Algebraic Geometry 6 (5) (2019) 530–547
doi:10.14231/AG-2019-024

Hecke correspondences for Hilbert schemes of reducible locally planar curves

Oscar Kivinen

Abstract

Let C be a complex, reduced, locally planar curve. We extend the results of Rennemo to reducible curves by constructing an algebra A acting on $V = \bigoplus_{n\geqslant 0} H_*^{\mathrm{BM}}(C^{[n]}, \mathbb{Q})$, where $C^{[n]}$ is the Hilbert scheme of n points on C. If m is the number of irreducible components of C, we realize A as a subalgebra of the Weyl algebra of $\mathbb{A}^{2m}$. We also compute the representation V in the simplest reducible example of a node.

1. Introduction

Let C be a complex, reduced, locally planar curve. We are interested in studying the homologies of the Hilbert schemes of points $C^{[n]}$. In the case when C is integral, work of Rennemo, Migliorini–Shende, and Maulik–Yun [Ren18, MS13, MY14] relates these homologies to the homology of the compactified Jacobian of C equipped with the perverse filtration. Furthermore, work of Migliorini–Shende–Viviani [MSV18] considers an extension of these results to reduced but possibly reducible curves.

Following Rennemo, we approach the problem of computing the homologies of the Hilbert schemes in question from the point of view of representation theory. In [Ren18], a Weyl algebra in two variables acting on $V := \bigoplus_{n\geqslant 0} H_*^{\mathrm{BM}}(C^{[n]})$ was constructed for integral locally planar curves, and V was described in terms of the representation theory of the Weyl algebra. The superscript BM denotes Borel–Moore homology. When C has m irreducible components, we construct an algebra A acting on V, where A is an explicit subalgebra of the Weyl algebra in $2m$ variables. The main result is the following.

Theorem 1.1. *If $C = \bigcup_{i=1}^m C_i$ is a decomposition of C into irreducible components, the space $V = \bigoplus_{n\geqslant 0} H_*(C^{[n]}, \mathbb{Q})$ carries a bigraded action of the algebra*

$$A = A_m := \mathbb{Q}\left[x_1, \ldots, x_m, \partial_{y_1}, \ldots, \partial_{y_m}, \sum_{i=1}^m y_i, \sum_{i=1}^m \partial_{x_i}\right],$$

where $V = \bigoplus_{n,d\geqslant 0} V_{n,d}$ is graded by the number of points n and homological degree d. Moreover, the operators x_i have degree $(1,0)$, and the operators ∂_{y_i} have degree $(-1,-2)$ in this bigrading. In effect, the operator $\sum y_i$ has degree $(1,2)$, and the operator $\sum \partial_{x_i}$ has degree $(-1,0)$.

Received 05 December 2017, accepted in final form 05 September 2018.
2010 Mathematics Subject Classification 14C05.
Keywords: Hilbert scheme of points, representation theory.

This work was partially funded by the Vilho, Yrjö and Kalle Visälä Foundation of the Finnish Academy of Science and Letters, a Fulbright Graduate Grant, and NSF grants DMS-1700814 and DMS-15593381.

Remark 1.2. The algebra A does not depend on C but only on the number of components m.

Remark 1.3. An argument similar to [Ren18, Theorem 1.2] shows that V is free over $\mathbb{Q}[x_i]$ for any $i = 1, \dots, m$ and also over $\mathbb{Q}\big[\sum_{i=1}^m y_i\big]$. Through the Oblomkov–Rasmussen–Shende (ORS) conjectures (see below), this may be seen as a version of Rasmussen's remark in [Ras15] that the triply graded homology of the link L of C is free over the homology of an unlink corresponding to a component of L.

In Section 2, we will discuss the relevant geometry, namely the deformation theory of locally planar curves. In particular, we prove that the relative families of (flag) Hilbert schemes have smooth total spaces, which is crucial for applying a bivariant homology formalism, described in Section 3.1. We then define the action of the generators of A on V by explicit geometric constructions in Section 3 and prove the commutation relations in Section 4. In Section 5, we describe the representation V of the algebra A_2 in the example of the node. More precisely, we have the following result.

Theorem 1.4. *When $C = \{x^2 = y^2\} \subset \mathbb{P}^2$, we have*

$$V \cong \frac{\mathbb{Q}[x_1, x_2, y_1, y_2]}{\mathbb{C}[x_1, x_2, y_1 + y_2](x_1 - x_2)} \tag{1.1}$$

as an A-module, where

$$A = \mathbb{Q}[x_1, x_2, \partial_{x_1} + \partial_{x_2}, y_1 + y_2, \partial_{y_1}, \partial_{y_2}] \subset \mathrm{Weyl}\left(\mathbb{A}^4\right).$$

Remark 1.5. Although seeing the algebra A for the first time immediately raises the question whether we can define the operators ∂_{x_i} or multiplication by y_i separately, that is, extend this action to the whole Weyl algebra, this example shows that it is in fact not possible to do this while retaining the module structure for V.

Locally planar curve singularities are connected naturally to topics ranging from the Hitchin fibration [Ngô06] to HOMFLY-PT homology of the links of the singularities [ORS18, GORS14]. For example, from [ORS18] we have the following.

Conjecture 1.6. If C has a unique singularity at 0, its *link* is by definition the intersection of C with a small three-sphere around 0. There is an isomorphism

$$V_0^c \cong \mathrm{HHH}_{a=0}(\text{Link of } C),$$

where $V_0^c = \bigoplus_{n \geqslant 0} H^*\big(C_0^{[n]}\big)$ is the cohomology of the punctual Hilbert scheme and $\mathrm{HHH}(-)$ is the triply graded HOMFLY-PT homology of Khovanov and Rozansky [KR08].

This conjecture is still wide open. Recently, advances on the knot homology side have been made by Hogancamp, Elias and Mellit [EH16, Hog17, Mel16], who compute the HOMFLY-PT homologies of, for example, (n, n)-torus links using algebraic techniques. As the (n, n)-torus links appear as the links of the curves $C = \{x^n = y^n\} \subset \mathbb{P}^2$, a partial motivation for this work was to study the Hilbert schemes of points on these curves.

Remark 1.7. There are many natural algebras acting on $\mathrm{HHH}(-)$, for example the positive half of the Witt algebra, as proven in [KR16]. It might be possible that the actions of the operators $\mu_+ = \sum_i \partial_{x_i}$ and x_i on V are related to this action.

In the case when $C = \{x^p = y^q\}$ for coprime p and q, there is an action of the spherical rational Cherednik algebra of SL_n with parameter $c = p/q$ on the cohomology of the compactified Jacobian of C [VV09, OY16], or rather its associated graded with respect to the perverse filtration, which is intimately related to the space V. For arbitrary torus links, it might still be true that V or its variants carry some form of an action of a rational Cherednik algebra.

2. Geometry of Hilbert schemes of points

We describe the general setup for this paper. Fix a locally planar reduced curve $C/\mathbb{C}$, and let $C = \bigcup_{i=1}^m C_i$ be a decomposition of C to irreducible components. We will be working with versal deformations of C.

Definition 2.1. If X is a projective scheme, a *versal deformation* of X is a map of germs $\pi\colon \mathcal{X} \to B$ such that B is smooth, $\pi^{-1}(0) = X$, and given $\pi'\colon \mathcal{X}' \to B'$ with $\pi'^{-1}(b') = X$ there exists a $\phi\colon B' \to B$ such that b' maps to 0 and π' is the pullback of π along ϕ. If $T_0 B$ coincides with the first-order deformations of X or, in other words, the base B is of minimal dimension, we call π a *miniversal* deformation.

We call a family of locally planar, reduced, complex algebraic curves over a smooth base B *locally versal* at $b \in B$ if the induced deformations of the germs of the singular points of $\pi^{-1}(b)$ are versal. We are interested in the smoothness of relative families of Hilbert schemes of points for such deformations, needed for example for Lemma 4.3.

Definition 2.2. If $\mathcal{X} \to B$ is any family of projective schemes and $P(t)$ is any Hilbert polynomial, we denote the *relative Hilbert scheme* of this family by $\mathcal{X}^{P(t)}$. By definition, Hilbert schemes are defined for families [Gro95], and we note here that at closed points $b \in B$, the fibers of the relative Hilbert scheme are exactly $\operatorname{Hilb}^{P(t)}(\mathcal{X}_b)$.

We now consider the tangent spaces to (relative) Hilbert schemes.

