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Kodaira dimension of algebraic fiber spaces over surfaces

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Abstract

In this short note, we prove the Iitaka C_{nm}-conjecture for algebraic fiber spaces over surfaces.

1. Introduction

Let $p\colon X \to Y$ be a fibration between two projective manifolds. A central problem in birational geometry is the *Iitaka conjecture*, stating that

$$\kappa(X) \geqslant \kappa(Y) + \kappa(X/Y)\,,$$

where $\kappa(X)$ is the Kodaira dimension of X and $\kappa(X/Y)$ is the Kodaira dimension of the generic fiber.

In this note, we prove that the log-version of the Iitaka conjecture holds true, provided that the base dimension $\dim Y$ is at most 2; this generalizes a result obtained by C. Birkar in [Bir09, Theorem 1.4] and a result of Y. Kawamata in [Kaw82]. More precisely, we have the following statement.

Theorem 1.1 (Theorem 3.1). *Let $p\colon X \to Y$ be a fibration between two projective manifolds. Let F be the generic fiber, and let Δ be a $\mathbb{Q}$-effective Kawamata log terminal (klt) divisor on X. Set $\Delta_F := \Delta|_F$. If $\dim Y \leqslant 2$, then we have*

$$\kappa(K_X + \Delta) \geqslant \kappa(K_F + \Delta_F) + \kappa(Y)\,. \tag{1.1}$$

We will next explain the main steps of the proof. Since $\dim Y \leqslant 2$, we can assume that K_Y is nef, by using the minimal model program. Three cases, as follows, may occur, according to the Kodaira dimension of Y.

If $\kappa(Y) \geqslant 1$, the inequality (1.1) is quickly verified by using [Kaw82].

If $\kappa(Y) = 0$, by classification theory, we know that Y is a torus or a K3 surface, modulo a finite étale cover. If Y is a torus, (1.1) is proved in [CH11] for the absolute case (that is, $\Delta = 0$) and in [CP17] for the klt pair case. Therefore, to prove (1.1), it is enough to assume that Y is a K3 surface. In order to treat this case, we rely on two main ingredients, namely the positivity of the direct images $p_\star(mK_{X/Y} + m\Delta)$ and the geometry of orbifold Calabi–Yau surfaces. Different

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aspects of the first topic were extensively studied in [Gri70, Fuj78, Kaw82, Kaw81, Kol87, Vie95, Ber09, BP10, PT18, BC15, Fuj16, HPS18, KP17], among many other articles. In our setup, this implies that $\det p_\star(mK_{X/Y} + m\Delta)$ is pseudo-effective (by using [PT18]). As Y is a K3 surface, the numerical dimension of $\det p_\star(mK_{X/Y} + m\Delta)$ coincides with its Iitaka dimension. If the numerical dimension $\mathrm{nd}(\det p_\star(mK_{X/Y} + m\Delta))$ is at least 1, we achieve our goals by standard arguments. If $\mathrm{nd}(\det p_\star(mK_{X/Y} + m\Delta)) = 0$, we can show that there exists a finite set of exceptional curves $\{C_i\}$ on Y such that $p_\star(mK_{X/Y} + m\Delta)$ is hermitian flat on $Y \setminus (\cup C_i)$, by using the results in [PT18, CP17, HPS18]. At this point we use the second ingredient, namely the uniformization theorem for compact Kähler orbifolds with trivial first Chern class; cf. [Cam04a]. We thus infer that the fundamental group of $Y \setminus (\cup C_i)$ is almost-abelian. Therefore, we can construct sufficient elements in $H^0(X, mK_{X/Y} + m\Delta)$ by using parallel transport, and (1.1) is proved.

2. Preparation

In this section, we recall the uniformization theorem for compact Kähler orbifolds with trivial first Chern class [Cam04a] (see also [CC14, GKP16]), as well as results concerning singular metrics on vector bundles and the positivity of direct images; see [BP08, PT18, Rau15, Pău18, HPS18] for more details.

First of all, we recall a few basic definitions concerning compact Kähler orbifolds, by following [Cam04a].

Definition 2.1 ([Cam04a, Definitions 3.1 and 5.1]). (i) A *compact Kähler orbifold* is a compact Kähler normal variety with only quotient singularities; that is, for every point $a \in X$, we can find a neighborhood U of a and a biholomorphism $\psi\colon U \to \widetilde{U}/G$, where $\widetilde{U}$ is an open set in $\mathbb{C}^n$ and $G \subset \mathrm{GL}(n, \mathbb{C})$ is a finite subgroup acting on $\widetilde{U}$ with $\psi(a) = 0$. For every $g \in G$, the set of the fixed points of g is of codimension at least 2.

(ii) Let $X^\star$ be the smooth locus of a compact Kähler orbifold X. We say that X is *simply connected in the sense of orbifolds* if $X^\star$ is simply connected.

(iii) A holomorphic morphism between two Kähler orbifolds $r\colon X' \to X$ is said to be an *orbifold cover* if it satisfies the following two conditions:

- The restriction of r to $r^{-1}(X^\star)$ is an étale cover.
- For every $a \in X$ with its neighborhood $\widetilde{U}/G$ (see part (i)), each component of $r^{-1}(\widetilde{U}/G)$ is of the form $\widetilde{U}/G'$ for some subgroup G' of G and the restricted morphism $r|_{\widetilde{U}/G'}$ is nothing but the natural quotient morphism $\widetilde{U}/G' \to \widetilde{U}/G$.

(iv) An m-dimensional compact Kähler orbifold X is called *Calabi–Yau* (respectively, *hyper-Kähler*) if it is simply connected in the sense of orbifolds (see part (ii)) and it admits a Ricci-flat Kähler metric such that the holonomy (when restricted to $X^\star$) is $\mathrm{SU}(m)$ (respectively, $\mathrm{Sp}(m/2)$).

We state here the uniformization theorem for compact Kähler orbifolds with trivial first Chern class, established in [Cam04a]. The statement parallels the classical case of smooth Kähler manifolds with trivial first Chern class.

