Algebraic Geometry 8 (3) (2021) 286–318
doi:10.14231/AG-2021-007

The integral Chow ring of $\overline{M}_2$

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Abstract

In this paper, we compute the Chow ring of the moduli stack $\overline{M}_2$ of stable curves of genus 2 with integral coefficients.

1. Introduction

Characteristic classes of vector bundles play an essential role in many areas of mathematics, from algebraic topology to differential geometry. Geometrically, characteristic classes of real or complex vector bundles are elements of the cohomology or Chow rings of the moduli stacks of real or complex vector spaces. These are the classifying spaces $B\mathrm{GL}_n \mathbb{R}$ and $B\mathrm{GL}_n \mathbb{C}$, whose cohomology or Chow rings are freely generated as a ring by the familiar Stiefel–Whitney and Chern classes, respectively.

Like vector spaces, algebraic curves are also ubiquitous in many areas of mathematics, so it is natural to try to understand their characteristic classes. Geometrically, this amounts to understanding the Chow ring (or cohomology ring) of the moduli stack $\overline{M}_g$ of stable curves of genus g. (Since the smooth Artin stack $\overline{M}_g$ is a global quotient stack, work of Edidin and Graham [EG98] implies that its Chow groups $\mathrm{CH}^*(\overline{M}_g)$ possess an intersection product. In fact, by work of Kresch, intersection products exist, more generally, on any smooth Artin stack that admits a stratification by global quotient stacks; cf. Section 5.1 of [Kre99].)

Question. What is the Chow ring $\mathrm{CH}^*(\overline{M}_g)$ of $\overline{M}_g$?

Despite much progress over the past half century, answers to this question are only known after making various simplifications—and even then only in small genus.

One such simplification is to study the Chow ring with rational coefficients $\mathrm{CH}^*(\overline{M}_g) \otimes \mathbb{Q}$; this removes the subtle torsion phenomenon that exists due to the presence of loci of curves with automorphisms. For example, the Chow ring of $\overline{M}_g$ with rational coefficients is known for $g = 2$ by work of Mumford [Mum83] and for $g = 3$ by work of Faber [Fab90].

Another such simplification is to replace $\overline{M}_g$ with a simpler but related space, for example M_g or $\overline{M}_{0,n}$ (with $n \geqslant 3$) or $\overline{M}_{1,n}$ (with $n \geqslant 1$). For example, the full Chow ring (that is, with integral coefficients) of M_g is known for $g = 2$ by work of Vistoli [Vis98].

Received 28 June 2020, accepted in final form 15 September 2020.
2020 Mathematics Subject Classification 14H10.
Keywords: Chow rings, moduli spaces.

The author would like to acknowledge the generosity of the National Science Foundation in supporting this research—both via the Research Experience for Undergraduates program where he began thinking about this problem in 2013 (DMS-1250467), and via the Mathematical Sciences Postdoctoral Research Fellowship program (DMS-1802908) that provided recent funding for the completion of this project.

However, to date, the full Chow ring $\mathrm{CH}^*(\overline{M}_g)$ is not known in a single case. The goal of the present paper is to give the first such example.

THEOREM 1.1. *Over any base field of characteristic distinct from 2 and 3, the Chow ring of the moduli space of stable curves of genus 2 is given by*

$$\mathrm{CH}^*(\overline{M}_2) = \mathbb{Z}[\lambda_1, \lambda_2, \delta_1]/\big(24\lambda_1^2 - 48\lambda_2, 20\lambda_1\lambda_2 - 4\delta_1\lambda_2, \delta_1^3 + \delta_1^2\lambda_1, 2\delta_1^2 + 2\delta_1\lambda_1\big)\,,$$

where λ_1 and λ_2 denote the Chern classes of the Hodge bundle and δ_1 denotes the class of the boundary substack with a disconnecting node.

In this basis, the class of the boundary substack with a self-node is given by $\delta_0 = 10\lambda_1 - 2\delta_1$.

One might hope to compute the Chow ring of $\overline{M}_2$ using a presentation of $\overline{M}_2$ as a quotient stack [EG98], as done for M_2 by Vistoli in [Vis98]. Unfortunately, finding a suitable presentation of $\overline{M}_2$ seems extremely difficult.

Instead, we stratify $\overline{M}_2$ into the boundary divisor Δ_1 and its complement $\overline{M}_2 \smallsetminus \Delta_1$, whose Chow rings can be computed using equivariant intersection theory. The usual localization sequence for Chow groups of schemes also holds for Artin stacks of finite type (cf. [Kre99, Proposition 2.3.6]), so we obtain an exact sequence

$$\mathrm{CH}^*(\Delta_1) \to \mathrm{CH}^*(\overline{M}_2) \to \mathrm{CH}^*(\overline{M}_2 \smallsetminus \Delta_1) \to 0\,.$$

One key observation is that a variant of Vistoli's argument in [Vis98] can not only compute $\mathrm{CH}^*(M_2)$, but in fact compute both the Chow ring and the first higher Chow groups with ℓ-adic coefficients (which control the failure of left-exactness in the above sequence) of $\overline{M}_2 \smallsetminus \Delta_1$. Unfortunately, the first higher Chow groups of the open stratum $\overline{M}_2 \smallsetminus \Delta_1$ do *not* vanish for $\ell = 2$; they are generated by 2-torsion classes in degrees 4 and 5.

For the degree 4 generator, we observe that all the 2-torsion in $\mathrm{CH}^3(\Delta_1)$ arises, in some sense, "locally" from those curves in Δ_1 which admit a bielliptic involution (exchanging the two components). Using the universal family of curves with a bielliptic involution as a test family, we show that the pushforward map $\mathrm{CH}^3(\Delta_1) \to \mathrm{CH}^4(\overline{M}_2)$ is injective, and so the image of the degree 4 generator in $\mathrm{CH}^3(\Delta_1)$ must vanish.

By contrast, the degree 5 generator can be thought of as arising, in some sense, "globally" from the presence of the hyperelliptic involution on every curve; in particular, it seems extremely difficult to study this class using a test family. Instead, we get a handle on this class using the action of $\mathbb{G}_m$ on $\overline{M}_2$ given by taking the "universal quadratic twist with respect to the hyperelliptic involution." This $\mathbb{G}_m$-action kills the degree 5 generator, which then corresponds to a "syzygy" between two relations of smaller degree in $\mathrm{CH}^*(\overline{M}_2/\mathbb{G}_m)$; using this, we can explicitly compute that the boundary map vanishes on this degree 5 generator.

A brief outline of the remainder of the paper is as follows: We begin in Section 2 by fixing some notation for some group representations that will appear throughout the paper. Next, in Section 3, we give presentations of the strata Δ_1 and $\overline{M}_2 \smallsetminus \Delta_1$, as quotients of open subsets in affine spaces by linear algebraic groups—a group we term G defined in the next section for Δ_1, respectively the group GL_2 for $\overline{M}_2 \smallsetminus \Delta_1$. In Section 4, we give formulas for pushforward maps between Chow groups of various projective bundles under multiplication maps, that will be used throughout the remainder of the paper. Then in Section 5, we compute $\mathrm{CH}^*(BG)$. This is enough to show in Section 6 that $\mathrm{CH}^*(\overline{M}_2)$ is generated by δ_1, λ_1, and λ_2, as claimed, and to establish all the relations appearing in the statement of Theorem 1.1 in characteristic zero.

Then in Section 7, we explicitly compute the pushforward and pullback maps between Chow groups along $BT \to BG$, where T is the maximal torus of G. In Sections 8 and 9, we compute the Chow ring of the stratum Δ_1, respectively the Chow ring and first higher Chow groups with ℓ-adic coefficients of the stratum $\overline{M}_2 \smallsetminus \Delta_1$; we also derive an explicit formula (which happens to evaluate to zero) for the image of the degree 5 generator mentioned above. In Section 10, we use this to reduce Theorem 1.1 to the nonvanishing of a finite number of classes in $\mathrm{CH}^*(\overline{M}_2) \otimes \mathbb{Z}/2\mathbb{Z}$. Finally, in Section 11, we compute the Chow ring of the universal family of bielliptic curves of genus 2, which completes the proof of Theorem 1.1 by serving as a test family to show the nonvanishing of these desired classes.

1.1 Results in cohomology

In characteristic zero, one may similarly ask for the singular (orbifold) cohomology with integral coefficients of $\overline{M}_2$. A variant of the arguments developed here for Chow groups implies that the even cohomology is isomorphic to the Chow ring via the cycle class map, and that the odd cohomology groups have exponent at most 96. Although determining the ring structure, or even the precise isomorphism type as groups, of the odd cohomology would require additional ideas, the orders of the groups $\mathrm{H}^{2n+1}(\overline{M}_2)$ can be determined explicitly (cf. equation (10.3)).

1.2 Remark

Upon completion of this manuscript, the author learned that Angelo Vistoli and Andrea Di Lorenzo are working on a different approach to this problem, which will hopefully yield an independent proof of Theorem 1.1.

2. Notation

In this section, we fix some notational conventions that we shall use for the remainder of the paper.

Assumptions on the characteristic

For the remainder of the paper, we work over a field k of characteristic distinct from 2 and 3.

Assumptions on stacks

For the remainder of the paper, we shall use the word *stack* to refer to an Artin stack which admits a stratification by global quotient stacks. (In fact, all the stacks we shall need are global quotient stacks.) As shown by Kresch in Theorem 2.1.12 of [Kre99], extending earlier work of Edidin and Graham for global quotient stacks [EG98], the Chow groups of such stacks satisfy the expected properties, including localization sequences, intersection products when they are smooth, and invariance under passing to the total space of a vector bundle.

Convention for projective bundles

Given a vector bundle $\mathcal{E} \to X$, we adopt the convention that its projective bundle $\mathbb{P}\mathcal{E}$ denotes the space of lines in $\mathcal{E}$.

Definition 2.1. Write V for the standard representation of GL_2, and define the representations

$$V_n = \mathrm{Sym}^n V^* \quad \text{and} \quad V_n(m) = V_n \otimes (\det V)^{\otimes m}.$$

Definition 2.2. We denote by G the wreath product

$$G = \mathbb{G}_m \wr \mathbb{Z}/2\mathbb{Z} := (\mathbb{G}_m \times \mathbb{G}_m) \rtimes \mathbb{Z}/2\mathbb{Z}\,,$$

where the action of $\mathbb{Z}/2\mathbb{Z}$ on $\mathbb{G}_m \times \mathbb{G}_m$ permutes the factors.

Given a representation L of $\mathbb{G}_m$, we observe that the direct sum $L \oplus L$ inherits a natural action of G (where the $\mathbb{Z}/2\mathbb{Z}$-action permutes the factors); we write $L \boxplus L$ for this.

Definition 2.3. Write L_n for the 1-dimensional representation of $\mathbb{G}_m$ where z acts as multiplication by z^n, and let

$$L_{a_1,a_2,\ldots,a_n} = L_{a_1} \oplus L_{a_2} \oplus \cdots \oplus L_{a_n}\,.$$

Let $W_{a_1,a_2,\ldots,a_n}$ be the representation of G defined by

$$W_{a_1,a_2,\ldots,a_n} = L_{a_1,a_2,\ldots,a_n} \boxplus L_{a_1,a_2,\ldots,a_n}\,,$$

and write $W = W_1$.

Additionally, let Γ be the representation of G arising from the sign representation of the $\mathbb{Z}/2\mathbb{Z}$-quotient of G.

Definition 2.4. Write

$$\alpha_i = c_i(V) \in \mathrm{CH}^*(B\mathrm{GL}_2)\,, \quad \beta_i = c_i(W) \in \mathrm{CH}^*(BG)\,, \quad \text{and} \quad \gamma = c_1(\Gamma) \in \mathrm{CH}^*(BG)\,.$$

3. Presentations of $\boldsymbol{\Delta_1}$ and $\overline{M}_2 \smallsetminus \boldsymbol{\Delta_1}$

In this section, we give presentations of the boundary stratum Δ_1 of curves with a disconnecting node, and of its complement $\overline{M}_2 \smallsetminus \Delta_1$, as the quotient of an affine space by the action of an algebraic group. These presentations will be used in subsequent sections to compute the Chow rings of these loci.

3.1 A presentation of $\boldsymbol{\Delta_1}$

Note that the curves parameterized by Δ_1 are in the form of two elliptic curves glued together at their marked points. In other words, the stack Δ_1 can be described as the symmetric square of $\overline{M}_{1,1}$:

$$\Delta_1 \simeq \mathrm{Sym}^2 \overline{M}_{1,1}\,,$$

where for any global quotient stack X/H, we write $\mathrm{Sym}^2[X/H] := [X \times X/(H \wr \mathbb{Z}/2\mathbb{Z})]$.

Note that $\overline{M}_{1,1}$ is indeed a global quotient stack. Namely, recall that an elliptic curve E can be written in Weierstrass form

$$y^2 = x^3 + ax + b\,,$$

which is stable if and only if a and b do not simultaneously vanish. The fiber of the Hodge bundle here is given by

$$H^0(\omega_E) = \left\langle \frac{dx}{y} \right\rangle.$$

Moreover, the isomorphisms between two curves in Weierstrass form are given by

$$(x, y) \mapsto \left(u^{-2}x, u^{-3}y\right) \quad \text{for} \quad u \in \mathbb{G}_m\,;$$

the isomorphism given by u defines an isomorphism between the curves $y^2 = x^3 + ax + b$ and $y^2 = x^3 + u^4ax + u^6b$ and acts on dx/y as multiplication by u.

Thus, $\overline{M}_{1,1} \simeq (L_{4,6} \smallsetminus 0)/\mathbb{G}_m$. Moreover, the Hodge bundle of $\overline{M}_{1,1}$ is the pullback of the representation L_1 from $B\mathbb{G}_m$. Consequently, we have the following fundamental presentation:

$$\Delta_1 \simeq \operatorname{Sym}^2 \overline{M}_{1,1} \simeq \left(W_{4,6} \smallsetminus (L_{4,6} \times 0 \cup 0 \times L_{4,6})\right)/G\,, \tag{3.1}$$

and the Hodge bundle is the pullback from BG of the representation W_1.

3.2 A presentation of $\overline{M}_2 \smallsetminus \Delta_1$

Here we show that curves in $\overline{M}_2 \smallsetminus \Delta_1$ admit a canonical degree 2 map to $\mathbb{P}^1$ and use this to give a presentation of $\overline{M}_2 \smallsetminus \Delta_1$ that mirrors Vistoli's presentation of M_2 in [Vis98] (subsequently generalized to other stacks of cyclic covers by Arsie and Vistoli in [AV04]). However, to make this paper more self-contained, we do not suppose familiarity with the above-mentioned results.

We first claim that a stable curve C of genus 2 satisfies $[C] \in \overline{M}_2 \smallsetminus \Delta_1$ if and only if $H^0(\omega_C)$ is basepoint-free.

Indeed, if $p \in C$ is smooth, then by the Riemann–Roch formula for $\mathcal{O}_C(p)$, the point p is a basepoint of $H^0(\omega_C)$ if and only if $\dim H^0(\mathcal{O}_C(p)) = 2$. In this case, the corresponding complete linear series defines a map $C \to \mathbb{P}^1$ which is of degree 1 on the irreducible component containing p and of degree 0 on all other irreducible components. In particular, the irreducible component of C containing p is isomorphic to $\mathbb{P}^1$ (that is, rational with no self-nodes) and meets every connected component of its complement in a unique point. This is impossible for a stable curve of genus 2, so $H^0(\omega_C)$ can never have a smooth basepoint.

If $p \in C$ is singular, then write $\tilde{C}$ for the partial normalization of C at p. Note that sections of ω_C vanishing at p are in bijection with sections of $\omega_{\tilde{C}}$. In particular, by the Riemann–Roch formula for $\mathcal{O}_{\tilde{C}}$, the point p is a basepoint of $H^0(\omega_C)$ if and only if $\dim H^0(\mathcal{O}_{\tilde{C}}) = 2$, that is, if and only if p is a disconnecting node.

Thus $H^0(\omega_C)$ is basepoint-free if and only if C has no disconnecting nodes, that is, if and only if $[C] \in \overline{M}_2 \smallsetminus \Delta_1$, as desired.

Consequently, every $[C] \in \overline{M}_2 \smallsetminus \Delta_1$ comes equipped with a canonical map to $\mathbb{P}^1$; by the Riemann–Hurwitz formula, this map is branched over six points with multiplicity. We conclude that the family of curves in weighted projective space given by

$$z^2 = ax^6 + bx^5y + cx^4y^2 + dx^3y^3 + ex^2y^4 + fxy^5 + gy^6\,, \tag{3.2}$$

over the locus of degree 6 polynomials $ax^6 + bx^5y + cx^4y^2 + dx^3y^3 + ex^2y^4 + fxy^5 + gy^6$ with no triple root, contains every isomorphism class $[C] \in \overline{M}_2 \smallsetminus \Delta_1$.

Moreover, since the map to $\mathbb{P}^1$ is canonical, we conclude that any isomorphism between two curves in this family is via a linear change of variables on x and y and scalar multiplication on z, that is, via the action of $\mathrm{GL}_2 \times \mathbb{G}_m$ with GL_2 acting on (x, y) and $\mathbb{G}_m$ acting on z. By inspection, the subgroup of $\mathrm{GL}_2 \times \mathbb{G}_m$ acting trivially is the image of $\mathbb{G}_m$ under the map $t \mapsto \left(t \cdot \mathbf{1}, t^3\right)$. We conclude that U is isomorphic to the quotient of the locus of degree 6 polynomials $ax^6 + bx^5y + cx^4y^2 + dx^3y^3 + ex^2y^4 + fxy^5 + gy^6$ with no triple root by the natural action of $(\mathrm{GL}_2 \times \mathbb{G}_m)/\mathbb{G}_m$ constructed above.

To recast this in a somewhat nicer form, we observe that $(\mathrm{GL}_2 \times \mathbb{G}_m)/\mathbb{G}_m$ is itself isomorphic to GL_2 via the map $\mathrm{GL}_2 \times \mathbb{G}_m \to \mathrm{GL}_2$ defined by

$$A \times t \mapsto \frac{\det A}{t} \cdot A\,,$$

whose kernel is, by inspection, the image of $\mathbb{G}_m$ under the map $t \mapsto \left(t \cdot \mathbf{1}, t^3\right)$. An inverse to this map is given by

$$A \mapsto A \times \det A\,.$$

In particular, it follows that

$$\overline{M}_2 \smallsetminus \Delta_1 \simeq (V_6(2) \smallsetminus \{\text{forms with triple roots}\})/\operatorname{GL}_2\,.$$

Moreover, since $H^0(\omega_C) = \langle x, y\rangle$ with the natural action of GL_2, it follows that the Hodge bundle is the pullback of the representation V_1 from $B\operatorname{GL}_2$.

4. Pushforwards along multiplication maps

Let X be any stack and $\mathcal{E} \to X$ be a rank 2 vector bundle on X. Write $c_i = c_i(\mathcal{E})$, and $x = c_1(\mathcal{O}_{\mathbb{P}\mathcal{E}}(1))$ for the tautological class of the projective bundle $\mathbb{P}\mathcal{E}$. In this section, we give formulas for the pushforward maps on Chow groups under multiplication maps, similar to those obtained by Vistoli in [Vis98].

Definition 4.1. For $j \leqslant r$, define the class $s_r^j \in \operatorname{CH}^j\big(\mathbb{P}\operatorname{Sym}^r\mathcal{E}\big)$ as the pushforward of $x_1x_2\cdots x_j$ under the multiplication map $(\mathbb{P}\mathcal{E})^j \times \mathbb{P}\big(\operatorname{Sym}^{r-j}\mathcal{E}\big) \to \mathbb{P}\operatorname{Sym}^r\mathcal{E}$, where x_i denotes the pullback of x under projection to the ith $\mathbb{P}\mathcal{E}$-factor.

Lemma 4.2. *The s_r^j can be calculated via the following recurrence relation:*

$$s_r^0 = 1 \quad\text{and}\quad s_r^{j+1} = (t + jc_1)\cdot s_r^j + j(r+1-j)c_2 \cdot s_r^{j-1}\,,$$

where t is the hyperplane class on $\mathbb{P}\operatorname{Sym}^r\mathcal{E}$.

Proof. By functoriality, it suffices to consider the case when $X = B\operatorname{GL}_2$, in which case the Chow ring of X is torsion-free. Moreover, recall that Chow groups of the projective bundles are free as modules over the Chow ring of the base. Therefore, we may work with rational coefficients.

