself. In Section 6, we focus on the distinguished ray $\langle\theta_0\rangle$ in the boundary of F.

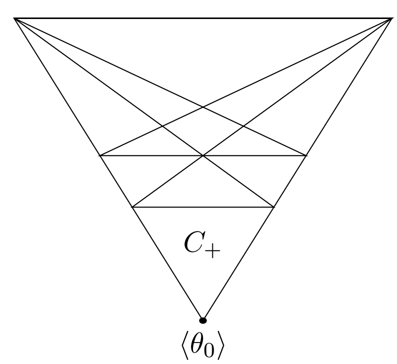


Figure 1. Wall-and-chamber structure inside the cone F for $\Gamma \cong \mu_3$ and $n = 3$.

We conclude this section with a lemma that identifies the key geometric fact that makes the chamber C^+ special; our argument depends crucially on this observation. For $\theta \in C_+$ and for any $\theta' \in \Theta_v$, we consider the line bundle $L_{C_+}(\theta') := \bigotimes_{0\leqslant i\leqslant r} \det(\mathcal{R}_i)^{\theta'_i}$ on $\mathfrak{M}_\theta$; the line bundle $L_J := L_{C_+}(\theta_J)$ will play a special role in particular.

Lemma 2.7. *Let $\theta \in C_+$.*

(i) *For each $\theta' \in \overline{C_+}$, the line bundle $L_{C_+}(\theta')$ on $\mathfrak{M}_\theta$ is globally generated.*
(ii) *For any $J \subseteq \{0, \dots, r\}$, after multiplying θ_J by a positive integer if necessary, the morphism to the linear series of L_J decomposes as the composition of π_J and a closed immersion:*

$$\begin{array}{ccc} \mathfrak{M}_\theta & \xrightarrow{\varphi_{|L_J|}} & |L_J|\,. \\ {\scriptstyle \pi_J}\downarrow & \nearrow & \\ \mathfrak{M}_{\theta_J} & & \end{array} \tag{2.3}$$

Proof. Since $\theta \in C_+$, the tautological bundles $\mathcal{R}_i$ on the quiver variety $\mathfrak{M}_\theta$ are globally generated for $i \in Q_0$ by [CIK18, Corollary 2.4]. Hence $L_{C_+}(\theta')$ is globally generated because $\theta'_i \geqslant 0$ for all $0 \leqslant i \leqslant r$. In particular, since $\theta_J \in \overline{C_+}$, the rational map $\varphi_{|L_J|}$ is defined everywhere. The line bundle L_J induces the morphism $\pi_J \colon \mathfrak{M}_\theta \to \mathfrak{M}_{\theta_J} \subset |L_J|$ by [BC20, Theorem 1.2], where we take a positive multiple of θ_J if necessary to ensure that the polarising ample bundle on $\mathfrak{M}_{\theta_J}$ is very ample. This proves the result. □

Remark 2.8. We choose a sufficiently high multiple of θ (and the same high multiple of each θ_J) to ensure that the polarising ample line bundle on $\mathfrak{M}_{\theta_J}$ is very ample for every subset $J \subseteq \{0, \ldots, r\}$.

3. Deleting an arrow and cornering the algebra

In the framed McKay quiver Q, let $b^* \in Q_1$ be the unique arrow with head at vertex ∞. Define a new $\mathbb{C}$-algebra as the quotient of the preprojective algebra Π by the two-sided ideal generated by the class of b^*:

$$A \coloneqq \Pi/(b^*)\,.$$

Equivalently, if we define a quiver Q^* to have vertex set $Q_0^* = \{\infty, 0, 1, \ldots, r\}$ and arrow set $Q_1^* = Q_1 \setminus \{b^*\}$, then A is the quotient of the path algebra $\mathbb{C}Q^*$ by the ideal of relations

$$\left(\sum_{\mathrm{h}(a)=i} \epsilon(a) a a^* \mid 0 \leqslant i, \mathrm{h}(a), \mathrm{h}(a^*) \leqslant r \right). \tag{3.1}$$

Since Q^* and Q share the same vertex set, we may consider $v \in \mathbb{Z}^{Q_0}$ as a dimension vector for A-modules and any parameter $\theta \in \Theta_v$ as a stability condition for A-modules of dimension vector v.

LEMMA 3.1. *For $\theta \in C_+$, the tautological $\mathbb{C}$-algebra homomorphism $\Pi \to \operatorname{End}(\mathcal{R})$ over $\mathfrak{M}_\theta$ factors through A. In particular, the tautological bundle $\mathcal{R}$ on $\mathfrak{M}_\theta$ is a flat family of θ-stable A-modules of dimension vector v.*

Proof. The image of the arrow b^* under the tautological homomorphism $\Pi \to \operatorname{End}(\mathcal{R})$ is a map of vector bundles $\mathcal{R}_0 \to \mathcal{R}_\infty \cong \mathcal{O}_{\mathfrak{M}_\theta}$. Under the isomorphism $\mathfrak{M}_\theta \cong n\Gamma\text{-}\operatorname{Hilb}(\mathbb{C}^2)$ from Theorem 2.3, we may regard the fibre of $\mathcal{R}$ over any closed point as the quotient $\mathbb{C}[x, y]/I$ for some Γ-invariant ideal I of $\mathbb{C}[x, y]$. In this language, the restriction to the 0-vertex is the Γ-invariant part of the quotient $\mathbb{C}[x, y]/I$, and it is well known that the induced map to the one-dimensional vector space at the vertex ∞ vanishes in this case. Thus, as $\mathfrak{M}_\theta$ is non-singular and in particular reduced, the corresponding map $\mathcal{R}_0 \to \mathcal{O}_{\mathfrak{M}_\theta}$ is the zero map, so the tautological $\mathbb{C}$-algebra homomorphism $\Pi \to \operatorname{End}(\mathcal{R})$ factors through A, as required. □

Lemma 3.1 allows us to work with the algebra A rather than Π. To illustrate why this is convenient, define the *unframed McKay quiver* Q_Γ to be the complete subquiver of Q^* on the vertex set $\{0, 1, \ldots, r\}$; that is, if $b \in Q_1^*$ is the unique arrow with tail at ∞, then Q_Γ has vertex set $\{0, \ldots, r\}$ and arrow set $Q_1^* \setminus \{b\}$. The preprojective algebra Π_Γ of the unframed McKay quiver is the quotient of $\mathbb{C}Q_\Gamma$ by the ideal generated by the preprojective relations in Q_Γ defined similarly to those in Q given in (2.1). Now, while Π_Γ is not a subalgebra of Π, it is isomorphic to a subalgebra of A, as we now show. Denote by $e_i \in A$ the vertex idempotents in the algebra A, and by a slight abuse of notation, let $b \in A$ denote the class of the arrow b.

Lemma 3.2. *The preprojective algebra* Π_Γ *is isomorphic to the subalgebra* $\bigoplus_{0\leqslant i,j\leqslant r} e_j A e_i$ *of* A. *Under this embedding, there is an isomorphism*

$$A \cong \Pi_\Gamma \oplus \Pi_\Gamma b \oplus \mathbb{C}e_\infty$$

of complex vector spaces.

Proof. The quiver Q^* has no arrow with head at vertex ∞, so the subalgebra $\bigoplus_{0\leqslant i,j\leqslant r} e_j A e_i$ of A is isomorphic to the quotient algebra $A/(e_\infty)$. The first statement follows since $A/(e_\infty) \cong \Pi_\Gamma$. The decomposition of A as a vector space is immediate from the structure of the quiver Q^*. □

Let $J \subseteq \{0, 1, \dots, r\}$ be non-empty. Define the idempotent $e_J := e_\infty + \sum_{j\in J} e_j$, and consider the subalgebra

$$A_J := e_J A e_J$$

of A spanned over $\mathbb{C}$ by the classes of paths in Q^* whose tail and head both lie in the set $\{\infty\}\cup J$. The process of passing from A to A_J is called *cornering*; see [CIK18, Remark 3.1].

We will study moduli spaces of certain finite-dimensional A_J-modules, and for this we must introduce a presentation of the algebra A_J in terms of a quiver with relations.

First, recall that the Γ-module $R := \mathbb{C}[x, y]$ decomposes into isotypical components $R = \bigoplus_{0\leqslant i\leqslant r} R_i$, where R_i is the sum of all Γ-submodules of R that are isomorphic to ρ_i. In particular, $R_0 = \mathbb{C}[x, y]^\Gamma$ and R_i is a reflexive R_0-module for each $0 \leqslant i \leqslant r$. Since $\Gamma \subset \mathrm{SL}(2, \mathbb{C})$, the Γ-invariant subring R_0 is well known to admit a presentation of the form

$$R_0 \cong \mathbb{C}[z_1, z_2, z_3]/(f)\,, \tag{3.2}$$

leading to the famous description of $\mathbb{C}^2/\Gamma$ as a hypersurface $(f = 0) \subset \mathbb{C}^3$. On the other hand, combining Auslander [Aus86] and Reiten–Van den Bergh [RVdB89] (compare with Buchweitz [Buc12] and Craw [Cra21, Lemma 3.1]) gives an isomorphism

$$\Pi_\Gamma \cong \mathrm{End}_{R_0}\Bigg(\bigoplus_{0\leqslant i\leqslant r} R_i\Bigg) \tag{3.3}$$

of $\mathbb{C}$-algebras. Note that for $0 \leqslant i, j \leqslant r$, the space $\mathrm{Hom}_{R_0}(R_i, R_j)$ is finitely generated as an R_0-module. One way to see this is to consider the reflexive sheaves $\widetilde{R}_i$ on $\mathbb{C}^2/\Gamma$ determined by the reflexive R_0-modules R_i; then $\mathrm{Hom}_{R_0}(R_i, R_j)$ is the space of sections of the coherent sheaf $\mathcal{H}om\big(\widetilde{R}_i, \widetilde{R}_j\big)$ on $\mathbb{C}^2/\Gamma$.

