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$\mathbb{P}$-functor versions of the Nakajima operators

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Abstract

For a smooth quasi-projective surface X, we construct a series of $\mathbb{P}$-functors between derived categories of Hilbert schemes of points on X using the derived McKay correspondence. They can be considered as analogues of the Nakajima operators. We also study the induced autoequivalences and, in particular, obtain a universal braid relation in the groups of derived autoequivalences of Hilbert squares of K3 surfaces. If we replace the surface X with a smooth curve, our functors become fully faithful and induce a semi-orthogonal decomposition of the derived category of the symmetric quotient stack of the curve.

1. Introduction

A central result in the theory of Hilbert schemes of points on surfaces is the identification of their cohomology with the Fock space representation of the Heisenberg algebra by means of the *Nakajima operators* $q_{\ell,n}\colon \mathsf{H}^*\left(X\times X^{[\ell]},\mathbb{Q}\right)\to \mathsf{H}^*\left(X^{[n+\ell]},\mathbb{Q}\right)$; see [Nak97] and [Gro96]. They are induced by the correspondences

$$X\times X^{[\ell]}\times X^{[n+\ell]}\supset Z^{\ell,n}:=\{(x,[\xi],[\xi'])\mid \xi\subset\xi',\, \xi \text{ and } \xi' \text{ only differ in } x\}\,. \tag{1.1}$$

Recently, there has been successful effort towards lifting this action from cohomology to other invariants of the Hilbert schemes, in particular to K-theory and the derived category; see [FT11, SV13, CL12, Kru18].

Also recently, autoequivalences of the (bounded) derived categories $\mathrm{D}^b\left(X^{[n]}\right)$ of Hilbert schemes were intensively studied; see [Plo07, Add16, PS14, Mea15, Kru15, CLS14, KS15b]. In particular, the notion of $\mathbb{P}^n$*-functors* was introduced in [Add16]. These are Fourier–Mukai transforms $F\colon \mathrm{D}^b(M)\to\mathrm{D}^b(N)$ between derived categories of varieties (or, more generally, orbifolds) having a right adjoint $R\colon \mathrm{D}^b(N)\to\mathrm{D}^b(M)$ and the main property that

$$R\circ F\cong \mathsf{id}\oplus D\oplus D^2\oplus\cdots\oplus D^n$$

for some autoequivalence $D\colon \mathrm{D}^b(M)\to\mathrm{D}^b(M)$, called the $\mathbb{P}$*-cotwist* of F. Every $\mathbb{P}^n$-functor F induces an autoequivalence of the target category $\mathrm{D}^b(N)$, called the $\mathbb{P}$*-twist*. In [Add16], the main example of a $\mathbb{P}$-functor is the Fourier–Mukai transform $F_n=\mathsf{FM}_{\mathcal{I}_\Xi}\colon \mathrm{D}^b(X)\to\mathrm{D}^b\left(X^{[n]}\right)$ along the ideal sheaf of the universal family $\Xi\subset X\times X^{[n]}$ for X a K3 surface and $n\geqslant 2$.

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An important tool for the investigation of derived categories of Hilbert schemes of points on surfaces is the *derived McKay correspondence* of Bridgeland–King–Reid [BKR01] and Haiman [Hai01]. It is given by an equivalence of triangulated categories $\mathrm{D}^b\left(X^{[n]}\right) \cong \mathrm{D}^b_{\mathfrak{S}_n}(X^n)$, where $\mathrm{D}^b_{\mathfrak{S}_n}(X^n)$ denotes the derived category of $\mathfrak{S}_n$-equivariant coherent sheaves on the product X^n or, equivalently, of coherent sheaves on the quotient stack $[X^n/\mathfrak{S}_n]$.

In [Kru15], it was shown that, for every smooth surface X, there is a natural $\mathbb{P}^{n-1}$-functor $H_{0,n}\colon \mathrm{D}^b(X) \to \mathrm{D}^b_{\mathfrak{S}_n}(X^n)$, namely the equivariant push-forward along the embedding of the small diagonal. Under the derived McKay correspondence, $H_{0,n}$ corresponds to a $\mathbb{P}^{n-1}$-functor $\mathrm{D}^b(X) \to \mathrm{D}^b\left(X^{[n]}\right)$ whose kernel is supported on $Z^{0,n} = \left\{(x,[\xi']) \mid \mathsf{supp}(\xi') = \{x\}\right\}$; compare with (1.1). Thus, one can regard $H_{0,n}$ as a lift of the Nakajima operator $q_{0,n}\colon \mathsf{H}^*(X,\mathbb{Q}) \to \mathsf{H}^*\left(X^{[n]},\mathbb{Q}\right)$.

1.1 Main results

The question is whether the other Nakajima operators $q_{\ell,n}\colon \mathsf{H}^*\left(X\times X^{[\ell]},\mathbb{Q}\right) \to \mathsf{H}^*\left(X^{[n]},\mathbb{Q}\right)$, for general ℓ, have analogues in the form of $\mathbb{P}^{n-1}$-functors

$$H_{\ell,n}\colon \mathrm{D}^b_{\mathfrak{S}_\ell}\left(X\times X^{\ell}\right) \cong \mathrm{D}^b\left(X\times X^{[\ell]}\right) \to \mathrm{D}^b\left(X^{[n+\ell]}\right) \cong \mathrm{D}^b_{\mathfrak{S}_{n+\ell}}\left(X^{n+\ell}\right).$$

To get an idea of what these functors should look like, consider the *Ith partial diagonal*

$$\Delta_I := \{(x_1,\dots,x_{n+\ell}) \mid x_i = x_j \text{ for } i,j\in I\} \subset X^{n+\ell} \quad \text{for } I\subset\{1,\dots,n+\ell\} \text{ with } |I| = n\,,$$

and note that Δ_I is isomorphic to $X\times X^{\ell}$, the variety which defines the source category of our desired functor $H_{\ell,n}$. In view of the shape of the known $\mathbb{P}$-functors $H_{0,n}$ and the correspondences $Z^{\ell,n}$, it makes sense to expect the functors $H_{\ell,n}\colon \mathrm{D}^b_{\mathfrak{S}_\ell}\left(X\times X^{\ell}\right) \to \mathrm{D}^b_{\mathfrak{S}_{n+\ell}}\left(X^{n+\ell}\right)$ to have the property that the objects in its image are supported on the union of partial diagonals

$$\nabla^{\ell,n} := \bigcup_{I\subset\{1,\dots,\ell+n\},\,|I|=n} \Delta_I = \left\{(x_1,\dots,x_{n+\ell}) \mid \text{at least } n \text{ of the } x_i \text{ coincide}\right\} \subset X^{n+\ell}\,.$$

In this paper, we construct such $\mathbb{P}$-functors $H_{\ell,n}$ for $n>\ell$. The construction for $\ell>0$, however, is considerably more complicated than the already known construction of $H_{0,n}$. In particular, the (equivariant) Fourier–Mukai kernel of the functor $H_{\ell,n}$ is, for higher ℓ, not concentrated in degree zero anymore, which means that it is a proper complex. The detailed definition and description of the functors $H_{\ell,n}$ and their right adjoints $R_{\ell,n}$ is given in Sections 3.2, 3.3, and 3.4.

As an interesting by-product, our construction gives fully faithful functors if we replace the surface X with a curve. Note, however, that in the case of a curve, there is no McKay correspondence, so we only get a statement about the derived categories of the symmetric quotient stacks, not of the Hilbert schemes of points. See [PVdB19] for related results as part of a more general conjecture concerning semi-orthogonal decompositions on quotient stacks.

THEOREM 1.1. *There is a family of functors* $H_{\ell,n}\colon \mathrm{D}^b_{\mathfrak{S}_\ell}\left(X\times X^{\ell}\right) \to \mathrm{D}^b_{\mathfrak{S}_{n+\ell}}\left(X^{n+\ell}\right)$, *for every smooth variety* X *and every* $n > \mathsf{max}\{\ell,1\}$, *such that every object in the image of* $H_{\ell,n}$ *is supported on* $\nabla^{\ell,n}$, *with the following further properties:*

(i) *Let* $X = C$ *be a smooth curve.*

 (a) *We have* $R_{\ell,n}\circ H_{\ell,n} \cong \mathsf{id}$, *which means that* $H_{\ell,n}$ *is fully faithful.*
 (b) *Let* ℓ', n' *be positive integers with* $n'+\ell' = n+\ell$ *and* $\ell' > \ell$. *Then* $R_{\ell',n'}\circ H_{\ell,n} = 0$.

In summary, there is a semi-orthogonal decomposition

$$\mathrm{D}^b_{\mathfrak{S}_m}(C^m) = \langle \mathcal{A}_{0,m}, \mathcal{A}_{1,m-1}, \dots, \mathcal{A}_{r,m-r}, \mathcal{B} \rangle \,,$$

where $\mathcal{A}_{\ell,m-\ell} := H_{\ell,m-\ell}\big(\mathrm{D}^b_{\mathfrak{S}_\ell}(C \times C^\ell)\big) \cong \mathrm{D}^b_{\mathfrak{S}_\ell}(C \times C^\ell)$ *and* $r = \lfloor (m-1)/2 \rfloor$.

(ii) *Let* X *be a smooth surface. Then* $H_{\ell,n}$ *is a* $\mathbb{P}^{n-1}$*-functor with* $\mathbb{P}$*-cotwist* $\bar{S}_X^{-1}$*, using the notation* $\bar{S}_X := (\_) \otimes (\omega_X \boxtimes \mathcal{O}_{X^\ell})[2]$*. In particular,*

$$R_{\ell,n} \circ H_{\ell,n} \cong \mathsf{id} \oplus \bar{S}_X^{-1} \oplus \cdots \oplus \bar{S}_X^{-(n-1)} \,.$$

We show that in both cases, that of surfaces and that of curves, there are induced autoequivalences of $\mathrm{D}^b_{\mathfrak{S}_{n+\ell}}(X^{n+\ell})$ which can be regarded as 'characteristic' or 'indicator functors' of the strata $\nabla^{\ell,n}$. More precisely, the autoequivalence induced by $H_{\ell,n}$ acts on skyscraper sheaves of generic $\mathfrak{S}_n$-orbits of $\nabla^{n,\ell}$ as degree shift by a fixed non-zero number ($2-n = 1 - \mathsf{codim}\, \nabla^{n,\ell}$ in the curve case and $2 - 2n = -\mathsf{codim}\, \nabla^{n,\ell}$ in the surface case) and acts as the identity on objects of $\mathrm{D}^b_{\mathfrak{S}_{n+\ell}}(X^{n+\ell})$ whose support is contained in the complement of $\nabla^{\ell,n}$; see Proposition 7.1.

1.2 Structure and content of the text

In Section 2, we collect basic notions and results concerning equivariant Fourier–Mukai transforms, spherical functors, and $\mathbb{P}$-functors.

In Section 3, we introduce the functors $H_{\ell,n}\colon \mathrm{D}^b_{\mathfrak{S}_\ell}(X \times X^\ell) \to \mathrm{D}^b_{\mathfrak{S}_{n+\ell}}(X^{n+\ell})$ as the Fourier–Mukai transforms along certain equivariant complexes $\mathcal{H}_{\ell,n} \in \mathrm{D}^b_{\mathfrak{S}_\ell \times \mathfrak{S}_{n+\ell}}(X \times X^\ell \times X^{n+\ell})$. In Proposition 3.2, we describe the relation between the left and the right adjoint of $H_{\ell,n}$ which already confirms that one of the defining properties of a $\mathbb{P}$-functor holds for $H_{\ell,n}$.

The main part of the proof of Theorem 1.1 consists of the computation of the composition $R_{\ell,n} \circ H_{\ell,n}$. For this purpose, in Sections 4.1 and 4.2, we collect some results on the compositions of equivariant pull-backs and push-forward. In more fancy words, Theorem 1.1 is a statement about the equivariant derived intersection theory of cartesian products, and Sections 4.1 and 4.2 treat the basics of that intersection theory.

The calculus of (equivariant) Fourier–Mukai transforms allows us to go back and forth between compositions of Fourier–Mukai transforms and the convolution products of their kernels. Using the composition of functors makes things a bit easier combinatorially, compared to the convolution product approach. The reason is that the kernels $\mathcal{H}_{\ell,n}$ carry a linearisation by the group $\mathfrak{S}_\ell \times \mathfrak{S}_{\ell+n}$, while objects in the image of the functor $H_{\ell,n}$ are only $\mathfrak{S}_{\ell+n}$-linearised. On the other hand, it is easier to trace the maps induced by the differentials of the complex $\mathcal{H}_{\ell,n}$ on the level of the convolution products than on the level of composition of functors. This is the reason why we stay on the level of the kernels and avoid using the functors throughout Section 5, where Theorem 1.1 is proved. However, before that, we carry out computations for small values of ℓ on the level of functors in Sections 4.3–4.8. The purpose of Sections 4.3–4.8 is threefold: They are supposed to prepare the reader for the combinatorially more involved computations of Section 5 for general ℓ. They show that the assumption $n > \mathsf{max}\{\ell, 1\}$ is really necessary for our functors $H_{\ell,n}$ to fulfil the properties described in Theorem 1.1 (see Section 4.8). Finally, they show that the $H_{\ell,n}$ only give a partial categorification of Nakajima's and Grojnowski's Heisenberg action; see Sections 4.7 and 6.2.

Let us sketch the structure of the proof of Theorem 1.1 as carried out in Section 5. First, for X smooth of arbitrary dimension, we compute, in Sections 5.1–5.3, the convolution products $\mathcal{R}^i_{\ell,n} \star \mathcal{R}^j_{\ell,n}$, where $\mathcal{R}^i_{\ell,n}$ is the right adjoint kernel of $\mathcal{H}^i_{\ell,n}$, the degree i term of the complex $\mathcal{H}_{\ell,n}$.

There are spectral sequences relating the convolution products $\mathcal{R}^i_{\ell,n}\star\mathcal{H}^j_{\ell,n}$ to $\mathcal{R}_{\ell,n}\star\mathcal{H}_{\ell,n}$, the convolution product which essentially encodes the statement of Theorem 1.1. We carry out the computation of $\mathcal{R}_{\ell,n}\star\mathcal{H}_{\ell,n}$ first, in Sections 5.6 and 5.7, in the curve case and then, in Sections 5.9 and 5.10, in the surface case. This completes the proof of part (i)(a) of Theorem 1.1, and for part (ii) it is only left to prove that $R_{\ell,n}\circ H_{\ell,n}$ has the correct monad structure in the surface case, which is proved in Section 5.11. For the proof of Theorem 1.1(i)(b), we also need to compute, in the curve case, the convolution product $\mathcal{R}_{\ell',n'}\star\mathcal{R}_{\ell,n}$ for $n'+\ell'=n+\ell$ and $\ell'>\ell$. Here, we only provide the results of the key steps of the computation in Section 5.8. The details of the computation of $\mathcal{R}_{\ell',n'}\star\mathcal{R}_{\ell,n}$ are left to the reader, which is justified by the fact that they are straightforward generalisations of the computations of the special case $(\ell',n')=(\ell,n)$, as carried out in detail in the earlier subsections.

In Section 6, we further study the similarities between our $\mathbb{P}$-functors $H_{\ell,n}$ and the Nakajima operators $q_{\ell,n}$. Concretely, the support of the kernel $\mathcal{H}_{\ell,n}$ coincides under the McKay correspondence with $Z^{\ell,n}$, the correspondence defining $q_{\ell,n}$; see Section 6.1. Furthermore, the fact that the $H_{\ell,n}$ are $\mathbb{P}$-functors gives a categorical analogue of many, though not all, of the relations between the generators of the Heisenberg algebra; see Section 6.2.

In Section 7.1, we study the twists induced by the $\mathbb{P}$-functors $H_{\ell,n}$ in the surface case, making the idea that they are 'indicator autoequivalences' for the strata $\nabla^{\ell,n}\subset X^{\ell+n}$ more precise. In Section 7.2, we show that, in the case of a K3 surface, the twist along $H_{0,2}$ satisfies, in $\mathsf{Aut}\left(\mathrm{D}^b\left(X^{[2]}\right)\right)$, a braid relation with the twist along the spherical functor that was constructed in [Add16]. In Section 7.3, we show that the induction functor, which can be considered as the extension of our series of functors $H_{\ell,n}$ to the case $n=1$, is a $\mathbb{P}$-functor too. Section 7.4 deals with the curve case. We identify a further piece of our semi-orthogonal decomposition from Theorem 1.1(i) as $\mathrm{D}^b\left(C^{(m)}\right)$, the derived category of the symmetric quotient variety, and construct autoequivalences of $\mathrm{D}^b_{\mathfrak{S}_m}(C^m)$ with properties similar to the $\mathbb{P}$-twists from the surface case. In Section 7.5, we point out that, for X an abelian surface, our functors $H_{\ell,n}$ also restrict to $\mathbb{P}$-functors to the derived categories of generalised Kummer varieties. In the final Section 7.6, we make a conjecture about certain cases in which we expect the twists along the $H_{\ell,n}$ to generate the full group of derived autoequivalences of the Hilbert schemes and give an idea of which kind of autoequivalences might still wait to be constructed.

Convention. We will work over the complex numbers throughout, though many parts remain true over more general ground fields.

2. Preliminaries

2.1 Equivariant Fourier–Mukai transforms

For further details on equivariant derived categories and Fourier–Mukai transforms, we refer to [BKR01, Section 4] and [Plo07]. Let G be a finite group acting on a variety M. Then we denote by $\mathrm{D}^b_G(M):=\mathrm{D}^b(\mathrm{Coh}_G(M))$ the bounded derived category of the category $\mathrm{Coh}_G(M)$ of coherent G-equivariant sheaves. Let $U\subset G$ be a subgroup. Then there is the *forgetful* or *restriction* functor $\mathsf{Res}^U_G\colon \mathrm{D}^b_G(M)\to\mathrm{D}^b_U(M)$. It has the *induction* functor $\mathsf{Ind}^G_U\colon\mathrm{D}^b_U(M)\to\mathrm{D}^b_G(M)$ as a left and right adjoint. For $E\in\mathrm{D}^b_U(M)$, we have $\mathsf{Ind}^G_U(E)=\oplus_{[g]\in U\backslash G}\,g^*E$ with the G-linearisation given as a combination of the U-linearisation of E and the permutation of the direct summands. In the following, we will often simply write Res and Ind for these functors when the groups G and U should be clear from the context. In the case that G acts trivially on M, there is also the functor

$\mathsf{triv}\colon \mathrm{D}^b(M)\to \mathrm{D}^b_G(M)$ which equips every object with the trivial G-linearisation. Its left and right adjoint is given by the functor of invariants $(\_)^G\colon \mathrm{D}^b_G(M)\to \mathrm{D}^b(M)$.

Let G' be a second finite group acting on M'. Then every object $\mathcal{P}\in \mathrm{D}^b_{G\times G'}(M\times M')$ induces the *equivariant Fourier–Mukai transform*

$$\mathsf{FM}_{\mathcal{P}} := \big[\mathsf{pr}_{M'*}\big(\mathsf{pr}_M^*(\_)\otimes \mathcal{P}\big)\big]^{G\times 1}\colon\ \mathrm{D}^b_G(M)\to \mathrm{D}^b_{G'}(M')\,. \tag{2.1}$$

For example, if $M=M'$ and G acts trivially, the functor $\mathsf{triv}\colon \mathrm{D}^b(M)\to \mathrm{D}^b_G(M)$ is the Fourier–Mukai transform along $\mathcal{O}_\Delta\in \mathrm{D}^b_{1\times G}(M\times M)$ and $(\_)^G\colon \mathrm{D}^b_G(M)\to \mathrm{D}^b(M)$ is the Fourier–Mukai transform along $\mathcal{O}_\Delta\in \mathrm{D}^b_{G\times 1}(M\times M)$.

Let G'' be a third finite group acting on M'', and let $\mathcal{Q}\in \mathrm{D}^b_{G'\times G''}(M'\times M'')$. Then we have $\mathsf{FM}_{\mathcal{Q}}\circ \mathsf{FM}_{\mathcal{P}}=\mathsf{FM}_{\mathcal{Q}\star\mathcal{P}}$, where $\mathcal{Q}\star\mathcal{P}$ is the *equivariant convolution product*

$$\mathcal{Q}\star\mathcal{P} = \big[\mathsf{pr}_{M\times M''*}\big(\mathsf{pr}^*_{M'\times M''}\,\mathcal{Q}\otimes \mathsf{pr}^*_{M\times M'}\,\mathcal{P}\big)\big]^{1\times G'\times 1}\in \mathrm{D}^b_{G\times G''}(M\times M'')\,. \tag{2.2}$$

Remark 2.1. The functors triv and Res do not really change objects when applied to them but only consider them canonically as equivariant objects with respect to a different group. Hence, one can usually omit them from formulae without ambiguity, and we will do this occasionally in order to keep formulae short. In fact, this is already the case in the definition of the equivariant Fourier–Mukai transform. Namely, in (2.1), strictly speaking, the notation pr_M^* stands, for the composition

$$\mathrm{D}^b_G(M)\xrightarrow{\mathsf{triv}} \mathrm{D}^b_{G\times G'}(M)\xrightarrow{\mathsf{pr}_M^*} \mathrm{D}^b_{G\times G'}(M\times M')\,.$$

Note, however, that one has to mind all the functors triv and Res when taking the adjoint of a composition of functors, as their adjoints, the functor of invariants and the induction functor, act non-trivially on objects. This becomes relevant in Section 3.4.

Remark 2.2. For $L\in \mathrm{D}^b_G(M)$, the tensor product functor $(\_)\otimes L\colon \mathrm{D}^b_G(M)\to \mathrm{D}^b_G(M)$ is given by $\mathsf{Ind}_{G_\Delta}^{G\times G}\,\delta_*L\cong \oplus_{g\in G}(1\times g)_*L$, where $\delta=(1\times 1)\colon M\to M\times M$ is the diagonal embedding. This can be confirmed quite easily using the principle for the computation of invariants explained in Section 2.2. In particular, the identity functor $\mathsf{id}\colon \mathrm{D}^b_G(M)\to \mathrm{D}^b_G(M)$ is the Fourier–Mukai transform along the kernel $\mathsf{Ind}_{G_\Delta}^{G\times G}\,\mathcal{O}_\Delta\cong \oplus_{g\in G}\mathcal{O}_{\Gamma_g}$.

2.2 Invariants of inductions

For the computation of the invariants of equivariant objects, we will frequently use the following principle; compare with [Dan01, Lemma 2.2] and [Sca09, Remark 2.4.2]. Let M be a variety with an action of a finite group G. Let $\mathcal{E}=(E,\lambda)\in \mathrm{D}^b_G(M)$ be such that $E=\oplus_{i\in I}E_i$ in $\mathrm{D}^b(M)$ for some finite index set I. Let us assume that there is an action of G on I such that $\lambda_g(E_i)=g^*E_{g(i)}$ for all $i\in I$. We say that the G-action on I *is induced by* the G-linearisation of E. We denote E_i together with the G_i-linearisation $(\lambda_{g|E_i})_{g\in G_i}$ by $\mathcal{E}_i\in \mathrm{D}^b_{G_i}(M)$, where $G_i=\mathrm{Stab}_G(i)$. The induced action of G on I is transitive if and only if $\mathcal{E}\cong \mathsf{Ind}_{G_i}^G\,\mathcal{E}_i$ for any $i\in I$; see [BL94, Section 8.2].

Let G act trivially on M, and assume that G acts transitively on I. Then, for every $i\in I$, the projection $E\to E_i$ induces an isomorphism $\mathcal{E}^G\cong \mathcal{E}_i^{G_i}$. The inverse is $s\mapsto \oplus_{[g]\in G_i\backslash G}\,\lambda_g(s)$.

Now, let the action of G on I be not transitive, and let $i_1,\dots,i_k$ be a system of representatives of the G-orbits. Then $\mathcal{E}\cong \mathsf{Ind}_{G_{i_1}}^G\,\mathcal{E}_{i_1}\oplus\cdots\oplus \mathsf{Ind}_{G_{i_k}}^G\,\mathcal{E}_{i_k}$ and

$$\mathcal{E}^G\cong \mathcal{E}_{i_1}^{G_{i_1}}\oplus\cdots\oplus \mathcal{E}_{i_1}^{G_{i_k}}\,. \tag{2.3}$$

2.3 Some standard identities for equivariant functors

In this subsection, we collect some isomorphisms of functors between equivariant derived categories for later use; see, in particular, the proof of Proposition 3.3.

Let G be a finite group acting on a smooth variety M, and let $U \leqslant G$ be a subgroup. We have

$$\mathsf{Ind}_U^G E \otimes F \cong \mathsf{Ind}_U^G \big(E \otimes \mathsf{Res}_G^U(\_)\big)\,. \tag{2.4}$$

This can be seen as a stacky version of the projection formula, using the identification of $\mathrm{D}^b_G(M)$ with the derived category of coherent sheaves on the quotient stack $[M/G]$, but it can also be deduced quite directly from the definitions of the functors Res and Ind.

Let N be a second smooth variety on which G acts, and let $f\colon M \to N$ be a G-equivariant proper morphism. Then

$$\begin{aligned} \mathsf{Res}_G^U \circ f^* \cong f^* \circ \mathsf{Res}_G^U\,, \quad \mathsf{Res}_G^U \circ f_* \cong f_* \circ \mathsf{Res}_G^U\,,\\ \mathsf{Ind}_G^U \circ f^* \cong f^* \circ \mathsf{Ind}_G^U\,, \quad \mathsf{Ind}_G^U \circ f_* \cong f_* \circ \mathsf{Ind}_G^U\,. \end{aligned} \tag{2.5}$$

The isomorphisms involving the restriction functor are quite obvious. The isomorphisms involving the induction functor follow from the ones involving the restriction functor by adjunction. If the G-action on M and N is the trivial one, we also have

$$\mathsf{triv} \circ f^* \cong f^* \circ \mathsf{triv}\,, \quad \mathsf{triv} \circ f_* \cong f_* \circ \mathsf{triv}\,, \tag{2.6}$$

$$(\_)^G \circ f^* \cong f^* \circ (\_)^G\,, \quad (\_)^G \circ f_* \cong f_* \circ (\_)^G\,. \tag{2.7}$$

Now, let H be another finite group, let H act trivially on M, and let $V \leqslant H \times G$ be a subgroup. We identify H with the subgroup $H \times 1 \leqslant H \times G$ and write $\pi_G\colon H \times G \to G$ for the projection to the second factor. Then, for an object $E \in \mathrm{D}^b_V(M)$, its invariants $E^{H\cap V}$ carry a canonical $\pi_G(V)$-linearisation, which means that we have a functor $(\_)^{H\cap V}\colon \mathrm{D}^b_V(M) \to \mathrm{D}^b_{\pi_G(V)}(M)$. There is an isomorphism

$$(\_)^H \circ \mathsf{Ind}_V^{H\times G} \cong \mathsf{Ind}_{\pi_G(V)}^G \circ (\_)^{V\cap H} \tag{2.8}$$

of functors from $\mathrm{D}^b_V(M)$ to $\mathrm{D}^b_G(M)$. This follows from the principle described in Section 2.2. Indeed, for $E \in \mathrm{D}^b_V(M)$, we have $\mathsf{Ind}_V^{H\times G}(E) \cong \oplus_{V\backslash(H\times G)} \sigma^* E$ with the H-linearisation of $\mathsf{Ind}_V^{H\times G}(E)$ inducing the action $h\colon V\sigma \mapsto V\sigma h^{-1}$ on the index set $V \setminus (H \times G)$. The stabiliser of V under this action is $V \cap H$, and the fibres of the natural map $V \setminus (H \times G) \to \pi_G(V) \setminus G$, $V\sigma \mapsto \pi_G(V)\pi_G(\sigma)$ are exactly the H-orbits. Hence, by (2.3), we get

$$\big(\mathsf{Ind}_V^{H\times G}(E)\big)^V \cong \Big(\bigoplus_{V\backslash(H\times G)} \sigma^* E\Big)^V \cong \bigoplus_{\pi_G(V)\backslash G} g^*\big(E^{H\cap V}\big) \cong \mathsf{Ind}_{\pi_G(V)}^G E^{V\cap H}\,.$$

2.4 ℙ-functors

Let G and H be finite groups acting on varieties M and N. Following [Add16], a $\mathbb{P}^n$-*functor* is an (equivariant) Fourier–Mukai transform $F\colon \mathrm{D}^b_G(M) \to \mathrm{D}^b_H(N)$ with right and left adjoints $F^R, F^L\colon \mathrm{D}^b_H(N) \to \mathrm{D}^b_G(M)$ such that the following hold:

(1) There is an autoequivalence D of $\mathrm{D}^b_G(M)$, called the ℙ-*cotwist* of F, such that

$$F^R \circ F \cong \mathsf{id} \oplus D \oplus D^2 \oplus \cdots \oplus D^n\,.$$

(2) Let $\varepsilon\colon F\circ F^R \to \mathsf{id}$ be the counit of the adjunction, and consider the map

$$D\oplus D^2\oplus\cdots\oplus D^{n+1} \cong D\circ F^R\circ F \hookrightarrow F^R\circ F\circ F^R\circ F \xrightarrow{F^R\varepsilon F} F^R\circ F \cong \mathsf{id}\oplus D\oplus\cdots\oplus D^n\,, \quad (2.9)$$

where the two isomorphisms and the embedding are given by the isomorphism of condition (1). The components $D^i \to D^j$ of (2.9) are isomorphisms for $i=j$ and are zero for $i<j$ (there is no condition on the components $D^i\to D^j$ with $i>j$).