Lemma 2.3. *For any projective scheme X and a flag of subschemes $X_1 \subset \cdots \subset X_k$ in X with fixed Hilbert polynomials $P_1(t), \ldots, P_k(t)$, the Zariski tangent space is given by*

$$T_{(X_1,\ldots,X_n)} \operatorname{Hilb}^{\overrightarrow{P(t)}}(X) \cong H^0(X, \mathcal{N}_{(X_1,\ldots,X_m)/X}),$$

where the sections of the normal sheaf $\mathcal{N}_{(X_1,\ldots,X_m)/X} \subseteq \bigoplus_{i=1}^k \mathcal{N}_{X_i/X}$ *are tuples $(\xi_1, \ldots, \xi_k)$ of normal vector fields such that $\xi_i|_{X_j} = \xi_j$ modulo $\mathcal{N}_{X_j/X_i}$ whenever $X_i \supseteq X_j$. The normal sheaf is, by definition, the sheaf of germs of commutative diagrams of homomorphisms of $\mathcal{O}_X$-modules of the form*

$$\begin{array}{ccccccc} \mathcal{I}_k & \subset & \mathcal{I}_{k-1} & \subset & \cdots & \subset & \mathcal{I}_1 \\ \downarrow{\scriptstyle \sigma_k} & & \downarrow{\scriptstyle \sigma_{k-1}} & & & & \downarrow{\scriptstyle \sigma_1} \\ \mathcal{O}_{X_k} & \to & \mathcal{O}_{X_{k-1}} & \to & \cdots & \to & \mathcal{O}_{X_1}\,. \end{array} \tag{2.1}$$

Proof. Note that from first-order deformation theory, it immediately follows that if $k = 1$, we have $T_{X_1} \operatorname{Hilb}^{P(t)}(X) \cong H^0(\mathcal{N}_{X_1/X}, X) = H^0\big(X, \operatorname{Hom}_{\mathcal{O}_{X_1}}(\mathcal{I}_1/\mathcal{I}_1^2, \mathcal{O}_{X_1})\big)$, where $\mathcal{I}_1$ is the ideal sheaf of X_1. For the proof of the result for flag Hilbert schemes, we refer to [Ser06, Proposition 4.5.3]. ☐

The following proposition is proved in, for example, [She12, Proposition 17], and we re-prove it here for the convenience of the reader.

Proposition 2.4. *Let $\pi\colon \mathcal{C} \to B'$ be a versal deformation of C, a reduced, locally planar curve. Then the total space of the family $\pi^{[n]}\colon \mathcal{C}^{[n]} \to B'$ is smooth.*

Proof. Let $B \subset \mathbb{C}[x,y]$ be a finite-dimensional, smooth family of polynomials containing the local equation for C and all polynomials of degree at most n, such that the associated deformation is versal. Consider the family of curves over B given by $\mathcal{C}_B := \{(f \in B,\, p \in \mathbb{C}^2) f(p) = 0\}$. Denote the fiber over f by C_f, and let $Z \subset C_f$ be a subscheme of length n. By, for example, [Ser06, Section 4], there is always an exact sequence

$$0 \to H^0(\mathcal{N}_{Z/C_f}, Z) \to T_Z C_B^{[n]} \to T_f B \to \operatorname{Ext}^1_{\mathcal{O}_{C_f}}(\mathcal{I}_Z, \mathcal{O}_Z)\,.$$

For squarefree f, there is always some open neighborhood U of f such that $C_U^{[n]}$ is reduced of pure dimension $n + \dim B$ (see [MY14, Proposition 3.5]). Since B is smooth and $H^0(\mathcal{N}_{Z/C_f}, Z)$ has dimension n (see, for example, [Che98]), it is enough to prove that the last Ext-group vanishes to get the smoothness of the total space $C_U^{[n]}$ at Z.

Now, from the short exact sequence $0 \to \mathcal{I}_Z \to \mathcal{O}_{C_f} \to \mathcal{O}_Z \to 0$, taking Hom to $\mathcal{O}_Z$, we have

$$\cdots \to \operatorname{Ext}^1_{\mathcal{O}_{C_f}}(\mathcal{O}_{C_f}, \mathcal{O}_Z) \to \operatorname{Ext}^1_{\mathcal{O}_{C_f}}(\mathcal{I}_Z, \mathcal{O}_Z) \to \operatorname{Ext}^2_{\mathcal{O}_{C_f}}(\mathcal{O}_Z, \mathcal{O}_Z) = 0 \to \cdots\,.$$

Since $\operatorname{Ext}^1_{\mathcal{O}_{C_f}}(\mathcal{O}_{C_f}, \mathcal{O}_Z) \cong H^1(\mathcal{O}_Z, C_f) = 0$ and the sequence is exact, we must also have $\operatorname{Ext}^1_{\mathcal{O}_{C_f}}(\mathcal{I}_Z, \mathcal{O}_Z) = 0$. So the total space is smooth.

Now, if $\overline{\mathcal{C}} \to \overline{B}$ is the miniversal deformation, by versality there are compatible isomorphisms $\mathcal{C} \cong \overline{\mathcal{C}} \times (\mathbb{C}^t, 0)$ and $B \cong \overline{B} \times (\mathbb{C}^t, 0)$ for some t; see, for example, [GLS07]. Hence, we have smoothness for any versal family. $\square$

We now consider the relative flag Hilbert scheme of a versal deformation. If S is a smooth complex algebraic surface, its nested Hilbert scheme of points $S^{[n,n+1]}$ is smooth by results of [Che98, Tik97].

Remark 2.5. This nested Hilbert scheme $S^{[n,n+1]}$ and the ordinary Hilbert scheme of n points $S^{[n]}$ are the only flag Hilbert schemes of points on S that are smooth, as shown in [Che98, Tik97].

We also have the following result.

Proposition 2.6. *The total space of the relative family $\mathcal{C}^{[n,n+1]} \to B$ is smooth.*

Proof. This is a local question, so we can assume that C is the germ of a plane curve singularity in $\mathbb{C}^2$.

From Lemma 2.3, we have that at $(J \subset I) \in \mathcal{C}^{[n,n+1]}$, the tangent space is $\ker(\phi - \psi)$, where

$$\phi\colon \operatorname{Hom}_{\mathbb{C}[x,y]}(I, \mathbb{C}[x,y]/I) \to \operatorname{Hom}_{\mathbb{C}[x,y]}(J, \mathbb{C}[x,y]/I)\,, \quad \text{and}$$
$$\psi\colon \operatorname{Hom}_{\mathbb{C}[x,y]}(J, \mathbb{C}[x,y]/J) \to \operatorname{Hom}_{\mathbb{C}[x,y]}(J, \mathbb{C}[x,y]/I)$$

are the induced maps given by restriction and further quotient. Here, $\phi - \psi$ is the difference of the maps from the direct sum. This is precisely the requirement needed for the normal vector fields in question.

Suppose again that $B \subset \mathbb{C}[x,y]$ is a finite-dimensional, smooth family of polynomials containing the local equation for C and all polynomials of degree at most $n+1$, such that the associated deformation is versal.

Consider the inclusion $\mathcal{C}_B^{[n,n+1]} \hookrightarrow B \times \left(\mathbb{C}^2\right)^{[n,n+1]}$. We have an exact sequence

$$0 \to T_{f,J\subset I}\mathcal{C}_B^{[n,n+1]} \to T_f B \times T_{J\subset I}\left(\mathbb{C}^2\right)^{[n,n+1]} \to \mathbb{C}[x,y]/J\,,$$

where the last map is given by

$$\left(f+\epsilon g, \begin{pmatrix}\eta_1\\ \eta_2\end{pmatrix}\right) \mapsto (\phi\eta_1)(f) - g \mod J\,.$$

By the assumption on B, the last map is surjective; hence, $T_{f,J\subset I}\mathcal{C}_B^{[n,n+1]}$ has dimension

$$\dim\left(\mathbb{C}^2\right)^{[n,n+1]} + \dim B - n + 1 = n + 1 + \dim B\,,$$

as expected, and the total space is smooth.

Again, if $\overline{\mathcal{C}} \to \overline{B}$ is the miniversal deformation, by versality there are compatible isomorphisms $\mathcal{C} \cong \overline{\mathcal{C}} \times \left(\mathbb{C}^t, 0\right)$ and $B \cong \overline{B} \times \left(\mathbb{C}^t, 0\right)$ for some t; see, for example, [GLS07]. Hence, we have smoothness for any versal family. □

Another result we will also need is the description of the components and dimensions of the irreducible components of $C^{[n]}$.

Proposition 2.7. *For any locally planar, reduced curve $C = \bigcup_{i=1}^m C_i$, the irreducible components of $C^{[n]}$ are given by*

$$\overline{(C_1^{sm})^{[r_1]} \times \cdots \times (C_m^{sm})^{[r_m]}}\,, \quad \sum_i r_i = n\,.$$

Here, C_i^{sm} denotes the smooth locus of C_i. In particular, there are $\binom{n+m-1}{n}$ irreducible components of $C^{[n]}$, all of dimension n.

Proof. That $(C^{sm})^{[n]}$ is dense in $C^{[n]}$ can be found, for example, in [MRV17, Fact 2.4]. The schemes $(C_1^{sm})^{[r_1]} \times \cdots \times (C_m^{sm})^{[r_m]}$ are disjoint. They are of dimension n, smooth and connected, so irreducible, and as $(r_1, \ldots, r_m)$ runs over all possibilities, cover $(C^{sm})^{[n]}$. Taking closures, we get the result. □

3. Definition of the algebra A

Let $V = \bigoplus_{i\geqslant 0} H_*(C^{[n]}, \mathbb{Q})$, where we take the singular homology in the analytic topology. This is mostly for simplicity; a majority of the results work with $\mathbb{Z}$-coefficients. A notable exception is Section 5, where $\mathbb{Q}$-coefficients are essential. From now on, we will be suppressing the coefficients from our notation. The space V is naturally a bigraded $\mathbb{Q}$-vector space, graded by the number of points n and homological degree d. We denote the (n,d)-graded piece of V by $V_{n,d}$. We define operators on V following ideas of Rennemo [Ren18] (that originally go back to Nakajima and Grojnowski [Nak97, Gro96]).

