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A Reider-type theorem for higher syzygies on abelian surfaces

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Abstract

Building on the theory of infinitesimal Newton–Okounkov bodies and previous work of Lazarsfeld–Pareschi–Popa, we present a Reider-type theorem for higher syzygies of ample line bundles on abelian surfaces.

1. Introduction

1.1 Background and motivation

A very effective way to study varieties is through their embeddings into projective space. For a projective variety X, a very ample line bundle L over X gives rise to an embedding

$$\phi_L \colon X \hookrightarrow \mathbb{P}^N = \mathbb{P}\big(H^0(X,L)\big)\,.$$

The homogeneous coordinate ring of the image, defined as $R(X,L) = \oplus_{m\in\mathbb{N}} H^0\big(X, L^{\otimes m}\big)$, is the main algebraic invariant associated with the pair (X,L). Quite naturally, the algebraic properties of $R(X,L)$ reveal a lot of information about the geometric picture it is attached to, and it has long been an object of central significance.

The algebraic behavior of $R(X,L)$ is best studied in the category of graded modules over the ring $S = \mathbb{C}[x_0,\ldots,x_N] = \operatorname{Sym}\big(H^0(X,L)\big)$. As an S-module, $R(X,L)$ admits a minimal graded free resolution $E_\bullet$

$$\cdots \to E_i \to E_{i-1} \to \cdots \to E_2 \to E_1 \to E_0 \to R_l \to 0\,,$$

where each $E_i = \oplus_j S(-a_{i,j})$ is a free graded S-module. The set of numbers $(a_{i,j})$ has an obvious geometric relevance; however, studying it is of paramount interest in algebra, combinatorics, and related areas as well.

Following the footsteps of Castelnuovo and later, Mumford [Mum70], Green and Lazarsfeld [GL86] introduced a sequence of properties asking that the first p terms in $E_\bullet$ be as simple as possible. More specifically, one says that the line bundle L *satisfies property* (N_p) if $E_0 = S$ and

$$a_{i,j} = i+1 \quad \text{for all } j \text{ and all } 1 \leqslant i \leqslant p\,.$$

As an illustration, translated to geometric terms, property (N_0) holds for L if and only if ϕ_L defines a projectively normal embedding, while property (N_1) is equivalent to requiring (N_0)

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and that the homogeneous ideal $\mathscr{I}_{X|\mathbb{P}^N}$ of X in $\mathbb{P}^N$ be generated by quadrics. Historically, property (N_0) on curves was first studied by Castelnuovo. Many years later Mumford completed the picture in the curve case for (N_0) and (N_1).

Due to its classical roots and its relevance for projective geometry, the area surrounding property (N_p) has generated a significant amount of work in the last thirty years or so, with some of the highlights being [Gre84a, Gre84b, BEL91, EL93c, EL15, Par00, Voi02]. Controlling higher syzygies has always been a notoriously difficult question. For more details about this circle of ideas, the reader is kindly referred to [Laz04a, Section 1.8.D] and [Eis05].

In the case of abelian varieties, Pareschi [Par00] proved a conjecture of Lazarsfeld claiming that for an ample line bundle L on an abelian variety X, the line bundle $L^{\otimes(p+3)}$ satisfies property (N_p). It is then natural to study property (N_p) for a given line bundle with certain numerics instead. Such a new line of attack in the case of abelian varieties has been recently initiated by Hwang–To [HT11], who used complex analytic techniques (more precisely, upper bounds on volumes of tubular neighborhoods of subtori of abelian varieties) and were able to control the projective normality of line bundles on abelian varieties in terms of Seshadri constants. Shortly thereafter, Lazarsfeld–Pareschi–Popa [LPP11] used multiplier ideal methods to extend the results of [HT11] to higher syzygies on abelian varieties. The essential contribution of [LPP11] is that property (N_p) can be guaranteed via constructing effective divisors with prescribed multiplier ideals and numerics. Our approach builds in part on the method of proof developed in [LPP11].

1.2 Description of the main result

The goal of this article is to study property (N_p) for line bundles on abelian surfaces over the complex numbers. Based on ideas coming from algebraic and differential geometry aided by the convex geometry of polygons in the plane, we are able to prove the following theorem.

Theorem 1.1. *Let $p \geqslant 0$ be a natural number, X a complex abelian surface, and L an ample line bundle on X with $(L^2) \geqslant 5(p+2)^2$. Then the following are equivalent:*

(i) *The surface X does not contain an elliptic curve C with $(C^2)=0$ and $1 \leqslant (L\cdot C) \leqslant p+2$.*

(ii) *The line bundle L satisfies property (N_p).*

The sequence of properties (N_p) is best considered as increasingly strong algebraic versions of positivity for line bundles along the lines of global generation and very ampleness. From this point of view, Theorem 1.1 is a natural generalization of Reider's celebrated result [Rei88].

In comparison with the proofs of Reider's and Pareschi's theorem, both relying heavily on vector bundle techniques, our approach in confirming property (N_p) relies on multiplier ideals and the associated vanishing theorems together with the theory of infinitesimal Newton–Okounkov polygons developed in [KL18]. The essential novelty of our work is the use of infinitesimal Newton–Okounkov polygons to construct effective $\mathbb{Q}$-divisors whose multiplier ideal coincides with the maximal ideal of the origin.

Theorem 1.2 (Theorem 2.10 and Corollary 2.11). *Let $p \geqslant 0$ be a positive integer, X a smooth projective surface, and L an ample line bundle on X with $(L^2) \geqslant 5(p+2)^2$. Let $x \in X$ be a very general point such that there is no irreducible curve $C \subseteq X$ smooth at x with $1 \leqslant (L\cdot C) \leqslant p+2$. Then there exists an effective $\mathbb{Q}$-divisor*

$$D \equiv \frac{(1-c)}{p+2}L \quad \textit{for some} \quad 0<c\ll 1$$

such that $\mathscr{J}(X;D) = \mathscr{I}_{X,x}$ in a neighborhood of the point x, where $\mathscr{J}(X;D)$ denotes the multiplier ideal of D.

The implication (ii) $\Rightarrow$ (i) of Theorem 1.1, on the other hand, is achieved by the method of restricting syzygies introduced in [GLP83] and developed further by the authors of [EGHP05]. More precisely, the strategy is that property (N_p) for the line bundle L implies the vanishing of certain higher cohomology groups of the ideal sheaf of the scheme-theoretical intersection $X \cap \Lambda$, where Λ is a plane of small dimension inside the projective space $\mathbb{P}\big(H^0(X,L)\big)$.

The main result of [LPP11] in dimension two follows from our Theorem 1.1. Part of the added value of our work lies in treating the cases where $\big(L^2\big)$ is large but $\epsilon(L;o)$ is small; as seen in [BS08] (cf. Example 3.9), such line bundles abound. Along the same lines, we give a criterion for the section ring $R(X,L)$ of an ample line bundle L on an abelian surface to be Koszul (see Corollary 3.7).

In [GP98], Gross and Popescu put forward the following conjecture: the homogeneous ideal of an embedded $(1,d)$-polarized abelian surface is generated by quadrics and cubics whenever $d \geqslant 9$. Our main result implies the conjecture for $d \geqslant 23$; this is shown in [KL17b, Section 5]. As pointed out by Agostini [Ago17], in fact our methods can be used to settle the conjecture for $d \geqslant 10$; in the same paper, Agostini solves the conjecture in its entirety.

Since the preprint [KL17b] appeared, Ito [Ito17]—relying heavily on the ideas of [KL17b] augmented with the theory of log canonical centers—has produced a streamlined version of our main result with slightly improved numerics.

1.3 Sketch of the proof

We outline the ideas behind the implication (1) $\Rightarrow$ (2) in Theorem 1.1 for the case of projective normality, that is, $p = 0$. To this end, let X be an abelian surface over the complex numbers and L an ample line bundle on X with $\big(L^2\big) \geqslant 20$. Suppose in addition that X does not contain an elliptic curve C with $\big(C^2\big) = 0$ and $1 \leqslant (L \cdot C) \leqslant 2$. To ease the presentation, we assume that there exists a Seshadri-exceptional curve $F \subseteq X$ through the origin $o \in X$ such that $r \stackrel{\text{def}}{=} (L \cdot F) \geqslant q = \operatorname{mult}_o(F) \geqslant 2$ and $\epsilon \stackrel{\text{def}}{=} \epsilon(L;o) = r/q$. It is worth pointing out here that it is the case $q = 1$ when elliptic curves of small L-degree occur.

Our starting point is the method of [LPP11], which builds on the following observation: consider the diagonal $\Delta \subseteq X \times X$ with ideal sheaf $\mathscr{I}_\Delta$; as shown in [Ina97], the projective normality of L is equivalent to the vanishing condition

$$H^1\big(X \times X, \operatorname{pr}_1^*(L) \otimes \operatorname{pr}_2^*\big(L^{\otimes h}\big) \otimes \mathscr{I}_\Delta\big) = 0 \quad \text{for all } h \geqslant 1\,. \tag{1.1}$$

The authors of [LPP11] then go on to show that in order to guarantee the vanishing in (1.1), it suffices to verify the existence of an effective $\mathbb{Q}$-divisor

$$D \equiv \frac{1-c}{2} L \quad \text{for some } 0 < c \ll 1$$

satisfying the additional property that $\mathscr{J}(X,D) = \mathscr{I}_{X,o}$. Assuming that one can do so, using the difference morphism $\delta\colon X \times X \to X$ given by $\delta(x,y) = x - y$, one deduces that $\mathscr{I}_\Delta = f^*(\mathscr{J}(X;D)) = \mathscr{J}(X \times X, f^*(D))$, which in turn leads to (1.1) via Nadel vanishing for multiplier ideals.