(3) There is an isomorphism $F^R \cong D^n\circ F^L$. If $\mathrm{D}^b_G(M)$ and $\mathrm{D}^b_H(N)$ have Serre functors, this is equivalent to $S_N\circ F\circ D^n \cong F\circ S_M$.

A $\mathbb{P}^1$-functor is the same as a split spherical functor. A general *spherical functor* is a Fourier–Mukai transform S such that $C:=\mathsf{cone}\big(\mathsf{id}\xrightarrow{\eta} S^R\circ S\big)$ is an autoequivalence and $S^R\cong C\circ S^L$. Here, η is the unit of the adjunction. The cone is well defined as a Fourier–Mukai transform since the natural transform η is induced by a morphism between the kernels; see [AL12]. This is the reason why we restrict ourself in the definition of spherical and $\mathbb{P}$-functors to Fourier–Mukai transforms between derived categories of coherent sheaves. More generally, one can work with dg-enhanced triangulated categories; see [AL17].

2.5 Spherical and $\mathbb{P}$-twists

For $S\colon \mathrm{D}^b_G(M)\to\mathrm{D}^b_H(N)$ a spherical functor, the associated *spherical twist* is defined as the cone $T_S:=\mathsf{cone}(S\circ S^R\xrightarrow{\varepsilon}\mathsf{id})$ of the counit. It is an autoequivalence of $\mathrm{D}^b_H(N)$ satisfying

$$T_S\circ S\cong S\circ C[1]\,,\quad T_S(B)=B\quad \text{if } S^R(B)=0\,; \qquad (2.10)$$

see [Add16, Section 2]. The construction of the $\mathbb{P}$-*twist* $P_F\in\mathsf{Aut}(\mathrm{D}^b_H(N))$ associated with a $\mathbb{P}^n$-functor $F\colon \mathrm{D}^b_G(M)\to\mathrm{D}^b_H(N)$ is a bit more complicated. As we do not need the concrete construction, we refer to [Add16, Section 4.3] for it. In analogy to (2.10), the twist P_F satisfies

$$P_F\circ F\cong F\circ D^{n+1}[2]\,,\quad P_F(B)=B\quad\text{if } F^R(B)=0\,. \qquad (2.11)$$

Given additional autoequivalences $\Psi\in\mathsf{Aut}(\mathrm{D}^b_G(M))$ and $\Phi\in\mathsf{Aut}(\mathrm{D}^b_H(N))$, we have

$$T_{S\circ\psi}\cong T_S\,,\quad P_{F\circ\Psi}\cong P_F\,,\quad T_{\Phi\circ S}\cong\Phi\circ T_S\circ\Phi^{-1}\,,\quad P_{\Phi\circ F}\cong\Phi\circ P_F\circ\Phi^{-1}\,; \qquad (2.12)$$

see [AA13, Proposition 13] and [Kru15, Lemma 2.3]. In the case that F is a $\mathbb{P}^1$-functor, that is, split spherical, the spherical and the $\mathbb{P}$-twist are related by $T_F^2\cong P_F$.

2.6 Braid relations between twists along spherical functors

We say that two elements a and b of a group satisfy the *braid relation* if $aba=bab$. Two twists T_E and T_F along spherical *objects* satisfy the braid relation if $\mathsf{Hom}^*(E,F)=\mathbb{C}[n]$ for some $n\in\mathbb{Z}$; see [ST01]. There is the following straightforward generalisation which gives a criterion for twists along spherical *functors* to satisfy the braid relation; compare with [AL17, Theorem 1.2].

Proposition 2.3. *Let $F=\mathsf{FM}_{\mathcal{F}}, H=\mathsf{FM}_{\mathcal{H}}\colon \mathrm{D}^b(M)\to\mathrm{D}^b(N)$ be two spherical functors such that $F^R\circ H\cong\mathsf{id}$ and $\mathsf{Hom}_{\mathrm{D}^b(M\times N)}(\mathcal{F},\mathcal{H})=\mathbb{C}$. Let $\mathcal{G}=\mathsf{cone}\big(\mathcal{F}\xrightarrow{\psi}\mathcal{H}\big)$ for $0\neq\psi\in\mathsf{Hom}(\mathcal{F},\mathcal{H})$, and set $G=\mathsf{FM}_{\mathcal{G}}$. Then G is also a spherical functor, and every pair of T_F, T_G, T_H spans $\langle T_F,T_G,T_H\rangle$ and satisfies the braid relation. In particular, $T_F\circ T_H\circ T_F\cong T_H\circ T_F\circ T_H$.*

Proof. Composing the triangle $F\circ F^R\to\mathsf{id}\to T_F$ with H and using that $F^R\circ H\cong\mathsf{id}$, we get the triangle $F\to H\to T_F\circ H$. The map $F\to H$ of this triangle is non-zero. Indeed, otherwise, we would have $T_F\circ H=H\oplus F[1]$, contradicting the fact that $T_F\circ H$ is again spherical. Because of $\mathsf{Hom}(\mathcal{F},\mathcal{H})=\mathbb{C}$, it follows that $G\cong T_F\circ H$. This shows that G is spherical and, by (2.12), there is the relation

$$T_G\cong T_F\circ T_H\circ T_F^{-1}\,. \tag{2.13}$$

The exact triangle $T_H^{-1}\to\mathsf{id}\to H\circ H^L$ induces the exact triangle $T_H^{-1}F\to F\to H$ since $H^L\circ F\cong\mathsf{id}$. The latter triangle shows that $T_H^{-1}F=G[-1]$ and, by (2.12),

$$T_G\cong T_H^{-1}\circ T_F\circ T_H\,. \tag{2.14}$$

The assertion follows from the equations (2.13) and (2.14). □

If M and N are projective, the second assumption of the proposition, namely $\mathsf{Hom}(\mathcal{F},\mathcal{H})=\mathbb{C}$, follows already from the first assumption, namely $F^R\circ H\cong\mathsf{id}$.

3. The functors $H_{\ell,n}$: Definition and first properties

3.1 Notation and conventions

(1) For a complex E, write $\mathscr{H}^i(E)$ for its ith cohomology, and set $\mathscr{H}^*(E):=\oplus_{i\in\mathbb{Z}}\mathscr{H}^i(E)[-i]$.

(2) The *alternating* or *sign representation* $\mathfrak{a}_n$ of the symmetric group $\mathfrak{S}_n$ is the 1-dimensional representation on which $\sigma\in\mathfrak{S}_n$ acts by multiplication by $\operatorname{sgn}(\sigma)$. If $\mathfrak{S}_n$ acts on a variety T, we set $\mathsf{M}_{\mathfrak{a}_n}:=(\_)\otimes\mathfrak{a}\colon \mathrm{D}^b_{\mathfrak{S}_n}(T)\to\mathrm{D}^b_{\mathfrak{S}_n}(T)$. It is the autoequivalence which sends an object (E,λ) to $(E,\bar\lambda)$, where the linearisation $\bar\lambda$ is given by $\bar\lambda_\sigma=\operatorname{sgn}(\sigma)\cdot\lambda_\sigma$.

(3) For $u\leqslant v$ positive integers, we use the notation $[u,v]:=\{u,u+1,\dots,v\}\subset\mathbb{N}$ and $[v]:=[1,v]=\{1,\dots,v\}\subset\mathbb{N}$.

(4) We set $[0]:=\emptyset$.

(5) For $A,B\subset\mathbb{N}$ two finite subsets of the same cardinality $|A|=|B|$, we let $e\colon A\to B$ denote the unique strictly increasing bijection.

3.2 The Fourier–Mukai kernel

Let X be a smooth variety of arbitrary dimension $d=\dim X$. In the following, we will construct the functors $H_{\ell,n}\colon \mathrm{D}^b_{\mathfrak{S}_\ell}\left(X\times X^\ell\right)\to\mathrm{D}^b_{\mathfrak{S}_{n+\ell}}\left(X^{n+\ell}\right)$ for $n,\ell\in\mathbb{N}$ with $n\geqslant 2$. For $i=1,\dots,\ell$, we set

$$\mathsf{Index}_{\ell,n}(i):=\left\{(I,J,\mu)\mid I\subset[\ell],\ |I|=i,\ J\subset[n+\ell],\ |J|=n+i,\ \mu\colon\bar I\to\bar J\text{ bijection}\right\},$$

where $\bar I:=[\ell]\setminus I$ and $\bar J:=[n+\ell]\setminus J$ denote the complements of I and J, respectively. For $(I,J,\mu)\in\mathsf{Index}_{\ell,n}(i)$, we consider the subvariety $\Gamma_{I,J,\mu}\subset X\times X^\ell\times X^{n+\ell}$ given by

$$\Gamma_{I,J,\mu}:=\left\{(x,x_1,\dots,x_\ell,y_1,\dots,y_{n+\ell})\mid x=x_a=y_b\ \forall\, a\in I,\ b\in J,\ x_c=y_{\mu(c)}\ \forall\, c\in\bar I\right\}.$$

This subvariety is invariant under the action of the subgroup

$$\mathfrak{S}_I\times\mathfrak{S}_{\bar I,\mu}\times\mathfrak{S}_J:=\{(\sigma,\tau)\mid\sigma(I)=I,\ \sigma(J)=J,\ (\mu\circ\sigma)_{|\bar I}=(\tau\circ\mu)_{|\bar I}\}\subset\mathfrak{S}_\ell\times\mathfrak{S}_{n+\ell}\,, \tag{3.1}$$

and thus $\mathcal{O}_{I,J,\mu}:=\mathcal{O}_{\Gamma_{I,J,\mu}}$ carries a canonical linearisation by this subgroup. Note that there is the isomorphism of groups $\mathfrak{S}_I\times\mathfrak{S}_{\bar I,\mu}\times\mathfrak{S}_J\cong\mathfrak{S}_i\times\mathfrak{S}_{\ell-i}\times\mathfrak{S}_{n+i}$ given by $(\sigma,\tau)\mapsto(\sigma_{|I},\sigma_{|\bar I},\tau_{|J})$. Let $\mathfrak{a}_J$ denote the 1-dimensional representation of $\mathfrak{S}_I\times\mathfrak{S}_{\bar I,\mu}\times\mathfrak{S}_J$ on which the factor $\mathfrak{S}_J=\{\sigma=\mathsf{id}\}$ acts by the sign of the permutations and the other factor $\mathfrak{S}_I\times\mathfrak{S}_{\bar I,\mu}=\{\tau_{|J}=\mathsf{id}_J\}$ acts trivially. We set $\mathcal{H}(I,J,\mu):=\mathcal{O}_{I,J,\mu}\otimes\mathfrak{a}_J$ and

$$\mathcal{H}^i_{\ell,n}:=\mathsf{Ind}^{\mathfrak{S}_\ell\times\mathfrak{S}_{n+\ell}}_{\mathfrak{S}_{[i]}\times\mathfrak{S}_{[i+1,\ell],e}\times\mathfrak{S}_{[n+i]}}\mathcal{H}([i],[n+i],e)=\bigoplus_{\mathsf{Index}_{\ell,n}(i)}\mathcal{H}(I,J,\mu)\,.$$

For $c \in \bar{I}$, we have $\Gamma_{I\cup\{c\},J\cup\{\mu(c)\},\mu_{|\bar{I}\setminus\{c\}}} \subset \Gamma_{I,J,\mu}$. This allows us to define a morphism $d^i\colon \mathcal{H}^i \to \mathcal{H}^{i+1}$ by letting the component $\mathcal{H}(I,J,\mu) \to \mathcal{H}(I\cup\{c\}, J\cup\{\mu(c)\}, \mu_{|\bar{I}\setminus\{c\}})$ be $(-1)^{\#\{b\in J \mid b<\mu(c)\}}$ times the map given by restriction of sections and setting all components $\mathcal{H}(I,J,\mu)\to\mathcal{H}(I',J',\mu')$ which are not of this form to be zero.

Lemma 3.1. *The morphisms d form a differential: $d\circ d = 0$.*

Proof. For a local section $s \in \mathcal{H}^i_{\ell,n}$ and $(I,J,\mu) \in \mathsf{Index}(j)$, we denote by $s(I,J,\mu)$ the component of s in $\mathcal{H}(I,J,\mu)$. We have to show that $d^2(s)(I,J,\mu) = 0$ for every $s \in \mathcal{H}^i$ and every $(I,J,\mu) \in \mathsf{Index}(i{+}2)$. By the definition of d, the only components of s possibly contributing to $d^2(s)(I,J,\mu)$ are of the form $s\big(I\setminus\{c,d\}, J\setminus\{e,f\}, \hat{\mu}\big)$ with $\hat{\mu}(\{c,d\}) = \{e,f\}$ and $\hat{\mu}_{|\bar{I}} = \mu$. In fact, $s\big(I\setminus\{c,d\}, J\setminus\{e,f\}, \hat{\mu}\big)$ contributes via two different compositions of components of d^i and d^{i+1}, namely

$$\mathcal{H}\big(I\setminus\{c,d\}, J\setminus\{e,f\}, \hat{\mu}\big) \to \mathcal{H}\big(I\setminus\{c\}, J\setminus\{\hat{\mu}(c)\}, \hat{\mu}_{|\bar{I}\cup c}\big) \to \mathcal{H}(I,J,\mu)\,,$$
$$\mathcal{H}\big(I\setminus\{c,d\}, J\setminus\{e,f\}, \hat{\mu}\big) \to \mathcal{H}\big(I\setminus\{d\}, J\setminus\{\hat{\mu}(d)\}, \hat{\mu}_{|\bar{I}\cup d}\big) \to \mathcal{H}(I,J,\mu)\,.$$

One can check that these two contributions are given by $\pm s\big(I\setminus\{c,d\}, J\setminus\{e,f\}, \hat{\mu}\big)_{|\Gamma_{I,J,\mu}}$ with opposite signs; hence, they cancel out. □

Accordingly, we have defined a complex of $(\mathfrak{S}_\ell \times \mathfrak{S}_{n+\ell})$-equivariant sheaves

$$\mathcal{H}_{\ell,n} := \big(0 \to \mathcal{H}^0 \to \cdots \to \mathcal{H}^\ell \to 0\big) \in \mathrm{D}^b_{\mathfrak{S}_\ell\times\mathfrak{S}_{n+\ell}}\left(X \times X^\ell \times X^{n+\ell}\right),$$

and we define the functor $H_{\ell,n}$ to be the equivariant Fourier–Mukai transform along this complex:

$$H_{\ell,n} := \mathsf{FM}_{\mathcal{H}_{\ell,n}}\colon\ \mathrm{D}^b_{\mathfrak{S}_\ell}\left(X\times X^\ell\right) \to \mathrm{D}^b_{\mathfrak{S}_{n+\ell}}\left(X^{n+\ell}\right).$$

3.3 Adjoint kernels

Even though we do not assume that X is projective, since $X \times X^\ell$ and $X^{n+\ell}$ are smooth and $\mathsf{supp}\,\mathcal{H}_{\ell,n} = \bigcup \Gamma_{I,J,\mu}$ is finite, hence projective, over $X\times X^\ell$ as well as over $X^{n+\ell}$, the functor $H_{\ell,n}$ has right and left adjoints $R_{\ell,n}, L_{\ell,n}\colon \mathrm{D}^b_{\mathfrak{S}_{n+\ell}}\left(X^{n+\ell}\right) \to \mathrm{D}^b_{\mathfrak{S}_\ell}\left(X\times X^\ell\right)$. They are the equivariant Fourier–Mukai transforms $R_{\ell,n} = \mathsf{FM}_{\mathcal{R}_{\ell,n}}$ and $L_{\ell,n} = \mathsf{FM}_{\mathcal{L}_{\ell,n}}$ with

$$\mathcal{R}_{\ell,n} = \mathcal{H}^\vee_{\ell,n}\otimes(\omega_{X\times X^\ell}\boxtimes\mathcal{O}_{X^{n+\ell}})[(\ell+1)d]\,,\quad \mathcal{L}_{\ell,n} = \mathcal{H}^\vee_{\ell,n}\otimes(\mathcal{O}_{X\times X^\ell}\boxtimes\omega_{X^{n+\ell}})[(n+\ell)d]\,;\quad (3.2)$$

see, for example, [Kuz06, Section 2.1]. The left and the right adjoints of $H_{\ell,n}$ are related as follows.

Proposition 3.2. (i) *If $\dim X$ is even, we have $R_{\ell,n} \cong \bar{S}_X^{-(n-1)} \circ L_{\ell,n}$.*

(ii) *If $\dim X$ is odd, we have $\mathsf{M}_{\mathfrak{a}_\ell}\circ R_{\ell,n} \cong \bar{S}_X^{-(n-1)}\circ L_{\ell,n}\circ \mathsf{M}_{\mathfrak{a}_{n+\ell}}$.*

Proof. For the underlying non-equivariant functors $\mathrm{D}^b\left(X^{\ell+n}\right) \to \mathrm{D}^b\left(X\times X^\ell\right)$, by (3.2), the assertion amounts to the invariance of $\mathcal{H}^\vee_{\ell,n}$ under tensor product by $\omega_X^n\boxtimes\omega_{X^\ell}\boxtimes\omega^{-1}_{X^{n+\ell}}$; compare with [Orl03, Proposition 2.1.6]. This invariance follows from the fact that

$$\big(\omega_X^n\boxtimes\omega_{X^\ell}\boxtimes\omega^{-1}_{X^{n+\ell}}\big)_{|\Gamma_{I,J,\mu}} \cong \mathcal{O}_{I,J,\mu}\quad \text{for all } 0\leqslant i\leqslant \ell \text{ and } (I,J,\mu)\in\mathsf{Index}_{\ell,n}(i)\,.$$

For the equivariant functors $\mathrm{D}^b_{\mathfrak{S}_{n+\ell}}\left(X^{n+\ell}\right) \to \mathrm{D}^b_{\mathfrak{S}_\ell}\left(X\times X^\ell\right)$, the difference between the cases of odd and of even dimension is explained by the difference in the sign of the linearisations of the equivariant canonical bundle of a product X^m. Indeed, on the fibre $\omega_{X^m}(x)$, the stabiliser of $x\in X^m$ acts by permuting blocks of length $d = \dim X$ in the wedge product. Hence, if X is even-dimensional, the $\mathfrak{S}_m$-equivariant canonical bundle of X^m equals $\omega_{X^m}\cong\omega_X^{\boxtimes m}$, where the linearisation is the one acting by permuting the factors, while in the odd-dimensional case, the linearisation is given by $\omega_{X^m}\cong\omega_X^{\boxtimes m}\otimes\mathfrak{a}_m$. □

This already proves one first piece of Theorem 1.1, namely that for X a smooth surface, the functor $H_{\ell,n}$ satisfies condition (3) of a $\mathbb{P}^{n-1}$-functor with cotwist $\bar S_X^{-1}:=(\_)\otimes\big(\omega_X^{-1}\boxtimes\mathcal{O}_{X^\ell}\big)[-2]$.

3.4 Description of the functor

For $I\subset[\ell]$ and $J\subset[n+\ell]$ with $|I|=i$ and $|J|=n+i$, we consider the closed embeddings

$$\iota_I\colon X\times X^{\ell-i}\hookrightarrow X\times X^\ell,\ (x,x_1,\dots,x_{\ell-i})\mapsto(x,y_1,\dots,y_\ell),\ y_i=x\text{ for }i\in I,\ y_i=x_{e(i)}\text{ for }i\notin I\,,$$
$$\delta_J\colon X\times X^{\ell-i}\hookrightarrow X^{n+\ell},\ (x,x_1,\dots,x_{\ell-i})\mapsto(z_1,\dots,z_{n+\ell}),\ z_j=x\text{ for }j\in J,\ z_j=x_{e(j)}\text{ for }j\notin J\,,$$

where e denotes the strictly increasing bijections $[\ell]\setminus I\to[\ell-i]$ and $[n+\ell]\setminus J\to[\ell-i]$ in the definitions of ι_I and δ_J, respectively. The images of these closed embeddings are the *partial diagonals* $D_I\subset X\times X^\ell$ and $\Delta_J\subset X\times X^{n+\ell}$ given by

$$D_I=\{(x,x_1,\dots,x_\ell)\mid x=x_a\,\forall\,a\in I\}\,,\quad \Delta_J=\{(y_1,\dots,y_{n+\ell})\mid y_a=y_b\,\forall\,a,b\in J\}\,.$$

We set $\iota_{[0]}=\iota_\emptyset:=\mathsf{id}\colon X\times X^\ell\to X\times X^\ell$.

PROPOSITION 3.3. *The functor $H^i_{\ell,n}:=\mathsf{FM}_{\mathcal{H}^i_{\ell,n}}$ is isomorphic to the composition*

$$\begin{aligned}&\mathrm{D}^b_{\mathfrak{S}_\ell}\big(X\times X^\ell\big)\xrightarrow{\mathsf{Res}}\mathrm{D}^b_{\mathfrak{S}_i\times\mathfrak{S}_{\ell-i}}\big(X\times X^\ell\big)\xrightarrow{\iota^*_{[i]}}\mathrm{D}^b_{\mathfrak{S}_i\times\mathfrak{S}_{\ell-i}}\big(X\times X^{\ell-i}\big)\\&\xrightarrow{(\_)^{\mathfrak{S}_i}}\mathrm{D}^b_{\mathfrak{S}_{\ell-i}}\big(X\times X^{\ell-i}\big)\xrightarrow{\mathsf{triv}}\mathrm{D}^b_{\mathfrak{S}_{n+i}\times\mathfrak{S}_{\ell-i}}\big(X\times X^{\ell-i}\big)\xrightarrow{\mathsf{M}_{\mathfrak{a}_{n+i}}}\mathrm{D}^b_{\mathfrak{S}_{n+i}\times\mathfrak{S}_{\ell-i}}\big(X\times X^{\ell-i}\big)\\&\xrightarrow{\delta_{[n+i]*}}\mathrm{D}^b_{\mathfrak{S}_{n+i}\times\mathfrak{S}_{\ell-i}}\big(X^{n+i}\times X^{\ell-i}\big)\xrightarrow{\mathsf{Ind}}\mathrm{D}^b_{\mathfrak{S}_{n+\ell}}\big(X^{n+\ell}\big)\,.\end{aligned}\tag{3.3}$$

Proof. Note that $\Gamma_{[i],[n+i],e}=(\iota_{[i]},\delta_{[n+i]})\big(X^{\ell-i}\big)$. Hence, by the projection formula,

$$\mathsf{pr}_{X^{n+\ell}*}(\mathcal{O}_{[i],[n+i],e}\otimes\mathsf{pr}^*_{X^\ell}(\_))\cong\delta_{[n+i]*}\circ\iota^*_{[i]}\,.\tag{3.4}$$

Setting $\mathfrak{S}(i):=\mathfrak{S}_{[i]}\times\mathfrak{S}_{[i+1,\ell],e}\times\mathfrak{S}_{[n+i]}$ (compare with (3.1)), we get

$$\begin{aligned}\mathsf{FM}_{\mathcal{H}^i_{\ell,n}}&\cong\Big[\mathsf{pr}_{X^{n+\ell}*}\Big(\mathsf{Ind}^{\mathfrak{S}_\ell\times\mathfrak{S}_{n+\ell}}_{\mathfrak{S}(i)}\mathcal{H}([i],[n+i],e)\otimes\mathsf{pr}^*_{X\times X^\ell}(\_)\Big)\Big]^{\mathfrak{S}_\ell}\\&\overset{(2.4)}{\cong}\Big[\mathsf{pr}_{X^{n+\ell}*}\Big(\mathsf{Ind}^{\mathfrak{S}_\ell\times\mathfrak{S}_{\ell+n}}_{\mathfrak{S}(i)}(\mathcal{H}([i],[n+i],e)\otimes\mathsf{Res}^{\mathfrak{S}(i)}_{\mathfrak{S}_\ell\times\mathfrak{S}_{\ell+n}}\mathsf{pr}^*_{X\times X^\ell}(\_))\Big)\Big]^{\mathfrak{S}_\ell}\\&\overset{(2.5)}{\cong}\Big[\mathsf{Ind}^{\mathfrak{S}_\ell\times\mathfrak{S}_{\ell+n}}_{\mathfrak{S}(i)}\Big(\mathsf{pr}_{X^{n+\ell}*}(\mathcal{H}([i],[n+i],e)\otimes\mathsf{Res}^{\mathfrak{S}(i)}_{\mathfrak{S}_i\times\mathfrak{S}_{\ell-i}\times\mathfrak{S}_{\ell+n}}\mathsf{pr}^*_{X\times X^\ell}\mathsf{Res}^{\mathfrak{S}_i\times\mathfrak{S}_{\ell-i}}_{\mathfrak{S}_\ell}(\_))\Big)\Big]^{\mathfrak{S}_\ell}\\&\overset{(3.4)}{\cong}\Big[\mathsf{Ind}^{\mathfrak{S}_\ell\times\mathfrak{S}_{\ell+n}}_{\mathfrak{S}(i)}\Big(\delta_{[n+i]*}\big(\mathfrak{a}_{[n+i]}\otimes\iota_{[i]*}\mathsf{Res}^{\mathfrak{S}_i\times\mathfrak{S}_{\ell-i}}_{\mathfrak{S}_\ell}(\_)\big)\Big)\Big]^{\mathfrak{S}_\ell}\\&\overset{(2.8)}{\cong}\mathsf{Ind}^{\mathfrak{S}_{\ell+n}}_{\mathfrak{S}_{[n+i]}\times\mathfrak{S}_{\ell-i}}\Big(\delta_{[n+i]*}\big(\mathfrak{a}_{[n+i]}\otimes\iota_{[i]*}\mathsf{Res}^{\mathfrak{S}_i\times\mathfrak{S}_{\ell-i}}_{\mathfrak{S}_\ell}(\_)\big)\Big)^{\mathfrak{S}_i}\\&\overset{(2.7)}{\cong}\mathsf{Ind}^{\mathfrak{S}_{\ell+n}}_{\mathfrak{S}_{[n+i]}\times\mathfrak{S}_{\ell-i}}\Big(\delta_{[n+i]*}\big(\mathfrak{a}_{[n+i]}\otimes\big[\iota_{[i]*}\mathsf{Res}^{\mathfrak{S}_i\times\mathfrak{S}_{\ell-i}}_{\mathfrak{S}_\ell}(\_)\big]^{\mathfrak{S}_i}\big)\Big)\,.\end{aligned}$$

