F-singularities in families

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Abstract

We study the behavior of test ideals and F-singularities in families. In particular, we obtain generic (and non-generic) restriction theorems for test ideals and non-F-pure ideals which imply, for example, openness of most of the F-singularity classes when the relative canonical sheaf is $\mathbb{Q}$-Cartier. Additionally, we study the global behavior of certain canonical linear systems (induced by the Frobenius morphism) associated to adjoint line bundles, in families. As a consequence, we obtain some positivity results for pushforwards of some adjoint line bundles and for certain subsheaves of these.

1. Introduction

In [Kun69], Kunz proved that a scheme over a field of characteristic $p > 0$ is regular if and only if the Frobenius (endo)morphism is flat, which initiated the study of singularities using the Frobenius morphism. Classes of singularities defined via the Frobenius morphism are usually referred to as F-singularities. Since Kunz's work, the study of F-singularities has become an active research area; see [HR76, GW77, Fed83, MR85, RR85, Sri91, Smi97, Har98, MS97, HY03, Tak04]. However, the methods of F-singularities have been widely applied to global geometry over a field of positive characteristic only recently [Sch14, Mus13, Hac15, MS14, Zha14, CHMS14, Pat14, HX15, Tan15], at least outside of special classes of varieties [MR85, BK05]. A large field within global geometry is moduli theory, which requires understanding how varieties behave in flat families. In this direction, there have not been any positive results on the behavior of test ideals in families. We fill this gap.

Previously, F-rational, Cohen–Macaulay, F-injective, and (Gorenstein) F-pure singularities have been studied in such a context [Has01, SZ09, Has10]. Additionally, [MY09, Example 4.7] showed that the test ideal τ does not satisfy the generic restriction theorem, at least as stated for multiplier ideals [Laz04b, Theorem 9.5.35]. We develop tools tackling this issue, obtaining generic (and non-generic) restriction theorems for test ideals; see Theorem A below.

In a flat family $f\colon X \longrightarrow V$, the absolute Frobenius morphism on X does not restrict to the Frobenius morphism on each fiber. Thus, we systematically study the relative Frobenius

morphism[1] $X' \to X \times_V V'$, where $X' \to X$ and $V' \to V$ are the Frobenius morphisms of X and V, respectively. Its fibers over V are morphisms between thickenings of the fibers of f. However, its fibers over V' are exactly the Frobenius morphisms of fibers of f (at least over points with perfect residue field).

We begin by stating our main result in the local setting. We denote by X^n and V^n the domains of n-iterated Frobenius morphisms of X and V, respectively. Let τ and σ denote the test ideal[2] [HH94] and non-sharply-F-pure ideal [FST11, BB11], respectively. These ideals play key roles in the theory of F-singularities, the test ideal τ being the unique smallest non-zero ideal fixed by p^{-e}-linear maps and σ being the unique largest. We define relative versions of these two ideals by using the relative instead of absolute Frobenius morphism. The iterated relative Frobenius morphism targets different spaces $X \times_V V^n$, and so we actually define a sequence of ideals, one on each of these spaces. We explain the setup.

Suppose that $f \colon X \to V$ is a finite-type, flat, equidimensional, reduced, and S2 and G1 morphism of Noetherian F-finite schemes with V integral. Additionally, suppose that Δ is a $\mathbb{Q}$-divisor on X satisfying suitable conditions such that it can be restricted to fibers and such that $K_{X/V} + \Delta$ is $\mathbb{Q}$-Cartier with index not divisible by $p > 0$, so that $(p^e - 1)(K_{X/V} + \Delta)$ is Cartier (see Remark 2.11 for a precise statement). Then, for each integer $n > 0$ divisible by e, we define ideals $\sigma_n(X/V, \Delta)$ and $\tau_n(X/V, \Delta) \subseteq \mathscr{O}_{X \times_V V^n}$ called the *relative non-F-pure* and *relative test ideals*, respectively.

Our main local theorem is that these ideals restrict to absolute non-F-pure and absolute test ideals on *all* of the geometric fibers and can be used to prove *generic* restriction theorems for the usual absolute test ideals on $X \times_V V^n$, at least if V is regular.

THEOREM A (Theorem 3.10, Corollaries 3.22, 4.8, and 4.14). *With notation as above, there exists an $N > 0$ such that for every perfect point[3] $s \in V$ and every $n \geqslant N$,*

$$\sigma_{ne}(X/V, \Delta) \cdot \mathscr{O}_{X_{s^{ne}}} = \sigma(X_s, \Delta|_{X_s})$$

and

$$\tau_{ne}(X/V, \Delta) \cdot \mathscr{O}_{X_{s^{ne}}} = \tau(X_s, \Delta|_{X_s}),$$

where $X_{s^{ne}}$ is the fiber of $X \times_V V^{ne} \to V^{ne}$ over $s^{ne} \in V^{ne}$, which is isomorphic to X_s since $k(s)$ is perfect. Additionally, both σ_n and τ_{ne} map surjectively onto their arbitrary base changes.

