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Explicit Schoen surfaces

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Abstract

We give an explicit construction for the 4-dimensional family of Schoen surfaces by computing equations for their canonical images, which are 40-nodal complete intersections of a quadric and the Igusa quartic in $\mathbb{P}^4$. We then study a particularly interesting example, with 240 automorphisms and maximal Picard number.

1. Introduction

While working on a problem related to the Hodge conjecture, Schoen [Sch07] used deformation theory to construct a family of surfaces S (from now on called *Schoen surfaces*) with some interesting features. The one that has interested Schoen most is that the Albanese map of S embeds it into its Albanese variety A, but for a generic Schoen surface, the cycle S in $H^4(A,\mathbb{Q})$ is not contained in the subspace generated by the intersections of two divisors of A; the existence of such cycles on an abelian variety A makes the Hodge conjecture more difficult to prove.

Another interesting property of S is that the natural map

$$\wedge^2 H^0(S,\Omega_S) \to H^0(S,K_S)$$

has a 1-dimensional kernel, and therefore S is a Lagrangian surface in A. Moreover, the kernel is not of the form $\omega_1\wedge\omega_2$, with $\omega_i \in H^0(S,\Omega_S)$; therefore, by the Castelnuovo–De Franchis theorem, the surface does not admit fibrations onto a curve of genus at least 2. Only a few Lagrangian surfaces without a fibration onto a curve of higher genus are known (see [BNP07, BPS10, BT00]). Such examples are interesting for people studying Kählerian groups; for example, one can ask whether their fundamental group is nilpotent (cf. [Cam95]).

By [Bea79], when the canonical map of a surface of general type has degree at least 2 onto a surface, that surface either has $p_g = 0$ or is itself canonically embedded, the latter case being rather exceptional (see [CPT03] for a list of the examples known so far). In [CMR15], Ciliberto, Mendes Lopes and the second author studied Schoen surfaces geometrically, proving that the canonical map of a Schoen surface S is 2-to-1 onto a 40-nodal, degree 8, complete intersection surface $X_{40} \subset \mathbb{P}^4$ and the ramification of the double cover $S \to X_{40}$ is the set of 40 nodes. They also show that Schoen surfaces are not universally covered by the bidisk (very few surfaces with $K^2 = 8\chi$ and this property are known).

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Miyaoka's bound tells us that on a degree 8 complete intersection surface in $\mathbb{P}^4$, there cannot be more than 40 nodes. The construction of Schoen surfaces gives the first theoretical proof that such a 40-nodal surface exists, but without providing any equations for it. In [Bea13], Beauville used Schoen surfaces in order to show the existence of 48-nodal degree 16 complete intersection surfaces X_{48} in $\mathbb{P}^6$ and surfaces $\tilde{S}$ whose canonical map is 2-to-1 onto X_{48}.

The main result of this paper is an explicit construction of the surfaces X_{40} by equations, and therefore an explicit construction of S by double cover. The idea of the construction of the surfaces X_{40} is the following. The Igusa quartic threefold $I_4 \subset \mathbb{P}^4$ is singular along 15 lines of A_1-singularities. Its intersection with a generic quadric thus gives a degree 8 complete intersection surface containing 30 nodes. The main question is therefore to find a quadric Q_2 which, while still transversal to the 15 singular lines, is tangent to the Igusa quartic at 10 more points, leading to a 40-nodal surface $X_{40} := I_4 \cap Q_2$. Our construction is very concrete since we have the explicit equation for the quadric Q_2, whose coefficients depend on four parameters.

In order to obtain that result, we use the knowledge of the above mentioned papers, computer algebra and the rich geometry of the Igusa quartic threefold and of its dual, the Segre cubic threefold $S_3 \subset \mathbb{P}^4$, which is the unique cubic threefold with 10 nodes. It is well known that the Igusa quartic threefold parametrizes quartic Kummer surfaces [Hun96, Theorem 3.3.8]: taking the intersection of I_4 by a hyperplane T_x tangent to a generic point $x \in I_4$, one obtains a 16-nodal Kummer surface, 15 nodes coming from the intersection of the 15 lines in I_4 with T_x, and one more node at x. Our construction of X uses the 15-nodal $K3$ surfaces obtained as the intersections of I_4 with a generic hyperplane, giving new interesting geometric features to the Igusa quartic I_4.

We then study a Schoen surface with a large group of symmetries (of order 240). We compute the isogeny class of its Albanese variety, and we prove that it has maximal Picard number. Although interesting, examples of surfaces with maximal Picard number are rather scarce; see, for example, [Bea14].

As a by-product of our work, we also obtain a geometric construction of a 3-dimensional subfamily of Schoen surfaces, as a bidouble cover of some particular Kummer surfaces. This construction interestingly matches some (theoretical) constructions of Lagrangian surfaces suggested by Bogomolov and Tschinkel in [BT00] (see Remark 13).

Since 15-nodal quartic surfaces, obtained as generic hyperplane sections of the Igusa quartic, play a key role in our construction, one may ask if an analogous construction could be done using a different family of 15-nodal quartics. The answer is negative: we show in the appendix that a generic quartic surface with 15 nodes can be realized as a hyperplane section of the Igusa quartic threefold.

The paper is organized as follows. In Section 2, we recall some known facts on Schoen surfaces. In Section 3, we construct the 40-nodal degree 8 surfaces in $\mathbb{P}^4$, we prove that their sets of 40 nodes are 2-divisible, their associated double covers are not universally covered by the bidisk and that they are Schoen surfaces. In Section 4, we study an example of a Schoen surface with a large automorphism group. The appendix is on the moduli of K3 surfaces with 15 nodes.

Notation. We work over the complex numbers. All varieties are assumed to be projective algebraic. For a smooth surface S, as usual, K_S is the canonical class, $p_g(S) := h^0(S, K_S)$ is the geometric genus, $q(S) := h^1(S, K_S)$ is the irregularity, and $\chi(\mathcal{O}_S) = 1 - q + p_g$ is the holomorphic

Euler characteristic. A $(-n)$-curve on a surface is a curve isomorphic to $\mathbb{P}^1$ with self-intersection $-n$. Linear equivalence of divisors is denoted by $\equiv$.

2. Schoen surfaces

Let C be a smooth genus 2 curve with Jacobian $J(C)$, and consider the union

$$V := C \times C \cup_C J(C)$$

glued along the diagonal of $C \times C$ and $C \hookrightarrow J(C)$. Notice that V is singular along C.

Theorem 1 ([Sch07]). *The reducible surface V can be deformed into a smooth surface of general type S with invariants*

$$c_1^2 = 16 = 2c_2\,, \quad q = 4\,, \quad p_g = 5\,.$$

The moduli space of these surfaces is 4-dimensional. The deformation space is locally smooth; thus, it is locally irreducible.

As said in the introduction, the canonical map of a Schoen surface S is of degree 2 onto a 40-nodal complete intersection X of a quadric and a quartic in $\mathbb{P}^4$. From [CMR15, Lemmas 4.5 and 4.6 and proof of Theorem 4.1], we deduce the following.

Proposition 2 ([CMR15]). *The above surfaces X degenerate to the union of a double quadric surface and a quartic Kummer surface, glued along a trope of the Kummer surface. These surfaces are given by the intersection of two hyperplanes and a quartic hypersurface in $\mathbb{P}^4$. Moreover, this degeneration induces the degeneration in Schoen's construction.*

3. The construction

In this section, we show the following.

Theorem 3. *Let I_4 be the Igusa quartic in $\mathbb{P}^4$. There exists a quadric on four parameters $Q_{a,b,c,d}$ such that, for generic values of the parameters, the surface*

$$X_{40} := I_4 \cap Q_{a,b,c,d}$$

has exactly 40 nodes. These nodes are 2-divisible in the Picard group, and the double cover $S \to X_{40}$ ramified over the nodes is a Schoen surface.