Proposition 3.3. *For any non-empty subset* $J \subseteq \{0, 1, \dots, r\}$, *the algebra* A_J *can be presented as the quotient of the path algebra of a quiver modulo a two-sided ideal of relations.*

Proof. If we regard Π_Γ as a subalgebra of A using Lemma 3.2, we see that $e'_J := \sum_{j\in J} e_j$ is the sum of vertex idempotents in Π_Γ, and the isomorphism (3.3) induces an isomorphism

$$e'_J(\Pi_\Gamma)e'_J \cong \mathrm{End}_{R_0}\Bigg(\bigoplus_{j\in J} R_j\Bigg). \tag{3.4}$$

Since Π_Γ is isomorphic to $\bigoplus_{0\leqslant i,j\leqslant r} e_j A e_i$ by Lemma 3.2, it follows that the algebra $e'_J(\Pi_\Gamma)e'_J$ from (3.4) is isomorphic to the subalgebra $e'_J A_J e'_J$ of A_J. For each $j \in J$, we choose three $\mathbb{C}$-algebra generators of $R_0 \subseteq \mathrm{Hom}_{R_0}(R_j, R_j)$ corresponding to the generators in the presentation (3.2), and we extend this to a set of $d_j \geqslant 3$ generators of $\mathrm{Hom}_{R_0}(R_j, R_j)$ as a $\mathbb{C}$-algebra.

Finally, for $i, j \in J$ with $i \neq j$, we choose a finite generating set for $\operatorname{Hom}_{R_0}(R_i, R_j)$ as an R_0-module comprising $d_{i,j} > 0$ generators.

We claim that there exist quivers Q_J and Q_J^* whose path algebras fit into a commutative diagram

$$\begin{array}{ccc} \mathbb{C}Q_J & \longrightarrow & \mathbb{C}Q_J^* \\ \downarrow{\scriptstyle \alpha_J} & & \downarrow{\scriptstyle \beta_J} \\ \operatorname{End}_{R_0}\left(\bigoplus_{j\in J} R_j\right) & \longrightarrow & A_J \end{array} \tag{3.5}$$

of $\mathbb{C}$-algebra homomorphisms where the vertical maps are surjective and the horizontal maps are injective. Given the claim, Proposition 3.2 follows because the required quiver is Q_J^* and the ideal of relations is $\ker(\beta_J)$.

To prove the claim, we consider two cases. First, suppose that $0 \in J$. Define the vertex set of Q_J to be J. For the arrow set of Q_J, introduce d_j loops at each vertex $j \in J$ corresponding to our chosen $\mathbb{C}$-algebra generators of $\operatorname{Hom}_{R_0}(R_j, R_j)$, including the three distinguished generators of its subalgebra R_0. Furthermore, for each $i, j \in J$ with $i \neq j$, introduce $d_{i,j}$ arrows from i to j corresponding to our chosen R_0-module generators of $\operatorname{Hom}_{R_0}(R_i, R_j)$. The concatenation of any arrow from i to j with loops at vertex j corresponding to the appropriate elements of $R_0 \subseteq \operatorname{Hom}_{R_0}(R_j, R_j)$ defines paths in Q_J that represent a spanning set for the vector space $e_j A_J e_i = \operatorname{Hom}_{R_0}(R_i, R_j)$. This determines by construction the left-hand epimorphism α_J in (3.5).

Next, define the vertex set of Q_J^* to be $\{\infty\} \cup J$, and define the arrow set by augmenting the arrow set of Q_J with one additional arrow b with tail at ∞ and head at 0. This is well defined since $0 \in J$ and, moreover, $\mathbb{C}Q_J$ is a subalgebra of $\mathbb{C}Q_J^*$ because Q_J^* has no arrows with head at ∞. The lower horizontal map in diagram (3.5) is simply the inclusion of the algebra from (3.4) as the subalgebra $e_J' A_J e_J'$ of A_J. To construct β_J, we need only extend α_J by sending the paths e_∞ and b in $\mathbb{C}Q_J^*$ to the classes of e_∞ and b in A_J, respectively. The surjectivity of β_J follows from the second statement of Lemma 3.2. This proves the claim for $0 \in J$.

It remains to consider the case $0 \notin J$. Define $\overline{J} := \{0\} \cup J$, and apply the construction for the case $0 \in \overline{J}$ to obtain the diagram (3.5) for $\overline{J}$. To define the quiver Q_J, we remove from $Q_{\overline{J}}$ the vertex 0 together with all arrows in $Q_{\overline{J}}$ that have head and/or tail at 0. Notice that for each $i, j \in J \subset \overline{J}$, the quiver $Q_{\overline{J}}$ already has $d_{i,j}$ arrows from i to j corresponding to a set of R_0-module generators of $\operatorname{Hom}_{R_0}(R_i, R_j)$, so the desired $\mathbb{C}$-algebra epimorphism α_J is obtained by restriction from the map $\alpha_{\overline{J}}$ in the diagram (3.5) for $\overline{J}$.

Next, for any $j \in J$, let $\{a'_{j,m} \mid 1 \leqslant m \leqslant d_{0,j}\}$ denote the arrows in $Q_{\overline{J}}$ corresponding to our chosen set of generators of $\operatorname{Hom}_{R_0}(R_0, R_j)$. Define the quiver Q_J^* to have vertex set $\{\infty\} \cup J$ and arrow set obtained by augmenting the arrow set of Q_J as follows: for each $j \in J$, introduce arrows $\{a_{j,m} \mid 1 \leqslant m \leqslant d_{0,j}\}$ from ∞ to j. Note that $\beta_{\overline{J}}(a'_{j,m} b) \in e_j A_{\overline{J}} e_\infty = e_j A_J e_\infty \subset A_J$. Therefore, if for any path p in $\mathbb{C}Q_J^*$, we define

$$\beta_J(p) = \begin{cases} \alpha_J(p) & \text{for } p \in \mathbb{C}Q_J\,, \\ e_\infty & \text{for } p = e_\infty\,, \\ \beta_{\overline{J}}(a'_{j,m} b) & \text{for } p = a_{j,m}\,, \end{cases}$$

then we determine uniquely a $\mathbb{C}$-algebra homomorphism $\beta_J \colon \mathbb{C}Q_J^* \to A_J$. To show that β_J is surjective, it suffices to check that the image of β_J contains $A_J e_\infty$ because α_J is surjective. For this, consider $\gamma \in A_J e_\infty \subseteq e_J A_{\overline{J}} e_\infty$. Since $\beta_{\overline{J}}$ is surjective, γ can be represented by a linear combination of paths $\gamma_i \in Q^*_{\overline{J}}$, each with tail at ∞ and head at a vertex in J. Every such path γ_i necessarily begins by traversing a path of the form $a'_{j,m}cb$ for some $j \in J$ and $1 \leqslant m \leqslant d_{0,j}$, where c is a (possibly empty) composition of loops at vertex 0. Crucially, as $\beta_{\overline{J}}(c) \in \mathrm{Hom}_{R_0}(R_0, R_0) \cong R_0$, the element $\beta_{\overline{J}}(a'_{j,m}c) \in e_j A_{\overline{J}} e_0 = \mathrm{Hom}_{R_0}(R_0, R_j)$ can be written as the image under $\beta_{\overline{J}}$ of a linear combination $\sum_{1\leqslant n\leqslant d_{0,j}} c_n a'_{j,n}$ for some $c_n \in R_0 \subseteq \mathrm{Hom}_{R_0}(R_j, R_j)$. Therefore the start $a'_{j,m}cb$ of the path γ_i satisfies

$$\beta_{\overline{J}}(a'_{j,m}cb) = \beta_{\overline{J}}\Bigg(\sum_{1\leqslant n\leqslant d_{0,j}} c_n a'_{j,n}\Bigg)\beta_{\overline{J}}(b) = \sum_{1\leqslant n\leqslant d_{0,j}} c_n \beta_{\overline{J}}(a'_{j,n}b) = \beta_J\Bigg(\sum_{1\leqslant n\leqslant d_{0,j}} \ell_n a_{j,n}\Bigg),$$

where each ℓ_n is a linear combination of loops in Q^*_J at vertex j satisfying $\alpha_J(\ell_n) = c_n$. Thus, the image in A_J of the beginning of our path γ_i lies in the image of β_J. It follows that each path γ_i arising in the linear combination of γ lies in the image of β_J because α_J is surjective. Therefore, the image of β_J contains $A_J e_\infty$, as required. □

Remarks 3.4. (1) The quiver Q^*_J that we construct in the proof above has many more arrows than necessary. For example, when $J = \{0, 1, \ldots, r\}$, the algorithm returns a quiver with arrow set containing at least three loops at each vertex, whereas Q^* contains no loops.

(2) An alternative proof for Proposition 3.3 could be given by exhibiting a finite number of paths in Q^* whose tail and head both lie in the set $\{\infty\} \cup J$, with the property that their classes, up to the preprojective relations, generate the cornered algebra A_J. While we believe this is indeed possible, the combinatorics of the situation gets rather intricate, especially in the case $0 \notin J$. The proof presented above has the advantage that it avoids case-by-case analysis of Dynkin diagrams.

4. Reconstructing quiver varieties via the cornered algebras

In general, the quiver variety $\mathfrak{M}_{\theta_J}$ is the coarse moduli space for S-equivalence classes of θ_J-semistable Π-modules of dimension vector v. However, in the special case $J = \{0, \ldots, r\}$, it may also be regarded as the fine moduli space of isomorphism classes of θ_J-stable A-modules of dimension vector v by Lemma 3.1. We now introduce an alternative, fine moduli space construction for each $\mathfrak{M}_{\theta_J}$ using the algebra A_J.