Furthermore, if V is regular and N is sufficiently large, then for all perfect points $s \in V$, the absolute non-F-pure ideal restricts to all of the fibers for $n \geqslant N$:

$$\sigma\big(X \times_V V^{ne}, \Delta \times_V V^{ne}\big) \cdot \mathscr{O}_{X_{s^{ne}}} = \sigma(X_s, \Delta|_{X_s}),$$

and at least for an open dense set of the base $U \subseteq V$, the same holds for the absolute test ideal:

$$\tau\big(X \times_V V^{ne}, \Delta \times_V V^{ne}\big) \cdot \mathscr{O}_{X_{s^{ne}}} = \tau(X_s, \Delta|_{X_s})$$

for all perfect points $s \in U$.

Additionally, we show that over a dense open subset U of V with $W = f^{-1}(U)$, the relative test ideal $\tau_{ne}(X/V, \Delta)|_W$ coincides with the absolute test ideal $\tau(X \times_V V^{ne}, \Delta \times_V V^{ne})|_W$ and likewise $\sigma_{ne}(X/V, \Delta)|_W = \sigma(X \times_V V^{ne}, \Delta \times_V V^{ne})|_W$, in Theorems 4.12 and 3.13, respectively.

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[1]This is also called the Radu–André morphism, especially in commutative algebra.

[2]Technically, we are working with what has recently become known as the *big test ideal* [Hoc07].

[3]By definition, a perfect point $s \in V$ is a morphism $\operatorname{Spec} K \to V$ from a perfect field K; see Definition 2.2.

The key method that allows us to prove this result (at least stated for σ_n) is that if $m > n$, then $\operatorname{im}\big(\sigma_{ne}(X/V, \Delta) \otimes_{\mathscr{O}_{X \times V^{ne}}} \mathscr{O}_{X \times V^{me}} \to \mathscr{O}_{X \times V^{me}}\big) \supseteq \sigma_{me}(X/V, \Delta)$ and there is a dense open subset of the base V over which equality holds for all sufficiently large n; this is Proposition 3.3. This stabilization result should be viewed as a relative version of [HS77, Proposition 1.11], [Lyu97, Proposition 4.4], and [Gab04, Remark 13.6].

Remark 1.1. If one replaces the divisor Δ with the language of principal Cartier algebras, then the previous result still holds without the technical assumptions about divisors from Remark 2.11.

Remark 1.2. There are a number of subtle issues in the statement above that we are suppressing. In particular, $\tau_n(X/V, \Delta)$ depends on the choice of some ideal I contained within the test ideal of every fiber and is shown to exist in Proposition 4.7.

A particularly important related question is that of deformation of sharply F-pure singularities in flat families with $\mathbb{Q}$-Cartier relative canonical divisors. This would be important for a positive-characteristic construction of the moduli of stable varieties, also known as the KSBA compactification. In characteristic zero, this is the moduli space given by the log-minimal model program. It classifies log-canonical models, hence birational equivalence classes of varieties of general type, and furthermore it contains some nodal varieties for the compactification. There is a conjectural framework for constructing this moduli space [Kol90]. One of the main ingredients in this framework is to prove that log-canonical singularity deforms in flat families with $\mathbb{Q}$-Cartier relative log-canonical divisor. An important step in this direction in positive characteristic is the corresponding statement for sharply F-pure singularities. It is also an important ingredient in an upcoming paper of the first author where he is planning to address the question of the existence of an algebraic space structure on the space of sharply F-pure stable varieties. In this paper, we handle the deformation of F-pure and F-regular singularities. Indeed, openness of F-pure and F-regular singularities is a direct consequence of Theorem A above.

Theorem B (Deformation of F-pure and F-regular singularities: Corollaries 3.29 and 4.19). *Let* $f \colon X \to V$ *and* Δ *be as in Theorem A, and, additionally, assume that f is proper. If $s \in V$ is a perfect point and $(X_s, \Delta|_{X_s})$ is sharply F-pure (respectively, strongly F-regular[4]), then there exists an open set $U \subseteq V$ such that for all $u \in U$, the pair $(X_u, \Delta|_{X_u})$ is also sharply F-pure (respectively, strongly F-regular).*

We also build relative test submodules and non-F-injective submodules of $\omega_{X \times_V V^{ne}/V^{ne}}$ and prove restriction theorems like Theorem A for them (Corollaries 5.4 and 5.8). As a consequence, in Theorem 5.13 we re-prove a result of M. Hashimoto [Has01], namely deformation for Cohen–Macaulay F-injectivity and F-rationality.

Furthermore, we apply our setup to global questions. One of the reasons for the recent global applications of the F-singularity theorem is the lifting theorem shown by the second author in [Sch14, Proposition 5.3]. This theorem can be used to replace some of the lifting arguments that use Kodaira vanishing in characteristic zero. One of the fundamental ideas in [Sch14] is to try to lift only a big enough set of sections of adjoint bundles instead of all the sections. This canonical set of sections for a pair (X, Δ) with $(1 - p^e)(K_X + \Delta)$ Cartier and for a line bundle M is defined as [Sch14, Definition 4.1]

$$S^0\big(X, \sigma(X, \Delta) \otimes M\big) := \bigcap_{m>0} \operatorname{im} \Big(H^0\big(X, F_*^{me} \mathscr{O}_X((1 - p^{me})(K_X + \Delta)) \otimes M\big) \to H^0\big(X, M\big)\Big).$$

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[4]In which case, you can even remove the "index not divisible by p" assumption from $K_{X/V} + \Delta$; see Corollary 4.20.