While directly constructing divisors with a given multiplier ideal is quite difficult in general, a simple observation from homological algebra (see Corollary 3.2) ensures that, at least in the case of projective normality, it is sufficient to exhibit such a divisor D with $\mathscr{J}(X,D) = \mathscr{I}_{X,o}$ locally

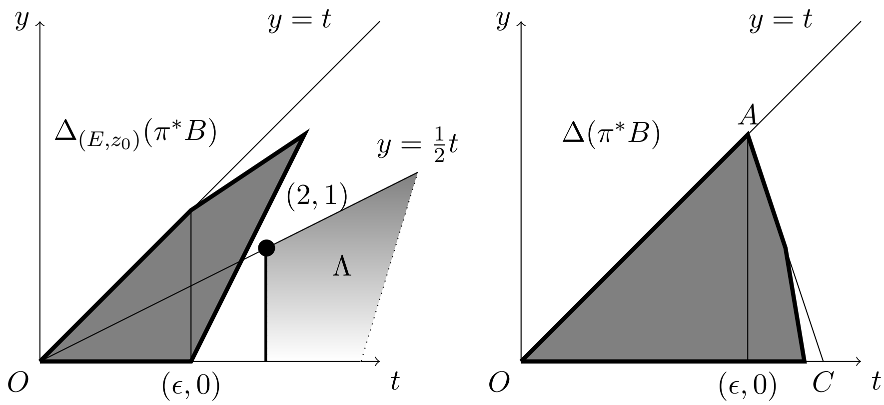

Figure 1. (a) $\Delta_{(E,z_0)}(\pi^*(B))$ and the region Λ (b) The containment $\Delta_{(E,z)}(\pi^*(B)) \subseteq \triangle OAC$

around the origin $o \in X$. This is where the main new ingredient of the paper comes into play: it turns out that one can use infinitesimal Newton–Okounkov polygons to show the existence of suitable $\mathbb{Q}$-divisors D with $\mathscr{J}(X, D) = \mathscr{I}_{X,o}$ over an open subset containing o.

Write $\pi\colon X' \to X$ for the blow-up of X at the origin o with exceptional divisor E, and let $B \stackrel{\text{def}}{=} \frac{1}{2}L$. The first step is to find a criterion in terms of infinitesimal Newton–Okounkov polygons that guarantees the existence of divisors D as above. In Theorem 2.5, we show that if

$$\text{interior}\left(\Delta_{(E,z)}(\pi^*(B)) \cap \underbrace{\left\{(t,y)\,|\, t \geqslant 2, 0 \leqslant y \leqslant \tfrac{1}{2}t\right\}}_{\Lambda}\right)$$

is non-empty for any $z \in E$, then one will always find an effective $\mathbb{Q}$-divisor $D = (1-c)B$ with $\mathscr{J}(X; D) = \mathscr{I}_{X,o}$ in a neighborhood of o.

Aiming at a contradiction, suppose that for some $z_0 \in E$, the Newton–Okounkov polygon $\Delta_{(E,z_0)}(\pi^*(B))$ does not intersect the interior of the region Λ (for an illustration see Figure 1(a)). This implies that the polygon $\Delta_{(E,z_0)}(\pi^*(D))$ sits above a certain line that passes through the point $(2,1)$. But since the area of $\Delta_{(E,z_0)}(\pi^*(B))$ is quite big (equal to $\frac{1}{2}(B^2) \geqslant \frac{5}{2}$), the Seshadri constant $\epsilon(B; o)$ is then forced to be small by convexity, for it is equal to the size of the largest inverted simplex inside $\Delta_{(E,z_0)}(\pi^*(B))$ by [KL18, Theorem 3.11]. A more precise computation gives the upper bound $\epsilon(B; o) \leqslant \frac{1}{2}(5 - \sqrt{5})$.

In order to obtain a contradiction, notice that X carries a transitive group action; thus, the origin $o \in X$ behaves like a very general point. Relying on [EKL95] and [Nak05], in [KL18, Proposition 4.2], the authors show the inclusion

$$\Delta_{(E,z)}(\pi^*(B)) \subseteq \triangle OAC \quad \text{for a generic point } z \in E\,,$$

where $O = (0,0)$, $A = (r/2q, r/2q)$, and $C = r/2(q-1)$ (see Figure 1(b)). The area of the polygon on the left-hand side is $\frac{1}{2}(B^2) \geqslant \frac{5}{2}$; hence, a simple area comparison gives $\epsilon(B; o) = \frac{1}{2}\epsilon(L; o) \geqslant \sqrt{5/2}$, which immediately leads to a contradiction since $\frac{1}{2}(5 - \sqrt{5}) < \sqrt{5/2}$.

1.4 Notation and terminology

In the course of this work, X stands for a smooth projective surface, which from Section 3 onwards will be required to be abelian. The morphism $\pi\colon X' \to X$ without exception denotes the blow-up of a point $x \in X$, which is taken to be the origin whenever X is abelian. A divisor means a $\mathbb{Q}$- or $\mathbb{R}$-Cartier divisor depending on the context. If we insist that a certain divisor D is integral, we explicitly say so.

The notation $\Delta_{(C,x)}(D)$ stands for the Newton–Okounkov body $\Delta_{Y_\bullet}(D)$ with $Y_1 = C$ and $Y_2 = \{x\}$, while $\Delta_x(D)$ denotes the generic infinitesimal Newton–Okounkov body $\Delta_{(E,z)}(\pi^*D)$ (see [KL18, Section 3]).

We work over the complex numbers, although some of our results might be valid over an arbitrary algebraically closed field. We do not strive for optimal hypotheses.

1.5 Organization of the paper

After a quick recap on infinitesimal Newton–Okounkov bodies, Section 2 deals with constructing singular divisors on arbitrary surfaces at arbitrary and, later, very general points. Section 3 is devoted to results specific to abelian surfaces, in particular to the proof of Theorem 1.1 along with a description of the method of [LPP11], and a short discussion of the case of projective normality. It is also here that we present a criterion for the Koszulness of section rings in terms of self-intersection numbers. In Section 4, we treat the converse direction of our main theorem: we prove a result that the existence of low-degree elliptic curves on X leads to property (N_p) not being met.

2. Newton-Okounkov polygons and singular divisors

In this section, we give a recipe for obtaining effective $\mathbb{Q}$-divisors with prescribed numerics and non-trivial multiplier ideal at a given point. After a quick overview of infinitesimal Newton–Okounkov polygons, we will focus on a sufficient condition for the existence of such divisors. We end this section with a similar statement for very general points.

Note that the results of this section are valid without exception for an arbitrary smooth projective surface X; we fix a point $x \in X$ and let $\pi\colon X' \to X$ be the appropriate blow-up with exceptional divisor E.

2.1 Infinitesimal Newton–Okounkov polygons

For the general theory of Newton–Okounkov bodies and basic facts, we kindly refer the reader to the writings [Bou14, KK12, LM09]; as far as the two-dimensional theory is concerned, the reader is invited to look at [KL18, KLM12, Roé16].

Thanks to [LM09, Theorem 6.2], Newton–Okounkov bodies on smooth projective surfaces are straightforward to determine assuming that one has full information on the variation of Zariski decomposition [BKS04, Theorem 1.1] in the Néron–Severi space of X. A more precise analysis using the results of [BKS04] shows that, in fact, $\Delta_{(C,x)}(D)$ is a polygon with rational slopes and, up to possibly two exceptions, rational vertices as well[1] (see [KLM12, Theorem B and Proposition 2.2]).

The theory of Newton–Okounkov polygons has been treated in [KL18]; here, we will work with so-called *infinitesimal Newton–Okounkov polygons*, convex bodies determined by flags coming from exceptional divisors (see [KL17a] for the higher-dimensional theory). With notation as above, an infinitesimal Newton–Okounkov polygon of a divisor D at a point $x \in X$ is a polygon of the form $\Delta_{(E,z)}(\pi^*D)$, where $z \in E$ is an arbitrary point. The basic convex-geometric objects that play a decisive role for local positivity in terms of infinitesimal data are the *inverted standard*

[1] One can in fact do better, assuming that one has control over the flag. It is shown in [AKL14, Proposition 11] that given a line bundle D, it is always possible to arrange for $\Delta_{(C,x)}(D)$ to be a rational polygon.

simplices

$$\Delta_\xi^{-1} \stackrel{\text{def}}{=} \left\{(t,y)\in\mathbb{R}^2 \,|\, 0\leqslant t\leqslant \xi,\, 0\leqslant y\leqslant t\right\}.$$

As it turns out, one has more control over infinitesimal Newton–Okounkov polygons than in general.

Proposition 2.1 ([KL18, Proposition 3.1]). *With notation as above,*

- (i) *one has* $\Delta_{(E,z)}(\pi^*D)\subseteq \Delta_{\mu'}^{-1}$ *for any* $z\in E$*;*
- (ii) *there exist finitely many points* $z_1,\dots,z_k\in E$ *such that the polygon* $\Delta_{(E,z)}(\pi^*D)$ *is independent of* $z\in E\setminus\{z_1,\dots,z_k\}$*, with base the whole line segment* $[0,\mu']\times\{0\}$*,*

where $\mu' \stackrel{\text{def}}{=} \sup\{t>0 \mid \pi^*D - tE \text{ is big}\}$*.*

The second part of the proposition implies that it makes sense to talk about the *generic infinitesimal Newton–Okounkov polygon* of D at the point $x\in X$, which we denote by $\Delta_x(D)$.

Local positivity can be described with the help of infinitesimal flags in a transparent way (cf. [KL17a, Theorem 3.1] and [KL17a, Theorem 4.1]).

Theorem 2.2 ([KL18, Theorem 3.8]). *Let* D *be a big* $\mathbb{R}$*-divisor on a smooth projective surface* X*. Then*

- (i) $x\notin \operatorname{Neg}(D)$ *if and only if* $(0,0)\in\Delta_{(E,z)}(\pi^*D)$ *for any* $z\in E$*;*
- (ii) $x\notin \operatorname{Null}(D)$ *if and only if there exists a* $\xi>0$ *such that* $\Delta_\xi^{-1}\subseteq \Delta_{(E,z)}(\pi^*D)$ *for any* $z\in E$*.*

By Theorem 2.2 and [KL18, Lemma 3.14], it makes sense to define the *largest inverted simplex constant* as follows:

$$\xi(D;x) \stackrel{\text{def}}{=} \sup\left\{\xi>0 \,|\, \Delta_\xi^{-1}\subseteq \Delta_{(E,z)}(\pi^*D)\right\}.$$

In fact, it is proven in [KL18] that the right-hand side does not depend on the choice of the point $z\in E$. Thus the notation makes sense. One of the main statements of [KL18] is that using these definitions, one can quickly recover the moving Seshadri constant of the divisor D at the point x.

Theorem 2.3 ([KL18, Theorem 3.11]). *Let* D *be a big* $\mathbb{R}$*-divisor on* X*. If* $x\notin\operatorname{Neg}(D)$*, then* $\epsilon(||D||;x)=\xi(D;x)$*.*

As we will see later, local positivity at very general points is somewhat less difficult to control. An important observation of Nakamaye's, based on the ideas from [EKL95], is the source of the following convex-geometric estimate, which has manifold applications (see [KL18, Section 4.1], for instance).