□

For $i=0$, the composition (3.3) reduces to

$$\begin{aligned}&\mathrm{D}^b_{\mathfrak{S}_\ell}\big(X\times X^\ell\big)\xrightarrow{\mathsf{triv}}\mathrm{D}^b_{\mathfrak{S}_n\times\mathfrak{S}_\ell}\big(X\times X^\ell\big)\xrightarrow{\mathsf{M}_{\mathfrak{a}_n}}\mathrm{D}^b_{\mathfrak{S}_n\times\mathfrak{S}_\ell}\big(X\times X^\ell\big)\\&\xrightarrow{\delta_{[n]*}}\mathrm{D}^b_{\mathfrak{S}_n\times\mathfrak{S}_\ell}\big(X^n\times X^\ell\big)\xrightarrow{\mathsf{Ind}}\mathrm{D}^b_{\mathfrak{S}_{n+\ell}}\big(X^{n+\ell}\big)\,.\end{aligned}\tag{3.5}$$

Using slightly shortened notation, the functor $H^i_{\ell,n}$ is on the level of objects given by

$$H^i_{\ell,n}\colon E \mapsto \bigoplus_{J\subset[n+\ell],\,\#J=n+i} \delta_{J*}\big(\mathfrak{a}_J \otimes \iota^*_{[i]}(E)^{\mathfrak{S}_{[i]}}\big)\,. \tag{3.6}$$

The right adjoint $R^i_{\ell,n}\colon \mathrm{D}^b_{\mathfrak{S}_{n+\ell}}\left(X^{n+\ell}\right) \to \mathrm{D}^b_{\mathfrak{S}_\ell}\left(X^\ell\right)$ is given by the composition

$$\begin{aligned}
&\mathrm{D}^b_{\mathfrak{S}_\ell}\left(X\times X^\ell\right) \xleftarrow{\mathsf{Ind}} \mathrm{D}^b_{\mathfrak{S}_i\times\mathfrak{S}_{\ell-i}}\left(X\times X^\ell\right) \xleftarrow{\iota_{[i]*}} \mathrm{D}^b_{\mathfrak{S}_i\times\mathfrak{S}_{\ell-i}}\left(X\times X^\ell\right) \qquad (3.7)\\
&\xleftarrow{\mathsf{triv}} \mathrm{D}^b_{\mathfrak{S}_{\ell-i}}\left(X\times X^{\ell-i}\right) \xleftarrow{(_)^{\mathfrak{S}_{n+i}}} \mathrm{D}^b_{\mathfrak{S}_{n+i}\times\mathfrak{S}_{\ell-i}}\left(X\times X^{\ell-i}\right) \xleftarrow{\mathsf{M}_{\mathfrak{a}_{n+i}}} \mathrm{D}^b_{\mathfrak{S}_{n+i}\times\mathfrak{S}_{\ell-i}}\left(X\times X^{\ell-i}\right)\\
&\xleftarrow{\delta^!_{[n+i]}} \mathrm{D}^b_{\mathfrak{S}_{n+i}\times\mathfrak{S}_{\ell-i}}\left(X^{n+i}\times X^{\ell-i}\right) \xleftarrow{\mathsf{Res}} \mathrm{D}^b_{\mathfrak{S}_{n+\ell}}\left(X^{n+\ell}\right),
\end{aligned}$$

which on the level of objects $F\in \mathrm{D}^b_{\mathfrak{S}_{n+\ell}}\left(X^{n+\ell}\right)$ means that

$$R^i_{\ell,n}\colon F \mapsto \bigoplus_{I\subset[\ell],\,\#I=i} \iota_{I*}\big(\mathfrak{a}_{[n+i]}\otimes \delta^!_{[n+i]}F\big)^{\mathfrak{S}_{[n+i]}}\,. \tag{3.8}$$

4. Techniques and examples

4.1 Derived intersections

Given a vector bundle E of rank c on a variety Z, we write $\wedge^* E := \oplus_{i=0}^c \wedge^i E[-i]$ and $\wedge^{-*}E := \oplus_{i=0}^c \wedge^i E[i]$ as objects in $\mathrm{D}^b(Z)$.

Theorem 4.1 ([AC12]). *Let $\iota\colon Z\hookrightarrow M$ be a regular embedding of codimension c such that the normal bundle sequence $0\to T_Z\to T_{M|Z}\to N_\iota\to 0$ splits. Then there is an isomorphism*

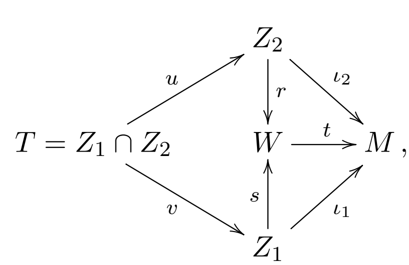
$$\iota^*\iota_*(\_)\simeq(\_)\otimes\wedge^{-*}N_\iota^\vee$$

of endofunctors of $\mathrm{D}^b(Z)$.

Recall that the right adjoint of ι_* is given by $\iota^! = \mathsf{M}_{\omega_\iota}\circ\iota^*[-\operatorname{codim}\iota]$, where $\omega_\iota = \wedge^{\operatorname{codim}\iota}N_\iota$; see [Har66, Corollary III 7.3]. We have $\wedge^{-*}N_\iota^\vee\otimes\omega_\iota[-\operatorname{codim}\iota]\cong\wedge^*N_\iota$.

Corollary 4.2. *Under the assumptions of the previous theorem, there is an isomorphism $\iota^!\iota_*(\_)\simeq(\_)\otimes\wedge^*N_\iota$.*

In particular, the *derived self-intersection* $\iota^*\iota_*\mathcal{O}_Z$ of Z in M is given by $\iota^*\iota_*\mathcal{O}_Z=\wedge^{-*}N_\iota^\vee$. More general results for *derived intersections*, that is, for $\iota_2^*\iota_{1*}\mathcal{O}_{Z_1}$ when $\iota_1\colon Z_1\to M$, $\iota_2\colon Z_2\to M$ are two different closed embeddings, are proven in [Gri14]. However, we will always be in the following situation where Theorem 4.1 is sufficient. Assume that there is a diagram



(4.1)

where all the maps are regular closed embeddings, t has a splitting normal bundle sequence, and the intersection of Z_1 and Z_2 inside of W is transversal.

Lemma 4.3. *Under the above assumptions, there is the isomorphism of functors* $\iota_2^*\iota_{1*}(\_) \cong u_*\big(v^*(\_)\otimes\wedge^{-*}N_{t|T}^\vee\big)$. *In particular,* $\iota_2^*(\iota_{1*}\mathcal{O}_{Z_1}) \cong u_*(\wedge^{-*}N_{t|T}^\vee)$.

Proof. Indeed, we have

$$\iota_2^*\iota_{1*} \cong r^*t^*t_*s_* \overset{(4.1)}{\cong} r^*\big(s_*(\_)\otimes\wedge^{-*}N_t^\vee\big) \cong r^*s_*(\_)\otimes\wedge^{-*}N_{t|Z_2}^\vee$$
$$\cong u_*v^*(\_)\otimes\wedge^{-*}N_{t|Z_2}^\vee \cong u_*\big(v^*(\_)\otimes\wedge^{-*}N_{t|T}^\vee\big)\,,$$

where the prior-to-last isomorphism is the base change theorem [Kuz06, Corollary 2.27]. □

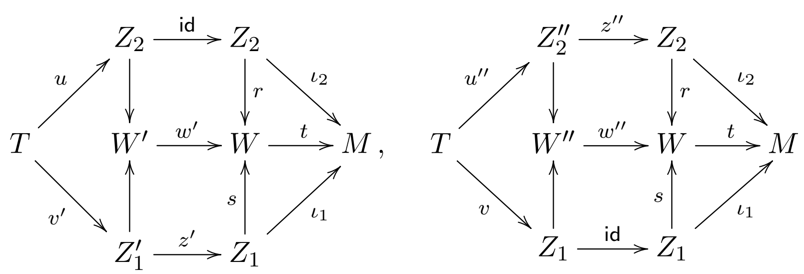
Corollary 4.4. *Under the same assumptions, we have*

$$\iota_2^!\iota_{1*}(\_) \cong u_*\big(v^*(\_)\otimes\wedge^{-*}N_{t|T}^\vee\big)\otimes\omega_{\iota_2}[-\operatorname{codim}\iota_2] \cong u_*(v^*(\_)\otimes\wedge^*N_{t|T}\otimes\omega_v)[-\operatorname{codim}v]\,.$$

In particular, $\iota_2^!\iota_{1*}\mathcal{O}_{Z_1} \cong u_*\big(\wedge^{-*}N_{t|T}^\vee\big)\otimes\omega_{\iota_2}[-\operatorname{codim}\iota_2] \cong u_*(\wedge^*N_{t|T}\otimes\omega_v)[-\operatorname{codim}v]$.

Note that, by Grothendieck duality, $\iota_{2*}\mathscr{H}^p\big(\iota_2^!\iota_{1*}\mathcal{O}_{Z_1}\big) \cong \mathcal{E}xt^p_{\mathcal{O}_M}(\mathcal{O}_{Z_2},\mathcal{O}_{Z_1})$.

Remark 4.5. In the above situation, consider in addition a variety W' with $Z_2 \subset W' \subset W$ such that $w'\colon W' \to W$ is a regular embedding and W' and Z_1 intersect transversally. We set $Z_1' = W'\cap Z_1$. We also consider $Z_1 \subset W'' \subset W$ such that $w''\colon W'' \to W$ is a regular embedding and W'' and Z_2 intersect transversally in $Z_2'' = W''\cap Z_2$. So we have the two diagrams of closed embeddings



We set $t' = t\circ w'$ and $t'' = t\circ w''$. The restriction map $\iota_{1*}\mathcal{O}_{Z_1} \to \iota'_{1*}\mathcal{O}_{Z_1'}$ induces, for every $q = 0,\dots,\operatorname{codim}(t)$, the map

$$u_*\big(\wedge^q N_{t|T}^\vee\big)\otimes\omega_{\iota_2} \cong \mathcal{E}xt^{\operatorname{codim}(\iota_2)-q}_{\mathcal{O}_M}(\mathcal{O}_{Z_2},\mathcal{O}_{Z_1}) \to \mathcal{E}xt^{\operatorname{codim}(\iota_2)-q}_{\mathcal{O}_M}(\mathcal{O}_{Z_2},\mathcal{O}_{Z_1'}) \cong u_*\big(\wedge^q N_{t'|T}^\vee\big)\otimes\omega_{\iota_2}\,.$$

As one can check locally using the Koszul resolutions, this map is given by the qth wedge power of the canonical map $N_{t|W'}^\vee \to N_{t'}^\vee$. Similarly, for $q = 0,\dots,\operatorname{codim}(t)$, the induced map

$$u_*\big(\wedge^q N_{t''|T}\otimes\omega_v\big) \cong \mathcal{E}xt^{\operatorname{codim}(v)+q}_{\mathcal{O}_M}(\mathcal{O}_{Z_2''},\mathcal{O}_{Z_1}) \to \mathcal{E}xt^{\operatorname{codim}(v)+q}_{\mathcal{O}_M}(\mathcal{O}_{Z_2},\mathcal{O}_{Z_1}) \cong u_*\big(\wedge^q N_{t|T}\otimes\omega_v\big)$$

is given by the qth wedge power of the canonical map $N_{t''} \to N_{t|W''}$.

Remark 4.6. Let G be a finite group acting on M such that all the subvarieties occurring above are invariant under this action. Then all the normal bundles carry a canonically induced G-linearisation. All the results of this subsection continue to hold as isomorphisms in the (derived) categories of G-equivariant sheaves when considering the normal bundles as G-bundles equipped with the canonical linearisations; compare with [LH09, Section 28].

4.2 Partial diagonals and the standard representation

Let I be a finite set of cardinality at least 2. The *standard representation* ϱ_I of the symmetric group $\mathfrak{S}_I$ can be considered either as the subrepresentation $\varrho_I \subset \mathbb{C}^I$ of the permutation representation consisting of all vectors whose components add up to zero, or as the quotient $\varrho_I = \mathbb{C}^I/\mathbb{C}$

by the 1-dimensional subspace of invariants. For $I \subset I'$, the first point of view gives a canonical $\mathfrak{S}_I$-equivariant inclusion $\varrho_I \to \varrho_{I'}$, while the second one gives a canonical $\mathfrak{S}_I$-equivariant surjection $\varrho_{I'} \to \varrho_I$. For X a smooth variety and $\delta_{[n]}\colon X \to X^n$ the embedding of the small diagonal, there is the $\mathfrak{S}_n$-equivariant isomorphism $N_{\delta_{[n]}} \cong T_X \otimes \varrho_n$; see [Kru15, Section 3]. More generally, for $I \subset [n]$, the normal bundle of the partial diagonal $\Delta_I \cong X \times X^{\bar I}$ is, as a $\mathfrak{S}_I$-bundle, given by

$$N_{\delta_I} \cong (T_X \otimes \varrho_I) \boxtimes \mathcal{O}_{X^{\bar I}}\,, \quad N^\vee_{\delta_I} \cong (\Omega_X \otimes \varrho_I) \boxtimes \mathcal{O}_{X^{\bar I}}\,. \tag{4.2}$$

Furthermore, the normal bundle sequence of δ_I splits since Δ_I is the fixed-point locus of the $\mathfrak{S}_I$-action on X^n; see [AC12, Section 1.20].

Remark 4.7. For $I \subset I' \subset [n]$, the embedding $\Delta_{I'} \to \Delta_I$ induces maps $N_{\delta_{I'}} \to N_{\delta_I|\Delta_{I'}}$ and $N^\vee_{\delta_I|\delta_{I'}} \to N^\vee_{\delta_{I'}}$. Under the isomorphisms (4.2), they are given by the canonical surjection $\varrho_{I'} \to \varrho_I$ and the canonical embedding $\varrho_I \to \varrho_{I'}$, respectively.

For $m \geqslant 2$ and X a smooth variety of dimension d, we set

$$\Lambda^*_m(X) := \left(\wedge^*(T_X \otimes \varrho_m)\right)^{\mathfrak{S}_m} = \bigoplus_{i=0}^{(m-1)d} \left(\wedge^i(T_X \otimes \varrho_m)\right)^{\mathfrak{S}_m}[-i]\,. \tag{4.3}$$

Lemma 4.8.

$$\Lambda^*_m(X) = \begin{cases} \mathcal{O}_X[0] & \text{for } X \text{ a curve}\,, \\ \mathcal{O}_X[0] \oplus \omega_X^{-1}[-2] \oplus \cdots \oplus \omega_X^{-(m-1)}[-2(m-1)] & \text{for } X \text{ a surface}\,. \end{cases}$$

Proof. For the curve case, since $\wedge^0(T_X \otimes \varrho_m) = \mathcal{O}_X$ is equipped with the trivial $\mathfrak{S}_m$-action, we only have to show that $\wedge^i(T_X \otimes \varrho_m)$ has no invariants for $i \geqslant 1$. For this, it is sufficient to consider the fibres which are given by $\wedge^i \varrho_m$. By [FH91, Proposition 2.12], the representations $\wedge^i \varrho_m$ are irreducible. They are non-trivial for $i \geqslant 1$; hence, their invariants vanish. For the surface case, see [Sca09, Lemma B.5] and [Kru15, Corollary 3.5]. □

Remark 4.9. For $d = \operatorname{dim} X \geqslant 3$, also vector bundles of higher rank occur as direct summands of $\Lambda^*_m(X)$. For example, for $m = 2$, we have

$$\Lambda^*_2(X) \cong \bigoplus_{0 \leqslant k \leqslant d/2} \wedge^{2k} T_X[-2k]\,.$$

This is the reason why the compositions $R_{\ell,n} \circ H_{\ell,n}$ are particularly simple for $\operatorname{dim} X \leqslant 2$ and Theorem 1.1 only makes statements for dimensions 1 and 2.

Definition 4.10. For X a smooth surface and $\ell \in \mathbb{N}$, we consider the autoequivalence

$$\bar S_X := (\_) \otimes (\omega_X \boxtimes \mathcal{O}_{X^{\bar I}})[-2]\colon\ \mathrm{D}^b_{\mathfrak{S}_\ell}\left(X \times X^\ell\right) \to \mathrm{D}^b_{\mathfrak{S}_\ell}\left(X \times X^\ell\right).$$

The reason for denoting this by $\bar S_X$ is that, for X projective and $\ell = 0$, this agrees with the Serre functor of $\mathrm{D}^b(X)$. For $0 \leqslant a \leqslant b$, we set $\bar S_X^{-[a,b]} := \bar S_X^{-a} \oplus \bar S_X^{-a-1} \oplus \cdots \oplus \bar S_X^{-b}$.

Remark 4.11. For $I \subset [n]$ of cardinality $m := |I| \geqslant 2$, consider the functor $G_I = \delta_{I*} \circ \mathsf{triv}$ and $G_I^R \circ G_I$. The latter is the composition

$$\mathrm{D}^b_{\mathfrak{S}_{\bar I}}\left(X \times X^{\bar I}\right) \xrightarrow{\mathsf{triv}} \mathrm{D}^b_{\mathfrak{S}_I \times \mathfrak{S}_{\bar I}}\left(X \times X^{\bar I}\right) \xrightarrow{\delta_{I*}} \mathrm{D}^b_{\mathfrak{S}_I \times \mathfrak{S}_{\bar I}}\left(X^n\right)$$
$$\xrightarrow{\delta_I^!} \mathrm{D}^b_{\mathfrak{S}_I \times \mathfrak{S}_{\bar I}}\left(X \times X^{\bar I}\right) \xrightarrow{(\_)^{\mathfrak{S}_I}} \mathrm{D}^b_{\mathfrak{S}_{\bar I}}\left(X \times X^{\bar I}\right).$$

Let $\mathsf{pr}_X\colon X \times X^{\bar I} \to X$ be the projection to the first factor. Corollary 4.2 and Lemma 4.8 give

$$G_I^R \circ G_I \cong (\_) \otimes \mathsf{pr}_X^* \Lambda_m^*(X) \cong \begin{cases} \mathsf{id} & \text{for } X \text{ a curve}, \\ \bar{S}_X^{-[0,m-1]} & \text{for } X \text{ a surface}. \end{cases} \tag{4.4}$$

4.3 The case $\ell = 0$

The fact that $H_{0,n}\colon \mathrm{D}^b(X) \to \mathrm{D}^b_{\mathfrak{S}_n}(X^n)$ is a $\mathbb{P}^{n-1}$-functor was already proved in [Kru15], but let us quickly recall one key part of the proof. Note that $G_{[n]} = \mathsf{M}_{\mathfrak{a}_n} \circ H_{0,n}$, and (4.4) gives

$$R_{0,n} \circ H_{0,n} \cong \begin{cases} \mathsf{id} & \text{for } X \text{ a curve}, \\ \bar{S}_X^{-[0,n-1]} & \text{for } X \text{ a surface}. \end{cases}$$

This proves the case $\ell = 0$ of part (i)(a) and most of part (ii) of Theorem 1.1 (for the only missing part of the proof, namely that condition (2) of a $\mathbb{P}^{n-1}$-functor holds for $H_{0,n}$ in the surface case, see [Kru15, Section 3]).

Let us have a quick first look at the $\ell > 0$ case, which we will treat in more detail throughout the rest of the article. For $E \in \mathrm{D}^b_{\mathfrak{S}_\ell}(X^\ell)$, note that $\delta_{[n]*}(E)$ is a direct summand of $H^0_{\ell,n}(E)$; see (3.5) and (3.6). Hence, $G^R_{[n]} G_{[n]}(E) \cong \bar{S}_X^{-[0,n-1]}(E)$ occurs as a direct summand of $R^0_{\ell,n} H^0_{\ell,n}(E)$. However, the other terms of $H^0_{\ell,n}(E)$ also have to be taken into account, which leads to an expression of the form

$$R^0_{\ell,n} H^0_{\ell,n}(E) \cong \bar{S}_X^{-[0,n-1]}(E) \oplus \big(\text{terms supported on partial diagonals of } X \times X^\ell\big). \tag{4.5}$$

The 'error terms' supported on the partial diagonals of $X \times X^\ell$ prevent the functor $H^0_{\ell,n}$ from being a $\mathbb{P}^{n-1}$-functor, which is the reason why we have to consider the more complicated functors $H_{\ell,n}$ instead. Basically, the higher terms of the complex $\mathcal{H}_{\ell,n}$ or, in other words, the functors $H^i_{\ell,n}$, lead to a cancellation of the error terms in (4.5).

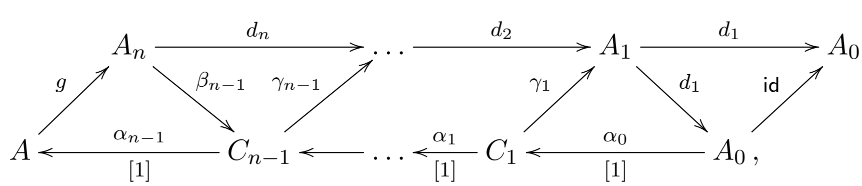
4.4 The approach for general ℓ

In order to establish Theorem 1.1, we need to compute the composition $R_{\ell,n} \circ H_{\ell,n}$ of the functor $H_{\ell,n}$ with its right adjoint, which amounts to the computation of its kernel $\mathcal{R}_{\ell,n} \star \mathcal{H}_{\ell,n}$.

Let us recall the notion of convolutions in triangulated categories; for details, we refer to [CS07, Section 3.1]. A *bounded complex* in a triangulated category $\mathcal{D}$ is a sequence

$$A_n \xrightarrow{d_n} A_{n-1} \xrightarrow{d_{n-1}} \dots \xrightarrow{d_2} A_1 \xrightarrow{d_1} A_0$$

such that $d_i \circ d_{i+1} = 0$ for all i. A *left convolution* of $A_\bullet$ is an object $A \in \mathcal{D}$ together with a morphism $g\colon A \to A_n$ such that there is a diagram



where the triangles involving an α_i as a horizontal arrow are exact and the triangles involving a d_i as a horizontal arrow are commutative. One may think of this as A having a filtration whose graded pieces are the A_i. There is also a dual notion of *right convolution*.

If $\mathcal{D} = \mathrm{D}^b(\mathcal{A})$ for some abelian category $\mathcal{A}$, every complex $A^\bullet$ of objects in $\mathcal{A}$ can be considered as a complex in $\mathcal{D}$, and a convolution of this complex is $A^\bullet$ itself, considered as *one* object in $\mathcal{D}$; see [CS07, Example 3.4]. In particular, $\mathcal{H}$ is a left convolution of $\mathcal{H}^0_{\ell,n} \to \cdots \to \mathcal{H}^\ell_{\ell,n}$. We set

$$\mathcal{R}^i_{\ell,n} := \big(\mathcal{H}^i_{\ell,n}\big)^\vee \otimes (\omega_{X\times X^\ell} \boxtimes \mathcal{O}_{X^{n+\ell}})[(\ell+1)d]\,, \quad d = \dim X\,.$$

Since exact functors, hence in particular $(\_)^\vee \otimes (\omega_{X\times X^\ell} \boxtimes \mathcal{O}_{X^{n+\ell}})[(\ell+1)d]$, preserve convolutions [CS07, Remark 3.1], we see that $\mathcal{R}_{\ell,n}$ is a left convolution of $\mathcal{R}^\ell_{\ell,n} \to \cdots \to \mathcal{R}^1_{\ell,n} \to \mathcal{R}^0_{\ell,n}$. We get a commutative diagram

$$\begin{array}{ccccccc}
\mathcal{R}^\ell_{\ell,n} \star \mathcal{H}_{\ell,n} & \longrightarrow & \mathcal{R}^\ell_{\ell,n} \star \mathcal{H}^0_{\ell,n} & \longrightarrow & \dots & \longrightarrow & \mathcal{R}^\ell_{\ell,n} \star \mathcal{H}^\ell_{\ell,n} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\vdots & \longrightarrow & \vdots & \longrightarrow & \ddots & \longrightarrow & \vdots \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\mathcal{R}^0_{\ell,n} \star \mathcal{H}_{\ell,n} & \longrightarrow & \mathcal{R}^0_{\ell,n} \star \mathcal{H}^0_{\ell,n} & \longrightarrow & \dots & \longrightarrow & \mathcal{R}^0_{\ell,n} \star \mathcal{H}^\ell_{\ell,n} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\mathcal{R}_{\ell,n} \star \mathcal{H}_{\ell,n} & \longrightarrow & \mathcal{R}_{\ell,n} \star \mathcal{H}^0_{\ell,n} & \longrightarrow & \dots & \longrightarrow & \mathcal{R}^\ell_{\ell,n} \star \mathcal{H}^\ell_{\ell,n}\,,
\end{array} \tag{4.6}$$

where the $\mathcal{R}^i_{\ell,n} \star \mathcal{H}_{\ell,n}$ and $\mathcal{R}_{\ell,n} \star \mathcal{H}^j_{\ell,n}$ are the left and right convolutions of the rows and columns, respectively. That means, in particular, that $\mathcal{R}_{\ell,n} \star \mathcal{H}^j_{\ell,n}$ can be written as a multiple cone

$$\mathsf{cone}\Big(\big(\cdots \mathsf{cone}\,\big(\,\mathsf{cone}\,\big(\mathcal{R}^\ell_{\ell,n} \star \mathcal{H}^j_{\ell,n} \to \mathcal{R}^{\ell-1}_{\ell,n} \star \mathcal{H}^j_{\ell,n}\big) \to \mathcal{R}^{\ell-2}_{\ell,n} \star \mathcal{H}^j_{\ell,n}\big) \cdots\big) \to \mathcal{R}^0_{\ell,n} \star \mathcal{H}^j_{\ell,n}\Big)\,.$$

The strategy of the proof of Theorem 1.1 is to start with the computation of the $\mathcal{R}^i_{\ell,n} \star \mathcal{H}^j_{\ell,n}$, then use the results to compute $\mathcal{R}_{\ell,n} \star \mathcal{H}^j_{\ell,n}$, and finally deduce the desired formulae for $\mathcal{R}^i_{\ell,n} \star \mathcal{H}_{\ell,n}$.

In the following subsections, which are intended to give gentle examples illustrating how the general proof works, we will do the computation in the case $\ell = 1$ (and some of it for $\ell = 2$) on the level of the functors instead of kernels. That means that we will compute the compositions $R^i_{\ell,n} \circ H^j_{\ell,n}$. We will see that the undesired terms (see (4.5)) are of a form which give them a good chance to cancel out when passing to the convolution $R_{\ell,n} \circ H_{\ell,n}$. However, in the present section, we will not compute the induced maps $R^i_{\ell,n} \circ H^j_{\ell,n} \to R^i_{\ell,n} \circ H^{j+1}_{\ell,n}$ and $R^i_{\ell,n} \circ H^j_{\ell,n} \to R^{i-1}_{\ell,n} \circ H^j_{\ell,n}$, which would be necessary to see that the terms really cancel. Later in Section 5, where the computations are performed for general ℓ on the level of the kernels, we will see that the appropriate terms cancel, which leads to a rigorous proof of Theorem 1.1.