First, we investigate questions about how this canonical space of sections behaves in families, namely semicontinuity, stabilization of the intersection, etc.

THEOREM C. *Let $f\colon X \to V$ and Δ be as in Theorem A, with f projective and V regular. Further, suppose that M is a line bundle on X such that $M - K_{X/V} - \Delta$ is f-ample (where by M we actually mean the Cartier divisor corresponding to M). Then the following statements hold:*

(a) *[Corollary 6.19, Examples 6.22, 6.23, and 6.25], cf. [Har98, Example 5.5], [Tan15, Theorem 8.3]. The function*

$$s \mapsto \dim_{k(s)} S^0(X_s, \sigma(X_s, \Delta_s) \otimes M_s) \tag{1.1}$$

is not semicontinuous (here $s \in V$ is a perfect point) in either direction; however, there is a dense open subset $U \in V$ such that the function (1.1) is constant on U.

(b) *[Theorem 6.18]. There exists an $n > 0$ such that for all integers $m \geqslant n$ and perfect points $s \in V$,*

$$\mathrm{im}\left(H^0\big(X_s, F^{me}_* \mathscr{O}_{X_s}((1 - p^{me})(K_{X_s} + \Delta_s)) \otimes M_s\big) \to H^0(X_s, M_s)\right)$$
$$= S^0(X_s, \sigma(X_s, \Delta_s) \otimes M_s).$$

(c) *[Proposition 6.20]. If there is a perfect point $s_0 \in V$ such that*

$$H^0(X_{s_0}, M_{s_0}) = S^0(X_{s_0}, \sigma(X_{s_0}, \Delta_{s_0}) \otimes M_{s_0}),$$

then there is an open neighborhood U of s_0 such that $f_* M|_U$ is locally free and compatible with base change and

$$H^0(X_s, M_s) = S^0(X_s, \sigma(X_s, \Delta_s) \otimes M_s)$$

for every perfect point $s \in U$. In particular, the function (1.1) is constant for $s \in U$.

We would like to also mention the following natural question left open by Theorem C.

QUESTION. *Can one remove the f-ampleness assumption from the statements of Theorem C?*

By Theorem C, it makes sense to talk about the general value of $\dim_{k(s)} S^0(X_s, \sigma(X_s, \Delta_s) \otimes M_s)$ in the following theorem.

THEOREM D. *With assumptions as in Theorem C, so that V is projective over a perfect field and $(p^e - 1)(K_X + \Delta)$ is Cartier, for every $n \gg 0$, there is a subsheaf $S^0_{\Delta, ne} f_*(M)$ of $(F^{ne}_V)^*(f_* M)$ for which the following statements hold:*

(a) *[Corollary 6.19]. The rank of $S^0_{\Delta, ne} f_*(M)$ is the general value of $\dim_{k(s)} S^0(X_s, \sigma(X_s, \Delta_s) \otimes M_s)$.*

(b) *[Proposition 6.26]. If $M - K_{X/V} - \Delta$ is ample, then $S^0_{\Delta, ne} f_*(M)$ is globally generated for every $n \gg 0$.*

(c) *[Theorem 6.31]. If $M - K_{X/V} - \Delta$ is nef, then $S^0_{\Delta, ne} f_*(M)$ is weakly positive for $n \gg 0$.*

(d) *[Corollary 6.14]. If $M = Q^l \otimes P$, where Q is f-ample, then for all $l \gg 0$ and every nef line bundle P, we have that $S^0_{\Delta, ne} f_*(M)$ is contained in $(f_{V^{ne}})_*(\sigma_{ne}(X/V, \Delta) \otimes M_{V^{ne}})$ as subsheaves of $(f_{V^{ne}})_* M_{V^{ne}}$, and, furthermore, these two subsheaves are generically equal.*

In fact, points (a) and (d) are true without the projectivity assumption on V.

Further, note that if V in Theorem D is a curve, if $M - K_{X/V} - \Delta$ is f-ample and nef, and if there is an $s \in V$ such that $H^0(X_s, M_s) = S^0(X_s, \sigma(X_s, \Delta_s))$, then points (a) and (c) of Theorem D together with point (c) of Theorem C imply that $f_* \mathscr{O}_X(M)$ is a nef vector bundle. This strengthens [Pat14, Proposition 3.6]; it removes the cohomology vanishing assumption made there. Of course, our statements are considerably stronger. For example, with the same assumptions except replacing $H^0(X_s, M_s) = S^0(X_s, \sigma(X_s, \Delta_s))$ with rk $f_*(\sigma(X, \Delta) \otimes M)$, which equals the general value of $S^0(X_s, \sigma(X_s, \Delta_s))$, we obtain that $f_*(\sigma(X, \Delta) \otimes M)$ is nef. In fact, the following even stronger statement can be made.

