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Satellites of spherical subgroups

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Abstract

Let G be a complex connected reductive algebraic group. Given a spherical subgroup $H \subset G$ and a subset I of the set of spherical roots of G/H, we define, up to conjugation, a spherical subgroup $H_I \subset G$ of the same dimension as H, called a satellite. We investigate various interpretations of the satellites. We also show a close relation between the Poincaré polynomials of the two spherical homogeneous spaces G/H and G/H_I.

1. Introduction

Let G be a complex connected reductive algebraic group. An irreducible algebraic G-variety X is said to be *spherical* if X is normal and if a Borel subgroup of G has an open orbit in X. An algebraic subgroup $H \subset G$ is called *spherical* if the homogeneous space G/H is spherical. In this article, we introduce and explore new combinatorial invariants attached to spherical subgroups of G from several perspectives.

From now on, we fix a Borel subgroup B of G and a spherical subgroup H of G. We first collect a few standard facts about the spherical homogeneous space G/H. There is no loss of generality in assuming that BH is Zariski dense in G. A *spherical embedding* of G/H is a normal G-variety X together with a G-equivariant open embedding $j\colon G/H \hookrightarrow X$ which allows us to identify H with the stabilizer of the point $x := j(H) \in X$ such that $Gx \cong G/H$. Therefore, we will always regard G/H as the open G-orbit in X whenever we write (X, x) for a G/H-embedding.

Let $U := BH/H$ be the open dense B-orbit in G/H. The *colors* of G/H are the irreducible components of the complement of the open set U in G/H, that is, the irreducible B-invariant divisors in G/H. Denote by $\mathscr{D} = \mathscr{D}(G/H)$ the set of colors of G/H. The stabilizer of U in G is a parabolic subgroup P of G containing B. Then $U \cong B/B \cap H$ is an affine variety isomorphic to $(\mathbb{C}^*)^r \times \mathbb{C}^s$, where s is the dimension of the unipotent radical P^u of P. The number r is the *rank* of the spherical homogeneous space G/H. Let $\mathscr{M}(U)$ be the free abelian group of rank r consisting of all invertible regular functions f in the affine coordinate ring $\mathbb{C}[U]$ such that $f(H) = 1$. Any such regular function $f \in \mathscr{M}(U)$ is a B-eigenfunction associated with some weight $\omega(f) \in \mathscr{X}^*(B)$, where $\mathscr{X}^*(B)$ is the lattice of characters of the Borel subgroup B. The map $f \mapsto \omega(f)$ yields a natural embedding of the lattice $\mathscr{M}(U)$ into the lattice $\mathscr{X}^*(B)$. The *weight lattice* of G/H is

$$M = \mathscr{X}^*(G/H) := \omega(\mathscr{M}(U))\,.$$

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We denote by $\langle -, - \rangle \colon M \times N \to \mathbb{Z}$ the natural pairing, where $N := \mathrm{Hom}(M, \mathbb{Z})$ is the dual lattice.

A spherical embedding X of G/H is said to be *elementary* if it consists of exactly two G-orbits: a dense orbit X^0 isomorphic to G/H and a closed orbit X' of codimension one. In this case, X is always smooth, and it is uniquely determined by the restriction of the divisorial valuation $v_{X'} \colon \mathbb{C}(G/H) \to \mathbb{Z}$ to the free abelian group $\mathscr{M}(U) \subset \mathbb{C}(G/H)$, where $\mathbb{C}(G/H)$ is the field of rational functions on G/H [LV83, §§ 8.3, 7.5 and 8.10]. Using the isomorphism $\omega \colon \mathscr{M}(U) \xrightarrow{\sim} M$, we may view this restriction as an element $n_{X'} \in N$. This induces a bijection between the set $\mathcal{V} = \mathcal{V}(G/H)$ of $\mathbb{Q}$-valued discrete G-invariant valuations of $\mathbb{C}(G/H)$ and a convex polyhedral cone in $N_{\mathbb{Q}} := N \otimes_{\mathbb{Z}} \mathbb{Q}$; see [LV83, Proposition 7.4] or [Kno91, Corollary 1.8]. The latter is usually referred to as the *valuation cone* of the spherical homogeneous space G/H. In this way, we may identify $\mathcal{V}$ with the valuation cone of G/H. According to the Luna–Vust theory [LV83, § 8], any G-equivariant embedding X of G/H can be combinatorially described by a *colored fan* in the r-dimensional vector space $N_{\mathbb{Q}}$. The classification of G-equivariant embeddings (the Luna–Vust theory) was first studied in a more general setting. For spherical varieties, this was later simplified and extended to any characteristic by Knop in [Kno91].

In this classification, elementary embeddings correspond to uncolored cones $\mathbb{Q}_{\geqslant 0} v$ in $N_{\mathbb{Q}}$ with v a nonzero lattice point in $N \cap \mathcal{V}$. We shall denote by (X_v, x_v) the elementary embedding of G/H with closed G-orbit X'_v corresponding to a nonzero lattice point v in $N \cap \mathcal{V}$. It is uniquely defined up to G-equivariant isomorphism.

It is known that $\mathcal{V}$ is a cosimplicial cone; that is, there exist $k \leqslant r$ linearly independent primitive lattice vectors $s_1, \ldots, s_k \in M$, called the *spherical roots* of G/H, such that

$$\mathcal{V} = \{ n \in N_{\mathbb{Q}} \colon \langle s_i, n \rangle \leqslant 0 \text{ for all } i \text{ with } 1 \leqslant i \leqslant k \} .$$

The definition of spherical roots goes back to Luna [Lun01, § 1.2]. The set $\Sigma = \Sigma(G/H)$ of spherical roots is one of the components of the *homogeneous spherical data associated with G/H*, a fundamental combinatorial invariant [Lun01, § 2.1]. The spherical roots $s_1, \ldots, s_k$ are nonnegative integral linear combinations of the positive roots with respect to the Borel subgroup $B \subset G$; see [BP87, § 4].

1.1. Main illustrating example. Symmetric homogeneous spaces are homogeneous spaces for which the stabilizer of a typical point is the fixed-point set of a nontrivial group involution. It is a standard fact that they are spherical [Vus74]. The lattice $N = \mathrm{Hom}(M, \mathbb{Z})$ of a symmetric homogeneous space G/H is naturally endowed with a root system, and the spherical roots of G/H can be expressed by means of this root system [Vus90, § 2]. The classification of all embeddings of symmetric homogeneous spaces was carried out in [Vus90].

In order to illustrate the content of our paper, we consider the well-known example (see, for example, [Bri97, Example 4.1]) of the group G viewed as a symmetric homogeneous space under $G \times G$ acting by left and right multiplication: $(a, b)c := acb^{-1}$ for $a, b, c \in G$. The stabilizer of the base point 1 is $\Delta(G)$, where Δ is the diagonal closed immersion. Let B^- be the unique Borel subgroup such that $T := B^- \cap B$ is a maximal torus of G, and let S be the set of simple roots of G relatively to (B, T). The spherical roots of G (relatively to the Borel subgroup $B^- \times B$) are the pairs $(-\alpha, \alpha)$, where $\alpha \in S$. Thus $M = \mathscr{X}^*(G)$ is identified with the root lattice of S, and the valuation cone is the antidominant Weyl chamber.

With every subset $I \subset S$, one associates a spherical subgroup $\Delta(G)_I \subset G \times G$ as follows.

Let P_I and P_I^- be the two opposite parabolic subgroups containing B and B^-, respectively, such that their common Levi component $L_I := P_I^- \cap P_I$ has I as a set of simple roots. Then set

$$\Delta(G)_I := \big(P_I^u \times (P_I^-)^u\big)\Delta(L_I)\,,$$

where P_I^u and $(P_I^-)^u$ are the unipotent radicals of P_I and P_I^-, respectively. We notice that $(B^- \times B)\Delta(G)_I$ is Zariski dense in $G \times G$. We call the subgroups $\Delta(G)_I$, where $I \subset S$, the *satellites of* $\Delta(G)$. All satellites of $\Delta(G)$ have the same dimension $\dim G = \dim \Delta(G)$. Since $(B^- \times B) \cap \Delta(G) = (B^- \times B) \cap \Delta(G)_I$ for all $I \subset S$, the weight lattices of the satellites are all equal to M:

$$M = \mathscr{X}^*(G) = \mathscr{X}^*((G \times G)/\Delta(G)_I) \quad \text{for all} \quad I \subset S\,.$$

We refer to Example 6.3 for another approach to this example for $G = \mathrm{GL}_n$.

When G is adjoint, that is, with trivial center, De Concini and Procesi constructed a wonderful compactification of the symmetric homogeneous space $G \cong (G\times G)/\Delta(G)$; see [DP83, § 3.4]. The datum of the satellites $\{\Delta(G)_I\colon\ I \subset S\}$ allows describing the stabilizers of points in the wonderful compactification (cf. Section 1.3) and computing its Poincaré polynomial (cf. Section 1.5 and Example 7.5).

Our purpose is to generalize the above construction to an arbitrary spherical homogeneous space G/H.

1.2. Brion subgroups. Choose a nonzero primitive lattice point v in the valuation cone $\mathcal{V} \subset N_{\mathbb{Q}}$. The total space $\tilde{X}_v$ of the normal bundle to the G-invariant divisor X'_v in the elementary embedding X_v has a natural G-action. One can show that $\tilde{X}_v$ is again a spherical G-variety containing exactly two G-orbits: the zero section of the normal bundle (it is isomorphic to X'_v) and its open complement $\tilde{X}_v^0$. The stabilizer H_v of a point in the open G-orbit $\tilde{X}_v^0$ will be referred to as the *Brion subgroup* corresponding to $v \in \mathcal{V}$. Such a subgroup was first studied by Brion in [Bri90, § 1.1]. [1] It is defined up to G-conjugation. The above construction of the subgroup H_v will be explained in greater detail in Section 2. In Section 3, we indicate an algebraic description of Brion subgroups.

We can alternatively define the Brion subgroup H_v by considering the total space $\tilde{X}_v^\vee$ of the conormal bundle of X'_v in X_v. To be more specific, if $\big(\tilde{X}_v^\vee\big)^0$ stands for the open complement of the zero section of $\tilde{X}_v^\vee$, then $\big(\tilde{X}_v^\vee\big)^0 \cong G/H_v$ and $\big(\tilde{X}_v^\vee\big)^0$ is spherical [Pan99, § 2.1]. Furthermore, $\tilde{X}_v^\vee$ is an elementary spherical embedding of G/H_v. Note that $\tilde{X}_v^\vee$ has a natural symplectic structure whose induced G-action is Hamiltonian.

If $v = 0$, we simply declare the Brion subgroup H_v to be H.

Our main observation is the following (see Theorem 4.4 for a more precise formulation).

Theorem 1.1. *The Brion subgroup $H_v \subset G$ depends, up to conjugation, only on the minimal face $\mathcal{V}(v)$ of the valuation cone $\mathcal{V}$ containing v.*

Theorem 1.1 is proved in Section 4. It follows from the description of the *homogeneous spherical data* introduced by Luna [Lun01] associated with the homogeneous space G/H_v (see Theorem 4.4). Set

$$I(v) := \{s_i \in \Sigma\colon\ \langle s_i, v\rangle = 0\}\,.$$

[1] In [Bri90], the subgroup H is assumed to be equal to its normalizer, but this assumption is superfluous in the construction of the subgroup H_v.

Then the minimal face $\mathcal{V}(v)$ of the valuation cone $\mathcal{V}$ containing v is determined by the equations $\langle s_i, v\rangle = 0$, where s_i runs through the set $I(v)$.

In particular, if v is an interior lattice point of the valuation cone $\mathcal{V}$, then $I(v) = \varnothing$ and the Brion subgroup H_v is known to be *horospherical*; that is, H_v contains a maximal unipotent subgroup of G; see [BP87, Corollary 3.8] and also Proposition 2.4(2). The horospherical satellite $H_\varnothing$ of H is closely related to a flat deformation of the spherical homogeneous space G/H to the horospherical one $G/H_\varnothing$ (this has already appeared in papers of Alexeev–Brion [AB04] and Kaveh [Kav05], using previous work of Popov [Pop87]).

We remark that for any subset $I \subset \Sigma$, there exists a primitive lattice point $v \in \mathcal{V}$ such that $I(v) = I$, unless $I = \Sigma$ and Σ is a basis of $M_\mathbb{Q} := M \otimes_\mathbb{Z} \mathbb{Q}$. This provides a natural reversing bijection between the faces of the valuation cone $\mathcal{V}$ and the subsets $I \subset \Sigma$. Therefore, the conjugacy class of a Brion subgroup H_v depends only on the subset $I(v)$. Given a subset $I \subset \Sigma$ with $I \neq \Sigma$, we define up to conjugacy the *spherical satellite* H_I of H as a Brion subgroup H_v, where $I = I(v)$. For $I = \Sigma$, we simply set $H_\Sigma := H$.

1.3. Stabilizers in toroidal compactifications. A spherical embedding X of G/H is called *simple* if X has a unique closed G-orbit, and a simple spherical embedding X is called *toroidal* if no color $D \in \mathscr{D}$ contains the unique closed G-orbit in its closure. Thus a spherical embedding X is simple and toroidal if and only if its fan has no color and contains a unique cone of maximal dimension.

If the number $k = |\Sigma|$ of spherical roots of G/H is equal to the rank r (that is, the maximal possible number of spherical roots), then the valuation cone $\mathcal{V}$ is strictly convex (that is, it contains no line). If so, according to Luna–Vust theory, G/H admits a canonical normal projective G-equivariant simple toroidal compactification X such that we have a natural bijection $I \leftrightarrow X_I$ between the subsets $I \subset \Sigma$ and the G-orbits $X_I \subset X$. We show that the normalizer of the stabilizer of any point in the G-orbit X_I is equal, up to G-conjugacy, to the normalizer $N_G(H_I)$ of the spherical satellite H_I (cf. Proposition 5.1). This stabilizer is exactly $N_G(H_I)$ if $I = \varnothing$. In particular, the unique closed G-orbit $X_\varnothing$ is the smooth projective homogeneous G-variety $G/N_G(H_\varnothing)$.