4.5 The case $\ell = 1$

We aim to compute $R_{1,n} \circ H_{1,n}\colon \mathrm{D}^b(X \times X) \to \mathrm{D}^b(X \times X)$ using the descriptions (3.6) and (3.8) of $H^j_{1,n}$ and $R^i_{1,n}$. For $E \in \mathrm{D}^b(X \times X)$, we have

$$R^0_{1,n} H^0_{1,n}(E) \cong \Big[\mathfrak{a}_{[n]} \otimes \delta^!_{[n]}\Big(\bigoplus_{a\in[n+1]} \delta_{[n+1]\setminus\{a\}*}(E \otimes \mathfrak{a}_{[n+1]\setminus\{a\}})\Big)\Big]^{\mathfrak{S}_{[n]}}\,.$$

For $\sigma \in \mathfrak{S}_{[n]}$, the $\mathfrak{S}_{[n]}$-linearisation of $\bigoplus_{a\in[n+1]} \delta^!_{[n]}\delta_{[n+1]\setminus\{a\}*}(E \otimes \mathfrak{a}_{[n+1]\setminus\{a\}})$ maps the summand $\delta^!_{[n]}\delta_{[n+1]\setminus\{a\}*}(E \otimes \mathfrak{a}_{[n+1]\setminus\{a\}})$ to $\delta^!_{[n]}\delta_{[n+1]\setminus\{\sigma(a)\}}(E \otimes \mathfrak{a}_{[n+1]\setminus\{\sigma(a)\}})$. Thus, the induced action on the index set $[n+1]$ is given by $a \mapsto \sigma(a)$. Hence, there are two $\mathfrak{S}_{[n]}$-orbits, namely $[n]$ and

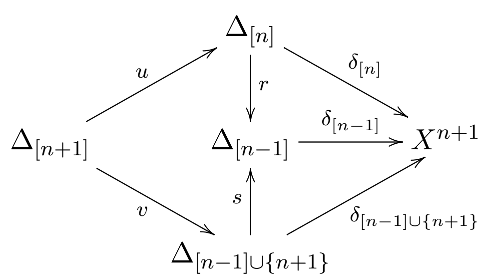
$\{n+1\}$. We have $\mathrm{Stab}_{\mathfrak{S}_{[n]}}(n+1)=\mathfrak{S}_{[n]}$ and $\mathrm{Stab}_{\mathfrak{S}_{[n]}}(n)=\mathfrak{S}_{[n-1]}$. As explained in Section 2.2, it follows that

$$R^0_{1,n}H^0_{1,n}(E)\cong \delta^!_{[n]}\delta_{[n]*}(E)^{\mathfrak{S}_{[n]}}\oplus \delta^!_{[n]}\delta_{[n-1]\cup\{n+1\}*}(E)^{\mathfrak{S}_{[n-1]}}\,. \tag{4.7}$$

The first direct summand equals $E\otimes \mathsf{pr}_1^*\Lambda^*_n(X)$ by (4.4).

CONVENTION 4.12. For $\{b\}\subset[m]$, a set with one element, we set $\Delta_{\{b\}}=X^m$. Furthermore, we set $\Lambda^*_1(X)=\mathcal{O}_X[0]$. This convention becomes relevant in this subsection in the case $n=2$ and later in the more general case $n=\ell+1$.

For the computation of the second summand of (4.7), consider the commutative diagram of closed embeddings



It fulfils the properties of diagram (4.1), which means that $\Delta_{[n+1]}=\Delta_{[n]}\cap\Delta_{[n-1]\cup\{n+1\}}$ and that this intersection is transversal inside $\Delta_{[n-1]}$. Furthermore, the normal bundle sequence of $\delta_{[n-1]}$ splits; see Section 4.2. This allows us to apply Corollary 4.4 to get

$$\delta^!_{[n]}\delta_{[n-1]\cup\{n+1\}*}(E)\cong u_*(v^*(\_)\otimes\wedge^*N_{\delta_{[n-1]}|\Delta_{[n+1]}}\otimes\omega_v)[-\operatorname{codim} v]\,. \tag{4.8}$$

Under the isomorphisms $\Delta_{[n+1]}\cong X$ and $\Delta_{[n]}\cong X\times X\cong\Delta_{[n-1]\cup\{n+1\}}$, the embeddings u and v equal the diagonal embedding $\iota\colon X\hookrightarrow X\times X$. Thus, $\operatorname{codim} v=\dim X=d$ and $\omega_v\cong\wedge^d N_v\cong\omega_X^{-1}$. Together with (4.2) and (4.3), this implies that after taking $\mathfrak{S}_{[n-1]}$-invariants in (4.8), we get $\delta^!_{[n]}\delta_{[n-1]\cup\{n+1\}*}(E)^{\mathfrak{S}_{[n-1]}}\cong\iota_*\big(\iota^*(E)\otimes\Lambda^*_{n-1}(X)\otimes\omega_X^{-1}\big)[-d]$. In summary, (4.7) gives

$$R^0_{1,n}H^0_{1,n}(E)\cong E\otimes\mathsf{pr}_1^*\Lambda^*_n(X)\oplus\iota_*\big(\iota^*(E)\otimes\Lambda^*_{n-1}(X)\otimes\omega_X^{-1}\big)[-d]\,. \tag{4.9}$$

The computation of the other three functor compositions is easier. Note that we have $\delta_{[n+1]}=\delta_{[n]}\circ u$ and $u^!\cong u^*(\_)\otimes\omega_X^{-1}[-d]$, and keep in mind the identification of u with the diagonal embedding $\iota\colon X\hookrightarrow X\times X$. Then,

$$\begin{aligned} R^0_{1,n}H^1_{1,n}(E)&\cong\big[\mathfrak{a}_{[n]}\otimes\delta^!_{[n]}\delta_{[n+1]*}(\iota^*E\otimes\mathfrak{a}_{[n+1]})\big]^{\mathfrak{S}_{[n]}}\\ &\cong\delta^!_{[n]}\delta_{[n]*}u_*\iota^*(E)^{\mathfrak{S}_{[n]}}\overset{(4.4)}{\cong}\iota_*\big(\iota^*(E)\otimes\Lambda^*_n(X)\big)\,, \end{aligned} \tag{4.10}$$

$$\begin{aligned} R^1_{1,n}H^0_{1,n}(E)&\cong\iota_*\Big[\mathfrak{a}_{[n+1]}\otimes\delta^!_{[n+1]}\Big(\bigoplus_{a=1}^{n+1}\delta_{[n+1]\setminus\{a\}*}(E\otimes\mathfrak{a}_{[n+1]\setminus\{a\}})\Big)\Big]^{\mathfrak{S}_{[n+1]}}\overset{(2.3)}{\cong}\iota_*\delta^!_{[n+1]}\delta_{[n]*}(E)^{\mathfrak{S}_{[n]}}\\ &\cong\iota_*u^!\delta^!_{[n]}\delta_{[n]*}(E)^{\mathfrak{S}_{[n]}}\overset{(4.4)}{\cong}\iota_*\big(\iota^*(E)\otimes\Lambda^*_n(X)\otimes\omega_X^{-1}\big)[-d]\,, \end{aligned} \tag{4.11}$$

$$R^1_{1,n}H^1_{1,n}(E)\cong\iota_*\big[\delta^!_{[n+1]}\delta_{[n+1]*}\iota^*(E)\big]^{\mathfrak{S}_{[n+1]}}\overset{(4.4)}{\cong}\iota_*\big(\iota^*(E)\otimes\Lambda^*_{n+1}(X)\big)\,. \tag{4.12}$$

Now, let $X=C$ be a smooth curve. By Lemma 4.8, we have $\Lambda^*_m(C)=\mathcal{O}_C[0]$ for all $m\geqslant 1$.

Plugging this into (4.9)–(4.12), we get

$$\begin{array}{ccc} R^1_{1,n}\circ H^0_{1,n} & \longrightarrow & R^1_{1,n}\circ H^1_{1,n} \\ \downarrow & & \downarrow \\ R^0_{1,n}\circ H^0_{1,n} & \longrightarrow & R^0_{1,n}\circ H^1_{1,n} \end{array} \quad\cong\quad \begin{array}{ccc} \iota_*\big(\iota^*(\_)\otimes\omega_C^{-1}\big)[-1] & \longrightarrow & \iota_*\iota^* \\ \downarrow & & \downarrow \\ \mathsf{id}\oplus\iota_*\big(\iota^*(\_)\otimes\omega_C^{-1}\big)[-1] & \longrightarrow & \iota_*\iota^*\,. \end{array} \tag{4.13}$$

We will see in Section 5.6 that the right-hand vertical map of this diagram as well as the component $\iota_*\big(\iota^*(\_)\otimes\omega_C^{-1}\big)[-1]\to\iota_*\big(\iota^*(\_)\otimes\omega_C^{-1}\big)[-1]$ of the left-hand vertical map are isomorphisms. Thus, by taking cones in the diagram (4.13), we get $R_{1,n}\circ H^0_{1,n}\cong\mathsf{id}$ and $R_{1,n}\circ H^1_{\ell,n}=0$; compare with (4.6). Considering the triangle $R_{1,n}\circ H_{1,n}\to R\circ H^0_{1,n}\to R_{1,n}\circ H^1_{1,n}$ shows $R_{1,n}\circ H_{1,n}=\mathsf{id}$. This amounts to the case $\ell=1$ of Theorem 1.1(i)(a).

For X a smooth surface, we have $(\_)\otimes\Lambda^*_m(X)\cong S_X^{-[0,m-1]}$; see (4.4). This gives

$$\begin{array}{ccc} R^1_{1,n}\circ H^0_{1,n} & \longrightarrow & R^1_{1,n}\circ H^1_{1,n} \\ \downarrow & & \downarrow \\ R^0_{1,n}\circ H^0_{1,n} & \longrightarrow & R^0_{1,n}\circ H^1_{1,n} \end{array} \quad\cong\quad \begin{array}{ccc} \iota_*S_X^{-[1,n]}\iota^* & \longrightarrow & \iota_*S_X^{-[0,n]}\iota^* \\ \downarrow & & \downarrow \\ \bar S_X^{-[0,n-1]}\oplus\iota_*S_X^{-[1,n-1]}\iota^* & \longrightarrow & \iota_*S_X^{-[0,n-1]}\iota^*\,, \end{array}$$

where $\bar S_X^{-1}=(\_)\otimes\mathsf{pr}_1^*\,\omega_X^{-1}[-2]$. Again, we will see later that all components of the maps in the diagram of the form $\iota_*S_X^{-k}\iota^*\to\iota_*S_X^{-k}\iota^*$ are isomorphisms, which by taking cones gives

$$R_{1,n}\circ H^0_{1,n}\cong\bar S_X^{-[0,n-1]}\oplus\iota_*S_X^{-n}\iota^*[1]\,,\quad R_{1,n}\circ H^1_{1,n}\cong\iota_*S_X^{-n}\iota^*[1]\,,$$

and, finally, $R_{1,n}\circ H_{1,n}\cong\bar S_X^{-[0,n-1]}$ as claimed in Theorem 1.1(b).

4.6 Orthogonality in the curve case

In this subsection, we compute that $R_{1,n}\circ H_{0,n+1}=0$ for $X=C$ a curve, which is one instance of Theorem 1.1(i)(b). We have

$$R^1_{1,n}H_{0,n+1}(E)\cong\iota_*\big(\delta^!_{[n+1]}\delta_{[n+1]*}(E)\big)^{\mathfrak S_{n+1}}\cong\iota_*(E\otimes\Lambda^*_{n+1}(X))\,, \tag{4.14}$$

$$R^0_{1,n}H_{0,n+1}(E)\cong\delta^!_{[n]}\delta_{[n+1]*}(E)^{\mathfrak S_n}\cong\delta^!_{[n]}\delta_{[n]*}u_*(E)^{\mathfrak S_n}\cong\iota_*(E\otimes\Lambda^*_n(X))\,. \tag{4.15}$$

For $X=C$ a curve, this gives $R^1_{1,n}\circ H_{0,n+1}\cong\iota_*$ and $R^0_{1,n}\circ H_{0,n+1}\cong\iota_*$. By the exact triangle

$$R^1_{1,n}\circ H_{0,n+1}\to R^0_{1,n}\circ H_{0,n+1}\to R_{1,n}\circ H_{0,n+1}\,, \tag{4.16}$$

we get the desired vanishing $R_{1,n}\circ H_{0,n+1}=0$.

4.7 Non-orthogonality in the surface case

For X a smooth surface, we have $R^1_{1,n}\circ H_{0,n+1}\cong\iota_*S_X^{-[0,n]}$ and $R^0_{1,n}\circ H_{0,n+1}\cong\iota_*S_X^{-[0,n-1]}$ by (4.14) and (4.15). Again, all the components $\iota_*S_X^{-k}\to\iota_*S_X^{-k}$ of the induced map $R^1_{1,n}\circ H_{0,n+1}\to R^0_{1,n}\circ H_{0,n+1}$ are isomorphisms for $k=0,\dots,n-1$. Thus, by triangle (4.16), we get

$$R_{1,n}\circ H_{0,n+1}\cong\iota_*S_X^{-n}[1]\,. \tag{4.17}$$

This non-orthogonality prevents the functors $H_{\ell,n}$ from giving a complete categorification of the Heisenberg action on the cohomology of the Hilbert schemes; compare with Section 6.2.

4.8 The case $\ell = 2$, $n = 2$

We make computations concerning the composition $R_{2,2} \circ H_{2,2} \colon \mathrm{D}^b_{\mathfrak{S}_2}\left(X \times X^2\right) \to \mathrm{D}^b_{\mathfrak{S}_2}\left(X \times X^2\right)$ in order to illustrate why the assumption $n > \ell$ is necessary for Theorem 1.1. We have

$$R^0_{2,2}H^0_{2,2}(E) \cong \left[\mathfrak{a}_{[2]} \otimes \delta^!_{[2]}\Big(\bigoplus_{J\subset[4],\,|J|=2} \delta_{J*}(E \otimes \mathfrak{a}_J)\Big)\right]^{\mathfrak{S}_{[2]}}.$$

For $J = [2]$, we get the direct summand $\delta^!_{[2]}\delta_{[2]*}(E)^{\mathfrak{S}_{[2]}} \cong E \otimes \mathsf{pr}_X^* \Lambda_2^*(X)$ of $R^0_{2,2}H^0_{2,2}(E)$. For $H_{2,2}$ to be fully faithful in the curve case and a ℙ-functor in the surface case, we would need $R_{2,2}H_{2,2}(E)$ to be isomorphic to that direct summand. For $J = [3,4]$, we consider the diagram

$$\begin{array}{ccc} \Delta_{[2]} \cap \Delta_{[3,4]} & \xrightarrow{u} & \Delta_{[2]} \\ {\scriptstyle v}\downarrow & & \downarrow{\scriptstyle \delta_{[2]}} \\ \Delta_{[3,4]} & \xrightarrow{\delta_{[3,4]}} & X^4\,, \end{array}$$

which is a transversal intersection. Under the isomorphism $\Delta_{[2]} \cong X \times X^2$, the subvariety $X \times X \cong \Delta_{[2]} \cap \Delta_{[3,4]} \subset \Delta_{[2]}$ equals $X \times \Delta_X$. Thus, for an appropriate choice of E, the direct summand $[\mathfrak{a}_{[2]} \otimes \delta^!_{[2]}\delta_{[3,4]*}(E \otimes \mathfrak{a}_{[3,4]})]^{\mathfrak{S}_{[2]}}$ of $R^0_{2,2}H^0_{2,2}(E)$ is supported on the whole $X \times \Delta_X$. On the other hand, one can easily see that all direct summands of $R^i_{2,2}H^j_{2,2}(E)$ for $(i,j) \neq (0,0)$ are supported on one of the subvarieties $D_{\{1\}}$, $D_{\{2\}}$, or $D_{[2]}$ of $X \times X^2$, none of them containing $X \times \Delta_X$. It follows that the direct summand $[\mathfrak{a}_{[2]} \otimes \delta^!_{[2]}\delta_{[3,4]*}(E \otimes \mathfrak{a}_{[3,4]})]^{\mathfrak{S}_{[2]}}$ of $R^0_{2,2}H^0_{2,2}(E)$ survives taking the multiple cones in the diagram induced by (4.6) which prevents $R_{2,2}H_{2,2}(E)$ from being isomorphic to $E \otimes \mathsf{pr}_X^* \Lambda_2^*(X)$.

5. Proof of the main results

Throughout this section, we fix a smooth variety X of dimension $d := \mathsf{dim}\, X$ (in later subsections, d will be specified to be 1 or 2), and we fix numbers $\ell, n \in \mathbb{N}$ with $n > \mathsf{max}\{\ell, 1\}$. In order to keep the formulae reasonably short, we will mostly omit these fixed numbers from the indices. For example, we write $\mathcal{H}$ instead of $\mathcal{H}_{\ell,n}$, $\mathcal{H}^i$ instead of $\mathcal{H}^i_{\ell,n}$, $\mathcal{R}$ instead of $\mathcal{R}_{\ell,n}$ and so on.

In this section, we will prove Theorem 1.1. In order to achieve this, we will first compute the convolution products $\mathcal{R}^i \star \mathcal{H}^j$ for $i, j \in \{0, \dots, \ell\}$. In the case that X is a curve or a surface, this will lead to the desired formulae for $\mathcal{R} \star \mathcal{H}$.

5.1 Computation of the direct summands

The sheaf $\mathcal{H}^i$ is given by a direct sum of the structure sheaves $\mathcal{O}_{I,J,\mu}$; see Section 3.2. Hence, for $0 \leqslant i, j \leqslant \ell$, a first important step in the computation of the equivariant convolution product $\mathcal{R}^i \star \mathcal{H}^j$ is the computation of the non-equivariant convolution product

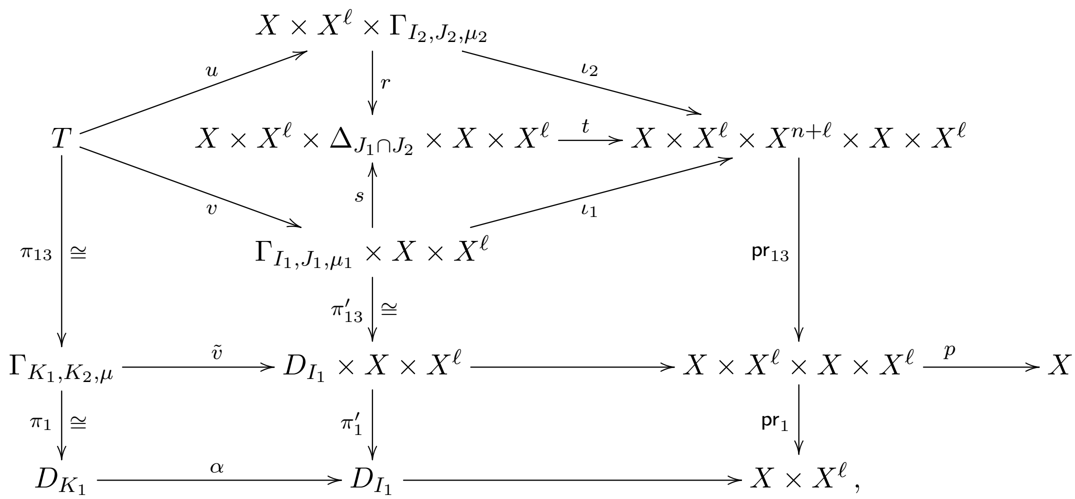
$$(\mathcal{O}_{I_2,J_2,\mu_2})^R \star \mathcal{O}_{I_1,J_1,\mu_1} = \mathsf{pr}_{13*}\big(\mathsf{pr}_{23}^*(\mathcal{O}_{I_2,J_2,\mu_2})^R \otimes \mathsf{pr}_{12}^*\, \mathcal{O}_{I_1,J_1,\mu_1}\big)\,,$$

where $(\mathcal{O}_{I_2,J_2,\mu_2})^R := (\mathcal{O}_{I_2,J_2,\mu_2})^\vee \otimes (\omega_{X\times X^\ell} \boxtimes \mathcal{O}_{X^{n+\ell}})[(\ell+1)d]$ for $(I_1, J_1, \mu_1) \in \mathsf{Index}(j)$ and $(I_2, J_2, \mu_2) \in \mathsf{Index}(i)$. We carry out this computation in this subsection.

We set $K_1 := I_1 \cup \mu_1^{-1}(J_2)$, $K_2 := I_2 \cup \mu_2^{-1}(J_1) \subset [\ell]$, and $k := |K_1| = |K_2|$ and consider the bijection $\mu := \mu^{-1}_{2|\overline{J_1 \cup J_2}} \circ \mu_{1|\bar K_1}$ between $\bar K_1 = [\ell] \setminus K_1$ and $\bar K_2 = [\ell] \setminus K_2$. Furthermore, we consider the subvariety

$$\Gamma_{K_1,K_2,\mu} := \left\{(x,x_1,\dots,x_\ell,z,z_1,\dots,z_\ell) \mid x = x_a = z_b = z\,\forall\, a\in K_1,\, b\in K_2,\, x_c = z_{\mu(c)}\,\forall\, c\in \bar{K}_1\right\}$$

of $X\times X^\ell \times X\times X^\ell$ and set $\mathcal{O}_{K_1,K_2,\mu} := \mathcal{O}_{\Gamma_{K_1,K_2,\mu}}$. There is the commutative diagram



(5.1)

where $T := \left(\Gamma_{I_1,J_1,\mu_1}\times X\times X^\ell\right)\cap\left(X\times X^\ell\times\Gamma_{I_2,J_2,\mu_2}\right)$, the maps π_{13} and π'_{13} are the restrictions of the projection pr_{13}, the maps π_1 and π'_1 are the restrictions of the projection pr_1, the map p is the projection to the third factor, and all the other arrows denote the appropriate closed embeddings. Note that $J_1\cap J_2\neq\emptyset$ because of the assumption $n>\ell$. We have

$$T = \left\{(x,x_1,\dots,x_\ell,y_1,\dots,y_{n+\ell},z,z_1,\dots,z_\ell)\,\middle|\, \begin{array}{l} x = x_a = y_b = z_c = z,\ x_d = y_{\mu_1(d)} = z_{\mu(d)} \\ \forall\, a\in K_1,\ b\in J_1\cup J_2,\ c\in K_2,\ d\in \bar{K}_1\end{array}\right\}.$$

We see that a point in T is determined by its $(x,x_1,\dots,x_\ell)$-component. Thus, π_{13} and π_1 are isomorphisms. Similarly, π'_{13} is an isomorphism. Let $\pi_2\colon \Gamma_{K_1,K_2,\mu}\to X\times X^\ell$ be the restriction of the projection $\mathsf{pr}_2\colon X\times X^\ell\times X\times X^\ell\to X\times X^\ell$ to the second factor. By the adjunction formula,

$$\pi_{13*}\omega_v \cong \omega_{\tilde v}\cong \omega_{\pi'_1\circ\tilde v}\otimes \tilde v^*\omega^{-1}_{\pi'_1}\cong \pi_1^*\omega_\alpha\otimes\pi_2^*\omega^{-1}_{X\times X^\ell}\cong p^*\omega_X^{-(k-j)}\otimes \pi_2^*\omega^{-1}_{X\times X^\ell}\,,$$

where the last isomorphism is due to the fact that on $\Gamma_{K_1,K_2,\mu}$ the projection to the first factor $X\times X^\ell\times X\times X^\ell\to X$ coincides with the projection p to the third factor. It follows that

$$\pi_{13*}\omega_v\otimes\pi_2^*\omega_{X\times X^\ell}\cong p^*\omega_X^{-(k-j)}\,. \tag{5.2}$$

Note that $|J_1\cup J_2| = n+k$ and $|J_1\cap J_2| = n+i+j-k$. Using this, one can check that diagram (5.1) with the two bottom lines removed satisfies the properties of diagram (4.1). Concretely, the square consisting of u, v, s, and r is a transversal intersection, with $\mathsf{codim}(t) = (n+i+j-k-1)d$ and $\mathsf{codim}(v) = (k-j+\ell+1)d$. By Corollary 4.4, we get

$$\mathcal{H}om(\mathsf{pr}_{23}^*\,\mathcal{O}_{I_2,J_2,\mu_2},\mathsf{pr}_{12}^*\,\mathcal{O}_{I_1,J_1,\mu_1})\cong \iota_2^!\iota_{1*}\mathcal{O}_{\Gamma_{I_1,J_1,\mu_1}\times X\times X^\ell}\cong \mathcal{O}_T\otimes\wedge^*N_t\otimes\omega_v[-(k-j+\ell+1)d]\,.$$

Combining this with (5.2) and, noting that $t = \mathsf{id}_{X\times X^\ell}\times\delta_{J_1\cup J_2}\times\mathsf{id}_{X\times X^\ell}$, with (4.2) gives

$$\begin{aligned}(\mathcal{O}_{I_2,J_2,\mu_2})^R\star\mathcal{O}_{I_1,J_1,\mu_1} &\cong \mathsf{pr}_{13*}\,\mathcal{H}om(\mathsf{pr}_{23}^*\,\mathcal{O}_{I_2,J_2,\mu_2},\mathsf{pr}_{12}^*\,\mathcal{O}_{I_1,J_1,\mu_1})\otimes\mathsf{pr}_2^*\,\omega_{X\times X^\ell}[(\ell+1)d]\\ &\cong \mathcal{O}_{K_1,K_2,\mu}\otimes p^*\left(\wedge^*(T_X\otimes\varrho_{J_1\cap J_2})\otimes\omega_X^{-(k-j)}\right)[-(k-j)d] &(5.3)\\ &\cong \mathcal{O}_{K_1,K_2,\mu}\otimes p^*\left(\wedge^{-*}(\Omega_X\otimes\varrho_{J_1\cap J_2})\otimes\omega_X^{-(n+i-1)}\right)[-(n+i-1)d]\,. &(5.4)\end{aligned}$$

5.2 The induced maps

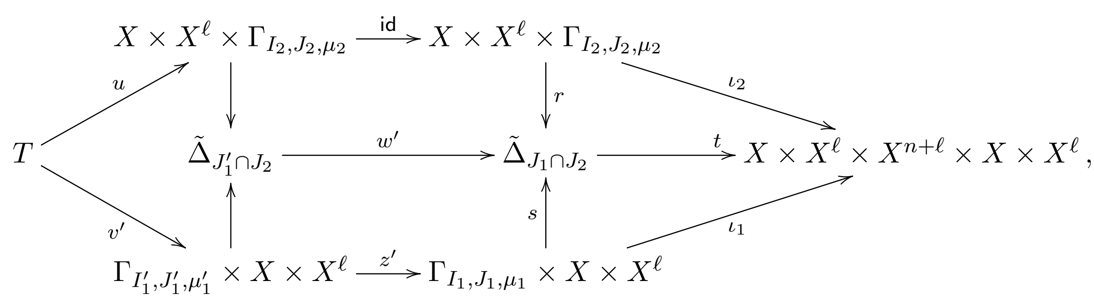
Let $c \in \bar{I}_1$ with $\mu_1(c) \in J_2$, and set $I_1' = I_1 \cup \{c\}$, $J_1' = J_1 \cup \{\mu_1(c)\}$, and $\mu_1' := \mu_{1|\bar{I}_1 \setminus \{c\}}$. The restriction $\mathcal{O}_{I_1,J_1,\mu_1} \to \mathcal{O}_{I_1',J_1',\mu_1'}$ induces, for $q = 0, \dots, (n+i+j-k)d$, a map

$$\mathscr{H}^{(n+i-1)d-q}\big((\mathcal{O}_{I_2,J_2,\mu_2})^R \star \mathcal{O}_{I_1,J_1,\mu_1}\big) \to \mathscr{H}^{(n+i-1)d-q}\big((\mathcal{O}_{I_2,J_2,\mu_2})^R \star \mathcal{O}_{I_1',J_1',\mu_1'}\big)\,,$$

which under the isomorphism (5.4) corresponds to a map

$$\mathcal{O}_{K_1,K_2,\mu} \otimes p^*\big(\wedge^q(\Omega_X \otimes \varrho_{J_1\cap J_2}) \otimes \omega_X^{-(n+i-1)}\big) \to \mathcal{O}_{K_1,K_2,\mu} \otimes p^*\big(\wedge^q(\Omega_X \otimes \varrho_{J_1'\cap J_2}) \otimes \omega_X^{-(n+i-1)}\big)\,.$$

By Remarks 4.5 and 4.7, this map is given by the canonical inclusion $\varrho_{J_1\cap J_2} \to \varrho_{J_1'\cap J_2} = \varrho_{(J_1\cap J_2)\cup\{\mu_1(c)\}}$. To see this, compare the diagram



where $\tilde{\Delta}_J := X \times X^\ell \times \Delta_J \times X \times X^\ell$, with the diagram of Remark 4.5.