THEOREM E (Corollary 6.39). *With assumptions as in Theorem C, so that V is projective over a perfect field, assume that $M - K_{X/V} - \Delta$ is nef and f-ample and that* rk $S^0 f_*(\sigma(X, \Delta) \otimes M)$ *equals the general value of $H^0(X_s, \sigma(X_s, \Delta_s) \otimes M_s)$. Then $S^0 f_*(\sigma(X, \Delta) \otimes M)$ is weakly positive. In particular, if V is a smooth curve, then it is a nef vector bundle.*

Further, we show how $S^0_{\Delta, ne} f_*(M)$ relates to the other similar notion $S^0 f_*(\sigma(X, \Delta) \otimes M)$ introduced in [HX15, Definition 2.14]. In particular, we obtain that $S^0 f_*(\sigma(X, \Delta) \otimes M)$ does not, in general, restrict to $S^0(X_s, \sigma(X_s, \Delta_s) \otimes M_s)$ for general $s \in S$. Intuitively, though $S^0 f_*(\sigma(X, \Delta) \otimes M)$ is a pushforward, it captures the global geometry of (X, Δ) rather than the geometry of the fibers. In positive characteristic, these two can differ considerably, essentially because the function field of V is not perfect. Of course, the relative and absolute $S^0 f_*$ are related. We study these similarities and differences in Section 6.6.

Organization. In Section 2, we set up the notation that we will follow throughout the paper, explain the interplay between p^{-e}-linear maps and $\mathbb{Q}$-Cartier divisors, and discuss the behavior of such maps and divisors under base changes. In Sections 3, 4, and 5, we introduce the relative non-F-pure ideals, the relative test ideals, and the relative test submodules, respectively. Applications to F-singularities in families are discussed in these three sections. Section 6 is devoted to the behavior of S^0 (introduced in [Sch14, Definition 4.1]) under base changes. Some semi-positivity results are also proved in this section. Finally, in the appendix, we collect some statements for which we could not find proper references in the generality needed in our paper.

2. Notation and setup

Throughout this paper, all schemes are Noetherian and all maps of schemes are separated. We fix the following notation, which is in effect for the entire paper. In particular, for simplicity we do not state it in every statement even though it is assumed.

NOTATION 2.1. Suppose that $f \colon X \to V$ is a flat, equidimensional, and geometrically reduced map[5] of finite type from a scheme X to an excellent integral scheme V of equal characteristic $p > 0$ with a dualizing complex. We write $F^e = F^e_V \colon V = V^e \to V$ for the absolute e-iterated Frobenius morphism on V, form the base change $f_{V^e} \colon X_{V^e} = X \times_V V^e \to V^e$, and define $F^e_{X^e/V^e} \colon X^e \to X_{V^e}$ to be the e-iterated relative Frobenius morphism. Furthermore, we often assume that V is F-finite, in which case we automatically assume that the Frobenius morphism $F_V \colon V \to V$ satisfies the identity[6] $(F_V)^! \omega_V^\bullet \simeq_{\mathrm{qis}} \omega_V^\bullet$. If we say that V is a variety, it is always of finite type over a perfect field k.

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[5]By geometrically reduced, we mean that $X_T = X \times_V T$ is reduced for all $T \to V$ with T integral.

[6]For some discussion of this identity, which always holds for varieties or schemes of essentially finite type over a local ring with a dualizing complex, see [BSTZ10, Section 2].

Because we will be considering numerous different sheaves on the same topological space $X = X^e = X \times_V V^e =$ etc. but with respect to different schemes, we will adopt the following somewhat non-standard notation. We use this because writing numerous operations $\left(F_{X^e/V^e}^e\right)_*$ and $\left(F_{X^e/V^e}^e\right)^{-1}$ that do not affect the underlying space is confusing. We would need notation for projections of the form $X^e \times_{V^e} V^{e+d} \to V^{e+d}, X^e$ in as well as more general relative Frobenius morphisms $F_{(X^e)_{V^{e+d}}/V^{e+d}}^e \colon (X^e)_{V^{e+d}} := X^e \times_{V^e} V^{e+d} \to X_{V^{e+d}}$, and maps $X^e \times_{V^{e+d+c}} \to X^e \times_{V^{e+d}}$. By using simply modules, this becomes more transparent.

(1) We will use R to denote $\mathscr{O}_X$.

(2) We will use A to denote $f^{-1}\mathscr{O}_V$.

(3) We will use A^{1/p^e} to denote $f^{-1}(F_V^e)_*\mathscr{O}_V$. We note that this is not an abuse of notation since V is integral.

(4) We will use R^{1/p^e} to denote $(F_X^e)_*R$. This is a slight abuse of notation if X is not reduced.

(5) We will use $R_{A^{1/p^e}}$ to denote $R \otimes_A A^{1/p^e}$.

(6) We will use $(R_{A^{1/p^e}})^{1/p^d}$ to denote $R^{1/p^d} \otimes_{A^{1/p^d}} A^{1/p^{e+d}}$. This may be a slight abuse of notation if X_{V^e} is not reduced.

(7) Given an R-module M, we will use M^{1/p^e} to denote $F_*^e M$. This will generally not cause any confusion because, typically, M will be locally free or even a line bundle.

(8) We use ω_R to denote ω_X and ω_A to denote $f^{-1}\omega_V$.