Write $\pi\colon (\mathbb{P}\mathcal{E})^r \to \mathbb{P}\operatorname{Sym}^r\mathcal{E}$ for the multiplication map. As $(\mathbb{P}\mathcal{E})^{r-j} \to \mathbb{P}\operatorname{Sym}^{r-j}\mathcal{E}$ is degree $(r-j)!$, we obtain

$$s_r^j = \frac{1}{(r-j)!}\pi_*(x_1\cdots x_j)\,.$$

By push-pull and the symmetry of π, this implies

$$\begin{aligned}
t\cdot s_r^j &= \frac{1}{(r-j)!}\pi_*\big(x_1\cdots x_j\cdot(x_1+\cdots+x_r)\big)\\
&= \frac{j}{(r-j)!}\pi_*\big(x_1\cdots x_{j-1}x_j^2\big) + \frac{r-j}{(r-j)!}\pi_*(x_1\cdots x_{j+1})\\
&= \frac{j}{(r-j)!}\pi_*\big(x_1\cdots x_{j-1}(-c_1x_j - c_2)\big) + \frac{r-j}{(r-j)!}\pi_*(x_1\cdots x_{j+1})\\
&= -\frac{jc_1}{(r-j)!}\pi_*(x_1\cdots x_j) - \frac{j(r+1-j)c_2}{(r-j+1)!}\pi_*(x_1\cdots x_{j-1}) + \frac{1}{(r-j-1)!}\pi_*(x_1\cdots x_{j+1})\\
&= -jc_1\cdot s_r^j - j(r+1-j)c_2\cdot s_r^{j-1} + s_r^{j+1}\,,
\end{aligned}$$

which yields the desired formulas. □

Lemma 4.3. *As a $\operatorname{CH}^*(X)$-module, $\operatorname{CH}^*\big(\mathbb{P}\operatorname{Sym}^r\mathcal{E}\big)$ is generated by the classes s_r^0, s_r^1, …, s_r^r.*

Proof. From the recurrence relations of Lemma 4.2, it is clear by induction on j that the sub-$\mathbb{Z}[c_1,c_2]$-module of $\operatorname{CH}^*\big(\mathbb{P}\operatorname{Sym}^r\mathcal{E}\big)$ generated by 1, t, t^2, …, t^j coincides with the sub-$\mathbb{Z}[c_1,c_2]$-module generated by s_r^0, s_r^1, s_r^2, …, s_r^j. □

The point of choosing this system of generators is that the pushforward along the multiplication map takes a particularly nice form in this basis.

Lemma 4.4. *The pushforward along the multiplication map*

$$\mathbb{P}\,\mathrm{Sym}^a\,\mathcal{E} \times \mathbb{P}\,\mathrm{Sym}^b\,\mathcal{E} \to \mathbb{P}\,\mathrm{Sym}^{a+b}\,\mathcal{E}$$

is given by

$$s_a^\alpha \times s_b^\beta \mapsto \binom{a-\alpha+b-\beta}{a-\alpha} \cdot s_{a+b}^{\alpha+\beta}\,.$$

Proof. Consider the commutative diagram, all of whose arrows are multiplication maps (plus permuting the factors):

$$\begin{array}{ccc} (\mathbb{P}\mathcal{E})^\alpha \times \big(\mathbb{P}\,\mathrm{Sym}^{a-\alpha}\,\mathcal{E}\big) \times (\mathbb{P}\mathcal{E})^\beta \times \mathbb{P}\big(\mathrm{Sym}^{b-\beta}\,\mathcal{E}\big) & \longrightarrow & (\mathbb{P}\mathcal{E})^{\alpha+\beta} \times \mathbb{P}\big(\mathrm{Sym}^{a+b-\alpha-\beta}\,\mathcal{E}\big) \\ \downarrow & & \downarrow \\ \mathbb{P}\big(\mathrm{Sym}^a\,\mathcal{E}\big) \times \mathbb{P}\big(\mathrm{Sym}^b\,\mathcal{E}\big) & \longrightarrow & \mathbb{P}\big(\mathrm{Sym}^{a+b}\,\mathcal{E}\big)\,. \end{array}$$

Then the left and right sides of the desired equality are the pushforwards around the left and right sides of the diagram, respectively, of the class $x_1x_2\cdots x_\alpha \cdot y_1y_2\cdots y_\beta$. □

Lemma 4.5. *The class of the diagonal $\mathbb{P}\mathcal{E} \subset \mathbb{P}\mathcal{E} \times \mathbb{P}\mathcal{E}$ is given by $x_1 + x_2 + c_1$.*

Proof. The result is given in Lemma 3.8 of [Vis98].

For completeness, we include a proof here: By functoriality, and considering the case when $X = B\mathrm{GL}_2$ and $\mathcal{E}$ is the standard representation, the desired class must be given by $ax_1+bx_2+cc_1$, for constants a, b, and c.

When X is a point, this formula asserts that the diagonal in $\mathbb{P}^1 \times \mathbb{P}^1$ has class (a,b); thus, $a = b = 1$. Moreover, when we tensor $\mathcal{E}$ by a line bundle $\mathcal{L}$ with Chern class $\ell = c_1(\mathcal{L})$, then the class of the diagonal is unaffected, but the x_i are replaced by $x_i - \ell$ while c_1 is replaced by $c_1 + 2\ell$. It follows that

$$(x_1 - \ell) + (x_2 - \ell) + c(c_1 + 2\ell) = x_1 + x_2 + cc_1\,,$$

and so $c = 1$, as desired. □

Lemma 4.6. *The class of the triple diagonal $\mathbb{P}\mathcal{E} \subset \mathbb{P}\mathcal{E} \times \mathbb{P}\mathcal{E} \times \mathbb{P}\mathcal{E}$ is given by*

$$(x_1x_2 + x_2x_3 + x_3x_1) + (x_1 + x_2 + x_3)\cdot c_1 + c_1^2 - c_2\,.$$

Proof. By Lemma 4.5, the image of the diagonal map $\mathbb{P}\mathcal{E} \to \mathbb{P}\mathcal{E} \times \mathbb{P}\mathcal{E}$ is $x_1 + x_2 + c_1$, where x_1 and x_1 are the pullbacks of the hyperplane classes from both factors of $\mathbb{P}\mathcal{E}$. Hence, the image of the triple diagonal map is given by

$$\begin{aligned} [\Delta_3] &= (x_1 + x_2 + c_1)(x_2 + x_3 + c_1) \\ &= (x_1x_2 + x_2x_3 + x_3x_1) + (x_1 + x_2 + x_3)\cdot c_1 + c_1^2 + x_2^2 + c_1x_2 \\ &= (x_1x_2 + x_2x_3 + x_3x_1) + (x_1 + x_2 + x_3)\cdot c_1 + c_1^2 - c_2\,. \end{aligned}$$ □

Lemma 4.7. *The pushforward along the symmetric square map $\mathbb{P}\mathcal{E} \to \mathbb{P}\,\mathrm{Sym}^2\,\mathcal{E}$ is given by*

$$s_1^0 \mapsto 2s_2^1 + 2c_1\,, \quad s_1^1 \mapsto s_2^2 - 2c_2\,.$$

Proof. By Lemma 4.5, the image of the diagonal map $\mathbb{P}\mathcal{E} \to \mathbb{P}\mathcal{E} \times \mathbb{P}\mathcal{E}$ is $[\Delta] = x_1 + x_2 + c_1$, where x_1 and x_1 are the pullbacks of the hyperplane classes from both factors of $\mathbb{P}\mathcal{E}$. Next, we multiply the above expression by x_1, to get

$$x_1 \cdot [\Delta] = x_1(x_1 + x_2 + c_1) = (-c_1x_1 - c_2) + x_1(x_2 + c_1) = x_1x_2 - c_2\,.$$

The desired formulas follow immediately. □

LEMMA 4.8. *The pushforward along the symmetric cube map $\mathbb{P}\mathcal{E} \to \mathbb{P}\operatorname{Sym}^3\mathcal{E}$ is given by*

$$s_1^0 \mapsto 3s_3^2 + 6c_1s_3^1 + 6\bigl(c_1^2 - c_2\bigr)\,, \quad s_1^1 \mapsto s_3^3 - 6c_2s_3^1 - 6c_1c_2\,.$$

Proof. By Lemma 4.6, the image of the triple diagonal map $\mathbb{P}\mathcal{E} \to \mathbb{P}\mathcal{E} \times \mathbb{P}\mathcal{E} \times \mathbb{P}\mathcal{E}$ is given by

$$[\Delta_3] = (x_1x_2 + x_2x_3 + x_3x_1) + (x_1 + x_2 + x_3)\cdot c_1 + c_1^2 - c_2\,.$$

Next, we multiply the above expression by x_1 to get

$$\begin{aligned} x_1 \cdot [\Delta_3] &= x_1^2(x_2 + x_3 + c_1) + x_1x_2x_3 + x_1(x_2 + x_3)c_1 + x_1\bigl(c_1^2 - c_2\bigr) \\ &= (-c_1x_1 - c_2)(x_2 + x_3 + c_1) + x_1x_2x_3 + x_1(x_2 + x_3)c_1 + x_1\bigl(c_1^2 - c_2\bigr) \\ &= x_1x_2x_3 - (x_1 + x_2 + x_3)\cdot c_2 - c_1 \cdot c_2\,. \end{aligned}$$

The desired formulas follow immediately. □

LEMMA 4.9. *Let $\mathcal{V} \subset \mathcal{W}$ be an inclusion of vector bundles over X. Then the fundamental class of $\mathbb{P}\mathcal{V} \subset \mathbb{P}\mathcal{W}$ in $\mathrm{CH}^*(\mathbb{P}\mathcal{W})$ is given by the Chern polynomial of the quotient bundle $\mathcal{W}/\mathcal{V}$. That is, if $\dim \mathcal{W}/\mathcal{V} = d$, then*

$$[\mathbb{P}\mathcal{V}] = x^d + c_1(\mathcal{W}/\mathcal{V})x^{d-1} + \cdots + c_d(\mathcal{W}/\mathcal{V})\,,$$

where $x = c_1(\mathcal{O}_{\mathbb{P}\mathcal{W}}(1))$.

Proof. The composition $\mathcal{O}_{\mathbb{P}\mathcal{W}}(-1) \to \pi^*\mathcal{W} \to \pi^*(\mathcal{W}/\mathcal{V})$, where $\pi\colon \mathcal{W} \to X$ denotes the structure map, gives a section of $\pi^*(\mathcal{W}/\mathcal{V}) \otimes \mathcal{O}_{\mathbb{P}\mathcal{W}}(1)$ whose vanishing locus is $\mathbb{P}\mathcal{V} \subset \mathbb{P}\mathcal{W}$. Thus

$$[\mathbb{P}\mathcal{V}] = c_d(\pi^*(\mathcal{W}/\mathcal{V}) \otimes \mathcal{O}_{\mathbb{P}\mathcal{W}}(1)) = x^d + c_1(\mathcal{W}/\mathcal{V})x^{d-1} + \cdots + c_d(\mathcal{W}/\mathcal{V})\,. \qquad \square$$

LEMMA 4.10. *For rank 2 vector bundles $\mathcal{E}_1$ and $\mathcal{E}_2$ on X, the pushforward along the Segre map*

$$\mathbb{P}\mathcal{E}_1 \times \mathbb{P}\mathcal{E}_2 \to \mathbb{P}(\mathcal{E}_1 \otimes \mathcal{E}_2)$$

is given by

$$\begin{aligned} 1 &\mapsto 2x + c_1(\mathcal{E}_1) + c_1(\mathcal{E}_2)\,, \\ x_1 &\mapsto x^2 + c_1(\mathcal{E}_2)x + c_2(\mathcal{E}_2) - c_2(\mathcal{E}_1)\,, \\ x_2 &\mapsto x^2 + c_1(\mathcal{E}_1)x + c_2(\mathcal{E}_1) - c_2(\mathcal{E}_2)\,, \\ x_1x_2 &\mapsto x^3 + (c_1(\mathcal{E}_1) + c_1(\mathcal{E}_2))x^2 + (c_2(\mathcal{E}_1) + c_1(\mathcal{E}_1)c_1(\mathcal{E}_2) + c_2(\mathcal{E}_2))x \\ &\qquad + c_1(\mathcal{E}_1)c_2(\mathcal{E}_2) + c_2(\mathcal{E}_1)c_1(\mathcal{E}_2)\,. \end{aligned}$$

Proof. To find the pushforward of 1, we note that by functoriality, the desired class must be given by $ax + bc_1(\mathcal{E}_1) + cc_1(\mathcal{E}_2)$ for constants a and b. When X is a point, this formula asserts that the Segre surface $\mathbb{P}^1 \times \mathbb{P}^1 \subset \mathbb{P}^3$ is of degree a; thus $a = 2$. Moreover, when we tensor $\mathcal{E}_1$ by a line bundle $\mathcal{L}$ with Chern class $\ell = c_1(\mathcal{L})$, then the pushforward is unaffected, but x is replaced by $x - \ell$ while $c_1(\mathcal{E}_1)$ is replaced by $c_1(\mathcal{E}_1) + 2\ell$ and $c_1(\mathcal{E}_2)$ is unchanged. It follows that

$$2(x - \ell) + b(c_1(\mathcal{E}_1) + 2\ell) + cc_1(\mathcal{E}_2) = 2x + bc_1(\mathcal{E}_1) + cc_1(\mathcal{E}_2)\,,$$

and so $b = 1$, as desired. By symmetry, $c = 1$ too.

To find the pushforward of x_1, we note that by functoriality, the desired class must be given by

$$ax^2 + bc_1(\mathcal{E}_1)x + cc_1(\mathcal{E}_2)x + dc_2(\mathcal{E}_1) + ec_2(\mathcal{E}_2) + fc_1(\mathcal{E}_1)^2 + gc_1(\mathcal{E}_2)^2 + hc_1(\mathcal{E}_2)c_1(\mathcal{E}_1)$$

for constants a, b, c, d, e, f, g, and h. Moreover, by symmetry, the pushforward of x_2 must be given by

$$ax^2 + cc_1(\mathcal{E}_1)x + bc_1(\mathcal{E}_2)x + ec_2(\mathcal{E}_1) + dc_2(\mathcal{E}_2) + gc_1(\mathcal{E}_1)^2 + fc_1(\mathcal{E}_2)^2 + hc_1(\mathcal{E}_2)c_1(\mathcal{E}_1)\,.$$

In particular, the pushforward of $x_1 + x_2$ must be given by

$$\begin{aligned}2ax^2 &+ (b+c)(c_1(\mathcal{E}_1) + c_1(\mathcal{E}_2))x + (d+e)(c_2(\mathcal{E}_1) + c_2(\mathcal{E}_2))\\ &+ (f+g)(c_1(\mathcal{E}_1)^2 + c_2(\mathcal{E}_2)^2) + 2hc_1(\mathcal{E}_2)c_1(\mathcal{E}_1)\,.\end{aligned}$$

But by push-pull and our previous calculation of the pushforward of 1, the pushforward of x_1+x_2 must also be

$$x \cdot (2x + c_1(\mathcal{E}_1) + c_1(\mathcal{E}_2)) = 2x^2 + x(c_1(\mathcal{E}_1) + c_1(\mathcal{E}_2))\,.$$

We conclude that $a = b + c = 1$ and $d + e = f + g = h = 0$. In other words, the pushforward of x_1 must be given by

$$x^2 + bc_1(\mathcal{E}_1)x + (1-b)c_1(\mathcal{E}_2)x + dc_2(\mathcal{E}_1) - dc_2(\mathcal{E}_2) + fc_1(\mathcal{E}_1)^2 - fc_1(\mathcal{E}_2)^2$$

for constants b, d, and f. When $\mathcal{E}_1$ is trivial, the pushforward of x_1 is simply the fundamental class of the diagonal $\mathbb{P}\mathcal{E}_2 \subset \mathbb{P}(\mathcal{E}_2 \oplus \mathcal{E}_2)$, so by Lemma 4.9,

$$x^2 + (1-b)c_1(\mathcal{E}_2)x - dc_2(\mathcal{E}_2) - fc_1(\mathcal{E}_2)^2 = x^2 + c_1(\mathcal{E}_2)x + c_2(\mathcal{E}_2)\,.$$

Thus $d = -1$ and $b = f = 0$, and so the class of the pushforward of x_1 is given by

$$x^2 + c_1(\mathcal{E}_2)x + c_2(\mathcal{E}_2) - c_2(\mathcal{E}_1)\,.$$

And by symmetry, the class of the pushforward of x_2 is given by

$$x^2 + c_1(\mathcal{E}_1)x + c_2(\mathcal{E}_1) - c_2(\mathcal{E}_2)\,.$$

Finally, to calculate the pushforward of x_1x_2, we note that by the projection formula, the pushforward of $(x_1 + x_2) \cdot x_2$ is

$$x \cdot (x^2 + c_1(\mathcal{E}_1)x + c_2(\mathcal{E}_1) - c_2(\mathcal{E}_2))\,.$$

On the other hand, $x_2^2 = -c_1(\mathcal{E}_2)x_2 - c_2(\mathcal{E}_2)$, so the pushforward of x_2^2 is

$$-c_1(\mathcal{E}_2) \cdot \big(x^2 + c_1(\mathcal{E}_1)x + c_2(\mathcal{E}_1) - c_2(\mathcal{E}_2)\big) - c_2(\mathcal{E}_2) \cdot (2x + c_1(\mathcal{E}_1) + c_1(\mathcal{E}_2))\,.$$

Subtracting, we obtain that the pushforward of x_1x_2 is

$$x^3 + (c_1(\mathcal{E}_1) + c_1(\mathcal{E}_2))x^2 + (c_2(\mathcal{E}_1) + c_1(\mathcal{E}_1)c_1(\mathcal{E}_2) + c_2(\mathcal{E}_2))x + c_1(\mathcal{E}_1)c_2(\mathcal{E}_2) + c_2(\mathcal{E}_1)c_1(\mathcal{E}_2)\,,$$

as desired. □

5. The Chow ring of BG

Let $G = (\mathbb{G}_m \times \mathbb{G}_m) \rtimes \mathbb{Z}/2$ as in Definition 2.2. Our goal in this section is to compute the Chow ring $\mathrm{CH}^*(BG)$ with integral coefficients. One approach would be to use general results on Chow rings of wreath products obtained by Totaro (cf. Section 2.8 of [Tot14]), which in particular determine $\mathrm{CH}^*(BG) \otimes \mathbb{Z}/2\mathbb{Z}$, and then leverage $\mathrm{CH}^*(BG) \otimes \mathbb{Z}/2\mathbb{Z}$ to compute $\mathrm{CH}^*(BG)$. Instead, we give here an independent argument which directly computes the integral Chow ring in this case.

To start with, we observe that we have a natural embedding $G \hookrightarrow \mathrm{GL}_2$ as the stabilizer of an unordered pair of distinct lines in V. In particular, BG is identified with the complement of the image of the squaring (Veronese) map

$$\mathbb{P}V_1/\,\mathrm{GL}_2 \to \mathbb{P}V_2/\,\mathrm{GL}_2\,.$$

Write $t = c_1(\mathcal{O}_{\mathbb{P}V_2}(1))$.

LEMMA 5.1. *The Chow ring of $\mathbb{P}V_2/\,\mathrm{GL}_2$ is given by*

$$\mathrm{CH}^*(\mathbb{P}V_2/\,\mathrm{GL}_2) \simeq \mathbb{Z}[\alpha_1,\alpha_2,t]/\big((t^2 - 2\alpha_1 t + 4\alpha_2)(t-\alpha_1)\big)\,.$$

Proof. Write a_1 and a_2 for the Chern roots of V. From Grothendieck's projective bundle formula, the Chow ring of $\mathbb{P}V_2$ is given by $\mathbb{Z}[\alpha_1,\alpha_2,t]/p$, where

$$p = (t-2a_1)(t-2a_2)(t-a_1-a_2) = \big(t^2 - 2\alpha_1 t + 4\alpha_2\big)(t-\alpha_1)\,. \qquad \square$$

From Lemma 4.7 and the localization exact sequence for Chow rings, we conclude that $\mathrm{CH}^*(BG)$ is the quotient of $\mathrm{CH}^*(\mathbb{P}V_2/\,\mathrm{GL}_2)$ by the relations $2s_2^1 - 2\alpha_1 = s_2^2 - 2\alpha_2 = 0$. Using Lemma 4.2, we calculate

$$s_2^0 = 1\,,\quad s_2^1 = t\,,\quad s_2^2 = t^2 - \alpha_1 t + 2\alpha_2\,.$$

Thus,

$$2s_2^1 - 2\alpha_1 = 2t - 2\alpha_1\,,\quad s_2^2 - 2\alpha_2 = t^2 - \alpha_1 t\,.$$

Note that these relations imply $(t^2 - 2\alpha_1 t + 4\alpha_2)(t-\alpha_1) = 0$. We conclude that

$$\mathrm{CH}^*(BG) \simeq \mathbb{Z}[\alpha_1,\alpha_2,t]/\big(2t - 2\alpha_1, t^2 - \alpha_1 t\big)\,. \tag{5.1}$$

THEOREM 5.2. *The Chow ring of BG is given by*

$$\mathrm{CH}^*(BG) \simeq \mathbb{Z}[\beta_1,\beta_2,\gamma]/\big(2\gamma, \gamma^2 + \beta_1\gamma\big)\,.$$

Proof. We have already done most of the work; it remains just to note that the representation V of GL_2 restricts to the representation W of G, so $\alpha_i = \beta_i$, and to identify γ and rewrite the presentation (5.1) in terms of γ.