The element

$$v_J := \rho_\infty + \sum_{j\in J} n \dim(\rho_j)\rho_j \in \mathbb{Z} \oplus \mathbb{Z}^J$$

is a dimension vector for A_J-modules, and we consider the stability condition $\eta_J \colon \mathbb{Z} \oplus \mathbb{Z}^J \to \mathbb{Q}$ given by

$$\eta_J(\rho_i) = \begin{cases} -\sum_{j\in J} n \dim(\rho_j) & \text{for } i = \infty\,, \\ 1 & \text{if } i \in J\,. \end{cases}$$

It follows directly from the definition that an A_J-module N of dimension vector v_J is η_J-stable if and only if there exists a surjective A_J-module homomorphism $A_J e_\infty \to N$. The vector v_J is indivisible, and η_J is a generic stability condition for A_J-modules.

The quiver moduli space construction of King [Kin94, Proposition 5.3] for finite-dimensional algebras can be adapted to any algebra presented as the quotient of a finite connected quiver by an ideal of relations. Thus, Proposition 3.3 allows us to define the fine moduli space $\mathcal{M}(A_J)$ of η_J-stable A_J-modules of dimension vector v_J. Let $T_J := \bigoplus_{i\in\{\infty\}\cup J} T_i$ denote the tautological bundle on $\mathcal{M}(A_J)$, where T_∞ is the trivial bundle and T_j has rank $n\dim(\rho_j)$ for $j \in J$. The line bundle

$$\mathcal{L}_J := \bigotimes_{j\in J} \det(T_j)$$

is the polarising ample bundle on $\mathcal{M}(A_J)$ given by the GIT construction.

Lemma 4.1. *Let $\theta \in C_+$, and let $J \subseteq \{0,\dots,r\}$ be any non-empty subset. There is a universal morphism*

$$\tau_J \colon \mathfrak{M}_\theta \to \mathcal{M}(A_J) \tag{4.1}$$

satisfying $\tau_J^*(T_i) \cong \mathcal{R}_i$ for $i \in \{\infty\} \cup J$.

Proof. In light of the universal property of $\mathcal{M}(A_J)$, it suffices to show that the locally free sheaf

$$\mathcal{R}_J := \bigoplus_{i\in\{\infty\}\cup J} \mathcal{R}_i$$

of rank $1 + \sum_{j\in J} n\dim(\rho_j)$ on the quiver variety $\mathfrak{M}_\theta$ is a flat family of η_J-stable A_J-modules of dimension vector v_J. We saw in Lemma 3.1 that we may delete one arrow from Q, giving rise to a $\mathbb{C}$-algebra homomorphism $\phi\colon A \to \operatorname{End}(\mathcal{R})$. Multiplying this on the left and right by the idempotent e_J determines a $\mathbb{C}$-algebra homomorphism $A_J \to \operatorname{End}(\mathcal{R}_J)$ which makes $\mathcal{R}_J$ into a flat family of A_J-modules of dimension vector v_J. To establish stability, write $\bigoplus_{i\in Q_0} \mathcal{R}_{i,y}$ for the fibre of $\mathcal{R}$ over a closed point $y \in \mathfrak{M}_\theta$. The fact that $\bigoplus_{i\in Q_0} \mathcal{R}_{i,y}$ is θ-stable is equivalent to the existence of a surjective A-module homomorphism $Ae_\infty \to \bigoplus_{i\in Q_0} \mathcal{R}_{i,y}$. Applying e_J on the left produces a surjective A_J-module homomorphism $A_J e_\infty \to \bigoplus_{i\in\{\infty\}\cup J} \mathcal{R}_{i,y}$ which in turn is equivalent to the η_J-stability of the fibre $\bigoplus_{i\in\{\infty\}\cup J} \mathcal{R}_{i,y}$ of $\mathcal{R}_J$ over $y \in \mathfrak{M}_\theta$. In particular, $\mathcal{R}_J$ is a flat family of η_J-stable A_J-modules of dimension vector v_J. □

Remarks 4.2. (1) An alternative proof of Lemma 4.1 uses the fact that the tautological bundles $\mathcal{R}_i$ on $\mathfrak{M}_\theta$ are globally generated for $i \in I$ by [CIK18, Corollary 2.4], in which case one can adapt the proof of [CIK18, Proposition 2.3] to deduce that $\mathcal{R}_J$ is a flat family of η_J-stable A_J-modules of dimension vector v_J. In particular, global generation is the key feature in Lemma 4.1, just as in the proof of Lemma 2.7. This is not a coincidence; see Theorem 4.8.

(2) Building on Remark 2.8, we now take an even higher multiple of θ if necessary (and the same high multiple of each η_J and each θ_J) to ensure that the polarising ample line bundles on $\mathcal{M}(A_J)$ and on $\mathfrak{M}_{\theta_J}$ are very ample for all relevant $J \subseteq \{0,\dots,r\}$.

Lemma 4.3. *Let $\theta \in C_+$, and assume that $J \subseteq \{0,\dots,r\}$ is non-empty. There is a commutative diagram*

$$\begin{array}{ccccc}
 & & \mathfrak{M}_\theta & & \\
 & \swarrow^{\pi_J} & \Big\downarrow{\scriptstyle \varphi_{|L_J|}} & \searrow^{\tau_J} & \\
\mathfrak{M}_{\theta_J} & & & & \mathcal{M}(A_J) \\
 & \searrow & & & \Big\downarrow{\scriptstyle \varphi_{|\mathcal{L}_J|}} \\
 & & |L_J| & \xrightarrow{\ \psi\ } & |\mathcal{L}_J|
\end{array} \tag{4.2}$$

of schemes over $\operatorname{Sym}^n\left(\mathbb{C}^2/\Gamma\right)$*, where ψ is an isomorphism.*

Proof. The commutative triangle on the left of (4.2) was constructed in Lemma 2.7. For the quadrilateral on the right, our choice of η_J ensures that the polarising line bundle $\mathcal{L}_J$ on $\mathcal{M}(A_J)$

is very ample, so the morphism $\varphi_{|\mathcal{L}_J|}$ is well defined. Since pullback commutes with tensor operations on the T_i, the isomorphisms $\tau_J^*(T_i) \cong \mathcal{R}_i$ for $i \in J$ imply that $L_J = \tau_J^*(\mathcal{L}_J)$. If $\mathcal{O}_{|\mathcal{L}_J|}(1)$ denotes the polarising ample bundle on $|\mathcal{L}_J|$, then

$$(\varphi_{|\mathcal{L}_J|} \circ \tau_J)^*(\mathcal{O}_{|\mathcal{L}_J|}(1)) = \tau_J^*(\mathcal{L}_J) = L_J = \varphi_{|L_J|}^*\big(\mathcal{O}_{|L_J|}(1)\big) \tag{4.3}$$

on $\mathfrak{M}_\theta$. The morphism to a complete linear series is unique up to an automorphism of the linear series, so there is an isomorphism $\psi\colon |L_J| \to |\mathcal{L}_J|$ such that $\varphi_{|\mathcal{L}_J|} \circ \tau_J = \psi \circ \varphi_{|L_J|}$, as required.

It remains to show that (4.2) is a diagram of schemes over $\operatorname{Sym}^n(\mathbb{C}^2/\Gamma)$. The Leray spectral sequence for the resolution $\pi\colon \mathfrak{M}_\theta \to \mathfrak{M}_0 \cong \operatorname{Sym}^n(\mathbb{C}^2/\Gamma)$ gives $H^0(\mathcal{O}_{\mathcal{M}_\theta}) \cong H^0(\mathcal{O}_{\mathcal{M}_0}) \cong \big(\mathbb{C}[V]^\Gamma\big)^{\mathfrak{S}_n}$ because $\operatorname{Sym}^n(\mathbb{C}^2/\Gamma)$ has rational singularities. It follows that $\pi = \varphi_{|\mathcal{O}_{\mathfrak{M}_\theta}|}$; that is, π is the structure morphism of $\mathfrak{M}_\theta$ as a variety over $\operatorname{Sym}^n(\mathbb{C}^2/\Gamma)$. Repeating the argument from (4.3), with the roles of $L_J, \mathcal{L}_J$ and $\mathcal{O}_{|\mathcal{L}_J|}(1)$ played instead by the trivial bundles on $\mathfrak{M}_\theta$, $\mathcal{M}(A_J)$ and $\operatorname{Sym}^n(\mathbb{C}^2/\Gamma)$, respectively, shows that $\mathcal{M}(A_J)$ is a scheme over $\operatorname{Sym}^n(\mathbb{C}^2/\Gamma)$. It follows that (4.2) is a diagram of schemes over $\operatorname{Sym}^n(\mathbb{C}^2/\Gamma)$. □

Our goal for the rest of this section is to add a morphism $\iota_J\colon \mathfrak{M}_{\theta_J} \to \mathcal{M}(A_J)$ to diagram (4.2) and to show that ι_J is an isomorphism on the underlying reduced schemes. Consider the functors

$$A\text{-mod} \underset{j_!}{\overset{j^*}{\rightleftarrows}} A_J\text{-mod}$$

defined by $j^*(-) := e_J A \otimes_A (-)$ and $j_!(-) := Ae_J \otimes_{A_J} (-)$. These are two of the six functors in a recollement of the module category A-mod [FP04]. In particular, j^* is exact, $j^* j_!$ is the identity functor, and for any A_J-module N, the A-module $j_!(N)$ is the maximal extension by $A/(Ae_JA)$-modules; see [CIK18, (3.4)].