(9) Etc.

Some of the main results of the paper concern restriction to fibers. These statements pertain only to a special set of fibers of f, the fibers over perfect points (see the definition below). For example, if V is a curve over an algebraically closed field, then we often restrict to fibers over all closed points and over the perfect closure of the generic point of V, but not over the generic point itself.

DEFINITION 2.2 (Perfect points). A *perfect point of V*, for $s \in V$, is a morphism $s = \operatorname{Spec} k(s) \to V$ from the spectrum of a perfect field $k(s)$ to V. It can also be viewed as a choice of a point $v \in V$ and a field extension $k(s) = K \supseteq k(v)$ such that K is perfect. Finally, a *neighborhood of a perfect point $s \in V$* is simply a neighborhood of the image v of s.

The fact that $f \colon X \to V$ is of finite type implies that the relative Frobenius morphism is a finite map. In some cases, this will allow us to avoid assuming that V is F-finite.

LEMMA 2.3. *With notation as above, since $f \colon X \to V$ is of finite type, the relative Frobenius map $f \colon X^e \to X_{V^e}$ is a finite map.*

Proof. We work locally on V and X. It is sufficient to show that R^{1/p^e} is a finite $R_{A^{1/p^e}}$-module. Write $S = A[x_1, \ldots, x_n]$ and $R = S/I$. We first observe that $S_{A^{1/p^e}} \to S^{1/p^e}$ is a finite map. However, the $x_1^{i_1} \cdots x_n^{i_n}$ for $0 \leqslant i_j < p^e$ clearly form a generating set for S^{1/p^e} over $S_{A^{1/p^e}}$.

Now, tensor the map $S_{A^{1/p^e}} \to S^{1/p^e}$ with $\otimes_S (S/I) = \otimes_S R$. We obtain

$$R_{A^{1/p^e}} = R \otimes_A A^{1/p^e} = (S/I) \otimes_S S_{A^{1/p^e}} \to (S/I) \otimes_S S^{1/p^e} = \left(S/I^{[p^e]}\right)^{1/p^e} \to (S/I)^{1/p^e},$$

where the final map is the canonical surjection of rings. The map is finite since each part is. $\square$

We also recall the result of Radu and André.

Theorem 2.4 ([Rad92, And93]). *Suppose that $f\colon X \to V$ is a flat map of Noetherian schemes. Then f has geometrically regular fibers if and only if the relative Frobenius $R_{A^{1/p^e}} \cong A^{1/p^e} \otimes_A R \to R^{1/p^e}$ is flat.*

We immediately obtain the following corollary, which will also be useful.

Corollary 2.5. *Suppose that $f\colon X \to V$ is as in Notation 2.1 and additionally has geometrically regular fibers. Then for any $R_{A^{1/p^e}}$-module M, the natural evaluation-at-1 map $\mathscr{H}\mathrm{om}_{R_{A^{1/p^e}}}\left(R^{1/p^e}, M\right) \to M$ surjects.*

Proof. The module R^{1/p^e} is a locally free $R_{A^{1/p^e}}$-module, and the result follows. $\square$

Now, we state the object which we will study for the majority of the paper.

Definition 2.6 (ϕ). From here on, we fix a line bundle L on $X \cong X^e$ and we also fix a (possibly zero) $R_{A^{1/p^e}}$-linear map $\phi\colon L^{1/p^e} \to R_{A^{1/p^e}}$.

Remark 2.7 (Reflexive sheaves for G1 and S2 morphisms). Suppose that $f\colon X \to V$ is G1 and S2. Note that there exists an open set $\iota\colon U \hookrightarrow X$ such that $X \setminus U$ has codimension at least 2 along each fiber and that $f|_U$ is a Gorenstein morphism.

Finally, suppose that M is any rank 1 reflexive R-module which is locally free on a set U as above. Then M^{1/p^e} is not only reflexive as an R^{1/p^e}-module, we claim that it is also reflexive as an $R_{A^{1/p^e}}$-module. Since we already have $i_*M|_U = M$ by Proposition A.7 because M is reflexive, it is sufficient to replace X by U. Thus, $\omega_{X/V}$ and M are both locally free as R-modules and $\omega_{X_{V^e}}$ is locally free as an $R_{A^{1/p^e}}$-module. We work locally so as to trivialize all these modules. Then

$$
\begin{aligned}
M^{1/p^e} &\cong \omega_{X/V}^{1/p^e} = \left(F_{X^e/V^e}^e\right)_* \omega_{X/V} \\
&\cong \mathscr{H}\mathrm{om}_{\mathscr{O}_{X_{V^e}}}\left(\left(F_{X^e/V^e}^e\right)_* \mathscr{O}_X, \omega_{X_V^e/V^e}\right) \\
&\cong \mathscr{H}\mathrm{om}_{R_{A^{1/p^e}}}\left(R^{1/p^e}, R_{A^{1/p^e}}\right),
\end{aligned}
$$

which is clearly reflexive (the second isomorphism follows from Grothendieck duality for the finite relative Frobenius map).