It is well known that in the case where G/H is horospherical, for each spherical embedding X of G/H, a conjugate of the stabilizer of any point $x \in X$ contains the subgroup H. In [BM13, Proposition 2.4], such stabilizers are explicitly described. Our notion of satellites allows us to extend this fact to the more general case of spherical varieties. More precisely, since all satellites H_I for $I \subset \Sigma$ have the same weight lattice M (cf. Propositions 2.4(1) and 3.1), we may view every lattice point $m \in M$ as a character of the normalizer $N_G(H_I)$ of H_I in G (cf. [Bri97, Theorem 4.3] or Lemma 5.2), and we obtain the following.

Theorem 1.2. *Let X be a simple toroidal spherical embedding of G/H corresponding to an uncolored cone σ in $N_\mathbb{Q}$. Denote by $I(\sigma)$ the set of all spherical roots in Σ that vanish on σ. Choose a point x' in the unique closed G-orbit X' of X, and let $G_{x'}$ be its stabilizer in G. Then, up to a conjugation, we have the inclusions*

$$H_{I(\sigma)} \subset G_{x'} \subset N_G(H_{I(\sigma)})\,.$$

Moreover, there is a homomorphism (given by Lemma 5.2) from $N_G(H_{I(\sigma)})$ to the torus $\mathrm{Hom}(\sigma^\perp \cap M, \mathbb{C}^*)$ whose kernel is $G_{x'}$.

Since every simple spherical embedding of G/H is dominated by a simple toroidal one, it follows that the stabilizer of a point in any spherical embedding of G/H contains, up to conjugation, one of the spherical satellites H_I (cf. Corollary 5.3).

1.4. Limits of stabilizers of points in arc spaces. Let $\mathscr{O} := \mathbb{C}[[t]]$ be the ring of formal power series, and let $\mathscr{K} := \mathbb{C}((t))$ be its field of fractions. We write $(G/H)(\mathscr{K})$ for the set of $\mathscr{K}$-valued points of the spherical homogeneous space G/H, and $G(\mathscr{O})$ for the group of $\mathscr{O}$-valued points of the reductive group G. Luna and Vust established in [LV83, § 4] a natural bijection between lattice points v in $\mathcal{V}$ and $G(\mathscr{O})$-orbits in $(G/H)(\mathscr{K})$. One can choose a representative $\widehat{\lambda}_v$ of the $G(\mathscr{O})$-orbit attached to v using an appropriate one-parameter subgroup in G. The reader is referred to Section 6 for more details.

Theorem 1.3. *The Brion subgroup H_v consists of limits as t goes to 0 of elements in the stabilizer in $G(\mathscr{O})$ of $\widehat{\lambda}_v$.*

Theorem 1.3 is helpful in practice to compute the satellites; cf. Examples 6.3 and 6.4. Its proof is carried out in Section 6.

1.5. Poincaré polynomial of satellites. With every complex algebraic variety X, we associate its *virtual Poincaré polynomial* $P_X(t)$, uniquely determined by the following properties:

(1) We have $P_X(t) = P_Y(t) + P_{X\setminus Y}(t)$ for every closed subvariety Y of X.
(2) If X is smooth and complete, then $P_X(t) = \sum_m \dim H^m(X)t^m$ is the usual Poincaré polynomial.

Then $P_X(t) = P_Y(t)P_F(t)$ for every fibration $F \hookrightarrow X \to Y$ which is locally trivial for the Zariski topology. The existence of the virtual Poincaré polynomial for every complex algebraic variety X follows from the existence of Deligne's mixed Hodge structure on the cohomology groups $H^\bullet_c(X)$ of X with compact supports and complex coefficients, which yields a polynomial $E_X(s,t)$ in two variables (see, for example, [BD96, § 3] or [DK87, § 1]). We have $P_X(t) = E_X(-t,-t)$.

In [BP02], Brion and Peyre investigated the virtual Poincaré polynomials of homogeneous spaces under complex connected linear algebraic groups. In particular, they showed that they are polynomials in t^2. They also obtained a factorization result for the virtual Poincaré polynomial of *regular embeddings* in the sense of [BDP90] of such homogeneous spaces, provided that the stabilizer of a typical point is connected.

Accordingly, if X is a spherical homogeneous space G/H, then the function $P_X\big(t^{1/2}\big)$ is a polynomial in t. By additivity, if X is a spherical embedding of a spherical homogeneous space G/H, then $P_X\big(t^{1/2}\big)$ is a polynomial since X is a disjoint finite union of G-orbits which are all spherical homogeneous spaces. Therefore, for our purpose, it will be convenient to set

$$\tilde{P}_X(t) := P_X\big(t^{1/2}\big)\,.$$

For example, $\tilde{P}_X(t) = (t-1)^r t^s$ if $X \cong (\mathbb{C}^*)^r \times \mathbb{C}^s$. Abusing the notation, we continue to call the function $\tilde{P}_X$ the *virtual Poincaré polynomial* of X.

A G-variety X is said to be *wonderful* (cf. [Lun96, Introduction]) of rank r if it is complete and smooth, with an open G-orbit whose complement is the union of r smooth irreducible G-divisors $Z_1, \ldots, Z_r$ such that any subset of these irreducible G-divisors has a transversal and nonempty intersection, and these intersections are exactly all G-orbit closures of X.

A wonderful embedding of G/H (that is, a G/H-embedding which is a wonderful G-variety) is unique, up to a G-equivariant isomorphism, if it exists (see, for example, [Lun01, § 1.3] or [Tim11,

§ 30]), and this happens for instance if $H = N_G(H)$, in which case X is the canonical toroidal embedding; see [Kno96, Corollary 7.2] and [Lun96, Introduction]. Wonderful embeddings were first introduced by De Concini and Procesi [DP83] for symmetric spaces, motivated by problems in enumerative geometry.

Assume that G/H admits a wonderful compactification X of rank r. Then the spherical homogeneous space G/H has rank r, and X contains exactly 2^r G-orbits X_I that are parameterized by all possible subsets $I \subset \{1, \dots, r\}$ such that the closure $\overline{X_I}$ is the intersection of all irreducible G-divisors Z_i with $i \notin I$.

As the wonderful embedding X is the disjoint union $\bigsqcup_I X_I$, we have

$$\tilde{P}_X(t) = \sum_I \tilde{P}_{X_I}(t)\,.$$

On the other hand, for $I \subset \{1, \dots, r\}$, the total space $\tilde{X}_I$ of the normal bundle to the smooth subvariety $\overline{X_I} \subset X$ is a simple spherical toroidal embedding of the spherical homogeneous space G/H_I. It is a direct sum of line bundles. Indeed, since $\overline{X_I} = \bigcap_{i \notin I} Z_i$, the normal bundle $N_{\overline{X_I}|X}$ of X along $\overline{X_I}$ is the direct sum of the restrictions to $\overline{X_I}$ of the line bundles $N_{Z_i|X}$, where $i \notin I$. One has a natural locally trivial torus fibration $f_I \colon G/H_I \to X_I$ for the Zariski topology whose fiber is isomorphic to $(\mathbb{C}^*)^{r-|I|}$. Hence, we get

$$\tilde{P}_X(t) = \sum_I \frac{\tilde{P}_{G/H_I}(t)}{(t-1)^{r-|I|}}\,. \tag{1.1}$$

Because the valuation cone $\mathcal{V} \subset N_{\mathbb{Q}}$ of G/H is generated by a basis $e_1, \dots, e_r$ of the lattice N, every lattice point $v \in N \cap \mathcal{V}$ can be written as a nonnegative integral linear combination $v = l_1 e_1 + \dots + l_r e_r$. The set of spherical roots $\{s_1, \dots, s_r\} \subset M$ forms a dual basis to the basis $\{e_1, \dots, e_r\}$ of the dual lattice N. Let $\mathcal{V}_I^\circ$ be the relative interior of the face $\mathcal{V}_I$ defined by the conditions $\langle x, s_i\rangle = 0$ for all $i \in I$. Using the power expansion

$$\frac{1}{t-1} = \frac{t^{-1}}{\left(1 - t^{-1}\right)} = \sum_{j \geqslant 1} t^{-j}\,,$$

we can rewrite (1.1) as

$$\tilde{P}_X(t) = \sum_I \frac{\tilde{P}_{G/H_I}(t)}{(t-1)^{r-|I|}} = \sum_I \tilde{P}_{G/H_I}(t) \sum_{v \in \mathcal{V}_I^\circ} t^{\kappa(v)}\,, \tag{1.2}$$

where κ is the linear function on N which takes value -1 on the lattice vectors $e_1, \dots, e_r$.

Formula (1.2) is an analog of the one we established for the Poincaré polynomial of a smooth projective toroidal horospherical variety [BM13]. The general version of both formulas for the stringy E-function of an arbitrary $\mathbb{Q}$-Gorenstein spherical embedding X will appear in an upcoming paper by the authors. It provides a motivic interpretation of the Brion–Peyre factorization result (cf. [BP02, Theorem 2]) for the virtual Poincaré polynomial of X. This has motivated us to investigate the ratio of the two virtual Poincaré polynomials,

$$\frac{\tilde{P}_{G/H_I}(t)}{\tilde{P}_{G/H}(t)}\,,$$

for general spherical homogeneous space G/H. Looking at some series of examples, we came to the following conjecture.

Conjecture 1.4. Let G/H be an arbitrary spherical homogeneous space. Then for any satellite H_I of the spherical subgroup $H \subset G$, the ratio of the two virtual Poincaré polynomials

$$\frac{\tilde{P}_{G/H_I}(t)}{\tilde{P}_{G/H}(t)}$$

is always a polynomial R_I in t^{-1} with integral coefficients.

We prove this conjecture in several examples, notably in the case where the spherical group H is connected (cf. Theorem 7.3) and in the case where the spherical homogeneous space G/H is of rank one (cf. Theorem 7.9).

Notation. Unless otherwise specified, we keep the notation used in Section 1 further on in the article.

2. Brion subgroups, first properties and examples

The section starts with a number of results of [LV83, BLV86] and [BP87] concerning elementary embeddings. Recall that P^u stands for the unipotent radical of the parabolic subgroup P.

Definition 2.1 ([BLV86, § 4.2] or [BP87, § 2.9]). Let L be a Levi subgroup of P, and let C be the neutral component of its center. We say that L is *H-adapted* if the following conditions are satisfied:

(1) We have $P \cap H = L \cap H$.

(2) The intersection $P \cap H$ contains the derived subgroup (L, L) of L.

(3) For any elementary embedding (X, x) of G/H, the action of P^u on $Y := Px \cup Px'$, with Px' the open P-orbit in the closed orbit of X, induces an isomorphism of algebraic varieties $P^u \times (\overline{Cx} \cap Y) \to Y$.

The existence of H-adapted Levi subgroups is established in [BLV86, §§ 3 and 4.2]. Fix such an H-adapted Levi subgroup L. Since P has an open dense orbit in G/H, any $f \in \mathscr{M}(U) \cong M$ is determined by its weight $\omega(f) \in \mathscr{X}^*(P)$ since f is a P-eigenvector in $\mathbb{C}(G/H)$. Furthermore, the restriction to $P \cap H$ of $\omega(f)$ is trivial. But the sublattice of $\mathscr{X}^*(P)$ consisting of characters whose restriction to $P \cap H$ is trivial identifies with the lattice of characters $\mathscr{X}^*(\mathbb{T})$ of the algebraic torus $\mathbb{T} := C/C \cap H$ since $P = P^u L$ and H contains (L, L). Here, C denotes, as in Definition 2.1, the neutral component of the center of L. As a result, we get $M \cong \mathscr{X}^*(\mathbb{T})$. Hence, by duality, the lattice $N_{\mathbb{Q}} \cong \operatorname{Hom}(M, \mathbb{Q})$ identifies with $\mathscr{X}_*(\mathbb{T}) \otimes_{\mathbb{Z}} \mathbb{Q}$, where $\mathscr{X}_*(\mathbb{T})$ is the free abelian group of one-parameter subgroups of $\mathbb{T}$.

Definition 2.2. A one-parameter subgroup λ of C is said to be *adapted to the elementary embedding* (X, x) if $\lim\limits_{t \to 0} \lambda(t)x$ exists and belongs to the open P-orbit of the closed orbit.

Through the identification $N_{\mathbb{Q}} \cong \mathscr{X}_*(\mathbb{T}) \otimes_{\mathbb{Z}} \mathbb{Q}$, the primitive lattice points in $N \cap \mathcal{V}$ are in bijection with the indivisible one-parameter subgroups of $\mathbb{T}$, adapted to the different elementary embeddings of G/H. More precisely, if v is a nonzero lattice point in $N \cap \mathcal{V}$, then any one-parameter subgroup of C adapted to (X_v, x_v) corresponds to a point of $N \cap \mathcal{V}$, which is equivalent to v; cf. [BP87, § 2.10].

We now give a concrete construction of the Brion subgroups H_v, following [Bri90, § 1.1]. Fix a nonzero lattice point $v \in N \cap \mathcal{V}$, and choose a one-parameter subgroup λ_v of C adapted to (X_v, x_v). Thus,

$$x'_v := \lim_{t \to 0} \lambda_v(t) x_v$$

belongs to the closed G-orbit X'_v. The stabilizer H'_v of x'_v in G acts on the one-dimensional normal space $T_{x'_v}(X_v)/T_{x'_v}(X'_v)$ via a character χ_v. The character χ_v is nontrivial since H'_v contains the image of λ_v, which acts nontrivially on $T_{x'_v}(Cx_v \cup Cx'_v)/T_{x'_v}(Cx'_v)$; cf. [Bri90, § 1.1]. We define H_v to be the kernel of χ_v in H'_v. If $v = 0$, we simply set $H_v = H$.

Definition 2.3. We call the subgroup H_v for $v \in N \cap \mathcal{V}$ a *Brion subgroup* of G.