Proposition 2.4 ([KL18, Proposition 4.2]). *Let* L *be an ample Cartier divisor on* X *and* $x\in X$ *be a very general point. Then the following mutually exclusive cases can occur:*

- (i) $\mu'(L;x)=\epsilon(L;x)$*: then* $\Delta_x(L)=\Delta^{-1}_{\epsilon(L;x)}$*.*
- (ii) $\mu'(L,x)>\epsilon(L,x)$*: then there exists an irreducible curve* $F\subseteq X$ *with* $(L\cdot F)=p$ *and* $\operatorname{mult}_x(F)=q$ *such that* $\epsilon(L;x)=p/q$*. Under these circumstances,*
  - (a) *if* $q\geqslant 2$*, then* $\Delta_x(L)\subseteq \triangle_{ODR}$*, where* $O=(0,0)$*,* $D=(p/q,p/q)$*, and* $R=(p/(q-1),0)$*.*
  - (b) *if* $q=1$*, then the polygon* $\Delta_x(L)$ *is contained in the area below the line* $y=t$ *and between the horizontal lines* $y=0$ *and* $y=\epsilon(L;x)$*.*

2.2 Singular divisors at arbitrary points

Our purpose here is to find an explicit connection between the Euclidean geometry of infinitesimal Newton–Okounkov polygons and the existence of singular divisors at a given point. Write

$$\Lambda \stackrel{\text{def}}{=} \big\{(t,y)\in\mathbb{R}^2 \,|\, t\geqslant 2\,, y\geqslant 0\,, \text{and } t\geqslant 2y\big\}\,.$$

Our main result is the following.

THEOREM 2.5. *Let B be an ample $\mathbb{Q}$-divisor on X.*

(i) *If*

$$\text{interior of } \big(\Delta_{(E,z)}(\pi^*B)\cap\Lambda\big)\neq\varnothing \quad \textit{for all } z\in E\,, \tag{2.1}$$

then there exists an effective $\mathbb{Q}$-divisor $D\equiv(1-c)B$ for any $0<c\ll 1$ such that $\mathscr{J}(X;D)=\mathscr{I}_x$ in a neighborhood of the point x.

(ii) *Let us assume $\big(B^2\big)\geqslant 5$. If*

$$\text{interior of } \big(\Delta_{(E,z_0)}(\pi^*B)\cap\Lambda\big)=\varnothing \tag{2.2}$$

for some point $z_0\in E$, then the Seshadri constant satisfies $\epsilon(B;x)\leqslant\frac{1}{2}(5-\sqrt{5})$.

Remark 2.6. Note that conditions (2.1) and (2.2) are complementary whenever $\big(B^2\big)\geqslant 5$ is met.

Remark 2.7. In his seminal work [Dem92], Demailly introduced Seshadri constants with the aim of controlling the asymptotic growth of the separation of jets by an ample line bundle at the given point. Our Theorem 2.5(i) can be viewed as a more effective version of Demailly's idea as explained in [Laz04a, Theorem 5.1.17] and [Laz04a, Proposition 5.1.19]. To see this, note that $\epsilon(B;x)$ equals the largest λ such that the inverted simplex Δ_λ^{-1} is contained in $\Delta_{(E,z)}(\pi^*B)$ for any $z\in E$, by Theorem 2.3.

Coupling Theorem 2.5 with Nadel vanishing, we obtain the following effective global generation result, reminiscent of Reider's theorem in the spirit of Demailly's original line of thought[2].

COROLLARY 2.8. *Let X be a smooth projective surface, $x\in X$, and L an ample line bundle on X with $\big(L^2\big)\geqslant 5$. If $\epsilon(L;x)\geqslant\frac{1}{2}(5-\sqrt{5})$, then x is not a base point of the adjoint linear series $|K_X+L|$.*

Proof of Theorem 2.5. (i) Let us fix a point $z\in E$, hence an infinitesimal flag (E,z). We first show that for any $0<c\ll 1$, condition (2.1) implies the existence of a $\mathbb{Q}$-effective divisor $D'\equiv(1-c)\pi^*B$ with $\mathscr{J}(X',D')|_U=\mathscr{O}_U(-2E)$ for some open neighborhood U of the exceptional divisor.

Note that (2.1) is an open condition; hence, it is also satisfied for the divisor class $\pi^*((1-c)B)$ whenever $0<c\ll 1$. In what follows, fix a rational number $c>0$ such that the above property holds, and set $B'=\pi^*((1-c)B)-2E$. Condition (2.1) yields that $\Delta_{(E,z)}(\pi^*B)$ contains an interior point $(t,y)\in\Lambda$ with $2\leqslant t<\mu_E(\pi^*B)$; therefore, π^*B-2E is a big $\mathbb{Q}$-divisor on X'. Also, by [KL18, Remark 1.7], we know that

$$\Delta_{(E,z)}(B')=\big(\Delta_{(E,z)}(\pi^*((1-c)B))\cap\{(t,y)\,|\,t\geqslant 2\}\big)-2\mathbf{e}_1\,,$$

where $\mathbf{e}_1$ stands for the point $(1,0)$.

[2] Corollary 2.8 also follows from Theorem 2.5 and [EL93a, Proposition 1.4], but the proof of the latter is more involved.

Write $B' = P + N$ for the Zariski decomposition of B'; we will look for the divisor D' in the form

$$D' = P' + N + 2E \quad (\equiv \pi^*((1-c)B))$$

with $P' \equiv P$ an effective divisor[3].

Since P is big and nef, [Laz04a, Theorem 2.3.9] shows that one can find an effective divisor N' and a sequence of ample $\mathbb{Q}$-divisors A_k with the property that $P = A_k + (1/k)N'$ for $k \gg 0$. Choose $P' \equiv P$ to be an effective $\mathbb{Q}$-divisor such that $A_k = P' - (1/k)N'$ is general and effective. This yields

$$\begin{aligned}\mathscr{J}(X', D') &= \mathscr{J}\big(X', A_k + (1/k)N' + N + 2E\big) = \mathscr{J}\big(X', (1/k)N' + N + 2E\big)\\ &= \mathscr{J}(X', N) \otimes \mathscr{O}_{X'}(-2E)\end{aligned}$$

for $k \gg 0$. The second equality is an application of the Kollár–Bertini theorem [Laz04b, Example 9.2.29], the third one comes from invariance under small perturbations (see [Laz04b, Example 9.2.30]) and [Laz04b, Proposition 9.2.31]. Thus, it remains to check that $\mathscr{J}(X', N)$ is trivial at any point $z \in E$. Since

$$\Delta_{(E,z)}(B') \cap \{0\} \times [0,1) \neq \varnothing \quad \text{for all } z \in E\,,$$

[LM09, Theorem 6.4] implies

$$1 > \operatorname{ord}_z(N|_E) \quad \text{for all } z \in E\,.$$

On the other hand, $\operatorname{ord}_z(N|E) \geqslant \operatorname{ord}_z(N)$ yields $\operatorname{ord}_z(N) < 1$ for all $z \in E$. In light of [Laz04b, Proposition 9.5.13], this implies that $\mathscr{J}(X', N)$ is trivial at any $z \in E$, as needed.

By Lemma 2.9, there exists an effective $\mathbb{Q}$-divisor $D \equiv (1-c)B$ on X with $D' = \pi^*D$. Hence, by the birational transformation rule for multiplier ideals (see [Laz04b, Theorem 9.2.33]), we have the sequence of equalities

$$\mathscr{J}(X, D) = \pi_*\big(\mathscr{O}_{X'}(K_{X'/X}) \otimes \mathscr{J}(X', D')\big) = \pi_*(\mathscr{J}(X', N)) \otimes \mathscr{I}_x\,,$$

where $K_{X'/X} = E$. Since $\mathscr{J}(X', N)$ is trivial at any $z \in E$, this means that $\mathscr{J}(X, D) = \mathscr{I}_x$ in a neighborhood of the point $x \in X$.

(ii) Set $\epsilon \stackrel{\text{def}}{=} \epsilon(B; x)$, and observe that [KL18, Theorem D] yields the containment $\triangle OAA' \subseteq \Delta_{(E,z_0)}(\pi^*B)$, where $O = (0,0)$, $A = (\epsilon, 0)$, and $A' = (\epsilon, \epsilon)$. If $\epsilon > 2$, then

$$\text{interior}\left(\Delta_{(E,z_0)}(\pi^*B) \cap \Lambda\right) \supseteq \text{interior}\left(\triangle OAA' \cap \Lambda\right) \neq \varnothing\,,$$

contradicting condition (2.2). The case $\epsilon = 2$ is equally impossible since that would imply via (2.2) that $\Delta_{(E,z_0)}(\pi^*B)$ lies to the left of the line $t = 2$; as it lies underneath the diagonal anyway, it would have volume at most 2, contradicting $\left(B^2\right) \geqslant 5$. Therefore, we can safely assume $\epsilon < 2$.

Now, condition (2.2) implies that the segment $\{2\} \times [0,1)$ does not intersect $\Delta_{(E,z_0)}(\pi^*B)$. Since the latter is convex, it must lie above some line ℓ that passes through the point $(2,1)$ and below the diagonal. Let $(\delta, 0)$ be the point of intersection of ℓ and the t-axis. It follows that $\delta \geqslant \epsilon$ since we know that the inverted simplex $\triangle OAA'$ is contained in $\Delta_{(E,z_0)}(\pi^*B)$. So, our goal is now to find an upper bound on δ.

We can assume that both $\epsilon, \delta > 1$ since $\epsilon \leqslant 1$ already implies our statement. In this case, we have $1/(2-\delta) > 1$ for the slope of the line ℓ; therefore, the diagonal and ℓ intersect at the point $(\delta/(\delta-1), \delta/(\delta-1))$ in the first quadrant. The triangle formed by ℓ, the diagonal, and the

[3] Note that the expression for D' is in general not its Zariski decomposition.

t-axis contains our Newton–Okounkov polygon; therefore, its area is no smaller than the area of $\Delta_{(E,z_0)}(\pi^*B)$, which is at least $\frac{5}{2}$. Hence, we obtain $\epsilon(B;x) \leqslant \delta \leqslant \frac{1}{2}(5-\sqrt{5})$. □

Lemma 2.9. *Let Y be a smooth projective variety, $Z \subseteq X$ a smooth subvariety, and $\pi\colon Y' \to Y$ the blow-up of Y along Z. Furthermore, let B be a Cartier divisor on Y and D' a Cartier divisor on Y'. If $D' \equiv \pi^*B$, then there exists a divisor $D \equiv B$ on Y such that $D' = \pi^*D$.*

Proof. This follows from the fact that $F' \equiv 0$ for a Cartier divisor F' on Y' (with integral, $\mathbb{Q}$-, or $\mathbb{R}$-coefficients) implies $F' = \pi^*F$ for a numerically trivial divisor F on Y. Indeed, apply this with $F' \overset{\text{def}}{=} D' - \pi^*B$ to obtain a numerically trivial divisor F on Y with $\pi^*F = D' - \pi^*B$, and set $D \overset{\text{def}}{=} B + F$. □

2.3 Singular divisors at very general points

Ever since the birth of the concept, it has been an important guiding principle that the local positivity of a line bundle is considerably easier to control at a general or a very general point. This observation is manifestly present in the work of Ein–Küchle–Lazarsfeld [EKL95] (see also [EL93b]), where the authors give a lower bound on Seshadri constants at very general points depending only on the dimension of the ambient space.