Conversely, if M is any R-module which is locally free on a set U and reflexive as an $R_{A^{1/p^e}}$-module, then it is also reflexive as an R^{1/p^e}-module. To see this, note that $M|_U$ is reflexive as an R-module, and because M is reflexive as an $R_{A^{1/p^e}}$-module, it satisfies $i_*M|_U = M$.

Definition 2.8 (ϕ versus divisors). Observe the following identifications:

$$
\begin{aligned}
&\mathscr{H}\mathrm{om}_{R_{A^{1/p^e}}}\left(L^{1/p^e}, R_{A^{1/p^e}}\right) \\
&\cong \underbrace{\left(\mathscr{H}\mathrm{om}_{R_{A^{1/p^e}}}\left(L^{1/p^e} \otimes_{R_{A^{1/p^e}}}\left(\omega_{R_{A^{1/p^e}}/A^{1/p^e}}\right), \omega_{R_{A^{1/p^e}}/A^{1/p^e}}\right)\right)^{**}}_{\text{(This clearly holds on } U\text{, note that both sheaves are reflexive and use Corollary A.8.)}} \\
&= \underbrace{\left(\mathscr{H}\mathrm{om}_{\mathscr{O}_{X_{V^e}}}\left(\left(F_{X^e/V^e}^e\right)_*\left(L \otimes \left(\left(F_{X^e/V^e}^e\right)^* \omega_{X_{V^e}/V^e}\right)\right), \omega_{X_{V^e}/V^e}\right)\right)^{**}}_{\text{(This just rewrites the previous line using different notation.)}} \\
&\cong \underbrace{\left(\left(F_{X^e/V^e}^e\right)_* \mathscr{H}\mathrm{om}_{\mathscr{O}_{X^e}}\left(L \otimes \left(\left(F_{X^e/V^e}^e\right)^* \omega_{X_{V^e}/V^e}\right), \omega_{X^e/V^e}\right)\right)^{**}}_{\text{(Grothendieck duality for a finite map)}}
\end{aligned}
$$

$$\cong \quad \underbrace{\left(F^e_{X^e/V^e}\right)_* \left(L^{-1} \otimes \omega_{X^e/V^e} \otimes (F^e_{X^e/V^e})^* \omega^{-1}_{X_{V^e}/V^e}\right)^{**}}. \tag{2.1}$$

(By Remark 2.7 and Corollary A.8, we may take the reflexive hull as $(F^e_{X^e/V^e})_* \mathscr{O}_{X^e}$-modules.)

Now, observe that $\omega_{X/V}$ is compatible with base change up to reflexification. In particular, if our base change is the Frobenius morphism $F^e_V \colon V^e \to V$, then writing $\pi_{V^e} \colon X_{V^e} = X \times_V V^e \to X$ as the projection, we have $\omega_{X_{V^e}/V^e} \cong (\pi^*_{V^e}\omega_{X/V})^{**}$ since both sheaves are reflexive and they certainly agree outside the non-relatively Cohen–Macaulay locus (which is of relative codimension at least 2) by Corollary A.8. In particular,

$$\left((F^e_{X^e/V^e})^* \omega_{X_{V^e}/V^e}\right)^{**} \cong \left((F^e_{X^e/V^e})^* \pi^*_{V^e}\omega_{X/V})\right)^{**} = \left((F^e_X)^* \omega_{X/V})\right)^{**} = \left(\omega^{p^e}_{X/V}\right)^{**}.$$

Plugging this into (2.1), we obtain

$$\mathscr{H}om_{R_{A^{1/p^e}}}\left(L^{1/p^e}, R_{A^{1/p^e}}\right) \cong \left(F^e_{X^e/V^e}\right)_* \left(L^{-1} \otimes \omega^{1-p^e}_{X^e/V^e}\right)^{**}. \tag{2.2}$$

If, additionally, X is absolutely (instead of relatively) G1 and S2 (for example, if V is regular), then any choice of non-degenerate[7] ϕ induces a non-zero, effective Weil divisorial sheaf D_ϕ such that $\mathscr{O}_X(D_\phi) \cong \left(L^{-1} \otimes \omega^{1-p^e}_{X^e/V^e}\right)^{**}$ by [Har94]. We would like to generalize this to the case that $X \to V$ is *relatively* G1 and S2.

DEFINITION 2.9. We say that ϕ is *relatively divisorial* if ϕ locally generates $\mathscr{H}om_{R_{A^{1/p^e}}}\left(L^{1/p^e}, R_{A^{1/p^e}}\right)$ as an R^{1/p^e}-module

(a) at the generic points of each fiber and

(b) at the generic point of every codimension 1 singular point of every geometric fiber.

In this case, by removing a set of relative codimension 2 so that f is relatively Gorenstein, we see that $\phi \cdot R^{1/p^e} \subseteq \mathscr{H}om_{R_{A^{1/p^e}}}\left(L^{1/p^e}, R_{A^{1/p^e}}\right)$ is a rank 1 free submodule of an invertible R^{1/p^e}-module. To be able to associate a divisor to this submodule in a sensible way, we should show that it is the trivial (full) submodule at every singular codimension 1 point. Indeed, let ξ be a singular codimension 1 point. Then one of the following cases holds:

○ $f(\xi)$ *is a codimension 1 point.* In this case, ξ is a general point of the fibers over $f(\xi)$, hence assumption (a) guarantees that ϕ generates a full submodule at ξ.