We explain how to compute the quadric $Q_{a,b,c,d}$. The corresponding computer code, implemented with Magma, is available at [Rit17].

3.1 Segre cubic, Igusa quartic

The linear system L of quadrics through points $p_1, \dots, p_5 \in \mathbb{P}^3$ in general position (that is, no four of them are contained in a hyperplane) is 4-dimensional. Let $\phi\colon \mathbb{P}^3 \dashrightarrow \mathbb{P}^4$ be the rational map corresponding to the linear system L on $\mathbb{P}^3$. The image $S_3 := \phi(\mathbb{P}^3)$ is the *Segre cubic*, the unique cubic threefold in $\mathbb{P}^4$ (up to projective equivalence) with singular set the union of 10 nodes (the images of the lines p_ip_j). The Segre cubic contains 15 planes: the "images" (after blowing up $\mathbb{P}^3$) of $p_1, \dots, p_5$ and of the 10 planes in $\mathbb{P}^3$ through exactly three of the points p_i. The dual variety I_4 (the image under the gradient map) of S_3 is the *Igusa quartic*. The dual map contracts the above 15 planes to singular lines of I_4, its singular set. The Igusa quartic has

10 *tropes*, that is, 10 hyperplane sections which are double quadrics. For more details, see, for example, [Hun96] or [Dol12].

3.2 The 40-nodal surface

Let $H \subset \mathbb{P}^4$ be a generic hyperplane. Then $Q_{15} := I_4 \cap H \subset \mathbb{P}^3$ is a quartic surface with 15 nodes. Since H is generic, we can choose five nodes $p_1, \dots, p_5$ in general position (that is, no four of them are in a hyperplane). Consider the map $\phi\colon H \dashrightarrow \mathbb{P}^4$ given by the linear system $|L|$ of quadrics which pass through $p_1, \dots, p_5$.

Proposition 4. *There exists a quadric $S_2 \subset \mathbb{P}^4$ such that*

$$Q_{10} := \phi(Q_{15}) \cong S_3 \cap S_2$$

and Q_{10} has 10 nodes which are disjoint from the nodes of S_3.

Proof. We have $\phi(H) \cong S_3$, and ϕ is of degree 1 outside of the lines p_ip_j; therefore, ϕ sends Q_{15} birationally to a surface Q_{10} contained in S_3, a 10-nodal $K3$ surface. Now consider the resolution of singularities $\widetilde{Q}_{15} \to Q_{15}$, and let $|L'|$ be the strict transform of $|L|$ in $\widetilde{Q}_{15}$. Since $L'^2 = 6$ and $|L'|$ has no base points, [Sai74, Theorem 6.1] implies that Q_{10} is a complete intersection of a quadric S_2 and a cubic in $\mathbb{P}^4$. This cubic can be assumed to be S_3 because $Q_{10} \subset S_3$.

The second assertion follows from the fact that the 10 nodes of Q_{10} correspond to the nodes of Q_{15} disjoint from the 10 lines p_ip_j for $i, j \in \{1, \dots, 5\}$, which are contracted to the nodes of S_3. □

Using Magma, we compute this 4-dimensional family of smooth quadrics S_2 (see [Rit17, Section A]). The dual of S_2 is a smooth quadric Q_2. Consider the dual maps

$$d_1\colon S_3 \dashrightarrow I_4\,,$$
$$d_2\colon S_2 \dashrightarrow Q_2\,,$$

and define $X_{40} := I_4 \cap Q_2$. Since S_2 is tangent to S_3 at 10 smooth points of S_3 and duality preserves tangencies, X_{40} has at least 10 singular points. The purpose of this construction is to find Q_2 meeting the 15 singular lines of I_4 transversally, so that X_{40} is a 40-nodal surface. We show below that, up to the symmetry of I_4, at most one choice of the nodes $p_1, \dots, p_5$ serves our aims. Notice that the quartic surface Q_{15} has 10 tropes (double conics), which are induced by the tropes of I_4.

Proposition 5. *If exactly three of the nodes $p_1, \dots, p_5$ are in a trope of Q_{15}, then the surface X_{40} is non-normal.*

Proof. Let T be the hyperplane of H which gives the trope of Q_{15} containing 3 of the nodes $p_1, \dots, p_5$. Since ϕ is injective outside the union of the lines p_ip_j, one has $\phi(T \cap Q_{15}) = \phi(T) \cap S_2$. Let C be the conic such that $T \cap Q_{15} = 2C$. When we blow up $\mathbb{P}^3$ at $p_1, \dots, p_5$, the strict transform of a quadric in $|L|$ meets the strict transform of C at exactly one point. This implies that the image $\phi(C)$ is a line. Then $\phi(T) \cap S_2$ is a double line and $\phi(T)$ is a plane in S_3, as stated in Section 3.1. When taking the duals, this plane is contracted to a singular line of I_4 (see, for example, [Hun96, Section 3.3.4]) and the quadric Q_2 contains this line. This implies that the singular set of X_{40} is of dimension 1. □

Therefore, a surface X_{40} is normal only if no three of the nodes $p_1, \dots, p_5$ are in a trope of Q_{15}. We compute that there are exactly six such sets of nodes (see [Rit17]). The 10 tropes of I_4 induce 10 tropes of Q_{15}, so we compute the sets of five singular lines of I_4 such that there is no trope containing three of them. Fixing one of these sets (the S_6 symmetry of I_4 gives the remaining sets), we have a choice of five nodes $p_1, \dots, p_5$ for each surface Q_{15}. The computations with Magma available in [Rit17] confirm that a generic surface X_{40} constructed as above has 40 nodes and no other singularities. Our computations are optimal in the sense that we construct the entire family at once: the output is a quadric on four parameters $Q_{a,b,c,d}$ such that, for generic values of the parameters, the surface $I_4 \cap Q_{a,b,c,d}$ has exactly 40 nodes.

$$\begin{array}{ccccccc} & & S_3 & \overset{d_1}{- - \to} & I_4 & & \\ I_4 \cap H_{a,b,c,d} & \overset{|L|}{- - - \to} & \supset & & \supset & =: X_{40} \\ & & S_2 & \overset{d_2}{- - \to} & Q_{a,b,c,d} & & \end{array}$$

Figure 1. Construction of X_{40}, where the symbol $\supset$ means intersection

3.3 2-divisibility of the nodes

In the previous section, we proved that with a generic point t in $\left(\mathbb{P}^4\right)^*$ (the dual space of $\mathbb{P}^4$), one can associate a surface X_t, the minimal resolution of a 40-nodal surface $X_{40}(t)$ in $\mathbb{P}^4$. Let o be a point in $\left(\mathbb{P}^4\right)^*$ such that the surface X_o is defined. There exist a small polydisk B centered at o, a smooth complex manifold $\mathcal{X}$ and a proper flat morphism $\mathcal{X} \to B$ such that the fiber at $t \in B$ is X_t. There are moreover 40 divisors $D_1, \dots, D_{40}$ on $\mathcal{X}$ such that the intersections of these divisors with X_t are the forty (-2)-curves of X_t over nodes in $X_{40}(t)$.