Brion subgroups are defined up to G-conjugacy. They have the same dimension as H, and they are spherical by [Bri90, Proposition 1.2] and the proof of [Bri90, Proposition 1.3].

Proposition 2.4. *Let $v \in N \cap \mathcal{V}$ be a lattice point in the valuation cone of G/H.*

(1) *The weight lattice $\mathscr{X}^*(G/H_v)$ of the spherical homogeneous space G/H_v is equal to the weight lattice M of G/H.*

(2) *The valuation cone of G/H_v is equal to $\mathcal{V} + \mathbb{Q}v$. In particular, $v \in \mathcal{V}^\circ$ if and only if the spherical subgroup H_v is horospherical, where $\mathcal{V}^\circ$ is the relative interior of $\mathcal{V}$.*

Proof. We can certainly assume that v is nonzero, since the statement is clear for $v = 0$.

(1) Denote by $\tilde{X}_v$ the total space of the normal bundle to the G-invariant divisor X'_v in X_v. We use the deformation of X_v to the normal bundle $\tilde{X}_v$, and we follow the ideas of the proof of [Bri90, Proposition 1.2].

Consider the product $X_v \times \mathbb{C}$ as a spherical embedding of $G/H \times \mathbb{C}^*$ endowed with the natural $(G \times \mathbb{C}^*)$-action. It is the toroidal simple spherical embedding corresponding to the two-dimensional cone in $N_{\mathbb{Q}} \oplus \mathbb{Q}$ generated by $(v, 0)$ and $(0, 1)$. Let

$$p\colon \tilde{Y}_v \longrightarrow X_v \times \mathbb{C}$$

be the blow-up of the codimension two closed $(G \times \mathbb{C}^*)$-orbit $X'_v \times \{0\}$ in $X_v \times \mathbb{C}$. The preimage by p of $X_v \times \{0\}$ contains a divisor which is isomorphic to X_v; denote its complement by $\tilde{Z}_v$. Let $\rho\colon \tilde{Z}_v \to \mathbb{C}$ be the restriction of the composition $\mathrm{pr}_2 \circ p$ to $\tilde{Z}_v \subset \tilde{Y}_v$, where pr_2 is the projection from $X_v \times \mathbb{C}$ to $\mathbb{C}$. Then ρ defines a fibration over $\mathbb{C}$ such that $\rho^{-1}(\mathbb{C}^*)$ identifies with $X_v \times \mathbb{C}^*$, and the fiber $\rho^{-1}(0)$ is isomorphic to $\tilde{X}_v$. The fibration ρ deforms the spherical variety X_v to $\tilde{X}_v$. We have a $(G \times \mathbb{C}^*)$-action on $\tilde{Z}_v$, so that $\big(\tilde{Z}_v, p^{-1}(x_v, 1)\big)$ is a smooth embedding of $(G \times \mathbb{C}^*)/H \times \{1\}$ containing a closed codimension one orbit isomorphic to $\tilde{X}^0_v \cong G/H_v$. Choose a point $\tilde{z}'_v \in \tilde{Z}_v$ in this closed $(G \times \mathbb{C}^*)$-orbit of $\tilde{Z}_v$, and let $\tilde{H}'_v$ be its stabilizer in $G \times \mathbb{C}^*$. Up to G-conjugacy, $\tilde{H}'_v$ is the image of H'_v by the map

$$j\colon\ H'_v \longrightarrow G \times \mathbb{C}^*, \quad h \longmapsto (h, \chi_v(h))\,.$$

As a G-variety, $(G \times \mathbb{C}^*)/\tilde{H}'_v$ is isomorphic to G/H_v, where G acts on $G \times \mathbb{C}^*$ by left multiplication on the left factor.

By the Luna–Vust correspondence [LV83, §§ 3.3, 7.5 and 8.10] between embeddings of $G \times \mathbb{C}^*$ and colored fans in the valuation cone of $(G \times \mathbb{C}^*)/H \times \{1\}$, the uncolored fan of the blow-up $\tilde{Y}_v$ is obtained from that of $X_v \times \mathbb{C}$ by adding the half-line generated by $(v, 0) + (0, 1) = (v, 1)$ in the lattice $N \oplus \mathbb{Z}$. Moreover, the ray generated by the lattice point $(v, 1)$ corresponds to the elementary embedding of $(G \times \mathbb{C}^*)/H \times \{1\}$ with closed $(G \times \mathbb{C}^*)$-orbit $\tilde{X}^0_v \cong G/H_v \cong (G \times \mathbb{C}^*)/\tilde{H}'_v$.

According to [BP87, § 3.6], we get

$$\mathscr{X}^*((G\times\mathbb{C}^*)/\tilde{H}'_v) = (M\oplus\mathbb{Z})\cap(v,1)^\perp\,,$$

since $\mathscr{X}^*(G\times\mathbb{C}^*)\cong\mathscr{X}^*(B)\oplus\mathbb{Z}$. The weight lattice of $(G\times\mathbb{C}^*)/\tilde{H}'_v$ as a G-variety is then the image of $(M\oplus\mathbb{Z})\cap(v,1)^\perp$ by the projection map

$$\mathscr{X}^*(B)\oplus\mathbb{Z}\to\mathscr{X}^*(B)\,.$$

This image is nothing but M. Therefore, the weight lattice of G/H_v is M since $G/H_v\cong(G\times\mathbb{C}^*)/\tilde{H}'_v$ as a G-variety.

(2) It is easily seen (see the proof of [BP87, Corollary 3.7]) that

$$\mathcal{V}\big((G\times\mathbb{C}^*)/H\times\{1\}\big)\cong\mathcal{V}\oplus\mathbb{Q}(0,1)\,.$$

On the other hand, by [BP87, Theorem 3.6], the valuation cone $\mathcal{V}((G\times\mathbb{C}^*)/\tilde{H}'_v)\cong\mathcal{V}(G/H_v)$ is the quotient of $\mathcal{V}((G\times\mathbb{C}^*)/H)$ by $\mathbb{Q}(v,1)$. But the image of $\mathcal{V}\oplus\mathbb{Q}(0,1)$ by the isomorphism $N_\mathbb{Q}\xrightarrow{\sim}(N_\mathbb{Q}+\mathbb{Q}(0,1))/\mathbb{Q}(v,1)$ is $\mathcal{V}+\mathbb{Q}v$, whence $\mathcal{V}(G/H_v)\cong\mathcal{V}+\mathbb{Q}v$. In particular, $v\in\mathcal{V}^\circ$ if and only if $\mathcal{V}(G/H_v)=\mathcal{V}+\mathbb{Q}v=N_\mathbb{Q}$. Since the equality $\mathcal{V}(G/H_v)=N_\mathbb{Q}$ holds if and only if H_v is horospherical by [Kno91, Corollary 6.2], the statement follows. □

Example 2.5. In the case where H is horospherical, all Brion subgroups associated with H are equal to H up to G-conjugacy. This can be proved by describing all elementary embeddings of G/H as induced from elementary embeddings of the torus $C/(C\cap H)$.

3. Algebraic approach to Brion subgroups

In this section, we investigate an algebraic approach to Brion subgroups, and we furnish another proof of Proposition 2.4(1). Let us fix a primitive lattice point v in $N\cap\mathcal{V}$.

Let $K:=\mathbb{C}(G/H)$ be the field of rational functions over G/H. The field K is a B-module, and it is the quotient field of the affine coordinate ring $A:=\mathbb{C}[U]$ of the open B-orbit $U\subset G/H$. From the restriction of the valuation $v\colon K\to\mathbb{Z}$ to the subring $A\subset K$, we define the subring

$$A'_v:=\{f\in A\colon\ v(f)\geqslant 0\text{ or }f=0\}\,.$$

By the local structure theorem for toroidal embeddings ([BP87, § 3.4] or [Tim11, Theorem 29.1]), the ring A'_v is isomorphic to the coordinate ring of a B-invariant open subset $U'\subset X_v$ containing U. Note that U' is the union of U and the open B-orbit in X'_v. Additionally, U' is isomorphic to $(\mathbb{C}^*)^{r-1}\times\mathbb{C}^{s+1}$, with r and s as in the introduction. Since U' is an affine variety with a nonempty intersection with X'_v, the vanishing ideal I_v in $A'_v=\mathbb{C}[U']$ of the divisor $X'_v\cap U'$ is generated by one element f_v. One has

$$I_v:=\{f\in A\colon\ v(f)>0\text{ or }f=0\}\,.$$

Then, for any $i\in\mathbb{Z}$, we set

$$I_v^i:=\{f\in A\colon\ v(f)\geqslant i\}\,.$$

We have a decreasing filtration

$$\cdots\supset I_v^{-j}\supset\cdots\supset I_v^{-2}\supset I_v^{-1}\supset A'_v\supset I_v\supset I_v^2\supset\cdots\supset I_v^i\supset\cdots$$

such that

$$A=\bigcup_{i\in\mathbb{Z}}I_v^i\,.$$

The group B acts on this filtration, and each I_v^i is a free A'_v-module of rank one generated by f_v^i. We set

$$\widetilde{A'_v} := \sum_{i\geqslant 0} I_v^i/I_v^{i+1} \quad \text{and} \quad \widetilde{A_v} := \sum_{i\in\mathbb{Z}} I_v^i/I_v^{i+1}\,,$$

so that $\widetilde{A'_v} = \operatorname{gr} A'_v$ and $\widetilde{A_v} = \operatorname{gr} A$ with respect to the above filtrations. Denote by $\overline{A'_v}$ the quotient ring A'_v/I_v, which is the affine coordinate ring of $X'_v \cap U'$. The rings $\widetilde{A'_v}$, A'_v and $\overline{A'_v}$ are naturally endowed with a B-action. Note that $\widetilde{A'_v}$ is the affine coordinate ring of the normal bundle to the divisor $X_v \cap U'$ in U'. Moreover, $\widetilde{A'_v}$ is the affine coordinate of a B-invariant open subset $\widetilde{U}'$ in $\tilde{X}_v$ which has a nonempty intersection with the closed G-orbit in $\tilde{X}_v$. As for $\widetilde{A_v}$, it is the affine coordinate ring of the open B-orbit $\widetilde{U}$ in $\widetilde{U}'$.

Proposition 3.1. *There is a natural B-equivariant isomorphism between the rings $\widetilde{A_v}$ and A which induces an isomorphism between the groups of B-eigenfunctions in $\widetilde{A_v}$ and in A. In particular, the lattices of weights of the spherical homogeneous spaces G/H and G/H_v are the same.*

Proof. Given any $m \in \mathscr{X}^*(B)$, the dimension of the set $K_m^{(B)}$ of B-eigenvectors of K associated with m is at most one for spherical H. For each $m \in M$, choose a generator f_m of $K_m^{(B)}$, and set

$$\hat{M} := \sum_{m\in M} \mathbb{C} f_m\,.$$

Then $\hat{M}$ is a $\mathbb{C}$-vector subspace of K. Similarly, let $K_v := \mathbb{C}(G/H_v)$, let M_v be the weight lattice of G/H_v, and set

$$\hat{M}_v := \sum_{m\in M_v} \mathbb{C} f_{v,m}\,,$$

where for each $m \in M_v$, the element $f_{v,m}$ is a generator of the set of B-eigenvectors of K_v associated with m. Let $T \in I_v/I_v^2$ be the class of the generator $f_v \in I_v$. Then we have

$$\widetilde{A'_v} \cong \overline{A'_v}[T] \cong \overline{A'_v} \oplus I_v/I_v^2 \oplus \cdots \oplus I_v^i/I_v^{i+1} \oplus \cdots \subset \overline{A'_v}\left[T, T^{-1}\right] \cong \sum_{i\in\mathbb{Z}} I_v^i/I_v^{i+1} = \widetilde{A_v}\,.$$

For each $i \in \mathbb{Z}$, write $\mathscr{M}_v^i$ for the image of the projection map $I_v^i \cap \hat{M} \longrightarrow I_v^i/I_v^{i+1}$, and set

$$\mathscr{M}_v := \sum_{i\in\mathbb{Z}} \mathscr{M}_v^i\,.$$

Let us first show that $\mathscr{M}_v = \hat{M}_v$. Observing that the intersection $I_v^i \cap \hat{M}$ is the direct sum of the lines generated by f_m, where $m \in M$ and $\langle m, v\rangle \geqslant i$, we deduce that $\mathscr{M}_v^i$ is the direct sum of the lines generated by f_m, where $m \in M$ and $\langle m, v\rangle = i$. By construction, $\mathscr{M}_v$ is contained in K_v and consists of B-eigenvectors, whence a first inclusion $\mathscr{M}_v \subset \hat{M}_v$.

To establish the converse inclusion, let us prove that for each $m \in M_v$, we have $f_{v,m} \in \mathscr{M}_v$. Note that K_v admits a $\mathbb{C}^*$-action induced from the $\mathbb{Z}_{\geqslant 0}$-grading on A_v which commutes with the G-action. Then for every $m \in M_v$, the B-eigenvector $f_{v,m}$ is also an eigenvector for the $\mathbb{C}^*$-action, which means that $f_{v,m}$ is homogeneous with respect to the $\mathbb{Z}$-grading on K_v. Hence, $f_{v,m}$ is in $\mathscr{M}_v^i$ for some i.

In conclusion,

$$\hat{M}_v = \mathscr{M}_v \cong \sum_{i\in\mathbb{Z}} \sum_{\substack{m\in M,\\ \langle m,v\rangle = i}} \mathbb{C} f_m \cong \hat{M}\,.$$

So the weight lattice in K_v is M. Because the above isomorphisms are B-equivariant, the proposition follows. □

Remark 3.2. Proposition 3.1 can also be deduced from the local structure theorem: $\operatorname{Spec} A \cong P^u \times Z$, where Z is an elementary embedding of $C/C \cap H$.

4. Homogeneous spherical data associated with satellites

The idea of classifying spherical homogeneous spaces G/H in combinatorial terms was proposed by Luna in 2001 [Lun01]. For this purpose, Luna invented in [Lun01, §§ 3 and 4] the *spherical systems* and *Luna diagrams* and conjectured that these combinatorial data can be used to classify spherical homogeneous spaces. This classification is carried out in two steps. The first one is to reduce the classification to the case of *wonderful subgroups*, that is, the subgroups $H \subset G$ such that G/H admits a wonderful completion. The second step is to describe all the wonderful G-varieties.