Theorem 2.2 ([Cam04a, Theorem 6.4]). *Let X be a compact Kähler orbifold with $c_1(X) = 0$. Then X admits a finite orbifold cover $\overline{X} = \overline{C} \times \overline{S} \times T$, where $\overline{C}$ (respectively, $\overline{S}$) is a finite product of Calabi–Yau Kähler (respectively, hyper-Kähler) orbifolds and T is a complex torus.*

Let Y be a K3 surface, and let $\cup C_i$ be a union of exceptional curves on Y. By Grauert's criterion [BHPvdV04, III, Theorem 2.1], there is a contraction morphism $\tau\colon Y \to Y_{\mathrm{can}}$ which contracts all C_i to points p_i in a normal space Y_{can}. As K_Y is trivial, we know that Y_{can} is in fact a compact Kähler orbifold ([KM98, Definition 4.4, Remark 4.21], [Cam04a, Example 3.2]) with $c_1(Y_{\mathrm{can}}) = 0$.

As a corollary of Theorem 2.2, we have the following statement.

Proposition 2.3 ([Cam04a, Corollary 6.7]). *Let Y be a K3 surface, and let $\{C_i\}$ be a finite set of exceptional curves on Y. Then $\pi_1(Y \setminus (\cup C_i))$ is almost-abelian.*

Moreover, let $\tau\colon Y \to Y_{\mathrm{can}}$ be the morphism which contracts the exceptional curves C_i to points $p_i \in Y_{\mathrm{can}}$. If $\pi_1(Y \setminus (\cup C_i))$ is not finite, there exists a finite orbifold cover $\sigma\colon T \to Y_{\mathrm{can}}$ from a complex torus T to Y_{can}. In particular, σ is a non-ramified cover over $Y_{\mathrm{can}} \setminus (\cup p_i)$ and $\sigma^{-1}(p_i)$ is of codimension 2 for every i.

The following non-vanishing property for pseudo-effective line bundles on $K3$ surfaces is an immediate consequence of the abundance theorem (which holds true in dimension 2).

Proposition 2.4. *Let Y be a $K3$ surface (in the smooth sense), and let L be a pseudo-effective line bundle on Y. Then L is $\mathbb{Q}$-effective.*

Proof. Since L is pseudo-effective, by using Zariski decomposition for surfaces [Fuj79, Theorem 1.12], we know that

$$L \equiv_{\mathbb{Q}} \sum_{i=1}^{s} a_i [C_i] + M\,,$$

where $a_i \in \mathbb{Q}^+$, the C_i are negative intersection curves, M is nef and $M \cdot C_i = 0$ for every i. Since Y is $K3$, all nef line bundles on Y are effective [BHPvdV04, VIII, Proposition 3.7]. Therefore, L is $\mathbb{Q}$-effective. □

Remark 2.5. It is well known that a nef line bundle M on a K3 surface is semi-ample. If its numerical dimension $\mathrm{nd}(M)$ is equal to 1, then it induces an elliptic fibration over $\mathbb{P}^1$.

In the second part of this section, we recall a few definitions and results about the singular metrics on vector bundles and the positivity of direct images. We refer to [BP08, Rau15, PT18, Pău18, HPS18] for more details.

Definition 2.6. Let $E \to X$ be a holomorphic vector bundle on a manifold X (which is not necessary compact). Locally, a *singular hermitian metric* h_E on E is a measurable map from X to the space of non-negative hermitian forms on the fibers. We say that (E, h_E) is *negatively curved* if $0 < \det h_E < +\infty$ almost everywhere and

$$x \mapsto \ln |u|_{h_E}(x)\,, \quad x \in X$$

is a plurisubharmonic (psh) function for any choice of a holomorphic local section u of E.

We say that the pair (E, h_E) is *positively curved* if the dual $(E^\star, h_E^\star)$ is negatively curved. We denote this by $i\Theta_{h_E}(E) \succeq 0$.

When h_E is smooth, "positively curved" is nothing but the classical Griffiths semi-positivity. The following result, proved in [PT18], plays an important role in this article.

Theorem 2.7 ([PT18, Theorem 5.1.2]). *Let $p\colon X \to Y$ be a fibration between two projective manifolds, and let L be a line bundle on X with a possibly singular metric h_L such that*

$i\Theta_{h_L}(L) \geqslant 0$. Let $m \in \mathbb{N}$ be such that the multiplier ideal sheaf $\mathcal{I}(h_L^{1/m}|_{X_y})$ is trivial over a generic fiber X_y, namely $\int_{X_y} |e_L|^{2/m}_{h_L} < +\infty$, where e_L is a basis of L.

Let Y_1 be the locally free locus of $p_\star(mK_{X/Y}+L)$. Then the vector bundle $p_\star(mK_{X/Y}+L)$ over Y_1 admits a possibly singular hermitian metric h such that $i\Theta_h(p_\star(mK_{X/Y}+L)) \succeq 0$ on Y_1. Moreover, h induces a possibly singular metric $\det h$ on the line bundle $\det p_\star(mK_{X/Y}+L)$ over Y such that

$$i\Theta_{\det h}(\det p_\star(mK_{X/Y}+L)) \geqslant 0$$

on Y in the sense of currents.

Remark 2.8. Let us recall briefly the construction of the metric h: Let h_B be the m-relative Bergman kernel metric on $K_{X/Y}+(1/m)L$ constructed in [BP10, A.2]. Set $L_1 := (m-1)K_{X/Y}+L$ and $h_{L_1} := (m-1)h_B + (1/m)h_L$. Thanks to [BP10, A.2], we know that

$$i\Theta_{h_{L_1}}(L_1) \geqslant 0 \quad \text{on} \quad X$$

in the sense of currents. Now, h_{L_1} induces a Hodge-type metric h on $\pi_\star(mK_{X/Y}+L)$ on the smooth locus Y_0 of π as follows: Let X_y be a smooth fiber, and let $f \in H^0(X_y, mK_{X/Y}+L)$. As $mK_{X/Y}+L = K_{X/Y}+L_1$, the norm

$$\|f\|_h^2 := \int_{X_y} |f|^2_{h_{L_1}}$$

is well defined. Since h_L is not necessarily smooth, h is a possibly singular hermitian metric on $(\pi_\star(mK_{X/Y}+L), Y_0)$. Thanks to [Ber09, BP08], we can prove that $(p_\star(mK_{X/Y}+L), h)$ is positively curved on Y_0. Finally, by studying the behavior of h near $Y_1 \setminus Y_0$ [PT18], we prove that h can be extended as a possibly singular hermitian metric on Y_1 with positive curvature in the sense of Definition 2.6.