Definition 3.1. (i) Let $c_i \in C_i^{sm}$ be fixed smooth points and $\iota_i\colon C^{[n]} \to C^{[n+1]}$ be the maps $Z \mapsto Z \cup c_i$. Let $x_i\colon V \to V$ be the operators given by $(\iota_i)_*$. These are homogeneous of degree $(1,0)$ and depend only on the component the points c_i lie in; see Lemma 3.2 below.

(ii) Let $d_i\colon V \to V$ be the operators given by the Gysin/intersection pullback map $(\iota_i)^!$. These are homogeneous of degree $(-1,-2)$ and well defined since the ι_i are regular embeddings. See Lemma 3.3 below for a proof of this latter fact.

Lemma 3.2. *The maps ι_{c_i} and $\iota_{c_i'}$ are homotopic whenever $c_i, c_i' \in C_i^{sm}$. In particular, the corresponding pushforwards induce the same operators x_i on V.*

Proof. Take any path $c_i(t)$ from c_i to c_i', and consider the homotopy $C^{[n]} \times [0,1] \to C^{[n+1]}$ given by $(Z,t) \mapsto Z \cup c_i(t)$. □

Lemma 3.3. *The map ι_x is a regular embedding.*

Proof. This is a property which is local in the analytic topology [ACG11, Chapter 2, Lemma 2.6]. Suppose that $Z \subset C$ is a subscheme of length n which contains x with multiplicity k.

If U is an analytic open set around x such that the only component of Z contained in $\bar{U}$ (the closure in the analytic topology) is x, then locally around Z, the morphism is isomorphic to

$$U^{[k]} \times (C \backslash \bar{U})^{[n-k]} \hookrightarrow (U)^{[k+1]} \times (C \backslash \bar{U})^{[n-k]},$$

where the map is given on factors by adding x and the identity map, respectively. In local coordinates, the first map looks exactly like the inclusion $\mathbb{C}^{[k]} \to \mathbb{C}^{[k+1]}$ given as follows. If we identify coordinates on $\mathbb{C}^{[k]}$ with symmetric functions a_i in the roots of some degree k polynomial, that is, coefficients of a monic polynomial of degree k, the map is given by $\sum_{i=0}^{k-1} a_i z^i \mapsto (z-x) \sum_{i=0}^{k-1} a_i z^i$. But this last map is linear in the a_i and of rank k, in particular a regular embedding. □

Consider the following diagram:

$$\begin{tikzcd} & C^{[n,n+1]} \arrow[dl, "p"'] \arrow[dr, "q"] & \\ C^{[n]} & & C^{[n+1]}. \end{tikzcd} \tag{3.1}$$

To define the operators μ_+ and μ_-, we want to define correspondences in homology between $C^{[n]}$ and $C^{[n+1]}$. This is done as follows. By Propositions 2.4 and 2.6, we may embed C into a smooth, locally versal family $\pi \colon \mathcal{C} \to B$ such that the relative family $\mathcal{C}^{[n]}$ is smooth and $\pi^{-1}(0) = C$. After possibly doing an étale base extension, we may also assume that the family also has sections $s_i \colon B \to \mathcal{C}$ hitting only the smooth loci of the fibers and such that $s_i(0) = c_i$.

Now, consider the diagram

$$\begin{tikzcd} & C^{[n,n+1]} \arrow[dl, "p"'] \arrow[dr, "q"] \arrow[dd, "i"] & \\ C^{[n]} \arrow[dd, "i"] & & C^{[n+1]} \arrow[dd, "i"] \\ & \mathcal{C}^{[n,n+1]} \arrow[dl, "\tilde{p}"'] \arrow[dr, "\tilde{q}"] & \\ \mathcal{C}^{[n]} & & \mathcal{C}^{[n+1]}, \end{tikzcd} \tag{3.2}$$

where i is the inclusion of the central fiber. Since $\mathcal{C}^{[n]}$ is smooth, from Property (7) in Section 3.1, we have

$$H^*\big(\mathcal{C}^{[n,n+1]} \to \mathcal{C}^{[n]}\big) \cong H^{\mathrm{BM}}_{*-2n-\dim B}\big(\mathcal{C}^{[n,n+1]}\big).$$

Denote the fundamental class of $\mathcal{C}^{[n,n+1]}$ under this isomorphism by $[\tilde{p}]$. Then pulling back $[\tilde{p}]$ along i to $H^*\big(C^{[n,n+1]} \to C^{[n]}\big)$ gives us a canonical orientation, using which we define $p^! \colon H_*\big(C^{[n]}\big)$

$\to H_*\big(C^{[n,n+1]}\big)$ as $p^!(\alpha) = \alpha \cdot i^*([\tilde{p}])$. The definition of $q^!$ is analogous, where we replace $C^{[n]}$ with $C^{[n+1]}$.

We are finally ready to define the operators μ_+ and μ_-.

Definition 3.4. Let $\mu_\pm \colon V \to V$ be the *Nakajima correspondences* $\mu_+ = q_* p^!$ and $\mu_- = p_* q^!$. These are operators of respective bidegrees $(1,2)$ and $(-1,0)$.

Remark 3.5. The n-degree in the above maps is easy to see from the definition. The homological degrees follow from the definition of the Gysin maps using $i^*[\tilde{p}]$, which sits in homological degree $2n+2$, and the fact that degrees are additive under the bivariant product.

We are now ready to define the algebra(s) A.

Theorem 3.6. *The operators from Definitions 3.1 and 3.4 satisfy the following commutation relations: $[d_i, \mu_+] = [\mu_-, x_i] = 1$, and the rest are trivial.*

Remark 3.7. For $m = 1$, we recover [Ren18, Theorem 1.2].

Definition 3.8. Fix $m \geqslant 1$. Let A_m be the $\mathbb{Q}$-algebra generated by the symbols

$$x_1, \dots, x_m, d_1, \dots, d_m, \mu_+, \mu_-$$

with the relations

$$[d_i, \mu_+] = [\mu_-, x_i] = 1\,, \quad [x_i, x_j] = [x_i, d_i] = [x_i, \mu_+] = [d_i, \mu_-] = 0\,.$$

Remark 3.9. We can realize A_m inside $\mathrm{Weyl}\big(\mathbb{A}^{2m}_{\mathbb{Q}}\big)$ as follows: Let $\mathbb{A}^{2m}$ have coordinates $x_1, \dots,$ x_m, $y_1, \dots, y_m$ and $d_i = \partial_{y_i}$, $k = \sum_{i=1}^m \partial_{x_i}$, $j = \sum_{i=1}^m y_i$. Then from the commutation relations, we immediately have that A_m is isomorphic to the subalgebra $\big\langle x_i, \partial_{y_i}, \sum_{i=1}^m \partial_{x_i}, \sum_{i=1}^m x_i \big\rangle \subset$ $\mathrm{Weyl}\big(\mathbb{A}^{2m}_{\mathbb{Q}}\big)$.

Remark 3.10. Although A_m depends on m, we will be suppressing the subscript from the notation from here on. It should be evident from the context which m we are considering.

Let us give an outline of the proof of Theorem 3.6. We first prove the trivial commutation relations in Subsections 4.1 and 4.2. In Subsection 4.3, we then prove that $[d_i, \mu_+] = 1$, with the aid of the bivariant homology formalism, and then in a similar vein that $[\mu_-, x_i] = 1$.

3.1 Bivariant Borel–Moore homology

We now describe the bivariant Borel–Moore homology formalism from [FM81]. Suppose that we are in a category of "nice" spaces, for example, those that can be embedded in some $\mathbb{R}^n$. We will not define bivariant homology here, but for us, the most essential facts about it are the following ones:

(1) The theory associates with maps $X \xrightarrow{f} Y$ a graded abelian group $H^*\big(X \xrightarrow{f} Y\big)$. We will be working over $\mathbb{Q}$ throughout and with bivariant homology.

(2) Given maps $X \xrightarrow{f} Y \xrightarrow{g} Z$, there is a product homomorphism

$$H^i\big(X \xrightarrow{f} Y\big) \otimes H^j\big(Y \xrightarrow{g} Z\big) \to H^{i+j}\big(X \xrightarrow{g\circ f} Z\big)\,.$$

For $\alpha \in H^i\big(X \xrightarrow{f} Y\big)$ and $\beta \in H^j\big(Y \xrightarrow{g} Z\big)$, we thus get a product $\alpha \cdot \beta \in H^{i+j}\big(X \xrightarrow{g\circ f} Z\big)$.

(3) For any proper map $X \xrightarrow{f} Y$ and any map $Y \xrightarrow{g} Z$, there is a pushforward homomorphism $f_* \colon H^*\big(X \xrightarrow{g\circ f} Z\big) \to H^*\big(Y \xrightarrow{g} Z\big)$.

(4) For any cartesian square

$$\begin{tikzcd} X' \arrow[r] \arrow[d, "g"] & X \arrow[d, "f"] \\ Y' \arrow[r] & Y \,, \end{tikzcd}$$

there is a pullback homomorphism $H^*\big(X \xrightarrow{f} Y\big) \to H^*\big(X' \xrightarrow{g} Y'\big)$. (Recall that a cartesian square is a square where $X' \cong X \times_Y Y'$.)