○ $f(\xi)$ *is the general point of V.* In this case, ξ is a codimension 1 point of the fiber over $f(\xi)$, and it is not in the smooth locus of f. Therefore, assumption (b) shows that again ϕ generates a full submodule at ξ.

Therefore, the submodule $\phi \cdot R^{1/p^e}$ determines a Cartier divisor and also an honest Weil divisor on the original X. We denote this divisor by D_ϕ as well.

DEFINITION 2.10 (ϕ as a divisor). If ϕ is relatively divisorial, we set Δ_ϕ to be the $\mathbb{Q}$-divisor $(1/(p^e-1))D_\phi$. This makes sense because D_ϕ is trivial along the codimension 1 components of the singular locus of X and so we avoid the pathological issues which occur for $\mathbb{Q}$-divisors on non-normal spaces.

We now explain how to recover ϕ from a $\mathbb{Q}$-divisor.

Remark 2.11 (Obtaining ϕ from divisors). We work under the conventions of Definition 2.10. Untangling Definition 2.8 yields a method to obtain ϕ from a divisor $\Delta \geqslant 0$ (which then coincides with Δ_ϕ) under the following assumptions:

---

[7]Here, non-degenerate means non-zero on any irreducible component.

(a) We have $\Delta = (1/m)D$ for some Weil divisor D, where $p \nmid m$.

(b) The Weil divisor D on X is Cartier in relative codimension 1.

(c) The support of D does not contain any generic point or any singular codimension 1 point of any geometric fiber.

(d) We have $(p^e - 1)/m \in \mathbb{Z}$ and $\left(\omega_{X/V}^{1-p^e} \otimes \mathscr{O}_X((1 - p^e)\Delta)\right)^{**} = L$ is a line bundle.

In such a case, the integral divisor $(p^e - 1)\Delta$ (which is well defined since $\operatorname{Supp}\Delta$ does not contain the codimension 1 components of the non-regular locus of X, cf. [KAAC92, pp. 171–173] or [MS12, Section 2.2]) induces an inclusion

$$\left(F_{X^e/V^e}^e\right)_* L \hookrightarrow \left(F_{X^e/V^e}^e\right)_* \left((\omega_{X/V}^{1-p^e})^{**}\right).$$

This composed with the natural Grothendieck trace map

$$\left(F_{X^e/V^e}^e\right)_* \left((\omega_{X/V}^{1-p^e})^{**}\right) \to \mathscr{O}_{X_{V^e}}$$

yields a map ϕ. It is easy to see that conditions (a)–(d) above guarantee that ϕ is relatively divisorial. Furthermore, we also have $\Delta = \Delta_\phi$.

For future reference we make the following definition.

DEFINITION 2.12. In the situation of Notation 2.1, (X, Δ) *is a pair if* f *is G1 and S2 and* Δ satisfies the assumptions of Remark 2.11.

2.1 Composing maps

Given ϕ as in Definition 2.8, we can compose $\phi \colon L^{1/p^e} \to R_{A^{1/p^e}}$ with itself (after twisting), similarly to [BS13, Section 4] or [Sch09], and thus obtain new maps

$$\phi^2 \in \operatorname{Hom}_{R_{A^{1/p^{2e}}}} \left(\left(L^{(p^e+1)}\right)^{1/p^{2e}}, R_{A^{1/p^{2e}}}\right)$$

and, more generally,

$$\phi^n \in \operatorname{Hom}_{R_{A^{1/p^{ne}}}} \left(\left(L^{\frac{p^{ne}-1}{p^e-1}}\right)^{1/p^{ne}}, R_{A^{1/p^{ne}}}\right).$$

We explain this construction.

Begin by tensoring ϕ by L over R, and then taking $1/p^e$th roots; we obtain

$$\left(L^{1+p^e}\right)^{1/p^{2e}} = \left(L^{1/p^e} \otimes_R L\right)^{1/p^e} \to \left(\left(R \otimes_A A^{1/p^e}\right) \otimes_R L\right)^{1/p^e} = L^{1/p^e} \otimes_{A^{1/p^e}} A^{1/p^{2e}}. \tag{2.3}$$

On the other hand, we can also tensor ϕ by $A^{1/p^{2e}}$ over A^{1/p^e} to obtain

$$L^{1/p^e} \otimes_{A^{1/p^e}} A^{1/p^{2e}} \to R_{A^{1/p^{2e}}}. \tag{2.4}$$

By composing (2.3) and (2.4), we obtain the desired map ϕ^2. We now define ϕ^n inductively as follows. Given

$$\phi^{n-1} \colon \left(L^{\frac{p^{(n-1)e}-1}{p^e-1}}\right)^{1/p^{(n-1)e}} \to R_{A^{1/p^{(n-1)e}}},$$

tensor with L over R and then take p^eth roots, which yields

$$\left(L^{\frac{p^{ne}-1}{p^e-1}}\right)^{1/p^{ne}} = \left(\left(L^{\frac{p^{(n-1)e}-1}{p^e-1}+p^{(n-1)e}}\right)^{1/p^{(n-1)e}}\right)^{1/p^e}$$

$$= \left(L \otimes_R \left(L^{\frac{p^{(n-1)e}-1}{p^e-1}}\right)^{1/p^{(n-1)e}}\right)^{1/p^e}$$

$$\longrightarrow \left(L \otimes_R R_{A^{1/p^{(n-1)e}}}\right)^{1/p^e} = L^{1/p^e} \otimes_{A^{1/p^e}} A^{1/p^{ne}}.$$