Luna's conjecture was recently solved due to the long efforts of several researchers. The uniqueness part of Luna's conjecture was proved by Losev in 2009 [Los09]. The existence part was recently completed in a series of papers by Bravi and Pezzini [BP14, BP15, BP16]. Another proof of the existence part, with different methods, has been proposed by Cupit-Foutou [Cup09]. We refer the reader to the introduction of [Avd15], and the references given there, for more about this topic.

In this section, we describe the homogeneous spherical data associated with Brion subgroups in order to prove Theorem 1.1. We start by recalling the definition and basic properties of the homogeneous spherical datum associated with the spherical homogeneous space G/H.

The Borel subgroup B acts on G/H by left multiplication, $(b, gH) \mapsto (bgH)$, and the spherical subgroup H acts on the generalized flag variety $B\backslash G$ by $(h, Bg) \mapsto Bgh^{-1}$. Then we have a natural bijection between the finite cosets $B\backslash(G/H)$ and $(B\backslash G)/H$ which induces a natural bijection $D_i \leftrightarrow D_i'$ sending D_i to $D_i' := B\backslash D_i H$ between the set $\mathscr{D}$ of B-invariant divisors in G/H and the set $\mathscr{D}' := \{D_1', \dots, D_m'\}$ of H-invariant divisors in $B\backslash G$. The Picard group of the generalized flag variety $B\backslash G$ is a free group whose base consists of the classes of line bundles $L_1, \dots, L_n$ such that the space of global sections $H^0(B\backslash G, L_j)$ is the jth fundamental representation (with the highest weight ϖ_j) of the semisimple part of the Lie algebra of G, for $1 \leqslant j \leqslant n$. Therefore, we can associate with every color $D_i \subset G/H$ a nonnegative linear combination of fundamental weights $\sum_{j=1}^n a_{ij}\varpi_j = [D_i'] \in \operatorname{Pic}(B\backslash G)$. Let $A := (a_{ij})_{\substack{1 \leqslant i \leqslant m \\ 1 \leqslant j \leqslant n}}$, and let $C := (c_1, \dots, c_n)$ be the sum of rows of A; that is, $c_j = \sum_{i=1}^m a_{ij}$ and

$$\sum_{i=1}^m [D_i'] = \sum_{j=1}^n c_j \varpi_j \,.$$

It is known (see, for example, [Lun01, Lemma 6.4.2] or [Tim11, Lemma 30.24]) that all entries c_j of the vector C belong to the set $\{0, 1, 2\}$ (so the same statement holds for all entries of the matrix A). The matrix A satisfies certain stronger additional conditions that are used in the classification of colors ([Lun97, §§ 2 and 3] and [Lun01, §§ 1 and 2]). Fix $i \in \{1, \dots, m\}$, and let $J(i)$ be the set of $j \in \{1, \dots, n\}$ such that $a_{ij} \neq 0$.

(1) If $a_{ij} = 2$ for some j, then $a_{il} = 0$ for all $l \neq j$; that is, $[D_i'] = 2\varpi_j$. In this case, the B-invariant divisor $D_i \subset G/H$ is said to be *of type* $2a$.

(2) If the color D_i is not of type $2a$, then

$$[D_i'] = \sum_{j\in J(i)} \varpi_j\,,$$

and either $c_j = 2$ for all $j \in J(i)$ (and the color D_i is called *of type a*), or $c_j = 1$ for all $j \in J(i)$ (and the color D_i is called *of type b*).

Denote by $\mathscr{D}^{2a}$, $\mathscr{D}^a$ and $\mathscr{D}^b$ the sets of colors of type $2a$, type a and type b, respectively.

Let us consider two maps

$$\delta\colon \mathscr{D} \to \operatorname{Pic}(B\backslash G) \quad \text{and} \quad \rho\colon\ \mathscr{D} \to N\,,$$

where $\delta(D_i) := [D_i'] \in \operatorname{Pic}(B\backslash G)$ for $1 \leqslant i \leqslant m$ and $\rho(D_i)$ is the lattice point in N corresponding to the restriction of the divisorial valuation ν_{D_i} of the field $\mathbb{C}(G/H)$ to the group of B-semi-invariants in $\mathbb{C}(G/H)$.

The following statement is the uniqueness part of Luna's program.

Theorem 4.1 ([Los09, Theorem 1]). *The triple $(M, \Sigma, \mathscr{D})$ consisting of the lattice of weights $M \subset \mathscr{X}^*(B)$, the set of spherical roots $\Sigma \subset M$ and the set $\mathscr{D}$ of all B-invariant divisors in G/H, together with the two maps $\delta\colon \mathscr{D} \to \operatorname{Pic}(B\backslash G)$ and $\rho\colon \mathscr{D} \to N$, uniquely determines the spherical subgroup $H \subset G$ up to conjugation.*

Using the natural bijection between the set $S = \{\alpha_1, \dots, \alpha_n\}$ of simple roots of G and the set of fundamental weights, $\alpha_j \leftrightarrow \varpi_j$, we can regard the set $J(i)$ as a subset in S. The set $\mathscr{D}^a$ of colors of type a can be characterized as the set of those B-invariant divisors $D_i \in \mathscr{D}$ for which the set $J(i)$ contains a spherical root; that is, $J(i) \cap \Sigma \neq \varnothing$. Set $S^p := \{\alpha \in S\colon\ c_\alpha = 0\}$; that is, S^p consists of those simple roots $\alpha \in S$ such that the corresponding fundamental weight is not a summand of $\delta(D_i)$ for all colors $D_i \in \mathscr{D}$. By [Lun01, § 2.3], the triple $(M, \Sigma, \mathscr{D})$ is uniquely determined by the quadruple $(M, \Sigma, S^p, \mathscr{D}^a)$, where the set $\mathscr{D}^a$ of colors of type a is equipped with only one map $\rho^a\colon \mathscr{D}^a \to N$, which is the restriction to $\mathscr{D}^a$ of ρ. The quadruple $(M, \Sigma, S^p, \mathscr{D}^a)$ is called the *homogeneous spherical datum* associated with the spherical homogeneous space G/H.

By the Luna general classification, the map $G/H \mapsto (M, \Sigma, S^p, \mathscr{D}^a)$ is a bijection between spherical homogeneous spaces of G (up to G-equivariant isomorphism) and homogeneous spherical data for G.

The following theorem has been proved by Gagliardi and Hofscheier.

Theorem 4.2 ([GH15, Theorem 1.1]). *Let X be a simple toroidal spherical embedding of G/H corresponding to an uncolored cone σ. Then the homogeneous spherical datum of the unique closed G-orbit in X (it is a spherical homogeneous space) is the quadruple*

$$\big(M_0, \Sigma_0, S^p, \mathscr{D}_0^a\big)$$

with

$$M_0 := M \cap \sigma^\perp, \quad \Sigma_0 := \Sigma \cap \sigma^\perp\,, \quad \mathscr{D}_0^a = \big\{D_i \in \mathscr{D}^a\colon\ J(i) \cap \Sigma_0 \neq \varnothing\big\}\,,$$

where the map $\rho_0\colon \mathscr{D}_0^a \to N/\langle\sigma\rangle$ is the restriction to $\mathscr{D}_0^a$ of the map $\rho\colon \mathscr{D}^a \to N$ composed with the natural homomorphism $N \to N/\langle\sigma\rangle$.

Corollary 4.3. *Let $v \in N \cap \mathcal{V}$ be a nonzero lattice point in the valuation cone of G/H. Then the homogeneous spherical datum of the closed divisorial G-orbit X_v' in the elementary spherical embedding X_v of G/H corresponding to v is the quadruple*

$$\big(M_0, \Sigma_0, S^p, \mathscr{D}_0^a\big)$$

with

$$M_0 := M \cap v^\perp\,, \quad \Sigma_0 := \Sigma \cap v^\perp\,, \quad \mathscr{D}_0^a = \big\{D_i \in \mathscr{D}^a\colon\ J(i) \cap \Sigma_0 \neq \varnothing\big\}\,,$$

where the map $\rho_0\colon \mathscr{D}_0^a \to N/\langle v\rangle$ *is the restriction to* $\mathscr{D}_0^a$ *of the map* $\rho^a\colon \mathscr{D}^a \to N$ *composed with the natural homomorphism* $N \to N/\langle v\rangle$*.*

We now compare the homogeneous spherical datum of the spherical homogeneous space G/H to that of the spherical homogeneous space G/H_v corresponding to the Brion subgroup $H_v \subset G$.

Theorem 4.4. *Let* $v \in N \cap \mathcal{V}$ *be a nonzero lattice point in the valuation cone of* G/H*. Then the homogeneous spherical datum of the spherical homogeneous space* G/H_v *is the quadruple*

$$\big(M, \Sigma \cap v^\perp, S^p, \mathscr{D}_v^a\big)\,,$$

where $\mathscr{D}_v^a = \big\{D_i \in \mathscr{D}^a\colon\ J(i) \cap \big(\Sigma \cap v^\perp\big) \neq \varnothing\big\}$ *and the map* $\rho_v^a\colon \mathscr{D}_v^a \to N$ *is the restriction of* $\rho^a\colon \mathscr{D}^a \to N$ *to the subset* $\mathscr{D}_v^a \subset \mathscr{D}^a$*. In particular, the* G*-variety* G/H_v *(and hence the conjugacy class of the Brion subgroup* H_v*) depends only on the minimal face of the valuation cone of* G/H *containing the lattice point* v*.*

Proof. We keep the notation of the proof of Proposition 2.4. Thus X'_v is the spherical homogeneous space G/H'_v, where H'_v is the stabilizer of a point in X'_v, and H_v is contained in H'_v. Let us consider the spherical homogeneous space $G/H \times \mathbb{C}^*$ together with the $(G \times \mathbb{C}^*)$-action. Its lattice of weights is equal to $M \oplus \mathbb{Z}$. Then $G/H_v \cong (G \times \mathbb{C}^*)/H'_v$ is the closed $(G \times \mathbb{C}^*)$-orbit in the elementary spherical embedding of $G/H \times \mathbb{C}^*$ corresponding to the primitive lattice vector $(v, 1) \in N \oplus \mathbb{Z}$. By the definition of colors as B-stable prime divisors, we get a natural bijection between the colors in $G/H \times \mathbb{C}^*$ and those in G/H, as well as a natural bijection between the colors in G/H'_v and those in $(G \times \mathbb{C}^*)/H'_v$ that both preserve the type of colors. On the other hand, by Corollary 4.3, we have a natural bijection between the set of a-colors $D_i \times \mathbb{C}^*$ in $G/H \times \mathbb{C}^*$ such that $J(i) \cap \big(\Sigma \cap v^\perp\big) \neq \varnothing$ and the set of a-colors in $G/H_v \cong (G \times \mathbb{C}^*)/H'_v$. Since the composition of the natural embedding $N \hookrightarrow N \oplus \mathbb{Z}$ and the epimorphism $N \oplus \mathbb{Z} \to (N \oplus \mathbb{Z})/\langle (v, 1)\rangle \cong N$ is the identity map on N, we obtain that the ρ_v^a-images in N of the a-colors in G/H_v and the ρ^a-images of the a-colors D_i in G/H such that $J(i) \cap \big(\Sigma \cap v^\perp\big) \neq \varnothing$ are the same; that is, the map $\rho_v^a\colon \mathscr{D}_v^a \to N$ is the restriction of $\rho^a\colon \mathscr{D}^a \to N$ to the subset $\mathscr{D}_v^a \subset \mathscr{D}^a$. □

According to Theorem 4.4, the following definition is legitimate.

Definition 4.5. Given a subset I of Σ, the *spherical satellite* H_I *of* H *associated with* I (or *with the face* $\mathcal{V}_I$) is the spherical subgroup H_v, where v is any point in the interior of $\mathcal{V}_I = \{n \in N_{\mathbb{Q}}\colon\ \langle s_i, n\rangle = 0 \text{ for all } s_i \in I\}$. Then H_I is well defined up to G-conjugation.

The spherical satellite H_Σ corresponding to the minimal face of $\mathcal{V}$ is G-conjugate to H. On the opposite side, there is a unique, up to a G-conjugation, horospherical satellite $H_\varnothing$ which corresponds to the whole cone $\mathcal{V}$ (cf. Proposition 2.4(2)). Recall that the valuation cone $\mathcal{V}$ is cosimplicial and that it has 2^k faces. Consequently, H has exactly 2^k spherical satellites.

Remark 4.6. When H_v is horospherical, that is, $v \in N \cap \mathcal{V}^\circ$, Theorem 4.4 can be easily proved by the arguments of Example 2.5.

As a consequence of Theorem 4.4, we derive the following description of the homogeneous spherical datum of the spherical homogeneous space G/H_I corresponding to a satellite H_I.

Corollary 4.7. *Given a subset I of Σ, the homogeneous spherical datum of the spherical homogeneous space G/H_I is the quadruple*

$$\left(M, I, S^p, \mathscr{D}_I^a\right),$$

where $\mathscr{D}_I^a = \{D_i \in \mathscr{D}^a\colon\ J(i) \cap I \neq \varnothing\}$ and the map $\rho_I^a\colon \mathscr{D}_I^a \to N$ is the restriction of $\rho^a\colon \mathscr{D}^a \to N$ to the subset $\mathscr{D}_I^a \subset \mathscr{D}^a$.