Later, Nakamaye [Nak05] elaborated some of the ideas of [EKL95], while translating them to the language of multiplicities. This thread was in turn picked up in [KL18] and further developed in the framework of infinitesimal Newton–Okounkov bodies of surfaces, as seen in Proposition 2.4. It is hence not surprising that one can expect stronger-than-usual results on singular divisors at very general points.

Theorem 2.10. *Let p be a positive integer, X a smooth projective surface, and L an ample line bundle on X with $\left(L^2\right) \geqslant 5(p+2)^2$. Let $x \in X$ be a very general point, and assume that there is no irreducible curve $C \subseteq X$ smooth at x with $1 \leqslant (L\cdot C) \leqslant p+2$.*

Then, for some (or, equivalently, every) point $z \in E$,

$$\operatorname{length}\left(\Delta_{(E,z)}(\pi^*B) \cap \{2\} \times \mathbb{R}\right) > 1\,, \tag{2.3}$$

where, as usual, we write $B \overset{\text{def}}{=} \left(1/(p+2)\right)L$.

Corollary 2.11. *Under the assumptions of Theorem 2.10, there always exists an effective $\mathbb{Q}$-divisor $D \equiv (1-c)B$ for some $0 < c \ll 1$ such that $\mathscr{J}(X;D) = \mathscr{I}_{X,x}$ in a neighborhood of the point x.*

Proof. By Proposition 2.4, any infinitesimal Newton–Okounkov polygon sits under the diagonal in $\mathbb{R}^2$. Therefore, (2.3) implies that condition (2.1) from Theorem 2.5 is satisfied, hence the claim. □

Remark 2.12. As abelian surfaces are homogeneous, Theorem 2.10 holds for all points on them.

Before proceeding with the proof, we make some preparations. Let $\epsilon = \epsilon(L;x)$, and write $\epsilon_1 \overset{\text{def}}{=} \epsilon(B;x) = \left(1/(p+2)\right)\epsilon$. As Zariski decompositions of the divisors $\pi^*B - tE$ along the line segment $t \in [\epsilon_1, 2)$ will play a decisive role, we will fix notation for them as well. By [KLM12, Proposition 2.1], there exist only finitely many curves $\overline{\Gamma}_1, \ldots, \overline{\Gamma}_r$ that occur in the negative part of $\pi^*B - tE$ for any $t \in [\epsilon_1, 2)$. Write ϵ_i for the value of t where $\overline{\Gamma}_i$ first appears. We can obviously assume $\epsilon_1 \leqslant \epsilon_2 \leqslant \cdots \leqslant \epsilon_r$; in addition, we put $\epsilon_{r+1} = 2$. Also set $m_i \overset{\text{def}}{=} (\overline{\Gamma}_i \cdot E) = \operatorname{mult}_x(\Gamma_i)$ for

$1 \leqslant i \leqslant r$, where $\Gamma_i = \pi(\overline{\Gamma}_i)$. For any $t \in [\epsilon_1, 2)$, let $\pi^*B - tE = P_t + N_{\pi^*(B)-tE}$ be the Zariski decomposition of the divisor.

Lemma 2.13. *With notation as above, we have* $N_{\pi^*B-tE} = \sum_{i=1}^{r} \operatorname{mult}_{\overline{\Gamma}_i}\big(\|\pi^*B - tE\|\big)\,\overline{\Gamma}_i$ *for all* $t \in [\epsilon_1, 2)$.

Proof. It follows from the definition of asymptotic multiplicity and the existence and uniqueness of the Zariski decomposition that for an arbitrary big and nef $\mathbb{R}$-divisor D and an irreducible curve C, the expression $\operatorname{mult}_C \|D\|$ picks up the coefficient of C in the negative part of D. $\square$

Lemma 2.14. *With notation as above, if $x \in X$ is a very general point, then*

$$\operatorname{length}\big(\Delta_{(E,z)}(\pi^*B) \cap \{t\} \times \mathbb{R}\big) = (P_t \cdot E) \leqslant l_i(t) \stackrel{\text{def}}{=} \Big(1 - \sum_{j=1}^{i} m_j\Big) t + \sum_{j=1}^{i} \epsilon_j m_j$$

for all $t \in [\epsilon_i, \epsilon_{i+1}]$ and all $z \in E$.

Remark 2.15. Lemma 2.14 is essentially a restatement of [Nak05, Lemma 1.3] in the context of Newton–Okounkov bodies. Note that Nakamaye's claim is strongly based on [EKL95, Section 2] (see also [Laz04a, Proposition 5.2.13]), providing a way of "smoothing divisors in affine families" using differential operators. This observation will play an important role in the proof of Theorem 2.10 and consequently in that of Theorem 1.1.

Returning to the verification of Lemma 2.14, let A_i stand for the point of intersection of the lines $t = \epsilon_i$ and l_i (here we identify the function l_i with its graph). Note that the function $\ell\colon [\epsilon_1, 2] \to \mathbb{R}_+$ given by $\ell(t) \stackrel{\text{def}}{=} l_i(t)$ on each interval $[\epsilon_i, \epsilon_{i+1}]$ is continuous and satisfies $(\epsilon_i, \ell(\epsilon_i)) = A_i$. Let T_i be the polygon spanned by the origin, the points $A_1, \dots, A_i$, and the intersection point F_i of the horizontal axis with the line l_i.

Remark 2.16. Since the upper boundary of the polygon $\Delta_x(B)$ is concave, we have $\Delta_x(B) \subseteq T_i$ for all $1 \leqslant i \leqslant r$. An explicit computation using the definition will convince one that the subsequent slopes of the functions l_i are decreasing; therefore, one has $T_i \supseteq T_{i+1}$ for all $1 \leqslant i \leqslant r$ as well.

Proof of Lemma 2.14. It follows from [KL18, Remark 1.9] that

$$\operatorname{length}\big(\Delta_{(E,z)}(\pi^*B) \cap \{t\} \times \mathbb{R}\big) = (P_t \cdot E) \quad \text{for all points} \quad z \in E\,.$$

The point $x \in X$ was chosen to be very general; therefore, one has $\operatorname{mult}_{\overline{\Gamma}_i}(\|\pi^*(B) - t\overline{\Gamma}_i\|) \geqslant t - \epsilon_i$ for all $1 \leqslant i \leqslant r$ and all $t \geqslant \epsilon_i$ by Nakamaye's lemma [KL18, Lemma 4.1] (see also [Nak05, Lemma 1.3]). As a consequence, we obtain via Lemma 2.13 that

$$\begin{aligned}(P_t \cdot E) &= \Big(\Big((\pi^*B - tE) - \sum_{i=1}^{r} \operatorname{mult}_{\overline{\Gamma}_i}\big(\|\pi^*B - tE\|\big)\,\overline{\Gamma}_i\Big) \cdot E\Big) \\ &\leqslant t - \sum_{j=1}^{i} (t - \epsilon_j) m_j = \Big(1 - \sum_{j=1}^{i} m_j\Big) t + \sum_{j=1}^{i} \epsilon_j m_j\end{aligned}$$

for all $\epsilon_i \leqslant t < 2$, as required. $\square$

Proof of Theorem 2.10. We will subdivide the proof into several cases depending on the size and nature of the Seshadri constant $\epsilon \stackrel{\text{def}}{=} \epsilon(L; x)$.

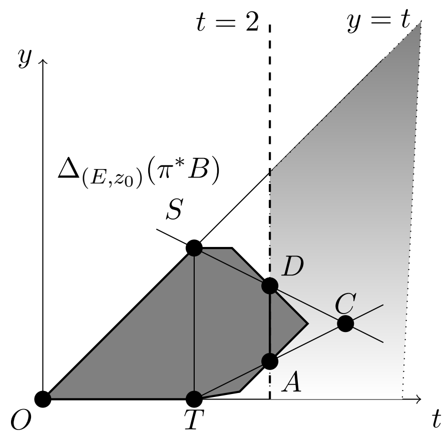

FIGURE 2. $\Delta_{(E,z_0)}(\pi^*B)$ and the triangles $\triangle STC$ and $\triangle ADC$

Case 1: *Assume* $\epsilon(L;x) \geqslant 2(p+2)$. Observe that [KL18, Theorem D] yields

$$\text{length}\left(\Delta_{(E,z)}(\pi^*B) \cap \{2\} \times \mathbb{R}\right) > 1$$

since $\epsilon_1 = \epsilon(B;x) = (1/(p+2))\epsilon \geqslant 2$ and $\Delta_{(E,z)}(\pi^*B) \supseteq \Delta^{-1}_{\epsilon(B;x)} \supseteq \Delta^{-1}_2$.

Case 2: *Assume that we have* $\epsilon(L;x) = (L.F)/\text{mult}_x(F)$ *for some irreducible curve* $F \subseteq X$ *with* $r \stackrel{\text{def}}{=\!=} (L.F) \geqslant q \stackrel{\text{def}}{=\!=} \text{mult}_x(F) \geqslant 2$ (*note that* $r \geqslant q$ *follows from the main result of [EL93b]*). The point $x \in X$ was chosen to be very general; therefore, Proposition 2.4 implies $\Delta_x(B) \subseteq \triangle OMN$, where $O = (0,0)$, $M = (r/(p+2)q, r/(p+2)q)$, and $N = (r/(p+2)(q-1), 0)$. Since the area of the first polygon is $\frac{1}{2}B^2 \geqslant \frac{5}{2}$, we obtain the inequality

$$\frac{1}{(p+2)^2} \cdot \frac{r^2}{q(q-1)} \geqslant 5\,.$$

Remembering that $q \geqslant 2$, we arrive at the following lower bound on the Seshadri constant:

$$\epsilon(B;x) = \frac{r}{(p+2)q} \geqslant \sqrt{5\left(1-\frac{1}{q}\right)} \geqslant \sqrt{\frac{5}{2}}\,. \tag{2.4}$$

Next, fix a point $z_0 \in E$, and let $S = (\epsilon_1, \epsilon_1)$, $T = (\epsilon_1, 0)$, $S' = (2,2)$, and $T' = (2,0)$. The triangle $\triangle OST$ is the largest inverted standard simplex inside $\Delta_{(E,z_0)}(\pi^*B)$. Let

$$[AD] \stackrel{\text{def}}{=\!=} \Delta_{(E,z_0)}(\pi^*B) \cap \{2\} \times \mathbb{R}\,,$$

and, aiming at a contradiction, suppose $||AD|| \leqslant 1$ (for a visual guiding, see Figure 2).