Because γ is the pullback to $\mathrm{CH}^1(BG)$ of a nontrivial 2-torsion element in $\mathrm{CH}^1(B(\mathbb{Z}/2\mathbb{Z}))$ and G splits as a semidirect product, γ is a nontrivial 2-torsion element in $\mathrm{CH}^1(BG)$. From (5.1), the only such element is $t - \alpha_1$. Substituting $t = \gamma + \alpha_1$ into (5.1) yields the desired result. $\square$

5.1 Results in cohomology

The analogous argument in singular cohomology (using the long exact sequence of a pair plus the Thom isomorphism in place of the localization sequence) shows that, in characteristic zero, the cycle class map $\mathrm{CH}^*(BG) \to \mathrm{H}^*(BG)$ is an isomorphism.

(The reader who is interested in results in cohomology, but does not find this analogy transparent, is encouraged to return to this section after reading Section 8.1, where the "analogous argument for cohomology" is spelled out completely in a significantly more involved case.)

6. Generators and some relations

In this section, we show that $\mathrm{CH}^*\big(\overline{M}_2\big)$ is generated by λ_1, λ_2, and δ_1, as per Theorem 1.1. We then establish several relations that these generators satisfy; in characteristic zero, this includes all relations claimed in Theorem 1.1.

Lemma 6.1. *We have* $\delta_1|_{\Delta_1} = \gamma - \lambda_1$.

Proof. The normal bundle of Δ_1 in $\overline{M}_2$ is given by the line bundle whose fiber over $E_1 \cup_p E_2$ is canonically identified with $T_pE_1 \otimes T_pE_2$. For any two line bundles L_1 and L_2, there is a natural isomorphism $L_1 \otimes L_2 \simeq \wedge^2(L_1 \oplus L_2)$; however, this isomorphism does not respect exchanging L_1 and L_2 (the two sides differ by a sign). Thus, $T_pE_1 \otimes T_pE_2$ differs from $\wedge^2(T_pE_1 \oplus T_pE_2)$ by the class of the sign representation γ. Since $T_pE_1 \oplus T_pE_2 \simeq H^0(\omega_{E_1 \cup_p E_2})^\vee$ has first Chern class $-\lambda_1$, this gives the desired formula. □

From Theorem 5.2 and the presentation given in Section 3.1, we see that the Chow ring of Δ_1 is generated by λ_1, λ_2, and γ; similarly, from the presentation given in Section 3.2, we see that the Chow ring of $\overline{M}_2 \smallsetminus \Delta_1$ is generated by λ_1 and λ_2. Applying the localization sequence

$$\mathrm{CH}^{*-1}(\Delta_1) \to \mathrm{CH}^*\big(\overline{M}_2\big) \to \mathrm{CH}^*\big(\overline{M}_2 \smallsetminus \Delta_1\big) \to 0$$

together with Lemma 6.1, we see that $\mathrm{CH}^*\big(\overline{M}_2\big)$ is generated by λ_1, λ_2, and δ_1, as promised.

6.1 Relations from $\mathrm{CH}^*(BG)$

Since the relations $2\gamma = \gamma^2 + \lambda_1\gamma = 0$ hold in $\mathrm{CH}^*(BG)$, they must also hold in $\mathrm{CH}^*(\Delta_1)$. Using Lemma 6.1, we may write these relations as

$$2\delta_1 + 2\lambda_1 = \delta_1^2 + \delta_1\lambda_1 = 0 \in \mathrm{CH}^*(\Delta_1)\,.$$

Pushing these relations forward to $\overline{M}_2$, we obtain relations in $\mathrm{CH}^*\big(\overline{M}_2\big)$:

$$2\delta_1^2 + 2\delta_1\lambda_1 = \delta_1^3 + \delta_1^2\lambda_1 = 0 \in \mathrm{CH}^*\big(\overline{M}_2\big)\,.$$

6.2 Relations from the Grothendieck–Riemann–Roch theorem

Here we recall Mumford's proof of several relations in $\mathrm{CH}^*\big(\overline{M}_2\big)$ in [Mum83], taking care to work with integral coefficients rather than rational coefficients where possible.

Let $\pi\colon \mathcal{C} \to \overline{M}_2$ be the universal curve. Write ω_π for the relative dualizing sheaf and Ω_π^1 for the sheaf of relative differentials; these are sheaves on $\mathcal{C}$. Applying the Grothendieck–Riemann–Roch theorem to calculate the pushforward of the structure sheaf of $\mathcal{C}$ under π, we obtain the following.

Lemma 6.2. *The following relations hold modulo torsion:*

$$2 = \pi_*\big(c_1\big(\Omega_\pi^1\big)\big)\,, \quad 12\lambda_1 = \pi_*\big(c_1\big(\Omega_\pi^1\big)^2 + c_2\big(\Omega_\pi^1\big)\big)\,, \quad 12\lambda_1^2 - 24\lambda_2 = \pi_*\big(c_1\big(\Omega_\pi^1\big)c_2\big(\Omega_\pi^1\big)\big)\,.$$

Remark 6.3. Pappas has shown that for maps of nonnegative relative dimension in characteristic zero, the relations given by the Grothendieck–Riemann–Roch theorem hold integrally when denominators are cleared [Pap07]. Consequently, these relations hold exactly in characteristic zero.

Write $\mathcal{S} = \mathcal{S}_0 \cup \mathcal{S}_1 \subset \mathcal{C}$ for the universal singular locus, where $\mathcal{S}_0$ denotes the universal self-node and $\mathcal{S}_1$ denotes the universal disconnecting node. Since $\Omega_\pi^1 \simeq \omega_\pi \otimes \mathcal{I}_{\mathcal{S}}$, we have

$$c_1(\Omega_\pi^1) = c_1(\omega_\pi) \quad \text{and} \quad c_2\big(\Omega_\pi^1\big) = [\mathcal{S}]\,.$$

The relations of Lemma 6.2 may thus be written as

$$2 = \pi_*(c_1(\omega_\pi))\,, \quad 12\lambda_1 = \pi_*\big(c_1(\omega_\pi)^2 + [\mathcal{S}]\big)\,, \quad 12\lambda_1^2 - 24\lambda_2 = \pi_*(c_1(\omega_\pi)\cdot[\mathcal{S}])\,.$$

Lemma 6.4. *We have the following relation in* $\mathrm{CH}^*(\mathcal{C})$*:*

$$c_1(\omega_\pi)^2 - c_1(\omega_\pi)\lambda_1 + \lambda_2 - [\mathcal{S}_1] = 0\,.$$

Proof. As shown in Section 3.2, the natural map $\pi^*\pi_*\omega_\pi \to \omega_\pi$ vanishes exactly along $\mathcal{S}_1$. Let $\mathcal{L}$ denote its kernel, so we have an exact sequence

$$0 \to \mathcal{L} \to \pi^*\pi_*\omega_\pi \to \omega_\pi \otimes \mathcal{I}_{\mathcal{S}_1} \to 0\,.$$

Since $\mathcal{S}_1$ is a local complete intersection of codimension 2, the projective dimension of $\omega_\pi \otimes \mathcal{I}_{\mathcal{S}_1}$ is 1; consequently, $\mathcal{L}$ is a line bundle, and so

$$\begin{aligned} 0 = c_2(\mathcal{L}) &= \left[\frac{1 + c_1(\pi^*\pi_*\omega_\pi) + c_2(\pi^*\pi_*\omega_\pi)}{1 + c_1(\omega_\pi \otimes \mathcal{I}_{\mathcal{S}_1}) + c_2(\omega_\pi \otimes \mathcal{I}_{\mathcal{S}_1})}\right]_2 = \left[\frac{1 + \lambda_1 + \lambda_2}{1 + c_1(\omega_\pi) + [\mathcal{S}_1]}\right]_2 \\ &= c_1(\omega_\pi)^2 - c_1(\omega_\pi)\lambda_1 + \lambda_2 - [\mathcal{S}_1]\,. \end{aligned}$$

□

Lemma 6.5. *We have the following relation in* $\mathrm{CH}^*\big(\overline{M}_2\big)$ *modulo torsion:* $24\lambda_1^2 - 48\lambda_2 = 0$.

Remark 6.6. In characteristic zero, the arguments of this section establish this relation exactly (not just modulo torsion); cf. Remark 6.3.

Proof. In light of the relation $12\lambda_1^2 - 24\lambda_2 = \pi_*(c_1(\omega_\pi) \cdot [\mathcal{S}])$, it remains to show that $c_1(\omega_\pi) \cdot [\mathcal{S}]$ is 2-torsion. But this is clear since the residue map gives a trivialization of $\omega_\pi|_{\mathcal{S}}$ up to sign. □

Lemma 6.7. *We have the following relation in* $\mathrm{CH}^*\big(\overline{M}_2\big)$ *modulo torsion:* $\delta_0 = 10\lambda_1 - 2\delta_1$.

Remark 6.8. In characteristic zero, the arguments of this section establish this relation exactly (not just modulo torsion); cf. Remark 6.3. Combined with Lemma 6.9 below, we obtain the relation $20\lambda_1\lambda_2 - 4\delta_1\lambda_2 = 0$.

Proof. Combining the relations $2 = \pi_*(c_1(\omega_\pi))$ and $12\lambda_1 = \pi_*\big(c_1(\omega_\pi)^2 + [\mathcal{S}]\big)$ with Lemma 6.4, we obtain

$$\begin{aligned} 12\lambda_1 &= \pi_*\big(c_1(\omega_\pi)^2 + [\mathcal{S}]\big) = \pi_*(c_1(\omega_\pi)\lambda_1 - \lambda_2 + [\mathcal{S}_1] + [\mathcal{S}_0] + [\mathcal{S}_1]) \\ &= \pi_*(c_1(\omega_\pi))\lambda_1 + 2\pi_*([\mathcal{S}_1]) + \pi_*(\mathcal{S}_0) = 2\lambda_1 + 2\delta_1 + \delta_0\,. \end{aligned}$$

This yields the desired relation upon rearrangement. □

Lemma 6.9. *We have the following relation in* $\mathrm{CH}^*\big(\overline{M}_2\big)$*:* $2\delta_0\lambda_2 = 0$.

Proof. The map $\overline{M}_{1,2} \to \overline{M}_2$ obtained by gluing together the marked points is 2-to-1 onto Δ_0, so the composition of pullback followed by pushforward $\mathrm{CH}^*\big(\overline{M}_2\big) \to \mathrm{CH}^*\big(\overline{M}_{1,2}\big) \to \mathrm{CH}^*\big(\overline{M}_2\big)$ is multiplication by $2\delta_0$.

It thus suffices to show that the pullback of λ_2 to $\overline{M}_{1,2}$ is zero. But this is clear since the residue map gives a surjection from the pullback of the Hodge bundle to the structure sheaf of $\overline{M}_{1,2}$. □

6.3 Results in cohomology

Since the cycle class map is a ring homomorphism, all relations in $\mathrm{CH}^*\big(\overline{M}_2\big)$ established here also hold in $\mathrm{H}^*\big(\overline{M}_2\big)$ in characteristic zero. However, we shall see later that $\mathrm{H}^*\big(\overline{M}_2\big)$ is *not* generated by λ_1, λ_2, and δ_1.

7. Pushforward and pullback along $B(\mathbb{G}_m \times \mathbb{G}_m) \to BG$

In this section, we calculate the pushforward and pullback maps along

$$\pi\colon B(\mathbb{G}_m \times \mathbb{G}_m) \to BG\,.$$

Write t_1 and t_2 in $\mathrm{CH}^*(\mathbb{G}_m \times \mathbb{G}_m)$ for the Chern classes of the standard representations of each $\mathbb{G}_m$-factor. Since pullback is induced by restrictions of representations, we immediately find

$$\pi^*(\beta_1) = t_1 + t_2\,, \quad \pi^*(\beta_2) = t_1 t_2\,, \quad \pi^*(\gamma) = 0\,.$$

LEMMA 7.1. *The pushforward map along* $\pi\colon B(\mathbb{G}_m \times \mathbb{G}_m) \to BG$ *may be described recursively as follows:*

$$\begin{aligned} \pi_*(1) &= 2\,, \quad \pi_*(t_1) = \beta_1 + \gamma\,,\\ \pi_*\big(t_1^a\big) &= \beta_1 \pi_*\big(t_1^{a-1}\big) - \beta_2 \pi_*\big(t_1^{a-2}\big) \quad \textit{for} \quad a \geqslant 2\,,\\ \pi_*\big(t_1^a t_2^b\big) &= \beta_2^{\min(a,b)} \pi_*\big(t_1^{|a-b|}\big)\,. \end{aligned}$$

Proof. Since π is of degree 2, we have $\pi_*(1) = 2$.

Suppose $\pi_*(t_1) = a\beta_1 + b\gamma$. Then by symmetry, $\pi_*(t_2) = a\beta_1 + b\gamma$. Moreover, by push-pull we have

$$\begin{aligned} 2\beta_1 = \pi_*\pi^*\beta_1 = \pi_*(t_1 + t_2) = (a\beta_1 + b\gamma) + (a\beta_1 + b\gamma) = 2a\beta_1\,,\\ (b-a)\gamma^2 = \gamma \cdot (a\beta_1 + b\gamma) = \gamma \cdot \pi_* t_1 = \pi_*(t_1 \cdot \pi^*\gamma) = \pi_*(t_1 \cdot 0) = 0\,. \end{aligned}$$

Thus, $a = b = 1$, that is, $\pi_*(t_1) = \beta_1 + \gamma$, as claimed.

Using push-pull, we have for $a \geqslant 2$

$$\pi_*\big(t_1^a\big) = \pi_*\big((t_1 + t_2)t_1^{a-1} - t_1 t_2 t_1^{a-2}\big) = \beta_1 \pi_*\big(t_1^{a-1}\big) - \beta_2 \pi_*\big(t_1^{a-2}\big)\,.$$

And finally, using push-pull and symmetry, we have

$$\pi_*\big(t_1^a t_2^b\big) = \pi_*\big((t_1 t_2)^{\min(a,b)} t_1^{|a-b|}\big) = \beta_2^{\min(a,b)} \pi_*\big(t_1^{|a-b|}\big)\,. \qquad \square$$

We conclude this section by calculating the Chern classes of the representations $V_n \boxplus V_n$, which are obtained by pushing forward the representation V_n of $\mathbb{G}_m \times \mathbb{G}_m$ with weight $(n, 0)$ to BG.

LEMMA 7.2. *The Chern classes of* W_n *are given by*

$$c_1(W_n) = n\beta_1 + (n+1)\gamma\,, \quad c_2(W_n) = n^2\beta_2\,.$$

Proof. We argue by induction on n. When $n = 0$, the representation W_0 is the regular representation of the $\mathbb{Z}/2\mathbb{Z}$-quotient of G, which splits as $\mathbf{1} \oplus \Gamma$; its Chern classes are thus γ and 0, respectively. When $n = 1$, the representation $W_1 = W$ has Chern classes β_1 and β_2 by definition.

For the inductive hypothesis, we suppose $n \geqslant 2$. Observe that we have a direct sum decomposition

$$W_{n-1} \otimes W_1 \simeq W_n \oplus \big(W_{n-2} \otimes \wedge^2 W_1 \otimes \Gamma\big)\,.$$

Writing a_n and b_n for the Chern roots of W_n, we obtain

$$\begin{aligned} &(1 + a_{n-1} + a_1)(1 + a_{n-1} + b_1)(1 + b_{n-1} + a_1)(1 + b_{n-1} + b_1)\\ &\qquad = (1 + a_n)(1 + b_n)(1 + a_{n-2} + \beta_1 + \gamma)(1 + b_{n-2} + \beta_1 + \gamma)\,. \end{aligned} \tag{7.1}$$

Comparing terms of degree 1 on both sides of (7.1), and using the relation $2\gamma = 0$, we have

$$2(a_{n-1} + b_{n-1}) + 2(a_1 + b_1) = a_n + b_n + a_{n-2} + b_{n-2} + 2\beta_1\,,$$

and so by our inductive hypothesis,

$$\begin{aligned} c_1(W_n) = a_n + b_n &= 2(a_{n-1} + b_{n-1}) + 2(a_1 + b_1) - (a_{n-2} + b_{n-2}) - 2\beta_1 \\ &= 2[(n-1)\beta_1 + n\gamma] + 2\beta_1 - [(n-2)\beta_1 + (n-1)\gamma] - 2\beta_1 \\ &= n\beta_1 + (n+1)\gamma\,. \end{aligned}$$

Similarly, comparing terms of degree 2 on both sides of (7.1), and using $2\gamma = 0$, we have

$$\begin{aligned} &(a_{n-1} + b_{n-1})^2 + 2a_{n-1}b_{n-1} + (a_1 + b_1)^2 + 2a_1 b_1 + 3(a_{n-1} + b_{n-1})(a_1 + b_1) \\ &\quad = a_n b_n + a_{n-2}b_{n-2} + (\beta_1 + \gamma)(a_{n-2} + b_{n-2}) + (\beta_1 + \gamma)^2 + (a_n + b_n)(a_{n-2} + b_{n-2} + 2\beta_1), \end{aligned}$$

and so by our inductive hypothesis, together with $2\gamma = \gamma^2 + \beta_1\gamma = 0$,

$$\begin{aligned} c_2(W_n) &= a_n b_n \\ &= (a_{n-1} + b_{n-1})^2 + 2a_{n-1}b_{n-1} + (a_1 + b_1)^2 + 2a_1 b_1 \\ &\qquad + 3(a_{n-1} + b_{n-1})(a_1 + b_1) - a_{n-2}b_{n-2} - (\beta_1 + \gamma)(a_{n-2} + b_{n-2}) \\ &\qquad - (\beta_1 + \gamma)^2 - (a_n + b_n)(a_{n-2} + b_{n-2} + 2\beta_1) \\ &= [(n-1)\beta_1 + n\gamma]^2 + 2(n-1)^2\beta_2 + \beta_1^2 + 2\beta_2 \\ &\qquad + 3[(n-1)\beta_1 + n\gamma]\beta_1 - (n-2)^2\beta_2 - (\beta_1 + \gamma)[(n-2)\beta_1 + (n-1)\gamma] \\ &\qquad - (\beta_1 + \gamma)^2 - [n\beta_1 + (n+1)\gamma]([(n-2)\beta_1 + (n-1)\gamma] + 2\beta_1) \\ &= n^2\beta_2 - (n-1)(\gamma^2 + \beta_1\gamma) \\ &= n^2\beta_2\,. \end{aligned}$$

□

8. The Chow ring of Δ_1

In this section, we calculate the Chow ring $\mathrm{CH}^*(\Delta_1)$. Recall the presentation of Δ_1 obtained in Section 3.1:

$$\Delta_1 \simeq (W_{4,6} \smallsetminus (L_{4,6} \times 0 \cup 0 \times L_{4,6}))\,/G\,.$$

In this section, we will use the calculations of $\mathrm{CH}^*(BG) \simeq \mathrm{CH}^*(W_{4,6}/G)$ from Section 5 to compute the Chow ring of Δ_1 using the localization sequence.

We first note that the formulas in Section 4 apply to calculate loci in the total space of vector bundles with the origin excised: simply pull back the corresponding classes from the projectivization. So to start with, we excise the origin of $W_{4,6}$ using the localization sequence

$$\mathrm{CH}^{*-4}(BG) \to \mathrm{CH}^*(W_{4,6}/G) \to \mathrm{CH}^*(W') \to 0\,, \quad \text{where} \quad W' = (W_{4,6} \smallsetminus (0 \times 0))/G\,.$$

In terms of the isomorphism $\mathrm{CH}^*(BG) \simeq \mathrm{CH}^*(W_{4,6}/G)$, this first map is multiplication by the Euler class $e(W_{4,6}) = c_4(W_{4,6})$, and so

$$\mathrm{CH}^*(W') \simeq \mathrm{CH}^*(BG)/c_4(W_{4,6})\,.$$

To compute $c_4(W_{4,6})$, we use Lemma 7.2, which gives

$$e(W_{4,6}) = c_4(W_{4,6}) = c_2(W_4) \cdot c_2(W_6) = 16\lambda_2 \cdot 36\lambda_2 = 576\lambda_2^2\,. \tag{8.1}$$

To get from W' to Δ_1, we have to excise the closed substack

$$Z = \big(((L_{4,6} \smallsetminus 0) \times 0) \cup (0 \times (L_{4,6} \smallsetminus 0))\big)/G \simeq ((L_{4,6} \smallsetminus 0) \times 0)/(\mathbb{G}_m \times \mathbb{G}_m)\,.$$

This isomorphism implies that $\mathrm{CH}^*(Z)$ is generated by $\{1, t_1\}$ as a module over $\mathrm{CH}^*(BG)$, so it suffices to determine the images of 1 and t_1 under the localization sequence

$$\mathrm{CH}^{*-2}(Z) \to \mathrm{CH}^*(W') \to \mathrm{CH}^*(\Delta_1) \to 0\,.$$

But from Lemma 4.9, the class of

$$Z'' = \big((L_{4,6} \smallsetminus 0) \times 0\big)/(\mathbb{G}_m \times \mathbb{G}_m) \subset \big(L_{4,6} \oplus L_{4,6} \smallsetminus (0 \times 0)\big)/(\mathbb{G}_m \times \mathbb{G}_m)\,,$$

which maps isomorphically onto Z, is given by $c_2(0 \times L_{4,6}) = 24t_2^2$. Consequently, the images of 1 and t_1 are the pushforwards of $24t_2^2$ and $24t_1t_2^2$ along $B(\mathbb{G}_m \times \mathbb{G}_m) \to BG$; by Lemma 7.1, these are $24\lambda_1^2 - 48\lambda_2$ and $24\lambda_1\lambda_2$, respectively. These imply our earlier relation $576\lambda_2^2 = 0$, and therefore the following result.