5. Normalizers of satellites and consequences

Let $N_G(H)$ be the normalizer of H in G. The homogeneous space $G/N_G(H)$ is spherical, and $N_G(H)$ is of finite index in $N_G(N_G(H))$.[2] Therefore, its valuation cone $\widehat{\mathcal{V}} := \mathcal{V}(G/N_G(H))$ is strictly convex by [BP87, Corollary 5.3]. Namely, we have $\widehat{\mathcal{V}} = \mathcal{V}/(\mathcal{V} \cap -\mathcal{V})$ (see, for example, the proof of [Tim11, Theorem 29.1]). Let $\widehat{X}$ be the unique, up to a G-equivariant isomorphism, complete simple toroidal embedding of $G/N_G(H)$. The corresponding uncolored cone of $N_{\mathbb{Q}}$ is $\widehat{\mathcal{V}}$. The G-orbits of $\widehat{X}$ are in bijection with the faces of $\widehat{\mathcal{V}}$. Thus we have a natural bijection $I \leftrightarrow \widehat{X}_I$ between the subsets $I \subset \Sigma$ and the G-orbits $\widehat{X}_I \subset \widehat{X}$ such that $\widehat{X}_\Sigma \cong G/N_G(H)$.

Proposition 5.1. *Let I be a subset of Σ. The normalizer of the stabilizer of any point in the G-orbit $\widehat{X}_I$ is equal to the normalizer $N_G(H_I)$ of the satellite H_I.*

Proof. Pick a nonzero point v in the interior $\mathcal{V}_I^\circ$ of $\mathcal{V}_I$. According to [Tim11, Theorem 15.10], the canonical map $G/H \to G/N_G(H)$ extends to a G-equivariant map $\pi\colon X_v \to \widehat{X}$. Choose a point x'_v in X'_v, and set $\widehat{x}_v := \pi(x'_v)$. Then $\widehat{x}_v$ belongs to the G-orbit of $\widehat{X}$ corresponding to the face $\widehat{\mathcal{V}}_I$ of $\widehat{\mathcal{V}}$, where $\widehat{\mathcal{V}}_I$ is the image of $\mathcal{V}_I$ by the projection map $N_{\mathbb{Q}} \to N_{\mathbb{Q}}/(\mathcal{V} \cap -\mathcal{V})$. Furthermore, since π is G-equivariant, the stabilizer H'_v of x'_v in G is contained in the stabilizer $G_{\widehat{x}_v}$ of $\widehat{x}_v$ in G. Hence, we have the following inclusions of spherical subgroups:

$$H_I = H_v \subset H'_v \subset G_{\widehat{x}_v}\,. \tag{5.1}$$

Let us show that $G_{\widehat{x}_v}$ is contained in $N_G(H'_v)$. We first observe that the fiber of π at x'_v is

$$\pi^{-1}(\pi(x'_v)) = (G_{\widehat{x}_v}).x'_v \cong G_{\widehat{x}_v}/H'_v\,. \tag{5.2}$$

Since $BN_G(H) = BH$ by [BP87, § 5], the set $BN_G(H)$ is dense in G. In addition, the parabolic subgroup P is the set of $s \in G$ such that $sBN_G(H) = BN_G(H)$. Fix an H-adapted Levi subgroup L of P; cf. Definition 2.1.

Let $\mathscr{D}$ and $\widehat{\mathscr{D}}$ be the sets of colors of G/H and $G/N_G(H)$, respectively. Set

$$X_v^\circ := X_v \setminus \bigcup_{D \in \mathscr{D}} \overline{D} \quad \text{and} \quad \widehat{X}^\circ := \widehat{X} \setminus \bigcup_{D \in \widehat{\mathscr{D}}} \overline{D}\,,$$

and apply the local structure theorem for toroidal embeddings (cf. [Tim11, § 29.1]) to $\widehat{X}$. There is a closed L-stable subvariety $\widehat{Z}$ of $\widehat{X}^\circ$ on which (L, L) acts trivially and such that the map

$$P^u \times \widehat{Z} \longrightarrow \widehat{X}^\circ$$

[2] In [BP87, § 5] and [Tim11, Lemma 30.2], it is claimed that $N_G(N_G(H)) = N_G(H)$, but this is not true in general. The simplest counterexample is the following, due to Avdeev. Let $G = \mathrm{GL}_2$, and let H be the set of diagonal matrices whose second entry is 1. Then $N_G(H)$ is the set of diagonal matrices, which is not self-normalizing (see [Avd13, Example 4] for another counterexample with $G = \mathrm{SL}_3$).

is an isomorphism. Moreover, $\widehat{Z}$ is an embedding of the torus $C/C \cap H$, and every G-orbits of $\widehat{X}$ meets $\widehat{Z}$ along L-orbits. Here C is the neutral component of the center of L. Note that $C \cong L/(L,L)$.

Since X_v° is the preimage of $\widehat{X}^\circ$ by π, the local structure for X_v holds with $Z := \pi^{-1}\big(\widehat{Z}\big)$ as an L-stable variety. Every G-orbit of X_v (respectively, $\widehat{X}$) meets Z (respectively, $\widehat{Z}$) and, hence, we may assume $x'_v \in Z$ and $\widehat{x}_v \in \widehat{Z}$.

Using the isomorphisms $X_v^\circ \cong P^u \times Z$ and $\widehat{X}^\circ \cong P^u \times \widehat{Z}$, we identify x_v with $(1, x_v)$ and $\widehat{x}_v$ with $(1, \widehat{x}_v)$ and identify the restriction of π to X_v° with the map

$$P^u \times Z \longrightarrow P^u \times \widehat{Z}\,, \quad (p, y) \longmapsto (p, \pi(y))\,.$$

The closed subvarieties Z and $\widehat{Z}$ are pointwise fixed by (L,L). Consequently, $\pi^{-1}(\pi(x'_v)) \cap X_v^\circ$ coincides through the above identifications with the algebraic subtorus

$$S := (C_{\widehat{x}_v}).x'_v \cong C_{\widehat{x}_v}/C_{x'_v}$$

of C which is contained in the closed subvariety Z. The Zariski closure $\overline{S}$ of S in X_v is contained in Z since Z is closed. On the other hand,

$$\overline{S} \cong \overline{\pi^{-1}(\pi(x'_v)) \cap X_v^\circ} = \pi^{-1}(\pi(x'_v)) \cong G_{\widehat{x}_v}/H'_v\,.$$

As a result, $G_{\widehat{x}_v}/H'_v \cong \overline{S} = S$. So by (5.2), the subgroup $G_{\widehat{x}_v}$ normalizes H'_v. On the other hand, we notice that H'_v normalizes H_v. Hence by [Tim11, Lemma 30.2] and the inclusions (5.1), we get[3] the expected equalities:

$$N_G(H_I) = N_G(H'_v) = N_G(G_{\widehat{x}_v})\,.$$

What is left is the case where $v = 0$. In this case, $I = \Sigma$ and $\widehat{\mathcal{V}}_\Sigma = \{0\}$, so that $G_{\widehat{x}_v} \cong N_G(H)$. But $H_0 = H_\Sigma = H$ up to G-conjugacy, whence the statement. □

We now present some applications of Proposition 5.1 to the description of stabilizers of points in spherical embeddings.

Let $\gamma \in N_G(H)$, and let f be a B-eigenvector of $\mathbb{C}(G/H)$ with weight $m \in M$. The map $g \mapsto f(g\gamma)$ is invariant by the right action of H, and it is a B-eigenvector of $\mathbb{C}(G/H)$ with weight m. Therefore, for some nonzero complex number $\omega_{H,f}(\gamma)$,

$$f(g\gamma) = \omega_{H,f}(\gamma) f(g) \quad \text{for all} \quad g \in G\,.$$

The map

$$\omega_{H,f} \colon N_G(H) \longrightarrow \mathbb{C}^*\,, \quad \gamma \longmapsto \omega_{H,f}(\gamma)$$

is a character whose restriction to H is trivial and which depends only on m; see [Bri97, § 4.3]. We denote it by $\omega_{H,m}$. The reader is referred to [Tim11, Theorem 21.5] for a proof of the following result.

Lemma 5.2. *The spherical subgroup H is the kernel of the homomorphism*

$$N_G(H) \longrightarrow \mathrm{Hom}(M, \mathbb{C}^*)\,, \quad \gamma \longmapsto (m \longmapsto \omega_{H,m}(\gamma))\,.$$

Now, fix an arbitrary simple embedding X of G/H, and choose a point x' in the unique closed G-orbit of X. There exist a simple toroidal (G/H)-embedding $\tilde{X}$ and a proper birational

[3] The equality $N_G(H_v) = N_G(H'_v)$ also results from [BP87, § 5.1].

G-equivariant morphism $\pi\colon \tilde{X} \to X$ such that $\pi(\tilde{x}') = x'$ for some $\tilde{x}'$ in the unique closed G-orbit of $\tilde{X}$; see, for example, [Bri97, Proposition 2]. Let $(\sigma, \mathscr{F})$ be the colored cone corresponding to the simple embedding X, and let $I(\sigma)$ be the set of all spherical roots in Σ that vanish on σ. Thus $\mathcal{V}_{I(\sigma)}$ is the minimal face of $\mathcal{V}$ containing σ. Note that $\sigma \cap \mathcal{V}$ is the uncolored cone corresponding to the simple toroidal embedding $\tilde{X}$.

We are now in a position to prove Theorem 1.2.

Proof of Theorem 1.2. By choosing $X = \tilde{X}$, we may actually assume that X is toroidal. Let v be a nonzero element in $\sigma \subset \mathcal{V}_{I(\sigma)}$. By [Tim11, Theorem 15.10], the identity map $G/H \to G/H$ and the canonical map $G/H \to G/N_G(H)$ extend to G-equivariant maps

$$\pi_v\colon X_v \longrightarrow X \quad \text{and} \quad \pi\colon X \longrightarrow \widehat{X}\,,$$

respectively. As in the proof of Proposition 5.1, we deduce from this the inclusions of spherical subgroups

$$H_{I(\sigma)} = H_v \subset H'_v \subset G_{x'} \subset G_{\widehat{x}}\,, \tag{5.3}$$

where H'_v is the stabilizer in G of a point x'_v in the closed G-orbit X'_v such that $x' = \pi_x(x'_v)$ and $\widehat{x} := \pi(x')$. By Proposition 5.1, we have $N_G(H_{I(\sigma)}) = N_G(G_{\widehat{x}})$. So by [Tim11, Lemma 30.2] and (5.3), it follows that $N_G(G_{x'}) = N_G(H_{I(\sigma)})$. Then Lemma 5.2 applied to the spherical subgroup $G_{x'}$ implies that

$$G_{x'} = \left\{\gamma \in N_G(H_{I(\sigma)})\colon\ \omega_{G_{x'},m}(\gamma) = 1 \text{ for all } m \in M \cap \sigma^{\perp}\right\}$$

since the weight lattice of $G/G_{x'}$ is $M \cap \sigma^{\perp}$. This finishes the proof of the theorem. □

Return to the case where X is a simple embedding of G/H (not necessarily toroidal). As a consequence of Theorem 1.2, we get the following result, announced in the introduction.

Corollary 5.3. *With the preceding notation, the stabilizer $G_{x'}$ of x' in G contains the satellite $H_{I(\sigma)}$.*

6. Limits of stabilizers of points in arc spaces

Let $\mathscr{K} := \mathbb{C}((t))$ be the field of formal Laurent series, and let $\mathscr{O} := \mathbb{C}[[t]]$ be the ring of formal power series. For a scheme X over $\mathbb{C}$, denote by $X(\mathscr{K})$ and $X(\mathscr{O})$ the sets of $\mathscr{K}$-valued points and $\mathscr{O}$-valued points of X, respectively. If X is a variety admitting an action of an algebraic group A, then $X(\mathscr{K})$ and $X(\mathscr{O})$ both admit a canonical action of the group $A(\mathscr{O})$ induced from the A-action on X. For example, the action of the group G on the homogeneous space G/H by left multiplication yields an action of the group $G(\mathscr{O})$ on $(G/H)(\mathscr{K})$.

As shown by Luna and Vust, the set $N \cap \mathcal{V}$ parameterizes the $G(\mathscr{O})$-orbits in $(G/H)(\mathscr{K})$. More precisely, with each $\lambda \in (G/H)(\mathscr{K}) \setminus (G/H)(\mathscr{O})$, one associates a valuation $v_\lambda\colon \mathbb{C}(G/H)^* \to \mathbb{Z}$ as follows; cf. [LV83, § 4.2]. The action of G on G/H induces a dominant morphism

$$G \times \operatorname{Spec} \mathscr{K} \longrightarrow G \times G/H \to G/H\,,$$

where the first map is $1 \times \lambda$, whence an injection of fields $\iota_\lambda\colon \mathbb{C}(G/H) \to \mathbb{C}(G)((t))$. Then we set

$$v_\lambda := v_t \circ \iota_\lambda\colon \mathbb{C}(G/H)^* \longrightarrow \mathbb{Z}\,,$$

where $v_t\colon \mathbb{C}(G)((t))^* \to \mathbb{Z}$ is the natural discrete valuation on $\mathbb{C}(G)((t))$ given by the order in t of the lowest term of formal series. If $\lambda \in (G/H)(\mathscr{O})$, we define v_λ to be the trivial valuation.

Then v_λ is a G-invariant discrete valuation of G/H, that is, an element of $N \cap \mathcal{V}$, by [LV83, §§ 4.4 and 4.6]. Furthermore, for every $\gamma \in G(\mathscr{O})$ and every $\lambda \in (G/H)(\mathscr{K})$, we have $v_{\gamma\lambda} = v_\lambda$.

By [LV83, Proposition 4.10] (see also [GN10, Theorem 3.2.1]), the mapping

$$(G/H)(\mathscr{K}) \longrightarrow N \cap \mathcal{V}\,, \quad \lambda \longmapsto v_\lambda\,, \tag{6.1}$$

is a surjective map, and the fiber over any $v \in N \cap \mathcal{V}$ is precisely one $G(\mathscr{O})$-orbit. Thus there is a one-to-one correspondence between the set of $G(\mathscr{O})$-orbits in $(G/H)(\mathscr{K})$ and the set $N \cap \mathcal{V}$. We write $\mathscr{C}_v := \{\lambda \in (G/H)(\mathscr{K})\colon\ v_\lambda = v\}$ for the $G(\mathscr{O})$-orbit in $(G/H)(\mathscr{K})$ corresponding to $v \in N \cap \mathcal{V}$.