(5) The product and pullback commute: given a tower of cartesian squares

$$\begin{tikzcd} X' \arrow[r, "h''"] \arrow[d, "f'"] & X \arrow[d, "f", "\alpha"'] \\ Y' \arrow[r, "h'"] \arrow[d, "g'"] & Y \arrow[d, "g", "\beta"'] \\ Z' \arrow[r, "h"] & Z \,, \end{tikzcd}$$

we have $h^*(\alpha \cdot \beta) = h'^*(\alpha) \cdot h^*(\beta)$ in $H^*\big(X' \xrightarrow{g' \circ f'} Z'\big)$.

(6) The product and pushforward commute: given

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} W$$

with $\alpha \in H^*\big(X \xrightarrow{g\circ f} Z\big)$ and $\beta \in H^*\big(Z \xrightarrow{h} W\big)$, we have $f_*(\alpha\cdot\beta) = f_*(\alpha)\cdot\beta$ in $H^*\big(Y \xrightarrow{h\circ g} W\big)$.

(7) For any space X, the groups $H^i(X \to \mathrm{pt})$ and $H^i\big(X \xrightarrow{\mathrm{id}} X\big)$ are by construction canonically identified with $H^{\mathrm{BM}}_{-i}(X)$ and $H^i(X)$, respectively. These are called the associated covariant and contravariant theories, respectively. Note that the three bivariant operations recover the usual homological operations of cup and cap product, proper pushforwards in homology and arbitrary pullbacks in cohomology.

(8) If Y is a nonsingular variety and $f\colon X \to Y$ is any morphism, the induced homomorphism

$$H^*\big(X \xrightarrow{f} Y\big) \to H^{*-2\dim Y}(X \to \mathrm{pt}) = H^{\mathrm{BM}}_{2\dim Y - *}(X)$$

given by taking the product with $[Y] \in H^{-2\dim Y}(Y \to \mathrm{pt})$ is an isomorphism. Again, the last equality is given by the associated covariant theory. In such a situation, we will frequently identify $H^*(X \to Y)$ with $H^{\mathrm{BM}}_{2\dim Y - *}(X)$. In particular, if X has a fundamental class $[X] \in H^{\mathrm{BM}}_{2\dim X}(X)$, this induces a class $[X] \in H^{2(\dim Y - \dim X)}(X \to Y)$.

(9) Any class $\alpha \in H^i\big(X \xrightarrow{f} Y\big)$ defines a Gysin pullback map $f^!\colon H^{\mathrm{BM}}_*(Y) \to H^{\mathrm{BM}}_{*-i}(X)$ by

$$f^!(\beta) := \alpha \cdot \beta\,, \quad \forall\, \beta \in H^{\mathrm{BM}}_*(Y)\,.$$

4. Proof of the commutation relations

4.1 Proof of the trivial commutation relations for x_i and d_i

We now show that $[x_i, x_j] = 0$ for all i, j. This is fairly easy; under either composition $\iota_i \circ \iota_j$ or $\iota_j \circ \iota_i$, we map $Z \mapsto Z \cup x_i \cup x_j$ and as $(\iota_i \circ \iota_j)_* = x_i x_j$, we get $x_i x_j = x_j x_i$.

The next step is to describe the Gysin maps and their commutation relations. Denote these as before by $d_i = (\iota_{x_i})^!\colon H_*\big(C^{[n]}\big) \to H_{*-2}\big(C^{[n-1]}\big)$.

Proposition 4.1. *We have $[d_i, d_j] = 0$ and $[d_i, x_j] = 0$.*

Proof. Let $\alpha \in H_*\big(C^{[n]}\big)$. As we saw before, $\iota_{x_i} \circ \iota_{x_j} = \iota_{x_j} \circ \iota_{x_i}$. By the functoriality of the Gysin maps, $[d_i, d_j] = 0$. Now, choosing a representative for α, we see that

$$d_i x_j(\alpha) = (\iota_j)^![\iota_i(\alpha)] = [Z \in \alpha | c_i, c_j \subset Z]\,.$$

However, we also have

$$x_j d_i(\alpha) = x_j[Z' \in \alpha | c_i \subset Z'] = [Z \in \alpha | c_i, c_j \subset Z]\,.$$

When the points are equal in the above, that is to say $i = j$, we pick a linearly equivalent point c_i' near c_i. Since the inclusion maps ι_{c_i} and $\iota_{c_i'}$ are homotopic in this case by Lemma 3.2, we still have $[d_i, x_j] = 0$. ☐

4.2 Proof of the trivial commutation relations for Nakajima operators

As we saw before, the definition of the Nakajima operators requires making sense of the Gysin morphisms $p^!$ and $q^!$, which is done using the bivariant homology formalism. Recall that we are working with a fixed family $\mathcal{C} \to B$ as in Section 3, guaranteeing the smoothness of $\mathcal{C}^{[n]}$ and $\mathcal{C}^{[n,n+1]}$. In this section and later on, all commutative diagrams should be thought of as commutative diagrams of topological spaces (corresponding to the analytic spaces of the varieties under consideration) and living over B, so that we may restrict to the central fiber and obtain similar squares with the calligraphic $\mathcal{C}$ replaced with the regular C, that is, our curve of interest. We will denote these restrictions in homology computations by the subscript 0.

Proposition 4.2. *We have $[x_i, \mu_+] = 0$ and $[d_i, \mu_-] = 0$.*

Proof. Consider the following commutative diagram:

$$\begin{tikzcd}
& \mathcal{C}^{[n,n+1]} \arrow[dl, "p"'] \arrow[dr, "q"] \arrow[rr, "\iota_i''"] & & \mathcal{C}^{[n+1,n+2]} \arrow[dl, "p'"'] \arrow[dr, "q'"] & \\
\mathcal{C}^{[n]} \arrow[rr, "\iota_i"] & & \mathcal{C}^{[n+1]} \arrow[rr, "\iota_i'"] & & \mathcal{C}^{[n+2]}\,.
\end{tikzcd} \tag{4.1}$$

Here, ι_i and ι_i' are defined as adding points at the sections s_i to $\mathcal{C}^{[n]}$ and $\mathcal{C}^{[n+1]}$, respectively, and ι_i'' is adding points at the section s_i as follows: $(Z_1 \subset Z_2) \mapsto (Z_1 \cup s_i \subset Z_2 \cup s_i)$.

We have, by definition, $x_i\mu_+ = \big((\iota_{s_i}')_* q_* p^!\big)_0$, where the subscript 0 denotes restriction to the central fiber.

In diagram (4.1), the square formed by the maps ι_i', q, q' and ι_i'' is commutative, so on homology, we have $(\iota_i')_* q_* p^! = q_*'(\iota_i'')_* p^!$. Similarly, the square formed by the maps ι_i, p', p, ι_i'' is commutative. Because of the fact that the pushforward and the Gysin maps in bivariant homology also commute in this case, as explained in Section 3.1 (Property (7)), we get $q_*'(\iota_i'')_* p^! = q_*'(p')^!(\iota_i)_*$. Restricting to C, we have $(q_*'(p')^!(\iota_i)_*)_0 = \mu_+ x_i$. Similarly, for the other commutation relation we have $\mu_- d_i = (p_* q^!(\iota_i')^!)_0 = (p_*(\iota_i'')^!(q')^!)_0 = ((\iota_i)^! p_*'(q')^!)_0 = d_i\mu_-$.

Let us explain the restriction to the central fiber once and for all. For example, in the last computation, if $\alpha \in H_*\big(C^{[n]}\big)$, we have $\mu_- d_i \alpha = i^*[\tilde{p}] \cdot \iota_i^! \alpha = i^*[\tilde{p}] \cdot i^*[\tilde{d}_i] \cdot \alpha$, where $i^*[\tilde{d}_i] = [d_i]$ is the fundamental class corresponding to the Gysin map $\iota_i^!$ and $[\tilde{d}_i]$ is the corresponding class in the family. Since the product and pullback commute in the bivariant theory, $i^*[\tilde{p}] \cdot i^*[\tilde{d}_i] \cdot \alpha = i^*\big([\tilde{p}] \cdot [\tilde{d}_i]\big) \cdot \alpha$. Similarly, $d_i \mu_- \alpha = [d_i] \cdot i^*[\tilde{p}] \cdot \alpha = i^*\big([\tilde{d}_i] \cdot [\tilde{p}]\big) \cdot \alpha$. So composing our operators in the family and then deducing the result for C is justified. ☐

4.3 Proof that $[d_i, \mu_+] = [\mu_-, x_i] = 1$

To compute the desired commutation relation, we compare the composition of the operators d_i and μ_+ on V in either order. By abuse of notation we will first consider d_i and μ_+ as operators acting on the space $\mathcal{V} = \bigoplus_{n\geqslant 0} H_*^{\mathrm{BM}}\big(\mathcal{C}^{[n]}\big)$ and then use the properties of the bivariant theory, more precisely, the ability to pull back in cartesian squares (Property (5) in Section 3.1), to restrict to the special fiber and get an action on V.