Theorem 8.1. *The Chow ring of Δ_1 is given by*

$$\mathrm{CH}^*(\Delta_1) = \mathbb{Z}[\lambda_1, \lambda_2, \gamma]/\big(2\gamma, \gamma^2 + \lambda_1\gamma, 24\lambda_1^2 - 48\lambda_2, 24\lambda_1\lambda_2\big)\,. \tag{8.2}$$

Remark 8.2. One observes from this that after inverting 2, the Chow ring is $\mathrm{Sym}^2\,\mathrm{CH}^*\big(\overline{M}_{1,1}\big)$.

8.1 Results in cohomology

(The reader interested only in the Chow ring of $\overline{M}_2$ may wish to skip this section.)

An essentially analogous argument in singular cohomology shows that in characteristic zero, the even singular cohomology of Δ_1 is isomorphic to its Chow ring via the cycle class map and also computes both the order and an upper bound for the exponent of the odd cohomology groups. Unlike many other arguments given in this paper, working with cohomology introduces two minor subtleties not present when working with Chow groups; we therefore spell this argument out explicitly for the interested reader.

We begin by computing the even cohomology of W', which is completely analogous to computing its Chow ring.

Lemma 8.3. *The even cohomology of W' is isomorphic to its Chow ring via the cycle class map*

$$\mathrm{H}^{2n}(W') \simeq \mathrm{CH}^n(W')\,,$$

and its odd cohomology is, as a group, isomorphic to

$$\mathrm{H}^{2n+1}(W') \simeq (\mathbb{Z}/2\mathbb{Z})^{\max(0, \lfloor (n-3)/2 \rfloor)}\,.$$

Proof. We wish to mimic, in cohomology, the Chow localization sequence for excision of the origin $0/G \simeq BG$ of $W_{4,6}$ to obtain its complement W'. For this, we use the long exact sequence for the pair $(W_{4,6}/G, W')$, which gives long exact sequences

$$\begin{aligned} \cdots \to \mathrm{H}^{2n+1}(W_{4,6}/G) \to \mathrm{H}^{2n+1}(W') \to \mathrm{H}^{2n}(W_{4,6}/G, W') \to \mathrm{H}^{2n}(W_{4,6}/G) \\ \to \mathrm{H}^{2n}(W') \to \mathrm{H}^{2n-1}(W_{4,6}/G, W') \to \cdots\,. \end{aligned}$$

Using the Thom isomorphism

$$\mathrm{H}^k(W_{4,6}/G, W') = \mathrm{H}^k(W_{4,6}/G, W_{4,6}/G \smallsetminus 0/G) \simeq \mathrm{H}^{k-8}(0/G) = \mathrm{H}^{k-8}(BG)\,,$$

we obtain the exact sequences

$$\begin{aligned} 0 = \mathrm{H}^{2n+1}(W_{4,6}/G) \to \mathrm{H}^{2n+1}(W') \to \mathrm{H}^{2n-8}(BG) \to \mathrm{H}^{2n}(W_{4,6}/G) \\ \to \mathrm{H}^{2n}(W') \to \mathrm{H}^{2n-7}(BG) = 0\,, \end{aligned}$$

which identify the even cohomology $\mathrm{H}^{2n}((W_{4,6} \smallsetminus (0 \times 0))/G)$ (respectively, the odd cohomology $\mathrm{H}^{2n+1}((W_{4,6} \smallsetminus (0 \times 0))/G)$) with the cokernel (respectively, kernel) of multiplication by $e(W_{4,6})$

from $\mathrm{H}^{2n-8}(BG)$ to $\mathrm{H}^{2n}(BG)$, just as the localization sequence for Chow rings identified $\mathrm{CH}^n(W')$ with the cokernel of multiplication by $e(W_{4,6})$ from $\mathrm{CH}^{n-4}(BG)$ to $\mathrm{CH}^n(BG)$.

We conclude that the even cohomology of W' is isomorphic to its Chow ring via the cycle class map, and its odd cohomology is isomorphic to the kernel of multiplication by $e(W_{4,6})$ on $\mathrm{H}^{2n-8}(BG)$:

$$\begin{aligned}\mathrm{H}^{2n+1}(W') &\simeq \left\{x \in \mathrm{CH}^{n-4}(BG)\colon 576\lambda_2^2 \cdot x = 0\right\} = \gamma \cdot (\mathbb{Z}/2\mathbb{Z})[\lambda_1, \lambda_2]_{n-5} \\ &\simeq (\mathbb{Z}/2\mathbb{Z})^{\max(0,\lfloor (n-3)/2\rfloor)}\,.\end{aligned}$$

□

The first subtlety here is that, in order to compute the cohomology of Δ_1 by excision of Z from W', we need Z to have no odd cohomology.

LEMMA 8.4. *The cohomology of Z is isomorphic to*

$$\mathrm{H}^*(Z) \simeq \mathbb{Z}[t_1, t_2]/\left(24t_1^2\right),$$

where t_1 and t_2 are of degree 2. In particular, the odd cohomology of Z vanishes.

Proof. When we excise the origin of $L_{4,6}$, we obtain the exact sequences

$$\begin{aligned}0 = \mathrm{H}^{2n+1}(L_{4,6}/(\mathbb{G}_m \times \mathbb{G}_m)) &\to \mathrm{H}^{2n+1}(Z) \to \mathrm{H}^{2n-4}(B(\mathbb{G}_m \times \mathbb{G}_m)) \\ \to \mathrm{H}^{2n}(L_{4,6}/(\mathbb{G}_m \times \mathbb{G}_m)) &\to \mathrm{H}^{2n}(Z) \to \mathrm{H}^{2n-3}(B(\mathbb{G}_m \times \mathbb{G}_m)) = 0\,,\end{aligned}$$

which identify the even cohomology $\mathrm{H}^{2n}(Z)$ (respectively, the odd cohomology $\mathrm{H}^{2n+1}(Z)$) with the cokernel (respectively, kernel) of the map $\mathrm{H}^{2n-4}(B(\mathbb{G}_m \times \mathbb{G}_m)) \to \mathrm{H}^{2n}(B(\mathbb{G}_m \times \mathbb{G}_m))$ given by multiplication by $e(L_{4,6})$. Because $e(L_{4,6}) = 24t_1^2$, which is not a zero divisor in the cohomology ring $\mathrm{H}^*(B(\mathbb{G}_m \times \mathbb{G}_m)) \simeq \mathbb{Z}[t_1, t_2]$, we obtain $\mathrm{H}^*(Z) \simeq \mathbb{Z}[t_1, t_2]/\left(24t_1^2\right)$, as claimed. □

When we excise Z from W' to obtain Δ_1, we obtain the exact sequences

$$\begin{aligned}0 = \mathrm{H}^{2n-3}(Z) \to \mathrm{H}^{2n+1}(W') \simeq (\mathbb{Z}/2\mathbb{Z})^{\max(0,\lfloor (n-3)/2\rfloor)} &\to \mathrm{H}^{2n+1}(\Delta_1) \to \mathrm{H}^{2n-4}(Z) \\ \to \mathrm{H}^{2n}(W') \to \mathrm{H}^{2n}(\Delta_1) \to \mathrm{H}^{2n-5}(Z) = 0\,,&\end{aligned}$$

which identify the even cohomology $\mathrm{H}^{2n}(\Delta_1)$ (respectively, the odd cohomology $\mathrm{H}^{2n+1}(\Delta_1)$) with the cokernel (respectively, an extension by $(\mathbb{Z}/2\mathbb{Z})^{\max(0,\lfloor (n-3)/2\rfloor)}$ of the kernel) of the pushforward map $\mathrm{H}^{2n-4}(Z) \to \mathrm{H}^{2n}(W')$. In particular, the even cohomology of Δ_1 is isomorphic to its Chow ring via the cycle class map

$$\mathrm{H}^{2n}(\Delta_1) \simeq \mathrm{CH}^n(\Delta_1)\,.$$

The second subtlety here is that computing the kernel of $\mathrm{H}^{2n-4}(Z) \to \mathrm{H}^{2n}(W')$ requires a bit of thought. Namely, to compute it, we first pull back to the degree 2 cover of W' given by

$$W'' := (L_{4,6} \oplus L_{4,6} \smallsetminus (0 \times 0))/(\mathbb{G}_m \times \mathbb{G}_m)\,.$$

The same argument used to compute $\mathrm{H}^*(W')$ shows

$$\mathrm{H}^*(W'') \simeq \mathrm{H}^*(B(\mathbb{G}_m \times \mathbb{G}_m))/c_4(W_{4,6}) = \mathbb{Z}[t_1, t_2]/\left(576t_1^2t_2^2\right).$$

For a cohomology class $x \in \mathrm{H}^*(B(\mathbb{G}_m \times \mathbb{G}_m))$, denote by $\overline{x}$ its image under the automorphism exchanging the two $\mathbb{G}_m$-factors; in these terms, the composition of pullback and pushforward

$$\mathrm{H}^*(B(\mathbb{G}_m \times \mathbb{G}_m)) \to \mathrm{H}^*(BG) \to \mathrm{H}^*(B(\mathbb{G}_m \times \mathbb{G}_m)) \quad \text{is given by} \quad x \mapsto x + \overline{x}\,.$$

Recall that the class of Z'' in W'' is given by $c_2(0 \times L_{4,6}) = 24t_2^2$; consequently, the composition $\mathrm{H}^{2n-4}(Z) \to \mathrm{H}^{2n}(W') \to \mathrm{H}^{2n}(W'')$ is given by

$$x \mapsto 24t_2^2 \cdot x + \overline{24t_2^2 \cdot x} = 24t_2^2 \cdot x + 24t_1^2 \cdot \overline{x} \in \mathbb{Z}[t_1, t_2]/\left(576t_1^2t_2^2\right).$$

In particular, an element $x \in \mathrm{H}^*(Z) = \mathbb{Z}[t_1, t_2]/\big(24t_1^2\big)$ in the kernel of $\mathrm{H}^{2n-4}(Z) \to \mathrm{H}^{2n}(W')$ must satisfy

$$24t_2^2 \cdot x + 24t_1^2 \cdot \overline{x} \in \big(576t_1^2t_2^2\big) \quad \Leftrightarrow \quad t_2^2 \cdot x + t_1^2 \cdot \overline{x} \in \big(24t_1^2t_2^2\big)$$

and, in particular, must satisfy $x \in (t_1^2)$. When we write $x = t_1^2 \cdot y$, the above condition becomes $t_2^2 \cdot t_1^2 \cdot y + t_1^2 \cdot t_2^2 \cdot \overline{y} \in \big(24t_1^2t_2^2\big)$, that is, $y + \overline{y} \in (24)$. As a module over the ring of symmetric functions in t_1 and t_2, the space of such y is generated by 12 and $t_1 - t_2$. Unwinding this, we see that the kernel of $\mathrm{H}^{2n-4}(Z) \to \mathrm{H}^{2n}(W')$ is contained in the sub-$\mathrm{H}^*(BG)$-module of $\mathrm{H}^{2n-4}(Z)$ generated by $12t_1^2$ and $(t_1 - t_2)t_1^2$. Conversely, applying Lemma 7.1, we see that both $12t_1^2$ and $(t_1 - t_2)t_1^2$ are contained in the kernel of $\mathrm{H}^{2n-4}(Z) \to \mathrm{H}^{2n}(W')$. Thus, the kernel of $\mathrm{H}^{2n-4}(Z) \to \mathrm{H}^{2n}(W')$ is exactly the sub-$\mathrm{H}^*(BG)$-module of $\mathrm{H}^{2n-4}(Z)$ generated by $12t_1^2$ and $(t_1 - t_2)t_1^2$. As a group, this is

$$\begin{aligned}(t_1 - t_2)t_1^2 \cdot (\mathbb{Z}/24\mathbb{Z})[t_1, t_2]_{n-5} &\oplus 12t_1^2 \cdot (\mathbb{Z}/2\mathbb{Z})[t_1]_{n-4} \\ &\simeq (\mathbb{Z}/24\mathbb{Z})^{\max(0,n-4)} \oplus (\mathbb{Z}/2\mathbb{Z})^{\epsilon} \quad \text{for} \quad \epsilon = \begin{cases} 1 & \text{if } n \geqslant 4\,, \\ 0 & \text{otherwise}\,. \end{cases}\end{aligned}$$

Putting this together, we have shown the following.

Proposition 8.5. *The even cohomology of Δ_1 is isomorphic to its Chow ring via the cycle class map $\mathrm{H}^{2n}(\Delta_1) \simeq \mathrm{CH}^n(\Delta_1)$, and its odd cohomology $H^{2n+1}(\Delta_1)$ is a group of exponent at most 48 and order*

$$24^{\max(0,n-4)} \cdot 2^{\max(0,\lfloor (n-3)/2 \rfloor)+\epsilon}\,, \quad \textit{where} \quad \epsilon = \begin{cases} 1 & \textit{if } n \geqslant 4\,, \\ 0 & \textit{otherwise}\,. \end{cases}$$

9. The Chow ring and higher Chow groups of $\overline{M}_2 \smallsetminus \Delta_1$

In this section, we compute the Chow ring and higher Chow groups with ℓ-adic coefficients (where ℓ is prime to the characteristic of the base field k) of $\overline{M}_2 \smallsetminus \Delta_1$ and of its quotient by the *twisting action* of $\mathbb{G}_m$—that is, the action of $\mathbb{G}_m$ on $\overline{M}_2 \smallsetminus \Delta_1$ obtained by scaling with weight 1 the coefficients a, b, c, d, e, f, and g in (3.2).

Bloch's higher Chow groups [Blo86] are defined as the homology of certain complexes $z^*(X, \bullet)$ (here we have one complex for each value of $*$, and $\bullet$ denotes the grading of the complexes). These groups are significantly easier to compute with torsion coefficients (in which case they can often be compared to étale cohomology) and when the base field is algebraically closed. So here we adopt the convention that by "higher Chow groups with ℓ-adic coefficients," we mean the groups

$$\mathrm{CH}^*(X, n; \mathbb{Z}_\ell) := H_n\left(\varprojlim_m z^*(X_{\overline{k}}, \bullet) \otimes^L \mathbb{Z}/\ell^m\mathbb{Z}\right),$$

where $X_{\overline{k}}$ denotes the base change of X to the algebraic closure $\overline{k}$ of our base field k. Note that when the $\mathrm{CH}^*(X_{\overline{k}}, n; \mathbb{Z}/\ell^m\mathbb{Z})$ are finitely generated, we have

$$\mathrm{CH}^*(X, n; \mathbb{Z}_\ell) \simeq \varprojlim_m \mathrm{CH}^*(X_{\overline{k}}, n; \mathbb{Z}/\ell^m\mathbb{Z})\,.$$

In particular, since we have shown that $\mathrm{CH}^*(\Delta_1)$ and $\mathrm{CH}^*\big(\overline{M}_2\big)$ are finitely generated, we still have

$$\mathrm{CH}^*(X, 0; \mathbb{Z}_\ell) = \mathrm{CH}^*(X_{\overline{k}}) \otimes \mathbb{Z}_\ell \quad \text{for} \quad X \in \big\{\Delta_1, \overline{M}_2, \overline{M}_2 \smallsetminus \Delta_1\big\}\,.$$

Moreover, Bloch's higher Chow groups satisfy $\mathrm{CH}^*(\operatorname{Spec}\overline{k}, 1; \mathbb{Z}/\ell^m\mathbb{Z}) = 0$ by Theorem 10.3 of [MVW06] (in combination with the isomorphism given in Lecture 19 of [MVW06] between higher Chow groups and motivic cohomology). Consequently, this definition of "higher Chow groups with ℓ-adic coefficients" satisfies $\mathrm{CH}^*(\operatorname{Spec} k, 1; \mathbb{Z}_\ell) = 0$.

The description of $\overline{M}_2 \smallsetminus \Delta_1$ given in Section 3.2 shows that

$$\begin{aligned}\left(\overline{M}_2 \smallsetminus \Delta_1\right)/\mathbb{G}_m &\simeq (\mathbb{P}V_6(2) \smallsetminus \{\text{forms with triple roots}\})/\operatorname{GL}_2\\ &\simeq (\mathbb{P}V_6 \smallsetminus \{\text{forms with triple roots}\})/\operatorname{GL}_2\,.\end{aligned}$$

Definition 9.1. Write $t = c_1(\mathcal{O}_{\mathbb{P}V_6}(1))$.

So by construction, $\overline{M}_2 \smallsetminus \Delta_1$ is a $\mathbb{G}_m$-bundle over $\left(\overline{M}_2 \smallsetminus \Delta_1\right)/\mathbb{G}_m$ given by the complement of the zero section in a line bundle with Chern class $t - 2\lambda_1$. (One could also take $2\lambda_1 - t$. Indeed, the complement of the zero section in a line bundle is isomorphic to the complement of the zero section in the dual line bundle, so there is no natural choice of sign here.)

Lemma 9.2. *The stack* $\mathbb{P}V_6/\operatorname{GL}_2$ *has Chow ring* $\mathrm{CH}^*(\mathbb{P}V_6/\operatorname{GL}_2) = \mathbb{Z}[\lambda_1, \lambda_2, t]/p(\lambda_1, \lambda_2, t)$, *where*

$$p(\lambda_1, \lambda_2, t) = \left(t^2 - 6\lambda_1 t + 36\lambda_2\right)\left(t^2 - 6\lambda_1 t + 5\lambda_1^2 + 16\lambda_2\right)\left(t^2 - 6\lambda_1 t + 8\lambda_1^2 + 4\lambda_2\right)(t - 3\lambda_1)\,.$$

Moreover, its first higher Chow groups with ℓ-adic coefficients vanish:

$$\mathrm{CH}^*(\mathbb{P}V_6/\operatorname{GL}_2, 1; \mathbb{Z}_\ell) = 0\,.$$

Proof. When we write α and β for the Chern roots of V^* (with $\alpha + \beta = -\lambda_1$ and $\alpha\beta = \lambda_2$), Grothendieck's formula for the Chow ring of a projective bundle gives

$$\mathrm{CH}^*(\mathbb{P}V_6/\operatorname{GL}_2) = \mathrm{CH}^*(B\operatorname{GL}_2)[t]/p(t) = \mathbb{Z}[\lambda_1, \lambda_2][t]/p(t)\,,$$

where

$$\begin{aligned}p(t) &= (t + 6\alpha)(t + 6\beta)(t + 5\alpha + \beta)(t + \alpha + 5\beta)(t + 4\alpha + 2\beta)(t + 2\alpha + 4\beta)(t + 3\alpha + 3\beta)\\ &= \left(t^2 - 6\lambda_1 t + 36\lambda_2\right)\left(t^2 - 6\lambda_1 t + 5\lambda_1^2 + 16\lambda_2\right)\left(t - 6\lambda_1 + 8\lambda_1^2 + 4\lambda_2\right)(t - 3\lambda_1)\,.\end{aligned}$$

Since $\mathrm{CH}^*(\operatorname{Spec} k, 1; \mathbb{Z}_\ell) = 0$ and higher Chow groups are preserved under taking vector bundles, we have $\mathrm{CH}^*(\mathbb{A}^n_k, 1; \mathbb{Z}_\ell) = 0$. Using the localization sequence and the standard decomposition of the Grassmanian into Schubert cells, we see that $\mathrm{CH}^*(\mathrm{Gr}_k(m, n), 1; \mathbb{Z}_\ell) = 0$, where $\mathrm{Gr}_k(m, n)$ denotes any Grassmanian over k. Consequently, $\mathrm{CH}^*(B\operatorname{GL}_2, 1; \mathbb{Z}_\ell) = 0$, and so the first higher Chow groups vanish for any projective bundle over $B\operatorname{GL}_2$ (see, for example, [Blo86, Theorem 7.1]), including for $\mathbb{P}V_6/\operatorname{GL}_2$. □

Definition 9.3. Write $T_1 \subset \mathbb{P}V_6/\operatorname{GL}_2$ for the locus with exactly one triple root, and write $T_2 \subset \mathbb{P}V_6/\operatorname{GL}_2$ for the locus of perfect cubes.