Let X be a spherical embedding of G/H associated with a colored fan $\mathbb{F}$. The *support* of $\mathbb{F}$ is the set

$$|\mathbb{F}| = \bigcup_{(\mathcal{C},\mathscr{F})\in\mathbb{F}} \mathcal{C} \cap \mathcal{V}\,.$$

The group $G(\mathscr{O})$ acts on the sets $X(\mathscr{O})$ and $(G/H)(\mathscr{K})$, which are both viewed as subsets of $X(\mathscr{K})$. Hence the group $G(\mathscr{O})$ acts on $X(\mathscr{O}) \cap (G/H)(\mathscr{K})$. By restriction to $X(\mathscr{O}) \cap (G/H)(\mathscr{K})$, the mapping (6.1) induces a surjective mapping

$$X(\mathscr{O}) \cap (G/H)(\mathscr{K}) \longrightarrow N \cap |\mathbb{F}|\,, \quad \lambda \longmapsto v_\lambda\,,$$

whose fiber over any $v \in N \cap |\mathbb{F}|$ is precisely one $G(\mathscr{O})$-orbit. Thus there is a one-to-one correspondence between the set of $G(\mathscr{O})$-orbits in $X(\mathscr{O}) \cap (G/H)(\mathscr{K})$ and the set $N \cap |\mathbb{F}|$.

Fix an H-adapted Levi subgroup L of P, and denote as usual by C the neutral component of its center. Let $x := eH$ be the base point in G/H. Any one-parameter subgroup λ of C yields a morphism of algebraic varieties

$$\widetilde{\lambda}\colon C^* \longrightarrow G/H\,, \quad t \longmapsto \lambda(t)x$$

whose comorphism gives rise to an element $\widehat{\lambda}$ of $(G/H)(\mathscr{K})$.

Pick a nonzero lattice point v in $N \cap \mathcal{V}$. Since the open orbit of X_v is G/H, one can assume $x_v = eH$. Then, λ is adapted to the elementary embedding (X_v, x_v) (cf. Definition 2.2) if and only if $\widehat{\lambda}$ lies in $X(\mathscr{O}) \cap (G/H)(\mathscr{K})$ and $\lim_{t\to 0} \widehat{\lambda}(t)$ exists in the closed G-orbit X'_v. One can choose $\lambda_v \in \mathscr{X}_*(C)$, adapted to (X_v, x_v), such that $v_{\lambda_v} = v$. By [LV83, § 5.4],

$$G(\mathscr{O})\widehat{\lambda}_v = \mathscr{C}_v\,,$$

and so $v = v_{\lambda_v} = v_{\widehat{\lambda}_v}$. Setting $\widetilde{\lambda}_v(0) := \lim_{t\to 0} \widetilde{\lambda}_v(t)$, one gets a morphism of algebraic varieties

$$\widetilde{\lambda}_v\colon \mathbb{C} \longrightarrow X_v\,.$$

One can view the differential $d\widetilde{\lambda}_v(0)$ of $\widetilde{\lambda}_v$ at 0 as an element of $T_{x'_v}(X_v)$ for the point $x'_v := \lim_{t\to 0} \lambda_v(t)x_v \in X'_v$.

Lemma 6.1. *The one-dimensional space $T_{x'_v}(X_v)/T_{x'_v}(X'_v)$ is generated by the class of $d\widetilde{\lambda}_v(0)$ in $T_{x'_v}(X_v)/T_{x'_v}(X'_v)$.*

Proof. According to the local structure theorem for toroidal embeddings ([BP87, § 3.4] or [Tim11, Theorem 29.1]), it suffices to show the statement for elementary embeddings of a torus. So we can assume that $X_v = \mathbb{T} \cup X'_v$ is an elementary embedding of the algebraic torus $\mathbb{T} := C/C \cap H$ in $\mathbb{C}^n$. Arguing in coordinates $x_1, \ldots, x_n$, we can assume furthermore that the divisor X'_v is given by the equation $x_1 = 0$. One can choose for λ_v the adapted one-parameter subgroup $t \mapsto (t, 1, \ldots, 1)$. The statement is now clear. □

Lemma 6.2. *We have $C \cap H_v = C \cap H$.*

Proof. It follows from the definition of the Brion subgroup H_v (cf. Definition 2.3) that $C \cap H_v$ is a Brion subgroup associated with the elementary embedding $Cx_v \cup Cx'_v$ of the algebraic torus $\mathbb{T} = C/C \cap H$. Since $\mathbb{T}$ is a torus, $C \cap H$ is equal to each of the Brion subgroups of $\mathbb{T}$. Therefore, we get $C \cap H = C \cap H_v$. □

Theorem 1.3 can now be proved.

Proof of Theorem 1.3. We have to show that $H_v = \left\{ \lim_{t\to 0} \gamma(t)\colon\ \gamma \in G(\mathscr{O})_{\widehat{\lambda}_v} \right\}$, where $G(\mathscr{O})_{\widehat{\lambda}_v}$ is the stabilizer of $\widehat{\lambda}_v$ in $G(\mathscr{O})$, that is, the set of $\gamma \in G(\mathscr{O})$ such that $\gamma(t)(\lambda_v(t)x_v) = \lambda_v(t)x_v$ for all $t \in \mathbb{C}^*$.

In order to establish the inclusion $H_v \supset \left\{ \lim_{t\to 0} \gamma(t)\colon\ \gamma \in G(\mathscr{O})_{\widehat{\lambda}_v} \right\}$, fix $\gamma \in G(\mathscr{O})_{\widehat{\lambda}_v}$ and write $\gamma(0)$ for its limit at 0. Let H'_v be the stabilizer of x'_v in G. One can assume that $H_v = \ker \chi_v \subset H'_v$ (cf. Definition 2.3). For any $t \in \mathbb{C}^*$,

$$\gamma(t)(\lambda_v(t)x_v) = \lambda_v(t)x_v\,. \tag{6.2}$$

Taking the limit at 0 on both sides, we get $\gamma(0)x'_v = x'_v$. Hence $\gamma(0) \in H'_v$. Since $\gamma(0) \in H'_v$, the element $\gamma(0)$ leaves invariant the tangent space $T_{x'_v}(X'_v)$. Differentiating the equality (6.2), we get

$$\gamma(0)\big(d\widetilde{\lambda}_v(0)\big) = d\widetilde{\lambda}_v(0) \mod T_{x'_v}(X'_v)\,.$$

Since H'_v acts on the one-dimensional space $T_{x'_v}(X_v)/T_{x'_v}(X'_v)$ by the character χ_v, we deduce from Lemma 6.1 that $\gamma(0)$ is in the kernel H_v of χ_v, whence the expected inclusion.

Let us prove the other inclusion. Pick $\gamma_0 \in H_v$. We have to show that $\gamma_0 = \lim_{t\to 0} \gamma(t)$ for some $\gamma \in G(\mathscr{O})_{\widehat{\lambda}_v}$. By [BP87, 5.2, Proposition and Corollary], we have $N_G(H_v) = H_v^0(C \cap N_G(H_v))$, with H_v^0 the neutral component of H_v, whence $H_v \subset H_v^0(C \cap H_v)$. Write $\gamma_0 = \tilde{\gamma}_0 c$ with $\tilde{\gamma}_0 \in H_v^0$ and $c \in C \cap H_v$. Since H_v^0 is connected, there exist $l \in \mathbb{Z}_{>0}$ and $\eta_1, \dots, \eta_l$ in $\mathfrak{h}_v := \mathrm{Lie}(H_v)$ such that

$$\tilde{\gamma}_0 = \exp_{H_v}(\eta_1) \cdots \exp_{H_v}(\eta_l)\,,$$

where $\exp_{H_v}\colon \mathfrak{h}_v \to H_v$ is the exponential map of H_v.

Denote by $(g, \xi) \mapsto g.\xi$ the adjoint action of the algebraic group G on its Lie algebra. By [Bri90, Lemma 1.2], the limit $\lim_{t\to 0} \lambda_v(t).\xi$ exists and lies in $\mathrm{Lie}(H'_v)$ for every ξ in the Lie algebra $\mathfrak{h}$ of H. Moreover, according to [Bri90, Proposition 1.2],[4] the Lie algebra of H_v is

$$\lim_{t\to 0} \lambda_v(t).\mathfrak{h} := \left\{ \lim_{t\to 0} \lambda_v(t).\xi\colon\ \xi \in \mathfrak{h} \right\}.$$

Therefore, for some $\xi_1, \dots, \xi_l \in \mathfrak{h}$, we have $\eta_i = \lim_{t\to 0} \lambda_v(t).\xi_i$. For any $t \in \mathbb{C}^*$, set

$$\tilde{\gamma}(t) := \lambda_v(t) \exp_H(\xi_1) \cdots \exp_H(\xi_l) \lambda_v\big(t^{-1}\big)\,,$$

where $\exp_H\colon \mathfrak{h} \to H$ is the exponential map of H. Then for any $t \in \mathbb{C}^*$,

$$\tilde{\gamma}(t)(\lambda_v(t)x_v) = \lambda_v(t) \exp_H(\xi_1) \cdots \exp_H(\xi_l) x_v = \lambda_v(t)x_v$$

since for $i = 1, \dots, l$, $\exp_H(\xi_i)$ lies in $H = G_{x_v}$. As a result, $\tilde{\gamma} \in G(\mathscr{O})_{\widehat{\lambda}_v}$.

[4]The result is stated in [Bri90, Proposition 1.2] for H_v equal to its normalizer. However, the proof does not use this hypothesis.

Using the natural embedding $G \hookrightarrow G(\mathscr{O})$, one can view the element c of $C \subset G$ as an element of $G(\mathscr{O})$. For any $t \in \mathbb{C}^*$, set $\gamma(t) := \tilde{\gamma}(t)c$. We have

$$\begin{aligned}\lim_{t\to 0}\tilde{\gamma}(t) &= \lim_{t\to 0}\big(\lambda_v(t)\exp_H(\xi_1)\cdots\exp_H(\xi_l)\lambda_v(t^{-1})\big)\\ &= \lim_{t\to 0}\big(\exp_G(\lambda_v(t).\xi_1)\cdots\exp_G(\lambda_v(t).\xi_l)\big) = \exp_{H_v}(\eta_1)\cdots\exp_{H_v}(\eta_l) = \tilde{\gamma}_0\,,\end{aligned}$$

with $\exp_G$ the exponential map of G, whence $\lim_{t\to 0}\gamma(t) = \tilde{\gamma}_0 c = \gamma_0$. By Lemma 6.2, the element c of $C \cap H_v$ is in $C \cap H$. In particular, c is in the stabilizer of x_v in G. Hence,

$$\gamma(t)(\lambda_v(t)x_v) = \tilde{\gamma}(t)c(\lambda_v(t)x_v) = \tilde{\gamma}(t)(\lambda_v(t)cx_v) = \tilde{\gamma}(t)(\lambda_v(t)x_v) = \lambda_v(t)x_v$$

because $\tilde{\gamma} \in G(\mathscr{O})_{\widehat{\lambda}_v}$ and c commutes with the image of λ_v. This proves that γ is in $G(\mathscr{O})_{\widehat{\lambda}_v}$. We have shown that $\gamma_0 \in \big\{\lim_{t\to 0}\gamma(t)\colon\ \gamma \in G(\mathscr{O})_{\widehat{\lambda}_v}\big\}$, as desired. □

Theorem 1.3 can be used in practice to compute the Brion subgroups H_v for $v \in N \cap \mathcal{V}$. According to Theorem 1.1, it suffices to compute finitely many of them to describe all the satellites H_I of H, where $I \subset \Sigma$. We illustrate this with some examples. In the following, no distinction is made between λ and $\widehat{\lambda}$, and so by a slight abuse of notation, we write $G(\mathscr{O})_\lambda$ for $G(\mathscr{O})_{\widehat{\lambda}}$.

Example 6.3. In the setting of Section 1.1, assume $G = \mathrm{GL}_n$ ($n \in \mathbb{N}^*$) and that $T = D_n$ is the set of diagonal matrices in G. Then $T \times T$ is a maximal torus of $G \times G$ and is adapted to $\Delta(G)$; cf. [Bri97, § 2.4, Example 2]. Hence,

$$M \cong \mathscr{X}^*(D_n) \cong \mathbb{Z}^n\,.$$

The dual lattice $N = \mathrm{Hom}(M, \mathbb{Z})$ identifies with the lattice of one-parameter subgroups of D_n, and $\mathcal{V}$ identifies with the set of sequences $(\lambda_1, \ldots, \lambda_n) \in \mathbb{Q}^n$ with $\lambda_1 \geqslant \cdots \geqslant \lambda_n$. Let T_n^+ and T_n^- be the subgroups of GL_n consisting of upper- and lower-triangular matrices, respectively. The identity matrix E lies in the open $(T_n^+ \times T_n^-)$-orbit since $T_n^+ T_n^-$ is open in GL_n. Consider the one-parameter subgroup λ of $T \times T$ given by

$$\lambda(t) = \big(\mathrm{diag}\big(t^{\lambda_1}, \ldots, t^{\lambda_n}\big), \mathrm{diag}\big(t^{-\lambda_1}, \ldots, t^{-\lambda_n}\big)\big) \quad \text{for all} \quad t \in \mathbb{C}^*\,,$$

where $(\lambda_1, \ldots, \lambda_n)$ in $\mathbb{Z}^n$ and $\lambda_1 \geqslant \cdots \geqslant \lambda_n$.

Write $(\lambda_1, \ldots, \lambda_n) = \big(\mu_1^{k_1}, \ldots, \mu_r^{k_r}\big)$ with $\mu_1 > \cdots > \mu_r$ and $k_1 + \cdots + k_r = n$, where $k_i \in \mathbb{N}^*$. Let P_λ be the (standard) parabolic subgroup of GL_n consisting of the block upper-triangular invertible matrices with diagonal blocks of sizes $k_1, \ldots, k_r$. Denote by L_λ its standard Levi factor, and let $P_{-\lambda}$ be the opposite parabolic subgroup to P_λ, so that $P_{-\lambda} \cap P_\lambda = L_\lambda$.