Consider the diagrams

$$\begin{tikzcd} & \mathcal{C}^{[n,n+1]} \arrow[dl, "\theta"', "p"] \arrow[d, "\kappa"', "q"] & X_i \arrow[l, hook', "\widetilde{\lambda}_i", "\widetilde{\iota}_i"'] \arrow[d, "\widetilde{\kappa}_i"', "\widetilde{q}_i"] \\ \mathcal{C}^{[n]} & \mathcal{C}^{[n+1]} & \mathcal{C}^{[n]} \arrow[l, hook', "\lambda_i", "\iota_i"'] \end{tikzcd} \tag{4.2}$$

and

$$\begin{tikzcd} & & \mathcal{C}^{[n-1,n]} \arrow[dl, "\theta'"', "p'"] \arrow[dr, "q'", "\kappa'"'] & \\ \mathcal{C}^{[n]} & \mathcal{C}^{[n-1]} \arrow[l, hook', "\lambda'_i", "\iota'_i"'] & & \mathcal{C}^{[n]}\,, \end{tikzcd} \tag{4.3}$$

where the two labels on the arrows denote the corresponding map $f\colon Y \to Z$ and a bivariant class $\alpha \in H^*\big(X \xrightarrow{f} Y\big)$.

In the first diagram, $X_i = \mathcal{C}^{[n,n+1]} \times_{\mathcal{C}^{[n+1]}} \mathcal{C}^{[n]}$ is the fiber product, and the square containing X_i is cartesian. The morphisms ι_i and ι'_i correspond to adding a point at the sections $s_i\colon B \to \mathcal{C}$. The bivariant classes θ, λ_i, κ and their primed versions are the ones defined by fundamental classes, using the fact that the targets are smooth. The classes $\widetilde{\lambda_i}$ and $\widetilde{\kappa_i}$ are the cartesian pullbacks of λ_i and κ, respectively.

Let $\alpha \in H_*\big(\mathcal{C}^{[n]}\big)$. We first compute

$$d_i\mu_+(\alpha) = \lambda'_i \cdot q_*(\theta\cdot\alpha) = \tilde{q}_*\big(\tilde{\lambda}_i \cdot \theta \cdot \alpha\big)\,, \tag{4.4}$$

$$\mu_+ d_i(\alpha) = q'_*(\theta' \cdot \lambda'_i \cdot \alpha)\,. \tag{4.5}$$

Let us elaborate a little bit on the first computation. Here, θ is the fundamental class, and the isomorphism $H^{\mathrm{BM}}_{*-n-1}\big(\mathcal{C}^{[n,n+1]}\big) \cong H^*\big(\mathcal{C}^{[n,n+1]} \to \mathcal{C}^{[n]}\big)$ of the Borel–Moore homology group with the bivariant one is given by the product with the fundamental class θ. On the other hand, the Gysin pullback $\iota_i^!$ is, by definition, equal to the product with λ_i in the bivariant theory. In the first equation of (4.4), we then use that the diagram (4.2) is cartesian.

Let $f_i\colon \mathcal{C}^{[n-1,n]} \to X_i$ be given by $(Z_1 \subset Z_2) \mapsto (Z_1\cup s_i \subset Z_2 \cup s_i, Z_1 \cup s_i)$ and $g_i\colon \mathcal{C}^{[n]} \to X_i$ be given by $Z \mapsto (Z \subset Z \cup s_i, Z)$.

Lemma 4.3. *For all i, we have $[X_i] = (f_i)_*\big(\big[\mathcal{C}^{[n-1,n]}\big]\big) + (g_i)_*\big(\big[\mathcal{C}^{[n]}\big]\big)$.*

Proof. By Propositions 2.4 and 2.6, the total spaces of the relative families $\mathcal{C}^{[n]} \to B$ and $\mathcal{C}^{[n,n+1]} \to B$ are smooth.

Consider the fiber product X_i. The images of f_i and g_i cover all of X_i. On the level of points (of the fibers) this is easy to see: We are looking at pairs consisting of a flag of subschemes of lengths n and $n+1$ and a subscheme of length n that project to the same length $n+1$ subscheme in the cartesian square (4.2). Since the points in the image contain s_i, the above pairs come either from adding s_i to both parts of the flag as well as taking the second factor to be $Z_1 \cup s_i$, or from creating a new flag by adding s_i to Z and taking Z to be the second factor.

By [Ren18, Lemma 3.4], the intersection of the images of f_i and g_i is of codimension one in X_i. Consider a point $(Z \subset Z \cup s_i, Z) \in \mathrm{Im}(f_i) \cap \mathrm{Im}(g_i)$, which can also be written as $(Z' \cup s_i \subset Z' \cup s_i \cup s_i, Z' \cup s_i)$. We can then remove the smooth point s_i from both of the factors unambiguously. So the intersection is isomorphic to $\mathcal{C}^{[n-1]}$. On the complement of the intersection, the maps f_i and g_i are scheme-theoretic isomorphisms because we can unambiguously remove the point s_i from $(Z \subset Z \cup s_i, Z)$ or $(Z_1 \cup s_i \subset Z_2 \cup s_i, Z_1 \cup s_i)$. Hence, the images of f_i and g_i yield a partition of X_i to irreducible components. In particular, the fundamental class $[X_i]$ is the sum of the fundamental classes of the images, which are by definition the pushforwards in question. □

Corollary 4.4. *We have* $[X_i] = (f_i)_*(\theta' \cdot \lambda_i') + (g_i)_*\big[\mathcal{C}^{[n]}\big]$.

Proof. By Lemma 4.3, we have $[X_i] = (f_i)_*\big(\big[\mathcal{C}^{[n-1,n]}\big]\big) + (g_i)_*\big(\big[\mathcal{C}^{[n]}\big]\big)$. Rewrite $\big[\mathcal{C}^{[n-1,n]}\big]$ as follows. First of all, note that

$$\theta' \cdot \lambda_i' \in H^{2\dim \mathcal{C}^{[n]} - 2\dim \mathcal{C}^{[n-1,n]}}\big(\mathcal{C}^{[n-1,n]} \to \mathcal{C}^{[n]}\big) \cong H^{\mathrm{BM}}_{2\dim \mathcal{C}^{[n-1,n]}}\big(\mathcal{C}^{[n,n+1]}\big)\,.$$

The last isomorphism is given by Property (8) in Section 3.1, that is, taking the product with $\big[\mathcal{C}^{[n]}\big] \in H^*\big(\mathcal{C}^{[n]} \to \mathrm{pt}\big)$. On the other hand, we know that $\theta' \cdot \lambda_i' \cdot \big[\mathcal{C}^{[n]}\big]$ has to be $\big[\mathcal{C}^{[n-1,n]}\big]$ by the same isomorphism. Plugging this into the result of Lemma 4.3 gives $[X_i] = (f_i)_*(\theta' \cdot \lambda_i') + (g_i)_*\big(\big[\mathcal{C}^{[n]}\big]\big)$. □

Using Corollary 4.4, we have

$$\begin{aligned} d_i\mu_+(\alpha) &= (\tilde{q}_i)_*\big(\tilde{\lambda}_i \cdot \theta \cdot \alpha\big) = (\tilde{q}_i)_*([X_i] \cdot \alpha) \\ &= (\tilde{q}_i)_*((f_i)_*(\theta' \cdot \lambda_i') \cdot \alpha) + (\tilde{q}_i)_*\big((g_i)_*\big[\mathcal{C}^{[n]}\big] \cdot \alpha\big)\,. \end{aligned} \tag{4.6}$$

The last equality follows by the linearity of the pushforward. From Property (6) in Section 3.1, we have that the pusforward is also functorial and commutes with products. Hence, we have

$$(\tilde{q}_i)_*((f_i)_*(\theta' \cdot \lambda_i') \cdot \alpha) + (\tilde{q}_i)_*\big((g_i)_*\big[\mathcal{C}^{[n]}\big] \cdot \alpha\big) = (\tilde{q}_i \circ (f_i))_*(\theta' \cdot \lambda_i' \cdot \alpha) + (\tilde{q}_i \circ (g_i))_*(\alpha)\,. \tag{4.7}$$

Since $\tilde{q}_i \circ f_i = q'$, we get $(\tilde{q}_i \circ (f_i))_*(\theta' \cdot \lambda_i' \cdot \alpha) = q'_*(\theta' \cdot \lambda_i' \cdot \alpha)$. Finally, since $\tilde{q}_i \circ (g_i) = \mathrm{id}$, we have $(\tilde{q}_i \circ (g_i))_*(\alpha) = \mathrm{id}_*(\alpha)$. Substituting these into (4.7), we get

$$(\tilde{q}_i \circ (f_i))_*(\theta' \cdot \lambda_i' \cdot \alpha) + (\tilde{q}_i \circ (g_i))_*(\alpha) = q'_*(\theta' \cdot \lambda_i' \cdot \alpha) + \mathrm{id}_*(\alpha) = \mu_+ d_i(\alpha) + \alpha\,. \tag{4.8}$$

Now, suppose that α_0 is a class in $H_*\big(C^{[n]}\big)$. Then by the fact that pushforward, pullback, and the product in the bivariant theory commute, $((\tilde{q}_i)_0)_*((\tilde{\lambda}_i)_0 \cdot \theta_0 \cdot \alpha_0) = (q_0')_*((\theta')_0 \cdot (\lambda_i')_0 \cdot \alpha_0) + \mathrm{id}_*(\alpha_0)$ and $d_i\mu_+ = \mu_+ d_i + \mathrm{id}\colon V \to V$, as desired.