First, note that by (2.4), one has $\epsilon_1 > 1$ and thus $||ST|| > 1$. Together with the assumption $||AD|| \leqslant 1$, this implies that the lines TA and SD intersect to the right of the vertical line $t = 2$; let us call the point of intersection C. Put $x = \text{dist}(C, t=2)$. Since both line segments $[TA]$ and $[SD]$ are contained in $\Delta_{(E,z_0)}(\pi^*B)$, convexity yields the inclusion

$$\triangle ADC \supseteq \Delta_{(E,z_0)}(\pi^*B)_{t \geqslant 2} \stackrel{\text{def}}{=\!=} \Delta_{(E,z_0)}(\pi^*B) \cap \{t \geqslant 2\} \times \mathbb{R}\,.$$

An area comparison yields the string of inequalities

$$\begin{aligned}\mathrm{Area}(\triangle ADC) &\geqslant \mathrm{Area}(\Delta_{(E,z_0)}(\pi^*(B))_{t\geqslant 2} = \mathrm{Area}(\Delta_{(E,z_0)}(\pi^*(B)) - \mathrm{Area}(\Delta_{(E,z_0)}(\pi^*(B))_{t\leqslant 2} \\ &\geqslant \frac{\mathrm{vol}_X(B)}{2} - \mathrm{Area}(\triangle OS'T') \geqslant \frac{5}{2} - \frac{4}{2} = \frac{1}{2}\,.\end{aligned}$$

By the similarity between $\triangle ADC$ and $\triangle TSC$, we see that $||AD|| = \epsilon_1 x/(x+2-\epsilon_1)$. Along with the condition $||AD|| \leqslant 1$, this implies

$$\frac{x+2}{x+1} \geqslant \epsilon_1\,. \tag{2.5}$$

On the other hand, by the above we also have $\mathrm{Area}(\triangle ADC) = \epsilon_1 x^2/2(x+2-\epsilon_1) \geqslant \frac{1}{2}$, which gives $\epsilon_1 \geqslant (x+2)/(x^2+1)$. Combining this inequality with (2.5), we arrive at

$$\frac{x+2}{x+1} \geqslant \frac{x+2}{x^2+1}\,,$$

forcing $x \geqslant 1$. Since the function $f(x) = (x+2)/(x+1)$ is decreasing for $x \geqslant 1$, the inequality (2.5) implies that $\epsilon_1 \leqslant f(1) = 3/2$, contradicting $\epsilon_1 \geqslant \sqrt{5/2}$ from (2.4). Thus. $||AD|| > 1$, as required.

Case 3: *Assume $\epsilon = \epsilon(L;x) \in \mathbb{N}$ with $1 \leqslant \epsilon \leqslant 2p+3$ and that there exists an irreducible curve $\Gamma_1 \subseteq X$ with $\mathrm{mult}_x(\Gamma_1) = 1$ and $\epsilon(L;x) = (L\cdot\Gamma_1) = \epsilon$.* We point out that when $1 \leqslant \epsilon \leqslant p+2$, our assumptions reduce to the condition in the statement of our theorem with Γ_1 playing the role of C. Hence, in what follows, we can assume that in addition $p+3 \leqslant \epsilon \leqslant 2p+3$. Second, since $\mathrm{mult}_x(\Gamma_1) = 1$, $(L\cdot\Gamma_1) = \epsilon \leqslant 2p+3$, and $L^2 \geqslant 5(p+2)^2$, the Hodge index theorem yields $\big(\Gamma_1^2\big) \leqslant 0$.

Since the point $x \in X$ is very general, Proposition 2.4(ii.b) shows that the slope of the linear function $t \mapsto (P_t.E)$ is non-positive. By (2.6), the only way this can happen is when $m_1 = 1$ and $\big(\Gamma_1^2\big) = 0$ (since $\big(\Gamma_1^2\big) \leqslant 0$).

We consider two sub-cases. Denote the proper transform of Γ_1 via π by $\overline{\Gamma}_1$.

Case 3(a): *Assume that $\overline{\Gamma}_1$ is the only curve appearing in the negative part of the divisor $\pi^*B - tE$ for any $t \in [\epsilon_1, 2)$.* First, note that by the above, we have that $\mathrm{mult}_x(\Gamma_1) = 1$ and thus $\overline{\Gamma}_1 = \pi^*\Gamma_1 - E$. Since the curve $\overline{\Gamma}_1$ is the only curve appearing in the negative part of $\pi^*B - tE$ for any $t \in [1,2)$, we can apply the algorithm for finding the negative part of the Zariski decomposition for each divisor $\pi^*B - tE$ and deduce that

$$\pi^*B - tE = P_t + (t-\epsilon_1)\overline{\Gamma}_1$$

is indeed the appropriate Zariski decomposition for any $t \in [\epsilon_1, 2)$. In particular, we obtain that

$$(P_t.E) = \epsilon_1 \quad \text{for all } t \in [\epsilon_1, 2)\,. \tag{2.6}$$

Having positive self-intersection, $\pi^*B - tE$ is big for all $t \in [\epsilon_1, 2]$; therefore, the Zariski decomposition along this line segment is continuous by [BKS04, Proposition 1.16]. Accordingly, (2.6) yields

$$\mathrm{length}\,\big(\Delta_{(E,z)}(\pi^*B) \cap \{2\}\times\mathbb{R}\big) = (P_2.E) = \epsilon_1 = \frac{\epsilon}{p+2} > 1$$

for we assumed $\epsilon \in \{p+3, \ldots, 2p+3\}$.

Case 3(b): *Assume that the negative part of $\pi^*B - tE$ contains another curve (or other curves) beside $\overline{\Gamma}_1$ for some $t \in [\epsilon_1, 2)$.* Denote these other curves by $\overline{\Gamma}_2, \ldots, \overline{\Gamma}_r$ for some $r \geqslant 2$. Our main tool is going to be the generic infinitesimal Newton–Okounkov polygon $\Delta_x(B)$. An important property of $\Delta_x(B)$ is that the vertical line segment $\Delta_x(B) \cap \{t\} \times \mathbb{R}$ starts on the t-axis for any $t \geqslant 0$ (see [KL18, Theorem 3.1]).

Aiming at a contradiction, suppose that length$(\Delta_x(B) \cap \{2\} \times \mathbb{R}) \leqslant 1$. First, note that this is equivalent to having the point $(2,1)$ outside the interior of the polygon $\Delta_x(B)$, as its lower boundary sits on the t-axis.

As $x \in X$ was chosen to be a very general point, by Remark 2.16, we have $\Delta_x(B) \subseteq T_2$, where the latter polygon denotes the convex hull of the vertices O, A_1, A_2, F_2, where $O = (0,0)$ and $A_1 = (\epsilon_1, \epsilon_1)$. We will focus on the slope of the line A_2F_2. If $A_2 = A_1$, then by Lemma 2.14, we know that

$$\text{slope of } A_2F_2 \leqslant \text{slope of } \ell(t) = (1 - m_1 - m_2)t + \epsilon_1 m_1 + \epsilon_2 m_2 \,.$$

In particular, the slope of A_2F_2 is at most -1 since $m_1 = 1$ and $m_2 \geqslant 1$. In the non-degenerate case $A_1 \neq A_2$, since we have $m_1 = 1$, we again have that

$$\text{slope of } A_2F_2 \leqslant \text{slope of } \ell(t) = -m_2 t + \epsilon_1 + \epsilon_2 m_2 \leqslant -1 \,.$$

Also note that in this case, based on the proof of Case 3(a), we are certain that the segment $[A_1A_2]$ is actually an edge of the convex polygon $\Delta_x(B)$.

Since the upper boundary of the polygon $\Delta_x(B)$ is concave by [KLM12, Theorem B], all supporting lines of the edges on this boundary, besides $[OA_1]$ and $[A_1A_2]$, then have slope at most -1. However, we initially assumed $(2,1) \notin \operatorname{interior} \Delta_x(B)$; thus, convexity yields that the first edge of the polygon $\Delta_x(B)$ intersecting the region $[2,\infty) \times \mathbb{R}$ will do so at a point on the line segment $\{2\} \times [0,1]$. Furthermore, this edge and all the other edges of the upper boundary in the region $[2,\infty) \times \mathbb{R}$ will have slope at most -1.

Now, set $A = (2,1)$, $C = (3,0)$, and $D = (2,0)$. Since the line AC has slope -1, by what we said just above, we have the following inclusions due to convexity reasons:

$$\Delta_x(B) \cap [2,\infty) \times \mathbb{R} \subseteq \triangle ADC \,.$$

Write $R = (2,2)$; since $\operatorname{Area}(\Delta_x(B)) \geqslant \frac{5}{2}$, the above inclusion implies

$$\operatorname{Area}(\triangle ADC) = \tfrac{1}{2} > \operatorname{Area}(\Delta_x(B)) - \operatorname{Area}(\triangle ODR) \geqslant \tfrac{5}{2} - 2 = \tfrac{1}{2} \,,$$

contradicting the statement $(2,1) \notin \operatorname{interior} \Delta_x(B)$. In particular, length$(\Delta_x(B) \cap \{2\} \times \mathbb{R}) > 1$, and we are done. □

3. Syzygies of abelian surfaces

3.1 Syzygies and singular divisors on abelian surfaces

The essential contribution of the work [LPP11] can be summarized in the following statement.

Theorem 3.1 ([LPP11]). *Let X be an abelian surface, L an ample line bundle on X, and p a positive integer such that there exists an effective $\mathbb{Q}$-divisor F_0 on X satisfying*

(i) *$F_0 \equiv \big((1-c)/(p+2)\big)L$ for some $0 < c \ll 1$, and*

(ii) *$\mathscr{J}(X; F_0) = \mathscr{I}_0$, the maximal ideal at the origin.*

Then L satisfies the property (N_p).