An easy computation allows us to determine the set of limits of points $(x(t), y(t)) \in G(\mathscr{O})_\lambda$, and applying Theorem 1.3, we get

$$H_\lambda = \big(P_{-\lambda}^u \times P_\lambda^u\big)\Delta(L_\lambda)\,,$$

where P_λ^u and $P_{-\lambda}^u$ are the unipotent radicals of P_λ and $P_{-\lambda}$, respectively. Thus we recover the description of the satellites of Section 1.1 for $G = \mathrm{GL}_n$.

Example 6.4. Let $n \in \mathbb{Z}$ with $n \geqslant 3$. Assume $G = \mathrm{SL}_n$ and that $H \cong \mathrm{SL}_{n-1}$ is the subgroup $H = \big\{\big(\begin{smallmatrix} A & 0 \\ 0 & 1\end{smallmatrix}\big)\colon\ A \in \mathrm{SL}_{n-1}\big\}$. The spherical homogeneous space G/H can be realized as

$$G/H \cong \big\{(x, y) \in \mathbb{C}^n \times (\mathbb{C}^n)^*\colon\ \langle y, x\rangle = 1\big\}\,.$$

The Levi subgroup

$$L = \left\{ \begin{pmatrix} a & 0 & 0 \\ 0 & B & 0 \\ 0 & 0 & c \end{pmatrix} : a, c \in \mathbb{C}^*,\ B \in \mathrm{GL}_{n-2},\ ac \det B = 1 \right\}$$

is adapted to the stabilizer $\tilde{H} \cong H$ of the point $((1,0,\dots,0,1),(1,0,\dots,0))$. Let λ be the one-parameter subgroup of the neutral component of the center of L, with $\lambda(t) = \mathrm{diag}\big(t, 1, \dots, 1, , t^{-1}\big)$ for all $t \in \mathbb{C}^*$. In order to compute $G(\mathscr{O})_\lambda$, we exploit the elementary embedding

$$X := \Big\{ [x_1; \dots; x_n; y_1; \dots; y_n; z_0] \colon \ \sum x_i y_i = z_0^2 \Big\} \subset \mathbb{P}^{2n}$$

of G/H. The closed G-orbit of X corresponds to the divisor $\{z_0 = 0\}$, and we have

$$\lambda(t)([1; 0; \dots, 0; 1; 1; 0; \dots; 0; 1]) = \big[t; 0; \dots; t^{-1}; t^{-1}; 0; \dots, 0; 1\big] \in \mathbb{P}^{2n}.$$

From this, we easily determine $G(\mathscr{O})_\lambda$, and by Theorem 1.3, we get

$$H_\varnothing \cong \left\{ \begin{pmatrix} a & 0 & 0 \\ u & B & 0 \\ v & w & a \end{pmatrix} \colon \ B \in \mathrm{SL}_{n-2},\ u, v, w, a \in \mathbb{C},\ a^2 = 1 \right\}.$$

7. Virtual Poincaré polynomials of G/H_I

This section is devoted to the proofs of Theorems 7.3 and 7.9. They are strongly inspired by the work of Brion and Peyre [BP02].

In order to state the results of [BP02] we need, let us assume for a while that G is any complex connected linear algebraic group and that H is any closed subgroup of G. We denote by r_H the rank of H and by u_H the dimension of a maximal unipotent subgroup of H. Choose maximal reductive subgroups $H^{\mathrm{red}} \subset H$ and $G^{\mathrm{red}} \subset G$ such that $H^{\mathrm{red}} \subset G^{\mathrm{red}}$.

Theorem 7.1 ([BP02, Theorem 1(b)]). *Under the above assumptions, there exists a polynomial $Q_{G/H}$ with nonnegative integer coefficients such that*

$$\tilde{P}_{G/H}(t) = t^{u_G - u_H}(t-1)^{r_G - r_H} Q_{G/H}(t) \quad \text{and} \quad Q_{G/H}(t) = Q_{G^{\mathrm{red}}/H^{\mathrm{red}}}(t).$$

In particular, if G is reductive, then

$$\tilde{P}_{G/H}(t) = t^{u_{H^{\mathrm{red}}} - u_H} \tilde{P}_{G/H^{\mathrm{red}}}(t).$$

Moreover, $Q_{G/H}(0) = 1$ if H is connected.

Let us now return to the case where G is reductive and H is a spherical subgroup of G. For each subset I of the set of spherical roots Σ of G/H, choose a maximal reductive subgroup $H_I^{\mathrm{red}} \subset H_I$ of the satellite H_I of H. We briefly denote by r_I, u_I and u_I^{red} the integers r_{H_I}, u_{H_I} and $u_{H_I^{\mathrm{red}}}$, respectively. Note that $r_I = r_{H_I^{\mathrm{red}}}$.

Lemma 7.2 ([BP02]). *Let $I \subset \Sigma$. Then H_I^{red} is contained in a G-conjugate of H.*

Proof. If $I = \Sigma$, the statement is clear. Let $I \subsetneq \Sigma$, and pick a lattice point v in $N \cap \mathcal{V}_I^\circ$. Without loss of generality, we may assume that $H_I^{\mathrm{red}} = H_v^{\mathrm{red}}$ is contained in H_v.

We now resume the arguments of [BP02, § 2]. Consider the action of H_v^{red} on the tangent space $T_{x_v'}(X_v)$, and choose an H_v^{red}-invariant complement N to the tangent space $T_{x_v'}(X_v')$. By construction, H_v^{red} fixes N pointwise. Then we can find an H_v^{red}-invariant subvariety Z of X_v

such that Z is smooth at x'_v and $T_{x'_v}(Z) = N$. Therefore, H_v^{red} fixes pointwise a neighborhood of x'_v in Z, and this neighborhood meets the open G-orbit G/H. Thus we may suppose that $H_v^{\mathrm{red}} = H_I^{\mathrm{red}}$ is contained in H, as expected. □

Since the satellite H_I is defined up to G-conjugacy, Lemma 7.2 allows us to assume from now on that H_I^{red} is contained in H for each subset $I \subset \Sigma$.

Theorem 7.3. *Assume that H is connected, and let $I \subset \Sigma$.*

(1) *We have $\tilde{P}_{G/H_I}(t) = \tilde{P}_{G/H}(t)\tilde{P}_{H/H_I^{\mathrm{red}}}(t)t^{-(u_I - u_I^{\mathrm{red}})}$. In particular, the ratio*

$$R_I(t^{-1}) := \frac{\tilde{P}_{G/H_I}(t)}{\tilde{P}_{G/H}(t)}$$

is a polynomial R_I in t^{-1} with integer coefficients and constant term 1.

(2) *The degree of $R_\varnothing$ is $u_G - u_H = u_\varnothing - u_H$.*

(3) *The degree of R_I is $u_I - u_H$ provided that H_I is connected.*

Proof. (1) Set $X = G/H_I^{\mathrm{red}}$, $Y = G/H$ and $F = H/H_I^{\mathrm{red}}$, and consider the locally trivial fibration $F \hookrightarrow X \xrightarrow{f} Y$ for the complex analytic topology.

Claim 7.4. *The local systems $R^j f_* \mathbb{C}_X$ are constant for each j.*

Proof. Recall that Y being a connected complex algebraic variety with the complex analytic topology, for any choice of a base point $y \in Y$, there in an equivalence of categories between the category of local systems on Y and the category of representations of the fundamental group $\pi_1(Y, y)$. So for each j, there is a natural $\pi_1(Y, y)$-action on the cohomology fiber $R^j f_*(\mathbb{C}_X)_y \cong H^j\big(f^{-1}(y), \mathbb{C}\big) \cong H^j(F, \mathbb{C})$. Hence, it is enough to show that this action is trivial. We are dropping, later on, the mention of the base point for the fundamental group.

Since the fiber $F = H/H_I^{\mathrm{red}}$ is connected (H being connected), we deduce from the long exact sequence of homotopy groups

$$\cdots \longrightarrow \pi_1(F) \longrightarrow \pi_1(X) \longrightarrow \pi_1(Y) \longrightarrow \pi_0(F) = 1 \longrightarrow \cdots$$

that the morphism $\pi_1(X) \to \pi_1(Y)$ is surjective.

Let us now make use of the pullback of the fiber bundle $X \to Y$ by the map $f\colon X \to Y$. We have $f^*(X) \cong X \times F$, and the following diagram commutes, where the bottom and left arrows are the natural projections:

$$\begin{array}{ccc} X & \xrightarrow{\quad f \quad} & Y \\ \uparrow & & \uparrow{\scriptstyle f} \\ f^*(X) \cong X \times F & \longrightarrow & X\,. \end{array}$$

The natural action of $\pi_1(X)$ on the cohomology fiber $H^j(F, \mathbb{C})$ furnishes a map $\rho\colon \pi_1(X) \to \mathrm{Aut}(H^j(F, \mathbb{C}))$ which is the composition of the surjective map $\pi_1(X) \to \pi_1(Y)$ and the representation $\pi_1(Y) \to \mathrm{Aut}(H^j(F, \mathbb{C}))$ induced by the local system $R^j f_* \mathbb{C}_X$. Since the fibration $f^*(X) \cong X \times F \to X$ is trivial, the action of $\pi_1(X)$ on $H^j(F, \mathbb{C})$ is trivial, that is, ρ is trivial. The map $\pi_1(X) \to \pi_1(Y)$ being surjective, we deduce that the representation $\pi_1(Y) \to \mathrm{Aut}(H^j(F, \mathbb{C}))$ is trivial too, as required. □

By Claim 7.4, we can apply [DL97, Theorem 6.1(ii)] to the algebraic morphism f to obtain

$$\tilde{P}_{G/H_I^{\mathrm{red}}}(t) = \tilde{P}_{G/H}(t)\tilde{P}_{H/H_I^{\mathrm{red}}}(t)\,.$$

Thus by Theorem 7.1, we get

$$\tilde{P}_{G/H_I}(t) = \tilde{P}_{G/H}(t)\tilde{P}_{H/H_I^{\mathrm{red}}}(t)t^{-(u_I-u_I^{\mathrm{red}})}\,.$$

In addition, $\tilde{P}_{H/H_I^{\mathrm{red}}}(t)$ is a polynomial with integer coefficients. Since $\dim G/H_I = \dim G/H$, the polynomials $\tilde{P}_{G/H_I}(t)$ and $\tilde{P}_{G/H}(t)$ have the same degree, and so the degree of $\tilde{P}_{H/H_I^{\mathrm{red}}}(t)$ must be equal to $d := u_I - u_I^{\mathrm{red}}$. Hence, for some integers $a_0, \dots, a_{d-1}, a_d$, we have

$$P_{H/H_I^{\mathrm{red}}}(t)t^{-(u_I-u_I^{\mathrm{red}})} = \big(a_0 + a_1 t + \cdots + a_{d-1}t^{d-1} + a_d t^d\big)t^{-d} = R_I\big(t^{-1}\big)\,,$$

where $R_I(X) = a_0X^d + a_1X^{d-1} + \cdots + a_{d-1}X + a_d$. It remains to show that the constant term a_d of R_I is 1. Since G/H_I and G/H are irreducible of the same dimension, the leading term of $\tilde{P}_{G/H_I}(t)$ and $\tilde{P}_{G/H}(t)$ is $t^{\dim G/H}$, whence $a_d = 1$.

(2) Because $H_\varnothing$ is horospherical, its normalizer $N_G(H_\varnothing)$ in G is a parabolic subgroup of G, and we have a locally trivial fibration $G/H_\varnothing \to G/N_G(H_\varnothing)$ for the Zariski topology, with fiber isomorphic to $N_G(H_\varnothing)/H_\varnothing$. The algebraic torus $N_G(H_\varnothing)/H_\varnothing$ has dimension the rank r of G/H. Therefore,

$$\tilde{P}_{G/H_\varnothing}(t) = (t-1)^r\tilde{P}_{G/N_G(H_\varnothing)}(t)\,. \tag{7.1}$$

Note that $\tilde{P}_{G/N_G(H_\varnothing)}(0) = 1$. On the other hand, by Theorem 7.1, the polynomial $\tilde{P}_{G/H}(t)$ is equivalent to $(-1)^{r_G-r_H}t^{u_G-u_H}$ when t goes to 0 since $Q_{G/H}(0) = 1$ for connected H. So, according to (7.1) and assertion (1), we obtain

$$(-1)^r = (-1)^{r_G-r_H}t^{u_G-u_H}t^{-\deg R_\varnothing}$$

since $R_\varnothing(0) = 1$, which forces $\deg R_\varnothing = u_G - u_H$. To conclude, notice that $u_G = u_\varnothing$ because $H_\varnothing$ is horospherical.

(3) Assume that H_I is connected. By Theorem 7.1, the polynomial $\tilde{P}_{G/H_v}(t)$ is equivalent to $(-1)^{r_G-r_I}t^{u_G-u_I}$ when t goes to 0. So, from assertion (1), we deduce that

$$(-1)^{r_G-r_I}t^{u_G-u_I} = (-1)^{r_G-r_H}t^{-\deg R_I}t^{u_G-u_H}$$

since $R_I(0) = 1$, which forces $\deg R_I = u_I - u_H$. □

Example 7.5. Keep the notation of the main illustrating example, Section 1.1. Given $I \subset S$, let Φ_I be the root system generated by I, and write Φ_I^+ for the set of its positive roots with respect to I. Note that $\dim G/P_I = \dim P_I^u = |\Phi_S^+ \setminus \Phi_I^+|$, where P_I^u is the unipotent radical of P_I. Simple computations show that

$$R_I\big(t^{-1}\big) = \tilde{P}_{G/P_I}(t)t^{-|\Phi_S^+\setminus\Phi_I^+|} = \prod_{\alpha\in\Phi_S^+\setminus\Phi_I^+}\left(\frac{t^{\mathrm{ht}(\alpha)+1}-1}{t^{\mathrm{ht}(\alpha)}-1}\right)t^{-|\Phi_S^+\setminus\Phi_I^+|}\,,$$

where $\mathrm{ht}(\alpha)$ denotes the height of α for $\alpha \in \Phi_S^+$. In particular, we obtain that the right-hand side is a polynomial in t^{-1}.