Similarly, consider $c \in \bar{I}_2$ with $\mu_2(c) \in J_1$, and set $I_2' := I_2 \cup \{c\}$, $J_2' := J_2 \cup \{\mu_2(c)\}$, and $\mu_2' := \mu_{2|\bar{I}_2\setminus\{c\}}$. Then the restriction $\mathcal{O}_{I_2,J_2,\mu_2} \to \mathcal{O}_{I_2',J_2',\mu_2'}$ induces a map

$$\mathscr{H}^{(k-j)d+q}\big((\mathcal{O}_{I_2',J_2',\mu_2'})^R \star \mathcal{O}_{I_1,J_1,\mu_1}\big) \to \mathscr{H}^{(k-j)d+q}\big((\mathcal{O}_{I_2,J_2,\mu_2})^R \star \mathcal{O}_{I_1,J_1,\mu_1}\big)$$

for $q = 0, \dots, (n+i+j-k)d$, which under the isomorphism (5.3) corresponds to a map

$$\mathcal{O}_{K_1,K_2,\mu} \otimes p^*\big(\wedge^q(T_X \otimes \varrho_{J_1\cap J_2'}) \otimes \omega_X^{-(k-j)}\big) \to \mathcal{O}_{K_1,K_2,\mu} \otimes p^*\big(\wedge^q(T_X \otimes \varrho_{J_1\cap J_2}) \otimes \omega_X^{-(k-j)}\big)\,.$$

This map is given by the canonical surjection $\varrho_{J_1\cap J_2'} = \varrho_{(J_1\cap J_2)\cup\{\mu_2(c)\}} \to \varrho_{J_1\cap J_2}$, again due to Remarks 4.5 and 4.7. In particular, the induced map

$$\mathscr{H}^{(k-j)d}\big((\mathcal{O}_{I_2',J_2',\mu_2'})^R \star \mathcal{O}_{I_1,J_1,\mu_1}\big) \to \mathscr{H}^{(k-j)d}\big((\mathcal{O}_{I_2,J_2,\mu_2})^R \star \mathcal{O}_{I_1,J_1,\mu_1}\big)$$

is given by the identity on $\mathcal{O}_{K_1,K_2,\mu} \otimes p^*\omega_X^{-(k-j)}$.

5.3 Computation of the $\mathcal{R}^i \star \mathcal{H}^j$

We make use of the principle explained in Section 2.2 to compute the convolution product $\mathcal{R}^i \star \mathcal{H}^j = \mathsf{pr}_{13*}\big(\mathsf{pr}_{23}^*\,\mathcal{R}^i \otimes \mathsf{pr}_{12}^*\,\mathcal{H}^j\big)^{1\times\mathfrak{S}_{n+\ell}\times 1}$. We set

$$\mathcal{R}(I,J,\mu) := \mathcal{H}(I,J,\mu)^\vee \otimes (\omega_{X\times X^\ell} \boxtimes \mathcal{O}_{X^{n+\ell}})[(\ell+1)d] = (\mathcal{O}_{I,J,\mu})^R \otimes \mathfrak{a}_J\,,$$

so that $\mathcal{R}^i = \oplus_{\mathsf{Index}(i)}\mathcal{R}(I,J,\mu)$. The $(\mathfrak{S}_\ell \times \mathfrak{S}_{n+\ell} \times \mathfrak{S}_\ell)$-linearisation of

$$\mathsf{pr}_{13*}\big(\mathsf{pr}_{23}^*\,\mathcal{R}^i \otimes \mathsf{pr}_{12}^*\,\mathcal{H}^j\big) \cong \bigoplus_{\mathsf{Index}(j)\times\mathsf{Index}(i)} \mathsf{pr}_{13*}\big(\mathsf{pr}_{23}^*\,\mathcal{R}(I_2,J_2,\mu_2) \otimes \mathsf{pr}_{12}^*\,\mathcal{H}(I_1,J_1,\mu_1)\big)$$

induces on the index set $\mathsf{Index}(j) \times \mathsf{Index}(i)$ the action

$$\sigma_1 \times \tau \times \sigma_2 \colon \big(I_1,J_1,\mu_1;I_2,J_2,\mu_2\big) \mapsto \big(\sigma_1(I_1),\tau(J_1),\tau\circ\mu_1\circ\sigma_1^{-1};\sigma(I_2),\tau(J_2),\tau\circ\mu_2\circ\sigma_2^{-1}\big)\,.$$

Let $O(i,j)$ be a set of representatives of the $(1\times \mathfrak{S}_{n+\ell}\times 1)$-orbits in $\mathsf{Index}(j)\times\mathsf{Index}(i)$. One can check that $O(i,j)$ is in bijection with

$$\mathsf{Index}(i,j):=\left\{(I_1,K_1,I_2,K_2,\mu)\,\middle|\,\begin{matrix}I_1\subset K_1\subset[\ell]\supset K_2\supset I_2,\ |I_1|=j,\ |I_2|=i,\\ |K_1|=|K_2|,\ \mu\colon \bar K_1\to\bar K_2 \text{ bijection}\end{matrix}\right\}$$

via the assignment $(I_1,J_1,\mu_1;I_2,J_2,\mu_2)\mapsto(I_1,K_1,I_2,K_2,\mu)$ where, exactly as in Section 5.1,

$$K_1=I_1\cup\mu_1^{-1}(J_2)\,,\quad K_2=I_2\cup\mu_2^{-1}(J_1)\,,\quad \mu=\mu^{-1}_{2|\overline{J_1\cup J_2}}\circ\mu_{1|\bar K_1}\,.$$

Furthermore, the $\mathfrak{S}_{n+\ell}$-stabiliser of $(I_1,J_1,\mu_1;I_2,J_2,\mu_2)$ is $\mathfrak{S}_{J_1\cap J_2}$. It follows by (2.3) and (5.3) that the equivariant convolution product $\mathcal{R}^i\star\mathcal{H}^j$ is given by

$$\begin{aligned}\mathcal{R}^i\star\mathcal{H}^j&\cong \mathsf{pr}_{13*}\left(\mathsf{pr}_{23}^*\,\mathcal{R}^i\otimes\mathsf{pr}_{12}^*\,\mathcal{H}^j\right)^{1\times\mathfrak{S}_{n+\ell}\times 1}\\ &\cong\bigoplus_{O(i,j)}\mathsf{pr}_{13*}\big(\mathsf{pr}_{23}^*\,\mathcal{R}(I_2,J_2,\mu_2)\otimes\mathsf{pr}_{12}^*\,\mathcal{H}(I_1,J_1,\mu_1)\big)^{1\times\mathfrak{S}_{J_1\cap J_2}\times 1} &(5.5)\\ &\cong\bigoplus_{\mathsf{Index}(i,j)}\mathcal{O}_{K_1,K_2,\mu}\otimes\mathfrak{a}_{K_1\setminus I_1}\otimes\mathfrak{a}_{K_2\setminus I_2}\otimes p^*\big(\Lambda^*_{n+i+j-k}(X)\otimes\omega_X^{-(k-j)}\big)[-(k-j)d]\,. &(5.6)\end{aligned}$$

We denote the direct summands of (5.6) by

$$\mathcal{P}(I_1,K_1,I_2,K_2,\mu):=\bigoplus_{\mathsf{Index}(i,j)}\mathcal{O}_{K_1,K_2,\mu}\otimes\mathfrak{a}_{K_1\setminus I_1}\otimes\mathfrak{a}_{K_2\setminus I_2}\otimes p^*\big(\Lambda^*_{n+i+j-k}(X)\otimes\omega_X^{-(k-j)}\big)[-(k-j)d]\,.$$

Note that the $(\mathfrak{S}_\ell\times\mathfrak{S}_\ell)$-linearisation of $\mathcal{R}^i\star\mathcal{H}^j$ induces on $O(i,j)\cong\mathsf{Index}(i,j)$ the action

$$\sigma_1\times\sigma_2\colon(I_1,K_1,I_2,K_2,\mu)\mapsto\big(\sigma_1(I_1),\sigma_1(K_1),\sigma_2(I_2),\sigma_2(K_2),\sigma_2\circ\mu\circ\sigma_1^{-1}\big)\,.$$

The $(\mathfrak{S}_\ell\times\mathfrak{S}_\ell)$-stabiliser of (I_1,K_1,I_2,K_2,μ) is $\mathfrak{S}_{I_1}\times\mathfrak{S}_{K_1\setminus I_1}\times\mathfrak{S}_{\bar K_1,\mu}\times\mathfrak{S}_{I_2}\times\mathfrak{S}_{K_2\setminus I_2}$. With this notation, we indicate the subgroup of $\mathfrak{S}_\ell\times\mathfrak{S}_\ell$ given by

$$\big\{(\sigma_1,\sigma_2)\mid\sigma_1(I_1)=I_1,\ \sigma_1(K_1)=K_1,\ \sigma_2(I_2)=I_2,\ \sigma_2(K_2)=K_2,\ (\sigma_2\circ\mu)_{|\bar K_1}=(\mu\circ\sigma_1)_{|\bar K_1}\big\}\,.$$

Furthermore, the orbits of the $(\mathfrak{S}_\ell\times\mathfrak{S}_\ell)$-action on $\mathsf{Index}(i,j)$ are given by

$$\mathsf{Index}(i,j)_k:=\{|K_1|=|K_2|=k\}\subset\mathsf{Index}(i,j)\quad\text{for } k=\mathsf{max}\{i,j\},\dots,\ell\,.$$

A representative of the orbit $\mathsf{Index}(i,j)_k$ is $([j],[k],[i],[k],e)$, where $e=\mathsf{id}_{[k+1,\ell]}$. We get

$$\begin{aligned}\mathcal{R}^i\star\mathcal{H}^j&\cong\bigoplus_{k=\mathsf{max}\{i,j\}}^{\ell}\mathcal{P}(i,j)_k\,,\\ \mathcal{P}(i,j)_k&:=\mathsf{Ind}^{\mathfrak{S}_\ell\times\mathfrak{S}_\ell}_{\mathfrak{S}_j\times\mathfrak{S}_{k-j}\times\mathfrak{S}_{\ell-k,e}\times\mathfrak{S}_i\times\mathfrak{S}_{k-i}}\mathcal{P}([j],[k],[i],[k],e)\,. &(5.7)\end{aligned}$$

5.4 Spectral sequences

For $j=0,\dots,\ell$, there is the spectral sequence $\mathcal{E}(j)$ associated with the complex $\mathsf{pr}_{23}^*\,\mathcal{H}$ and the functor $\mathcal{H}om\left(\_,\mathsf{pr}_{12}^*\,\mathcal{H}^j\right)$ given by

$$\mathcal{E}(j)_1^{p,q}=\mathcal{E}xt^q\left(\mathsf{pr}_{23}^*\,\mathcal{H}^{-p},\mathsf{pr}_{12}^*\,\mathcal{H}^j\right)\implies\mathcal{E}(j)^{p+q}=\mathcal{E}xt^{p+q}\left(\mathsf{pr}_{23}^*\,\mathcal{H},\mathsf{pr}_{12}^*\,\mathcal{H}^j\right);$$

see, for example, [Huy06, Remark 2.67]. By Section 5.1, every term of this spectral sequence is finitely supported over $X\times X^\ell\times X\times X^\ell$, hence pr_{13*}-acyclic. Since the functors $(\_)^{\mathfrak{S}_{n+\ell}}$ and $(\_)\otimes\mathsf{pr}_2^*\,\omega_{X\times X^\ell}$ are exact, we can apply the functor $\mathsf{pr}_{13*}(\_)^{1\times\mathfrak{S}_{n+\ell}\times 1}\otimes\mathsf{pr}_2^*\,\omega_{X\times X^\ell}$ to every level of the spectral sequence $\mathcal{E}(j)$ to get a spectral sequence with values in $\mathrm{Coh}_{\mathfrak{S}_\ell\times\mathfrak{S}_\ell}\big(X\times X^\ell\times X\times X^\ell\big)$.

Shifting this spectral sequence by $(\ell+1)d$ in the q-direction, we get the spectral sequence $E(j)$ with the property

$$E(j)_1^{p,q} = \mathscr{H}^q\left(\mathcal{R}^{-p}\star\mathcal{H}^j\right) \implies E(j)^{p+q} = \mathscr{H}^{p+q}\left(\mathcal{R}\star\mathcal{H}^j\right).$$

Similarly, we get a spectral sequence

$$E_1^{p,q} = \mathscr{H}^q\left(\mathcal{R}\star\mathcal{H}^p\right) \implies E^{p+q} = \mathscr{H}^{p+q}(\mathcal{R}\star\mathcal{H}). \tag{5.8}$$

5.5 Čech-type exact complex for the sign representation

For $k \geqslant 1$, there is an exact complex of $\mathfrak{S}_k$-representations

$$0 \to \mathbb{C} \to \cdots \to \mathsf{Ind}_{\mathfrak{S}_i\times\mathfrak{S}_{k-i}}^{\mathfrak{S}_k} \mathfrak{a}_i \xrightarrow{\check{d}^i} \mathsf{Ind}_{\mathfrak{S}_{i+1}\times\mathfrak{S}_{k-i-1}}^{\mathfrak{S}_k} \mathfrak{a}_{i+1} \to \cdots \to \mathfrak{a}_k \to 0,$$

which we denote by $\check{\mathcal{C}}_k^\bullet$. We consider $\check{\mathcal{C}}_k^\bullet$ as a complex in degrees $[0,k]$. The terms are

$$\check{\mathcal{C}}_k^i = \mathsf{Ind}_{\mathfrak{S}_i\times\mathfrak{S}_{k-i}}^{\mathfrak{S}_k} \mathfrak{a}_i \cong \bigoplus_{I\subset[k],\,|I|=i} \mathfrak{a}_I .$$

Under this identification, the differential $\check{d}^i$ is determined by its components $\mathfrak{a}_I \to \mathfrak{a}_J$, which are given by $\varepsilon_{I,b} = (-1)^{\#\{a\in I\mid a<b\}}$ if $J = I\cup\{b\}$ and which are zero if $I \not\subset J$. That the sequence is exact can be checked either by hand or by considering it as a special case of a Čech complex. We also set $\hat{\mathcal{C}}_k^\bullet := \check{\mathcal{C}}_k^\bullet \otimes \mathfrak{a}_k$. Then $\hat{\mathcal{C}}_k^\bullet$ is the exact complex

$$0 \to \mathfrak{a}_k \to \cdots \to \mathsf{Ind}_{\mathfrak{S}_i\times\mathfrak{S}_{k-i}}^{\mathfrak{S}_k} \mathfrak{a}_{k-i} \xrightarrow{\hat{d}^i} \mathsf{Ind}_{\mathfrak{S}_{i+1}\times\mathfrak{S}_{k-i-1}}^{\mathfrak{S}_k} \mathfrak{a}_{k-i-1} \to \cdots \to \mathbb{C} \to 0.$$

Let M be a variety on which we consider $\mathfrak{S}_k$ to act trivially. For $E \in \mathrm{Coh}(M)$, we set $\check{\mathcal{C}}_k^\bullet(E) := E\otimes_{\mathbb{C}}\check{\mathcal{C}}_k^\bullet$. This is an exact complex in $\mathrm{Coh}_{\mathfrak{S}_k}(M)$ given by

$$E \to \cdots \to \mathsf{Ind}_{\mathfrak{S}_i\times\mathfrak{S}_{k-i}}^{\mathfrak{S}_k}(E\otimes\mathfrak{a}_i) \xrightarrow{\check{d}^i(E)} \mathsf{Ind}_{\mathfrak{S}_{i+1}\times\mathfrak{S}_{k-i-1}}^{\mathfrak{S}_k}(E\otimes\mathfrak{a}_{i+1}) \to \cdots \to E\otimes\mathfrak{a}_k .$$

There also is the exact complex $\hat{\mathcal{C}}_k^\bullet(E) := E\otimes_{\mathbb{C}}\hat{\mathcal{C}}_k^\bullet \cong \check{\mathcal{C}}_k^\bullet(E)\otimes\mathfrak{a}_k$.

LEMMA 5.1. *Let $E \in \mathrm{Coh}(M)$ be simple; that is, $\mathsf{Hom}(E,E) = \mathbb{C}$. Then*

$$\mathsf{Hom}_{\mathfrak{S}_k}\left(\check{\mathcal{C}}_k^i(E), \check{\mathcal{C}}_k^{i+1}(E)\right) \cong \mathbb{C} \cong \mathsf{Hom}_{\mathfrak{S}_k}\left(\hat{\mathcal{C}}_k^i(E), \hat{\mathcal{C}}_k^{i+1}(E)\right).$$

Proof. By the adjunction $\mathsf{Ind} \dashv \mathsf{Res}$, we have

$$\begin{aligned}\mathsf{Hom}_{\mathfrak{S}_k}\left(\check{\mathcal{C}}_k^i(E), \check{\mathcal{C}}_k^{i+1}(E)\right) &\cong \left[\bigoplus_{|I|=i+1} \mathsf{Hom}(E\otimes\mathfrak{a}_{[i]}, E\otimes\mathfrak{a}_I)\right]^{\mathfrak{S}_i\times\mathfrak{S}_{k-i}} \\ &\cong \left[\bigoplus_{|I|=i+1} \mathsf{Hom}(E,E)\otimes\mathfrak{a}_{[i]\setminus I}\otimes\mathfrak{a}_{I\setminus[i]}\right]^{\mathfrak{S}_i\times\mathfrak{S}_{k-i}}.\end{aligned}$$

For $[i]\not\subset I$, we have $|I\setminus[i]| \geqslant 2$, and hence $(\mathsf{Hom}(E,E)\otimes\mathfrak{a}_{[i]\setminus I}\otimes\mathfrak{a}_{I\setminus[i]})^{\mathfrak{S}_{I\setminus[i]}} = 0$. It follows by Section 2.2 that

$$\mathsf{Hom}_{\mathfrak{S}_k}\left(\check{\mathcal{C}}_k^i(E), \check{\mathcal{C}}_k^{i+1}(E)\right) \cong \mathsf{Hom}(E\otimes\mathfrak{a}_{[i]}, E\otimes\mathfrak{a}_{[i+1]})^{\mathfrak{S}_{[i]}\times\mathfrak{S}_{[i+2,k]}} \cong \mathsf{Hom}(E,E) \cong \mathbb{C}.$$

The second assertion follows from the first one since $\hat{\mathcal{C}}_k^\bullet(E) \cong \check{\mathcal{C}}_k^\bullet(E)\otimes\mathfrak{a}_k$. □

COROLLARY 5.2. *Let $E \in \mathrm{Coh}(M)$ be simple. Then, up to isomorphism, every non-zero $\mathfrak{S}_k$-equivariant morphism $\check{\mathcal{C}}_k^i(E) \to \check{\mathcal{C}}_k^{i+1}(E)$ equals $\check{d}^i(E)$, and every non-zero $\mathfrak{S}_k$-equivariant morphism $\hat{\mathcal{C}}_k^i(E) \to \hat{\mathcal{C}}_k^{i+1}(E)$ equals $\hat{d}^i(E)$.*

Convention 5.3. We also define $\check{\mathcal{C}}^\bullet_0 := \mathbb{C}[0] =: \hat{\mathcal{C}}^\bullet_0$ to be the one-term complex with $\mathbb{C}$ in degree zero. Obviously, the complexes $\check{\mathcal{C}}^\bullet_0$ and $\hat{\mathcal{C}}^\bullet_0$ are not exact, in contrast to the case $k \geqslant 1$ described above.

5.6 The curve case: Induced maps

We will need the following easy fact further on.

Lemma 5.4. *Let $\iota\colon Z \to M$ be a closed embeddings of an irreducible subvariety. Then we have $\mathsf{Hom}_M(E, \iota_* L) = 0$ for all $L \in \mathrm{Pic}(Z)$ and $E \in \mathrm{Coh}(M)$ with $\mathsf{supp}\, E \not\supset Z$.*

For the present and the next two subsections, let $X = C$ be a smooth curve. By Lemma 4.8, we have $\Lambda^*_m(C) = \mathcal{O}_C[0]$. Using the notation of Section 5.3, we get isomorphisms

$$\begin{aligned}\mathcal{P}(i,j)_k &\cong \mathsf{Ind}^{\mathfrak{S}_\ell\times\mathfrak{S}_\ell}_{\mathfrak{S}_j\times\mathfrak{S}_{k-j}\times\mathfrak{S}_{\ell-k,e}\times\mathfrak{S}_i\times\mathfrak{S}_{k-i}} \mathcal{O}_{[k],[k],e}\otimes\mathfrak{a}_{[j+1,k]}\otimes\mathfrak{a}_{[i+1,k]}\otimes p^*\omega_C^{-(k-j)}[-(k-j)]\\ &\cong \mathsf{Ind}^{\mathfrak{S}_\ell\times\mathfrak{S}_\ell}_{\mathfrak{S}_j\times\mathfrak{S}_{k-j}\times\mathfrak{S}_{\ell-k,e}\times\mathfrak{S}_k} \check{\mathcal{C}}^{k-i}_k\big(\mathcal{O}_{[k],[k],e}\otimes p^*\omega_C^{-(k-j)}\big)\otimes\mathfrak{a}_{[j+1,k]}[-(k-j)] \qquad (5.9)\end{aligned}$$

for $k = \max\{i,j\},\dots,\ell$. The second isomorphism is due to the general fact that for subgroups $V \subset U \subset G$ of a finite group G, there is an isomorphism of functors $\mathsf{Ind}_V^G \cong \mathsf{Ind}_U^G \circ \mathsf{Ind}_V^U$.

In particular, $\mathcal{P}(i,j)_k$ is concentrated in degree $k-j$, so that (5.7) induces an isomorphism $\mathscr{H}^{k-j}\left(\mathcal{R}^i \star \mathcal{H}^j\right) \cong \mathscr{H}^{k-j}(\mathcal{P}(i,j)_k)$.

Lemma 5.5. *Let $\max\{i,j\} \leqslant k \leqslant \ell$. Under the isomorphism (5.9), the morphism*

$$\mathscr{H}^{k-j}\left(\mathcal{R}^i\star\mathcal{H}^j\right)\cong\mathscr{H}^{k-j}(\mathcal{P}(i,j)_k)\to\mathscr{H}^{k-j}(\mathcal{P}(i-1,j)_k)\cong\mathscr{H}^{k-j}\left(\mathcal{R}^{i-1}\star\mathcal{H}^j\right)$$

induced by the differential $\mathcal{H}^{i-1}\to\mathcal{H}^i$ is given by $\mathsf{Ind}^{\mathfrak{S}_\ell\times\mathfrak{S}_\ell}_{\mathfrak{S}_j\times\mathfrak{S}_{k-j}\times\mathfrak{S}_{\ell-k,e}\times\mathfrak{S}_k} \check{d}^{k-i}\big(\mathcal{O}_{[k],[k],e}\otimes p^*\omega_C^{-(k-j)}\big)$.

Here, $\check{d}^{k-i}\big(\mathcal{O}_{[k],[k],e}\otimes p^*\omega_C^{-(k-j)}\big)$ denotes the differential in degree $k-i$ of the complex $\check{\mathcal{C}}^\bullet\big(\mathcal{O}_{[k],[k],e}\otimes p^*\omega_C^{-(k-j)}\big)$.