For the sake of clarity, we give a quick outline of the argument in [LPP11]; to this end, we first recall some terminology. As above, p will denote a natural number; one studies sheaves on the $(p+2)$-fold self-product $X^{\times(p+2)}$. Based on the philosophy of Green [Gre84b, Section 3] and as established later by Inamdar in [Ina97], the N_p property for the line bundle L holds provided that

$$H^i\big(X^{\times(p+2)}, \boxtimes^{p+2}L \otimes N \otimes \mathscr{I}_\Sigma\big) = 0 \quad \text{for all } i > 0 \tag{3.1}$$

and for any nef line bundle N, where $\mathscr{I}_\Sigma$ is the ideal sheaf of the reduced algebraic subset

$$\Sigma \overset{\text{def}}{=\!=} \{(x_0, \dots, x_{p+1}) \mid x_0 = x_i \text{ for some } 1 \leqslant i \leqslant p+1\}\,.$$

This is the vanishing condition one intends to verify.

Observe that $\Sigma \subseteq X^{\times(p+2)}$ can be realized in the following manner. Upon forming the self-product $Y \overset{\text{def}}{=\!=} X^{\times(p+1)}$ with projection maps $\mathrm{pr}_i \colon Y \to X$, one considers the subvariety

$$\Lambda \overset{\text{def}}{=\!=} \bigcup_{i=1}^{p+1} \mathrm{pr}_i^{-1}(0) = \{(x_1, \dots, x_{p+1}) \,|\, x_i = 0 \text{ for some } 1 \leqslant i \leqslant p+1\}\,.$$

Next, look at the morphism

$$\delta \colon X^{\times(p+2)} \to X^{\times(p+1)}\,, \quad (x_0, \dots, x_{p+1}) \mapsto (x_0 - x_1, \dots, x_0 - x_{p+1})\,;$$

then $\Sigma = \delta^{-1}(\Lambda)$ scheme-theoretically. Consider the divisors

$$E_0 \overset{\text{def}}{=\!=} \sum_{i=1}^{p+1} \mathrm{pr}_i^* F_0 \quad \text{and} \quad E \overset{\text{def}}{=\!=} \delta^* E_0\,.$$

As forming multiplier ideals commutes with taking pullbacks by smooth morphisms (cf. [Laz04b, Example 9.5.45]), one observes that

$$\mathscr{J}\big(X^{\times p+2}; E\big) = \mathscr{J}\big(X^{\times p+2}; \delta^* E_0\big) = \delta^* \mathscr{J}\big(X^{\times p+1}; E_0\big) = \delta^* \mathscr{I}_\Lambda = \mathscr{I}_\Sigma\,.$$

The divisor $\big(\boxtimes_{i=1}^{p+2} L\big) - E$ is ample by [LPP11, Proposition 1.3]; therefore,

$$H^i\big(X^{\times(p+2)}, \boxtimes^{p+2}L \otimes N \otimes \mathscr{I}_\Sigma\big) = H^i\big(X^{\times(p+2)}, \boxtimes^{p+2}L \otimes \mathscr{J}(X^{\times p+2}; E)\big) = 0 \quad \text{for all} \quad i > 0$$

by Nadel vanishing, where one sets $N = 0$.

3.2 Projective normality on abelian surfaces

As a toy example, we provide an easy proof of Theorem 1.1 for projective normality via the convex geometric results given in Theorems 2.5 and 2.10(ii). What sets this apart from the general case is that it suffices to find a divisor whose multiplier ideal is the maximal ideal of the origin in a small neighborhood. This is of course much simpler to check than the global statement of [LPP11]. Historically, this was the first case that was dealt with by Hwang and To [HT11]. Observe that our methods provide a precise characterization of projective normality in terms of the existence of elliptic curves with given numerical properties.

Corollary 3.2. *Let X be an abelian surface and L an ample line bundle with $\big(L^2\big) \geqslant 20$. Then the following are equivalent:*

(i) *The surface X does not contain an elliptic curve C with $(L \cdot C) \leqslant 2$ and $\big(C^2\big) = 0$.*

(ii) *The line bundle L defines a projectively normal embedding.*

Remark 3.3. As remarked in the introduction, there seems to be a discrepancy between Corollary 3.2 and Reider's theorem from [Rei88]. In the case of checking the very ampleness condition for an ample line bundle L on an abelian surface X, the latter says that if $L^2 \geqslant 10$, then L is very ample on X if and only if there is no elliptic curve $C \subseteq X$ with $C^2 = 0$ and $1 \leqslant (L.C) \leqslant 2$. So, now consider a general abelian surface X given by a principal polarization L of type $(1, 6)$ and hence with $L^2 = 12$. By Reider's theorem, L is very ample. In order for L to be projectively normal, the map $\mathrm{Sym}^2\big(H^0(X, L)\big) \to H^0\big(X, L^{\otimes 2}\big)$ needs to be surjective. By the Riemann–Roch theorem, the dimension of the first vector space is 21, while that of the image is 24; therefore, L is very ample but not projectively normal.

A similar observation holds for the cases $(1, 5)$ and $(2, 2)$; note that in the second case, the line bundle is not very ample, but it still defines a $2:1$ morphism from X to the Kummer surface. For the cases $(2, d)$ with $d \geqslant 4$, a result of Pareschi and Popa [PP04, Theorem 6.1] yields that L is in fact projectively normal when X is a generic choice of an abelian surface with the given polarization. It turns out that it is much harder to say when a very ample line bundle is not projectively normal, especially that by Reider's theorem and Corollary 3.2, the only difference in these two cases is the lower bound on $\big(L^2\big)$.

Proof of Corollary 3.2. (i) $\Rightarrow$ (ii) The main observation is that in order to prove the projective normality on an abelian surface, it is enough to show $\mathscr{J}(X; D) = \mathscr{I}_o$ in a neighborhood $U \subseteq X$ of the origin for some effective $\mathbb{Q}$-divisor

$$D \equiv (1-c)B \overset{\text{def}}{=\!=} \frac{1}{2}(1-c)L\,.$$

To see this, consider the map $\sigma\colon X \times X \to X$ given by $\sigma(x, y) = x - y$. Then

$$\mathscr{J}(X \times X, \sigma^*(D)) = \sigma^*(\mathscr{J}(X; D)) = I_\Delta \cdot \mathscr{I},$$

where Δ stands for the diagonal and $\mathscr{I} \subseteq \mathscr{O}_{X\times X}$ is an ideal sheaf whose cosupport V is contained in $\sigma^{-1}(X \setminus U)$, hence disjoint from Δ. Now, one has the short exact sequences

$$0 \to I_\Delta \cdot \mathscr{I} \to \mathscr{O}_{X\times X} \to \mathscr{O}_{X\times X}/(I_\Delta \cdot \mathscr{I}) \to 0\,,$$
$$0 \to I_\Delta \cdot \mathscr{I} \to I_\Delta \to I_\Delta/(I_\Delta \cdot \mathscr{I}) \to 0$$

since $\Delta \cap V = \varnothing$, which also implies by straightforward commutative algebra that $I_\Delta/(I_\Delta \cdot \mathscr{I})$ is a direct summand of $\mathscr{O}_{X\times X}/(I_\Delta \cdot \mathscr{I})$.

Therefore, in order to prove the vanishing of the higher cohomology of shifts of I_Δ, one shifts the first sequence and uses Nadel vanishing for the first two terms to obtain the vanishing of the cohomology of the shifts of the quotient. A second application of Nadel vanishing to the second sequence and the fact that taking cohomology commutes with direct sums gives the required vanishing.

To finish the proof, we need to present a divisor $D \equiv (1-c)B$ with $\mathscr{J}(X; D) = \mathscr{I}_{X,o}$, where the equality takes place in a neighborhood of the origin. Since we work on an abelian surface, we can assume that $x \in X$ is a very general point. As in our previous proofs, we approach the question through an analysis of the Seshadri constant $\epsilon(B; x) = \frac{1}{2}\epsilon(L; x)$.

If $\epsilon(L; x) = r/q$ with $r \geqslant q \geqslant 2$, then case (ii) of Theorem 2.10 and Theorem 2.5 imply the existence of such a divisor. If $\epsilon(L; x) \geqslant 4$, then again condition (2.1) in Theorem 2.5 applies, and we are done. Thus, it remains to tackle the cases when there exists a curve $C \subseteq X$ with $\mathrm{mult}_x(C) = 1$ and $(L \cdot C) = \epsilon(L; x)$ for which $\epsilon(L; x) = 1, 2$, or 3. If $\epsilon(L; x) = 1$ or 2, then the Hodge index theorem yields $\big(C^2\big) \leqslant 0$, and since X is abelian, $\big(C^2\big) = 0$. The adjunction formula then shows that C must be an elliptic curve as abelian varieties do not contain rational curves.

If $\epsilon(L;x) = 3$, then $\epsilon(B;x) \geqslant \frac{3}{2}$. Note that $\frac{3}{2} \geqslant \frac{1}{2}(5-\sqrt{5})$; thus, by Theorem 2.5(ii), we deduce the existence of a divisor with the desired properties.

(ii) $\Rightarrow$ (i) In the opposite direction, suppose that L satisfies property N_0, that is, it is projectively normal, and that there exists an elliptic curve $C \subseteq X$ with $(L \cdot C) = 1$ or 2. The first condition implies that L is a very ample divisor and defines an embedding $X \subseteq \mathbb{P}(H^0(X,L))$. On the other hand, using [Laz04a, Theorem 2.2.15] and the fact that $L^2 \geqslant 20$, we know that $L - C$ is big; then it is automatically nef since X is an abelian surface. Kawamata–Viehweg vanishing then implies that the restriction map $H^0(X,L) \to H^0(C, L|_C)$ is surjective. Since L is very ample, $L|_C$ defines an embedding of C into some projective space as well; however, as $1 \leqslant (L \cdot C) \leqslant 2$, this can never be the case. □

3.3 Property (N_p) in the absence of elliptic curves of low degree

In this subsection, we give a proof of the direct implication of Theorem 1.1. As opposed to the case of projective normality, it is no longer clear whether finding an effective divisor D with $\mathscr{J}(X;D) = \mathscr{I}_o$ locally suffices to verify (N_p). We show, however, that with a bit more work, one can in fact control the multiplier ideal of the divisor found in Theorem 2.5 over the whole abelian surface X. As always, $\pi\colon X' \to X$ denotes the blow-up of o with exceptional divisor E.

The main goal of this subsection is to prove the following theorem.