Lemma 4.4. *Let N be an A_J-module of dimension vector v_J.*

(i) *If there exists a surjective A_J-module homomorphism $A_Je_\infty \to N$, then there exists a surjective A-module homomorphism $Ae_\infty \to j_!(N)$.*
(ii) *The A-module $j_!(N)$ is finite-dimensional and satisfies $\dim_i j_!(N) = \dim_i N$ for all $i \in \{\infty\} \cup J$.*

Proof. The construction of the quiver from Proposition 3.3 shows that A is a finitely generated module over the algebra $R_0 \cong e_0Ae_0$. Armed with this observation, the proof of [CIK18, Lemma 3.6] applies verbatim (the notation differs slightly for part (i): our map $A_Je_\infty \to N$ is written $A_Ce_0' \to N$ in ibid.). □

Lemma 4.5. *Let N be an η_J-stable A_J-module of dimension vector v_J. The A-module $j_!(N)$ is θ_J-semistable.*

Proof. Since N is η_J-stable, there is a surjective A_J-module homomorphism $A_Je_\infty \to N$. Lemma 4.4 gives a surjective A-module homomorphism $Ae_\infty \to j_!(N)$, and, moreover, the finite-dimensional A-module $j_!(N)$ satisfies $\dim_i j_!(N) = \dim_i N$ for $i \in \{\infty\} \cup J$. Recall that $\theta_J(\rho_i) = 0$ for $i \notin \{\infty\} \cup J$, so

$$\theta_J\big(j_!(N)\big) = \theta_J\left(\sum_{i\in\{\infty\}\cup J} \dim_i(j_!(N))\rho_i\right) = \eta_J\left(\sum_{i\in\{\infty\}\cup J} \dim_i(N)\rho_i\right) = \eta_J(N) = 0\,.$$

Now let $M \subset j_!(N)$ be a proper submodule. If $\dim_\infty M = 1$, then the surjectivity of the map $Ae_\infty \to j_!(N)$ gives $M = j_!(N)$, which is absurd, so $\dim_\infty M = 0$. But $\theta_J(\rho_i) \geqslant 0$ for all $i \neq \infty$, so $\theta_J(M) \geqslant 0$, as required. □

Lemma 4.6. *Let N be an η_J-stable A_J-module of dimension vector v_J. Then there exists a θ_J-semistable A-module M such that $j^*M \cong N$ and $\dim_i M \leqslant n\dim(\rho_i)$ for all $i \notin \{\infty\} \cup J$.*

Proof. By Lemma 4.5, the A-module $j_!(N)$ is θ_J-semistable. If $\dim_i j_!(N) \leqslant n\dim(\rho_i)$ for $i \notin \{\infty\} \cup J$, then we can simply set $M := j_!(N)$, as $j^*j_!$ is the identity. Otherwise, consider the θ_J-polystable module $\bigoplus_\lambda M_\lambda$ that is S-equivalent to $j_!(N)$. Let M_{λ_∞} denote the unique summand satisfying $\dim_\infty M_{\lambda_\infty} = 1$. Since M_{λ_∞} is by construction a θ_J-stable A-module, it follows that $\dim_i M_{\lambda_\infty} = n\dim(\rho_i)$ for all $i \in J$, and hence $\dim_i M_\lambda = 0$ for all $\lambda \neq \lambda_\infty$ and all $i \in \{\infty\} \cup J$. For each index λ and for all $i \in \{\infty\} \cup J$, we have

$$\dim_i j^*M_\lambda = \dim e_i\big(e_J A \otimes_A (M_\lambda)\big) = \dim e_i A \otimes_A M_\lambda = \dim_i M_\lambda\,.$$

It follows that $\dim_i j^*M_\lambda = 0$ for all $\lambda \neq \lambda_\infty$ and $i \in \{\infty\} \cup J$, and hence $j^*M_\lambda = 0$ for $\lambda \neq \lambda_\infty$.

We claim that $j^*M_{\lambda_\infty}$ is isomorphic to N. Indeed, the A-module $j_!(N)$ is θ_J-semistable by Lemma 4.5, and the θ_J-stable A-modules M_λ are by construction the factors in the composition series of $j_!(N)$ in the category of θ_J-semistable A-modules. It follows from the exactness of j^* that the A_J-modules j^*M_λ are the factors in the composition series of $j^*j_!(N) \cong N$ in the category of η_J-semistable A_J-modules. But $j^*M_\lambda = 0$ for $\lambda \neq \lambda_\infty$, so the only non-zero factor of the composition series is $j^*M_{\lambda_\infty}$. It follows that $j^*M_{\lambda_\infty} \cong N$ because the factor $j^*M_{\lambda_\infty}$ can only appear once in the composition series.

As a result, the θ_J-stable A-module M_{λ_∞} satisfies $j^*M_{\lambda_\infty} \cong N$ and $\dim_i M_{\lambda_\infty} = n\dim(\rho_i)$ for all $i \in J$. Therefore, M_{λ_∞} is the required A_J-module as long as $\dim_i M_{\lambda_\infty} \leqslant n\dim(\rho_i)$ for $i \notin \{\infty\} \cup J$. We establish this key inequality in the appendix. □

Remark 4.7. The modules M_λ for $\lambda \neq \lambda_\infty$ in the proof of Lemma 4.6 are in fact all one-dimensional vertex simple A-modules. To see this, note that removing any non-empty set of vertices and their incident edges from an extended Dynkin diagram gives a diagram in which every connected component is Dynkin of finite type. Thus removing the vertices $\{\infty\} \cup J$ and all incident edges from the framed extended diagram leaves us with a collection of Dynkin diagrams of finite type. Choose $\lambda \neq \lambda_\infty$. Since $\dim_j M_\lambda = 0$ for all $j \in \{\infty\} \cup J$, the module M_λ is a simple module of the preprojective algebra of a quiver of finite type. But such modules are one-dimensional by [ST11, Lemma 2.2].

Theorem 4.8. *For any non-empty $J \subseteq \{0, \dots, r\}$, there is a commutative diagram of morphisms*

$$\begin{array}{ccc} & \mathfrak{M}_\theta & \\ {}^{\pi_J}\swarrow & & \searrow^{\tau_J} \\ \mathfrak{M}_{\theta_J} & \xrightarrow{\quad \iota_J \quad} & \mathcal{M}(A_J)\,, \end{array} \tag{4.4}$$

where ι_J is an isomorphism of the underlying reduced schemes. In particular, $\mathcal{M}(A_J)$ is irreducible, and its underlying reduced scheme is normal and has symplectic singularities.

Proof. Let $\sigma_J\colon \mathfrak{M}_{\theta_J} \to |\mathcal{L}_J|$ be the composition of the isomorphism ψ of Lemma 4.3 with the closed immersion $\mathfrak{M}_{\theta_J} \hookrightarrow |L_J|$ from diagram (4.2). Since σ_J is a closed immersion, it identifies $\mathfrak{M}_{\theta_J}$ with $\operatorname{Im}(\sigma_J)$. The surjectivity of π_J and the commutativity of diagram (4.2) then imply that $\mathfrak{M}_{\theta_J}$ is isomorphic to the subscheme $\operatorname{Im}(\sigma_J \circ \pi_J) = \operatorname{Im}(\varphi_{|\mathcal{L}_J|} \circ \tau_J)$ of $|\mathcal{L}_J|$. Since $\mathcal{L}_J$ is the polarising very ample line bundle on the GIT quotient $\mathcal{M}(A_J)$, the closed immersion $\varphi_{|\mathcal{L}_J|}$ induces an isomorphism $\lambda_J\colon \operatorname{Im}(\varphi_{|\mathcal{L}_J|}) \to \mathcal{M}(A_J)$. The morphism

$$\iota_J := \lambda_J \circ \sigma_J \colon \mathfrak{M}_{\theta_J} \longrightarrow \mathcal{M}(A_J)$$

is therefore a closed immersion. Note that

$$\iota_J \circ \pi_J = \lambda_J \circ \sigma_J \circ \pi_J = \lambda_J \circ \varphi_{|\mathcal{L}_J|} \circ \tau_J = \tau_J \,,$$

so diagram (4.4) commutes. In order to prove that ι_J is an isomorphism of the underlying reduced schemes, it suffices to show that ι_J is surjective on closed points.

Consider a closed point $[N] \in \mathcal{M}(A_J)$, where N is an η_J-stable A_J-module of dimension vector v_J. Let M be the θ_J-semistable A-module from Lemma 4.6. For $i \notin \{\infty\} \cup J$, define $m_i := n\dim(\rho_i) - \dim_i M \geqslant 0$, and let $S_i := \mathbb{C}e_i$ denote the vertex simple A-module at vertex $i \in Q_0$. The A-module

$$\overline{M} := M \oplus \bigoplus_{i\in\{0,\dots,r\}\setminus J} S_i^{\oplus m_i}$$

is θ_J-semistable of dimension vector v by construction, and it satisfies $j^*\big(\overline{M}\big) = j^*(M) = N$. Write $\big[\,\overline{M}\,\big] \in \mathfrak{M}_{\theta_J}$ for the corresponding closed point, and let $\widetilde{M}$ be any θ-stable A-module of dimension vector v such that the closed point $\big[\widetilde{M}\big] \in \mathfrak{M}_\theta$ satisfies $\pi_J\big(\big[\widetilde{M}\big]\big) = \big[\,\overline{M}\,\big] \in \mathfrak{M}_{\theta_J}$. Then $j^*\big(\widetilde{M}\big) = j^*\big(\overline{M}\big) = N$, hence $\tau_J\big(\big[\widetilde{M}\big]\big) = [N]$, and the commutativity of diagram (4.4) gives that

$$\iota_J\big(\big[\,\overline{M}\,\big]\big) = (\iota_J \circ \pi_J)\big(\big[\widetilde{M}\big]\big) = \tau_J\big(\big[\widetilde{M}\big]\big) = [N]\,,$$

so ι_J is indeed surjective. The last statement of Theorem 4.8 follows from Lemmas 2.1 and 4.3. □

Remarks 4.9. (1) If $J \neq \{0,\dots,r\}$, then the stability parameter θ_J lies in the boundary of the GIT chamber C_+, so $\mathfrak{M}_{\theta_J}$ does not admit a universal family of θ_J-semistable Π-modules of dimension vector v. However, the fine moduli space $\mathcal{M}(A_J)$ does carry a universal family T_J of η_J-stable A_J-modules of dimension vector v_J, and hence under the isomorphism of Theorem 4.8, the bundle $\iota_J^*(T_J)$ on $\mathfrak{M}_{\theta_J}$ pulls back along π_J to the summand $\bigoplus_{i\in\{\infty\}\cup J}\mathcal{R}_i$ of the tautological bundle on $\mathfrak{M}_\theta$.