Proof. Recall that $\mathcal{P}(i,j)_k = \oplus_{\mathsf{Index}(i,j)_k}\mathcal{P}(I_1,K_1,I_2,K_2,\mu)$; see Section 5.3. Under the isomorphism (5.9), this gives the direct-sum decomposition

$$\check{\mathcal{C}}^{k-i}_k\big(\mathcal{O}_{[k],[k],e}\otimes p^*\omega_C^{-(k-j)}\big)\otimes\mathfrak{a}_{[j+1,k]}[-(k-j)]\cong\bigoplus_{I_2\subset[k],\,|I_2|=i}\mathcal{P}([j],[k],I_2,[k],e)\,.$$

By Lemma 5.4, all the components $\mathcal{P}([j],[k],I_2,[k],e)\to\mathcal{P}(I_1',K_1',I_2',K_2',\mu)$ of the map

$$\mathcal{P}([j],[k],I_2,[k],e)\hookrightarrow\mathcal{P}(i,j)_k\to\mathcal{P}(i-1,j)_k \qquad (5.10)$$

are zero unless $K_1' = K_2' = [k]$ and $\mu = e$. They are also zero for $I_1' \neq [j]$ since our morphism $\mathcal{R}^i\star\mathcal{H}^j\to\mathcal{R}^{i-1}\star\mathcal{H}^j$ is given by the identity on the factor $\mathcal{H}^j$, and $\mathcal{P}(I_1,K_1,I_2,K_2,\mu)$ arises as $\mathsf{pr}_{13*}\big(\mathsf{pr}_{23}^*\mathcal{R}(I_2,J_2,\mu_2)\otimes\mathsf{pr}_{12}^*\mathcal{H}(I_1,J_1,\mu_1)\big)^{1\times\mathfrak{S}_{J_1\cap J_2}\times 1}$; see (5.5) and (5.6). In summary, all the non-zero components of (5.10) are of the form $\mathcal{P}([j],[k],I_2,[k],e)\to\mathcal{P}([j],[k],I_2',[k],e)$. By the adjunction $\mathsf{Res}\dashv\mathsf{Ind}$, it follows that the map $\mathscr{H}^{k-j}(\mathcal{P}(i,j)_k)\to\mathscr{H}^{k-j}(\mathcal{P}(i-1,j)_k)$ is of the form $\mathsf{Ind}^{\mathfrak{S}_\ell\times\mathfrak{S}_\ell}_{\mathfrak{S}_j\times\mathfrak{S}_{k-j}\times\mathfrak{S}_{\ell-k,e}\times\mathfrak{S}_k}(f)$ for some

$$f\colon\check{\mathcal{C}}^{k-i}_k\big(\mathcal{O}_{[k],[k],e}\otimes p^*\omega_C^{-(k-j)}\big)\to\check{\mathcal{C}}^{k-i+1}_k\big(\mathcal{O}_{[k],[k],e}\otimes p^*\omega_C^{-(k-j)}\big)\,,$$

and we need to prove that $f = \check{d}^{k-i}\big(\mathcal{O}_{[k],[k],e}\otimes p^*\omega_C^{-(k-j)}\big)$. For this purpose, by Corollary 5.2, it is sufficient to show that the component

$$\mathscr{H}^{k-j}\big(\mathcal{P}([j],[k],[i],[k],e)\big)\to\mathscr{H}^{k-j}\big(\mathcal{P}([j],[k],[i-1],[k],e)\big) \qquad (5.11)$$

of $\mathscr{H}^{k-j}(\mathcal{P}(i,j)_k) \to \mathscr{H}^{k-j}(\mathcal{P}(i-1,j)_k)$ is non-zero. By (5.5) and (5.6), we have

$$\mathcal{P}\big([j],[k],[i],[k],e\big) \cong \big[\mathcal{R}\big([i],[n+i],e\big) \star \mathcal{H}\big([j],J_1,e\big)\big]^{\mathfrak{S}_{[n+i+j-k]}},$$

where a possible choice of J_1 is $J_1 = [n+i+j-k] \cup [n+i+1, n+k]$. In degree $k-j$, the $\mathfrak{S}_{[n+i+j-k]}$-action on $\mathcal{R}\big([i],[n+i],e\big) \star \mathcal{H}\big([j],J_1,e\big)$ is trivial because given by the representation $\wedge^0 \varrho_{[n+i+j-k]}$; see (5.3). Hence,

$$\mathscr{H}^{k-j}\big(\mathcal{P}([j],[k],[i],[k],e)\big) \cong \mathscr{H}^{k-j}\big(\mathcal{R}\big([i],[n+i],e\big) \star \mathcal{H}\big([j],J_1,e\big)\big). \tag{5.12}$$

Analogously, we get

$$\mathscr{H}^{k-j}\big(\mathcal{P}([j],[k],[i-1],[k],e)\big) \cong \mathscr{H}^{k-j}\big(\mathcal{R}\big([i-1],[2,n+i],e\big) \star \mathcal{H}\big([j],J_1,e\big)\big). \tag{5.13}$$

Under (5.12) and (5.13), the morphism (5.11) corresponds to the morphism

$$\mathscr{H}^{k-j}\big(\mathcal{R}\big([i],[n+i],e\big) \star \mathcal{H}\big([j],J_1,e\big)\big) \longrightarrow \mathscr{H}^{k-j}\big(\mathcal{R}\big([i-1],[2,n+i],e\big) \star \mathcal{H}\big([j],J_1,e\big)\big)$$

induced by the restriction $\mathcal{O}_{[i-1],[2,n+i],e} \to \mathcal{O}_{[i],[n+i],e}$. As pointed out at the end of Section 5.2, this is an isomorphism. □

5.7 The curve case: Full faithfulness

Proposition 5.6. *For $X = C$ a curve, we have $\mathcal{R} \star \mathcal{H} \cong \mathsf{Ind}_{\mathfrak{S}_{\ell,e}}^{\mathfrak{S}_\ell,\mathfrak{S}_\ell} \mathcal{O}_{\Delta_{C\times C^\ell}}$.*

Proof. Consider the spectral sequences $E(j)_1^{p,q} = \mathscr{H}^q\left(\mathcal{R}^{-p} \star \mathcal{H}^j\right) \implies \mathscr{H}^{p+q}\left(\mathcal{R} \star \mathcal{H}^j\right)$; see Section 5.4. By (5.9) and Lemma 5.5, for $k = j,\dots,\ell$, the $(k-j)$th row of $E(j)_1$ is given by the complex $\mathsf{Ind}_{\mathfrak{S}_j\times\mathfrak{S}_{k-j}\times\mathfrak{S}_{\ell-k,e}\times\mathfrak{S}_k}^{\mathfrak{S}_\ell\times\mathfrak{S}_\ell} \check{\mathcal{C}}_k^\bullet\big(\mathcal{O}_{[k],[k],e} \otimes p^*\omega_X^{-(k-j)}\big) \otimes \mathfrak{a}_{[j+1,k]}$ shifted into degrees $[-k,0]$. The induction functor is exact. Thus, all the rows of the spectral sequences are exact with one exception: the zero row of $E(0)_1$ is given by the single non-zero object $E(0)_1^{0,0} = \mathscr{H}^0(\mathcal{P}(0,0)_0)$; see Convention 5.3. It follows that $\mathcal{R} \star \mathcal{H}^j = 0$ for $j \geqslant 1$ and $\mathcal{R} \star \mathcal{H}^0 \cong \mathscr{H}^0(\mathcal{P}(0,0)_0)$. Now, by the spectral sequence (5.8) or, alternatively, by the fact that $\mathcal{R} \star \mathcal{H}$ is a left convolution of $\mathcal{R}\star\mathcal{H}^0 \to \mathcal{R}\star\mathcal{H}^1 \to \cdots \to \mathcal{R}\star\mathcal{H}^\ell$, it follows that $\mathcal{R}\star\mathcal{H} \cong \mathscr{H}^0(\mathcal{P}(0,0)_0) \cong \mathsf{Ind}_{\mathfrak{S}_{\ell,e}}^{\mathfrak{S}_\ell,\mathfrak{S}_\ell} \mathcal{O}_{\Delta_{C\times C^\ell}}$. □

Proof of Theorem 1.1(i)(a). The identity functor $\mathsf{id}\colon \mathrm{D}^b_{\mathfrak{S}_\ell}\left(C\times C^\ell\right) \to \mathrm{D}^b_{\mathfrak{S}_\ell}\left(C\times C^\ell\right)$ equals the equivariant FM transform with kernel $\mathsf{Ind}_{(\mathfrak{S}_\ell)_\Delta}^{\mathfrak{S}_\ell\times\mathfrak{S}_\ell} \Delta_{C\times C^\ell}$; see Remark 2.2. Note that we have $(\mathfrak{S}_\ell)_\Delta = \mathfrak{S}_{\ell,e} \subset \mathfrak{S}_\ell\times\mathfrak{S}_\ell$. Thus, the assertion $R_{\ell,n}\circ H_{\ell,n} \cong \mathsf{id}$ follows from Proposition 5.6. □

5.8 The curve case: Orthogonality

In this section, we will outline the proof of Theorem 1.1(i)(b). Lemmas 5.7 and 5.8 below state formulae for the convolution products $\mathcal{R}^i_{\ell',n'} \star \mathcal{H}^j_{\ell,n}$ for $n+\ell = n'+\ell'$ and the induced maps between them. The proofs, which are completely analogous to the computations of Sections 5.1–5.3 and 5.6, are left to the reader. The author decided to explicitly write down the computations of $\mathcal{R}^i_{\ell',n'} \star \mathcal{H}^j_{\ell,n}$ in the Sections 5.1 and 5.3 only in the special case $(\ell,n) = (\ell',n')$ in order to avoid the heavier notation that the general case would have required and because the special case $(\ell,n) = (\ell',n')$ is entirely sufficient for parts (i)(a) and (ii) of Theorem 1.1. In particular, the reader mainly interested in the surface case may safely skip the rest of the current subsection.

Let $\ell,n,\ell',n' \in \mathbb{Z}$ be integers such that $n > \max\{1,\ell\}$, $n' > \max\{1,\ell'\}$, and $n+\ell = n'+\ell'$. We introduce some notation which, for $(\ell,n) = (\ell',n')$, specialises to the notation of the previous subsections. For $I_1 \subset K_1 \subset [\ell]$, $I_2 \subset K_2 \subset [\ell']$, and $\mu\colon \bar K_1 \to \bar K_2$ a bijection, we consider

$$\Gamma_{K_1,K_2,\mu} := \big\{(x,x_1,\dots,x_\ell,z,z_1,\dots,z_{\ell'}) \mid x = x_a = z_b\,\forall\, a\in K_1,\ b\in K_2,\ x_c = z_{\mu(c)}\,\forall\, c\in \bar K_1\big\}$$

as a subvariety of $X\times X^\ell\times X\times X^{\ell'}$. We set $\mathcal{O}_{K_1,K_2,\mu}:=\mathcal{O}_{\Gamma_{K_1,K_2,\mu}}$ and

$$\mathcal{P}(I_1,K_1,I_2,K_2,\mu):=\mathcal{O}_{K_1,K_2,\mu}\otimes\mathfrak{a}_{K_1\setminus I_1}\otimes\mathfrak{a}_{K_2\setminus I_2}\otimes p^*\big(\Lambda^*_{n'+i+j-k}(X)\otimes\omega_X^{-(k-j)}\big)[-(k-j)d]\,,$$

where $j:=|I_1|$ and $k:=|K_1|$. Again, $p\colon X\times X^\ell\times X\times X^{\ell'}\to X$ denotes the projection to the third factor. For $0\leqslant i\leqslant \ell'$, $0\leqslant j\leqslant \ell$, and $\max\{n'-n+i,j\}\leqslant k\leqslant\ell$, we set

$$\mathcal{P}(i,\ell',j,\ell)_k:=\mathsf{Ind}_{\mathfrak{S}_j\times\mathfrak{S}_{k-j}\times\mathfrak{S}_{\ell-k,e}\times\mathfrak{S}_i\times\mathfrak{S}_{k+n-n'-i}}^{\mathfrak{S}_\ell\times\mathfrak{S}_\ell}\mathcal{P}([j],[k],[i],[k+n-n'],e)\,.$$

Lemma 5.7. *$\mathcal{R}^i_{\ell',n'}\star\mathcal{H}^j_{\ell,n}\cong\bigoplus_{k=\max\{n'-n+i,j\}}^{\ell}\mathcal{P}(i,\ell',j,\ell)_k$.*

Proof. This follows from computations analogous to those of Sections 5.1 and 5.3. □

Lemma 5.8. *Let $X=C$ be a curve and $\ell'>\ell$. Then, for $0\leqslant j\leqslant k\leqslant\ell$, we have the vanishing $\mathscr{H}^{k-j}\left(\mathcal{R}^i_{\ell',n'}\star\mathcal{H}^j_{\ell,n}\right)=0$ for $i>k+n-n'$, and the sequence*

$$0\to\mathscr{H}^{k-j}\left(\mathcal{R}^{k+n-n'}_{\ell',n'}\star\mathcal{H}^j_{\ell,n}\right)\to\cdots\to\mathscr{H}^{k-j}\left(\mathcal{R}^{0}_{\ell',n'}\star\mathcal{H}^j_{\ell,n}\right)\to 0\,,$$

whose differentials are induced by the differentials of $\mathcal{H}_{\ell',n'}$, is isomorphic to

$$\mathsf{Ind}_{\mathfrak{S}_j\times\mathfrak{S}_{k-j}\times\mathfrak{S}_{\ell-k,e}\times\mathfrak{S}_{k+n-n'}}^{\mathfrak{S}_\ell\times\mathfrak{S}_\ell}\check{\mathcal{C}}^\bullet_{k+n-n'}\big(\mathcal{O}_{[k],[k+n-n'],e}\otimes p^*\omega_C^{-(k-j)}\big)\otimes\mathfrak{a}_{[j+1,k]}\,.$$

In particular, it is an exact sequence.

Proof. This follows from computations analogous to those of Sections 5.2 and 5.6. □

Proposition 5.9. *Let $X=C$ be a curve and $\ell'>\ell$. Then $\mathcal{R}_{\ell',n'}\star\mathcal{H}_{\ell,n}=0$.*

Proof. This follows from Lemma 5.8 together with spectral sequences analogous to those of Section 5.4. □

5.9 The surface case: Induced maps

For the remainder of this section, let X be a smooth surface. Recall that $p\colon X\times X^\ell\times X\times X^\ell\to X$ denotes the projection to the third factor, and set

$$\tilde S_X:=(\_)\otimes p^*\omega_X[2]\in\mathsf{Aut}\left(\mathrm{D}^b_{\mathfrak{S}_\ell\times\mathfrak{S}_\ell}\left(X\times X^\ell\times X\times X^\ell\right)\right)$$

and $\tilde S_X^{-[a,b]}:=\tilde S_X^{-a}\oplus\tilde S_X^{-(a+1)}\oplus\cdots\oplus\tilde S_X^{-b}$ for $a\leqslant b$ two integers. By Lemma 4.8, we have $p^*\Lambda^*_m(X)=\tilde S_X^{-[0,m-1]}(\mathcal{O}_{X\times X^\ell\times X\times X^\ell})$. Hence, by the results of Section 5.3, for $k=\max\{i,j\},\dots,\ell$, we get

$$\begin{aligned}\mathcal{P}(i,j)_k&\cong\tilde S_X^{-[k-j,n+i-1]}\,\mathsf{Ind}_{\mathfrak{S}_j\times\mathfrak{S}_{k-j}\times\mathfrak{S}_{\ell-k,e}\times\mathfrak{S}_i\times\mathfrak{S}_{k-i}}^{\mathfrak{S}_\ell\times\mathfrak{S}_\ell}\mathcal{O}_{[k],[k],e}\otimes\mathfrak{a}_{[j+1,k]}\otimes\mathfrak{a}_{[i+1,k]}\\&\cong\tilde S_X^{-[k-j,n+i-1]}\,\mathsf{Ind}_{\mathfrak{S}_k\times\mathfrak{S}_{\ell-k,e}\times\mathfrak{S}_k}^{\mathfrak{S}_\ell\times\mathfrak{S}_\ell}\hat{\mathcal{C}}^j_k\check{\mathcal{C}}^{k-i}_k(\mathcal{O}_{[k],[k],e})\,.\end{aligned}\tag{5.14}$$

Here, the inner term $\check{\mathcal{C}}^{k-i}_k$ is interpreted by considering $\mathfrak{S}_k$ as a subgroup of $\mathfrak{S}_\ell\times\mathfrak{S}_\ell$ by the embedding into the second factor, while the outer term $\hat{\mathcal{C}}^j_k$ is interpreted by considering $\mathfrak{S}_k$ as a subgroup of $\mathfrak{S}_\ell\times\mathfrak{S}_\ell$ by the embedding into the first factor. In particular, we have

$$\mathscr{H}^{2r}\big(\mathcal{P}(i,j)_k\big)\cong\mathsf{Ind}_{\mathfrak{S}_k\times\mathfrak{S}_{\ell-k,e}\times\mathfrak{S}_k}^{\mathfrak{S}_\ell\times\mathfrak{S}_\ell}\hat{\mathcal{C}}^j_k\check{\mathcal{C}}^{k-i}_k\big(\mathcal{O}_{[k],[k],e}\otimes p^*\omega_X^{-r}\big)\quad\text{for } k-j\leqslant r\leqslant n+i-1\,.\tag{5.15}$$

Lemma 5.10. *Let $k,k'\in[\max\{i,j\},\ell]$. The components $\mathscr{H}^q(\mathcal{P}(i,j)_k)\to\mathscr{H}^q(\mathcal{P}(i-1,j)_{k'})$ of the morphism $\mathscr{H}^q\left(\mathcal{R}^i\star\mathcal{H}^j\right)\to\mathscr{H}^q(\mathcal{R}^{i-1}\star\mathcal{P}^j)$ which is induced by the differential $\mathcal{H}^{i-1}\to\mathcal{H}^i$ are zero for $k\neq k'$. Furthermore, under the isomorphism (5.15), for $k-j\leqslant r\leqslant n+i-2$, the component $\mathscr{H}^{2r}(\mathcal{P}(i,j)_k)\to\mathscr{H}^{2r}(\mathcal{P}(i-1,j)_k)$ is given by $\mathsf{Ind}_{\mathfrak{S}_k\times\mathfrak{S}_{\ell-k,e}\times\mathfrak{S}_k}^{\mathfrak{S}_\ell\times\mathfrak{S}_\ell}\hat{\mathcal{C}}^j_k\check{d}^{k-i}_k\big(\mathcal{O}_{[k],[k],e}\otimes p^*\omega_X^{-r}\big)$.*

Here, $d_k^{k-i}\big(\mathcal{O}_{[k],[k],e}\otimes p^*\omega_X^{-r}\big)$ denotes the differential of the complex $\mathcal{C}^\bullet\big(\mathcal{O}_{[k],[k],e}\otimes p^*\omega_X^{-r}\big)$. Furthermore, we regard $\hat{\mathcal{C}}_k^j$ as the functor $\mathsf{Ind}_{\mathfrak{S}_i\times\mathfrak{S}_{k-i}}^{\mathfrak{S}_k}\big((\_)\otimes\mathfrak{a}_i\big)$, which is applied to the morphism $d_k^{k-i}\big(\mathcal{O}_{[k],[k],e}\otimes p^*\omega_X^{-r}\big)$ to give a morphism $\hat{\mathcal{C}}_k^j\mathcal{C}_k^{k-i}\big(\mathcal{O}_{[k],[k],e}\otimes p^*\omega_X^{-r}\big)\to\hat{\mathcal{C}}_k^j\mathcal{C}_k^{k-i+1}\big(\mathcal{O}_{[k],[k],e}\otimes p^*\omega_X^{-r}\big)$.

Proof. The components $\mathcal{H}(I_2',J_2',\mu_2')\to\mathcal{H}(I_2,J_2,\mu_2)$ of the differential $\mathcal{H}^{i-1}\to\mathcal{H}^i$ are non-zero only if $I_2'\subset I_2$; compare with Section 3.2. Thus, following the computations of Section 5.3, the only components $\mathcal{P}(i,j)_k\to\mathcal{P}(i-1,j)_{k'}$ of $\mathcal{R}^i\star\mathcal{H}^j\to\mathcal{R}^{i-1}\star\mathcal{H}^j$ which are possibly non-zero are those with $k=k'$ or $k-1=k'$. But $\mathscr{H}^q(\mathcal{P}(i,j)_k)\to\mathscr{H}^q(\mathcal{P}(i-1,j)_{k-1})$ is zero by Lemma 5.4.

Exactly as in the curve case, we can reduce the proof of the second assertion to the claim that

$$\mathscr{H}^{2r}\big(\mathcal{P}([j],[k],[i],[k],e)\big)\to\mathscr{H}^{2r}\big(\mathcal{P}([j],[k],[i-1],[k],e)\big)\tag{5.16}$$

is non-zero; see Corollary 5.2 and the proof of Lemma 5.5. By (5.5) and (5.6), we have

$$\mathscr{H}^{2r}\big(\mathcal{P}([j],[k],[i],[k],e)\big)\cong\mathscr{H}^{2r}\big(\mathcal{R}\big([i],[n+i],e\big)\star\mathcal{H}\big([j],J_1,e\big)\big)^{\mathfrak{S}_{[n+i+j-k]}},$$

where a possible choice of J_1 is $J_1=[n+i+j-k]\cup[n+i+1,n+k]$. Also,

$$\begin{aligned}\mathscr{H}^{2r}\big(\mathcal{P}([j],[k],[i-1],[k],e)\big)&\cong\mathscr{H}^{2r}\big(\mathcal{R}\big([i-1],[2,n+i],e\big)\star\mathcal{H}\big([j],J_1,e\big)\big)^{\mathfrak{S}_{[2,n+i+j-k]}}\\&\cong\Bigg[\bigoplus_{a\in[n+i+j-k]}\mathscr{H}^{2r}\big(\mathcal{R}\big([i-1],[n+i]\setminus\{a\},e\big)\star\mathcal{H}\big([j],J_1,e\big)\big)\Bigg]^{\mathfrak{S}_{[n+i+j-k]}},\end{aligned}$$

where the second isomorphism is due to Section 2.2. As explained in Section 5.2, under the isomorphism (5.3), the components of the induced map

$$\begin{aligned}&\mathscr{H}^{2r}\big(\mathcal{R}\big([i],[n+i],e\big)\star\mathcal{H}\big([j],J_1,e\big)\big)\\&\longrightarrow\bigoplus_{a\in[n+i+j-k]}\mathscr{H}^{2r}\big(\mathcal{R}\big([i-1],[n+i]\setminus\{a\},e\big)\star\mathcal{H}\big([j],J_1,e\big)\big)\end{aligned}\tag{5.17}$$

are given by the canonical surjections $\varrho_{[n+i+j-k]}\to\varrho_{[n+i+j-k]\setminus\{a\}}$. It follows by [Sca09, Lemma B.6(3)] that the map induced by (5.17) on the $\mathfrak{S}_{[n+i+j-k]}$-invariants, which is exactly (5.16), is an isomorphism. □

Lemma 5.11. *Let $k,k'\in[\max\{i,j+1\},\ell]$. The components $\mathscr{H}^q(\mathcal{P}(i,j)_k)\to\mathscr{H}^q(\mathcal{P}(i,j+1)_{k'})$ of the morphism $\mathscr{H}^q\left(\mathcal{R}^i\star\mathcal{H}^j\right)\to\mathscr{H}^q\left(\mathcal{R}^i\star\mathcal{H}^{j+1}\right)$ which is induced by the differential $\mathcal{H}^j\to\mathcal{H}^{j+1}$ are zero for $k'\notin\{k,k+1\}$. For $k-j\leqslant r\leqslant n+i-1$, the component $\mathscr{H}^{2r}(\mathcal{P}(i,j)_k)\to\mathscr{H}^{2r}(\mathcal{P}(i,j+1)_k)$ is given by $\mathsf{Ind}_{\mathfrak{S}_k\times\mathfrak{S}_{\ell-k,e}\times\mathfrak{S}_k}^{\mathfrak{S}_\ell\times\mathfrak{S}_\ell}\,\check{d}_k^j\big(\check{\mathcal{C}}_k^{k-i}\big(\mathcal{O}_{[k],[k],e}\otimes p^*\omega_X^{-r}\big)\big)$.*

Proof. The proof is analogous to that of Lemma 5.10. The first assertion follows from the fact that the only non-zero components of $\mathcal{H}^j\to\mathcal{H}^{j+1}$ are of the form $\mathcal{H}(I_1,J_1,\mu_1)\to\mathcal{H}(I_1',J_1',\mu_1')$ with $I_1\subset I_1'$. The second assertion can be reduced to the non-vanishing of the component

$$\mathscr{H}^{2r}\big(\mathcal{P}([j],[k],[i],[k],e)\big)\to\mathscr{H}^{2r}\big(\mathcal{P}([j+1],[k],[i],[k],e)\big).$$

Set $J_2:=[n+i+j+1-k]\cup[n+j+2,n+k]$. There are isomorphisms

$$\mathscr{H}^{2r}\big(\mathcal{P}([j+1],[k],[i],[k],e)\big)\cong\mathscr{H}^{2r}\big(\mathcal{R}\big([i],J_2,e\big)\star\mathcal{H}\big([j+1],[n+j+1],e\big)\big)^{\mathfrak{S}_{[n+i+j+1-k]}},$$

$$\begin{aligned}&\mathscr{H}^{2r}\big(\mathcal{P}([j],[k],[j],[k],e)\big)\\&\cong\Bigg[\bigoplus_{b\in[n+i+j+1-k]}\mathscr{H}^{2r}\big(\mathcal{R}\big([i],J_2,e\big)\star\mathcal{H}\big([j],[n+j+1]\setminus\{b\},e\big)\big)\Bigg]^{\mathfrak{S}_{[n+i+j+1-k]}}.\end{aligned}$$

Again following Section 5.2, under the isomorphism (5.4), the components of the induced map

$$\bigoplus_{b\in[n+i+j+1-k]} \mathscr{H}^{2r}\big(\mathcal{R}([i],J_2,e)\star\mathcal{H}([j],[n+j+1]\setminus\{b\},e)\big) \longrightarrow \mathscr{H}^{2r}\big(\mathcal{R}([i],J_2,e)\star\mathcal{H}([j+1],[n+j+1],e)\big) \tag{5.18}$$

are given by the canonical injections $\varrho_{[n+i+j+1-k]\setminus\{b\}} \to \varrho_{[n+i+j+1-k]}$. It follows by [Sca09, Lemma B.6(4)] that (5.18) induces an isomorphism on the $\mathfrak{S}_{[n+i+j+1-k]}$-invariants. □

In fact, one can compute that the component $\mathscr{H}^{2r}(\mathcal{P}(i,j)_k)\to\mathscr{H}^{2r}(\mathcal{P}(i,j+1)_{k+1})$ is induced by the restriction $\mathcal{O}_{[k],[k],e}\to\mathcal{O}_{[k+1],[k+1],e}$. But this will not be relevant for our purposes.

5.10 The surface case: The cohomology of $\mathcal{R}\star\mathcal{H}$

Recall that, for $0\leqslant m<k$, there is the *stupid truncation* $\sigma^{\leqslant m}\check{\mathcal{C}}^\bullet_k$ with

$$\sigma^{\leqslant m}\check{\mathcal{C}}^\bullet_k = \big(0\to\check{\mathcal{C}}^0_k\to\cdots\to\check{\mathcal{C}}^m_k\to 0\big)\,,\quad \mathscr{H}^\alpha\left(\sigma^{\leqslant m}\check{\mathcal{C}}^\bullet_k\right) = \begin{cases}\operatorname{coker}\check{d}^{m-1} & \text{for } \alpha=m\,,\\ 0 & \text{else}\,,\end{cases}$$

where, for $m=0$, we have $\operatorname{coker}\check{d}^{-1}=\check{\mathcal{C}}^0_k$. For $\max\{i,j\}\leqslant k\leqslant \ell$, we set

$$\mathcal{Q}(i,j,k) := \operatorname{Ind}^{\mathfrak{S}_\ell\times\mathfrak{S}_\ell}_{\mathfrak{S}_k\times\mathfrak{S}_{\ell-k,e}\times\mathfrak{S}_k}\hat{\mathcal{C}}^j_k\big(\mathcal{T}(i,k)\big)\,,\quad \mathcal{T}(i,k):=\mathscr{H}^{k-i}\big(\sigma^{\leqslant k-i}\check{\mathcal{C}}^\bullet_k\big(\mathcal{O}_{[k],[k],e}\otimes p^*\omega_X^{-(n+i-1)}\big)\big).$$

Lemma 5.12. *For $0\leqslant j\leqslant \ell$ and $1\leqslant i\leqslant\ell$, we have*

$$E(j)_2^{-i,2(n+i-1)}\cong\oplus_{k=\max\{i,j\}}^{\ell}\mathcal{Q}(i,j,k)\,.$$

For $j\geqslant 1$, these are the only non-vanishing terms on the 2-level of $E(j)$. For $j=0$, there are the additional non-vanishing terms $E(0)_2^{0,2r}=\operatorname{Ind}^{\mathfrak{S}_\ell\times\mathfrak{S}_\ell}_{\mathfrak{S}_{\ell,e}}\mathcal{O}_{\Delta_{X\times X^\ell}}\otimes p^*\omega_X^{-r}$ for $r=0,\dots,n-1$.