The following proposition comes from the standard extension theorem.

Proposition 2.9. *In the setting of Theorem 2.7, we suppose moreover that there exists a fibration $q\colon Y \to Z$ to some projective manifold Z. Let H be a pseudo-effective line bundle on Y with a possible singular metric h_H such that $i\Theta_{h_H}(H) \geqslant 0$ in the sense of currents. Let A_Z be an ample line bundle on Z. Then for $c \in \mathbb{N}$ large enough (depending only on A_Z and Z), the following extension property holds.*

Let $z \in Z$ be a generic point, and let X_z and Y_z be the fibers of $p \circ q$ and q, respectively, over z. Let $e \in \mathcal{O}_{Z,z}(c \cdot A_Z)$, and let $s \in H^0(Y_z, K_Y \otimes H \otimes p_\star(mK_{X/Y}+L))$ be such that

$$\int_{Y_z} |s|^2_{h_H,h} < +\infty\,, \tag{2.1}$$

where h is the metric on $p_\star(mK_{X/Y}+L)$ in Theorem 2.7.[1] *Then there exists a section*

$$S \in H^0(Y, K_Y \otimes H \otimes p_\star(mK_{X/Y}+L) \otimes q^\star(cA_Z))$$

such that $S|_{Y_z} = s \otimes q^\star e$.

Proof. Let (L_1, h_1) be the line bundle constructed in Remark 2.8. Then s induces a section $u \in H^0(X_z, K_X + H + L_1)$ and (2.1) implies that $\int_{X_z} |s|^2_{h_H,h_1} < +\infty$. For $c \in \mathbb{N}$ large enough

[1] As $q^\star K_Z$ is a trivial bundle on Y_z, modulo this trivial bundle, $|s|^2_{h_H,h}$ can be seen as a volume form on Y_z. Therefore, the integral (2.1) is well defined.

(depending only on Z and A_Z), by the standard Ohsawa–Takegoshi extension theorem (see, for example, [Dem12, Chapter 13]), we can find a section

$$U \in H^0(X, K_X + H + L_1 + (p \circ q)^\star cA_Z)$$

such that $U|_{X_z} = u \otimes (p \circ q)^\star e$. Then U induces a section

$$S \in H^0(Y, K_Y \otimes H \otimes p_\star(mK_{X/Y} + L) \otimes q^\star(cA_Z))$$

such that $S|_{Y_z} = s \otimes q^\star e$, and the proposition is proved. □

As another direct consequence of the Ohsawa–Takegoshi extension, the following proposition will be important for us.

PROPOSITION 2.10 ([BP10, A.2]). *In the setting of Theorem 2.7, let U be a small Stein open subset of X, and let $V \Subset U$ be some open set of compact support in U. Let e be a basis of $mK_{X/Y} + L$ over U. Then there exists a uniform constant $C(U, V, e)$ depending only on U, V, e such that for every $t \in \pi(V)$ and every $s \in H^0(X_t, mK_{X/Y} + L)$, we have*

$$\left\| \frac{s}{e} \right\|_{C^0(V \cap \pi^{-1}(t))} \leqslant C(U, V, e) \cdot \|s\|_h \,.$$

Proof. As explained in Remark 2.8, the line bundle $L_1 := (m-1)K_{X/Y} + L$ can be equipped with a possibly singular metric h_{L_1} such that

$$i\Theta_{h_{L_1}}(L_1) \geqslant 0 \quad \text{on} \quad X \,.$$

Since U is a small open set, we can find a Stein open set $B \subset Y$ such that $U \subset p^{-1}(B)$. As $mK_{X/Y} + L = K_{X/Y} + L_1$, by applying the Ohsawa–Takegoshi extension theorem to the fibration $p^{-1}(B) \to B$, we can find an $\widetilde{s} \in H^0(p^{-1}(B), K_X + L_1)$ such that

$$\int_{p^{-1}(B)} |\widetilde{s}|^2_{h_{L_1}} \leqslant C \int_{X_t} |s|^2_{h_{L_1}} = C \cdot \|s\|_h^2 \tag{2.2}$$

and

$$\widetilde{s}|_{X_t} = s \wedge p^\star(e_B) \,, \tag{2.3}$$

where e_B is a basis of K_Y over B.

On the open set U, the element $\widetilde{s}$ can be written as $\widetilde{s} = \widetilde{w} \cdot e \wedge p^\star(e_B)$ for some holomorphic function $\widetilde{w}$ on U. Note that $V \Subset p^{-1}(B)$; thus (2.2) implies that

$$\|\widetilde{w}\|_{C^0(V)} \leqslant C(U, V, e) \cdot \|s\|_h$$

for some constant $C(U, V, e)$ depending only on U, V and e. Thanks to (2.3), we have $\widetilde{w}|_{X_t} = s/e$. Therefore,

$$\left\| \frac{s}{e} \right\|_{C^0(V \cap X_t)} \leqslant C(U, V, e) \cdot \|s\|_h \,.$$

The proposition is proved. □

The last result of this section concerns the regularity of the metric h.

PROPOSITION 2.11 ([CP17, Corollary 2.8]). *Let $E \to X$ be a holomorphic vector bundle on a manifold X (which is not necessary compact). Let h_E be a possibly singular hermitian metric on E such that (E, h_E) is positively curved. Let U be a topological open set of X. If*

$$i\Theta_{\det h_E}(\det E) \equiv 0 \quad \text{on} \quad U \,,$$

then h_E is a smooth metric on $E|_U$ and $(E|_U, h_E)$ is hermitian flat.

3. Proof of the main theorem

We now prove the main theorem of the article.