Theorem 3.4. *Let X be an abelian surface and B an ample $\mathbb{Q}$-divisor on X with $\left(B^2\right) > 4$. Suppose that*

$$\operatorname{length}\left(\Delta_{(E,z_0)}(\pi^*(B)) \cap \{2\} \times \mathbb{R}\right) > 1 \tag{3.2}$$

for some point $z_0 \in E$. Then there exists an effective $\mathbb{Q}$-divisor $D \equiv (1-c)B$ for some $0 < c < 1$ such that $\mathscr{J}(X,D) = \mathscr{I}_{X,o}$ over the whole of X.

Proof of Theorem 1.1. (i) $\Rightarrow$ (ii) By Theorem 3.1, it suffices to find a divisor as produced by Theorem 3.4. Since X is abelian, it is enough to treat the case when the origin o behaves like a very general point. By Theorem 2.10, the condition (3.2) is automatically satisfied whenever X does not contain an elliptic curve C with $\left(C^2\right) = 0$ and $1 \leqslant (L \cdot C) \leqslant p+2$.

It remains to show that the exceptions in Theorem 2.10 correspond to the exceptions in the statement in Theorem 1.1. For a curve $C \subseteq X$ which is smooth at the point x and satisfies $1 \leqslant (L \cdot C) \leqslant p+2$, one has $\left(C^2\right) \leqslant 0$ by the Hodge index theorem since $\left(L^2\right) \geqslant 5(p+2)^2$. Since we are on an abelian surface, we then automatically have $\left(C^2\right) = 0$ and, by adjunction, this indeed forces C to be an elliptic curve. □

Proof of Theorem 3.4. To start, [KL18, Proposition 3.1] gives the inclusion

$$\Delta_{(E,z)}(\pi^*(B)) \cap \{2\} \times \mathbb{R} \subseteq \{2\} \times [0,2] \quad \text{for any } z \in E\,.$$

Hence, according to [KL18, Remark 1.9], condition (3.2) implies (2.1) in Theorem 2.5 for any $z \in E$ as $\left(B^2\right) > 4$ by assumption.

By Theorem 2.5, we know how to find a divisor $D \equiv B$ such that $\mathscr{J}(X,D) = \mathscr{I}_{X,0}$ locally around a point. It remains to show that this equality in fact holds over the whole of X. Recall that D is the image of a divisor D' on X', where (revisiting the proof of Theorem 2.5)

$$D' \equiv P + \sum_{i=1}^{i=r} a_i E_i' + 2E\,,$$

and $\pi^*(B) - 2E = P + \sum a_i E_i'$ is the appropriate Zariski decomposition.

Writing E_i for the image of E'_i under π, the first step in the proof is to show the following claim.

Claim. Let $y \neq o$ be an point of X such that $\mathcal{J}(X,D)_y \neq \mathcal{O}_{X,y}$. Then

$$\sum_{i=1}^{i=r} a_i \operatorname{mult}_o(E_i) < \sum_{i=1}^{i=r} a_i \operatorname{mult}_y(E_i)\,. \tag{3.3}$$

Proof. First, observe that by [Laz04b, Proposition 9.5.13], the condition $\mathcal{J}(X,D)_y \neq \mathcal{O}_{X,y}$ implies $\operatorname{mult}_y(D) \geqslant 1$. Since the morphism π is an isomorphism around the point y, considering y as a point on X', we actually have $\operatorname{mult}_y(D') \geqslant 1$. In the proof of Theorem 2.5, we were able to write $P = A_k + (1/k)N$, where A_k is ample and N' is an effective $\mathbb{Q}$-divisor for any $k \gg 0$. Thus, we chose $D' = P' + \sum a_i E'_i + 2E$, where $P' = A + (1/k)N'$, $A \equiv A_k$ is a generic choice, and $k \gg 0$. Since A_k is ample, a generic choice of A does not pass through the point y; in particular, $\operatorname{mult}_y(P') \to 0$ as $k \to \infty$. Since $\operatorname{mult}_y(D') \geqslant 1$, this implies that

$$\operatorname{mult}_y\left(\sum_i a_i E_i\right) = \sum_i a_i \operatorname{mult}_y(E_i) \geqslant 1\,.$$

As a consequence, it suffices to check that $\sum_i a_i \operatorname{mult}_o(E_i) < 1$. To this end, recall that $E'_i = \pi^* E_i - \operatorname{mult}_o(E_i)E$, and therefore

$$2 = ((\pi^*B - 2E)\cdot E) = (P\cdot E) + \sum_i a_i(E'_i \cdot E) = (P\cdot E) + \sum_i a_i \operatorname{mult}_o(E_i)\,.$$

Because $(P\cdot E)$ is equal to the length of the vertical segment $\Delta_{(E,z)}(\pi^*B) \cap \{2\} \times \mathbb{R}$ for any $z \in E$, inequality (3.2) implies that $\sum_i a_i \operatorname{mult}_o(E_i) < 1$, as we wanted. □

We return to the proof of the main statement. We will argue by contradiction and suppose that there exists a point $y \in Y$ for which $\mathcal{J}(X;D)_x \neq \mathcal{O}_{X,y}$. In other words, by the claim above, we have the inequality

$$\sum_i a_i \operatorname{mult}_o(E_i) < \sum_i a_i \operatorname{mult}_y(E_i)\,.$$

This yields the existence of a curve E_i with $\operatorname{mult}_o(E_i) < \operatorname{mult}_y(E_i)$.

By our assumptions E'_i is a negative curve on X'; let E^y_i denote the proper transform of E_i with respect to the blow-up $\pi_y\colon X_y \to X$. Since $\operatorname{mult}_o(E_i) < \operatorname{mult}_y(E_i)$, we obtain

$$\left(E^y_i\right)^2 = E_i^2 - (\operatorname{mult}_y(E_y))^2 < E_i^2 - (\operatorname{mult}_o(E_i))^2 = E_i'^2 < 0\,;$$

in particular, we deduce that E^y_i remains a negative curve on X_y, just as E'_i on X'. However, Lemma 3.5 shows that E_i must then be a smooth elliptic curve passing through o and y. Therefore, $\operatorname{mult}_o E_i = \operatorname{mult}_y E_i = 1$, which contradicts the inequality above. □

Lemma 3.5. *Let X be an abelian surface and $C \subsetneq X$ a curve passing through two distinct points $x_1, x_2 \in X$. If the proper transforms of C for the respective blow-ups of X at the x_i both become negative curves, then C must be the smooth elliptic curve that is invariant under the translation maps $T_{x_1-x_2}$ and $T_{x_2-x_1}$.*

Proof. Denote by C_1 and C_2 the proper transforms of C with respect to the blow-up of X at x_1 and x_2, respectively. Aiming at a contradiction, suppose $T_{x_2-x_1}(C) \neq C$. Since both proper transforms C_1 and C_2 are negative curves, one has

$$0 > (C_1^2) = (C^2) - (\operatorname{mult}_{x_1}(C))^2 \quad \text{and} \quad 0 > (C_2^2) = (C^2) - (\operatorname{mult}_{x_2}(C))^2$$

or, equivalently, $\left(C^2\right) < \min\{\mathrm{mult}_{x_1}(C)^2, \mathrm{mult}_{x_2}(C)^2\}$. On the other hand, observe that

$$\left(C^2\right) = (C \cdot T_{x_2-x_1}(C)) \geqslant \mathrm{mult}_{x_2}(C) \cdot \mathrm{mult}_{x_2}(T_{x_2-x_1}(C)) = \mathrm{mult}_{x_2}(C) \cdot \mathrm{mult}_{x_1}(C)\,,$$

for C is algebraically equivalent to its translate $T_{x_2-x_1}(C)$. This gives a contradiction, so we conclude that C is invariant under both of the translation maps $T_{x_1-x_2}$ and $T_{x_2-x_1}$ and is indeed a smooth elliptic curve. □

3.4 On the Koszul property of section rings

Still making use of the observations of Lazarsfeld–Pareschi–Popa, we produce a strong numerical criterion for the section ring $R(X;L)$ of an ample line bundle on an abelian surface to be Koszul. Summarizing the proof of [LPP11, Proposition 3.1], one obtains the following.

Proposition 3.6 ([LPP11, Proposition 3.1]). *With notation as above, assume that X is an abelian surface and L an ample line bundle on X. If there exist a rational number $0 < c < 1$ and an effective $\mathbb{Q}$-divisor F_0 such that*

(i) $F_0 \equiv \frac{1}{3}(1-c)L$,

(ii) $\mathscr{J}(X;F_0) = \mathscr{I}_o$,

then $R(X,L)$ is Koszul.

Corollary 3.7. *Let X be an abelian surface and L an ample line bundle on X with $\left(L^2\right) \geqslant 45$. If X does not contain an elliptic curve C with $\left(C^2\right) = 0$ and $1 \leqslant (L \cdot C) \leqslant 3$, then $R(X,L)$ is Koszul.*

Proof. Our reasoning is essentially the same as in the proof of Theorem 1.1: take $p = 1$ in Corollary 2.11, and apply Theorem 3.4 to obtain the required divisor F_0. We conclude the proof with Proposition 3.6. □

Remark 3.8. It is a classical result of Kempf [Kem89] that $4L$ is Koszul for any ample line bundle L on an abelian variety. As pointed out following Corollary B of [LPP11], the main interest for our result lies in the case when L is not a (large) multiple of an ample line bundle, where for instance Kempf's theorem is not available. Furthermore, if $L^2 \geqslant 3$, then Corollary 3.7 implies that $R(X,4L)$ is Koszul, thus leading to the recovery of Kempf's result.

Regarding the bound of [LPP11], our methods seem to imply a slightly weaker version. In reality, looking carefully at the proof of Theorem 2.10, we see that the results of [LPP11] are covered by Case 1. The self-intersection L^2 was chosen to be large in order to force that only elliptic curves of small degree encode the syzygies of L and the Koszul property of $R(X,L)$. It is only due to the large lower bound on L^2 that we do not seem to recover the bounds in [LPP11] straight from Corollary 3.7 or Theorem 1.1.

More importantly, note that our result confirms the Koszulness and the (N_p) property of many ample line bundles with a small Seshadri constant and large self-intersection; the criterion we give is easily checked in concrete examples.

Example 3.9. Here, we present a class of ample line bundles on a self-product of an elliptic curve where our criterion verifies the Koszulness of the section ring, but [LPP11] does not. For this, we will rely on computations from [BS08].