Definition 9.4. Write $\mathrm{cub}_1\colon (\mathbb{P}V_1/\operatorname{GL}_2) \to (\mathbb{P}V_3/\operatorname{GL}_2)$ and $\mathrm{cub}_2\colon (\mathbb{P}V_2/\operatorname{GL}_2) \to (\mathbb{P}V_6/\operatorname{GL}_2)$ for the cubing maps. Write $S_{ij} = s_1^i \times s_3^j$ (recall Definition 4.1), and denote by s_{ij} the pushforward of S_{ij} under the map

$$\theta\colon (\mathbb{P}V_1 \times \mathbb{P}V_3)/\operatorname{GL}_2 \to \mathbb{P}V_6/\operatorname{GL}_2 \quad \text{defined by} \quad (f, g) \mapsto f^3 \cdot g\,.$$

Observe that we have the following commutative diagram:

$$\begin{array}{ccc}(\mathbb{P}V_1 \times \mathbb{P}V_1)/\operatorname{GL}_2 & \xrightarrow{\mathbf{1}\times \mathrm{cub}_1} & (\mathbb{P}V_1 \times \mathbb{P}V_3)/\operatorname{GL}_2\\ {\scriptstyle\text{multiplication}}\big\downarrow & & \big\downarrow{\scriptstyle\theta}\\ \mathbb{P}V_2/\operatorname{GL}_2 & \xrightarrow{\mathrm{cub}_2} & \mathbb{P}V_6/\operatorname{GL}_2\,.\end{array}$$

In particular, taking the images of the generators $s_1^i \times s_1^j$, we obtain from Lemmas 4.4 and 4.8 (with $c_1 = -\lambda_1$ and $c_2 = \lambda_2$)

$$\begin{aligned} 2\mathrm{cub}_2\big(s_2^0\big) &= 3s_{02} - 6\lambda_1 s_{01} + 6\big(\lambda_1^2 - \lambda_2\big)s_{00}\,, \\ \mathrm{cub}_2\big(s_2^1\big) &= 3s_{12} - 6\lambda_1 s_{11} + 6\big(\lambda_1^2 - \lambda_2\big)s_{10}\,, \\ \mathrm{cub}_2\big(s_2^1\big) &= s_{03} - 6\lambda_2 s_{01} + 6\lambda_1\lambda_2 s_{00}\,, \\ \mathrm{cub}_2\big(s_2^2\big) &= s_{13} - 6\lambda_2 s_{11} + 6\lambda_1\lambda_2 s_{10}\,. \end{aligned}$$

The middle two relations give two expressions for $\mathrm{cub}_2\big(s_2^1\big)$, which must therefore be equal. Note also that this forces the coefficients of s_{02} to be even; we can thus write

$$s'_{02} = \tfrac{1}{2}\, s_{02}\,.$$

Lemma 9.5. *The Chow ring $\mathrm{CH}^*((\mathbb{P}V_6 \smallsetminus T_2)/\mathrm{GL}_2)$ is generated as a module over $\mathbb{Z}[\lambda_1, \lambda_2]$ by the elements s_6^0, s_6^1, s_6^2, s_6^3, s_6^4, s_6^5, and s_6^6, subject to the relations*

$$\begin{aligned} &3 \cdot \big[s'_{02} - \lambda_1 s_{01} + \big(\lambda_1^2 - \lambda_2\big)s_{00}\big] = 0\,, \\ &3 \cdot \big[s_{12} - 2\lambda_1 s_{11} + 2\big(\lambda_1^2 - \lambda_2\big)s_{10}\big] = 0\,, \\ &s_{13} - 6\lambda_2 s_{11} + 6\lambda_1\lambda_2 s_{10} = 0\,. \end{aligned}$$

Moreover, assuming that these three elements are linearly independent over $\mathbb{Z}[\lambda_1, \lambda_2]$, the first higher Chow group vanishes; that is, $\mathrm{CH}^*((\mathbb{P}V_6 \smallsetminus T_2)/\mathrm{GL}_2, 1; \mathbb{Z}_\ell) = 0$.

Proof. Using the first of our two expressions for $\mathrm{cub}_2\big(s_2^1\big)$, this is immediate from Lemmas 9.2 and 4.3 together with the localization exact sequence

$$\begin{aligned} \cdots \to \mathrm{CH}^*(\mathbb{P}V_6/\mathrm{GL}_2, 1; \mathbb{Z}_\ell) = 0 \to \mathrm{CH}^*((\mathbb{P}V_6 \smallsetminus T_2)/\mathrm{GL}_2, 1; \mathbb{Z}_\ell) \to \mathrm{CH}^{*-4}(T_2)_{\overline{k}} \otimes \mathbb{Z}_\ell \\ \to \mathrm{CH}^*(\mathbb{P}V_6/\mathrm{GL}_2)_{\overline{k}} \otimes \mathbb{Z}_\ell \to \mathrm{CH}^*((\mathbb{P}V_6 \smallsetminus T_2)/\mathrm{GL}_2)_{\overline{k}} \otimes \mathbb{Z}_\ell \to 0\,. \end{aligned}$$

□

Lemma 9.6. *The Chow ring $\mathrm{CH}^*\big(\big(\overline{M}_2 \smallsetminus \Delta_1\big)/\mathbb{G}_m\big)$ is generated as a module over $\mathbb{Z}[\lambda_1, \lambda_2]$ by the elements s_6^0, s_6^1, s_6^2, s_6^3, s_6^4, s_6^5, and s_6^6, subject to the relations*

$$s_{00} = s_{10} = s_{01} = s_{11} = s'_{02} = s_{12} = s_{13} = 0\,.$$

Moreover, assuming that these seven elements are linearly independent over $\mathbb{Z}[\lambda_1, \lambda_2]$, the first higher Chow group vanishes; that is, $\mathrm{CH}^*\big(\big(\overline{M}_2 \smallsetminus \Delta_1\big)/\mathbb{G}_m, 1; \mathbb{Z}_\ell\big) = 0$.

Proof. From the localization exact sequence

$$\cdots \to \mathrm{CH}^*((\mathbb{P}V_1 \times \mathbb{P}V_1)/\mathrm{GL}_2) \to \mathrm{CH}^*((\mathbb{P}V_1 \times \mathbb{P}V_3)/\mathrm{GL}_2) \to \mathrm{CH}^*(T_1) \to 0\,,$$

it follows that $\mathrm{CH}^*(T_1)$ is generated as a module over $\mathbb{Z}[\lambda_1, \lambda_2]$ by S_{00}, S_{01}, S_{02}, S_{03}, S_{10}, S_{11}, S_{12}, and S_{13}, with the relations (cf. Lemma 4.8):

$$\begin{aligned} 3S_{02} - 6\lambda_1 S_{01} + 6\big(\lambda_1^2 - \lambda_2\big)S_{00} = 0\,, &\quad S_{03} - 6\lambda_2 S_{01} + 6\lambda_1\lambda_2 S_{00} = 0\,, \\ 3S_{12} - 6\lambda_1 S_{11} + 6\big(\lambda_1^2 - \lambda_2\big)S_{10} = 0\,, &\quad S_{13} - 6\lambda_2 S_{11} + 6\lambda_1\lambda_2 S_{10} = 0\,. \end{aligned}$$

Equivalently, eliminating S_{03} and S_{13} via the two rightmost relations and writing

$$x = S_{02} - 2\lambda_1 S_{01} + 2\big(\lambda_1^2 - \lambda_2\big)S_{00} \quad \text{and} \quad y = S_{12} - 2\lambda_1 S_{11} + 2\big(\lambda_1^2 - \lambda_2\big)S_{10}\,,$$

we have that the Chow ring $\mathrm{CH}^*(T_1)$ is generated as a module over $\mathbb{Z}[\lambda_1, \lambda_2]$ by S_{00}, S_{01}, S_{10}, S_{11}, x and y, with relations

$$3x = 3y = 0\,.$$

Note that the assumed independence of s_{00}, s_{10}, s_{01}, s_{11}, s'_{02}, s_{12}, and s_{13} implies the independence assumed in Lemma 9.5. By Lemma 9.5, the Chow ring $\mathrm{CH}^*((\mathbb{P}V_6 \smallsetminus T_2)/\,\mathrm{GL}_2)$ is generated as a module over $\mathbb{Z}[\lambda_1, \lambda_2]$ by s_6^0, s_6^1, s_6^2, s_6^3, s_6^4, s_6^5, and s_6^6, subject to the relations

$$3x' = 0\,, \quad 3 \cdot \theta(y) = 0\,, \quad s_{13} - 6\lambda_2 s_{11} + 6\lambda_1\lambda_2 s_{10} = 0\,,$$

where $\theta(x) = 2x'$, and $\mathrm{CH}^*((\mathbb{P}V_6 \smallsetminus T_2)/\,\mathrm{GL}_2, 1; \mathbb{Z}_\ell) = 0$. Using the localization exact sequence

$$\mathrm{CH}^{*-2}(T_1) \to \mathrm{CH}^*((\mathbb{P}V_6 \smallsetminus T_2)/\,\mathrm{GL}_2) \to \mathrm{CH}^*\big((\overline{M}_2 \smallsetminus \Delta_1)/\mathbb{G}_m\big) \to 0\,,$$

we conclude that the Chow ring $\mathrm{CH}^*\big((\overline{M}_2 \smallsetminus \Delta_1)/\mathbb{G}_m\big)$ is generated as a module over $\mathbb{Z}[\lambda_1, \lambda_2]$ by the elements s_6^0, s_6^1, s_6^2, s_6^3, s_6^4, s_6^5, and s_6^6, subject to the relations

$$s_{00} = s_{10} = s_{01} = s_{11} = x' = \theta(y) = s_{13} - 6\lambda_2 s_{11} + 6\lambda_1\lambda_2 s_{10} = 0\,.$$

Moreover, assuming that these seven elements are linearly independent over $\mathbb{Z}[\lambda_1, \lambda_2]$, the localization sequence

$$\begin{aligned} \cdots &\to \mathrm{CH}^*((\mathbb{P}V_6 \smallsetminus T_2)/\,\mathrm{GL}_2, 1; \mathbb{Z}_\ell) \to \mathrm{CH}^*\big((\overline{M}_2 \smallsetminus \Delta_1)/\mathbb{G}_m, 1; \mathbb{Z}_\ell\big) \\ &\to \mathrm{CH}^{*-2}(T_1)_{\overline{k}} \otimes \mathbb{Z}_\ell \to \mathrm{CH}^*((\mathbb{P}V_6 \smallsetminus T_2)/\,\mathrm{GL}_2)_{\overline{k}} \otimes \mathbb{Z}_\ell \to \cdots \end{aligned}$$

implies $\mathrm{CH}^*\big((\overline{M}_2 \smallsetminus \Delta_1)/\mathbb{G}_m, 1; \mathbb{Z}_\ell\big) = 0$. Since

$$x' = s'_{02} - \lambda_1 s_{01} + \big(\lambda_1^2 - \lambda_2\big)s_{00} \quad \text{and} \quad y = s_{12} - 2\lambda_1 s_{11} + 2\big(\lambda_1^2 - \lambda_2\big)s_{10}\,,$$

the submodule generated by these seven elements is the same as the submodule generated by the seven elements given in the statement of the lemma. ☐

We now combine Lemmas 4.4 and 4.8 to calculate s_{ij} by pushing forward along θ. For this, we factor θ as cubing the first factor followed by multiplication:

$$(\mathbb{P}V_1 \times \mathbb{P}V_3)/\,\mathrm{GL}_2 \to (\mathbb{P}V_3 \times \mathbb{P}V_3)/\,\mathrm{GL}_2 \to \mathbb{P}V_6/\,\mathrm{GL}_2\,.$$

The results are as follows:

$$\begin{aligned}
s_{10}\colon s_1^1 \times s_3^0 &\mapsto \big[s_3^3 - 6\lambda_2 s_3^1 + 6\lambda_1\lambda_2 s_3^0\big] \times s_6^0 \mapsto s_6^3 - 60\lambda_2 s_6^1 + 120\lambda_1\lambda_2 s_6^0\,,\\
s_{11}\colon s_1^1 \times s_3^1 &\mapsto \big[s_3^3 - 6\lambda_2 s_3^1 + 6\lambda_1\lambda_2 s_3^0\big] \times s_6^1 \mapsto s_6^4 - 36\lambda_2 s_6^2 + 60\lambda_1\lambda_2 s_6^1\,,\\
s_{12}\colon s_1^1 \times s_3^2 &\mapsto \big[s_3^3 - 6\lambda_2 s_3^1 + 6\lambda_1\lambda_2 s_3^0\big] \times s_6^2 \mapsto s_6^5 - 18\lambda_2 s_6^3 + 24\lambda_1\lambda_2 s_6^2\,,\\
s_{13}\colon s_1^1 \times s_3^3 &\mapsto \big[s_3^3 - 6\lambda_2 s_3^1 + 6\lambda_1\lambda_2 s_3^0\big] \times s_6^3 \mapsto s_6^6 - 6\lambda_2 s_6^4 + 6\lambda_1\lambda_2 s_6^3\,,\\
s_{00}\colon s_1^0 \times s_3^0 &\mapsto \big[3s_3^2 - 6\lambda_1 s_3^1 + 6\big(\lambda_1^2 - \lambda_2\big)s_3^0\big] \times s_3^0 \mapsto 12 s_6^2 - 60\lambda_1 s_6^1 + 120\big(\lambda_1^2 - \lambda_2\big)s_6^0\,,\\
s_{01}\colon s_1^0 \times s_3^1 &\mapsto \big[3s_3^2 - 6\lambda_1 s_3^1 + 6\big(\lambda_1^2 - \lambda_2\big)s_3^0\big] \times s_3^1 \mapsto 9 s_6^3 - 36\lambda_1 s_6^2 + 60\big(\lambda_1^2 - \lambda_2\big)s_6^1\,,\\
&\Rightarrow s'_{02} = 3s_6^4 - 9\lambda_1 s_6^3 + 12\big(\lambda_1^2 - \lambda_2\big)s_6^2\,.
\end{aligned}$$

LEMMA 9.7. *We have* $\mathrm{CH}^*\big((\overline{M}_2 \smallsetminus \Delta_1)/\mathbb{G}_m, 1; \mathbb{Z}_\ell\big) = 0$.

Proof. We just have to check that s_{10}, s_{11}, s_{12}, s_{13}, s_{00}, s_{01}, and s'_{02} are linearly independent over $\mathbb{Z}[\lambda_1, \lambda_2]$. For this, we use the above expressions for them in the basis s_6^0, s_6^1, s_6^2, s_6^3, s_6^4, s_6^5, and s_6^6, which reduces our problem to checking that the following matrix is nonsingular:

$$\begin{matrix} 120\lambda_1\lambda_2 & -60\lambda_2 & 0 & 1 & 0 & 0 & 0 \\ 0 & 60\lambda_1\lambda_2 & -36\lambda_2 & 0 & 1 & 0 & 0 \\ 0 & 0 & 24\lambda_1\lambda_2 & -18\lambda_2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 6\lambda_1\lambda_2 & -6\lambda_2 & 0 & 1 \\ 120\left(\lambda_1^2-\lambda_2\right) & -60\lambda_1 & 12 & 0 & 0 & 0 & 0 \\ 0 & 60\left(\lambda_1^2-\lambda_2\right) & -36\lambda_1 & 9 & 0 & 0 & 0 \\ 0 & 0 & 12\left(\lambda_1^2-\lambda_2\right) & -9\lambda_1 & 3 & 0 & 0 \end{matrix}$$

It thus remains to note that this matrix has determinant $86400\left(\lambda_1^2-4\lambda_2\right)^3 \neq 0$. □

Lemma 9.8. *The Chow ring of* $\left(\overline{M}_2 \smallsetminus \Delta_1\right)/\mathbb{G}_m$ *is given by*

$$\mathrm{CH}^*\left(\left(\overline{M}_2 \smallsetminus \Delta_1\right)/\mathbb{G}_m\right) = \mathbb{Z}[\lambda_1,\lambda_2,t]/(s_{00}, s_{10}, s_{02}') \,.$$

Proof. By Lemmas 9.6 and 9.2,

$$\mathrm{CH}^*\left(\left(\overline{M}_2 \smallsetminus \Delta_1\right)/\mathbb{G}_m\right) = \mathbb{Z}[\lambda_1,\lambda_2,t]/(p, s_{00}, s_{10}, s_{01}, s_{11}, s_{02}', s_{12}, s_{13}) \,.$$

As $\theta^* t$ restricts to a hyperplane class on the $\mathbb{P}^3$-factor, push-pull implies that $s_{ij} \in (s_{00}, s_{10})$ for any i and j (but not that $s_{02}' \in (s_{00}, s_{10})$!), as well as that $p \in (s_{00}, s_{10})$. Thus, s_{00}, s_{10}, and s_{02}' generate the ideal of relations. □

To compute the Chow ring of $\left(\overline{M}_2 \smallsetminus \Delta_1\right)/\mathbb{G}_m$ more explicitly, we first use the recursion of Lemma 4.2. to compute the first few s_6^j:

$$\begin{aligned} s_6^0 &= 1\,, \quad s_6^1 = t\,, \quad s_6^2 = t^2 - \lambda_1 t + 6\lambda_2\,, \\ s_6^3 &= t^3 - 3\lambda_1 t^2 + \left(2\lambda_1^2 + 16\lambda_2\right)t - 12\lambda_1\lambda_2\,, \\ s_6^4 &= t^4 - 6\lambda_1 t^3 + \left(11\lambda_1^2 + 28\lambda_2\right)t^2 + \left(-6\lambda_1^3 - 72\lambda_1\lambda_2\right)t + 36\lambda_1^2\lambda_2 + 72\lambda_2^2\,. \end{aligned}$$

Substituting these into the expressions for s_{ij} given above, we obtain

$$\begin{aligned} s_{10} &= t^3 - 3\lambda_1 t^2 + \left(2\lambda_1^2 - 44\lambda_2\right)t + 108\lambda_1\lambda_2 = 20\lambda_1\lambda_2 + \left(t^2 - \lambda_1 t - 44\lambda_2\right)\cdot(t-2\lambda_1)\,, \\ s_{00} &= 12t^2 - 72\lambda_1 t + 120\lambda_1^2 - 48\lambda_2 = 24\lambda_1^2 - 48\lambda_2 + (12t - 48\lambda_1)\cdot(t-2\lambda_1)\,, \\ s_{02}' &= 3t^4 - 27\lambda_1 t^3 + \left(72\lambda_1^2 + 72\lambda_2\right)t^2 - \left(48\lambda_1^3 + 348\lambda_1\lambda_2\right)t + 288\lambda_1^2\lambda_2 + 144\lambda_2^2 \\ &= -60\left(\lambda_1^2 - 4\lambda_2\right)(t - 3\lambda_1)\cdot(t-2\lambda_1) + (3t - 6\lambda_1)s_{10} - \left(\lambda_1 t - 3\lambda_1^2 + 3\lambda_2\right)s_{00}\,. \end{aligned}$$

In other words, the Chow ring of $\left(\overline{M}_2 \smallsetminus \Delta_1\right)/\mathbb{G}_m$ is generated by λ_1, λ_2, and t with relations

$$\begin{aligned} R_1 &= 20\lambda_1\lambda_2 + \left(t^2 - \lambda_1 t - 44\lambda_2\right)\cdot(t-2\lambda_1) = 0\,, \\ R_2 &= 24\lambda_1^2 - 48\lambda_2 + (12t - 48\lambda_1)\cdot(t-2\lambda_1) = 0\,, \\ R_3 &= 60\left(\lambda_1^2 - 4\lambda_2\right)(t-3\lambda_1)\cdot(t-2\lambda_1) = 0\,. \end{aligned}$$

Theorem 9.9. *The Chow ring of* $\overline{M}_2 \smallsetminus \Delta_1$ *is given by*

$$\mathrm{CH}^*\left(\overline{M}_2 \smallsetminus \Delta_1\right) = \mathbb{Z}[\lambda_1,\lambda_2]/\left(24\lambda_1^2 - 48\lambda_2, 20\lambda_1\lambda_2\right).$$

In addition, $\mathrm{CH}^*\left(\overline{M}_2 \smallsetminus \Delta_1, 1; \mathbb{Z}_\ell\right)$ *is generated freely as a module over* $(\mathbb{Z}/2\mathbb{Z})[\lambda_1,\lambda_2]$ *by two 2-torsion classes in degrees* 4 *and* 5*, respectively.*

Proof. Note that $\overline{M}_2 \smallsetminus \Delta_1$ is a $\mathbb{G}_m$-bundle over $(\overline{M}_2 \smallsetminus \Delta_1)/\mathbb{G}_m$, with Chern class $t - 2\lambda_1$. From the localization sequence

$$\mathrm{CH}^*\big((\overline{M}_2 \smallsetminus \Delta_1)/\mathbb{G}_m\big) \to \mathrm{CH}^*\big((\overline{M}_2 \smallsetminus \Delta_1)/\mathbb{G}_m\big) \to \mathrm{CH}^*\big(\overline{M}_2 \smallsetminus \Delta_1\big) \to 0\,,$$

we learn that $\mathrm{CH}^*(\overline{M}_2 \smallsetminus \Delta_1)$ is given by the cokernel of multiplication by $t - 2\lambda_1$ on the Chow ring $\mathrm{CH}^*\big((\overline{M}_2 \smallsetminus \Delta_1)/\mathbb{G}_m\big)$, which is evidently the ring given in the statement of the theorem.