Theorem 3.1. *Let $p\colon X \to Y$ be a fibration between two projective manifolds. Let F be the generic fiber, and let Δ be a $\mathbb{Q}$-effective klt divisor on X. Set $\Delta_F := \Delta|_F$. If $\dim Y \leqslant 2$, then*

$$\kappa(K_X + \Delta) \geqslant \kappa(K_F + \Delta_F) + \kappa(Y) \,. \tag{3.1}$$

Proof. Since the theorem is well known for the case $\dim Y = 1$ [Kaw82, CP17], we can suppose $\dim Y = 2$. By taking its minimal model, we can moreover assume that Y is a smooth projective surface with nef canonical bundle. We show next that it will be enough to treat the case where Y is a K3 surface.

Indeed, if $\kappa(Y) \geqslant 1$, as the klt version of $C_{n,1}$ is known (see [Kaw82, CP17]), we have (3.1). We refer to Proposition A.2 in the appendix for a detailed proof. If $\kappa(Y) = 0$, by using the classification of minimal surfaces [BHPvdV04, Theorem 1.1], we have $c_1(Y) = 0 \in H^{1,1}_{\mathbb{Q}}(Y)$. After a finite étale cover (observe that (3.1) is invariant under this operation), the base Y is either a torus or a K3 surface. If Y is a torus, [CP17, Theorem 1.1] implies (3.1). We assume for the rest of our proof that Y *is a K3 surface.*

Let $m \in \mathbb{N}$ be sufficiently divisible, and let Y_1 be the locally free locus of the direct image sheaf $p_\star(mK_{X/Y} + m\Delta)$. By Theorem 2.7, there exists a possibly singular hermitian metric h on $(p_\star(mK_{X/Y} + m\Delta), Y_1)$ such that

$$i\Theta_h(p_\star(mK_{X/Y} + m\Delta)) \succeq 0 \quad \text{on} \quad Y_1 \,,$$

and h induces a hermitian metric $\det h$ on $(\det p_\star(mK_{X/Y} + m\Delta), Y)$ such that

$$i\Theta_{\det h}(\det p_\star(mK_{X/Y} + m\Delta)) \geqslant 0 \quad \text{on} \quad Y$$

in the sense of currents. In particular, the bundle $\det p_\star(mK_{X/Y} + m\Delta)$ is pseudo-effective.

By Proposition 2.4, we have a Zariski decomposition

$$\det p_\star(mK_{X/Y} + m\Delta) \equiv_{\mathbb{Q}} \sum_{i=1}^{s} a_i[C_i] + L_m \,,$$

where $a_i \in \mathbb{Q}^+$, the C_i are negative intersection curves, L_m is nef and $L_m \cdot C_i = 0$ for every i. Let $\mathrm{nd}(L_m)$ be the numerical dimension of L_m. Next, we distinguish among three cases, according to the numerical dimension of L_m.

Case 1: The numerical dimension of L_m equals 2. We infer that the bundle $\det p_\star(mK_{X/Y} + m\Delta)$ is big on Y, and (3.1) is thus proved by using [Cam04b] or [CP17, Theorem 5.1].

Case 2: The numerical dimension of L_m equals 1. Thanks to Remark 2.5, we know that L_m is semi-ample. Then L_m induces a fibration $\pi\colon Y \to \mathbb{P}^1$. As $L_m \cdot [C_i] = 0$ for every i, we have

$$L_m \cdot \det p_\star(mK_{X/Y} + m\Delta) = 0 \,. \tag{3.2}$$

By using [Vie83, Lemma 7.3], we can find a birational morphism $Y' \to Y$ from a projective manifold Y' and a desingularization X' of $Y' \times_Y X$ satisfying

$$\begin{array}{ccc} X' & \xrightarrow{\pi_X} & X \\ {\scriptstyle p'}\downarrow & & \downarrow{\scriptstyle p} \\ Y' & \xrightarrow[\pi_Y]{} & Y \\ & & \downarrow{\scriptstyle \pi} \\ & & \mathbb{P}^1 . \end{array}$$

Each divisor $W \subset X'$ such that $\operatorname{codim}_{Y'} p'(W) \geqslant 2$ is π_X-contractible. Since Δ is klt, we can find a klt $\mathbb{Q}$-effective divisor Δ' on X' and an effective π_X-exceptional divisor D' such that

$$\pi_X^\star(K_X + \Delta) + D' = K_{X'} + \Delta' . \tag{3.3}$$

CLAIM 3.2. *The bundle*

$$\det p'_\star(mK_{X'/Y'} + m\Delta') - c(\pi_Y \circ \pi)^\star \mathcal{O}_{\mathbb{P}^1}(1)$$

is pseudo-effective on Y' for some constant $c > 0$.

We will verify this claim later; for now, we finish the proof of the theorem. By using [CP17, Theorem 3.4], the claim implies the existence of a divisor $E \subset X'$ such that $\operatorname{codim}_{Y'} p'(E) \geqslant 2$ and

$$D := K_{X'/Y'} + \Delta' + E - \epsilon(p' \circ \pi_Y \circ \pi)^\star \mathcal{O}_{\mathbb{P}^1}(1) \tag{3.4}$$

is $\mathbb{Q}$-pseudo-effective on X' for some $\epsilon > 0$.

Let $m_1 \gg m_2 \gg 1$. Thanks to (3.4), we have

$$\begin{aligned} &(m_1 + m_2)(K_{X'/Y'} + \Delta' + E) \\ &\quad = m_1\Big(K_{X'/Y'} + \Delta' + \frac{m_2}{m_1} D + E\Big) + \varepsilon m_2 (p' \circ \pi_Y \circ \pi)^\star \mathcal{O}_{\mathbb{P}^1}(1) . \end{aligned} \tag{3.5}$$

As $m_1 \gg m_2$, we can apply Theorem 2.7 to $m_1(K_{X'/Y'} + \Delta' + (m_2/m_1)D + E)$. In particular, we can find a possibly singular metric h_{m_1} on

$$\mathcal{V}_1 := p'_\star\Big(m_1\Big(K_{X'/Y'} + \Delta' + \frac{m_2}{m_1} D + E\Big)\Big)$$

such that $i\Theta_{h_{m_1}}(\mathcal{V}_1) \succeq 0$. Set $T := i\Theta_{\det h_{m_1}}(\det \mathcal{V}_1)$. Then $T \geqslant 0$ in the sense of currents. Let Y'_t be a generic fiber of $\pi_Y \circ \pi$.