Proof. The terms $E(j)_1^{p,q}=\mathscr{H}^q\left(\mathcal{R}^{-p}\star\mathcal{H}^j\right)$ are described by (5.7) together with (5.14). We see that the only non-vanishing rows on the 1-level of $E(j)$ have $q=2r$, where $r=0,\dots,n+\ell-1$. By Lemma 5.10, for $i\geqslant 1$, row $q=2(n+i-1)$ is the complex

$$\sigma^{\leqslant -i}\Bigg(\bigoplus_{k=\max\{i,j\}}^{\ell}\operatorname{Ind}^{\mathfrak{S}_\ell\times\mathfrak{S}_\ell}_{\mathfrak{S}_k\times\mathfrak{S}_{\ell-k,e}\times\mathfrak{S}_k}\hat{\mathcal{C}}^j_k\big(\check{\mathcal{C}}^\bullet_k\big(\mathcal{O}_{[k],[k],e}\otimes p^*\omega_X^{-(n+i-1)}\big)\big)[k]\Bigg)$$
$$\cong\bigoplus_{k=\max\{i,j\}}^{\ell}\sigma^{\leqslant k-i}\Big(\operatorname{Ind}^{\mathfrak{S}_\ell\times\mathfrak{S}_\ell}_{\mathfrak{S}_k\times\mathfrak{S}_{\ell-k,e}\times\mathfrak{S}_k}\hat{\mathcal{C}}^j_k\big(\check{\mathcal{C}}^\bullet_k\big(\mathcal{O}_{[k],[k],e}\otimes p^*\omega_X^{-(n+i-1)}\big)\big)\Big)[k]\,. \tag{5.19}$$

Since the functor $\operatorname{Ind}^{\mathfrak{S}_\ell\times\mathfrak{S}_\ell}_{\mathfrak{S}_k\times\mathfrak{S}_{\ell-k,e}\times\mathfrak{S}_k}\hat{\mathcal{C}}^j_k(\_)$ is exact, it follows that the cohomology of row $q=2(n+i-1)$ is concentrated in degree $-i$ and equal to $\oplus_{k=\max\{i,j\}}^{\ell}\mathcal{Q}(i,j,k)$. This proves the first assertion. For $r=0,\dots,n-1$, row $q=2r$ of $E(j)_1$ is given by

$$\bigoplus_{k=j}^{\ell}\operatorname{Ind}^{\mathfrak{S}_\ell\times\mathfrak{S}_\ell}_{\mathfrak{S}_k\times\mathfrak{S}_{\ell-k,e}\times\mathfrak{S}_k}\hat{\mathcal{C}}^j_k\big(\check{\mathcal{C}}^\bullet_k\big(\mathcal{O}_{[k],[k],e}\otimes p^*\omega_X^{-r}\big)\big)[k]\,.$$

Thus, it is an exact complex with one exception: in the case $j=0$, the one-term complex $\operatorname{Ind}^{\mathfrak{S}_\ell\times\mathfrak{S}_\ell}_{\mathfrak{S}_{\ell,e}}\hat{\mathcal{C}}^0_0\big(\check{\mathcal{C}}^\bullet_0\big(\mathcal{O}_{\emptyset,\emptyset,e}\otimes p^*\omega_X^{-r}\big)\big)[0]\cong\operatorname{Ind}^{\mathfrak{S}_\ell\times\mathfrak{S}_\ell}_{\mathfrak{S}_{\ell,e}}\mathcal{O}_{\Delta_{X\times X^\ell}}\otimes p^*\omega_X^{-r}[0]$ occurs as a direct summand; compare with Convention 5.3. □

COROLLARY 5.13. *For $j = 1, \dots, \ell$, we have*

$$\mathscr{H}^q\left(\mathcal{R} \star \mathcal{H}^j\right) = \begin{cases} \oplus_{k=\max\{i,j\}}^{\ell} \mathcal{Q}(i,j,k) & \text{for } q = 2(n-1)+i,\ i = 1, \dots, \ell\,, \\ 0 & \text{else}\,. \end{cases}$$

Furthermore,

$$\mathscr{H}^q\left(\mathcal{R} \star \mathcal{H}^0\right) = \begin{cases} \oplus_{k=\max\{i,j\}}^{\ell} \mathcal{Q}(i,0,k) & \text{for } q = 2(n-1)+i,\ i = 1, \dots, \ell\,, \\ \mathsf{Ind}_{\mathfrak{S}_{\ell,e}}^{\mathfrak{S}_\ell \times \mathfrak{S}_\ell} \mathcal{O}_{\Delta_{X \times X^\ell}} \otimes p^*\omega_X^{-r} & \text{for } q = 2r,\ r = 0, \dots, n-1\,, \\ 0 & \text{else}\,. \end{cases}$$

Proof. By the positioning of the non-vanishing terms, we see that all the $E(j)$ degenerate at the 2-level. The result follows since $E(j)^q = \mathscr{H}^q\left(\mathcal{R} \star \mathcal{H}^j\right)$; see Section 5.4. ☐

LEMMA 5.14. *Let $\mathcal{A}$ be an abelian category and, for $\alpha = 1, \dots, m$, let $(C_\alpha^\bullet, d_\alpha)$ be complexes in $\mathcal{A}$. Let $C^\bullet$ be a complex with terms $C^j = C_1^j \oplus \cdots \oplus C_m^j$ and differentials of the form*

$$d^j = \begin{pmatrix} d_1^j & 0 & \cdots & 0 \\ * & d_2^j & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ * & \cdots & * & d_m^j \end{pmatrix},$$

where the stars stand for arbitrary morphisms. Then, if all the $C_\alpha^\bullet$ are exact, $C^\bullet$ is exact too.

Proof. Let $B^\bullet$ be the complex with terms $B^j = C_1^j \oplus \cdots \oplus C_{m-1}^j$ and differentials

$$d_B^j = \begin{pmatrix} d_1^j & 0 & \cdots & 0 \\ * & d_2^j & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ * & \cdots & * & d_{m-1}^j \end{pmatrix}.$$

By induction, we can assume that $B^\bullet$ is an exact complex. There is the short exact sequence of complexes $0 \to C_m^\bullet \to C^\bullet \to B^\bullet$, where the first map is given by the inclusion of the last direct summand and the second map is the projection to the first $m-1$ direct summands. The exactness of $C^\bullet$ follows from the associated long exact cohomology sequence. ☐

For $r \in \mathbb{Z}$, we set $\bar{\mathcal{S}}_X^r := \tilde{S}_X^r\big(\mathsf{Ind}_{\mathfrak{S}_{\ell,e}}^{\mathfrak{S}_\ell \times \mathfrak{S}_\ell} \mathcal{O}_{\Delta_{X \times X^\ell}}\big) = \mathsf{Ind}_{\mathfrak{S}_{\ell,e}}^{\mathfrak{S}_\ell \times \mathfrak{S}_\ell} \mathcal{O}_{\Delta_{X \times X^\ell}} \otimes p^*\omega_X^r[2r]$. We have $\bar{\mathcal{S}}_X^r = \left(\bar{\mathcal{S}}_X^1\right)^{\star r}$ and

$$\mathsf{FM}_{\bar{\mathcal{S}}_X^r} = \bar{S}_X^r = (\_) \otimes \left(\omega_X^r \boxtimes \mathcal{O}_{X^\ell}\right)[2r] \colon \mathrm{D}^b_{\mathfrak{S}_\ell}\left(X \times X^\ell\right) \to \mathrm{D}^b_{\mathfrak{S}_\ell}\left(X \times X^\ell\right).$$

PROPOSITION 5.15. $\mathscr{H}^*(\mathcal{R} \star \mathcal{H}) = \bar{\mathcal{S}}_X^{-[0,n-1]} := \bar{\mathcal{S}}_X^0 \oplus \bar{\mathcal{S}}_X^{-1} \oplus \cdots \oplus \bar{\mathcal{S}}_X^{-(n-1)}$.

Proof. We consider the spectral sequence $E_1^{p,q} = \mathscr{H}^q(\mathcal{R} \star \mathcal{H}^p) \implies \mathscr{H}^{p+q}(\mathcal{R} \star \mathcal{H})$; see (5.8). By Corollary 5.13, the only non-vanishing rows of E_1 are

$$q = 0, 2, \dots, 2(n-1), 2(n-1)+1, \dots, 2(n-1)+\ell\,.$$

Note that the terms of row $q = 2(n-1)+i$, for $i = 1, \dots, \ell$, equal those of the exact complex $\bigoplus_{k=i}^{\ell} \mathsf{Ind}_{\mathfrak{S}_k \times \mathfrak{S}_{\ell-k,e} \times \mathfrak{S}_k}^{\mathfrak{S}_\ell \times \mathfrak{S}_\ell} \hat{\mathcal{C}}_k^\bullet(\mathcal{T}(i,k))$. For $j > k$, we set $\mathcal{Q}(i,j,k) = 0$ and $\hat{d}_k^j = 0$. By Lemma 5.11, the map $d^j\colon E_1^{j,q} = \oplus_{k=i}^{\ell}\mathcal{Q}(i,j,k) \to E_1^{j+1,q} = \oplus_{k=i}^{\ell}\mathcal{Q}(i,j+1,k)$ is given by

$$d^j = \begin{pmatrix} d_i^j & 0 & \cdots & 0 \\ * & d_{i+1}^j & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ 0 & 0 & * & d_\ell^j \end{pmatrix}, \quad d_k^j = \mathsf{Ind}_{\mathfrak{S}_k\times\mathfrak{S}_{\ell-k,e}\times\mathfrak{S}_k}^{\mathfrak{S}_\ell\times\mathfrak{S}_\ell}\hat{d}_k^j(\mathcal{T}(i,k)).$$

It follows by Lemma 5.14 that row $q = 2(n-1)+i$ is exact for all $i = 1,\dots,\ell$.

For $r = 0,\dots,n-1$, row $q = 2r$ has only one non-vanishing term, namely $E_1^{0,2r} = \bar{\mathcal{S}}_X^{-r}$. In summary, the only non-zero terms on the 2-level are $E_2^{0,2r} = \bar{\mathcal{S}}_X^{-r}$ for $r = 0,\dots,n-1$, from which the proposition follows. □

5.11 The surface case: Splitting and monad structure

Lemma 5.16. *Let $\mathcal{A}$ be an abelian category with enough injectives, and consider $A^\bullet, B^\bullet, C^\bullet \in \mathrm{D}^b(\mathcal{A})$ together with morphisms $f\colon A^\bullet \to C^\bullet$ and $g\colon B^\bullet \to C^\bullet$ in $\mathrm{D}^b(\mathcal{A})$. Let there be an $m \in \mathbb{Z}$ such that the cohomology of $A^\bullet$ and $B^\bullet$ is concentrated in degrees smaller than m and such that $\mathscr{H}^i(f)$ as well as $\mathscr{H}^i(g)$ are isomorphisms for all $i < m$. Then $A^\bullet \cong B^\bullet$ in $\mathrm{D}^b(\mathcal{A})$.*

Proof. We may assume $A^i = B^i = 0$ for all $i \geqslant m$; see [Huy06, Example 2.31]. Choose an injective complex $I^\bullet$ which is quasi-isomorphic to $C^\bullet$. Then f and g are represented by morphisms of complexes $f^\bullet\colon A^\bullet \to I^\bullet$ and $g^\bullet\colon B^\bullet \to I^\bullet$; see [Huy06, Lemma 2.39]. These morphisms factor through the smart truncation $\tau^{\leqslant m-1}I^\bullet$ and, by the hypothesis, these factorisations are quasi-isomorphisms. Thus, they are isomorphisms in $\mathrm{D}^b(\mathcal{A})$, which proves the assertion. □

Proposition 5.17. $\mathcal{R}\star\mathcal{H} \cong \bar{\mathcal{S}}_X^{-[0,n-1]}$.

Proof. This follows by applying the previous lemma to the situation that $f\colon \mathcal{R}\star\mathcal{H} \to \mathcal{R}\star\mathcal{H}^0$ is the map induced by the canonical map $\mathcal{H}\to\mathcal{H}^0$ and g is the composition

$$\bar{\mathcal{S}}_X^{-[0,n-1]} \to \mathcal{R}^0\star\mathcal{H}^0 \to \mathcal{R}\star\mathcal{H}^0,$$

where the first map is the inclusion of the direct summand $\mathcal{P}(0,0)_0 \cong \bar{\mathcal{S}}_X^{-[0,n-1]}$ under the isomorphism (5.7) and the second map is induced by $\mathcal{R}^0 \to \mathcal{R}$. □

Proof of Theorem 1.1(ii). By Proposition 5.17, the functor $H_{\ell,n}$ fulfils condition (1) of a $\mathbb{P}^{n-1}$-functor.

Set $F_{\ell,n} := \delta_{[n]*}\circ \mathsf{M}_{\mathfrak{a}_n}\circ \mathsf{triv}\colon \mathrm{D}^b_{\mathfrak{S}_\ell}(X\times X^\ell) \to \mathrm{D}^b_{\mathfrak{S}_n\times\mathfrak{S}_\ell}(X^n\times X^\ell)$, so that $H^0_{\ell,n} = \mathsf{Ind}\circ F_{\ell,n}$; compare with (3.5). By (4.4), we have $F^R_{\ell,n}\circ F_{\ell,n} = \bar{S}_X^{-[0,n-1]}$. The unit of adjunction $\eta\colon \mathsf{id} \to \mathsf{Res}\circ\mathsf{Ind}$ gives a map of monads $F^R_{\ell,n}\eta F_{\ell,n}\colon F^R_{\ell,n}\circ F_{\ell,n} \to R^0_{\ell,n}\circ H^0_{\ell,n}$. On the level of the kernels, it coincides with the inclusion $\bar{\mathcal{S}}_X^{-[0,n-1]} \to \mathcal{R}^0\star\mathcal{H}^0$. Since $F_{\ell,n} = F_{0,n}\boxtimes \mathsf{id}_{\mathrm{D}^b_{\mathfrak{S}_\ell}(X^\ell)}$, the monad multiplication $\mu(F_{\ell,n})\colon F^R_{\ell,n}\circ F_{\ell,n}\circ F^R_{\ell,n}\circ F_{\ell,n} \to F^R_{\ell,n}\circ F_{\ell,n}$ equals $\mu(F_{0,n})\boxtimes\mathsf{id}$. By [Kru15], the functor $F_{0,n}$ is a $\mathbb{P}^{n-1}$-functor. In particular, the monad structure of $F_{0,n}$ has the right shape, which means that the components $S_X^{-1}\circ S_X^{-k} \to S_X^{-(k+1)}$ of $\mu(F_{0,n})$ are isomorphisms for $k = 0,\dots,n-2$. Thus, also the components $\bar{S}_X^{-1}\circ\bar{S}_X^{-k} \to \bar{S}_X^{-(k+1)}$ of $\mu(F_{\ell,n})$ are isomorphisms for $k = 0,\dots,n-2$. Equivalently, on the level of the kernels, the components $\bar{\mathcal{S}}_X^{-1}\circ\bar{\mathcal{S}}_X^{-k} \to \bar{\mathcal{S}}_X^{-(k+1)}$ of the monad multiplication

$$\mathcal{R}(\emptyset,[n],e)\star\mathcal{H}(\emptyset,[n],e)\star\mathcal{R}(\emptyset,[n],e)\star\mathcal{H}(\emptyset,[n],e) \to \mathcal{R}(\emptyset,[n],e)\star\mathcal{H}(\emptyset,[n],e),$$

which we denote again by $\mu(F_{\ell,n})$, are isomorphisms. Let $U := \left(X \times X^\ell\right) \setminus (\cup_{\emptyset \neq I \subset [\ell]} D_I)$, and let $u\colon U \to X \times X^\ell$ be the open embedding. Then $F_{\ell,n}^R \eta F_{\ell,n}\colon F_{\ell,n}^R \circ F_{\ell,n} \to R_{\ell,n}^0 \circ H_{\ell,n}^0$ is an isomorphism over $U \times U$, and $H_{\ell,n} \circ u_* \cong H_{\ell,n}^0 \circ u_*$. It follows that the components $\bar{\mathcal{S}}_X^{-1} \circ \bar{\mathcal{S}}_X^{-k} \to \bar{\mathcal{S}}_X^{-(k+1)}$ of

$$\mu(H_{\ell,n})\colon \mathcal{R} \star \mathcal{H} \star \mathcal{R} \star \mathcal{H} \to \mathcal{R} \star \mathcal{H}$$

are isomorphisms over $U \times U$. Since the $\mathcal{S}_X^{-k}[2k]$ are direct sums of line bundles on the graphs of the $\mathfrak{S}_\ell$-action on $X \times X^\ell$ and the codimension of the complement of U in $X \times X^\ell$ is 2, it follows that the components $\bar{\mathcal{S}}_X^{-1} \circ \bar{\mathcal{S}}_X^{-k} \to \bar{\mathcal{S}}_X^{-(k+1)}$ of $\mu(H_{\ell,n})$ are isomorphisms over the whole $X \times X^\ell \times X \times X^\ell$. Together with the fact that, for $i < k$, the components $\bar{\mathcal{S}}_X^{-1} \circ \bar{\mathcal{S}}_X^{-i} \to \bar{\mathcal{S}}_X^{-(k+1)}$ are zero for degree reasons, this amounts to condition (2) of a $\mathbb{P}^{n-1}$-functor.

That the $H_{\ell,n}$ satisfy condition (3) of a $\mathbb{P}^{n-1}$-functor was already shown in Section 3.3. □

6. Similarities to the Nakajima operators

In this section, in order to justify the title of our paper, we will explain some similarities between the $\mathbb{P}^{n-1}$-functors $H_{\ell,n}$ and the Nakajima operators $q_{\ell,n}$.

Let us quickly recall Nakajima's construction [Nak97]. Throughout this section, X will be a smooth quasi-projective surface. The *Nakajima operator* $q_{\ell,n}\colon \mathsf{H}^*(X \times X^{[\ell]}, \mathbb{Q}) \to \mathsf{H}^*\left(X^{[n+\ell]}, \mathbb{Q}\right)$ is the linear map induced by the correspondence

$$X \times X^{[\ell]} \times X^{[n+\ell]} \supset Z^{\ell,n} := \{(x, [\xi], [\xi']) \mid \xi \subset \xi',\ \mu([\xi']) = \mu([\xi]) + n \cdot x\}, \tag{6.1}$$

where $\mu\colon X^{[n]} \to X^{(m)} = X^m/\mathfrak{S}_m$ denotes the Hilbert–Chow morphism and points in the symmetric product are written as formal sums. For every $\alpha \in \mathsf{H}^*(X, \mathbb{Q})$, by the Künneth formula, there is the map

$$i_\alpha\colon \mathsf{H}^*\left(X^{[\ell]}, \mathbb{Q}\right) \to \mathsf{H}^*\left(X \times X^{[\ell]}, \mathbb{Q}\right) \cong \mathsf{H}^*(X, \mathbb{Q}) \otimes \mathsf{H}^*\left(X^{[\ell]}, \mathbb{Q}\right), \quad i_\alpha(\beta) = \alpha \otimes \beta.$$

The operators $q_{\ell,n}(\alpha) := q_{\ell,n} \circ i_\alpha\colon \mathsf{H}^*\left(X^{[\ell]}, \mathbb{Q}\right) \to \mathsf{H}^*\left(X^{[n+\ell]}, \mathbb{Q}\right)$ are again called Nakajima operators. Furthermore, $q_{\ell,-n}(\alpha)\colon \mathsf{H}^*\left(X^{[n+\ell]}, \mathbb{Q}\right) \to \mathsf{H}^*\left(X^{[\ell]}, \mathbb{Q}\right)$ is defined as the adjoint of $q_{\ell,n}(\alpha)$ with respect to the intersection pairing. One usually considers all of these operators for varying values of ℓ together as operators on $\mathbb{H} := \oplus_{\ell \geqslant 0} \mathsf{H}^*\left(X^{[\ell]}, \mathbb{Q}\right)$ by setting

$$q_n(\alpha) := \oplus_\ell\, q_{\ell,n}(\alpha)\colon\ \oplus_\ell\, \mathsf{H}^*\left(X^{[\ell]}, \mathbb{Q}\right) \to \oplus_\ell\, \mathsf{H}^*\left(X^{[n+\ell]}, \mathbb{Q}\right),$$
$$q_{-n}(\alpha) := \oplus_\ell\, q_{\ell,-n}(\alpha)\colon\ \oplus_\ell\, \mathsf{H}^*\left(X^{[n+\ell]}, \mathbb{Q}\right) \to \oplus_\ell\, \mathsf{H}^*\left(X^{[\ell]}, \mathbb{Q}\right).$$

Then, as shown in [Nak97], the commutator relations between these operators are given by

$$[q_n(\alpha), q_{n'}(\beta)] = n \cdot \delta_{n,-n'} \langle \alpha, \beta \rangle \cdot \mathsf{id}_{\mathbb{H}}\ . \tag{6.2}$$

This agrees with the relations between the generators of the Heisenberg algebra associated with $\mathsf{H}^*(X, \mathbb{Q})$, which shows that the Nakajima operators induce an action of the Heisenberg algebra on $\mathbb{H}$, the cohomology of the Hilbert schemes. Taking $n = -n'$ and considering the degree ℓ piece of formula (6.2) for $\ell < n$, we get

$$q_{\ell,-n}(\alpha) \circ q_{\ell,n}(\beta) = n \cdot \langle \alpha, \beta \rangle \cdot \mathsf{id}\colon\ \mathsf{H}^*\left(X^{[\ell]}, \mathbb{Q}\right) \to \mathsf{H}^*\left(X^{[\ell]}, \mathbb{Q}\right). \tag{6.3}$$

6.1 Support of the image under the McKay correspondence

For every $m \in \mathbb{N}$, by [BKR01] and [Hai01], there is the derived McKay correspondence

$$\Phi_m = \mathsf{FM}_{\mathcal{O}_{I^m X}}\colon \mathrm{D}^b\left(X^{[m]}\right) \xrightarrow{\cong} \mathrm{D}^b_{\mathfrak{S}_m}\left(X^m\right).$$

It is the Fourier–Mukai transform along the structure sheaf of the *isospectral Hilbert scheme*

$$I^m X = \left(X^{[m]} \times_{X^{(m)}} X^m\right)_{\mathsf{red}} = \left\{([\xi], x_1, \dots, x_m) \mid \mu([\xi]) = x_1 + \dots + x_m\right\} \subset X^{[m]} \times X^m .$$

We can translate our $\mathbb{P}$-functors $H_{\ell,n}$ from the equivariant side to the Hilbert scheme side of the McKay correspondence by setting

$$\widetilde{H}_{\ell,n} := \Phi_{n+\ell}^{-1} \circ H_{\ell,n} \circ (\mathsf{id} \boxtimes \Phi_\ell)\colon \mathrm{D}^b\left(X \times X^{[\ell]}\right) \to \mathrm{D}^b\left(X^{[n+\ell]}\right).$$

Recall that the Fourier–Mukai kernel of $H_{\ell,n}$ is supported on

$$\mathsf{supp}\,\mathcal{H}_{\ell,n} = \bigcup_{\mathsf{Index}_{\ell,n}(i)} \Gamma_{I,J,\mu} \subset X \times X^\ell \times X^{n+\ell};$$

compare with Section 3.2. Using [Orl03, Proposition 2.1.6] or, more precisely, its equivariant analogue [KS15a, Lemma 2.7], one can deduce quite easily that the kernel of $\widetilde{H}_{\ell,n}$ is supported on $Z^{\ell,n} \subset X \times X^{[\ell]} \times X^{[\ell+n]}$, the correspondence defining the Nakajima operator $q_{\ell,n}$. Clearly, it would be desirable to have a more concrete description of the kernel as an object in $\mathrm{D}^b\big(X \times X^{[\ell]} \times X^{[n+\ell]}\big)$, but for the time being, we are unable to provide one except for the case $\ell = 0$ and $n = 2$; see [Kru15, Section 4] and Section 7.2.

6.2 The functors $H_{\ell,n}$ as a partial Heisenberg categorification

In this subsection, let X be a projective surface with trivial canonical bundle $\omega_X \cong \mathcal{O}_X$, that is, an abelian or a K3 surface. In this case, we will see that the functors $\widetilde{H}_{\ell,n}$, together with their right adjoints $\widetilde{R}_{\ell,n}$, fulfil some categorical versions of the Heisenberg relations (6.2).

Let $X \xleftarrow{q} X \times X^{[\ell]} \xrightarrow{p} X^{[\ell]}$ be the projections. For $E \in \mathrm{D}^b(X)$, we consider the functor

$$I_E := q^*E \otimes p^*(\_)\colon \mathrm{D}^b\left(X^{[\ell]}\right) \to \mathrm{D}^b\left(X \times X^{[\ell]}\right), \quad I_E(F) = E \boxtimes F .$$

Its right adjoint is $I_E^R = p_*(q^*E^\vee \otimes (\_))$. We set

$$\widetilde{H}_{\ell,n}(E) := \widetilde{H}_{\ell,n} \circ I_E\colon \mathrm{D}^b\left(X^{[\ell]}\right) \to \mathrm{D}^b\left(X^{[n+\ell]}\right),$$
$$\widetilde{R}_{\ell,n}(E) := I_E^R \circ \widetilde{R}_{\ell,n}\colon \mathrm{D}^b\left(X^{[\ell+n]}\right) \to \mathrm{D}^b\left(X^{[\ell]}\right).$$

For $E, F \in \mathrm{D}^b(X)$, by Theorem 1.1(ii), we get

$$\begin{aligned}\widetilde{R}_{\ell,n}(E) \circ \widetilde{H}_{\ell,n}(F) &\cong I_E^R \circ \widetilde{R}_{\ell,n} \circ \widetilde{H}_{\ell,n} \circ I_F \cong I_E^R \circ I_F([0] \oplus [-2] \oplus \dots \oplus [-2(n-1)]) \\ &\cong (\_) \otimes_{\mathbb{C}} \mathrm{Ext}^*(E,F)([0] \oplus [-2] \oplus \dots \oplus [-2(n-1)]).\end{aligned}$$

On the level of the Grothendieck group, this gives

$$\widetilde{R}_{\ell,n}(E) \circ \widetilde{H}_{\ell,n}(F) = n \cdot \chi(E,F) \cdot \mathsf{id}\colon \mathrm{K}\left(X^{[\ell]}\right) \to \mathrm{K}\left(X^{[\ell]}\right), \tag{6.4}$$

which fits nicely with (6.3).

Remark 6.1. Note that the assumption $n > \ell$ is necessary for (6.3) to hold, as, for $n \leqslant \ell$, the composition $q_{\ell,n}(\beta) \circ q_{\ell,-n}(\alpha)$ contributes non-trivially to the commutator $[q_n(\alpha), q_n(\beta)]$. Hence, from the standpoint of categorification, we would not necessarily expect the Nakajima operators $q_{\ell,n}$ with $n \leqslant \ell$ to lift to $\mathbb{P}$-functors on the level of derived categories.

Another special case of (6.2) is the relation $q_{1,-n}(\alpha)\circ q_{0,n+1}(\beta)=0$. However, by (4.17), for general $E,F\in \mathrm{D}^b(X)$, the composition $\widetilde{R}_{1,n}(E)\circ \widetilde{H}_{0,n+1}(F)$ does not induce the zero morphism on the level of the Grothendieck groups. Thus, it seems like the collection of the $H_{\ell,n}$ does not give rise to a categorified action of the Heisenberg algebra.

Categorified Heisenberg actions on the Hilbert schemes were constructed in [CL12] and [Kru18]. However, these categorical actions are constructed by lifting generators of the Heisenberg algebra different from the $q_n(\alpha)$. Hence, it would still be of interest to somehow adapt the functors $H_{\ell,n}$ such that they provide a categorification of the Heisenberg action.

7. Further remarks

7.1 Induced autoequivalences on the Hilbert schemes

Let X be a smooth quasi-projective surface. In this section, we study, for $n\geqslant \mathsf{max}\{1,\ell\}$, the $\mathbb{P}$-twists $P_{H_{\ell,n}}\in \mathsf{Aut}\left(\mathrm{D}^b_{\mathfrak{S}_n}\left(X^n\right)\right)\cong \mathsf{Aut}\left(\mathrm{D}^b\left(X^{[n]}\right)\right)$ associated with the $\mathbb{P}$-functors $H_{\ell,n}$. Note that $H_{0,2}$ and $H_{1,2}$ are $\mathbb{P}^1$-functors, which means that they are spherical. Hence, there are also the associated spherical twist $T_{H_{0,2}}$ and $T_{H_{1,2}}$, which satisfy $T^2_{H_{0,2}}\cong P_{H_{0,2}}$ and $T^2_{H_{1,2}}\cong P_{H_{0,2}}$.

All objects in the image of $H_{\ell,n}$ are supported on

$$\nabla^{\ell,n}:=\bigcup_{\substack{I\subset\{1,\dots,\ell+n\}\\ |I|=n}}\Delta_I=\left\{(x_1,\dots,x_{n+\ell})\mid \text{at least } n \text{ of the } x_i \text{ coincide}\right\}\subset X^{n+\ell}\,.$$

This follows from the fact that $\nabla^{\ell,n}$ is the image of $\mathsf{supp}\,\mathcal{H}_{\ell,n}=\bigcup\Gamma_{I,J,\mu}$ under the projection $X\times X^{\ell}\times X^{\ell+n}$ or, alternatively, from Proposition 3.3. Inside $\nabla^{\ell,n}$, there is the dense open subset

$$\nabla^{\ell,n}_0:=\left\{x\in X^{\ell+n}\mid \pi(x)=n\cdot y_0+y_1+\dots+y_\ell \text{ with pairwise distinct } y_i\in X\right\}\subset\nabla^{\ell,n}\subset X^{n+\ell}\,,$$

where $\pi\colon X^{n+\ell}\to X^{(n+\ell)}$ denotes the quotient morphism. For $x\in X^{n+\ell}$, we denote the orbit of x under the $\mathfrak{S}_{n+\ell}$-action on $X^{n+\ell}$ by $\mathsf{orb}(x)\subset X^{\ell+n}$. We set $\bar{\mathbb{C}}(x):=\mathcal{O}_{\mathsf{orb}(x)}\otimes\mathfrak{a}_{n+\ell}\in\mathrm{D}^b_{\mathfrak{S}_m}\left(X^m\right)$.