Proposition 6. *Suppose that the sum of the forty (-2)-curves on X_o is 2-divisible. Then, up to shrinking B, the sum of the forty (-2)-curves on the surface X_t $(t \in B)$ is 2-divisible.*

Proof. Let $\mathcal{L} = \mathcal{O}_{\mathcal{X}}(\sum D_i)$. By hypothesis, there exists a line bundle $\mathcal{M}_o$ on X_o such that $\mathcal{L}_{|X_o} = \mathcal{M}_o^{\otimes 2}$. The results follows from [CMR15, Lemma 2.2]. □

According to our computations (see [Rit17, Section B]), one of the 40-nodal surfaces constructed above is projectively equivalent to the Σ_5-invariant surface $\overline{X}_{40}$ given in $\mathbb{P}^5$ by

$$\begin{gathered} x+y+z+w+t+h=0\,, \\ 5\bigl(x^2+\cdots+t^2\bigr)-7(x+\cdots+t)^2=0\,, \\ 4\bigl(x^4+\cdots+t^4+h^4\bigr)-\bigl(x^2+\cdots+t^2+h^2\bigr)^2=0\,. \end{gathered}$$

Proposition 7. *The nodes of $\overline{X}_{40}$ are 2-divisible.*

Proof. One can verify that $\overline{X}_{40}$ has a $(40,12)$ configuration: 40 tropes and 40 nodes, each trope contains 12 nodes, through each node pass 12 tropes. Using Magma (see [Rit17, Section C]), we show the existence of tropes $T_1, \dots, T_4$ such that

- we have $T_i = 2C_i$, with C_2, C_3, C_4 smooth and C_1 the union of two conics;
- the singular points of C_1 are not in $C_2 \cup C_3 \cup C_4$;
- $C_1 \cup C_2$ contains exactly 20 nodes of $\overline{X}_{40}$ which are not in $C_1 \cap C_2$;
- $C_3 \cup C_4$ contains exactly 20 nodes of $\overline{X}_{40}$ which are not in $C_3 \cap C_4$;
- the above two sets of 20 nodes are disjoint.

Let $\widehat{X}_{40}$ be the smooth minimal model of $\overline{X}_{40}$. Denote by $\widetilde{T}_i$ the total transform of T_i in $\widehat{X}_{40}$ and by $\widehat{C}_i$ the strict transform of C_i in $\widehat{X}_{40}$, for $i = 1, \ldots, 4$. There are (-2)-curves $A_1, \ldots, A_{22} \subset \widehat{X}_{40}$ such that

$$\widetilde{T}_1 2\widehat{C}_1 + \sum_1^{10} n_i A_i + n_{21}A_{21} + n_{22}A_{22}\,,$$

$$\widetilde{T}_2 2\widehat{C}_2 + \sum_{11}^{20} n_i A_i + n'_{21}A_{21} + n'_{22}A_{22}$$

for some integers $n_1, \ldots, n_{22}, n'_{21}, n'_{22}$. From $0 = \widetilde{T}_j A_i = 2\widehat{C}_j A_i - 2n_i = 2 - 2n_i$, we get $n_i = 1$, $i = 1, \ldots, 22$, and also $n'_{21} = n'_{22} = 1$. So, we have

$$2\widetilde{T} \equiv \widetilde{T}_1 + \widetilde{T}_2 = 2\widehat{C}_1 + 2\widehat{C}_2 + \sum_1^{20} A_i + 2A_{21} + 2A_{22}\,,$$

where $\widetilde{T}$ is the pullback of a general hyperplane section of $\overline{X}_{40}$. This shows that $C_1 \cup C_2$ contains 20 nodes of $\overline{X}_{40}$ which are 2-divisible. Analogously, the remaining 20 nodes of $\overline{X}_{40}$, contained in $C_3 \cup C_4$, are also 2-divisible. □

Proposition 8. *The 40 nodes of a generic surface X_{40} are 2-divisible.*

Proof. This is immediate from Propositions 6 and 7. □

3.4 Surfaces with $K_S^2 = 2c_2 = 16$ and $q = 4$

From the previous section, for a surface X_{40} with exactly 40 nodes as constructed above, there is a double covering $\pi \colon S \to X_{40}$ ramified over the nodes.

Proposition 9. *We have $p_g(S) = 5$, $q(S) = 4$ and $K_S^2 = 16$.*

Proof. The canonical line bundle K_S is the pullback of $K_{X_{40}}$, which is nef; thus, S is minimal and $K_S^2 = 2K_{X_{40}}^2 = 16$.

Let $\widehat{X}_{40}$ be the smooth minimal model of X_{40}, let $A_1, \ldots, A_{40}$ be the (-2)-curves which contract to the nodes of X_{40}, and let $S' \to \widehat{X}_{40}$ be the double covering with branch locus $\sum_1^{40} A_i$. The minimal model of S' is isomorphic to S.

Let L be the divisor such that $\sum_1^{40} A_i \equiv 2L$. The double covering formulas (see, for example, [BHPvdV04, Section V.22]) give

$$\chi(S) = 2\chi\big(\widehat{X}_{40}\big) + \frac{1}{2}L\big(K_{\widehat{X}_{40}} + L\big) = 12 - 10 = 2\,.$$

Let us compute $p_g(S)$. We have $p_g(S) \geqslant p_g\,(X_{40}) = 5$, thus $q(S) \geqslant 4$. Suppose $q(S) \geqslant 5$. We know from [Deb82, Appendice] that one always has $p_g(S) \geqslant 2q(S) - 4$, with equality only if S is the product of a curve of genus 2 and a curve of genus $q(S) - 2 \geqslant 2$. Thus $p_g(S) = q(S) + 1$ implies that $q(S) = 5$ and $p_g(S) = 6$, and S is the product of a genus 2 curve with a genus 3 curve. The restriction of the canonical map of S to a genus 2 fiber F is a map of degree 2 to $\mathbb{P}^1$, the canonical map of F. Hence, the map $\pi|_F$ is of degree at least 2 to $\mathbb{P}^1$. This gives a contradiction because, since X_{40} is of general type, it is not a ruled surface. □

Proposition 10. *The surface S is not covered by the bidisk $\mathbb{H} \times \mathbb{H}$.*

Proof. If S is universally covered by $\mathbb{H}\times\mathbb{H}$, then it is the quotient of $\mathbb{H}\times\mathbb{H}$ by a discrete cocompact subgroup Γ of $\mathrm{Aut}(\mathbb{H}\times\mathbb{H})=\mathrm{Aut}(\mathbb{H})^2\rtimes(\mathbb{Z}/2\mathbb{Z})$ acting freely. Let

$$\Gamma_0 := \Gamma\cap\mathrm{Aut}(\mathbb{H})^2\,,$$

and let Γ_0' and Γ_0'' be the projections of Γ_0 to the factors of $\mathrm{Aut}(\mathbb{H})\times\mathrm{Aut}(\mathbb{H})$. By [Shi63, Theorem 1], if one of Γ_0' and Γ_0'' is discrete, so is the other. In that case, we say that Γ is *reducible*.

If Γ is irreducible, we know from [MS63, Introduction] that

$$b_1(\mathbb{H}\times\mathbb{H}/\Gamma_0)=b_1\big(\mathbb{P}^1\times\mathbb{P}^1\big)=0\,.$$

This is impossible because $2q=b_1$ and $q(S)=4$.

So, Γ is reducible, and then Γ_0 is a finite index subgroup of $\Gamma_0'\times\Gamma_0''$. It follows that $\mathbb{H}\times\mathbb{H}/\Gamma_0$ is a covering of the product of two curves $\mathbb{H}/\Gamma_0'\times\mathbb{H}/\Gamma_0''$. We *claim* that S is *isogenous* to a product of curves (that is, it is a quotient of a product of curves by a fixed-point-free group action). In fact, there exists a normal sublattice Γ_1 of Γ_0, of finite index, of the form

$$\Gamma_1'\times\Gamma_1''\subset\Gamma_0\subset\Gamma_0'\times\Gamma_0''\,.$$

This implies the existence of an étale map

$$\mathbb{H}\times\mathbb{H}/\Gamma_1=\mathbb{H}/\Gamma_1'\times\mathbb{H}/\Gamma_1''\longrightarrow\mathbb{H}\times\mathbb{H}/\Gamma_0\,,$$

the action being given by Γ_0/Γ_1.