If $T|_{Y'_t}$ is not identically 0, as Y'_t is of dimension 1, the restriction $T|_{Y'_t}$ is strictly positive at a generic point of Y'_t. Together with (3.5), this implies that $\det p'_\star((m_1 + m_2)(K_{X'/Y'} + \Delta' + E))$ is big on Y'. By applying [CP17], we get

$$\kappa(X', K_{X'} + \Delta' + E) \geqslant \kappa(F, K_F + \Delta_F) + 2 . \tag{3.6}$$

As E and D' are π_X-contractible, (3.3) and (3.6) imply (3.1).

If $T|_{Y'_t} \equiv 0$, by Proposition 2.11, the pair $(\mathcal{V}_1|_{Y'_t}, h_{m_1})$ is hermitian flat on Y'_t. In particular, $h_{m_1}|_{Y'_t}$ is a smooth metric. Note that $H^0(Y', K_{Y'})$ is of dimension 1. It defines a canonical metric $h_{Y'}$ on $K_{Y'}$, and the restriction of $h_{Y'}$ on Y'_t is smooth. As a consequence, we have

$$\int_{Y'_t} |s|^2_{(m_1-1)h_{Y'}, h_{m_1}} < +\infty \quad \text{for every} \quad s \in H^0(Y'_t, K_{Y'} \otimes (m_1 - 1)K_{Y'} \otimes \mathcal{V}_1) .$$

Combining this with Proposition 2.9 (we take $H = (m_1 - 1)K_{Y'}$ and $h_H = (m_1 - 1)h_{Y'}$), we get

$$\begin{aligned} &h^0(Y', K_{Y'} \otimes (m_1 - 1)K_{Y'} \otimes \mathcal{V}_1 \otimes \varepsilon m_2(\pi_Y \circ \pi)^\star \mathcal{O}_{\mathbb{P}^1}(1)) \\ &\qquad \geqslant h^0(Y'_t, K_{Y'} \otimes (m_1 - 1)K_{Y'} \otimes \mathcal{V}_1) = h^0\left(X'_t, m_1\left(K_{X'_t} + \Delta' + \frac{m_2}{m_1}D + E\right)\right). \end{aligned}$$

Together with (3.5), this gives

$$\begin{aligned} &h^0(X', m_1 \cdot (p')^\star K_{Y'} + (m_1 + m_2)(K_{X'/Y'} + \Delta' + E)) \\ &\qquad \geqslant h^0\left(X'_t, m_1\left(K_{X'_t} + \Delta' + \frac{m_2}{m_1}D + E\right)\right). \end{aligned} \tag{3.7}$$

Finally, by applying [Kaw82, CP17] to $X'_t \to Y'_t$, we have

$$\kappa\left(X'_t, K_{X'_t} + \Delta' + \frac{m_2}{m_1}D + E\right) \geqslant \kappa(F, K_F + \Delta_F).$$

Together with (3.7) and the fact that $K_{Y'}$ is $\mathbb{Q}$-effective, this gives

$$\kappa(X', K_{X'} + \Delta' + E) \geqslant \kappa(F, K_F + \Delta_F). \tag{3.8}$$

As E and D' are π_X-contractible, (3.3) and (3.8) imply (3.1).

Case 3: The numerical dimension of L_m equals 0. In this case, L_m is trivial (as it is semi-ample), and we have

$$\det p_\star(mK_{X/Y}) \equiv_{\mathbb{Q}} \sum_{i=1}^{s} a_i[C_i],$$

where the C_i are negative curves. As $i\Theta_{\det h}(\det p_\star(mK_{X/Y} + m\Delta))$ is a positive current in the same class as $\sum_{i=1}^{s} a_i[C_i]$, we get

$$i\Theta_{\det h}(\det p_\star(mK_{X/Y} + m\Delta)) = \sum_{i=1}^{s} a_i[C_i] \quad \text{on} \quad Y$$

in the sense of currents. In particular, we have

$$i\Theta_{\det h}(\det p_\star(mK_{X/Y} + m\Delta)) \equiv 0 \quad \text{on} \quad Y \setminus (\cup C_i).$$

By using Proposition 2.11, we see that $(p_\star(mK_{X/Y} + m\Delta), h)$ is hermitian flat on $Y_1 \setminus (\cup C_i)$.

Let $\tau\colon Y \to Y_{\mathrm{can}}$ be the morphism which contracts the union of negative curves $\cup C_i$. There are two possible cases: $\pi_1(Y \setminus (\cup C_i))$ is finite or infinite. We will analyze each possibility.

Subcase 3.1: The fundamental group $\pi_1\big(Y \setminus (\cup C_i)\big)$ is finite. As $\operatorname{codim}_Y(Y \setminus Y_1) \geqslant 2$, we know that $\pi_1(Y_1 \setminus (\cup C_i)) = \pi_1(Y \setminus (\cup C_i))$ is finite. Let r be the number of elements of the finite group $\pi_1(Y_1 \setminus (\cup C_i))$. Fix a generic point $y \in Y_1 \setminus (\cup C_i)$. As the direct image vector bundle $(p_\star(mK_{X/Y} + m\Delta), h)$ is hermitian flat on $Y_1 \setminus (\cup C_i)$, the parallel transport induces a representation

$$\rho\colon \pi_1(Y_1 \setminus (\cup C_i)) \to \operatorname{Aut}\left(H^0(X_y, mK_{X/Y} + m\Delta)\right).$$

Let $f \in H^0(X_y, mK_{X/Y} + m\Delta)$ be an element with unit norm. Although the parallel transport of f cannot induce a global section over $Y_1 \setminus (\cup C_i)$, the corresponding parallel transport of

$$\prod_{a \in \pi_1(Y_1 \setminus (\cup C_i))} \rho(a)(f) \in H^0(X_y, mr(K_{X/Y} + \Delta))$$

induces a section $\widetilde{f} \in H^0(p^{-1}(Y_1 \setminus (\cup C_i)), mr(K_{X/Y} + \Delta))$.