Moreover, from the localization sequence

$$\begin{aligned}\cdots \to \mathrm{CH}^*\big((\overline{M}_2 \smallsetminus \Delta_1)/\mathbb{G}_m, 1; \mathbb{Z}_\ell\big) = 0 \to \mathrm{CH}^*\big(\overline{M}_2 \smallsetminus \Delta_1, 1; \mathbb{Z}_\ell\big)\\ \to \mathrm{CH}^{*-1}\big((\overline{M}_2 \smallsetminus \Delta_1)/\mathbb{G}_m\big)_{\overline{k}} \otimes \mathbb{Z}_\ell \to \mathrm{CH}^*\big((\overline{M}_2 \smallsetminus \Delta_1)/\mathbb{G}_m\big)_{\overline{k}} \otimes \mathbb{Z}_\ell \to \cdots\,,\end{aligned}$$

we learn that $\mathrm{CH}^*(\overline{M}_2 \smallsetminus \Delta_1, 1; \mathbb{Z}_\ell)$ is given by the kernel of multiplication by $t - 2\lambda_1$ on $\mathrm{CH}^{*-1}\big((\overline{M}_2 \smallsetminus \Delta_1)/\mathbb{G}_m\big)_{\overline{k}} \otimes \mathbb{Z}_\ell$. This kernel is evidently generated by the classes

$$\begin{aligned} c_4 &= 60(\lambda_1^2 - 4\lambda_2)(t - 3\lambda_1)\,,\\ c_5 &= 5\lambda_1\lambda_2 \cdot (12t - 48\lambda_1) - \big(6\lambda_1^2 - 12\lambda_2\big) \cdot \big(t^2 - \lambda_1 t - 44\lambda_2\big)\,. \end{aligned}$$

Finally, by inspection, both of these classes, when multiplied by 2 or t, are in the ideal (R_1, R_2, R_3). To see that these classes generate the kernel freely over $(\mathbb{Z}/2\mathbb{Z})[\lambda_1, \lambda_2]$, assume to the contrary that some linear combination $p_4 \cdot c_4 + p_5 \cdot c_5$, with p_4, p_5 not both in $(2, t)$, was in (R_1, R_2, R_3).

As $c_4, c_5, R_2, R_3 \in (2)$ but $R_1 \notin (2)$, we have $p_4 \cdot c_4 + p_5 \cdot c_5 \in (2R_1, R_2, R_3) \subset (2t^2, 4)$. Since $c_4 \in (2t^2, 4)$, we obtain $p_5 \cdot c_5 \in (2t^2, 4)$. As $c_5 \notin (2t^2, 4)$, this forces $p_5 \in (2, t)$.

In particular, $p_5 \cdot c_5 \in (R_1, R_2, R_3)$, so $p_4 \cdot c_4 \in (R_1, R_2, R_3)$. As $c_5, R_2, R_3 \in (4)$, but $R_1 \notin (2)$, we have $p_4 \cdot c_4 + p_5 \cdot c_5 \in (4R_1, R_2, R_3) \subset (4t, 8)$. Since $c_4 \notin (4t, 8)$, this forces $p_5 \in (2, t)$. □

Observe that the twisting $\mathbb{G}_m$-action extends over the whole of $\overline{M}_2$. Indeed, consider the universal family of curves over $\overline{M}_2$, pulled back to $\overline{M}_2 \times \mathbb{G}_m$. There is a natural action of $\mathbb{Z}/2\mathbb{Z}$ via the hyperelliptic involution on the universal family over $\overline{M}_2$ composed with multiplication by -1 on $\mathbb{G}_m$; the quotient of this family by this action gives a family of curves over $\overline{M}_2 \times (\mathbb{G}_m/(\mathbb{Z}/2\mathbb{Z}))$— or, using the isomorphism $\overline{M}_2 \times (\mathbb{G}_m/(\mathbb{Z}/2\mathbb{Z})) \simeq \overline{M}_2 \times \mathbb{G}_m$ given by the squaring map, a family of curves over $\overline{M}_2 \times \mathbb{G}_m$, that is, an action of $\mathbb{G}_m$ on $\overline{M}_2$. By inspection, this restricts to the twisting action defined above on $\overline{M}_2 \smallsetminus \Delta_1$.

Our next task is to leverage the fact that the $\mathbb{G}_m$-action extends over the whole of $\overline{M}_2$ to compute the image of the degree 5 generator in $\mathrm{CH}^5(\overline{M}_2 \smallsetminus \Delta_1, 1; \mathbb{Z}_\ell)$ in $\mathrm{CH}^4(\Delta_1)_{\overline{k}} \otimes \mathbb{Z}_\ell$ under the boundary map in the localization sequence

$$\cdots \to \mathrm{CH}^5\big(\overline{M}_2 \smallsetminus \Delta_1, 1; \mathbb{Z}_\ell\big) \to \mathrm{CH}^4(\Delta_1)_{\overline{k}} \otimes \mathbb{Z}_\ell \to \mathrm{CH}^5\big(\overline{M}_2\big)_{\overline{k}} \otimes \mathbb{Z}_\ell \to \cdots\,.$$

First observe that since $\mathrm{CH}^*(\overline{M}_2 \smallsetminus \Delta_1, 1; \mathbb{Z}_\ell)$ is generated in degrees 4 and 5, any degree 2 or 3 class $x \in \mathrm{CH}^*(\overline{M}_2)_{\overline{k}} \otimes \mathbb{Z}_\ell$ whose image in $\mathrm{CH}^*(\overline{M}_2 \smallsetminus \Delta_1)_{\overline{k}} \otimes \mathbb{Z}_\ell$ vanishes is the image of a unique degree 1 or 2 class in $\mathrm{CH}^*(\Delta_1)_{\overline{k}} \otimes \mathbb{Z}_\ell$, which we denote by $\overline{x}$.

LEMMA 9.10. *The image of the degree* 5 *generator in* $\mathrm{CH}^5(\overline{M}_2 \smallsetminus \Delta_1, 1; \mathbb{Z}_\ell)$ *in* $\mathrm{CH}^4(\Delta_1)_{\overline{k}} \otimes \mathbb{Z}_\ell$ *is given by*

$$5\lambda_1\lambda_2 \cdot \big(\overline{24\lambda_1^2 - 48\lambda_2}\big) - \big(6\lambda_1^2 - 12\lambda_2\big) \cdot \overline{20\lambda_1\lambda_2}\,.$$

Proof. Consider the diagram

$$\begin{array}{ccccccccc}
0 \longrightarrow & \mathrm{CH}^*(\Delta_1/\mathbb{G}_m)_{\overline{k}} \otimes \mathbb{Z}_\ell & \longrightarrow & \mathrm{CH}^*\big(\overline{M}_2/\mathbb{G}_m\big)_{\overline{k}} \otimes \mathbb{Z}_\ell & \longrightarrow & \mathrm{CH}^*\big(\big(\overline{M}_2 \smallsetminus \Delta_1\big)/\mathbb{G}_m\big)_{\overline{k}} \otimes \mathbb{Z}_\ell & \longrightarrow 0 \\
& \downarrow & & \downarrow & & \downarrow & \\
0 \longrightarrow & \mathrm{CH}^*(\Delta_1/\mathbb{G}_m)_{\overline{k}} \otimes \mathbb{Z}_\ell & \longrightarrow & \mathrm{CH}^*\big(\overline{M}_2/\mathbb{G}_m\big)_{\overline{k}} \otimes \mathbb{Z}_\ell & \longrightarrow & \mathrm{CH}^*\big(\big(\overline{M}_2 \smallsetminus \Delta_1\big)/\mathbb{G}_m\big)_{\overline{k}} \otimes \mathbb{Z}_\ell & \longrightarrow 0\,,
\end{array}$$

where the vertical maps are multiplication by $t - 2\lambda_1$; since $\mathrm{CH}^*\big(\big(\overline{M}_2 \smallsetminus \Delta_1\big)/\mathbb{G}_m, 1; \mathbb{Z}_\ell\big) = 0$, the rows are exact. The snake lemma then gives us a map from the kernel of multiplication by $t - 2\lambda_1$ on $\mathrm{CH}^*\big(\big(\overline{M}_2 \smallsetminus \Delta_1\big)/\mathbb{G}_m\big)_{\overline{k}} \otimes \mathbb{Z}_\ell$ to the cokernel of multiplication by $t - 2\lambda_1$ on $\mathrm{CH}^*(\Delta_1/\mathbb{G}_m)_{\overline{k}} \otimes \mathbb{Z}_\ell$; under the identification of the former with $\mathrm{CH}^*\big(\overline{M}_2 \smallsetminus \Delta_1, 1; \mathbb{Z}_\ell\big) = 0$ and the latter with $\mathrm{CH}^*(\Delta_1)_{\overline{k}} \otimes \mathbb{Z}_\ell$, this map corresponds to the boundary map appearing in the localization exact sequence.

We have a natural lift of $t - 2\lambda_1$ (and thus t) corresponding to the extension of the $\mathbb{G}_m$-action. Given a class in $x \in \mathrm{CH}^*(\overline{M}_2/\mathbb{G}_m)_{\overline{k}} \otimes \mathbb{Z}_\ell$ whose image in $\mathrm{CH}^*\big(\big(\overline{M}_2 \smallsetminus \Delta_1\big)/\mathbb{G}_m\big)_{\overline{k}} \otimes \mathbb{Z}_\ell$ is zero, we also write $\overline{x}$ for the class in $\mathrm{CH}^*(\Delta_1/\mathbb{G}_m)_{\overline{k}} \otimes \mathbb{Z}_\ell$ mapping to x.

Now the map arising from the snake lemma can be described as follows: Given a class in the kernel of multiplication by $t - 2\lambda_1$ on $\mathrm{CH}^*\big(\big(\overline{M}_2 \smallsetminus \Delta_1\big)/\mathbb{G}_m\big)_{\overline{k}} \otimes \mathbb{Z}_\ell$, we pick a lift of this class to $\mathrm{CH}^*(\overline{M}_2/\mathbb{G}_m)_{\overline{k}} \otimes \mathbb{Z}_\ell$; we multiply that lift by $t - 2\lambda_1$; we write that (uniquely) as the image of a class in $\mathrm{CH}^*(\Delta_1/\mathbb{G}_m)_{\overline{k}} \otimes \mathbb{Z}_\ell$; and finally, we reduce that class modulo $t - 2\lambda_1$ to obtain a class in the cokernel of multiplication by $t - 2\lambda_1$ on $\mathrm{CH}^*(\Delta_1/\mathbb{G}_m)_{\overline{k}} \otimes \mathbb{Z}_\ell$. Following that recipe for our degree 5 class, we obtain

$$\overline{(t - 2\lambda_1) \cdot \big[5\lambda_1\lambda_2 \cdot (12t - 48\lambda_1) - \big(6\lambda_1^2 - 12\lambda_2\big) \cdot \big(t^2 - \lambda_1 t - 44\lambda_2\big)\big]} \mod t - 2\lambda_1\,.$$

Writing

$$\begin{aligned}
(t - 2\lambda_1) \cdot \big[5\lambda_1\lambda_2 \cdot (12t - 48\lambda_1) &- \big(6\lambda_1^2 - 12\lambda_2\big) \cdot \big(t^2 - \lambda_1 t - 44\lambda_2\big)\big] \\
&= (5\lambda_1\lambda_2) \cdot \big[24\lambda_1^2 - 48\lambda_2 + (12t - 48\lambda_1)(t - 2\lambda_1)\big] \\
&\quad - \big(6\lambda_1^2 - 12\lambda_2\big) \cdot \big[20\lambda_1\lambda_2 + \big(t^2 - \lambda_1 t - 44\lambda_2\big)(t - 2\lambda_1)\big]\,,
\end{aligned}$$

we obtain

$$\begin{aligned}
&(5\lambda_1\lambda_2) \cdot \big[\overline{24\lambda_1^2 - 48\lambda_2 + (12t - 48\lambda_1)(t - 2\lambda_1)} \mod t - 2\lambda_1\big] \\
&\quad - \big(6\lambda_1^2 - 12\lambda_2\big) \cdot \big[\overline{20\lambda_1\lambda_2 + \big(t^2 - \lambda_1 t - 44\lambda_2\big)(t - 2\lambda_1)} \mod t - 2\lambda_1\big]\,.
\end{aligned}$$

Since $\overline{24\lambda_1^2 - 48\lambda_2 + (12t - 48\lambda_1)(t - 2\lambda_1)} \mod t - 2\lambda_1$ is a class in degree 2 and the map $\mathrm{CH}^2(\Delta_1)_{\overline{k}} \otimes \mathbb{Z}_\ell \to \mathrm{CH}^2\big(\overline{M}_2\big)_{\overline{k}} \otimes \mathbb{Z}_\ell$ is injective, this class is determined by its image in the Chow group $\mathrm{CH}^2\big(\overline{M}_2\big)_{\overline{k}} \otimes \mathbb{Z}_\ell$, which is just $24\lambda_1^2 - 48\lambda_2$. Consequently, this class can be written simply as $\overline{24\lambda_1^2 - 48\lambda_2}$. Reasoning similarly for the second term, we deduce the formula given in the statement of the lemma. □

9.1 Results in cohomology

The analogous argument in singular cohomology shows that in characteristic zero, the even singular cohomology of $\overline{M}_2 \smallsetminus \Delta_1$ is isomorphic to its Chow ring via the cycle class map

$$\mathrm{H}^{2n}\big(\overline{M}_2 \smallsetminus \Delta_1\big) \simeq \mathrm{CH}^n\big(\overline{M}_2 \smallsetminus \Delta_1\big)$$

and that the odd singular cohomology of $\overline{M}_2 \smallsetminus \Delta_1$ is isomorphic to its first higher Chow groups via the cycle class map

$$\mathrm{H}^{2n+1}\big(\overline{M}_2 \smallsetminus \Delta_1, \mathbb{Z}_\ell\big) \simeq \mathrm{CH}^n\big(\overline{M}_2 \smallsetminus \Delta_1, 1; \mathbb{Z}_\ell\big)\,.$$

In particular, $\mathrm{H}^{2n+1}\big(\overline{M}_2 \smallsetminus \Delta_1\big)$ is freely generated as a module over $(\mathbb{Z}/2\mathbb{Z})[\lambda_1, \lambda_2]$ by two 2-torsion classes in degrees $2\cdot 4+1 = 9$ and $2 \cdot 5 + 1 = 11$. So as a group,

$$\begin{aligned}\mathrm{H}^{2n+1}\big(\overline{M}_2 \smallsetminus \Delta_1\big) &\simeq (\mathbb{Z}/2\mathbb{Z})[\lambda_1, \lambda_2]_{n-4} \oplus (\mathbb{Z}/2\mathbb{Z})[\lambda_1, \lambda_2]_{n-5} \\ &\simeq (\mathbb{Z}/2\mathbb{Z})^{\max(0, \lfloor (n-2)/2 \rfloor) + \max(0, \lfloor (n-3)/2\rfloor)} = (\mathbb{Z}/2\mathbb{Z})^{\max(0,n-3)}\,.\end{aligned}$$

10. Reduction to three nonvanishing statements

In this section, we show that Theorem 1.1 holds provided that the classes

$$\delta_1(\delta_1+\lambda_1)\lambda_1^2, \quad \delta_1(\delta_1+\lambda_1)\lambda_2, \quad \text{and} \quad \delta_1(\delta_1+\lambda_1)\big(\lambda_1^2+\lambda_2\big)$$

are all nonvanishing in $\mathrm{CH}^*\big(\overline{M}_2\big) \otimes \mathbb{Z}/2\mathbb{Z}$ (which we assume for the remainder of this section). Since

$$\mathrm{CH}^*(\Delta_1) = \mathbb{Z}[\lambda_1, \lambda_2, \gamma]/\big(2\gamma, \gamma^2 + \lambda_1\gamma, 24\lambda_1^2 - 48\lambda_2, 24\lambda_1\lambda_2\big)$$

is invariant under extension of the base field from k to $\overline{k}$ and only has torsion coprime to the characteristic (as we have assumed the characteristic is distinct from 2 and 3), the injectivity of

$$\mathrm{CH}^*(\Delta_1)_{\overline{k}} \otimes \mathbb{Z}_\ell \to \mathrm{CH}^{*+1}\big(\overline{M}_2\big)_{\overline{k}} \otimes \mathbb{Z}_\ell$$

for ℓ coprime to the characteristic implies the injectivity of $\mathrm{CH}^*(\Delta_1) \to \mathrm{CH}^{*+1}\big(\overline{M}_2\big)$.

10.1 Injectivity in degrees 0, 1, and 2

By Theorem 9.9, the morphism $\mathrm{CH}^*(\Delta_1)_{\overline{k}} \otimes \mathbb{Z}_\ell \to \mathrm{CH}^{*+1}\big(\overline{M}_2\big)_{\overline{k}} \otimes \mathbb{Z}_\ell$ is injective in degrees 0, 1, and 2, for all ℓ coprime to the characteristic. Consequently, the map $\mathrm{CH}^*(\Delta_1) \to \mathrm{CH}^{*+1}\big(\overline{M}_2\big)$ is injective in degrees 0, 1, and 2.

10.2 Relations from the Grothendieck–Riemann–Roch theorem

Here we show that the relations established in Section 6.2 in characteristic zero, and modulo torsion in arbitrary characteristic, in fact hold exactly in arbitrary characteristic distinct from 2 and 3.

The injectivity of $\mathrm{CH}^0(\Delta_1) \to \mathrm{CH}^1\big(\overline{M}_2\big)$, combined with Theorems 8.1 and 9.9, implies that $\mathrm{CH}^1\big(\overline{M}_2\big)$ is torsion-free. Thus, the formula in Lemma 6.7 holds exactly, not merely modulo torsion. Combined with Lemma 6.9, this implies the relation

$$20\lambda_1\lambda_2 - 4\delta_1\lambda_2 = 0\,. \tag{10.1}$$

Next, the relation $24\lambda_1^2 - 48\lambda_2 = 0$ holds on $\overline{M}_2 \smallsetminus \Delta_1$ by Theorem 9.9 and holds on $\overline{M}_2$ modulo torsion by Lemma 6.5. Since $\mathrm{CH}^1(\Delta_1) \to \mathrm{CH}^2\big(\overline{M}_2\big)$ is injective, $24\lambda_1^2 - 48\lambda_2$ must equal the pushforward of a torsion class in $\mathrm{CH}^1(\Delta_1)$. But the only nonzero torsion class in $\mathrm{CH}^1(\Delta_1)$ is γ, whose pushforward is $\delta_1(\delta_1+\lambda_1)$ by Lemma 6.1. Thus, either

$$24\lambda_1^2 - 48\lambda_2 = 0\,, \quad \text{or} \quad 24\lambda_1^2 - 48\lambda_2 = \delta_1(\delta_1+\lambda_1)\,.$$

This second relation implies $\delta_1(\delta_1+\lambda_1) = 0 \in \mathrm{CH}^2\big(\overline{M}_2\big) \otimes \mathbb{Z}/2\mathbb{Z}$, which contradicts our assumption that $\delta_1(\delta_1+\lambda_1)\lambda_1^2 \neq 0 \in \mathrm{CH}^4\big(\overline{M}_2\big)\otimes\mathbb{Z}/2\mathbb{Z}$. Consequently the first relation must hold; that is,

$$24\lambda_1^2 - 48\lambda_2 = 0\,. \tag{10.2}$$

10.3 Injectivity in degree 3

By Lemma 6.1, the kernel of $\mathrm{CH}^3(\Delta_1) \to \mathrm{CH}^4\big(\overline{M}_2\big)$ is contained in the kernel of multiplication by $\gamma - \lambda_1$. Any element of this latter kernel can be written as a polynomial $p(\lambda_1, \lambda_2, \gamma)$ which satisfies

$$(\gamma - \lambda_1)\cdot p \in \big(2\gamma, \gamma^2 + \lambda_1\gamma, 24\lambda_1^2 - 48\lambda_2, 24\lambda_1\lambda_2\big) \subset (\gamma, 24) \Rightarrow -\lambda_1 p \in (\gamma, 24) \Rightarrow p \in (\gamma, 24)\,.$$

As multiplication by 24 is zero on $\mathrm{CH}^3(\Delta_1)$, any such element is thus a multiple of γ; using the relations $2\gamma = \gamma^2 + \lambda_1\gamma = 0$, we see that there are only three such nonzero elements: $\gamma\lambda_1^2$, $\gamma\lambda_2$, and $\gamma\big(\lambda_1^2 + \lambda_2\big)$.

Pushing these classes forward to $\mathrm{CH}^4\big(\overline{M}_2\big)$ using Lemma 6.1, we obtain exactly the three classes whose reduction modulo 2 we have assumed is nonvanishing.