Surfaces with $p_g=5$ and $q=4$ isogenous to a product of curves are classified in [BNP07]. They are of the form $(C\times H)/(\mathbb{Z}/2\mathbb{Z})$, where

a) C and H are curves of genus 3 with fixed-point-free involutions, or

b) C is a curve of genus 5 with a fixed-point-free involution and H is a bielliptic curve of genus 2.

We know from [Pol06, Theorem 3.4] that the curves in case a) are hyperelliptic; hence, in both cases, the canonical map factors through a double covering of a ruled surface, and then the canonical image is not of general type. This implies that S is not isogenous to a product of curves, giving a contradiction. □

3.5 The degeneration

Proposition 11. *The family of surfaces S constructed above coincides with the family of surfaces constructed by Schoen in* [Sch07].

Proof. The deformation space in Schoen's construction is locally irreducible (see Theorem 1); hence, we get from Proposition 2 that it suffices to show that the 4-dimensional family of surfaces X_{40} degenerates to a 3-dimensional family of reducible surfaces which are each the union of a double quadric surface and a quartic Kummer surface, glued along a trope of the Kummer surface.

Let us consider the Igusa quartic given by the equation

$$4\big(x^4+y^4+z^4+w^4+t^4+h^4\big)=\big(x^2+y^2+z^2+w^2+t^2+h^2\big)^2\,,$$

where $h:=-x-y-z-w-t$, in $\mathbb{P}^4(x,y,z,w,t)$. Recall from Section 3.2 that to a generic hyperplane section $H_{a,b,c,d}:=ax+by+cz+dw-t$ of the Igusa quartic corresponds a quadric $Q_{a,b,c,d}$ such that $X_{40}:=I_4\cap Q_{a,b,c,d}$ (notice that the two I_4 that appear in Figure 1 denote isomorphic surfaces that are given by different equations in our computations). Let

$$F_{a,b,c,d} = c_1 x^2 + \cdots + c_{15} wt\,, \quad c_i = c_i(a,b,c,d)\,,$$

be the defining polynomial of $Q_{a,b,c,d}$ in $\mathbb{P}^4$. The correspondence $H_{a,b,c,d} \mapsto Q_{a,b,c,d}$ is a rational map

$$\varphi\colon \mathbb{A}^4 \dashrightarrow \mathbb{P}^{14} = \mathbb{P}\big(H^0\big(\mathbb{P}^4, \mathcal{O}(2)\big)\big)\,, \quad (a,b,c,d) \longmapsto (c_1 : \cdots : c_{15})\,.$$

From our computations (see [Rit17]), if $H_{a,b,c,d}$ gives a trope of I_4, then $F_{a,b,c,d}$ vanishes identically. This happens, for instance, for $(a,b,c,d) = (0,0,-1,-1)$. We resolve the corresponding indeterminacy of φ by blowing up: locally, this is done by evaluating the coefficients c_i at $(a, ab, ac-1, ad-1)$. The computations give that

$$F_{a,ab,ac-1,ad-1} = a^3 \cdot G_{a,b,c,d}$$

with G of degree 2. Moreover, there exists a linear form $J_{b,c,d}$ such that

$$G_{0,b,c,d} = x \cdot J_{b,c,d}\,.$$

The hyperplane $\{x = 0\}$ gives a trope of I_4 (a double quadric). For generic values of the parameters, the hyperplane given by $J_{b,c,d}$ is tangent to I_4 at a point (it gives a quartic Kummer surface), and the quadric given by $G_{a,b,c,d}$ meets I_4 at a 40-nodal surface. One can verify from the equations obtained in [Rit17] that this quadric and the Kummer surface are glued along a trope of the Kummer surface. □

4. The surface $\overline{X}_{40}$ with Σ_5-symmetries

In this section, we study a surface $\overline{S}$ which is the double cover of a particular 40-nodal, degree 8, complete intersection surface with a large group of symmetries. Using these symmetries, we prove that its Picard number is maximal, and we find the isogeny class of its Albanese variety. We moreover describe another construction of a 3-dimensional subfamily of Schoen surfaces given as bidouble covers of some special Kummer surfaces.

4.1 Some Schoen surfaces as bidouble covers

Recall from Section 3.3 the complete intersection $\overline{X}_{40} \subset \mathbb{P}^4$ of the following quadric and quartic:

$$5\big(x^2 + \cdots + t^2\big) - 7(x + \cdots + t)^2 = 0\,,$$
$$4\big(x^4 + \cdots + t^4 + h^4\big) - \big(x^2 + \cdots + t^2 + h^2\big)^2 = 0\,,$$

where $h = -(x+y+z+w+t)$. The surface $\overline{X}_{40}$ has 40 nodes (defined over the field $\mathbb{Q}(\sqrt{-15})$). The permutation group Σ_5 is a subgroup of $\mathrm{Aut}(\overline{X}_{40})$, the automorphism group of $\overline{X}_{40}$.

Let $\overline{S} \to \overline{X}_{40}$ be the double cover branched over the 40 nodes, and let σ be the corresponding involution of $\overline{S}$. Let $\hat{X}_{40}$ be the minimal resolution of $\overline{X}_{40}$. By the argument given in the proof of Proposition 6, the integral cohomology group $H^2(\hat{X}_{40}, \mathbb{Z})$ is torsion free. Thus, the Néron–Severi group $\mathrm{NS}(\hat{X}_{40})$, which is a subgroup of $H^2(\hat{X}_{40}, \mathbb{Z})$, is also torsion free. We also note that any automorphism in Σ_5 preserves the set of nodes. Then by Theorem 1e) in Section 1.3 of R. A. Livne's Ph.D. thesis, Harvard, 1981, each element of Σ_5 lifts to an automorphism of $\overline{S}$.

Proposition 12. *Let $\tau \in \Sigma_5$ be a transposition. The quotient surface $Q := \overline{X}_{40}/\tau$ is a K3 surface with 15 nodes containing in the smooth locus two (-2)-curves A_{16} and A'_{16} such that $A_{16}A'_{16} = 10$. The double cover $\overline{X}_{40} \to Q$ is branched over $A_{16} + A'_{16}$.*

Let $A_1, \dots, A_{15}$ be the fifteen (-2)-curves in the resolution $\hat{Q}$ of Q. The divisors $A_{16}+\sum_{i=1}^{15} A_i$ and $A'_{16} + \sum_{i=1}^{15} A_i$ are 2-divisible. The bidouble cover $\hat{S} \to \hat{Q}$ associated with the divisors

$$D_1 = \sum_{i=1}^{15} A_i \,, \quad D_2 = A_{16} \,, \quad D_3 = A'_{16}$$

gives the blow-up $\hat{S} \to \overline{S}$ at the 40 fixed points of σ; the bidouble cover decomposes as

$$\begin{array}{ccccc} & & \hat{S} & & \\ & \swarrow & \downarrow & \searrow & \\ \hat{B}_1 & & \hat{X}_{40} & & \hat{B}_2 \\ & \searrow & \downarrow & \swarrow & \\ & & \hat{Q} \,, & & \end{array}$$

where $\hat{B}_1$ and $\hat{B}_2$ are Abelian surfaces B_1 and B_2 blown up at their 2-torsion points, each map $\hat{S} \to \hat{B}_i$ is a double cover branched over a curve of genus 4 and the maps $\hat{B}_i \to \hat{Q}$ for $i = 1, 2$ are branched over $D_1 + D_2$, and $D_1 + D_3$, respectively.