We now prove that $\widetilde{f}$ can be extended to the total space X. Let U be an arbitrary small Stein open subset of X and $V \Subset U$ some arbitrary open set with compact support in U. Let e be a basis of $mK_{X/Y} + m\Delta$ on U. We have $\widetilde{f} = \widetilde{l} \cdot e^{\otimes r}$ for some holomorphic function

$$\widetilde{l} \in H^0\big(V \cap p^{-1}(Y_1 \setminus (\cup C_i)), \mathcal{O}_{V \cap p^{-1}(Y_1 \setminus (\cup C_i))}\big) .$$

By construction, on every fiber X_t, we have $\widetilde{f} = \prod_{i=1}^{r} f_i$ for some $f_i \in H^0(X_t, mK_{X/Y} + m\Delta)$ with unit norm. By Proposition 2.10, the C^0-norm $\|f_i/e\|_{C^0(V \cap X_t)}$ is bounded by a constant $C(U, V, e)$ independent of t. Therefore,

$$\|\widetilde{l}\|_{C^0(V \cap X_t)} = \left\| \prod_{i=1}^{r} \frac{f_i}{e} \right\|_{C^0(V \cap X_t)} \leqslant C(U, V, e)^r .$$

In particular, $|\widetilde{l}|$ is bounded on $V \cap p^{-1}(Y_1 \setminus (\cup C_i))$ and $\widetilde{f}$ can be thus extended as a holomorphic section on V. Since V is an arbitrary small open set in X, the section $\widetilde{f}$ can be extended to the total space X.

In conclusion, for any element $f \in H^0(X_y, mK_{X/Y} + m\Delta)$, we can find a section

$$\widetilde{f} \in H^0(X, mr(K_{X/Y} + \Delta))$$

such that $\widetilde{f}|_{X_y} = \prod_{a \in \pi_1(Y_1 \setminus (\cup C_i))} \rho(a)(f)$. In particular, we have

$$\operatorname{div}(\widetilde{f}|_{X_y}) = \sum_{a \in \pi_1(Y_1 \setminus (\cup C_i))} \operatorname{div}(\rho(a)(f)) .$$

Therefore, $\kappa(K_X + \Delta) \geqslant 1$ if $\kappa(K_F + \Delta_F) \geqslant 1$. In other words, we have $\kappa(K_X + \Delta) \geqslant \min\{1, \kappa(K_F + \Delta_F)\}$. Combining this with a standard argument (see Proposition A.1), we get $\kappa(K_X + \Delta) \geqslant \kappa(K_F + \Delta_F)$, and the first subcase is completely proved.

Subcase 3.2: The fundamental group $\pi_1\big(Y \setminus (\cup C_i)\big)$ is not finite. As a consequence of Proposition 2.3, there exists a orbifold cover $\tau_Y : T \to Y_{\text{can}}$ from a complex torus T to Y_{can}. Let X' be a desingularization of $X \times_{Y_{\text{can}}} T$. We thus have a commutative diagram

$$\begin{array}{ccc} X' & \xrightarrow{\tau_X} & X \\ {\scriptstyle p'}\big\downarrow & & \big\downarrow{\scriptstyle p} \\ T & \xrightarrow[\tau_Y]{} & Y_{\text{can}} . \end{array}$$

Set $T_1 := \tau_Y^{-1}(\tau(Y_1 \setminus (\cup C_i)))$, where $\tau : Y \to Y_{\text{can}}$ is the contraction morphism. By Proposition 2.3, the cover τ_Y is non-ramified on T_1 and $\operatorname{codim}_T(T \setminus T_1) \geqslant 2$. As Δ is klt, we can find a klt $\mathbb{Q}$-effective divisor Δ' on X' and some $\mathbb{Q}$-divisor D' supported in $(p')^{-1}(T \setminus T_1)$ such that

$$\pi_X^\star(K_X + \Delta) + D' = K_{X'} + \Delta' . \tag{3.9}$$

Since T is a torus, by applying [CP17], we obtain $\kappa(K_{X'} + \Delta') \geqslant \kappa(K_F + \Delta_F)$.

Let $m \in \mathbb{N}$ be a sufficiently divisible number, and let $s \in H^0(X', mK_{X'/T} + m\Delta')$. Thanks to (3.9) and the fact that D' is supported in $(p')^{-1}(T \setminus T_1)$, we know that s induces an element

$$s_T \in H^0(T_1, \tau_Y^\star(p_\star(mK_{X/Y} + m\Delta))) .$$

Since $(p_\star(mK_{X/Y}+m\Delta), h)$ is hermitian flat on Y_1, the function $\|s_T\|^2_{(\pi_Y)^\star h}(t)$ is a psh function on $t \in T_1$. As $\operatorname{codim}_T(T\backslash T_1) \geqslant 2$, the function $\|s_T\|_{\pi_Y^\star h}(t)$ is thus constant with respect to $t \in T_1$. Let r be the degree of the cover τ_Y. Since τ_Y is a non-ramified cover on T_1, the section s_T induces an element $\widetilde{s} \in H^0(p^{-1}(Y_1\backslash(\cup C_i)), mrK_{X/Y}+mr\Delta)$. As $\|s_T\|_{\tau_Y^\star h}(t)$ is constant, by using the same argument as in Subcase 3.1, the element $\widetilde{s}$ can be extended to an element of $H^0(X, mrK_{X/Y}+mr\Delta)$. Inequality (3.1) is thus proved by using the same argument as at the end of Subcase 3.1. □

Our next job is to establish the claim used in the proof of our main result, which is a consequence of the volume estimate inequality (or the holomorphic Morse inequalities).