Remark 7.6. When R_I verifies the statement of Conjecture 1.4, it would be interesting to find an algorithm for computing the polynomial R_I using combinatorial properties of the spherical roots in $I \subset \Sigma$, as in Example 7.5.

Example 7.7. Assume $G = \mathrm{SL}_2 \times \mathrm{SL}_2 \times \mathrm{SL}_2$ and that H is SL_2, diagonally embedded in G. The spherical homogeneous space G/H admits three spherical roots s_1, s_2, s_3, and so its valuation cone has $2^3 = 8$ faces. By symmetry, $H_{\{s_1\}} \cong H_{\{s_2\}} \cong H_{\{s_2\}}$ and $H_{\{s_1,s_2\}} \cong H_{\{s_2,s_3\}} \cong H_{\{s_1,s_3\}}$. Applying Theorem 1.3, we can compute the satellites $H_{\{s_1\}}$, $H_{\{s_1,s_2\}}$ and $H_\varnothing$, and we find that

$$H_{\{s_1\}} = \left\{ \left(\begin{pmatrix} a_1 & 0 \\ c_1 & a_1^{-1} \end{pmatrix}, \begin{pmatrix} a_1 & b_2 \\ 0 & a_1^{-1} \end{pmatrix}, \begin{pmatrix} a_1 & 0 \\ 0 & a_1^{-1} \end{pmatrix} \right) : \ a_1 \in \mathbb{C}^*, \ c_1, b_2 \in \mathbb{C} \right\},$$

$$H_{\{s_1,s_2\}} = \left\{ \left(\begin{pmatrix} a_1 & 0 \\ c_1 & a_1^{-1} \end{pmatrix}, \begin{pmatrix} a_1 & b_2 \\ 0 & a_1^{-1} \end{pmatrix}, \begin{pmatrix} a_1 & b_2 \\ 0 & a_1^{-1} \end{pmatrix} \right) : \ a_1 \in \mathbb{C}^*, \ c_1, b_2 \in \mathbb{C} \right\},$$

$$H_\varnothing = \left\{ \left(\begin{pmatrix} a_1 & 0 \\ c_1 & a_1 \end{pmatrix}, \begin{pmatrix} a_1 & b_2 \\ 0 & a_1 \end{pmatrix}, \begin{pmatrix} a_1 & b_3 \\ 0 & a_1 \end{pmatrix} \right) : \ c_1, b_2, b_3 \in \mathbb{C} \text{ and } a_1^2 = 1 \right\}.$$

We observe that $G/H_\varnothing$ is a locally trivial fibration for the Zariski topology over $(\mathbb{P}^1)^3$ with fiber $(\mathbb{C}^*)^3$. So we get

$$\tilde{P}_{G/H_\varnothing}(t) = (t-1)^3(t+1)^3 .$$

On the other hand, $H_{\{s_1\}}$ and $H_{\{s_1,s_2\}}$ are connected. By Theorem 7.1, we get

$$\tilde{P}_{G/H_{\{s_1\}}}(t) = \tilde{P}_{G/H_{\{s_1,s_2\}}}(t) = t(t-1)^2(t+1)^3 .$$

In conclusion, $R_\varnothing(t^{-1}) = 1 - t^{-2}$ and $R_{\{s_1\}}(t) = R_{\{s_1,s_2\}}(t^{-1}) = 1 + t^{-1}$.

Example 7.8. Let $G = \mathrm{SL}_n$, and let H be a maximal standard Levi factor of semisimple type $\mathrm{SL}_{n-1} \times \mathbb{C}^*$. The homogeneous space G/H is spherical and admits only one elementary embedding, up to isomorphism, $X := \mathbb{P}(\mathbb{C}^n) \times \mathbb{P}((\mathbb{C}^n)^*) \cong \mathbb{P}^{n-1} \times \mathbb{P}^{n-1}$. So G/H has a unique satellite $H_\varnothing$ which is horospherical. The unique closed G-orbit of X is

$$X' = \left\{ ([x_1 : \cdots : x_n], [y_1 : \cdots : y_n]) \in \mathbb{P}^{n-1} \times \mathbb{P}^{n-1} : \ \sum_{i=1}^{n} x_i y_i = 0 \right\}.$$

The variety X' is a locally trivial fibration for the Zariski topology over $\mathbb{P}^{n-1}$ with a fiber isomorphic to $\mathbb{P}^{n-2}$. In addition, denoting by H' the stabilizer of a point in X', we know that $G/H_\varnothing$ is a locally trivial fibration for the Zariski topology over $G/H' \cong X'$ with fiber $\mathbb{C}^*$. Hence,

$$\tilde{P}_{G/H_\varnothing}(t) = \left(1 + t + \cdots + t^{n-1}\right)\left(1 + t + \cdots + t^{n-2}\right)(t-1) .$$

Observing that $\tilde{P}_{G/H} = \tilde{P}_X(t) - \tilde{P}_{X'}(t)$, we get

$$\tilde{P}_{G/H}(t) = \left(1 + t + \cdots + t^{n-1}\right)t^{n-1} .$$

In conclusion, $R_\varnothing(t^{-1}) = 1 - t^{-(n-1)}$.

Theorem 7.9. *Assume that the spherical homogeneous space G/H is of rank one. Then $R_\varnothing$ is a polynomial with integer coefficients.*

Proof. Spherical homogeneous spaces G/H of rank one were classified by Akhiezer [Ahi83] and Brion [Bri89]. Such a homogeneous space G/H either is horospherical or has a wonderful compactification. Theorem 7.9 is obvious if G/H is horospherical. So we are interested only in those homogeneous spaces G/H of rank one that admit a wonderful compactification. These spaces are listed in [Tim11, Table 30.1]. In this case, the spherical subgroup H has a unique satellite subgroup different from H; this is the horospherical subgroup $H_\varnothing$. For each homogeneous space G/H from the list in [Tim11, Table 30.1], we compute the Poincaré polynomials $\tilde{P}_{G/H}(t)$

and $\tilde{P}_{G/H_\varnothing}(t)$ and then the ratio

$$R_\varnothing\big(t^{-1}\big) = \frac{\tilde{P}_{G/H_\varnothing}(t)}{\tilde{P}_{G/H}(t)}\,.$$

The obtained results are described in Table 1. Our calculations show that the ratio $R_\varnothing\big(t^{-1}\big)$ is a polynomial in t^{-1} containing only two terms, with integer coefficients.

Let us explain our computations. Let $G/H \hookrightarrow X$ be a wonderful embedding of G/H with closed G-orbit X'. We have

$$\tilde{P}_X(t) = \tilde{P}_{G/H}(t) + \tilde{P}_{X'}(t) = \tilde{P}_{G/H}(t) + \frac{\tilde{P}_{G/H_\varnothing}(t)}{(t-1)}\,. \tag{7.2}$$

Note that X' is a projective homogeneous space G/P, where P is certain parabolic subgroup of G such that $\dim G/P = \dim G/H - 1$.

In some cases, like in Example 7.8, from the knowledge of X and X', we compute $\tilde{P}_X(t)$ and $\tilde{P}_{X'}(t)$ and so $\tilde{P}_{G/H}(t)$ and $\tilde{P}_{G/H_\varnothing}(t)$ by (7.2). We can proceed in this way for cases 1, 3 (which corresponds to Example 7.8), 7a, 7b and 10 of Table 1.

If H is connected, it is sometimes easier to compute directly $\tilde{P}_{G/H}(t)$ using Theorem 7.1 and [BP02, Theorem 1(c)] instead of computing $\tilde{P}_{X'}(t)$. Then we get $\tilde{P}_{G/H_\varnothing}(t)$ from $\tilde{P}_X(t)$ and $\tilde{P}_{G/H}(t)$ by (7.2). We can argue in this way for cases 2, 5, 6, 7a, 7b, 9, 10, 11 and 13.

Still, if H is connected, it is alternatively possible to compute $\tilde{P}_{G/H_\varnothing}(t)$ even without the knowledge of X. Indeed, it is sometimes possible to deduce the conjugacy class of the parabolic stabilizer P of a point in X' just by dimension reasons. Then we get $\tilde{P}_{G/H_\varnothing}(t)$ since

$$\tilde{P}_{G/H_\varnothing}(t) = \tilde{P}_{G/P}(t)(t-1)\,.$$

Consider, for example, the case 12 of Table 1, where $G = \mathbf{F}_4$ and $H = \mathbf{B}_4$. Then $\dim G/H = 52 - 36 = 16$. So $\dim G/P = 15$ and $\dim P = 37$. Hence, for dimension reasons, P is conjugate to a parabolic subgroup whose semisimple Levi part is of type either $\mathbf{B}_3$ or $\mathbf{C}_3$. In both cases, we get (assuming that P contains the standard Borel subgroup B)

$$\tilde{P}_{G/P}(t) = \frac{\tilde{P}_{G/B}(t)}{\tilde{P}_{P/B}(t)} = \frac{\big(t^2-1\big)\big(t^6-1\big)\big(t^8-1\big)\big(t^{12}-1\big)}{(t-1)\big(t^2-1\big)\big(t^4-1\big)\big(t^6-1\big)}$$

since $\mathbf{B}_3$ and $\mathbf{C}_3$ share the same exponents 1, 3, 5. Hence,

$$\tilde{P}_{G/H_\varnothing}(t) = \tilde{P}_{G/P}(t)(t-1) = \big(t^4+1\big)\big(t^{12}-1\big)\,.$$

On the other hand, since H is connected, we have by [BP02]

$$\tilde{P}_{G/H} = \frac{\big(t^2-1\big)\big(t^6-1\big)\big(t^8-1\big)\big(t^{12}-1\big)t^{24}}{\big(t^2-1\big)\big(t^4-1\big)\big(t^6-1\big)\big(t^8-1\big)t^{16}} = \frac{\big(t^{12}-1\big)t^8}{t^4-1}\,.$$

In conclusion,

$$R_\varnothing\big(t^{-1}\big) = \frac{t^8-1}{t^8} = 1 - t^{-8}\,.$$

We can proceed similarly for cases 4, 8a, 8b, 14 and 15. □

n°	G	H	$\tilde{P}_{G/H}(t)$	$\tilde{P}_{G/H_\varnothing}(t)$	$R_\varnothing(t^{-1})$
1	$\mathrm{SL}_2 \times \mathrm{SL}_2$	SL_2	$t(t^2-1)$	$(t-1)(1+t)^2$	$1+t^{-1}$
2	$\mathrm{PSL}_2 \times \mathrm{PSL}_2$	PSL_2	$t(t^2-1)$	$(t-1)(1+t)^2$	$1+t^{-1}$
3	SL_n	$\mathrm{S}(\mathrm{L}_1 \times \mathrm{L}_{n-1})$	$\frac{t^{n-1}(t^n-1)}{t-1}$	$\frac{(t^{n-1}-1)(t^n-1)}{t-1}$	$1-t^{-(n-1)}$
4	PSL_2	PO_2	t^2	t^2-1	$1-t^{-2}$
5	Sp_{2n}	$\mathrm{Sp}_2 \times \mathrm{Sp}_{2n-2}$	$\frac{t^{2n-2}(t^{2n}-1)}{t^2-1}$	$\frac{(t^{2n-2}-1)(t^{2n}-1)}{t^2-1}$	$1-t^{-(2n-2)}$
6	Sp_{2n}	$B(\mathrm{Sp}_2) \times \mathrm{Sp}_{2n-2}$	$\frac{t^{2n-1}(t^{2n}-1)}{t-1}$	$\frac{(t^{2n-1}-1)(t^{2n}-1)}{t-1}$	$1-t^{-(2n-1)}$
7a	SO_{2n+1}	SO_{2n}	$t^n(t^n+1)$	$t^{2n}-1$	$1-t^{-n}$
7b	SO_{2n}	SO_{2n-1}	$t^{n-1}(t^n-1)$	$(t^{n-1}+1)(t^n-1)$	$1+t^{-(n-1)}$
8a	SO_{2n+1}	$\mathrm{S}(\mathrm{O}_1 \times \mathrm{O}_{2n})$	t^{2n}	$t^{2n}-1$	$1-t^{-2n}$
8b	SO_{2n}	$\mathrm{S}(\mathrm{O}_1 \times \mathrm{O}_{2n-1})$	$t^{n-1}(t^n-1)$	$(t^{n-1}+1)(t^n-1)$	$1+t^{-(n-1)}$
9	SO_{2n}	$\mathrm{GL}_n \ltimes \wedge^2\mathbb{C}^n$	$t\prod_{i=1}^n(t^i+1)$	$(t-1)\prod_{i=1}^n(t^i+1)$	$1-t^{-1}$
10	Spin_7	$\mathbf{G}_2$	$t^3(t^4-1)$	$(t^3+1)(t^4-1)$	$1+t^{-3}$
11	SO_7	$\mathbf{G}_2$	$t^3(t^4-1)$	$(t^3+1)(t^4-1)$	$1+t^{-3}$
12	$\mathbf{F}_4$	$\mathbf{B}_4$	$\frac{t^8(t^{12}-1)}{t^4-1}$	$(t^4+1)(t^{12}-1)$	$1-t^{-8}$
13	$\mathbf{G}_2$	SL_3	$t^3(t^3+1)$	$(t^3+1)(t^3-1)$	$1-t^{-3}$
14	$\mathbf{G}_2$	$N(\mathrm{SL}(3))$	t^6	t^6-1	$1-t^{-6}$
15	$\mathbf{G}_2$	$\mathrm{GL}_2 \ltimes \left(\mathbb{C} \oplus \mathbb{C}^2\right) \otimes \wedge^2\mathbb{C}^2$	$\frac{t^2(t^6-1)}{t-1}$	$\frac{(t^2-1)(t^6-1)}{t-1}$	$1-t^{-2}$

TABLE 1. Data for homogeneous spherical spaces of rank one.

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