The group generated by the lifts of τ on $\overline{S}$ is $(\mathbb{Z}/2\mathbb{Z})^2$, and it contains σ.

Proof. Let $\tau \in \Sigma_5$ be a transposition (for example, the one exchanging the coordinates x and y). Using Magma, we compute that the fixed-point set of τ is a union of two smooth genus 0 curves meeting at 10 points which are 10 nodes of $\overline{X}_{40}$. Moreover, the quotient of the surface $\overline{X}_{40}$ by τ is a quartic $K3$ surface $Q \hookrightarrow \mathbb{P}^3$ which has 15 nodes (see [Rit17, Section D]).

The image of the fixed-point set of τ by the quotient map is $A_{16} + A'_{16}$, where A_{16} and A'_{16} are two (-2)-curves which are disjoint from the 15 nodes and meet transversally at 10 points. It is the intersection of Q with a quadric in $\mathbb{P}^3$.

Let $A_1, \dots, A_{15}$ be the fifteen (-2)-curves above the nodes on the minimal resolution $\hat{Q}$ of Q. Let us keep the same notation for the strict transforms of A_{16} and A'_{16} on $\hat{Q}$. The 16 curves $A_1, \dots, A_{15}, A_{16}$ are disjoint, and so are the 16 curves $A_1, \dots, A_{15}, A'_{16}$. Thus by [Nik75, Theorem 1], the divisors $A_{16} + \sum_{i=1}^{15} A_i$ and $A'_{16} + \sum_{i=1}^{15} A_i$ are 2-divisible. Using the 3 divisors D_1, D_2, D_3, the associated bidouble cover $\hat{S} \to \hat{Q}$ gives the blow-up of $\overline{S}$ at the 40 fixed points of σ (see [Par91] or [Cat99] for information on bidouble covers); the remaining assertions follow. $\square$

Remark 13. More generally, one can prove that there exists a 3-dimensional family of quartic $K3$ surfaces with 15 nodes, containing on their smooth locus two (-2)-curves A_{16} and A'_{16} such that $A_{16}A'_{16} = 10$ (cf. [Rem07, Pia23]). Their associated bidouble covers as above give a 3-dimensional subfamily of Schoen surfaces.

It is interesting to compare this construction of Schoen surfaces by bidouble covers with the construction of Lagrangian surfaces done by Bogomolov and Tschinkel in [BT00, Sections 3 and 4].

4.2 The 240 automorphisms of $\overline{S}$

We will use standard results in representation theory for which we refer the reader to [FH91]. The permutation group Σ_5 has seven irreducible representations (up to isomorphism), which we denote by

$$U \,, \quad U' \,, \; V \,, \quad V' = V \otimes U' \,, \quad W \,, \quad W' = W \otimes U' \,, \quad \wedge^2 V \,,$$

of respective dimension 1, 1, 4, 4, 5, 5, 6, where U is the trivial representation, U' is the signature, the 4-dimensional representation V satisfies $\mathrm{Tr}(\tau)=2$, and the 5-dimensional representation W is determined by $\mathrm{Tr}(\tau)=1$ (Tr is the trace and $\tau \in \Sigma_5$ is a transposition).

One has $K_{\overline{X}_{40}}=\mathcal{O}(1)$. By looking at the symmetries of the equations of $\overline{X}_{40}$, we see that the representation of Σ_5 on $H^0\big(\overline{X}_{40}, K_{\overline{X}_{40}}\big)$ is faithful. On $\mathbb{P}^4$, the point $(1:1:1:1:1)$ is invariant; thus, the corresponding vector space is stable, and the representation is not irreducible. The only non-irreducible 5-dimensional faithful representations are

$$U+V\,,\quad U+V'\,,\quad U'+V\,,\quad U'+V'\,.$$

Let $\mathrm{Aut}(\overline{S})^o$ be the subgroup of $\mathrm{Aut}(\overline{S})$ generated by the lifts of the elements of $\Sigma_5 \subset \mathrm{Aut}(\overline{X}_{40})$. There is a natural exact sequence

$$0 \to \mathbb{Z}/2\mathbb{Z} \to \mathrm{Aut}(\overline{S})^o \to \Sigma_5 \to 0\,,$$

where the morphism $\mathbb{Z}/2\mathbb{Z} \to \mathrm{Aut}(\overline{S})^o$ is obtained by the inclusion of σ. By Schur theory, the group extensions $0 \to \mathbb{Z}/2\mathbb{Z} \to H \to \Sigma_5 \to 0$ of Σ_5 by $\mathbb{Z}/2\mathbb{Z}$ are classified by the second homology group $H^2(\Sigma_5, \mathbb{Z}/2\mathbb{Z})$, which is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^2$; therefore, $\mathrm{Aut}(\overline{S})^o$ is one of the groups

$$\mathbb{Z}/2\mathbb{Z} \times \Sigma_5\,,\quad 2.\Sigma_5^-\,,\quad 2.\Sigma_5^+\,,\quad 4.A_5\,,$$

which we will describe later.

Theorem 14. *The group* $\mathrm{Aut}(\overline{S})^o$ *is* $2.\Sigma_5^+$.

We prove this result by showing that $\mathrm{Aut}(\overline{S})^o$ cannot be $\mathbb{Z}/2\mathbb{Z} \times \Sigma_5$, $2.\Sigma_5^-$ or $4.A_5$. We need the following lemma.

Lemma 15. *The trace of the involution* σ *on* $H^0\big(\overline{S}, \Omega_{\overline{S}}\big)$ *is* -4.

Proof. The minimal resolution $\hat{X}_{40}$ of the quotient surface $\overline{S}=\overline{X}_{40}/\sigma$ is regular. By [Bea96, Lemma VI.11 and Example VI.12 3)], the space of σ-invariant 1-forms on $\overline{S}$ and the space of 1-forms on $\hat{X}_{40}$ have the same dimension. Therefore, σ acts on $H^0\left(\overline{S}, \Omega_{\overline{S}}\right)$ by multiplication by -1; the result follows. □

We moreover remark that the morphism

$$\varphi_{2,0}\colon \wedge^2 H^0\big(\overline{S}, \Omega_{\overline{S}}\big) \longrightarrow H^0\big(\overline{S}, K_{\overline{S}}\big) \simeq H^0\big(\overline{X}_{40}, K_{\overline{X}_{40}}\big)$$

is equivariant under $\mathrm{Aut}(\overline{S})^o$, and we know that it has a 1-dimensional kernel (since it is a Schoen surface). The group $\mathrm{Aut}(\overline{S})^o$ acts on $H^0\big(\overline{S}, K_{\overline{S}}\big)=H^0\big(\overline{X}_{40}, K_{\overline{X}_{40}}\big)$ through $\mathrm{Aut}(\overline{S})^o/\sigma=\Sigma_5$.

Proof of Theorem 14. Suppose that we have $\mathrm{Aut}(\overline{S})^o=\mathbb{Z}/2\mathbb{Z} \times \Sigma_5$. If the 4-dimensional representation $H^0\big(\overline{S}, \Omega_{\overline{S}}\big)$ of Σ_5 is faithful, then it is V or V', and $\wedge^2 H^0\big(\overline{S}, \Omega_{\overline{S}}\big)=\wedge^2 V=\wedge^2 V'$ is an irreducible (6-dimensional) representation; this gives a contradiction. Therefore, $H^0\big(\overline{S}, \Omega_{\overline{S}}\big)=U^a+U'^b$, but then the representation of $\Sigma_5=\mathrm{Aut}(\overline{S})^o/\sigma$ on $\wedge^2 H^0\big(\overline{S}, \Omega_{\overline{S}}\big)$ is not faithful, which again gives a contradiction.