Proof of Claim 3.2. Thanks to [PT18], we know that $\det p'_\star(mK_{X'/Y'}+m\Delta')$ is pseudo-effective on Y'. Let A be the nef part of the Zariski decomposition of $\det p'_\star(mK_{X'/Y'}+m\Delta')$. Set $B := (\pi_Y \circ \pi)^\star \mathcal{O}_{\mathbb{P}^1}(1)$. As B is semi-ample, we have

$$\begin{aligned} 0 \leqslant A \cdot B &\leqslant c_1(\det p'_\star(mK_{X'/Y'}+m\Delta')) \cdot c_1(B) \\ &= (\pi_Y)_\star(c_1(\det p'_\star(mK_{X'/Y'}+m\Delta'))) \cdot \pi^\star c_1(\mathcal{O}_{\mathbb{P}^1}(1)) \\ &= c_1(\det p_\star(mK_{X/Y}+m\Delta)) \cdot \pi^\star c_1(\mathcal{O}_{\mathbb{P}^1}(1)) = 0\,, \end{aligned}$$

where the last equality is a consequence of (3.2). Then, we have

$$A \cdot B = 0\,. \tag{3.10}$$

Let L be an ample line bundle on Y', and set $c := (L\cdot A)/(2L \cdot B) \in \mathbb{Q}^+$. For any $\tau \in \mathbb{Q}^+$ small enough, thanks to (3.10) and the choice of c, the basic volume estimate (see, for example, [Dem12, 8.4] or [Laz04, Theorem 2.2.15]) implies that

$$\begin{aligned} \operatorname{vol}(A+\tau L - cB) &\geqslant (A+\tau L)^2 - 2c(A+\tau L)\cdot B \\ &\geqslant 2\tau(L\cdot A - cL\cdot B) + o(\tau) > 0\,. \end{aligned}$$

Therefore, $A+\tau L - cB$ is big for any $\tau \in \mathbb{Q}^+$. As $\tau \to 0^+$, the divisor $A - cB$ is pseudo-effective. Then $\det p'_\star(mK_{X'/Y'}+m\Delta') - cB$ is pseudo-effective and the claim is proved. □

Appendix

In this appendix, we will gather two standard results which should be well known to experts.

Proposition A.1 ([Kaw82, CH11]). *Let $p\colon X \to Y$ be a fibration from an n-dimensional projective manifold to a K3 surface, and let Δ be an effective klt $\mathbb{Q}$-divisor on X. Assume that Theorem 3.1 holds for $\dim X \leqslant n-1$. If $\kappa(K_X+\Delta) \geqslant 1$, then*

$$\kappa(K_X+\Delta) \geqslant \kappa(K_F+\Delta_F)\,,$$

where F is the generic fiber of p and $\Delta_F = \Delta|_F$.

Proof. We use the argument in [CP17, Proposition 3.7]. Modulo desingularization, we can assume that the Iitaka fibration of $K_X+\Delta$ is a morphism between two projective manifolds $\varphi\colon X \to W$:

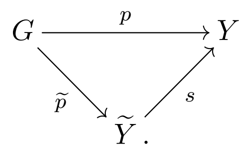
$$\begin{array}{ccc} X & \xrightarrow{\ \varphi\ } & W \\ & \searrow^{p} & \\ & & Y\,. \end{array}$$

Let G be the generic fiber of φ, and set $\Delta_G := \Delta|_G$. Then

$$\kappa(K_G + \Delta_G) = 0\,. \tag{A.11}$$

Let $p\colon G \to p(G)$ be the restriction of p to G. Next, we analyze three cases which may occur.

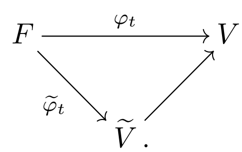
Case 1: Assume that $p(G)$ projects onto Y. Let $\widetilde{p}\colon G \to \widetilde{Y}$ be the Stein factorization of $p\colon G \to Y$,



After desingularization of $\widetilde{p}$, we can assume that $\widetilde{Y}$ is smooth. Let G_t be the generic fiber of $\widetilde{p}$. By assumption, Theorem 3.1 holds for $G \to \widetilde{Y}$. Therefore, (A.11) implies that

$$\kappa(K_{G_t} + \Delta_{G_t}) = 0\,. \tag{A.12}$$

Next, we estimate the dimension of G. Let F be the generic fiber of $p\colon X \to Y$. By restricting φ to F, we obtain a morphism $\varphi_t\colon F \to V$, where V is a subvariety of W. Let $\widetilde{V} \to V$ be the Stein factorization of φ_t,



Since G is generic, we infer that the generic fiber of $\widetilde{p}$ coincides with the generic fiber of $\widetilde{\varphi}_t$. Combining this with (A.12), we see that [Uen75, Theorem 5.11] implies that

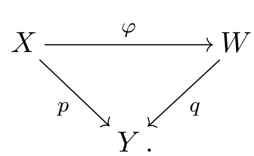
$$\kappa(K_F + \Delta_F) \leqslant \dim \widetilde{V} = \dim F - \dim G_t\,.$$

Therefore, we have $\dim G_t \leqslant \dim F - \kappa(K_F + \Delta_F)$ and thus we infer that

$$\dim G = \dim G_t + \dim \widetilde{Y} \leqslant \dim F - \kappa(K_F + \Delta_F) + \dim Y = \dim X - \kappa(K_F + \Delta_F)\,.$$

Finally, by construction of the Iitaka fibration, $\dim G = \dim X - \kappa(K_X + \Delta)$; we obtain the inequality $\dim X - \kappa(K_X+\Delta) \leqslant \dim X - \kappa(K_F+\Delta_F)$ and in conclusion $\kappa(K_X+\Delta) \geqslant \kappa(K_F+\Delta_F)$.

Case 2: Assume that the image $p(G)$ has dimension 0. Since G is connected, $p(G)$ is a point in Y. This means that we can define a map $W \to Y$, which can be assumed to be regular by blowing up W. We thus have the commutative diagram



Set $t := p(G)$. Let F be the fiber of p over t. Then F is a generic fiber of p and G is a generic fiber of $\varphi\colon F \to \varphi(F)$, and by [Uen75, Theorem 5.11], we infer that

$$\kappa(K_F + \Delta_F) \leqslant \kappa(K_G + \Delta_G) + \dim \varphi(F) = \dim \varphi(F)\,.$$

Note that $\varphi(F)$ is the fiber of q over $t \in Y$. We have $\dim W = \dim \varphi(F) + \dim Y$. Therefore, $\dim W \geqslant \kappa(K_F + \Delta_F) + \dim Y$. Combining this with the fact that φ is the Iitaka fibration, we thus have $\kappa(K_X + \Delta) = \dim W \geqslant \kappa(K_F + \Delta_F) + \dim Y$, and we are done.