(2) In the course of the proof of Theorem 4.8, we deduce directly that τ_J is surjective on closed points.

(3) For $J = \varnothing$, we have $\mathfrak{M}_{\theta_J} \cong \operatorname{Sym}^n\big(\mathbb{C}^2/\Gamma\big)$. On the other hand, $e_\infty A e_\infty = \mathbb{C}e_\infty$, which does not provide enough information with which to reconstruct $\operatorname{Sym}^n\big(\mathbb{C}^2/\Gamma\big)$.

5. Identifying the posets for the coarse and fine moduli problems

We now establish that the morphisms $\iota_J\colon \mathfrak{M}_{\theta_J} \to \mathcal{M}(A_J)$ from Theorem 4.8 are compatible with the morphisms $\pi_{J,J'}\colon \mathfrak{M}_{\theta_J} \to \mathfrak{M}_{\theta_{J'}}$ that feature in the poset introduced in Proposition 2.4.

Lemma 5.1. *For non-empty subsets $J' \subset J \subset \{0,1,\dots,r\}$, there is a commutative diagram*

$$\begin{array}{ccc} \mathfrak{M}_{\theta_J} & \xrightarrow{\iota_J} & \mathcal{M}(A_J) \\ {\scriptstyle \pi_{J,J'}}\big\downarrow & & \big\downarrow{\scriptstyle \tau_{J,J'}} \\ \mathfrak{M}_{\theta_{J'}} & \xrightarrow{\iota_{J'}} & \mathcal{M}(A_{J'}) \end{array} \qquad (5.1)$$

in which the horizontal arrows are isomorphisms on the underlying reduced schemes and the vertical arrows are surjective, projective, birational morphisms.

Proof. The subbundle $\bigoplus_{i\in\{\infty\}\cup J'} T_i$ of the tautological bundle T_J on $\mathcal{M}(A_J)$ is a flat family of $\eta_{J'}$-stable $A_{J'}$-modules of dimension vector $v_{J'}$, so there is a universal morphism

$$\tau_{J,J'}\colon \mathcal{M}(A_J) \longrightarrow \mathcal{M}(A_{J'})$$

satisfying $\tau_{J,J'}^*(T_i') = T_i$ for $i \in \{\infty\} \cup J'$, where $\bigoplus_{i\in\{\infty\}\cup J'} T_i'$ is the tautological bundle on $\mathcal{M}(A_{J'})$. Now

$$\big(\tau_{J,J'} \circ \tau_J\big)^*(T_i') = \tau_J^*(T_i) = \mathcal{R}_i = \tau_{J'}^*(T_i')$$

for all $i \in \{\infty\} \cup J'$, and since this property characterises the morphism $\tau_{J'}$, we have a commutative diagram

$$\begin{array}{ccc} & \mathfrak{M}_\theta & \\ {}^{\tau_J}\swarrow & & \searrow^{\tau_J} \\ \mathcal{M}(A_J) & \xrightarrow{\quad\tau_{J,J'}\quad} & \mathcal{M}(A_{J'})\,. \end{array} \tag{5.2}$$

Proposition 2.4 gives a similar commutative diagram expressing the identity $\pi_{J,J'} \circ \pi_J = \pi_{J'}$ for morphisms between quiver varieties, while Theorem 4.8 establishes the identities $\iota_J \circ \pi_J = \tau_J$ and $\iota_{J'} \circ \pi_{J'} = \tau_{J'}$. Taken together, these identities show that the maps in all four triangles in the following pyramid diagram commute:

$$\begin{array}{ccccc} & & \mathfrak{M}_\theta & & \\ & {}^{\pi_J}\swarrow & {}^{\pi_{J'}}\swarrow \quad \searrow^{\tau_{J'}} & \searrow^{\tau_J} & \\ \mathfrak{M}_{\theta_J} & & \xrightarrow{\quad\iota_J\quad} & & \mathcal{M}(A_J) \\ {\scriptstyle \pi_{J,J'}}\downarrow & & & & \downarrow{\scriptstyle \tau_{J,J'}} \\ \mathfrak{M}_{\theta_{J'}} & & \xrightarrow{\quad\iota_{J'}\quad} & & \mathcal{M}(A_{J'})\,. \end{array} \tag{5.3}$$

To show that the morphisms around the pyramid's square base commute, choose for any closed point $x \in \mathfrak{M}_{\theta_J}$ a lift $y \in \pi_J^{-1}(x) \subset \mathfrak{M}_\theta$. The commutativity of the triangles in the diagram gives

$$\big(\iota_{J'} \circ \pi_{J,J'}\big)(x) = \iota_{J'}\big(\pi_{J'}(y)\big) = \tau_{J'}(y) = \tau_{J,J'}\big(\tau_J(y)\big) = \big(\tau_{J,J'} \circ \iota_J\big)(x)\,,$$

and since $x \in \mathfrak{M}_\theta$ was arbitrary and π_J is surjective, we have that $\iota_{J'} \circ \pi_{J,J'} = \tau_{J,J'} \circ \iota_J$, as required. □

We deduce the following.

Proposition 5.2. *The face poset of the cone $\overline{C_+}$ can be identified with the poset on the set of fine moduli spaces $\mathcal{M}(A_J)$ for non-empty subsets $J \subseteq \{0, \ldots, r\}$ together with $\mathbb{C}^{2n}/\Gamma_n$, where edges in the Hasse diagram of the poset indicating inequalities $\mathfrak{M}(A_J) > \mathfrak{M}(A_{J'})$ and $\mathfrak{M}(A_J) > \mathbb{C}^{2n}/\Gamma_n$ are realised by the universal morphisms $\tau_{J,J'}$ and the structure morphisms $\varphi_{|\mathcal{O}_{\mathcal{M}(A_J)}|}$, respectively.*

6. Punctual Hilbert schemes for Kleinian singularities

In this section, we specialise to the case $J = \{0\}$ and study the algebra A_J, before establishing the link between the fine moduli space $\mathcal{M}(A_J)$ and the Hilbert scheme of n points on $\mathbb{C}^2/\Gamma$. It will be convenient to write dimension vectors of A_J-modules as pairs (v_∞, v_0) in this case.

As we saw in Proposition 3.3, the algebra A_J can be presented as the path algebra of a quiver modulo an ideal of relations. The relations appear to be fairly complicated, but for $J = \{0\}$ it is possible to give an explicit presentation of A_J; this will turn out to be sufficient for our purposes. To spell this out, recall the construction of the quiver Q_J^* from Proposition 3.3 that has vertex set $\{\infty, 0\}$ and arrow set comprising one arrow b from ∞ to 0 and loops α_1, α_2, α_3 at vertex 0 corresponding to a set of minimal $\mathbb{C}$-algebra generators of $\mathrm{Hom}_{R_0}(R_0, R_0) = R_0 \cong \mathbb{C}[x,y]^\Gamma$ as shown in Figure 2:

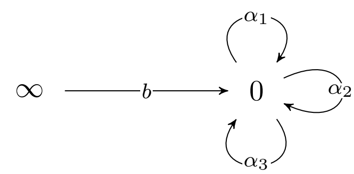


Figure 2. The quiver Q_J^* used in the presentation of A_J for $J = \{0\}$.

To state the presentation of A_J in this case, recall the presentation of the algebra $\mathbb{C}[x,y]^\Gamma$ from (3.2).

Lemma 6.1. *For $J = \{0\}$, the algebra A_J is isomorphic to the quotient of $\mathbb{C}Q_J^*$ by the two-sided ideal*

$$K = \big(f(\alpha_1,\alpha_2,\alpha_3), \alpha_1\alpha_2 - \alpha_2\alpha_1, \alpha_1\alpha_3 - \alpha_3\alpha_1, \alpha_2\alpha_3 - \alpha_3\alpha_2\big), \tag{6.1}$$

where $f \in \mathbb{C}[z_1,z_2,z_3]$ is the defining equation of the hypersurface $\mathbb{C}^2/\Gamma \subseteq \operatorname{Spec}\mathbb{C}[z_1,z_2,z_3]$.

Proof. In Proposition 3.3, we constructed a $\mathbb{C}$-algebra epimorphism $\beta_J\colon \mathbb{C}Q_J^* \to A_J$. The images under β_J of the arrows α_1, α_2, α_3 correspond to minimal $\mathbb{C}$-algebra generators of $R_0 \cong e_0Ae_0$, so the ideal K lies in the kernel of β_J. We claim that the induced $\mathbb{C}$-algebra epimorphism

$$\beta_J\colon \mathbb{C}Q_J^*/K \longrightarrow A_J$$

is an isomorphism. Define a $\mathbb{C}$-algebra homomorphism $\gamma_J\colon A_J \to \mathbb{C}Q_J^*/K$ by sending the chosen minimal $\mathbb{C}$-algebra generators of $R_0 \cong e_0A_Je_0$ to the classes of the arrows α_1, α_2, α_3 in $\mathbb{C}Q_J^*/K$ and by sending the classes of e_∞ and b in A_J to the classes of the paths e_∞ and b in $\mathbb{C}Q_J^*/K$. This defines a $\mathbb{C}$-algebra homomorphism because $e_0A_Je_0$ is a subalgebra of A with quotient $A/(e_\infty)$. Clearly $\gamma_J = \beta_J^{-1}$, as required. □

Proposition 6.2. *For the subset $J = \{0\}$, there is an isomorphism of schemes*

$$\omega_n\colon \mathrm{Hilb}^{[n]}\left(\mathbb{C}^2/\Gamma\right) \to \mathcal{M}(A_J)$$

over $\mathrm{Sym}^n\left(\mathbb{C}^2/\Gamma\right)$.