The case of $[\mu_-, x_i]$ is very similar; here, we have

$$\begin{aligned} \mu_- x_i(\alpha) &= (p)_*(\kappa \cdot (\iota_i)_*(\alpha)) = (p \circ \widetilde{\iota}_i)_*(\widetilde{\kappa}_i \cdot \alpha)\,, \\ x_i \mu_-(\alpha) &= (\iota_i' \circ p')_*(\kappa' \cdot \alpha)\,. \end{aligned}$$

Under the identification of $H^*\big(X \xrightarrow{\tilde{q}} \mathcal{C}^{[n]}\big)$ with $H^{\mathrm{BM}}_{*+2\dim \mathcal{C}^{[n]}}(X)$, we have $\widetilde{\kappa}_i = [X_i]$. This follows from

$$\widetilde{\kappa}_i \cdot \big[\mathcal{C}^{[n]}\big] = \widetilde{\kappa}_i \cdot \lambda_i \cdot \big[\mathcal{C}^{[n+1]}\big] = \widetilde{\lambda}_i \cdot \kappa\big[\mathcal{C}^{[n+1]}\big] = \widetilde{\lambda}_i \cdot \big[\mathcal{C}^{[n,n+1]}\big] = [X_i]\,,$$

where the last equality is the fact that X_i is a Cartier divisor in $\mathcal{C}^{[n,n+1]}$. Using Lemma 4.3, we get

$$\widetilde{\kappa}_i = [X_i] = (f_i)_*\big[\mathcal{C}^{[n-1,n]}\big] + (g_i)_*\big[\mathcal{C}^{[n]}\big] = (f_i)_*(\kappa') + (g_i)_*\big[\mathcal{C}^{[n]}\big]\,.$$

A computation similar to (4.6) now shows that we have $\mu_{-}x_i(\alpha) = x_i\mu_{-}(\alpha) + \alpha$, as needed, and the restriction to the special fiber works exactly the same way. This finishes the proof of Theorem 3.6 and thus of Theorem 1.1.

5. Example: The node

In this section, we describe the representation V for the curve $\{xy = 0\} \subseteq \mathbb{P}^2_{\mathbb{C}}$, which is the first nontrivial curve singularity with two components.

5.1 Geometric description of $C^{[n]}$

One first thing we may ask is how the components in Proposition 2.7 look. Ran [Ran05b] describes the geometry of the Hilbert scheme of points on (germs of) nodal curves very thoroughly. For $n = 0, 1$, we get a point and C itself, whereas $C^{[2]}$ is a chain of three rational surfaces that intersect their neighbors transversely along projective lines. More generally, $C^{[n]}$ is a chain of $n+1$ irreducible components of dimension n, consecutive members of which meet along codimension one subvarieties.

Lemma 5.1. *Denote by $M_{n,k}$ the irreducible component of $C^{[n]}$, where generically we have k points on the component $y = 0$ of C and $n-k$ points on the component $x = 0$. Then*

$$M_{n,k} \cong \mathrm{Bl}_{\mathbb{P}^{k-1}\times\mathbb{P}^{n-k-1}}\left(\mathbb{P}^k \times \mathbb{P}^{n-k}\right).$$

Proof. First of all, $\mathbb{P}^k \times \mathbb{P}^{n-k}$ has natural coordinates given by the coefficients of polynomials $(a(x), b(y))$ of degrees k and $n-k$. It is also natural to identify the roots of these polynomials with the corresponding subschemes in $C_1, C_2 \cong \mathbb{P}^1$. From $(a(x), b(x))$, we construct an ideal in the homogeneous coordinate ring of C by taking the product

$$I = (y, a(x))(x, b(x)) = (xy, xa(x), yb(y), a(x)b(y)).$$

This determines a length n subscheme, so a point in $M_{n,k}$. Note that this map is invertible outside the locus where we have at least one point from each axis at the origin.

We can further write $a(x)b(y) = a_0b_0 + a_0b'(y) + b_0a'(x) \bmod (xy)$, where $a(x) = a_0 + a'(x)$, $b(y) = b_0 + b'(y)$ and $a'(x)$, $b'(x)$ have no constant term. Now, consider the limit of $I = I_1$ as the products of the coordinates of the roots of $a(x)$ and $b(y)$ separately go to zero linearly; that is, let $t \to 0$ in $a_0 = At$, $b_0 = Bt$ and in the corresponding family of ideals I_t. Since this is a flat family, the limiting ideal $I_0 = \lim_{t\to 0} I_t$ has the same colength and support on the locus where at least one point from each axis is at the origin. In particular, $(a_0b_0 + a_0b'(y) + b_0a'(x))/t \to Ab'(y) + Ba'(x)$ as $t \to 0$, and

$$\lim_{t\to 0} I_t = \left(xy, xa'(x), yb'(y), Ab'(y) + Ba'(x)\right).$$

Since all ideals in the locus of $M_{n,k}$ with at least one point from each axis at the origin can be written in this form and $(A : B) \in \mathbb{P}^1$ determines the limiting ideal completely, we can identify $(A : B)$ with the normal coordinates $(a_0 : b_0)$ and the natural map

$$\pi\colon\ M_{n,k} = \overline{(C_1^{sm})^{[k]} \times (C_2^{sm})^{[n-k]}} \to \mathbb{P}^k \times \mathbb{P}^{n-k}$$

is the blowup along the locus where both $a(x)$ and $b(y)$ have zero as a root.

See also [Ran05a] for a similar blowup description. ☐

The intersections of the components can also be seen in this description.

Lemma 5.2. *We have $E^n_{k,k+1} = M_{n,k} \cap M_{n,k+1} \cong \mathbb{P}^{n-k-1} \times \mathbb{P}^k$, and all the other intersections are trivial.*

Proof. We continue in the notation of the proof of Lemma 5.1. Denote the locus of $M_{n,k}$ where at least one point from either axis is at the origin by $L_{n,k}$. Then, suppose that we are outside $L_{n,k} \cup L_{n,\tilde{k}}$ inside $M_{n,k} \cap M_{n,\tilde{k}}$. Then only one point is at the origin, and this locus is naturally identified with the complement of the corresponding locus in $\mathbb{P}^{n-k-1} \times \mathbb{P}^k$ if $k+1 = \tilde{k}$ and is empty otherwise. We are thus left to studying the loci $L_{n,k}$.

Consider again points $I = (xy, xa'(x), yb'(y)), Ab'(y) + Ba'(x))$ in $L_{n,k}$ and points $\tilde{I} = (xy, x\tilde{a}'(x), y\tilde{b}'(y)), \tilde{A}\tilde{b}'(y) + \tilde{B}\tilde{a}'(x))$ in $L_{n,\tilde{k}}$. First, restrict I to the x-axis; that is, let $y = 0$. Then $I|_{y=0} = (xa'(x), Ba'(y))$. If $B = 0$, this has colength $k+1$ since $a'(x)$ is of degree k. If $B \neq 0$, the colength is k. Similarly, we get the colengths of $\tilde{I}|_{y=0}$ to be $\tilde{k}$ or $\tilde{k}+1$ depending on whether $\tilde{B}$ is nonzero or not. Without loss of generality, we can assume $\tilde{k} > k$. In this case, the only possibility for I and $\tilde{I}$ to be in the intersection $M_{n,k} \cap M_{n,k+1}$ is to have $k+1 = \tilde{k}, B = 0$ and $\tilde{B} \neq 0$. A similar analysis for the y-axis shows that we must have $\tilde{A} = 0$ and $A \neq 0$. So in particular, the intersections $E^n_{k,\tilde{k}}$ are isomorphic to $\mathbb{P}^{n-k-1} \times \mathbb{P}^k$ if $k+1 = \tilde{k}$ and empty otherwise. $\square$

Now, one may compute V. There is a natural stratification of a blowup to the exceptional divisor and its complement. These both come with affine pavings, so a particularly easy way to compute the cohomologies of $C^{[n]}$, or at least the Betti numbers, is to count these cells.

Proposition 5.3. *The bigraded Poincaré series for the space $V = \bigoplus_{n\geqslant 0} H_*\big(C^{[n]}\big)$ is given by*

$$P_V(q,t) = \frac{q^2t^2 - q + 1}{(1-q)^2(1-qt^2)^2}\,.$$

The grading corresponding to t is the homological degree, whereas q keeps track of the grading given by the number of points.

Proof. It is easily confirmed that the Poincaré polynomials of the components are given by

$$P_{M_{n,k}}(t) = t^2\left(\sum_{i=0}^{k-1} t^{2i}\right)\left(\sum_{i=0}^{n-k-1} t^{2i}\right) + \left(\sum_{i=0}^{n-k} t^{2i}\right)\left(\sum_{i=0}^{k} t^{2i}\right).$$

Similarly, the Poincaré polynomials of the intersections are given by

$$Q_{E^n_{k,k+1}}(t) = \left(\sum_{i=0}^{k} t^{2i}\right)\left(\sum_{i=0}^{n-k-1} t^{2i}\right), \quad k \leqslant n-1\,,$$

and by the Mayer–Vietoris sequence, $\sum_{k=0}^{n} P_{M_{n,k}}(t) - \sum_{k=0}^{n-1} Q_{E^n_{k,k+1}}(t)$ is the Poincaré polynomial of $C^{[n]}$. It is easy to see that $\sum_{n\geqslant 0} q^n \left(\sum_{k=0}^{n} P_{M_{n,k}}(t) - \sum_{k=0}^{n-1} Q_{E^n_{k,k+1}}(t)\right) = P_V(q,t)$. $\square$

Figure 1 shows the graded dimension of V as a bigraded vector space.