Case 3 (remaining case): The image $p(G)$ is a proper subvariety of Y. Let $p(G)'$ be the normalization of $p(G)$. If $p(G)'$ is a curve of general type, then $\kappa(K_G + \Delta_G) \geqslant 1$ and we get a contradiction with (A.11). If $p(G)'$ is $\mathbb{P}^1$, as G is generic, Y is thus covered by rational curves. We therefore get a contradiction with the assumption that Y is K3. As a consequence, $p(G)'$ is a torus. Then $[p(G)]$ is a semi-ample class of numerical dimension 1 in Y. Therefore, $p(G)$ is a generic fiber of a fibration $\pi\colon Y \to \mathbb{P}^1$. We thus have the commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{\ \varphi\ } & W \\ & {\scriptstyle p\circ\pi}\searrow \quad \swarrow{\scriptstyle q} & \\ & \mathbb{P}^1 . & \end{array}$$

Set $t := p\circ\pi(G)$. Let X_t be the fiber of $p\circ\pi$ over t. Then G is the generic fiber of $\varphi|_{X_t}\colon X_t \to \varphi(X_t)$, and by [Uen75, Theorem 5.11], we infer that

$$\kappa(K_{X_t} + \Delta_{X_t}) \leqslant \kappa(K_G + \Delta_G) + \dim \varphi(X_t) = \dim \varphi(X_t)\,.$$

Note that $\varphi(X_t)$ is the fiber of q over $t \in \mathbb{P}^1$. We have $\dim W = \dim \varphi(X_t) + 1$. Therefore, $\dim W \geqslant \kappa(K_{X_t} + \Delta_{X_t}) + 1$. Combining this with the fact that φ is the Iitaka fibration, we thus have

$$\kappa(K_X + \Delta_X) = \dim W \geqslant \kappa(K_{X_t} + \Delta_{X_t}) + 1 \geqslant \kappa(K_F + \Delta_F) + 1\,,$$

where the last inequality comes from the fact that X_t is a fibration over a torus with generic fiber F. The proposition is thus proved. □

Proposition A.2. *Let $p\colon X \to Y$ be a fibration between two projective manifolds. Let F be the generic fiber, and let Δ be a $\mathbb{Q}$-effective klt divisor on X. Set $\Delta_F := \Delta|_F$. If $\dim Y = 2$ and $\kappa(Y) \geqslant 1$, then*

$$\kappa(K_X + \Delta) \geqslant \kappa(K_F + \Delta_F) + \kappa(Y)\,. \tag{A.13}$$

Proof. Since Y is of dimension 2, we can consider its minimal model and assume that Y is smooth with semi-ample canonical bundle.

If $\kappa(Y) = 2$, then K_Y is big and it is known that (A.13) holds.

If $\kappa(Y) = 1$, we can suppose that K_Y is semi-ample. Then K_Y induces a fibration $\pi\colon Y \to Z$ to a 1-dimensional variety Z and $K_Y = \pi^\star A$ for some ample line bundle A on Z. Let Y_z be a generic fiber of π. Then Y_z is a 1-torus. Let $m \in \mathbb{N}$ be a sufficiently large number, and let h be the possibly singular hermitian metric on $p_\star(mK_{X/Y} + m\Delta)$ defined in Theorem 2.7. There are two cases.

Case 1: $i\Theta_{\det h}(\det p_\star(mK_{X/Y} + m\Delta))|_{Y_z} \equiv 0$. By Proposition 2.11, the vector bundle $(p_\star(mK_{X/Y} + m\Delta)|_{Y_z}, h)$ is hermitian flat. Therefore,

$$\int_{Y_z} |s|_h^2 < +\infty \quad \text{for every} \quad s \in H^0(Y_z, p_\star(mK_{X/Y} + m\Delta))\,. \tag{A.14}$$

As $K_Y = \pi^\star A$ for some ample line bundle on Z, Proposition 2.9 and the L^2-condition (A.14) imply that

$$\kappa(X, K_X + \Delta) \geqslant \kappa(X_z, K_{X/Y} + \Delta|_{X_z}) + 1\,. \tag{A.15}$$

Moreover, by applying [CP17, Kaw82] to $X_z \to Y_z$ and using the fact that Y_z is a torus, we have $\kappa(X_z, K_{X/Y} + \Delta|_{X_z}) \geqslant \kappa(K_F + \Delta_F)$. Together with (A.15), this implies (A.13).

Case 2: $i\Theta_{\det h}(\det p_\star(mK_{X/Y}+m\Delta))|_{Y_z} \gneqq 0$. Since Y_z is of dimension 1, the line bundle $\det p_\star(mK_{X/Y}+m\Delta)|_{Y_z}$ is ample on Y_z. As K_Y is semi-ample, we can find some non-negative $\mathbb{Q}$-div Δ' in the class of $c \cdot p^\star K_Y$ for some $c>0$ small enough such that $\Delta+\Delta'$ is klt. Then

$$\det p_\star(mK_{X/Y}+m\Delta+m\Delta') = \det p_\star(mK_{X/Y}+m\Delta)+m\Delta'$$

is big on Y. By applying, for example, [Cam04b, CP17], we have

$$\kappa(K_{X/Y}+\Delta+\Delta') \geqslant \kappa(K_F+\Delta_F)+2\,.$$

As $c<1$, we know that $K_X+\Delta-(K_{X/Y}+\Delta+\Delta')=(1-c)\cdot p^\star K_Y$ is $\mathbb{Q}$-effective. Therefore, $\kappa(K_X+\Delta)\geqslant \kappa(K_F+\Delta_F)+2$, and (A.13) is proved. □

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