10.4 Injectivity in higher degrees

Combining the injectivity we have established in degrees 0, 1, 2, and 3 with Theorem 9.9 and Lemma 9.10, it remains to show

$$5\lambda_1\lambda_2\cdot\big(\overline{24\lambda_1^2 - 48\lambda_2}\big) - \big(6\lambda_1^2 - 12\lambda_2\big)\cdot\overline{20\lambda_1\lambda_2} = 0 \in \mathrm{CH}^4(\Delta_1)\,.$$

But using the relations (10.2) and (10.1), we have

$$\begin{aligned}&5\lambda_1\lambda_2\cdot\big(\overline{24\lambda_1^2 - 48\lambda_2}\big) - \big(6\lambda_1^2 - 12\lambda_2\big)\cdot\overline{20\lambda_1\lambda_2}\\ &\qquad = 5\lambda_1\lambda_2\cdot 0 - \big(6\lambda_1^2 - 12\lambda_2\big)\cdot 4\lambda_2 = -\lambda_2\cdot\big(24\lambda_1^2 - 48\lambda_2\big) = 0\,.\end{aligned}$$

10.5 Conclusion of the proof

The injectivity established thus far implies that we have a short exact sequence

$$0 \to \mathrm{CH}^{*-1}(\Delta_1) \to \mathrm{CH}^*\big(\overline{M}_2\big) \to \mathrm{CH}^*\big(\overline{M}_2 \smallsetminus \Delta\big) \to 0\,,$$

and we have previously (Theorems 8.1 and 9.9) established that

$$\begin{aligned}\mathrm{CH}^*(\Delta_1) &= \mathbb{Z}[\lambda_1, \lambda_2, \gamma]/\big(2\gamma, \gamma^2 + \lambda_1\gamma, 24\lambda_1^2 - 48\lambda_2, 24\lambda_1\lambda_2\big)\,,\\ \mathrm{CH}^*(\overline{M}_2 \smallsetminus \Delta) &= \mathbb{Z}[\lambda_1, \lambda_2]/\big(24\lambda_1^2 - 48\lambda_2, 20\lambda_1\lambda_2\big)\,.\end{aligned}$$

Given Lemma 6.1, and the relations (10.2) and (10.1) that we have established and which extend the relations $24\lambda_1^2 - 48\lambda_2 = 0$ and $20\lambda_1\lambda_2 = 0$ from $\overline{M}_2 \smallsetminus \Delta_1$ to all of $\overline{M}_2$, it follows that $\mathrm{CH}^*\big(\overline{M}_2\big)$ is generated by λ_1, λ_2, and δ_1 with relations given by (10.2), (10.1), and the pushforwards of the relations that hold in $\mathrm{CH}^*(\Delta_1)$, that is,

$$\begin{gathered}24\lambda_1^2 - 48\lambda_2 = 0\,,\quad 20\lambda_1\lambda_2 - 4\delta_1\lambda_2 = 0\,,\quad [2(\delta_1 + \lambda_1)]\cdot\delta_1 = 0\,,\\ \big[(\delta_1 + \lambda_1)^2 + \lambda_1(\delta_1 + \lambda_1)\big]\cdot\delta_1 = 0\,,\quad \big[24\lambda_1^2 - 48\lambda_2\big]\cdot\delta_1 = 0\,,\quad [24\lambda_1\lambda_2]\cdot\delta_1 = 0\,.\end{gathered}$$

By inspection, these relations generate exactly the ideal of relations claimed in Theorem 1.1.

10.6 Results in cohomology

Since the cycle class map $\mathrm{CH}^n\big(\overline{M}_2 \smallsetminus \Delta_1\big) \to \mathrm{H}^{2n}\big(\overline{M}_2 \smallsetminus \Delta_1\big)$ is an isomorphism (cf. Section 9.1) and $\mathrm{CH}^n\big(\overline{M}_2\big) \to \mathrm{CH}^n\big(\overline{M}_2 \smallsetminus \Delta_1\big)$ is surjective, it follows that $\mathrm{H}^{2n}\big(\overline{M}_2\big) \to \mathrm{H}^{2n}\big(\overline{M}_2 \smallsetminus \Delta_1\big)$ is surjective.

And since the cycle class map $\mathrm{CH}^n(\overline{M}_2 \smallsetminus \Delta_1, 1; \mathbb{Z}_\ell) \to \mathrm{H}^{2n+1}(\overline{M}_2 \smallsetminus \Delta_1, \mathbb{Z}_\ell)$ is an isomorphism (cf. Section 9.1) and the boundary maps $\mathrm{CH}^n(\overline{M}_2 \smallsetminus \Delta_1, 1; \mathbb{Z}_\ell) \to \mathrm{CH}^{n-1}(\Delta_1) \otimes \mathbb{Z}_\ell$ are zero, the boundary maps $\mathrm{H}^{2n+1}(\overline{M}_2 \smallsetminus \Delta_1, \mathbb{Z}_\ell) \to \mathrm{H}^{2n-2}(\Delta_1) \otimes \mathbb{Z}_\ell$ are all zero.

Putting this together, we see that all of the boundary maps in the associated long exact sequence for cohomology groups are zero. The five lemma thus implies that the even cohomology of $\overline{M}_2$ is isomorphic to its Chow ring via the cycle class map

$$\mathrm{H}^{2n}(\overline{M}_2) \simeq \mathrm{CH}^n(\overline{M}_2)$$

and that the odd cohomology of $\overline{M}_2$ is an extension of the odd cohomology of $\overline{M}_2 \smallsetminus \Delta_1$ by the odd cohomology of Δ_1. Combining the results of Section 8.1 with Section 9.1, this implies the odd cohomology of $\overline{M}_2$ has exponent at most 96 and is of order

$$\begin{aligned}\#\,\mathrm{H}^{2n+1}(\overline{M}_2) &= \#\,\mathrm{H}^{2n+1}(\overline{M}_2 \smallsetminus \Delta_1) \cdot \#\,\mathrm{H}^{2n-1}(\Delta_1) \\ &= 24^{\max(0,n-5)} \cdot 2^{\max(0,n-3)+\max(0,\lfloor (n-4)/2 \rfloor)+\epsilon}, \text{ where } \epsilon = \begin{cases} 1 & \text{if } n \geqslant 5\,, \\ 0 & \text{otherwise}\,. \end{cases}\end{aligned} \tag{10.3}$$

11. Chow ring of the bielliptic stack

As shown in Section 10, in order to complete the proof of Theorem 1.1, all that remains is to show that the classes

$$\delta_1(\delta_1 + \lambda_1)\lambda_1^2\,, \quad \delta_1(\delta_1 + \lambda_1)\lambda_2\,, \quad \text{and} \quad \delta_1(\delta_1 + \lambda_1)(\lambda_1^2 + \lambda_2)$$

are all nonvanishing in $\mathrm{CH}^*(\overline{M}_2) \otimes \mathbb{Z}/2\mathbb{Z}$. We will do this by pulling these classes back to a certain test family, given explicitly in Section 11.2.

11.1 Motivation for our test family

(This section is, strictly speaking, not logically necessary.)

Recall that these classes are pushforwards of classes on Δ_1, which are multiples of the pullback to Δ_1 of the generator of $\mathrm{CH}^*(B(\mathbb{Z}/2\mathbb{Z}))$, where $\mathbb{Z}/2\mathbb{Z}$ acts by exchanging an ordered pair of elliptic curves. The fixed-point locus of this $\mathbb{Z}/2\mathbb{Z}$-action corresponds to curves in Δ_1 consisting of two isomorphic elliptic curves joined at a point, and on this locus, the $\mathbb{Z}/2\mathbb{Z}$-action gives rise to the automorphism that exchanges the two components, which is a bielliptic involution. This consideration suggests that the universal family of bielliptic curves would serve as a good test family. To be precise, we have the following.

Definition 11.1. A *bielliptic curve of genus* 2 is a stable curve C of genus 2, together with an unordered pair $\{\iota, \iota'\}$ of bielliptic involutions which differ by multiplication by the hyperelliptic involution.

Lemma 11.2. *Let C be a bielliptic curve of genus* 2. *Then* $\dim H^0(\omega_C^2) = 3$, *and the hyperelliptic involution acts trivially on* $H^0(\omega_C^2)$. *The bielliptic involution* (*which is well defined modulo the hyperelliptic involution*) *has a* 2-*dimensional trivial eigenspace and a* 1-*dimensional nontrivial eigenspace.*

Proof. Since $H^0(\omega_C^{-1}) = 0$ when C is a stable curve, the Riemann–Roch formula implies $\dim H^0(\omega_C^2) = 3$ as desired. The remaining conditions are then visibly both open and closed, so it suffices to check them when C is smooth.

In this case, any section of ω_C^2 sent to its negative under the hyperelliptic involution must vanish at the six Weierstrass points; since ω_C^2 is of degree 4, there are no such sections, and so the hyperelliptic involution acts trivially.

For the action of the bielliptic involution, write $f\colon C \to E$ for the quotient map. Then there is an evident filtration $H^0\big(f^*\omega_E^2\big) \subset H^0(f^*\omega_E \otimes \omega_C) \subset H^0\big(\omega_C^2\big)$, whose quotients are 1-dimensional, on which the bielliptic involution acts as $+1$, -1, and $+1$, respectively. $\square$

Consequently, every bielliptic curve comes equipped with a canonical map to $\mathbb{P}^1$ defined by the sections of $H^0\big(\omega_C^2\big)$ invariant under the bielliptic involution. By the Riemann–Hurwitz formula, this map is branched over five points with multiplicity.

Moreover, taking the ramification points of the quotients of C by the two bielliptic involutions mapping to $\mathbb{P}^1$ defines two subsets of four of these five points. Their intersection then gives a canonically defined subset of three points. We conclude that the family of curves in weighted projective space given by

$$z^2 = (ax+by)\cdot\big(ex^3+fx^2y+gxy^2+hy^3\big)\,,$$
$$w^2 = (cx+dy)\cdot\big(ex^3+fx^2y+gxy^2+hy^3\big)\,,$$

over the locus of triples of two degree 1 polynomials and one degree 3 polynomial whose product has no triple roots, contains every isomorphism class of bielliptic curves.

Moreover, since the map to $\mathbb{P}^1$ is canonical, we conclude that any isomorphism between two curves in this family is via a linear change of variables on x and y, scalar multiplication and exchanging on z and w, and rescaling the cubic polynomial. In other words, any such isomorphism is via the action of $\mathrm{GL}_2 \times G \times \mathbb{G}_m$, with GL_2 acting on (x,y), and G acting on (z,w), and $\mathbb{G}_m$ acting on a, b, c, d with weight -1 and on e, f, g, h with weight $+1$. By inspection, the subgroup of $\mathrm{GL}_2 \times G \times \mathbb{G}_m$ acting trivially is the image of $\mathbb{G}_m$ under the map $t \mapsto \big(t\cdot\mathbf{1}, t^4\cdot\mathbf{1}, t^3\big)$.

We conclude that the stack of bielliptic curves is isomorphic to the quotient of the locus of triples of polynomials of degrees 1, 1, and 3, whose product has no triple root, by the natural action of $(\mathrm{GL}_2 \times G \times \mathbb{G}_m)/\mathbb{G}_m$ constructed above.

To recast this in a somewhat nicer form, we observe that $(\mathrm{GL}_2 \times G \times \mathbb{G}_m)/\mathbb{G}_m$ is itself isomorphic to $\mathrm{GL}_2 \times G$ via the map $\mathrm{GL}_2 \times G \times \mathbb{G}_m \to \mathrm{GL}_2 \times G$ defined by

$$A \times B \times t \mapsto \frac{\det A}{t}\cdot A \times \left(\frac{\det A}{t}\right)^4 \cdot B\,,$$

whose kernel is, by inspection, the image of $\mathbb{G}_m$ under the map $t \mapsto \big(t\cdot\mathbf{1}, t^4\cdot\mathbf{1}, t^3\big)$. An inverse to this map is given by $A \times B \mapsto A \times B \times \det A$.

11.2 Our test family

Our test family will be an open substack (obtained by excising the triple root loci) of the space

$$[V_3(1) \times (V_1(-1) \boxtimes W_{-2})]/(G \times \mathrm{GL}_2)\,;$$

here, we view elements of $V_3(1)$ as cubic polynomials $ex^3+fx^2y+gxy^2+hy^3$ and elements of $V_1(-1) \boxtimes W_{-2}$ as pairs of linear polynomials $(ax+by, cx+dy)$. There is a family of genus 2 curves defined over this quotient by

$$z^2 = (ax+by)\cdot\big(ex^3+fx^2y+gxy^2+hy^3\big)\,,$$
$$w^2 = (cx+dy)\cdot\big(ex^3+fx^2y+gxy^2+hy^3\big)\,,$$

where GL_2 acts naturally on (x, y) and G acts naturally on (z, w). Away from the triple root loci, these curves are stable. This will serve as our test family; the discussion of Section 11.1 shows that this is the universal family of bielliptic curves.

11.3 The zero sections

As in Section 8, we will use the formulas in Section 4 to calculate loci in the total space of vector bundles with the origin excised, by pulling back the corresponding classes from the projectivization.

So we begin by excising the origins of $V_3(1)$ and $V_1(-1) \boxtimes W_{-2}$; as in Section 4, this imposes relations given by the vanishing of the Euler classes $e(V_3(1)) = c_4(V_3(1))$ and $e(V_1(-1) \boxtimes W_{-2}) = c_4(V_1(-1) \boxtimes W_{-2})$, respectively.

Write a_1 and a_2 for the Chern roots of V and b_1 and b_2 for the Chern roots of W_2. Then the Euler class of $V_3(1)$ is

$$\begin{aligned} e(V_3(1)) &= (\alpha_1 - 3a_1)(\alpha_1 - 3a_2)(\alpha_1 - 2a_1 - a_2)(\alpha_1 - a_1 - 2a_2) \\ &= \big(\alpha_1^2 - 3(a_1 + a_2)\alpha_1 + 9a_1a_2\big)\big(\alpha_1^2 - 3(a_1 + a_2)\alpha_1 + 2(a_1 + a_2)^2 + a_1a_2\big) \\ &= \big(\alpha_1^2 - 3\alpha_1^2 + 9\alpha_2\big)\big(\alpha_1^2 - 3\alpha_1^2 + 2\alpha_1^2 + \alpha_2\big) \\ &= 9\alpha_2^2 - 2\alpha_1^2\alpha_2\,. \end{aligned}$$

Similarly, the Euler class of $V_1(-1) \boxtimes W_{-2}$ is

$$\begin{aligned} e(V_1(-1) \boxtimes W_{-2}) &= (-\alpha_1 - a_1 - b_1)(-\alpha_1 - a_2 - b_1)(-\alpha_1 - a_1 - b_2)(-\alpha_1 - a_2 - b_2) \\ &= \big((\alpha_1 + b_1)^2 + (\alpha_1 + b_1)(a_1 + a_2) + a_1a_2\big) \\ &\quad \cdot \big((\alpha_1 + b_2)^2 + (\alpha_1 + b_2)(a_1 + a_2) + a_1a_2\big) \\ &= \big((\alpha_1 + b_1)^2 + (\alpha_1 + b_1)\alpha_1 + \alpha_2\big) \cdot \big((\alpha_1 + b_2)^2 + (\alpha_1 + b_2)\alpha_1 + \alpha_2\big) \\ &= (b_1b_2)^2 + 3\alpha_1(b_1 + b_2)b_1b_2 + \big(2\alpha_1^2 + \alpha_2\big)(b_1 + b_2)^2 \\ &\quad + \big(5\alpha_1^2 - 2\alpha_2\big)b_1b_2 + \big(6\alpha_1^3 + 3\alpha_1\alpha_2\big)(b_1 + b_2) + \big(2\alpha_1^2 + \alpha_2\big)^2 \\ &= (4\beta_2)^2 + 3\alpha_1(2\beta_1 + \gamma)(4\beta_2) + \big(2\alpha_1^2 + \alpha_2\big)(2\beta_1 + \gamma)^2 \\ &\quad + \big(5\alpha_1^2 - 2\alpha_2\big)(4\beta_2) + \big(6\alpha_1^3 + 3\alpha_1\alpha_2\big)(2\beta_1 + \gamma) + \big(2\alpha_1^2 + \alpha_2\big)^2 \\ &= 4\alpha_1^4 + 12\alpha_1^3\beta_1 + 8\alpha_1^2\beta_1^2 + 4\alpha_1^2\alpha_2 + 6\alpha_1\alpha_2\beta_1 + 4\alpha_2\beta_1^2 \\ &\quad + 20\alpha_1^2\beta_2 + 24\alpha_1\beta_1\beta_2 + \alpha_1\alpha_2\gamma + \alpha_2\beta_1\gamma + \alpha_2^2 - 8\alpha_2\beta_2 + 16\beta_2^2 \\ &\quad + \big(2\alpha_1^2 + a2\big)\big(\gamma^2 + \beta_1\gamma\big) + \big(3\alpha_1^3 + 3\alpha_1^2\beta_1 + \alpha_1\alpha_2 + \alpha_2\beta_1 + 6\alpha_1\beta_2\big)(2\gamma) \\ &= 4\alpha_1^4 + 12\alpha_1^3\beta_1 + 8\alpha_1^2\beta_1^2 + 4\alpha_1^2\alpha_2 + 6\alpha_1\alpha_2\beta_1 + 4\alpha_2\beta_1^2 \\ &\quad + 20\alpha_1^2\beta_2 + 24\alpha_1\beta_1\beta_2 + \alpha_1\alpha_2\gamma + \alpha_2\beta_1\gamma + \alpha_2^2 - 8\alpha_2\beta_2 + 16\beta_2^2\,. \end{aligned}$$

Next we excise the locus in $V_1(-1) \boxtimes W_{-2}$ where one of the linear forms is zero. In other words, we excise from $(V_1(-1) \boxtimes W_{-2} \smallsetminus (0 \times 0))/(G \times \mathrm{GL}_2)$ the closed substack Z given by

$$\begin{aligned} &\big(V_1(-1) \boxtimes (L_{-2} \times 0) \cup V_1(-1) \boxtimes (0 \times L_{-2})\big)/(G \times \mathrm{GL}_2) \\ &\qquad \simeq (V_1(-1) \boxtimes (L_{-2} \times 0))/(\mathbb{G}_m \times \mathbb{G}_m \times \mathrm{GL}_2)\,. \end{aligned}$$

As a module over $\mathrm{CH}^*(BG \times B\mathrm{GL}_2)$, the Chow ring $\mathrm{CH}^*(Z)$ is generated by 1 and t_1, so it suffices to determine the pushforwards of 1 and t_1 to $\mathrm{CH}^*([V_1(-1) \boxtimes W_{-2} \smallsetminus (0 \times 0)]/(G \times \mathrm{GL}_2))$.