5.2 Computation of the A-action

We will now investigate the action of the algebra $A = \langle x_1, x_2, \mu_+, \mu_-, d_1, d_2\rangle_{\mathbb{Q}}$ on V.

Consider $V_{\bullet,2n} = \bigoplus_i V_{i,2n} \subset V$, that is, all the classes in homological degree $2n$. Denote by $[M_{n,k}]$ the fundamental class of the irreducible component $M_{n,k}$ of $C^{[n]}$ as described in the previous subsection.

Theorem 5.4. *The fundamental classes $[M_{n,k}] \in V_{n,2n}$ generate $V_{\bullet,2n}$ as a $\mathbb{Q}[x_1,x_2]$-module.*

	0	**2**	**4**	**6**	**8**	**10**	$\cdots$
0	1	0	0	0	0	0	$\cdots$
1	1	2	0	0	0	0	$\cdots$
2	1	3	3	0	0	0	$\cdots$
3	1	4	5	4	0	0	$\cdots$
4	1	5	7	7	5	0	$\cdots$
5	1	6	9	10	9	6	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

FIGURE 1. The dimensions $V_{n,d}$, that is, the Betti numbers of $C^{[n]}$; the columns are labeled by homological degree d and the rows by the number of points n

Proof. This is equivalent to proving that the maps $x_i|_{V_{k,2n}}$ are jointly surjective for $k \geqslant n$. If we dualize the maps to pullbacks $x_i^*|_{V_{k,2n}}$ in cohomology, this condition says that the operators $x_i^*\colon H^{<2k+2}(C^{[k+1]}) \to H^*(C^{[k]})$ must satisfy $\bigcap \ker x_i^* = 0$.

We have the following diagram for the components of $C^{[n]}$ and their intersections:

$$\begin{tikzcd}
& E^n_{0,1} \arrow[dl, hook] \arrow[dr, hook] & & E^n_{1,2} \arrow[dl, hook] \arrow[dr, hook] & & E^n_{2,3} \arrow[dl, hook] \arrow[dr, hook] & \cdots & \\
M_{n,0} & & M_{n,1} & & M_{n,2} & & M_{n,3} & \cdots
\end{tikzcd}$$

Since $C^{[n]}$ is a chain of the components $M_{k,n-k}$ intersecting transversally, without triple intersections, the Mayer–Vietoris sequence in homology for unions splits to short exact sequences:

$$0 \to \bigoplus_{k=0}^{n-1} H_i\big(E^n_{k,k+1}\big) \to \bigoplus_{k=0}^{n} H_i(M_{n,k}) \to H_i\big(C^{[n]}\big) \to 0\,.$$

Dually, we have an exact sequence the other way around in cohomology. By our blowup description of $\pi\colon M_{n,k} \mapsto \mathbb{P}^{n-k} \times \mathbb{P}^k$, we have the following equality of graded vector spaces (see, for example, [GH78, Chapter 6]):

$$H^*(M_{n,k}) = \frac{\pi^* H^*(\mathbb{P}^{n-k} \times \mathbb{P}^k) \oplus H^*(\mathbb{P}(\mathcal{N}_{\mathbb{P}^{n-k}\times\mathbb{P}^k/\mathbb{P}^{n-k-1}\times\mathbb{P}^{k-1}}))}{\pi^* H^*(\mathbb{P}^{n-k-1} \times \mathbb{P}^{k-1})}\,.$$

In particular, we can write

$$H^*\big(\mathbb{P}^{n-k} \times \mathbb{P}^k\big) = \mathbb{Q}[a_{n,k}, b_{n,k}]/\big(a_{n,k}^{n-k+1}, b_{n,k}^{k+1}\big)$$

as well as

$$H^*\big(\mathbb{P}(\mathcal{N}_{\mathbb{P}^{n-k}\times\mathbb{P}^k/\mathbb{P}^{n-k-1}\times\mathbb{P}^{k-1}})\big) = \mathbb{Q}[a'_{n,k}, b'_{n,k}, \zeta_{n,k}]/\big(a'^{\,n-k}_{n,k}, b'^{\,k}_{n,k}, (\zeta_{n,k} - a'_{n,k})(\zeta_{n,k} - b'_{n,k})\big)\,,$$

where $\zeta_{n,k} = c_1(\mathcal{O}(1))$.

Since the classes

$$a''^{\,i}_{n,k} b''^{\,j}_{n,k} \in \pi^* H^*\big(\mathbb{P}^{n-k-1} \times \mathbb{P}^{k-1}\big) \cong \mathbb{Q}[a''_{n,k}, b''_{n,k}]/\big(a''^{\,n-k}_{n,k}, b''^{\,k}_{n,k}\big)$$

are identified with $a^i_{n,k} b^j_{n,k}$ in $\pi^* H^*\big(\mathbb{P}^{n-k} \times \mathbb{P}^k\big)$, where $i < n-k$ and $j < k$, and with $a'^{\,i}_{n,k} b'^{\,j}_{n,k}$ in the exceptional divisor, the quotient map as graded $\mathbb{Q}$-modules is the one identifying $a^i_{n,k} b^j_{n,k}$

with $a'^i_{n,k} b'^j_{n,k}$.

All in all, we can write that, as graded $\mathbb{Q}$-vector spaces,

$$H^*(M_{n,k}) = \left\langle 1, a_{n,k}, b_{n,k}, \ldots, a_{n,k}^{n-k} b_{n,k}^k, \zeta_{n,k}, \ldots, \zeta_{n,k} a_{n,k}^{n-k-1} b^{k-1} \right\rangle_{\mathbb{Q}} . \tag{5.1}$$

Example 5.5. We have $H^*(M_{2,0}) = \operatorname{span}\left\{1, a_{3,0}, a_{3,0}^2\right\}$, as expected, since $M_{2,0} \cong \mathbb{P}^2_{\mathbb{C}}$. We also have $H^*(M_{2,1}) = \operatorname{span}\left\{1, a_{2,1}, a_{2,1}^2, b_{2,1}, a_{2,1}b_{2,1}, a_{2,1}^2 b_{2,1}, \zeta_{2,1}, \zeta_{2,1}a_{2,1}\right\}$.

Having described the cohomology of the components $M_{n,k}$, we can get back to our exact sequence. Identify

$$H^*\left(\mathbb{P}^{n-k-1} \times \mathbb{P}^k\right) \cong \mathbb{Q}[\mu_{n,k}, \nu_{n,k}] / \left(\mu_{n,k}^{n-k}, \nu_{n,k}^{k+1}\right) .$$

Lemma 5.6. *Under the inclusion $E^n_{k,k+1} \hookrightarrow M_{n,k}$, we have $\mu_{n,k} \mapsto a_{n,k} - \zeta_{n,k}$ and $\nu_{n,k} \mapsto b_{n,k} - \zeta_{n,k}$ in cohomology. Similarly, under the inclusion $E^n_{k,k+1} \hookrightarrow M_{n,k+1}$, we have $\mu_{n,k} \mapsto a_{n,k+1} - \zeta_{n,k+1}$ and $\nu_{n,k} \mapsto b_{n,k+1} - \zeta_{n,k+1}$.*

Proof. The class of $\mu_{n,k}$ in the intersection is the class dual to the line $L^y_{n,k}$, where we fix all points in $E^n_{k,k+1}$ at the origin except for one at the y-axis. Similarly, the class of $\nu_{n,k}$ is the line $L^x_{n,k}$, where we have but one point on the x-axis. Under the blowup $\pi_{n,k}\colon M_{n,k} \to \mathbb{P}^{n-k} \times \mathbb{P}^k$, the class of $L^y_{n,k}$ in $M_{n,k}$ is given by the total transform, which satisfies $\left[L^y_{n,k}\right] + \zeta_{n,k} = a_{n,k}$. The computation for the other three cases is nearly identical, and we omit it. $\square$

Example 5.7. When $n = 2$, the Hilbert scheme $C^{[2]}$ has the following components: $M_{2,0} \cong M_{2,2} \cong \mathbb{P}^2$ and $M_{2,1} \cong \mathrm{Bl}_{\mathrm{pt}}\left(\mathbb{P}^1 \times \mathbb{P}^1\right)$. The intersections are $E^2_{0,1} \cong E^2_{1,2} \cong \mathbb{P}^1$. The fundamental class of the first intersection is denoted by $\mu_{2,0}$ and that of the second one is denoted by $\mu_{2,1}$. Under the inclusion $E^2_{0,1} \hookrightarrow M_{2,0}$, the class $\mu_{2,0}$ is identified with $a_{2,0}$, and under the inclusion $E^2_{0,1} \hookrightarrow M_{2,1}$, it is identified with $a_{2,1} - \zeta_{2,1}$. Similarly, under the inclusion $E^2_{1,2} \hookrightarrow M_{2,1}$, the class $\mu_{2,1}$ is identified with $a_{2,1} - \zeta_{2,1}$, whereas under the inclusion $E^2_{1,2} \hookrightarrow M_{2,2}$, it is identified with $a_{2,2}$.