The group $2.\Sigma_5^-$ is group number 89 among the order 240 groups in the Magma database. It contains a unique involution. But by Proposition 12, the automorphisms of $\overline{S}$ lifting the transpositions of Σ_5 acting on $\overline{X}_{40}$ are involutions; thus, $\mathrm{Aut}(\overline{S})^o$ cannot be $2.\Sigma_5^-$.

The group $A_5.4$ (group number 91 in the Magma database) has 14 irreducible representations χ_i, for $i=1,\dots,14$, of respective dimensions 1^4, 4^4, 5^4 and 6^2, where a^b means a repeated b times.

Let W_4 be a non-irreducible 4-dimensional representation of $A_5.4$. It is easy to see that $\wedge^2 W_4$ is not a faithful representation of $\Sigma_5 = \mathrm{Aut}(\overline{S})^o/\sigma$; thus, $\wedge^2 H^0(\overline{S}, \Omega_{\overline{S}})$ cannot be such a representation W_4.

Looking at the character table (for instance given by Magma), we see that the representation $H^0(\overline{S}, \Omega_{\overline{S}})$ cannot be χ_5 or χ_6 since the trace of the involution σ must be -4. The two remaining 4-dimensional representations χ_7 and χ_8 satisfy $\wedge^2\chi_7 = \wedge^2\chi_8 = \chi_{14}$ (for the computation of the wedge product of a representation, see [FH91]), which is an irreducible representation of $\Sigma_5 = \mathrm{Aut}(\overline{S})^o/\sigma$; thus, $A_5.4$ is not $\mathrm{Aut}(\overline{S})^o$. The only possibility is thus $\mathrm{Aut}(\overline{S})^o = 2.\Sigma_5^+$. □

4.3 The group $2.\Sigma_5^+$ and its action on $\overline{S}$

The group $2.\Sigma_5^+$ (number 90 among groups of order 240 in the Magma database) has 12 irreducible representations $\chi_1, \dots, \chi_{12}$, of respective dimensions 1^2, 4^5, 5^2 and 6^3. It has 21 involutions, divided into two conjugacy classes, one containing a unique element σ, which is the involution of the double cover $\overline{S} \to \overline{X}_{40}$. Since the trace of σ on χ_3 and χ_5 is not -4, the only possibilities are $H^0(\overline{S}, \Omega_{\overline{S}}) = \chi_4$, χ_6 or χ_7. One has

$$\wedge^2\chi_4 = \chi_1 + \chi_2 + \chi_3 \quad \text{and} \quad \wedge^2\chi_6 = \wedge^2\chi_7 = \chi_2 + \chi_9\,.$$

The representation of $2.\Sigma_5^+$ on χ_9 gives an irreducible 5-dimensional representation of $2.\Sigma_5^+/\sigma = \Sigma_5$, which is impossible since $H^0(\overline{X}_{40}, K_{\overline{X}_{40}})$ is not irreducible. We have thus proved that $H^0(\overline{S}, \Omega_{\overline{S}}) = \chi_4$, which has character

Order	1	2	2	3	4	5	6	6	6	8	8	10
Trace	4	-4	0	-2	0	-1	0	0	2	0	0	1

We conclude the following.

Proposition 16. *The representation of the group $2.\Sigma_5^+$ on $H^0(\overline{S}, \Omega_{\overline{S}})$ is χ_4. Moreover, one has $\wedge^2 H^0(\overline{S}, \Omega_{\overline{S}}) = \chi_1 + \chi_2 + \chi_3$ and $H^{1,1}(A) = \chi_4 \otimes \chi_4 = \chi_1 + \chi_2 + \chi_3 + \chi_5 + \chi_{10}$, where A is the Albanese variety of $\overline{S}$.*

The group $\Sigma_5 = \mathrm{Aut}(\overline{S})^o/\sigma$ acts on $\wedge^2\chi_4$ and $\wedge^2\chi_4 = U + U' + V$.

Proposition 17. *The representation of $2.\Sigma_5^+$ on $H^0(\overline{S}, K_{\overline{S}})$ is $\chi_2 + \chi_3$.*

Proof. The trace of an involution $\iota \neq \sigma$ in $2.\Sigma_5^+$ acting on χ_4 equals 0; thus, the eigenvalues of ι on the space of holomorphic 1-forms are 1, 1, -1, -1. Moreover, since $\wedge^2\chi_4 = \chi_1 + \chi_2 + \chi_3$, the involution ι acts on $H^0(\overline{S}, K_{\overline{S}})$ with trace -1 or -3 according to whether

$$H^0(\overline{S}, K_{\overline{S}}) = U + V \quad \text{or} \quad H^0(\overline{S}, K_{\overline{S}}) = U' + V\,.$$

Then the eigenvalues of ι on $H^0(\overline{S}, K_{\overline{S}})$ are, respectively, 1, 1, -1, -1, -1 and 1, -1, -1, -1, -1. By [Bea96, Lemma VI.11 and Example VI.12, 3)], the quotient surface has invariants $q = 2$ and $p_g = 2$ or $p_g = 1$, respectively. By Proposition 12, that quotient surface is (birational to) an Abelian surface, and it is the second case that is actually occurring; thus, $H^0(\overline{S}, K_{\overline{S}}) = U' + V$, which corresponds to the representation $\chi_2 + \chi_3$ for $2.\Sigma_5^+$. □

There is a basis $\omega_1, \dots, \omega_4$ of $H^0(\overline{S}, \Omega_{\overline{S}})$ such that the action of $2.\Sigma_5^+$ is generated by the following matrices of order 2 and 8:

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} \frac{1}{2}\sqrt{2}(1+I) & 0 & -\frac{1}{2}\sqrt{2}(1+I) & -I \\ 0 & 0 & -1 & 0 \\ 0 & 1 & -\sqrt{2} & -1 \\ 0 & 0 & 0 & \frac{1}{2}\sqrt{2}(1-I) \end{pmatrix},$$

where $I^2 = -1$. We have $\wedge^2 H^0(\overline{S}, \Omega_{\overline{S}}) = \chi_1 + \chi_2 + \chi_3$, where the trivial representation χ_1 is generated by the indecomposable vector $v = \omega_1 \wedge \omega_4 + \omega_2 \wedge \omega_3$, which generates the kernel of $\wedge^2 H^0(\overline{S}, \Omega_{\overline{S}}) \to H^0(\overline{S}, K_{\overline{S}})$. By the theorem of Castelnuovo–De Franchis, this gives another proof that $\overline{S}$ has no fibration onto a curve of genus at least 2.

4.4 The periods of the Albanese variety of $\overline{S}$

Let us study the Albanese variety of $\overline{S}$.

Proposition 18. *The Albanese variety A of $\overline{S}$ is isogenous to E^4, where E is an elliptic curve with complex multiplication by $\mathbb{Z}[\sqrt{-15}]$.*

Proof. The Albanese variety A of $\overline{S}$ is $A = H^0(\overline{S}, \Omega_{\overline{S}})^*/\Lambda$, where $\Lambda = H_1(\overline{S}, \mathbb{Z}) \subset H^0(\overline{S}, \Omega_{\overline{S}})^*$. Since $2.\Sigma_5^+$ acts on A, the lattice Λ is a $2.\Sigma_5^+$-stable lattice in $\chi_4 = H^0(\overline{S}, \Omega_{\overline{S}})^*$. The representation χ_4 has Schur index 2, and one computes that there exists a non-trivial $2.\Sigma_5^+$-invariant anti-symmetric bilinear form on $V_4 = \chi_4$. By [PZ06, Theorem 4.1(ii$_4$)], this implies that A is isogenous to E^4, where E is an elliptic curve with CM.