If we write a_1 and a_2 for the Chern roots of V, then applying Lemma 4.9, we see that the class of

$$\begin{aligned}&[V_1(-1) \boxtimes (L_{-2} \times 0) \smallsetminus (0 \times 0)]/(\mathbb{G}_m \times \mathbb{G}_m \times \mathrm{GL}_2) \\ &\qquad \subset [V_1(-1) \boxtimes (L_{-2} \oplus L_{-2}) \smallsetminus (0 \times 0)]/(\mathbb{G}_m \times \mathbb{G}_m \times \mathrm{GL}_2)\end{aligned}$$

is given by

$$\begin{aligned}c_2(V_1(-1) \boxtimes (0 \times L_{-2})) &= (-\alpha_1 - 2t_2 - a_1)(-\alpha_1 - 2t_2 - a_2) \\ &= (\alpha_1 + 2t_2)^2 + (\alpha_1 + 2t_2)(a_1 + a_2) + a_1 a_2 \\ &= (\alpha_1 + 2t_2)^2 + (\alpha_1 + 2t_2)\alpha_1 + \alpha_2 = 4t_2^2 + 6\alpha_1 t_2 + 2\alpha_1^2 + \alpha_2 \,.\end{aligned}$$

Excising this locus therefore introduces relations given by the pushforwards of

$$4t_2^2 + 6\alpha_1 t_2 + 2\alpha_1^2 + \alpha_2 \quad \text{and} \quad \big(4t_2^2 + 6\alpha_1 t_2 + 2\alpha_1^2 + \alpha_2\big)t_1$$

along $B(\mathbb{G}_m \times \mathbb{G}_m) \to BG$; by Lemma 7.1, these are

$$\begin{aligned}&4 \cdot \big(\beta_1^2 + \beta_1\gamma - 2\beta_2\big) + 6\alpha_1 \cdot (\beta_1 + \gamma) + \big(2\alpha_1^2 + \alpha_2\big) \cdot 2 \\ &\quad = 4\alpha_1^2 + 6\alpha_1\beta_1 + 4\beta_1^2 + 2\alpha_2 - 8\beta_2 + (3\alpha_1 + 2\beta_1) \cdot 2\gamma \\ &\quad = 4\alpha_1^2 + 6\alpha_1\beta_1 + 4\beta_1^2 + 2\alpha_2 - 8\beta_2 \,,\end{aligned}$$

respectively

$$\begin{aligned}&4 \cdot (\beta_1\beta_2 + \gamma\beta_2) + 6\alpha_1 \cdot 2\beta_2 + \big(2\alpha_1^2 + \alpha_2\big) \cdot (\beta_1 + \gamma) \\ &\quad = 2\alpha_1^2\beta_1 + \alpha_2\beta_1 + 12\alpha_1\beta_2 + 4\beta_1\beta_2 + \alpha_2\gamma + \big(\alpha_1^2 + 2\beta_2\big) \cdot 2\gamma \\ &\quad = 2\alpha_1^2\beta_1 + \alpha_2\beta_1 + 12\alpha_1\beta_2 + 4\beta_1\beta_2 + \alpha_2\gamma \,.\end{aligned}$$

To summarize, the relations obtained by excising the loci where one form is zero are thus

$$\begin{aligned}&9\alpha_2^2 - 2\alpha_1^2\alpha_2 = 0 \,, \\ &4\alpha_1^2 + 6\alpha_1\beta_1 + 4\beta_1^2 + 2\alpha_2 - 8\beta_2 = 0 \,, \\ &2\alpha_1^2\beta_1 + \alpha_2\beta_1 + 12\alpha_1\beta_2 + 4\beta_1\beta_2 + \alpha_2\gamma = 0 \,, \\ &4\alpha_1^4 + 12\alpha_1^3\beta_1 + 8\alpha_1^2\beta_1^2 + 4\alpha_1^2\alpha_2 + 6\alpha_1\alpha_2\beta_1 + 4\alpha_2\beta_1^2 + 20\alpha_1^2\beta_2 \\ &\qquad + 24\alpha_1\beta_1\beta_2 + \alpha_1\alpha_2\gamma + \alpha_2\beta_1\gamma + \alpha_2^2 - 8\alpha_2\beta_2 + 16\beta_2^2 = 0 \,.\end{aligned} \tag{11.1}$$

11.4 The tautological classes

Since

$$H^0(\omega_C) = \left\langle \frac{xdy - ydx}{z}, \frac{xdy - ydx}{w} \right\rangle,$$

with the natural action of GL_2 on (x, y) and G on (z, w), the Hodge bundle is just the dual of the standard representation of G tensored with the dual of the determinant representation of GL_2. In particular, its Chern classes are

$$\lambda_1 = -\beta_1 - 2\alpha_1 \quad \text{and} \quad \lambda_2 = \alpha_1^2 + \alpha_1\beta_1 + \beta_2 \,.$$

To find the pullback of δ_1 to our family, we note that the preimage of the boundary stratum Δ_1 is the image of the Segre map $V_1(-1) \times W_{-2} \to V_1(-1) \boxtimes W_{-2}$. So by Lemma 4.10,

$$\delta_1 = c_1(V_1(-1)) + c_1(W_{-2}) = -3\alpha_1 - 2\beta_1 + \gamma \,.$$

11.5 Excision of triple root loci

To excise the triple root loci, we will find the classes of the triple root loci in the product of projectivizations

$$[\mathbb{P}V_3(1) \times \mathbb{P}(V_1(-1) \boxtimes W_{-2})]/(G \times \mathrm{GL}_2) \simeq [\mathbb{P}V_3 \times \mathbb{P}(V_1 \boxtimes W_{-2})]/(G \times \mathrm{GL}_2)$$

and pull back to $[V_3(1) \times (V_1(-1) \boxtimes W_{-2})]/(G \times \mathrm{GL}_2)$ by substituting the hyperplane class on $\mathbb{P}V_3$ for α_1 and the hyperplane class on $\mathbb{P}(V_1 \boxtimes W_{-2})$ for $-\alpha_1$.

We begin by excising the locus where the cubic form has a triple root. When we substitute the hyperplane class on $\mathbb{P}V_3$ for α_1, the recursion relation from Lemma 4.2 yields

$$s_3^0 = 1\,, \quad s_3^1 = \alpha_1\,, \quad s_3^2 = 3\alpha_2\,, \quad s_3^3 = \alpha_1\alpha_2\,. \tag{11.2}$$

Consequently, we can apply Lemma 4.8 to calculate that excising this locus imposes relations given by the vanishing of

$$3s_3^2 - 6\alpha_1 s_3^1 + 6\big(\alpha_1^2 - \alpha_2\big) = 3\alpha_2 \quad \text{and} \quad s_3^3 - 6\alpha_2 s_3^1 + 6\alpha_1\alpha_2 = \alpha_1\alpha_2\,.$$

We next excise the locus where all three forms share a common factor; this is the image of the closed immersion

$$[\mathbb{P}V_2 \times \mathbb{P}V_1 \times \mathbb{P}W_{-2}]/(G \times \mathrm{GL}_2) \to [\mathbb{P}V_3 \times \mathbb{P}(V_1 \boxtimes W_{-2})]/(G \times \mathrm{GL}_2)$$

defined by $(f, g, h) \mapsto (f \cdot g, g \boxtimes h)$.

Note that the hyperplane class on $\mathbb{P}V_3$, respectively on $\mathbb{P}(V_1 \boxtimes W_{-2})$, pulls back to the sum of the hyperplane classes on $\mathbb{P}V_2$ and $\mathbb{P}V_1$, respectively on $\mathbb{P}V_1$ and $\mathbb{P}W_{-2}$. Since these, along with the hyperplane class on $\mathbb{P}W_{-2}$, generate $\mathrm{CH}^*([\mathbb{P}V_2 \times \mathbb{P}V_1 \times \mathbb{P}W_{-2}]/(G \times \mathrm{GL}_2))$ as a module over $\mathrm{CH}^*(BG \times B\mathrm{GL}_2)$, the relations obtained by excision are all generated by the pushforwards of the fundamental class 1 and hyperplane class $w = c_1(\mathcal{O}_{\mathbb{P}W_{-2}}(1))$ under this map.

To compute these two pushforwards, we factor this map as the composition of the diagonal map on the $\mathbb{P}V_1$-factor

$$[\mathbb{P}V_2 \times \mathbb{P}V_1 \times \mathbb{P}W_{-2}]/(G \times \mathrm{GL}_2) \to [\mathbb{P}V_2 \times (\mathbb{P}V_1 \times \mathbb{P}V_1) \times \mathbb{P}W_{-2}]/(G \times \mathrm{GL}_2) \tag{11.3}$$

followed by multiplication and the Segre map

$$[(\mathbb{P}V_2 \times \mathbb{P}V_1) \times (\mathbb{P}V_1 \times \mathbb{P}W_{-2})]/(G \times \mathrm{GL}_2) \to [\mathbb{P}V_3 \times \mathbb{P}(V_1 \boxtimes W_{-2})]/(G \times \mathrm{GL}_2)\,. \tag{11.4}$$

When we write x_1 and x_2 for the hyperplane sections on the two $\mathbb{P}V_1$-factors, Lemma 4.5 gives that the pushforwards of 1 and w under (11.3) are $x_1 - \alpha_1 + x_2$ and $x_1 w - \alpha_1 w + x_2 w$. Using Lemmas 4.4 and 4.10, and writing $x = c_1(\mathcal{O}_{\mathbb{P}(V_1 \boxtimes W_{-2})}(1))$, we see that the pushforwards of these classes under (11.4) are

$$\begin{aligned} &s_3^1 \cdot [2x + c_1(V_1) + c_1(W_{-2})] - \alpha_1 \cdot 3s_3^0 \cdot [2x + c_1(V_1) + c_1(W_{-2})] \\ &+ 3s_3^0 \cdot \big[x^2 + c_1(W_{-2})x + c_2(W_{-2}) - c_2(V_1)\big]\,, \end{aligned}$$

respectively

$$\begin{aligned} &s_3^1 \cdot \big[x^2 + c_1(V_1)x + c_2(V_1) - c_2(W_{-2})\big] - \alpha_1 \cdot 3s_3^0 \cdot \big[x^2 + c_1(V_1)x + c_2(V_1) - c_2(W_{-2})\big] \\ &+ 3s_3^0 \cdot \big[x^3 + (c_1(V_1) + c_1(W_{-2}))x^2 + (c_2(V_1) + c_1(V_1)c_1(W_{-2}) + c_2(W_{-2}))x \\ &\qquad + c_1(V_1)c_2(W_{-2}) + c_2(V_1)c_1(W_{-2})\big]\,. \end{aligned}$$

From Lemma 4.2, we have that $s_3^0 = 1$ and that s_3^1 is the hyperplane class on $\mathbb{P}V_3$, which we must substitute for α_1. Substituting x for $-\alpha_1$, the relations we obtain from excising this locus are the vanishing of

$$\alpha_1 \cdot [-2\alpha_1 + c_1(V_1) + c_1(W_{-2})] - 3\alpha_1 \cdot [-2\alpha_1 + c_1(V_1) + c_1(W_{-2})] \\ + 3\big[\alpha_1^2 - c_1(W_{-2})\alpha_1 + c_2(W_{-2}) - c_2(V_1)\big]\,,$$

respectively the vanishing of

$$\alpha_1 \cdot \big[\alpha_1^2 - c_1(V_1)\alpha_1 + c_2(V_1) - c_2(W_{-2})\big] - 3\alpha_1 \cdot \big[\alpha_1^2 - c_1(V_1)\alpha_1 + c_2(V_1) - c_2(W_{-2})\big] \\ + 3\big[-\alpha_1^3 + (c_1(V_1) + c_1(W_{-2}))\alpha_1^2 - (c_2(V_1) + c_1(V_1)c_1(W_{-2}) + c_2(W_{-2}))\alpha_1 \\ + c_1(V_1)c_2(W_{-2}) + c_2(V_1)c_1(W_{-2})\big]\,.$$

Since $V_1 = V^*$, we have $c_1(V_1) = -\alpha_1$ and $c_2(V_1) = \alpha_1$; similarly, since $W_{-2} = W_2^*$, Lemma 7.2 gives $c_1(W_{-2}) = \gamma - 2\beta_1$ and $c_2(W_{-2}) = 4\beta_2$. Substituting these in and collecting like terms, we see that our relations are simply the vanishing of

$$9\alpha_1^2 + 10\alpha_1\beta_1 + \alpha_1\gamma - 3\alpha_2 + 12\beta_2 - 3\alpha_1 \cdot 2\gamma = 9\alpha_1^2 + 10\alpha_1\beta_1 + \alpha_1\gamma - 3\alpha_2 + 12\beta_2\,,$$

respectively the vanishing of

$$\alpha_2\gamma - 10\alpha_1^3 - 12\alpha_1^2\beta_1 - 5\alpha_1\alpha_2 - 6\alpha_2\beta_1 - 16\alpha_1\beta_2 + \big(3\alpha_1^2 + \alpha_2\big)\cdot 2\gamma \\ = \alpha_2\gamma - 10\alpha_1^3 - 12\alpha_1^2\beta_1 - 5\alpha_1\alpha_2 - 6\alpha_2\beta_1 - 16\alpha_1\beta_2\,.$$

Finally, we excise the locus where the cubic form is divisible by the square of one of the linear forms. This locus is the norm from $\mathbb{G}_m \times \mathbb{G}_m$ to G of the locus where the square of the first linear form divides the cubic form (this latter locus is only $(\mathbb{G}_m \times \mathbb{G}_m \times \mathrm{GL}_2)$-equivariant).

Away from the locus where one of the linear forms is zero, we have a $(\mathbb{G}_m \times \mathbb{G}_m \times \mathrm{GL}_2)$-equivariant natural map $\mathbb{P}(V_1 \boxtimes W_{-2}) \to \mathbb{P}V_1 \times \mathbb{P}V_1$ induced by projection from the subspaces $V_1 \boxtimes (L_{-2} \times 0)$ and $V_1 \boxtimes (0 \times L_{-2})$. This induces a map

$$\mathbb{P}V_3 \times \mathbb{P}(V_1 \boxtimes W_{-2}) \to \mathbb{P}V_3 \times \mathbb{P}V_1 \times \mathbb{P}V_1\,; \tag{11.5}$$

the locus we want to excise can then be described as the pullback under this map of the image of the map

$$\mathbb{P}V_1 \times \mathbb{P}V_1 \times \mathbb{P}V_1 \to \mathbb{P}V_3 \times \mathbb{P}V_1 \times \mathbb{P}V_1 \tag{11.6}$$

defined by $(f, g, h) \mapsto \big(fg^2, g, h\big)$.

Since the pullback under (11.5) of the hyperplane class on the first $\mathbb{P}V_1$-factor differs from the hyperplane class on $\mathbb{P}(V_1 \boxtimes W_{-2})$ by $2t_2$, we just need to push forward the generators of $\mathrm{CH}^*([\mathbb{P}V_1 \times \mathbb{P}V_1 \times \mathbb{P}V_1]/(\mathbb{G}_m \times \mathbb{G}_m \times \mathrm{GL}_2))$ along (11.6), substitute the hyperplane class on $\mathbb{P}V_3$ for α_1, substitute the hyperplane class on the first $\mathbb{P}V_1$-factor for $-\alpha_1 - 2t_2$, and take norms along $B(\mathbb{G}_m \times \mathbb{G}_m) \to BG$. Moreover, since pullback along (11.6) is surjective by inspection, the only generators we need to use are 1 and t_1 (which generate $\mathrm{CH}^*(B(\mathbb{G}_m \times \mathbb{G}_m))$ as a module over $\mathrm{CH}^*(BG)$).

To implement this, we factor (11.6) as a composition of a triple diagonal map on the middle $\mathbb{P}V_1$-factor

$$\mathbb{P}V_1 \times \mathbb{P}V_1 \times \mathbb{P}V_1 \to \mathbb{P}V_1 \times (\mathbb{P}V_1 \times \mathbb{P}V_1 \times \mathbb{P}V_1) \times \mathbb{P}V_1 \tag{11.7}$$

followed by multiplication

$$(\mathbb{P}V_1 \times \mathbb{P}V_1 \times \mathbb{P}V_1) \times \mathbb{P}V_1 \times \mathbb{P}V_1 \to \mathbb{P}V_3 \times \mathbb{P}V_1 \times \mathbb{P}V_1\,. \tag{11.8}$$

Write x_i (with $1 \leqslant i \leqslant 5$) for the hyperplane class on the ith $\mathbb{P}V$-factor. From Lemma 4.6, the pushforward of the fundamental class is

$$x_2x_3 + x_3x_4 + x_4x_2 - (x_2 + x_3 + x_4)\alpha_1 + \alpha_1^2 - \alpha_2 .$$

Using Lemma 4.4 and (11.2), and writing x for the hyperplane class on $\mathbb{P}V_1$ that we must substitute for $-\alpha_1 - 2t_2$, we deduced that the pushforward of the fundamental class along (11.8) is then

$$\begin{aligned} 1 \mapsto s_3^2 &+ 2x \cdot 2s_3^1 - 2\alpha_1 \cdot 2s_3^1 - \alpha_1 x \cdot 6s_3^0 + \left(\alpha_1^2 - \alpha_2\right) \cdot 6s_3^0 \\ &= 3\alpha_2 + 2(-\alpha_1 - 2t_2) \cdot 2\alpha_1 - 2\alpha_1 \cdot 2\alpha_1 - \alpha_1(-\alpha_1 - 2t_2) \cdot 6 + \left(\alpha_1^2 - \alpha_2\right) \cdot 6 \\ &= 4\alpha_1 t_2 + 4\alpha_1^2 - 3\alpha_2 . \end{aligned}$$

Our relations are thus given by the vanishing of the norms along $B(\mathbb{G}_m \times \mathbb{G}_m) \to BG$ of the classes

$$4\alpha_1 t_2 + 4\alpha_1^2 - 3\alpha_2 \quad \text{and} \quad \left(4\alpha_1 t_2 + 4\alpha_1^2 - 3\alpha_2\right)t_1 .$$

By Lemma 7.1, these are

$$4\alpha_1(\beta_1 + \gamma) + 8\alpha_1^2 - 6\alpha_2 = 4\alpha_1\beta_1 + 8\alpha_1^2 - 6\alpha_2 + 2 \cdot 2\gamma = 4\alpha_1\beta_1 + 8\alpha_1^2 - 6\alpha_2 ,$$

respectively

$$\begin{aligned} 4\alpha_1 \cdot 2\beta_2 + \left(4\alpha_1^2 - 3\alpha_2\right)(\beta_1 + \gamma) &= 8\alpha_1\beta_2 + 4\alpha_1^2\beta_1 - 3\alpha_2\beta_1 + \alpha_2\gamma + \left(2\alpha_1^2 - 2\alpha_2\right) \cdot 2\gamma \\ &= 8\alpha_1\beta_2 + 4\alpha_1^2\beta_1 - 3\alpha_2\beta_1 + \alpha_2\gamma . \end{aligned}$$

To summarize, the relations obtained by excising the triple root loci are thus:

$$\begin{aligned} &3\alpha_2 = 0 , \quad \alpha_1\alpha_2 = 0 , \quad 4\alpha_1\beta_1 + 8\alpha_1^2 - 6\alpha_2 = 0 , \\ &8\alpha_1\beta_2 + 4\alpha_1^2\beta_1 - 3\alpha_2\beta_1 + \alpha_2\gamma = 0 , \\ &9\alpha_1^2 + 10\alpha_1\beta_1 + \alpha_1\gamma - 3\alpha_2 + 12\beta_2 = 0 , \\ &\alpha_2\gamma - 10\alpha_1^3 - 12\alpha_1^2\beta_1 - 5\alpha_1\alpha_2 - 6\alpha_2\beta_1 - 16\alpha_1\beta_2 = 0 . \end{aligned} \tag{11.9}$$

11.6 Chow ring of our test family

From our expressions for the tautological classes on $\overline{M}_2$ in Section 11.4, we can express α_1, β_1, and β_2 in terms of γ, δ_1, λ_1, and λ_2:

$$\begin{aligned} &\alpha_1 = -2\lambda_1 + \delta_1 - \gamma , \quad \beta_1 = 3\lambda_1 - 2\delta_1 + 2\gamma , \\ &\beta_2 = \lambda_2 - \alpha_1^2 - \alpha_1\beta_1 = 2\lambda_1^2 - 3\lambda_1\delta_1 + \delta_1^2 + 3\lambda_1\gamma - 2\delta_1\gamma + \gamma^2 + \lambda_2 . \end{aligned}$$

Subtracting the relation $4\alpha_1^2 + 6\alpha_1\beta_1 + 4\beta_1^2 + 2\alpha_2 - 8\beta_2 = 0$ (see (11.1)) from the relation $3\alpha_2 = 0$ (see (11.9)), we obtain

$$\alpha_2 = 4\alpha_1^2 + 6\alpha_1\beta_1 + 4\beta_1^2 - 8\beta_2 = 2\lambda_1\delta_1 - 2\lambda_1\gamma - 8\lambda_2 .$$

Substituting these expressions into all of our previous relations in (11.1), (11.9), and Theorem 5.2, we check the ideal the resulting relations generate is as follows.

THEOREM 11.3. *The Chow ring of the moduli space B of bielliptic curves is generated by γ, δ_1, λ_1, and λ_2, subject to the relations*

$$\begin{aligned} &2\gamma = 0 , \quad \gamma^2 + \lambda_1\gamma = 0 , \quad \delta_1^2 + \delta_1\gamma + 8\lambda_1^2 - 12\lambda_2 = 0 , \quad 24\lambda_1^2 - 48\lambda_2 = 0 , \\ &2\delta_1^2 + 2\lambda_1\delta_1 = 0 , \quad 20\lambda_1\lambda_2 - 4\delta_1\lambda_2 = 0 , \quad 8\lambda_1^3 - 8\lambda_1\lambda_2 = 0 . \end{aligned}$$

In particular, reducing modulo 2, we obtain

$$\mathrm{CH}^*(B) \otimes \mathbb{Z}/2\mathbb{Z} \simeq \mathbb{Z}[\gamma, \delta_1, \lambda_1, \lambda_2]/\left(\gamma^2 + \lambda_1\gamma, \delta_1^2 + \delta_1\gamma\right),$$

from which we conclude that the classes

$$\delta_1(\delta_1 + \lambda_1)\lambda_1^2, \quad \delta_1(\delta_1 + \lambda_1)\lambda_2, \quad \text{and} \quad \delta_1(\delta_1 + \lambda_1)\left(\lambda_1^2 + \lambda_2\right)$$

are all nonvanishing in $\mathrm{CH}^*(B) \otimes \mathbb{Z}/2\mathbb{Z}$ and thus in $\mathrm{CH}^*\left(\overline{M}_2\right) \otimes \mathbb{Z}/2\mathbb{Z}$, as desired.

Acknowledgements

First and foremost, the author would like to profusely thank Akhil Mathew for many extremely fruitful conversations (as well as comments on the manuscript and assistance locating references). In these conversations, he suggested several mathematical insights that were critical to the success of this project; these include the technique used to calculate $\mathrm{CH}^*(\Delta_1)$ in Section 8 and the idea of using the higher Chow groups of $\overline{M}_2 \smallsetminus \Delta_1$. This paper would not have existed without his help!

The author would also like to thank Ken Ono, Sam Payne, Burt Totaro, Ravi Vakil, David Zureick-Brown, and the anonymous referee, for various helpful discussions and comments on this manuscript.

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