PROPOSITION 7.1. *Let X be a smooth quasi-projective surface. For $n>\mathsf{max}\{\ell,1\}$, we have*

$$P_{H_{\ell,n}}(\bar{\mathbb{C}}(x))\cong\begin{cases}\bar{\mathbb{C}}(x)[-2(n-1)] & \text{for } x\in\nabla^{\ell,n}_0\,,\\ \bar{\mathbb{C}}(x) & \text{for } x\in X^{\ell+n}\setminus\nabla^{\ell,n}\,,\end{cases}$$

$$T_{H_{0,2}}\big(\bar{\mathbb{C}}(x)\big)\cong\begin{cases}\bar{\mathbb{C}}(x)[-1] & \text{for } x\in\Delta\,,\\ \bar{\mathbb{C}}(x) & \text{for } x\in X^2\setminus\Delta\,,\end{cases}\qquad T_{H_{1,2}}\big(\bar{\mathbb{C}}(x)\big)\cong\begin{cases}\bar{\mathbb{C}}(x)[-1] & \text{for } x\in\nabla^{1,2}\,,\\ \bar{\mathbb{C}}(x) & \text{for } x\in X^{\ell+n}\setminus\nabla^{\ell,n}\,.\end{cases}$$

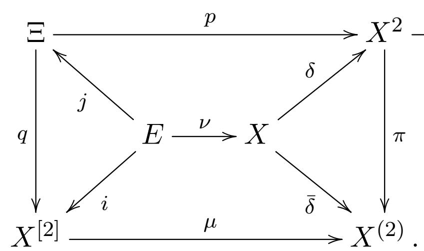
Proof. Every $x\in\nabla^{\ell,n}_0$ has a point of the form $y=(y,\dots,y,y_1,\dots,y_\ell)$ in its $\mathfrak{S}_{n+\ell}$-orbit. Then, by Proposition 3.3, we have $H_{\ell,m-\ell}(\mathbb{C}(y,y_1,\dots,y_\ell))\cong\bar{\mathbb{C}}(y)\cong\bar{\mathbb{C}}(x)$, hence $\bar{\mathbb{C}}(x)\in\mathsf{im}\,H_{\ell,n}$. Furthermore, by (3.7), we have $R_{\ell,m-\ell}\big(\bar{\mathbb{C}}(x)\big)=0$ for $x\in X^{n+\ell}\setminus\nabla^{\ell,n}$. The assertion for the $\mathbb{P}$-twist follows by (2.11), and the assertion for the spherical twists follows by (2.10). □

7.2 Braid relation on $(K3)^{[2]}$

As mentioned in the introduction, the first example of a $\mathbb{P}$-functor was given by Addington [Add16]. For X a K3 surface, it is the Fourier–Mukai transform $F_n=\mathsf{FM}_{\mathcal{I}_{\Xi_n}}\colon\ \mathrm{D}^b(X)\to\mathrm{D}\left(X^{[n]}\right)$ along the ideal sheaf of the universal subscheme $\Xi_n\subset X\times X^{[n]}$. For $n=2$, we can describe the relation to the Nakajima-type $\mathbb{P}$-functor $H_{0,2}$. We set $F:=F_2$ and $H:=\Phi_2^{-1}\circ H_{0,2}$, both of which are $\mathbb{P}^1$-functors, hence spherical, from $\mathrm{D}^b(X)$ to $\mathrm{D}^b(X^{[2]})$.

Proposition 7.2. *We have the isomorphism $L \circ F \cong \mathsf{id}_{\mathrm{D}^b(X)}$, where L is the left adjoint of H.*

Proof. We consider the commutative diagram with cartesian squares



Here, π and q are $\mathfrak{S}_2$-quotients, δ and $\bar\delta$ are the diagonal embeddings, and p and μ are the blow-ups along the respective diagonals. Furthermore, i and j are the embeddings of the exceptional divisor of the blow-ups, and ν is the $\mathbb{P}^1$-bundle morphism of the exceptional divisor to the centre X of the blow-ups. We have an exact triangle of Fourier–Mukai transforms $F \to F' \to F''$ with

$$F' = \mathsf{FM}_{\mathcal{O}_{X\times X^{[2]}}} \cong \mathcal{O}_{X^{[2]}} \otimes \mathsf{H}^*(X, \_)\,, \quad F'' = \mathsf{FM}_{\mathcal{O}_\Xi} \cong q_* \circ p^* \circ \mathsf{pr}_1^*\,.$$

By [Kru15, Proposition 4.2], we have $H \cong i_*\big(\nu^*(\_) \otimes \mathcal{O}_\nu(-1)\big)$. Hence,

$$L \cong \nu_!\big(i^*(\_) \otimes \mathcal{O}_\nu(1)\big) \cong \nu_*\big(i^*(\_) \otimes \mathcal{O}_\nu(-1)\big)[1]\,.$$

As $\mathsf{H}^*(E, \mathcal{O}_\nu(-1)) = 0$, we get $L(\mathcal{O}_{X^{[2]}}) = 0$, hence $L \circ F' \cong 0$. Furthermore, we compute

$$L \circ F'' \cong \nu_*\big(i^* q_* p^* \,\mathsf{pr}_{1*}(\_) \otimes \mathcal{O}_\nu(-1)\big)[1] \cong \nu_*\big(j^* q^* q_* p^* \,\mathsf{pr}_{1*}(\_) \otimes \mathcal{O}_\nu(-1)\big)[1]\,. \tag{7.1}$$

As the morphism ω_q is a two-to-one cover branched over E, we have an exact triangle of Fourier–Mukai transforms $\tau(\_)^* \otimes \mathcal{O}_\Xi(-E) \to q^* q_* \to \mathsf{id}_{\mathrm{D}^b(\Xi)}$, where $\tau \in \mathfrak{S}_2$ is the transposition; see [Add16, Section 2.2(6)]. Combining this with (7.1) gives the exact triangle

$$\nu_*\Big(j^*\big(\tau^* p^* \,\mathsf{pr}_{1*}(\_) \otimes \mathcal{O}_\Xi(-E)\big) \otimes \mathcal{O}_\nu(-1)\Big)[1] \to L \circ F'' \to \nu_*\big(j^* p^* \,\mathsf{pr}_{1*}(\_) \otimes \mathcal{O}_\nu(-1)\big)[1]\,. \tag{7.2}$$

We have $j^* \circ p^* \cong \nu^* \circ \delta^*$ and $\delta^* \circ \mathsf{pr}_1^* \cong \mathsf{id}^* \cong \mathsf{id}$. Hence, the rightmost term of the triangle (7.2) is given by $\nu_*(\nu^*(\_)\mathcal{O}_\nu(-1))[1]$, which vanishes by the projection formula as $\nu_*\mathcal{O}_\nu(-1) = 0$. Similarly, since $j^* \circ \tau^* \cong j^*$ and $j^*\mathcal{O}_\Xi(-E) \cong \mathcal{O}_\nu(1)$, the leftmost term of (7.2) is given by $[1] \circ \nu_* \circ \nu^* \cong [1]$. In summary, we have $L \circ F' \cong 0$ and $L \circ F'' \cong [1]$, and the assertion follows by the exact triangle $L \circ F \to L \circ F' \to L \circ F''$. □

The proposition implies that $F^R \circ H \cong \mathsf{id}$. Hence, by Proposition 2.3, we get the braid relation $T_H \circ T_F \circ T_H \cong T_F \circ T_H \circ T_F$ in $\mathsf{Aut}\big(\mathrm{D}^b\big(X^{[2]}\big)\big)$.

7.3 The case $n = 1$: Induction as a $\mathbb{P}$-functor

In Section 3, we defined the functors $H_{\ell,n}\colon \mathrm{D}^b_{\mathfrak{S}_\ell}\big(X \times X^\ell\big) \to \mathrm{D}^b\big(X^{n+\ell}\big)$ for $n \geqslant 2$. Regarding (3.5), it is a natural extension to the case $n = 1$ to set

$$H^0_{\ell,1} := \mathsf{Ind}_{\mathfrak{S}_\ell}^{\mathfrak{S}_{\ell+1}}\colon \mathrm{D}^b_{\mathfrak{S}_\ell}\big(X \times X^\ell\big) \to \mathrm{D}^b_{\mathfrak{S}_{\ell+1}}\big(X^{\ell+1}\big)\,.$$

While the functors $H_{\ell,n}$ are $\mathbb{P}^{n-1}$-functors for $n \geqslant 2$ (in the surface case), the functor $H^0_{\ell,1}$ is a $\mathbb{P}^\ell$-functor (for $\dim X$ arbitrary) which can be seen as a special case of the following observation.

Let G be a finite group, and let $H \leqslant G$ be a subgroup such that there is an element $g \in G$ of order $n = [G : H]$ such that $1, g, \ldots, g^{n-1}$ forms a system of representatives of the right cosets.

Let G act on a variety M. Recall that, in this case, the induction functor is given by

$$\mathsf{Ind} := \mathsf{Ind}_H^G\colon \mathrm{D}^b_H(M) \to \mathrm{D}^b_G(M)\,, \quad \mathsf{Ind}(A) = \bigoplus_{k=0}^{n-1} g^{k*}A \tag{7.3}$$

with the linearisation of $\mathsf{Ind}(A)$ given by permutation of the summands.

Lemma 7.3. *The induction functor* Ind *is a* $\mathbb{P}^{n-1}$*-functor with* $\mathbb{P}$*-cotwist* g^**.*

Proof. The left and right adjoint of Ind is the restriction functor Res. By (7.3), we indeed have $\mathsf{Res}\circ\mathsf{Ind} = \mathsf{id}\oplus g^*\cdots\oplus g^{(n-1)*}$, which is condition (1) of a $\mathbb{P}^{n-1}$-functor. Condition (3) of a $\mathbb{P}^{n-1}$-functor amounts to the fact that $\mathsf{Res}\cong g^{(n-1)*}\,\mathsf{Res}$. For $(B,\lambda)\in \mathrm{D}^b_G(M)$, the counit map $\varepsilon\colon \mathsf{Ind}\circ\mathsf{Res}(B)=\bigoplus_{k=0}^{n-1} g^{k*}B\to B$ is given by the components $\lambda_{g^k}^{-1}\colon g^{k*}B\to B$; compare with [Ela14, Section 3]. Using this, one can compute that the monad structure has the desired form. □

However, the induced twists are not very interesting.

Lemma 7.4. *For* $n=[G:H]=2$*, we have* $T_{\mathsf{Ind}}\cong \mathsf{M}_{\mathfrak{a}}[1]$*.*

Proof. Let $(B,\lambda)\in \mathrm{D}^b_G(M)$, and consider the G-equivariant morphism $\varphi\colon B\otimes\mathfrak{a}\to \mathsf{Ind}\,\mathsf{Res}(B)$ with components $\mathsf{id}\colon B\to B$ and $-\lambda_g\colon B\to g^*B$. This gives the exact triangle

$$B\otimes\mathfrak{a}\xrightarrow{\varphi}\mathsf{Ind}\,\mathsf{Res}(B)\xrightarrow{\varepsilon} B\,.$$

Since T_F is defined by the exact triangle $\mathsf{Ind}\,\mathsf{Res}(B)\xrightarrow{\varepsilon} B\to T_F$, it follows that $T_F\cong\mathsf{M}_{\mathfrak{a}}[1]$. □

By similar computations, one can show that, for $n=[G:H]$ arbitrary, the associated $\mathbb{P}$-twist is given by $P_{\mathsf{Ind}}\cong[2]$.

7.4 Semi-orthogonal decomposition and induced autoequivalences in the curve case

Let $X=C$ be a smooth curve. By Theorem 1.1(i), there is the semi-orthogonal decomposition

$$\mathrm{D}^b_{\mathfrak{S}_m}(C^m)=\langle\mathcal{A}_{0,m},\mathcal{A}_{1,m-1},\dots,\mathcal{A}_{r,m-r},\mathcal{B}\rangle\,,\ \mathcal{A}_{\ell,m-\ell}=H_{\ell,m-\ell}\big(\mathrm{D}^b_{\mathfrak{S}_\ell}\left(C\times C^\ell\right)\big)\,,\ r=\left\lfloor\frac{m-1}{2}\right\rfloor.$$

We can identify one part of the category $\mathcal{B}$ with the bounded derived category $\mathrm{D}^b\left(C^{(m)}\right)$ of the symmetric product as follows. Since C is a smooth curve, the symmetric product $C^{(m)}$ is smooth. Hence, the pull-back along $\pi\colon C^m\to C^{(m)}$ preserves boundedness of complexes, which means that we have a well-defined functor $\pi^*\circ\mathsf{triv}\colon \mathrm{D}^b\left(C^{(m)}\right)\to\mathrm{D}^b_{\mathfrak{S}_m}(C^m)$. Since $(\pi_*\mathcal{O}_{C^m})^{\mathfrak{S}_m}=\mathcal{O}_{C^{(m)}}$, it follows by the projection formula that $(\_)^{\mathfrak{S}_m}\pi_*\pi^*\,\mathsf{triv}\cong\mathsf{id}$, which means that $\pi^*\,\mathsf{triv}$ is fully faithful. For $I\subset[m]$ with $|I|\geqslant 2$, we have $(\_)^{\mathfrak{S}_I}\circ\mathsf{M}_{\mathfrak{a}_I}\circ\delta_I^*\circ\mathsf{Res}\circ\pi^*\circ\mathsf{triv}=0$. Hence, $L_{\ell,m-\ell}\pi^*\,\mathsf{triv}=0$ for all $\ell\geqslant 2$, which shows that $\pi^*\,\mathsf{triv}(\mathrm{D}^b\left(C^{(m)}\right))\subset\mathcal{B}$. A similar but, for $m\geqslant 4$, finer, semi-orthogonal decomposition of $\mathrm{D}^b_{\mathfrak{S}_m}(C^m)$ is constructed in [PVdB19] by very different methods.

It turns out that, also in the curve case, the functors $H_{\ell,n}$, for $n>\max\{\ell,1\}$ induce autoequivalences of $\mathrm{D}^b_{\mathfrak{S}_{n+\ell}}\left(C^{n+\ell}\right)$, similar to the $\mathbb{P}$-twists from the surface case. To see this, let $\mathfrak{A}_{n+\ell}\subset\mathfrak{S}_{n+\ell}$ be the alternating group. The functor $\mathsf{Res}:=\mathsf{Res}^{\mathfrak{A}_{n+\ell}}_{\mathfrak{S}_{n+\ell}}\colon \mathrm{D}^b_{\mathfrak{S}_{n+\ell}}\left(X^{n+\ell}\right)\to\mathrm{D}^b_{\mathfrak{A}_{n+\ell}}\left(X^{n+\ell}\right)$ is spherical with cotwist $\mathsf{M}_{\mathfrak{a}}$ and twist $\tau^*[1]$, where τ is any element of $\mathfrak{S}_{n+\ell}\setminus\mathfrak{A}_{n+\ell}$. This follows by Lemmas 7.3 and 7.4 together with the fact that a functor is spherical if and only if its right adjoint is spherical with the roles of the twist and cotwists exchanged; see [AL17, Theorem 1.1].

The composition $\mathsf{Res}\circ H_{\ell,n}$ with the fully faithful functor $H_{\ell,n}$ is again spherical. This follows by [HS16, Theorem 4.14], whose assumptions are fulfilled due to Proposition 3.2(ii). We have $\tau^*\circ\mathsf{Res}\circ H_{\ell,n}\cong\mathsf{Res}\circ H_{\ell,n}$. Thus, the spherical twist $\widetilde{T}_{\ell,n}:=T_{\mathsf{Res}\circ H_{\ell,n}}\in\mathsf{Aut}\left(\mathrm{D}^b_{\mathfrak{A}_{n+\ell}}\left(X^{n+\ell}\right)\right)$ is τ-invariant, which means that $\tau^*\circ\widetilde{T}_{\ell,n}\cong\widetilde{T}_{\ell,n}\circ\tau^*$, by (2.12). Hence, $\widetilde{T}_{\ell,n}$ descends to an autoequivalence $T_{\ell,n}\in\mathsf{Aut}\left(\mathrm{D}^b_{\mathfrak{S}_{n+\ell}}\left(X^{n+\ell}\right)\right)$; see [Plo07, Theorem 6] or [KS15a, Theorem 1.1].

One can check that the behaviour of the autoequivalences $T_{\ell,n}$ is similar to that of the $P_{H_{\ell,n}}$ as described in Proposition 7.1. Namely,

$$T_{\ell,n}(\bar{\mathbb{C}}(x))=\begin{cases}\bar{\mathbb{C}}(x)[-(n-2)] & \text{for } x\in\nabla^{\ell,n}_0\,,\\ \bar{\mathbb{C}}(x) & \text{for } x\in C^{\ell+n}\setminus\nabla^{\ell,n}\,.\end{cases}\tag{7.4}$$

7.5 Restriction to generalised Kummer varieties

In this subsection, let $X=A$ be an abelian variety. For $\ell,n\in\mathbb{N}$, we consider the morphisms

$$\Sigma_{n+\ell}\colon A^{n+\ell}\to A\,,\quad (a_1,\dots,a_{n+\ell})\mapsto\sum_{1=1}^{n+\ell}a_i\,,$$

$$\Sigma_{n,\ell}\colon A\times A^{\ell}\to A\,,\quad (b,a_1,\dots,a_\ell)\mapsto n\cdot b+\sum_{i=1}^{\ell}a_i\,.$$

Let $F:=\left(A\times A^\ell\right)\times_A A^{n+\ell}$ be the fibre product with respect to these two summation morphisms. Note that, for every $i\in[0,\ell]$ and every $(I,J,\mu)\in\mathsf{Index}_{\ell,n}(i)$, we have $\Gamma_{I,J,\mu}\subset F$. It follows that $H_{\ell,n}\colon\mathrm{D}^b_{\mathfrak{S}_\ell}\left(A\times A^\ell\right)\to\mathrm{D}^b_{\mathfrak{S}_{n+\ell}}\left(A^{n+\ell}\right)$ is a relative Fourier–Mukai transform over A. This means that its kernel is of the form $\mathcal{H}_{\ell,n}\cong\iota_*\hat{\mathcal{H}}_{\ell,n}$ for an object $\hat{\mathcal{H}}_{\ell,n}\in\mathrm{D}^b_{\mathfrak{S}_\ell\times\mathfrak{S}_{\ell+n}}(F)$, where $\iota\colon F\hookrightarrow A\times A^\ell\times A^{n+\ell}$ is the embedding of the fibre product. Now, let $\left(A\times A^\ell\right)_0:=\Sigma_{n,\ell}^{-1}(0)$ and $A_0^{n+\ell}:=\Sigma^{-1}_{n+\ell}(0)$, and consider the closed embedding $i\colon\left(A\times A^\ell\right)_0\times_A A_0^{n+\ell}\hookrightarrow F$. We set

$$\overline{\mathcal{H}}_{\ell,n}=i^*\hat{\mathcal{H}}_{\ell,n}\,,\quad \overline{H}_{\ell,n}:=\mathsf{FM}_{\overline{\mathcal{H}}_{\ell,n}}\colon\mathrm{D}^b_{\mathfrak{S}_\ell}\left(\left(A\times A^\ell\right)_0\right)\to\mathrm{D}^b_{\mathfrak{S}_{\ell+n}}\left(A_0^{n+\ell}\right).$$

Using the calculus of relative Fourier–Mukai transforms and their restrictions (see, for example, [LST13, Section 1]), it is not hard to deduce from Theorem 1.1 that, for $n>\max\{\ell,1\}$, the functor $\overline{H}_{\ell,n}$ is fully faithful if A is an elliptic curve and a $\mathbb{P}^{n-1}$-functor if A is an abelian surface.

If A is an abelian surface, there is a variant of the Bridgeland–King–Reid–Haiman equivalence as an equivalence $\mathrm{D}^b(K_{m-1}A)\cong\mathrm{D}^b_{\mathfrak{S}_m}(A_0^m)$, where $K_{m-1}A\subset A^{[m]}$ is the generalised Kummer variety; see [Nam02] or [Mea15]. Hence, we also get induced $\mathbb{P}$-functors and autoequivalences on the derived categories of the generalised Kummer varieties.

As shown in [Mea15], the Fourier–Mukai transform $\overline{F}_m\colon\mathrm{D}^b(A)\to\mathrm{D}^b(K_{m-1})$ along the ideal sheaf of the universal family is again a $\mathbb{P}^{m-2}$ functor for $m\geqslant 3$. In analogy to Section 7.2, one can show that the spherical twists $T_{\overline{F}_3},T_{\overline{H}_{1,2}}\in\mathsf{Aut}\left(\mathrm{D}^b(K_2A)\right)$ satisfy the braid relation.

7.6 Some conjectures

The twist autoequivalences induced by the functors $\widetilde{H}_{\ell,n}$ on the derived category $\mathrm{D}^b\left(X^{[m]}\right)$ of the Hilbert scheme of points exist for every smooth quasi-projective surface X. In contrast, all of the other autoequivalences on Hilbert schemes of points on surfaces constructed in the literature [Plo07, Add16, PS14, Mea15, Kru15, CLS14, KS15b] depend crucially on the type of surface (often a K3) and properties of its derived category $\mathrm{D}^b(X)$. Hence, it seems very difficult to make a reasonable general conjecture on the shape of the group $\mathsf{Aut}\left(\mathrm{D}^b\left(X^{[m]}\right)\right)$ for a surface X, let alone prove it.

However, if the canonical bundle ω_X is ample or anti-ample, there are no non-standard autoequivalences coming from the surface [BO01], and we expect the following to hold.

Conjecture 7.5. Let X be a smooth projective surface with ω_X ample or anti-ample. Then for $m = 2, 3$, the group $\mathsf{Aut}\left(\mathrm{D}^b\left(X^{[m]}\right)\right)$ is generated by standard autoequivalences and the twists along the functors $\widetilde{H}_{\ell,n}$:

$$\mathsf{Aut}\left(\mathrm{D}^b\left(X^{[2]}\right)\right) = \left\langle \mathsf{Aut}^{st}\left(\mathrm{D}^b\left(X^{[2]}\right)\right), T_{\widetilde{H}_{0,2}}\right\rangle,$$
$$\mathsf{Aut}\left(\mathrm{D}^b\left(X^{[3]}\right)\right) = \left\langle \mathsf{Aut}^{st}\left(\mathrm{D}^b\left(X^{[3]}\right)\right), P_{\widetilde{H}_{0,3}}, T_{\widetilde{H}_{1,2}}\right\rangle.$$

The *group of standard autoequivalences* $\mathsf{Aut}^{st}\left(\mathrm{D}^b(Y)\right) \cong \mathbb{Z} \times \left(\mathsf{Aut}(Y) \ltimes \mathrm{Pic}(Y)\right)$, for Y a smooth projective variety, is the group generated by degree shifts $[k]$, pull-backs φ^* along automorphisms $\varphi \in \mathsf{Aut}(Y)$, and tensor products $(\_) \otimes L$ by line bundles $L \in \mathrm{Pic}(Y)$.

The reason why we restrict this conjecture to $m = 2, 3$ is the condition $n > \max\{\ell, 1\}$ in Theorem 1.1. It implies that, for fixed m, we have twist autoequivalences $P_{\widetilde{H}_{i,m-i}} \in \mathsf{Aut}\left(\mathrm{D}^b\left(X^{[m]}\right)\right)$ only for $i \leqslant r = \lfloor (m-1)/2 \rfloor$. For $m \geqslant 4$, we expect there to be further autoequivalences

$$P_{r+1,m-r-1}, \dots, P_{m-3,3}, T_{m-2,2} \in \mathsf{Aut}\left(\mathrm{D}^b_{\mathfrak{S}_m}\left(X^m\right)\right) \cong \mathsf{Aut}\left(\mathrm{D}^b\left(X^{[m]}\right)\right),$$

yet to be constructed, with similar properties to the ones of the twists along the $H_{\ell,n}$ as described in Proposition 7.1. Namely, their values on skyscraper sheaves of orbits should be

$$P_{\ell,m-\ell}\big(\bar{\mathbb{C}}(x)\big) \cong \begin{cases} \bar{\mathbb{C}}(x)[-2(m-\ell-1)] & \text{for } x \in \nabla_0^{\ell,m-\ell}, \\ \bar{\mathbb{C}}(x) & \text{for } x \in X^m \setminus \nabla^{\ell,m-\ell}, \end{cases}$$
$$T_{m-2,2}\big(\bar{\mathbb{C}}(x)\big) \cong \begin{cases} \bar{\mathbb{C}}(x)[-1] & \text{for } x \in \nabla_0^{m-2,2}, \\ \bar{\mathbb{C}}(x) & \text{for } x \in X^m \setminus \nabla^{m-2,2}. \end{cases}$$

These missing autoequivalences could play an important role in a description of the behaviour of the tensor product under the derived McKay correspondence; compare with [KPS18, Section 4.6].

The main evidence for the existence of these additional autoequivalences is that there is, in fact, an autoequivalence that one can consider to be the desired $T_{m-2,2}$, namely the composition $\mathsf{M}_{\mathfrak{a}} \circ \Phi_m \circ \mathsf{M}_{\mathcal{O}(D_m/2)} \circ \Phi_m^{-1}$, where $D_m \subset X^{[m]}$ denotes the boundary divisor. Indeed, for $x \in \nabla_0^{m-2,2}$, we have $\Phi_m^{-1}\big(\bar{\mathbb{C}}(x)\big) \cong \mathcal{O}_{\mu^{-1}(\pi(x))}(-1)$ and $\Phi_m^{-1}(\bar{\mathbb{C}}(x) \otimes \mathfrak{a}) \cong \mathcal{O}_{\mu^{-1}(\pi(x))}(-2)[1]$. This is shown in the case $m = 2$ in [Kru15, Proposition 4.2], and the proof for general m is the same. Furthermore, we have $\mathcal{O}(D/2)_{|\mu^{-1}(\pi(x))} \cong \mathcal{O}_{\mu^{-1}(\pi(x))}(-1)$. We get

$$\mathsf{M}_{\mathcal{O}(D_m/2)}\, \Phi_m^{-1}\big(\bar{\mathbb{C}}(x)\big) \cong \mathcal{O}_{\mu^{-1}(\pi(x))}(-2) \cong \Phi_m^{-1}\, \mathsf{M}_{\mathfrak{a}}\left(\bar{\mathbb{C}}(x)\right)[-1],$$

hence $\mathsf{M}_{\mathfrak{a}}\, \Phi_m\, \mathsf{M}_{\mathcal{O}(D_m/2)}\, \Phi_m^{-1}(\bar{\mathbb{C}}(x)) \cong \bar{\mathbb{C}}(x)[-1]$, as desired.

Also for smooth curves, there are autoequivalences of $\mathrm{D}^b_{\mathfrak{S}_m}(C^m)$, constructed in Section 7.4, that act as 'characteristic functors' of the strata $\nabla_0^{\ell,n}$; see (7.4). Thus, one might ask whether such autoequivalences exist in $\mathrm{D}^b_{\mathfrak{S}_m}\left(X^m\right)$ for X a smooth variety of arbitrary dimension. Note, however, that for $\dim X \geqslant 3$, the functors $H_{\ell,n}$ are far from being fully faithful or $\mathbb{P}$-functors; compare with Remark 4.9. Hence, these autoequivalences would have to be constructed by different methods.

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