Proof. We begin by constructing the morphism of schemes ω_n. Let $\mathcal{T}$ denote the tautological rank n bundle on $\mathrm{Hilb}^{[n]}\left(\mathbb{C}^2/\Gamma\right)$, and write $\mathcal{O}$ for the trivial bundle. In light of the universal property of $\mathcal{M}(A_J)$, it suffices to show that $\mathcal{O} \oplus \mathcal{T}$ carries a natural structure of a flat family of η_J-stable A_J-modules of dimension vector $v_J = (1,n)$ on $\mathrm{Hilb}^{[n]}\left(\mathbb{C}^2/\Gamma\right)$. A closed point of $\mathrm{Hilb}^{[n]}\left(\mathbb{C}^2/\Gamma\right)$ corresponds to a codimension n ideal $I \lhd \mathbb{C}[x,y]^\Gamma \cong \mathbb{C}[z_1,z_2,z_3]/(f)$. The quotient vector space $\mathbb{C}[x,y]^\Gamma/I$ is of dimension n, it carries the action of commuting arrows α_1, α_2, α_3 satisfying the relation f, and it has a distinguished generator $[1] \in \mathbb{C}[x,y]^\Gamma/I$, which can be thought of as the image of a map w from a one-dimensional vector space. Lemma 6.1 now shows that we get the data of an A_J-module of dimension vector $(1,n)$. This module is moreover cyclic with generator at vertex ∞, so it is η_J-stable, as required. This construction works relatively over the whole of $\mathrm{Hilb}^{[n]}\left(\mathbb{C}^2/\Gamma\right)$, equipping $\mathcal{O} \oplus \mathcal{T}$ with the structure of a family of η_J-stable A_J-modules, as claimed. Moreover, since the bundle $\mathcal{O} \oplus \mathcal{T}$ inducing ω_n has $\mathcal{O}$ as a summand, and since the trivial bundle on any scheme induces the structure morphism, we see that ω_n commutes with the structure morphisms to $\mathrm{Sym}^n\left(\mathbb{C}^2/\Gamma\right)$.

Reading Lemma 6.1 in the opposite direction, an η_J-stable A_J-module of dimension vector $(1,n)$ defines a cyclic $\mathbb{C}[x,y]^\Gamma$-module of dimension n over $\mathbb{C}$. The universal property of $\mathrm{Hilb}^{[n]}\left(\mathbb{C}^2/\Gamma\right)$ then ensures that the flat family $T_\infty \oplus T_0$ of η_J-stable A_J-modules of dimension vector $(1,n)$ over $\mathcal{M}(A_J)$ determines a morphism $\mathcal{M}(A_J) \to \mathrm{Hilb}^{[n]}\left(\mathbb{C}^2/\Gamma\right)$, which is by construction the inverse of the morphism ω_n. □

We deduce Theorem 1.1 announced in the introduction.

Corollary 6.3. *For any $n \geqslant 1$, the reduced scheme underlying $\mathrm{Hilb}^{[n]}\left(\mathbb{C}^2/\Gamma\right)$ is isomorphic to the quiver variety $\mathfrak{M}_{\theta_0}$ for the parameter $\theta_0 = (-n,1,0,\dots,0)$ (compare with Figure 1). In particular, $\mathrm{Hilb}^{[n]}\left(\mathbb{C}^2/\Gamma\right)_{\mathrm{red}}$ is a normal, irreducible scheme over $\mathbb{C}^{2n}/\Gamma_n$ with symplectic singularities that admits a unique projective symplectic resolution, namely the morphism*

$$n\Gamma\text{-}\mathrm{Hilb}\left(\mathbb{C}^2\right) \to \mathrm{Hilb}^{[n]}\left(\mathbb{C}^2/\Gamma\right)_{\mathrm{red}}$$

that sends an ideal I in $\mathbb{C}[x,y]$ to the ideal $I \cap \mathbb{C}[x,y]^\Gamma$.

Proof. The first statement follows from Theorem 4.8 and Proposition 6.2, while the geometric properties of $\mathrm{Hilb}^{[n]}\left(\mathbb{C}^2/\Gamma\right)_{\mathrm{red}}$ are all inherited from its manifestation as $\mathfrak{M}_{\theta_0}$ via Lemma 2.1.

Next we prove the statement about the resolution. In the notation of [BC20, Theorem 1.2], the extremal ray $\rho_1^\perp \cap \cdots \cap \rho_r^\perp$ of the cone F that contains $\theta_0 = (-n,1,0,\dots,0)$ lies in the closure of precisely one chamber, namely the chamber C_+. Under the isomorphism L_F from ibid., it follows that there is exactly one projective symplectic resolution of $\mathfrak{M}_{\theta_0}$, namely the fine moduli space $\mathfrak{M}_\theta$ for $\theta \in C_+$. By Theorem 2.3, this resolution is indeed $\mathfrak{M}_\theta \cong n\Gamma\text{-}\mathrm{Hilb}\left(\mathbb{C}^2\right)$.

To see the last statement of the corollary, consider the morphism $\tau_J \colon \mathfrak{M}_\theta \to \mathcal{M}(A_J)$ constructed in Lemma 4.1. This is obtained by restricting a representation of the framed preprojective algebra Π to the vertices 0 and ∞, noting that the map of vector bundles $\mathcal{R}_0 \to \mathcal{R}_\infty$ is the zero map, and thus we indeed get a representation of A_J. On the other hand, as we discussed before, the isomorphism $\mathfrak{M}_\theta \cong n\Gamma\text{-}\mathrm{Hilb}\left(\mathbb{C}^2\right)$ identifies a Π-module with the quotient $\mathbb{C}[x,y]/I$ for a Γ-invariant ideal I of $\mathbb{C}[x,y]$. In this language, the restriction to the 0-vertex is the Γ-invariant part of the quotient $\mathbb{C}[x,y]/I$. The statement follows. □

Remark 6.4. (1) The irreducibility of $\mathrm{Hilb}^{[n]}\left(\mathbb{C}^2/\Gamma\right)$ was first established by Zheng [Zhe17] through the study of maximal Cohen–Macaulay modules on Kleinian singularities using a case-by-case analysis following the ADE classification.

(2) The uniqueness of the symplectic resolution of $\mathrm{Hilb}^{[n]}\left(\mathbb{C}^2/\Gamma\right)$ was previously known in the special case $n=2$ by the work of Yamagishi [Yam17, Proposition 2.10].

(3) Our approach does not shed light on whether $\mathrm{Hilb}^{[n]}\left(\mathbb{C}^2/\Gamma\right)$ is reduced in its natural scheme structure coming from its moduli space interpretation.

Remark 6.5. For $n=1$, the statement of Theorem 1.1 is well known because $\mathrm{Hilb}^{[1]}\left(\mathbb{C}^2/\Gamma\right) \cong \mathbb{C}^2/\Gamma$, while the statement of Theorem 4.8 is a framed version of [CIK18, Theorem 1.2] for $\Gamma \subset \mathrm{SL}(2,\mathbb{C})$. Nevertheless, the approach of the current paper is valid for $n=1$ and shows in particular that $\mathfrak{M}_{\theta_J} \cong \mathbb{C}^2/\Gamma$ for $J=\{0\}$. In fact, this result follows from [BC20, Proposition 7.11]. Indeed, ibid. constructs a surjective linear map $L_{C_+} \colon \Theta_v \to N^1\big(S/\left(\mathbb{C}^2/\Gamma\right)\big)$ with kernel equal to the subspace spanned by $(-1,1,0,\dots,0)$, such that $L_{C_+}(C_+)$ is the ample cone of S over $\mathbb{C}^2/\Gamma$. Since $\theta_J = (-1,1,0,\dots,0)$ for $J=\{0\}$ and $n=1$, it follows that $\mathfrak{M}_{\theta_J} \cong \mathbb{C}^2/\Gamma$ in that case. In addition, this explicit description of the kernel of L_{C_+} for $n=1$ shows that the morphisms $\pi_{J,J'}$ and $\tau_{J,J'}$ from Propositions 2.4 and 5.2 are isomorphisms if and only if $J' \setminus J = \{0\}$.

Appendix. Bounding the dimension vectors of θ_J-stable modules

A.1 The key statement

We use the term 'diagram' to mean 'framed extended Dynkin diagram' and use the notation A_i, D_i, E_i for the framed extended versions of these Dynkin diagrams. An A-module M of the appropriate type naturally determines a representation V of the corresponding quiver Q^* that satisfies the relations from equation (3.1); below we will call these simply 'quiver representations'. The notion of θ_J-stability for M defines a notion of θ_J-stability for V.

For $i \in Q_0^* = \{\infty, 0, 1, \dots, r\}$ we write $v_i := \dim_i V$, and for $0 \leqslant i \leqslant r$ we write $\delta_i := \dim(\rho_i)$, so that the regular representation $\delta = \sum_{0\leqslant i\leqslant r} \delta_i \rho_i$ coincides with the minimal imaginary root of the affine Lie algebra associated to the extended Dynkin diagram.

The goal of this appendix is to prove the following result, which we require in the proof of Lemma 4.6.

Proposition A.1. *Let $J \subseteq \{0, 1, \dots, r\}$ be a non-empty subset. Assume that V is a θ_J-stable quiver representation with $v_\infty = 1$ and $v_i = n\delta_i$ for $i \in J$ and some fixed natural number n. Then $v_j \leqslant n\delta_j$ for $j \notin J \cup \{\infty\}$.*

The proof splits into two cases according to whether or not $0 \in J$. We first treat the case $0 \in J$ that is required for our conclusions about $\mathrm{Hilb}^{[n]}\left(\mathbb{C}^2/\Gamma\right)$. We then go on to study the case $0 \notin J$ in a lengthy case-by-case analysis beginning in Section A.2.