As follows from the definition of the maps $\iota_i\colon C^{[n]} \to C^{[n+1]}$, we can consider them as restricted to $M_{n,k}$. They induce, by abuse of notation, maps in cohomology $x_i^*\colon H^*(M_{n+1,k+i-1}) \to H^*(M_{n,k})$. We can describe these maps explicitly.

Lemma 5.8. *In the basis of (5.1), the map x_1^* is*

$$a^i_{n+1,k} b^j_{n+1,k} \mapsto a^i_{n,k} b^j_{n,k}\,, \quad \zeta_{n+1,k} a^i_{n+1,k} b^j_{n+1,k} \mapsto \zeta_{n,k} a^i_{n,k} b^j_{n,k}\,.$$

Similarly, the map x_2^* is

$$a^i_{n+1,k+1} b^j_{n+1,k+1} \mapsto a^i_{n,k} b^j_{n,k}\,, \quad \zeta_{n+1,k+1} a^i_{n+1,k+1} b^j_{n+1,k+1} \mapsto \zeta_{n,k} a^i_{n,k} b^j_{n,k}\,.$$

Proof. We are adding one fixed smooth point, that is, an embedding $C^{[n]} \hookrightarrow C^{[n+1]}$, as a divisor. When we blow down the components, it is immediate that the a-classes go to the a-classes and the b-classes go to the b-classes. We can treat the classes in the exceptional divisor separately; there, everything again reduces to embedding products of projective spaces as above. In addition, we need that $x_1^* \zeta_{n+1,k+1} = \zeta_{n,k}$, which is saying that the normal bundle of the exceptional divisor of $M_{n+1,k+1}$ restricts to that of the exceptional divisor of $M_{n,k}$ under the embedding $\iota_1\colon M_{n,k} \to M_{n+1,k+1}$. In the notation of Lemma 5.1, on $M_{n,k}$, the map ι_1 is given by multiplying $a(x)$ by $x - c$ for some fixed $c \neq 0$. In particular, the centers of the blowups become identified, and the restriction of the normal bundle of the exceptional divisor of $M_{n+1,k+1}$ is the normal bundle of $M_{n,k}$. $\square$

Having the above lemmas at our hands, we want to prove that the intersections of the kernels of the x_i^* are only the fundamental classes.

The basic object of study here is the commutative diagram

$$\begin{tikzcd} \bigoplus_{k=0}^{n} H^{<2n}(M_{n,k}) \arrow[d] & H^{<2n}\big(C^{[n]}\big) \arrow[l] \arrow[d] \\ \bigoplus_{i=0}^{n-1} H^{*}(M_{n-1,i}) & H^{*}\big(C^{[n-1]}\big) . \arrow[l] \end{tikzcd} \tag{5.2}$$

We can explicitly describe the kernels on the left: for each $M_{n,k}$, only the classes $\zeta_{n,k}a_{n,k}^{k-1}b_{n,k}^{n-k-1}$ are in their intersection. In particular, the intersection of the kernels is nonempty. But this can be remedied on the right, as follows. By Lemma 5.6 and the Mayer–Vietoris sequence, inside the intersection, we can check that

$$\begin{aligned} x_2^*\left(\sum_k \lambda_k\zeta_{n,k}a_{n,k}^{k-1}b_{n,k}^{n-k-1}\right) &= \sum_k \lambda_k x_2^*(\zeta_{n,k})a_{n-1,k}^{k-1}b_{n-1,k}^{n-k-1} \\ &= \sum_k \lambda_k x_2^*(a_{n,k}-a_{n,k+1}+\zeta_{n,k+1})a_{n-1,k}^{k-1}b_{n-1,k}^{n-k-1} \\ &= \sum_k \lambda_k (a_{n-1,k-1}-a_{n-1,k})a_{n-1,k}^{k-1}b_{n-1,k}^{n-k-1} \\ &= \sum_k \lambda_k a_{n-1,k-1}a_{n-1,k}^{k-1}b_{n-1,k}^{n-k-1} , \end{aligned}$$

which is 0 if and only if $\lambda_k = 0$ for all k. Repeating this for x_1^*, we have

$$x_1^*\left(\sum_k \lambda_k\zeta_{n,k}a_{n,k}^{k-1}b_{n,k}^{n-k-1}\right) = \sum_k \lambda_k b_{n-1,k-1}a_{n-1,k}^{k-1}b_{n-1,k}^{n-k-1} = 0$$

if and only if $\lambda_k = 0$ for all k. In particular, we see that the image of the fundamental class of the exceptional divisor is also nonzero; that is, it is not in the kernel and $\bigcap_i \ker x_i^* = 0$ on the right. This finishes the proof of Theorem 5.4. □

Having Theorem 5.4 at our hands, we can finally restate Theorem 1.4.

Theorem 5.9. *Consider the bigraded vector space $V'' = \mathbb{Q}[x_1, x_2, y_1, y_2]$ with x_i in degree $(1,0)$ and y_i in degree $(1,2)$. Consider the action of $A' = \mathbb{Q}\langle x_i, \partial_{y_i}, \sum y_i, \sum \partial_{x_i}\rangle$ on this space as differential operators, and let U be the submodule $\mathbb{Q}[x_1, x_2, y_1+y_2](x_1-x_2)$. Define $V' = V''/U$. Then $V \cong V'$ as A-modules, where $A = \mathbb{Q}[x_1, x_2, d_1, d_2, \mu_+, \mu_-] \cong A'$.*

Proof. That A' is isomorphic to A in this case follows from the commutation relations when we map $x_i \mapsto x_i$, $\partial_{y_i} \mapsto d_i$ and $\sum y_i \mapsto \mu_+$, $\sum \partial_{x_i} \mapsto \mu_-$. We can identify V' and V as A-modules by letting the monomial $y_1^i y_2^j/i!j!$ correspond to the fundamental class of $M_{i+j,i}$. It is then clear that on the diagonal $\bigoplus_{n\geqslant 0} V_{n,2n}$, the operators d_1, d_2, μ_+, μ_- act as the corresponding differential operators in A'. Namely, the Gysin maps d_1 and d_2 are given by intersection, from which it follows that $d_1[M_{i+j,i}] = [M_{i+j-1,i-1}]$ and $d_2[M_{i+j,i}] = [M_{i+j-1,i}]$. This can be compared to the fact that, for example, $\partial_{y_1} y_1^i y_2^j/i!j! = y_1^{i-1}y_2^j/(i-1)!j!$.

By the commutation relations, $[d_1^{i+1}, \mu_+] = (i+1)d_1^i$, so

$$[d_1^{i+1}, \mu_+]\, y_1^i y_2^j/i!j! = (i+1)y_2^j/j!\,, \quad [d_2^{j+1}, \mu_+]\, y_1^i y_2^j/i!j! = (j+1)y_1^i/i!$$

and, in particular,

$$d_1^{i+1}\mu_+ y_1^i y_2^j/i!j! = (i+1)y_2^j/j!\,, \quad d_2^{j+1}\mu_+ y_1^i y_2^j/i!j! = (j+1)y_1^i/i!\,.$$

Since $\mu_+ y_1^i y_2^j/i!j! = \sum_{k=0}^{i+j+1} c_k y_1^k y_2^{i+j+1-k}$ for some constants c_k, we must have $c_k = 0$ unless $k = i$ or $k = i+1$, in which case we have $c_k = 1/i!j!$. This shows that μ_+ can be identified with multiplication by $y_1 + y_2$ on the diagonal; the same holds below the diagonal since μ_+ commutes with the action of x_1 and x_2. On the diagonal, the operator μ_- acts as zero by degree reasons. An argument similar to the above shows that below the diagonal, μ_- acts by $\partial_{x_1} + \partial_{x_2}$.

Since the maps x_i are jointly surjective on the rows, by Theorem 5.4, we get a surjection $\phi\colon \mathbb{Q}[x_1, x_2, y_1, y_2] \twoheadrightarrow V$. This is an A-module homomorphism, by the above. Its kernel contains U since $(x_1 - x_2)\cdot 1 = 0$ and the actions of x_i and μ_+ commute with the x_i.

Then, consider the graded dimensions/Poincaré series of V' and V. We have

$$P_{V'}(q,t) = P_{V''}(q,t) - P_U(q,t) = \frac{1}{(1-q)^2(1-qt^2)} - \frac{q(1-qt^2)}{(1-q)^2(1-qt^2)^2} = P_V(q,t)\,,$$

and since $\ker\phi \supseteq U$, we must have $\ker\phi = U$. □

Acknowledgements

I would like to thank my advisor Eugene Gorsky for his keen interest in this work and his continuing support, as well as for suggesting to study homological correspondences on Hilbert schemes. In addition, I thank Eric Babson, Brian Osserman, Vivek Shende and Zhiwei Yun for helpful discussions.

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Oscar Kivinen okivinen@ucdavis.edu
Department of Mathematics, University of California, Davis, One Shields Avenue, 95616 Davis, CA, USA