In order to be in the situation of [BS08, Theorem 1], let C be an elliptic curve without complex multiplication, and let $L = a_1F_1 + a_2F_2 + a_3\Delta$ be an ample line bundle on $C \times C$, where the F_i are fibers of the two natural projections and $\Delta \subseteq C \times C$ stands for the class the diagonal. The self-intersection is then computed by

$$\left(L^2\right) = 2a_1a_2 + 2a_1a_3 + 2a_2a_3\,.$$

We will take $a_1, a_2, a_3 > 0$; hence, [BS08, Example 2.1] applies. Set $a_2 = 3$ and $a_3 = 2$; our plan is to take $a_1 \gg a_2, a_3$. In any case, if $a_1 \geqslant 4$, then

$$\epsilon(L; o) = \min\{a_1 + a_2, a_1 + a_3, a_2 + a_3\} = 5\,,$$

and there is no elliptic curve of L-degree less than 5 on $C \times C$. Our choice of a_2 and a_3 forces the line bundle L to be primitive; that is, it is not a multiple of any other line bundle on $C \times C$. Thus, neither [Kem89] nor [LPP11] applies. Moreover, if $a_1 \geqslant 4$, then $\left(L^2\right) \geqslant 52 \geqslant 45$, and therefore $R(C \times C, L)$ is Koszul according to Corollary 3.7.

4. Low-degree elliptic curves and syzygies

In this section, we give the proof of the implication (ii) $\Rightarrow$ (i) of Theorem 1.1.

THEOREM 4.1. *Let X be an abelian surface and L a very ample line bundle on X. Suppose $\left(L^2\right) \geqslant 4p + 5$ and that there exists an elliptic curve $C \subseteq X$ with $1 \leqslant (L \cdot C) \leqslant p + 2$. Then L does not satisfy property (N_p).*

Proof of (ii) $\Rightarrow$ (i) *of Theorem* 1.1. Assume that X contains an elliptic curve C with $C^2 = 0$ and $1 \leqslant (L.C) \leqslant p+2$. Then either L is not very ample, in which case it cannot satisfy property (N_p) for $p \geqslant 0$, or it is, but then Theorem 4.1 applies. □

Remark 4.2. Property (N_p) is predominantly studied via vector bundle techniques. This turns out to be a feasible approach in our case as well, by making use of syzygy bundles (cf. [Laz86]) and Lazarsfeld–Mukai bundles (introduced simultaneously in [Laz86] and [Muk89]). An extensive cohomology computation exploiting these vector bundles on X and C leads to a proof that L does not satisfy property (N_p) under the given conditions, primarily since the restricted line bundle $L|_C$ does not satisfy the same property on the elliptic curve C either.

Along the way, one also needs to rely on some ideas from [Par00] via the observation that $L|_C$ is actually numerically equivalent to the $(p + 2)$nd power of a line bundle on the curve C. For a recent overview on the techniques involving syzygy bundles and Lazarsfeld–Mukai bundles and their manifold applications, the reader is kindly referred to Aprodu's surveys [Apr13, Apr14].

In order to give a much shorter proof of Theorem 4.1, we will follow a path suggested by Rob Lazarsfeld via restricting syzygies and invoke ideas of Eisenbud–Green–Hulek–Popescu from [EGHP05]. The kernel of the proof is the same as in Remark 4.2, namely that low-degree elliptic curves have bad syzygies. Assuming by contradiction that there are good syzygies on X, one can take a plane and restrict the resolution of the ideal sheaf of X to the elliptic curve C. This way, one obtains a complex that is exact away from a set of dimension one; diagram chasing will lead to the desired contradiction. This strategy was first introduced in [GLP83] and further developed in [EGHP05].

We start by recalling the following lemma.

LEMMA 4.3 ([GLP83, Lemma 1.6]). *Let*

$$E_\bullet : \ \cdots \to E_{r-1} \to \cdots \to E_1 \to E_0 \xrightarrow{\epsilon} F \to 0$$

be a complex of coherent sheaves on a projective variety V of dimension r, with ϵ surjective. Assume that

(i) *$E_\bullet$ is exact away from a set of dimension 1;*

(ii) *for a given integer $1 \leqslant m \leqslant r$, one has $H^i(V, E_0) = \cdots = H^i(V, E_{r-m}) = 0$ for $i > 0$.*

Then $H^i(V, F) = 0$ for $i \geqslant m$.

Proof of Theorem 4.1. The case of projective normality was already discussed in Corollary 3.2, so we can suppose $p \geqslant 1$. Note that in Corollary 3.2, we assumed $\big(L^2\big) \geqslant 20$, but in fact the argument for the direction (ii) $\Rightarrow$ (i) works whenever $\big(L^2\big) \geqslant 5$. This way, we will get a contradiction by assuming that L satisfies property (N_p) on X.

First, observe that L is, in particular, projectively normal, hence very ample, and so L gives rise to an embedding $X \subseteq \mathbb{P}^N = \mathbb{P}\big(H^0(X, L)\big)$. As this embedding is defined by the complete linear series $|L|$ so, in particular, $H^0\big(\mathbb{P}^N, \mathscr{I}_{X|\mathbb{P}^N}(1)\big) = 0$, the image is non-degenerate.

Second, we point out that the natural restriction map

$$H^0(X, L) \xrightarrow{\text{restr.}} H^0(C, L|_C) \tag{4.1}$$

is surjective. For this, $H^1(X, L - C) = 0$ would suffice, which follows from Kawamata–Viehweg vanishing once we show that $L - C$ is big and nef. Since $\big(L^2\big) > 2(L \cdot C)$, by the condition in Theorem 4.1, Theorem 2.2.15 of [Laz04a] implies that $L - C$ is big. Nefness then follows since X is an abelian surface, and therefore any effective divisor is nef.

Next, being the restriction of a very ample line bundle, $L|_C$ defines an embedding $C \subseteq \mathbb{P}^{p+1} = \mathbb{P}\big(H^0(C, L|_C)\big)$. Because the restriction map in (4.1) is surjective, $\mathbb{P}^{p+1}$ can be seen as a $(p+1)$-dimensional plane $\Lambda \subseteq \mathbb{P}^N$. Here, we can assume $(L \cdot C) = p + 2$ by the induction hypothesis. The embedding $X \subseteq \mathbb{P}^N$ was non-degenerate; therefore, the scheme-theoretic intersection $X \cap \Lambda$ is a subset of dimension one (recall that $p + 1 < N$). Furthermore, by analyzing the exact sequence

$$0 \longrightarrow H^0\left(X, L - C\right) \longrightarrow H^0\left(X, L\right) \longrightarrow H^0\left(C, L - C|_C\right) \longrightarrow 0\,,$$

we see that $\mathfrak{b}(|L - C|) = (\mathscr{I}_{X\cap\Lambda/X} \colon \mathscr{I}_{C/X}) \subseteq \mathscr{O}_X$; in particular, $X \cap \Lambda = C$ if and only if the complete linear series $|L - C|$ is base-point free.

We have already dealt with the case $p = 0$; hence, we will for the moment assume $p \geqslant 1$. As shown in [GP98, Section 1.2], the line bundle $L - C$ is base-point free by Reider's theorem whenever it is big and nef with $(L-C)^2 \geqslant 5$. This is implied by the conditions $p \geqslant 1$, $\big(L^2\big) \geqslant 4p+5$, $(L \cdot C) \leqslant p + 2$, and $\big(C^2\big) = 0$.

The rest of the proof follows that of [EGHP05, Theorem 1.1], with the difference that there the plane is of dimension at most p (and hence one obtains strong statements about the regularity of the ideal sheaf $\mathscr{I}_{X\cap\Lambda|\Lambda}$ of the embedding $X \cap \Lambda \subseteq \Lambda$), whereas in our case the plane is of larger dimension. In any case, by the same token, we will obtain that $H^2\big(\mathbb{P}^{p+1}, \mathscr{I}_{X\cap\Lambda|\Lambda}\big) = 0$, which will give us the required contradiction.

Recall that, aiming at a contradiction, we assumed that L satisfies property (N_p) for some $p \geqslant 1$. By definition, one has a resolution of the ideal sheaf

$$E_\bullet :\ \cdots \to E_{p+1} \to E_p \to \cdots \to E_2 \to E_1 \to \mathscr{I}_{X|\mathbb{P}^N} \to 0\,,$$

where $E_i = \oplus\mathscr{O}_{\mathbb{P}^N}(-i-1)$ for any $i = 1, \ldots, p$. We tensor this resolution by $\mathscr{O}_\Lambda$ and obtain a complex $E_\bullet|_\Lambda$:

$$\cdots \to E_{p+1} \otimes \mathscr{O}_\Lambda \to \oplus\mathscr{O}_\Lambda(-p-1) \to \cdots \to \oplus\mathscr{O}_\Lambda(-3) \to \oplus\mathscr{O}_\Lambda(-2) \to \mathscr{I}_{X|\mathbb{P}^N} \otimes \mathscr{O}_\Lambda \to 0$$

that is exact outside the intersection $\Lambda \cap X$, that is, away from a set of dimension 1. Directly applying Lemma 4.3 and the automatic vanishing we have of the higher cohomology of line bundles on projective space yields $H^m(\Lambda, \mathscr{I}_{X|\mathbb{P}^N} \otimes \mathscr{O}_\Lambda) = 0$ for any $m \geqslant 2$. Now, consider the short exact sequence

$$0 \to \mathscr{I}_X \cap \mathscr{I}_\Lambda / \mathscr{I}_X \cdot \mathscr{I}_\Lambda \to \mathscr{I}_{X|\mathbb{P}^N} \otimes \mathscr{O}_\Lambda \to \mathscr{I}_{X\cap\Lambda|\Lambda} \to 0\,.$$

The kernel $\mathscr{K} \stackrel{\text{def}}{=} \mathscr{I}_X \cap \mathscr{I}_\Lambda / \mathscr{I}_X \cdot \mathscr{I}_\Lambda$ is supported on the one-dimensional scheme $X \cap \Lambda$, so $H^m(\Lambda, \mathscr{K}) = 0$ for any $m \geqslant 2$. We then deduce that $H^m(\Lambda, \mathscr{I}_{X\cap\Lambda|\Lambda}) = 0$ for any $m \geqslant 2$. Under the assumption $p \geqslant 1$, the short exact sequence

$$0 \to \mathscr{I}_{X\cap\Lambda|\Lambda} \to \mathscr{O}_\Lambda \to \mathscr{O}_{X\cap\Lambda} \to 0$$

gives that $H^1(X \cap \Lambda, \mathscr{O}_{X\cap\Lambda}) = 0$. In particular, one arrives at $H^1(C, \mathscr{O}_C) = 0$, which contradicts the fact that C is an elliptic curve. □

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