Our main tool for proving Proposition A.1 is the following estimate, the proof of which is inspired by a result of Crawley-Boevey [Cra01, Lemma 7.2]. This inequality is the only consequence of θ_J-stability that we use in the subsequent numerical argument.

Lemma A.2. *Let V be a θ_J-stable quiver representation. If $i \notin J$, then $2v_i \leqslant \sum_{\{a\in Q_1 \mid \mathrm{h}(a)=i\}} v_{\mathrm{t}(a)}$.*

Proof. Define

$$V_\oplus := \bigoplus_{\substack{a\in Q_1,\\ \mathrm{h}(a)=i}} V_{\mathrm{t}(a)}\,.$$

The maps in V determined by arrows with tail and head at vertex i combine to define maps $f\colon V_i \to V_\oplus$ and $g\colon V_\oplus \to V_i$ satisfying $g \circ f = 0$.

If $\ker(f) \neq 0$, then V admits a non-zero subrepresentation W such that $W_i = \ker(f)$ and $W_j = 0$ for $j \neq i$. But then W is a proper, non-zero subrepresentation of V satisfying $\theta_J(W) = 0$, thereby contradicting the θ_J-stability of V. Thus f is injective. Similarly, if $\mathrm{Im}(g) \subsetneq V_i$, then V admits a subrepresentation U such that $U_i = \mathrm{Im}(g)$ and $U_j = V_j$ for $j \neq i$. Then U is a proper, non-zero subrepresentation of V satisfying $\theta_J(U) = \theta_J(V) = 0$, which again contradicts the θ_J-stability of V, so g is surjective. It follows that the complex

$$0 \longrightarrow V_i \xrightarrow{\ f\ } V_\oplus \xrightarrow{\ g\ } V_i \longrightarrow 0 \tag{A.2}$$

has non-zero homology only at $V_\oplus$, so $\dim V_\oplus \geqslant 2 \dim V_i$. □

Proof of Proposition A.1 in the case $0 \in J$. Let v' be the restriction of the stable dimension vector v to the underlying extended Dynkin diagram, and let C be the Cartan matrix of the same extended diagram. Define $u = v' - n\delta$. We will be done once we show that $u_i \leqslant 0$ for all i.

We can rephrase Lemma A.2 as saying that $(Cv')_i < 0$ for $i \notin J$. Since $C\delta = 0$, this also implies that $(Cu)_i \leqslant 0$ for all i. As in Remark 4.7, removing $J \cup \{\infty\}$ from the diagram leaves a collection of finite-type Dynkin diagrams. Let Q' be any such subdiagram, let $C_{Q'}$ be its Cartan matrix, and let $u_{Q'}$ be the restriction of u to Q'. As $u_i = 0$ for any $i \in J$, it follows that $(C_{Q'}u_{Q'})_i \leqslant 0$ for all i. Now $C_{Q'}^{-1}$ has only positive coefficients (see, for example, [Ros97, Matrices (1.157), (1.158)]), and so

$$u_{Q'} = C_{Q'}^{-1} C_{Q'} u_{Q'}$$

also satisfies $u_{Q',i} \leqslant 0$ for all $i \in Q'$. Then $u_i \leqslant 0$ for all i, giving $v_i' \leqslant n\delta_i$ for all i, as required. □

Remark A.3. Our original proof of Proposition A.1 in the case $0 \in J$ used a lengthy case-by-case argument, similar to that which follows for the case $0 \notin J$. We are grateful to the referee for suggesting this more elegant approach (see also [Nak20, Section 1.2]). Unfortunately, we were unable to extend this technique to the case $0 \notin J$. Indeed, if we define $V_\oplus$ as the sum of all vector spaces indexed by adjacent vertices in the McKay quiver – that is, excluding the framing vertex – then the complex (A.2) can have non-zero homology at the second V_i.

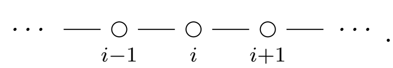
A.2 Strategy and preparatory results for the case $0 \notin J$

We now lay the foundation for the proof of Proposition A.1 in the case $0 \notin J$. We argue by contradiction, performing a case-by-case analysis on Dynkin diagrams. The basic idea is as follows. First, if the inequality $v_i > n\delta_i$ holds for a vertex i but not its neighbour j, we deduce a basic inequality (A.3) and show that this inequality can be 'pushed along' the branches of the diagram (see Lemma A.4). If the diagram branches at a trivalent vertex, then we push the inequality further along at least one branch (see Lemma A.5). This leads either to a contradiction or to strong constraints on $\dim V$.

Lemma A.4. (i) *Let $i, i-1$ be adjacent vertices of the diagram. If $v_i > n\delta_i$ and $v_{i-1} \leqslant n\delta_{i-1}$, then*

$$\delta_{i-1} v_i > \delta_i v_{i-1}\,. \tag{A.3}$$

(ii) *Suppose that the vertex $i \notin J$ is bivalent and neither of its neighbours is ∞:*

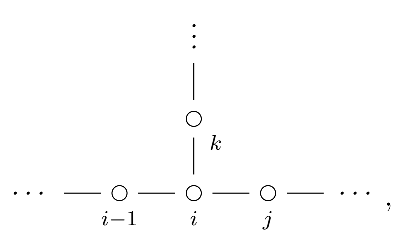


(A.4)

Then $\delta_{i-1}v_i > \delta_i v_{i-1}$ implies $\delta_i v_{i+1} > \delta_{i+1} v_i$. If in addition $v_i > n\delta_i$, then $v_{i+1} > n\delta_{i+1}$.

Proof. Part (i) is immediate. Since i and ∞ are not neighbours, $2\delta_i = \delta_{i-1} + \delta_{i+1}$ holds. Part (ii) follows by combining this equality with the assumed inequality $\delta_{i-1}v_i > \delta_i v_{i-1}$ and $2v_i \leqslant v_{i-1} + v_{i+1}$ coming from Lemma A.2. The last statement is again immediate. □

Lemma A.5. *Suppose that the diagram has a trivalent vertex $i \notin J$, not adjacent to the vertex ∞:*



(A.5)

and assume that $\delta_{i-1}v_i > \delta_i v_{i-1}$.

(i) *At least one of the inequalities $\delta_j v_i < \delta_i v_j$ and $\delta_k v_i < \delta_i v_k$ must hold.*

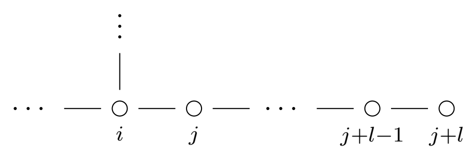
Now, suppose that $v_i > n\delta_i$*, that* $\delta_j v_i < \delta_i v_j$ *holds, and furthermore that the branch starting at* j *does not branch further. Then*

(ii) *the branch starting at* j *does not contain any vertices in* J*, and*

(iii) *the same branch must terminate at the framing vertex* ∞*, and in this case* $\delta_i v_j = \delta_j v_i + 1$*.*

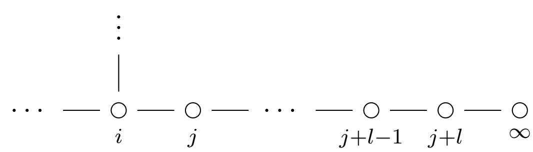
Remark A.6. The only framed extended Dynkin diagrams where a trivalent vertex is adjacent to the framing vertex are of type A_i for $i > 1$. We handle the case of such a vertex not being in J in Lemma A.8.

Proof of Lemma A.5*.* For part (i), combining $2\delta_i = \delta_{i-1} + \delta_j + \delta_k$ with $2v_i \leqslant v_{i-1} + v_k + v_j$ and $\delta_{i-1}v_i > \delta_i v_{i-1}$ leads to $\delta_j v_i + \delta_k v_i < \delta_i v_j + \delta_i v_k$, which implies the result. For parts (ii) and (iii), we number the vertices as



(A.6)

if the branch does not contain the framing vertex and as



(A.7)

if it does. To simplify notation, we take $j - 1 = i$ in the following argument. One of the following must occur:

(1) *The branch contains another vertex in* J*.* Suppose that j' is the node with smallest index on the branch such that $j' \neq i$ and $j' \in J$. Lemma A.4(ii) gives $\delta_{j'-1}v_{j'} > \delta_{j'}v_{j'-1}$ and $v_{j'} > n\delta_{j'}$, contradicting $j' \in J$.

(2) *The branch contains no vertices in* $J \cup \infty$*.* Repeated applications of Lemma A.4(ii) show that $\delta_{j+l-1}v_{j+l} > \delta_{j+l}v_{j+l-1}$. However, since $2\delta_{j+l} = \delta_{j+l-1}$, this implies $2v_{j+l} > v_{j+l-1}$, contradicting Lemma A.2.

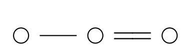
(3) *The branch contains no vertices of* J *and terminates at* ∞*.* We prove a slightly stronger statement, namely that for any vertex $m \neq \infty$ on the branch, we have $\delta_{m-1}v_m = \delta_m v_{m-1} + 1$. We proceed by induction on the number of edges that lie between ∞ and m. For the base case $m = j + l$, note that $\delta_{j+l-1}v_{j+l} > \delta_{j+l}v_{j+l-1}$ implies $2v_{j+l} > v_{j+l-1}$. However, since $2v_{j+l} \leqslant v_{j+l-1} + 1$ by Lemma A.2, we must have $2v_{j+l} = v_{j+l-1} + 1$. If there is more than one edge between ∞ and m, then the induction hypothesis gives $\delta_m v_{m+1} = \delta_{m+1}v_m + 1$. Combining this with $2v_m \leqslant v_{m-1} + v_{m+1}$ from Lemma A.2 and $2\delta_m = \delta_{m+1} + \delta_{m-1}$ shows that $\delta_{m-1}v_m \leqslant \delta_m v_{m-1} + 1$. Lemma A.4(ii) gives $\delta_{m-1}v_m > \delta_m v_{m-1}$, and the result follows.