Let τ be the involution acting on $\overline{X}_{40}$ by exchanging the first two coordinates. The line

$$L = \left\{ X + Z = Y + \frac{1}{4}(-1 + I\sqrt{15})W = 0 \right\}, \quad I^2 = -1,$$

is contained in the quotient surface $Q = \overline{X}_{40}/\tau$, the equation of which is given in [Rit17]. This line contains three nodes a_1, a_2, a_3 and cuts the two (-2)-curves disjoint from the 15 nodes in points denoted by a_4 and a_5. By Proposition 12 and its proof, the surface Q is birational to two Kummer surfaces $B_i/[-1]$, for $i = 1, 2$, where each B_i is an Abelian surface. The four points $a_1, \dots, a_4$ are the branch points of a degree 2 cover $E \to L$, where E is therefore an elliptic curve on B_1 (say). The line L is also the image of an elliptic curve on B_2, with branch points a_1, a_2, a_3, a_5. Using the cross ratio for the points $a_1, \dots, a_4$, one finds that

$$E = \left\{ y^2 = x(x-1)(x-\lambda) \right\},$$

where

$$\lambda = \frac{1}{64}\left(17 + 21\sqrt{5} + I\left(7\sqrt{15} - 17\sqrt{3}\right)\right), \quad I^2 = -1.$$

The j-invariant of E is

$$-\frac{3^3 5}{2}\left(5 \cdot 283 + 7^2 13\sqrt{5}\right),$$

and using Magma again (see [Rit17, Section E]), one obtains that E has CM by the order $\mathbb{Z}[\sqrt{-15}]$ (taking the cross ratio for a_1, a_2, a_3, a_5 gives an elliptic curve with j-invariant conjugated to $j(E)$ and CM by the same order). We know by Proposition 12 that $\overline{S}$ admits a map onto B_1, giving the result. □

4.5 The surface $\overline{S}$ has maximal Picard number

Finally, we prove the following.

Theorem 19. *The surface $\overline{S}$ and the minimal resolution $\hat{X}_{40}$ of $\overline{X}_{40}$ have maximal Picard number, equal to, respectively,* 12 *and* 52.

Proof. The Albanese variety A of $\overline{S}$ is isogeneous to E^4, where E is an elliptic curve with CM; therefore, A has maximal Picard number. Moreover, the map

$$H^{2,0}(A) = \wedge^2 H^0\left(\overline{S}, \Omega_{\overline{S}}\right) \longrightarrow H^{2,0}\left(\overline{S}\right) = H^0\left(\overline{S}, K_{\overline{S}}\right)$$

is surjective; thus, by [Bea14, Proposition 2(a)], the surface $\overline{S}$ has maximal Picard number. There is a dominant rational map $\overline{S} \dashrightarrow \hat{X}_{40}$, and $\overline{S}$ has maximal Picard number; thus, by [Bea14, Proposition 2(b)], the surface $\hat{X}_{40}$ has maximal Picard number. It is easy to check that $h^{1,1}(\overline{S}) = 12$ and $h^{1,1}(\hat{X}_{40}) = 52$. □

Appendix. Quartics with 15 nodes

Let Q_{15} be a $K3$ surface in $\mathbb{P}^3(\mathbb{C})$ with 15 nodes. In this section we show the following:

- The moduli space of quartic $K3$ surfaces with 15 nodes can be described as the moduli space of $K3$ surfaces polarized by some lattice N that we describe below, and it is irreducible.
- A generic $K3$ surface with 15 nodes can be realized as a section of the Igusa quartic threefold, generalizing a similar result for Kummer quartic surfaces.

The K3 surfaces as Q_{15} above are described in [GS16, Theorem 8.6] and in [Gar17, Section 5]. We recall here the following result for convenience.

Theorem 20 ([GS16]). *Let $\tilde{Q}_{15}$ be a projective $K3$ surface with 15 disjoint smooth rational curves M_i, for $i = 1, \dots, 15$.*

1) *The Néron–Severi group of $\tilde{Q}_{15}$ contains the lattice $M_{(\mathbb{Z}/2\mathbb{Z})^4}$ (which is the smallest primitive sublattice of the $K3$ lattice containing the 15 rational curves M_i).*
2) *There exists a $K3$ surface X with a symplectic action by $G = (\mathbb{Z}/2\mathbb{Z})^4$ such that $\tilde{Q}_{15}$ is the minimal resolution of the quotient X/G.*

With the same notation as in Theorem 20, assume that $\tilde{Q}_{15}$ is the minimal resolution of Q_{15}. By [GS16, Theorem 8.3], the Néron–Severi group $\mathrm{NS}(\tilde{Q}_{15})$ contains the sublattice $\langle 4\rangle \oplus \langle -2\rangle^{\oplus 15}$ of rank 16. We denote by $M_1, \dots, M_{15}$ the fifteen (-2)-curves that are the exceptional divisors on $\tilde{Q}_{15}$. In Section A.1, we show that $\mathrm{NS}(\tilde{Q}_{15})$ must contain a special overlattice of $\langle 4\rangle \oplus \langle -2\rangle^{\oplus 15}$, which is described in details in [GS16, Theorem 8.3] and is generated by

- a pseudo-ample class L with $L^2 = 4$ (and $L \cdot M_i = 0$ for $i = 1, \dots, 15$),
- the lattice $M := M_{(\mathbb{Z}/2\mathbb{Z})^4}$ (that we recall below),
- a class $(L - v)/2$, where v contains exactly six of the M_i in its support (these are not chosen arbitrarily; we recall them below).

A.1 The lattice M and the class v

The lattice M has discriminant 2^7, and it is described by Nikulin [Nik76, § 7]. Let K denote the Kummer lattice, that is, the smallest primitive sublattice of the $K3$ lattice that contains sixteen (-2)-classes. This is negative definite and has rank 16 and discriminant 2^6; see [Nik75]. We identify the 16 classes of the Kummer lattice with the elements of $(\mathbb{Z}/2\mathbb{Z})^4$, so we denote the curves by K_{ijkh} with $i, j, k, h \in \{0, 1\}$. One can identify $M = K_{0000}^{\perp} \cap K$. By using the description of K (see, for example, [GS16]), we see that the following hold:

- The 15 classes K_{ijkh} with $(i, j, k, h) \in (\mathbb{Z}/2\mathbb{Z})^4 \backslash \{(0, 0, 0, 0)\}$ are contained in M.
- Let W be an hyperplane in the affine space $(\mathbb{Z}/2\mathbb{Z})^4$ with an equation $\sum_{i=1}^{4} \alpha_i x_i = 1$, with $\alpha_i \in \{0, 1\}$. Then the 15 classes $(1/2)\sum_{p\in W} K_p$ are contained in M. Each of these classes contains exactly eight distinct (-2)-classes of the K_{ijkh}.

Finally, as explained in [GS16, Theorem 8.3], the class v such that $(L-v)/2 \in \mathrm{NS}(Y)$ can be taken as the sum

$$K_{0001}+K_{0010}+K_{0011}+K_{1000}+K_{0100}+K_{1100}\,.$$

Notation. For the rest of the section, we will denote the fifteen (-2)-classes by M_i with $i = 1,\ldots,15$ or by K_{ijkh} with $(i,j,k,h) \in (\mathbb{Z}/2\mathbb{Z})^4\backslash\{(0,0,0,0)\}$, depending on whether or not it is important to specify the indices.

A.2 The Néron–Severi group

Let N denote the abstract lattice generated by $\mathbb{Z}L \oplus M$ and by a class $(L-v)/2$. The following result is contained in the paper [Gar17, Proposition 5.1] in a more general context; for convenience, we give here a specific proof for our situation.

Proposition 21. *Let $\tilde{Q}_{15}$ be the minimal resolution of a $K3$ quartic surface with 15 nodes. Then $\tilde{Q}_{15}$ is pseudo-ample N-polarized; that is, there is a primitive embedding of N in $\mathrm{NS}(\tilde{Q}_{15})$, and the image of N in $\mathrm{NS}(\tilde{Q}_{15})$ contains a pseudo-ample class.*

Proof. We use a similar argument as in the proof of [GS16, Theorem 8.6]. By construction and by [GS16, Theorem 8.6, 1)], we know that $\mathbb{Z}L\oplus M$ is a sublattice of $\mathrm{NS}(\tilde{Q}_{15})$ (and L is pseudo-ample). Let Q be the orthogonal complement of $\mathbb{Z}L\oplus M$ in $\mathrm{NS}(\tilde{Q}_{15})$, and let $R := (\mathbb{Z}L\oplus M)\oplus Q$. Then $\mathrm{NS}(\tilde{Q}_{15})$ is an overlattice of finite index of R, and $R^\vee/R$ has number of generators $\ell(R)$ equal to $1+7+\ell(Q)$, where $\ell(Q)$ denotes the number of generators of $Q^\vee/Q$ (recall that M has discriminant 2^7 and discriminant group isomorphic to $(\mathbb{Z}/2\mathbb{Z})^7$). If k denotes the index of R in $\mathrm{NS}(\tilde{Q}_{15})$, then we have

$$\ell(\mathrm{NS}(Y)) = 8+\ell(Q)-2k\,.$$

Let $T_{\tilde{Q}_{15}}$ be the transcendental lattice. Since the $K3$ lattice is unimodular, we have

$$\ell\big(\mathrm{NS}(\tilde{Q}_{15})\big) = \ell\big(T_{\tilde{Q}_{15}}\big) \leqslant \mathrm{rk}\,\big(T_{\tilde{Q}_{15}}\big) = 22-\mathrm{rk}\,\big(\mathrm{NS}(\tilde{Q}_{15})\big) = 6-\mathrm{rk}(Q)\,.$$

This gives $8+\ell(Q)-2k \leqslant 6-\mathrm{rk}(Q)$ and then

$$k \geqslant \frac{1}{2}(\ell(Q)+\mathrm{rk}(Q))+1\,.$$

Observe that k is the minimum number of classes we have to add to R to obtain the lattice $\mathrm{NS}(\tilde{Q}_{15})$. The classes can be of two types: either they are classes in $(\mathbb{Z}L\oplus M)^\vee/(\mathbb{Z}L\oplus M)$, or they are sums $\nu+\nu'$ with $\nu \in (\mathbb{Z}L\oplus M)^\vee/(\mathbb{Z}L\oplus M)$ and $\nu' \in Q^\vee/Q$. The maximum number of classes of the second kind is bounded by $\ell(Q)$, so we must have at least $(\mathrm{rk}(Q)-\ell(Q))/2+1$ classes of the first type. Since $\mathrm{rk}(Q)-\ell(Q) \geqslant 0$, we have at least one class of the first kind, that is, contained in $(\mathbb{Z}L\oplus M)^\vee/(\mathbb{Z}L\oplus M)$. The discriminant group here is $\mathbb{Z}/4\mathbb{Z}\oplus M^\vee/M = \mathbb{Z}/4\mathbb{Z}\oplus(\mathbb{Z}/2\mathbb{Z})^7$. Observe that, here, a class ν is then of the form $(aL/4+w/2)$, and we have $2(aL/4+w/2)-w \in \mathrm{NS}(Y)$ so that $a=\pm 2$. This shows that the class can be assumed to be $(L+w)/2$. Moreover, the square of this class must be in $2\mathbb{Z}$, which gives $L^2+w^2 = 0 \mod 8$. If h is the number of curves contained in the support of w, we get $2-h=0 \mod 4$. By the description of the discriminant group of M [GS16, Proposition 8.2], we get that $h=6$ or $h=10$, so that we may assume that the class is of the form $(L-v)/2$ as in the statement (since by [GS16, Proposition 8.2], if we take a class with $h=10$, we get the same lattice N). This concludes the proof. ☐

Remark 22. One can easily show that if a $K3$ surface has Néron–Severi group exactly isometric to N, then it admits a projective model as a quartic surface with 15 nodes (that is, N contains a pseudo-ample class), so the corresponding moduli space X_Γ is 4-dimensional, and (see [Hun96, Section 2.3]) it is an arithmetic quotient by some subgroup Γ of the isometries of the $K3$ lattice of the domain

$$\mathcal{D}_N = \left\{\omega \in \mathbb{P}(T \otimes \mathbb{C}) \,|\, \omega^2 = 0,\ \omega\bar{\omega} = 0\right\},$$

where T is the orthogonal complement of N in the $K3$ lattice $U^3 \oplus E_8(-1)^2$. This has rank 4, and it is the transcendental lattice of the generic $K3$ surface in the family.

A.3 The moduli space

Let $\mathcal{M}_N$ be the moduli space of $K3$ surfaces that are pseudo-ample N-polarized. This moduli space is described, for example, in [Dol96, Section 1], where it is shown that it is isomorphic to the space X_Γ from Remark 22.

Proposition 23. *The moduli space $\mathcal{M}_N$ is irreducible.*

Proof. The embedding of N into the $K3$ lattice is unique by [Nik80, Theorem 1.14.4 and Remark 1.14.5] (see also [GS16, Theorem 8.3]). By the construction of [Dol96, Section 3], the domain $\mathcal{D}_N$ has two connected components, both isomorphic to a bounded Hermitian domain of type $\mathrm{IV}_{19-(\mathrm{rk}(N)-1)} = \mathrm{IV}_4$. Observe that by [Nik80, Theorems 1.13.2 and 1.14.2], the orthogonal complement of N in the $K3$ lattice is uniquely determined by the signature and discriminant form. We compute, as in [GS16, Theorem 8.3], that the discriminant group of N is $(\mathbb{Z}/4\mathbb{Z}) \oplus (\mathbb{Z}/2\mathbb{Z})^5$. If we denote by q_2 the discriminant form of the lattice $U(2)$ (this is the lattice U with the bilinear form multiplied by 2), then the discriminant form is the same as $q_2 \oplus q_2$ on $(\mathbb{Z}/2\mathbb{Z})^4$ and takes value $1/4$ and $1/2$ on the remaining part $(\mathbb{Z}/4\mathbb{Z}) \oplus (\mathbb{Z}/2\mathbb{Z})$. Hence, we can identify $N^\perp$ (modulo isometries) with the lattice

$$U(2) \oplus U(2) \oplus \langle -2\rangle \oplus \langle -4\rangle\,.$$

By [Dol96, Proposition 5.6 and Lemma 5.4], there is an involution in Γ that exchanges the two connected components of $\mathcal{D}_N$, so that $X_\Gamma \simeq \mathcal{M}_N$ is irreducible. □

Since the hyperplane sections of the Igusa quartic give a 4-dimensional family of quartic surfaces with 15 nodes, Proposition 23 implies the following.

Theorem 24. *A generic quartic $K3$ surface with 15 nodes can be realized as a section of the Igusa quartic.*

Remark 25. An interesting locus in the moduli space $\mathcal{M}_N$ corresponds to quartic Kummer surfaces with 16 nodes that can be described as tangent sections of the Igusa quartic; see [Hun96, Chapter 3, Section 3.3.3].

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