This concludes the proof. □

A.3 Proof for the case $0 \notin J$, types A_1 and D_4

Lemma A.7. *Proposition* A.1 *holds for* A_1 *and* D_4*.*

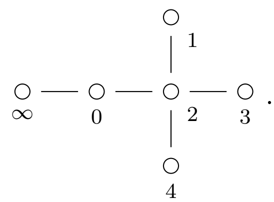
Proof. For type A_1, we have the diagram



, (A.8)

where the symbol $=\!=$ indicates that the diagram has two edges. The only remaining case is $J = \{1\}$. A straightforward adaptation of Lemma A.2 shows that if $J = \{1\}$, then $2v_0 \leqslant 2v_1 + 1 = 2n + 1$, giving $v_0 \leqslant n$.

For type D_4, the diagram is



(A.9)

Suppose without loss of generality that $1 \in J$. Then any other vertex i with $v_i > n\delta_i$ will, by Lemma A.5 or A.2, give that $v_2 > 2n$. The same lemmas show that

$$4v_2 \leqslant 2v_0 + 2v_1 + 2v_3 + 2v_4 \leqslant 2n + 3v_2 + 1 \tag{A.10}$$

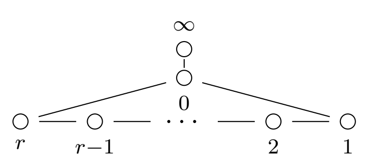
and thus $v_2 \leqslant 2n+1$. So $v_2 = 2n+1$, but then Lemma A.2 gives that $v_1 = v_3 = v_4 = n$. Plugging this into (A.10) gives $6n + 3 = 3v_2 \leqslant 6n + 1$, which leads to a contradiction. □

A.4 Proof when $0 \notin J$, the general case

For the rest, we need to handle each diagram type separately.

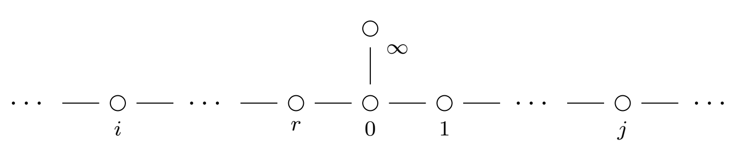
LEMMA A.8. *Proposition A.1 holds for any diagram of type A_i with $i > 1$.*

Proof. We number the vertices as follows:



(A.11)

Assume that some vertex $k' \neq \infty$ has $v_{k'} > n\delta_{k'} = n$. Since $0 \notin J$, we may consider a subdiagram



(A.12)

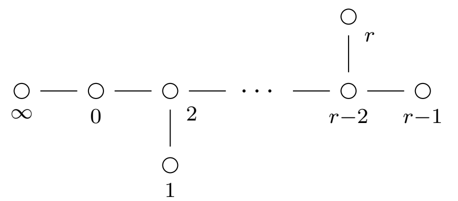
where i, j (possibly equal) are the only vertices in J, with k' some vertex in this subdiagram. We can without loss of generality assume that $0 \leqslant k' < j$. Then there are adjacent vertices k, $k+1$ such that $k' \leqslant k$ and $k+1 \leqslant j$ with $v_k > n \geqslant v_{k+1}$. Repeatedly applying Lemma A.4 gives

$$v_0 > v_1 > n\,. \tag{A.13}$$

There must also be adjacent vertices l, $l+1$ between i and 0 such that $v_{l+1} > v_l$. In a similar way, this leads to $v_0 > v_r$. Combining with (A.13), we deduce $2v_0 > v_1 + v_r + 1$, contradicting Lemma A.2. □

LEMMA A.9. *Proposition A.1 holds for diagrams of type D_i with $i > 4$.*

Proof. We number the vertices as follows:



(A.14)

Our proof of Proposition A.1 for the case $0 \in J$ leaves only three possible configurations for the nodes in J when $0 \notin J$, up to symmetry of the diagram. We prove Proposition A.1 by contradiction in each case.

(1) *There is an i such that $2 \leqslant i \leqslant r-1$, $v_i > n\delta_i = 2n$, and all $j \in J$ have $i < j$.* Let k be maximal among the vertices such that $v_k > n\delta_k$. If $k \leqslant r-2$, we have $\delta_{k+1}v_k > \delta_k v_{k+1}$. Otherwise, $k = r-1$. We must have $J = \{r\}$, and by Lemma A.5, we get $\delta_{r-3}v_{r-2} > \delta_{r-2}v_{r-3}$. By symmetry, the case $k = r$ also leads to $\delta_{r-3}v_{r-2} > \delta_{r-2}v_{r-3}$.
Both cases lead, by Lemma A.4, to $v_2 > v_3$; that is, $v_2 - 1 \geqslant v_3$. Then Lemma A.5 gives $2v_0 = v_2 + 1$ and $2v_1 \leqslant v_2$. Combining these with Lemma A.2 leads to

$$4v_2 \leqslant 2v_3 + 2v_1 + 2v_0 \leqslant 4v_2 - 1\,,$$

which is absurd.

(2) *We have $v_1 > n\delta_1 = n$, and all $j \in J$ have $j > 2$.* This implies $v_2 > 2n$. Let j be the least vertex such that $v_j \leqslant n\delta_j$. Applying Lemmas A.4 and A.5 to the vertices $j-1$, j (or if $j = r$, the vertices r, $r-2$), we again find $v_2 > v_3$. Then the conclusion of case (1) applies.

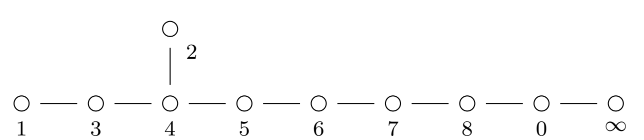
(3) *We have $v_0 > n\delta_0$, and all $j \in J$ have $j \geqslant 2$.* If $2 \in J$, we have $v_2 = 2n$, and then $v_1 > n$ leads to $2v_1 > 2n + 1$, contradicting Lemma A.2. If $2 \notin J$, we can again take j as the least vertex with $v_j \leqslant n\delta_j$ and argue as in case (2).

Therefore, if D_i has $i > 4$, then Proposition A.1 holds. □

To conclude, we consider the diagrams E_6, E_7 and E_8. As the proof strategies for these are very similar, we only include the full argument for the E_8 case.

Lemma A.10. *Proposition A.1 holds for type E_8.*

Proof. We number the vertices as follows:



(A.15)

This time, we split the possible configurations for J across the diagram in the case $0 \notin J$ into four possibilities. Let k be the minimal vertex with $v_k > n\delta_k$. The possible configurations are:

(1) *We have $k > 4$ and all $j \in J$ have $j < k$, or $k = 0$.* By Lemmas A.4 and A.5, we find that $v_0 > n\delta_0 = n$. The same lemmas show that $\delta_{k+1}v_k + 1 = \delta_k v_{k+1}$. Let us temporarily use the designation 9 for the vertex marked 0. By Lemma A.2, we get

$$2\delta_k v_k \leqslant \delta_k v_{k+1} + \delta_k v_{k-1} \leqslant \delta_{k+1}v_k + 1 + \delta_k n\delta_{k-1}\,,$$

implying $\delta_{k-1}(v_k - n\delta_k) \leqslant 1$. But this contradicts $v_k > n\delta_k$.

(2) *We have $k = 4$, and all $j \in J$ have $j < k$.* We have

$$2v_4 \leqslant v_2 + v_3 + v_5 \leqslant n\delta_2 + n\delta_3 + v_5 = 7n + v_5 \,.$$

Since we also have $5v_4 + 1 = 6v_5$ by Lemma A.5, this implies that $7v_5 - 2 \leqslant 35n$. But since $v_5 > 5n$, this is impossible.

(3) *We have $k = 2$, and at least one of the vertices 1 and 3 is in J.* By Lemma A.2, we must have $v_4 \geqslant 6n+1$. By Lemma A.4 applied to the vertex chain 1, 3, 4, we find $6v_3 < 4v_4$. Then Lemma A.5 shows that $6v_5 = 5v_4 + 1$. Now, if $v_3 \leqslant n\delta_3$, the same lemma and Lemma A.2 imply

$$12v_4 \leqslant 6v_2 + 6v_3 + 6v_5 \leqslant 8v_4 + 24n + 1 \,,$$

leading to $24n + 4 \leqslant 4v_4 \leqslant 24n + 1$, which gives a contradiction. So suppose that $1 \in J$ and $v_3 > 4n$. By Lemma A.4, we get $6v_3 < 4v_4$. As above, we find

$$12v_4 \leqslant 6v_2 + 6v_3 + 6v_5 \leqslant 8v_4 + 6v_3 + 1 \,,$$

leading to $4v_4 \leqslant 6v_3 + 1$. This implies that $4v_4 = 6v_3 + 1$, which has no integer solutions. Hence we have a contradiction.

(4) *We have $k = 1$ or $k = 3$, and so J only consists of 2.* Suppose that $v_2 = n\delta_2 = 3n$. Then, by Lemmas A.5 and A.4, we get $v_4 > 4n$, say $v_4 = 4n + t$ with $t > 0$. But then Lemmas A.5 and A.2 give

$$12v_4 \leqslant 6v_2 + 6v_3 + 6v_5 \leqslant 18n + 4v_4 + 5v_4 + 1 \,,$$

leading to $18n + 3t = 3v_4 \leqslant 18n + 1$, which gives a contradiction.

Therefore, Proposition A.1 holds for E_8. □

Proof of Proposition A.1 in the case $0 \notin J$. Our case-by-case analysis is given in Lemmas A.7–A.10 where, as noted above, the E_6- and E_7-cases are similar to that of E_8 from Lemma A.10. □

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