We then apply ϕ to the first term in the final tensor product to obtain

$$\left(L^{\frac{p^{ne}-1}{p^e-1}}\right)^{1/p^{ne}} \longrightarrow L^{1/p^e} \otimes_{A^{1/p^e}} A^{1/p^{ne}} \xrightarrow{\phi\otimes\cdots} \left(R \otimes_A A^{1/p^e}\right) \otimes_{A^{1/p^e}} A^{1/p^{ne}} \cong R_{A^{1/p^{ne}}},$$

which we take as the official definition of ϕ^n. On the other hand, for every $0 < m < n$, one can look at the following composition of ϕ^m and ϕ^{n-m}:

$$\gamma \colon \left(L^{\frac{p^{ne}-1}{p^e-1}}\right)^{1/p^{ne}} \cong \left(L^{\frac{p^{me}-1}{p^e-1}} \otimes_R L^{\frac{p^{ne}-p^{me}}{p^e-1}}\right)^{1/p^{ne}} \cong \left(\left(L^{\frac{p^{me}-1}{p^e-1}}\right)^{1/p^{me}} \otimes_R L^{\frac{p^{(n-m)e}-1}{p^e-1}}\right)^{1/p^{(n-m)e}}$$

$$\xrightarrow{\left(\phi^m\otimes\mathrm{id}_L\cdots\right)^{1/p^{(n-m)e}}} \left(\left(R \otimes_A A^{\frac{1}{p^{me}}}\right) \otimes_R L^{\frac{p^{(n-m)e}-1}{p^e-1}}\right)^{1/p^{(n-m)e}}$$

$$\cong \left(L^{\frac{p^{(n-m)e}-1}{p^e-1}}\right)^{1/p^{(n-m)e}} \otimes_{A^{1/p^{(n-m)e}}} A^{1/p^{ne}}$$

$$\xrightarrow{\phi^{n-m}\otimes_A\cdots A\cdots} \left(R \otimes_A A^{1/p^{(n-m)e}}\right) \otimes_{A^{1/p^{(n-m)e}}} A^{1/p^{ne}} \cong R \otimes_A A^{1/p^{ne}}. \qquad (2.5)$$

In particular, taking $m = 1$ gives an a priori different map which we could also define as ϕ^n. We now explain why this map is actually equal to the *official* ϕ^n.

LEMMA 2.13. *With notation as above, $\gamma = \phi^n$.*

Proof. The statement is local, and so we may suppose $L = R$. With the (obscuring) powers of L removed, ϕ^n is described as the following composition:

$$R^{1/p^{ne}} \xrightarrow{\phi^{1/p^{(n-1)e}}} R^{1/p^{(n-1)e}} \otimes_{A^{1/p^{(n-1)e}}} A^{1/p^{ne}}$$

$$\xrightarrow{\phi^{1/p^{(n-2)e}}\otimes\cdots} \left(R^{1/p^{(n-2)e}} \otimes_{A^{1/p^{(n-2)e}}} A^{1/p^{(n-1)e}}\right) \otimes_{A^{1/p^{(n-1)e}}} A^{1/p^{ne}}$$

$$\xrightarrow{\phi^{1/p^{(n-3)e}}\otimes\cdots} \cdots\cdots\cdots$$

$$\vdots$$

$$\xrightarrow{\phi^{1/p^e}\otimes\cdots} R^{1/p^e} \otimes_{A^{1/p^e}} \cdots \otimes_{A^{1/p^{(n-1)e}}} A^{1/p^{ne}}$$

$$\xrightarrow{\phi\otimes\cdots} R \otimes_A \cdots \otimes_{A^{1/p^{(n-1)e}}} A^{1/p^{ne}}.$$

The first m entries in the composition make up $(\phi^m \otimes \cdots)^{1/p^{(n-m)e}}$ in (2.5), and the last $n - m$ entries clearly yield $\phi^{n-m} \otimes_{A^{1/p^{(n-m)e}}} A^{1/p^n}$, as desired. The result is then obvious. $\qquad\square$

LEMMA 2.14. *With notation as in Definition 2.8, assume additionally that ϕ is relatively divisorial; then ϕ^n is relatively divisorial and $\Delta_\phi = \Delta_{\phi^n}$ for every integer $n \geqslant 1$.*

Proof. For showing any of the two statements, we may remove the non–relatively Gorenstein locus. That is, by possibly further restricting R, we may assume that R is relatively Gorenstein

over V and that L is trivial (since our map $f\colon X \to V$ is geometrically G1 and S2). Hence, by Remark 2.7,

$$\operatorname{Hom}_{R \otimes_A A^{1/p^{ie}}}\left(R^{1/p^{ie}}, R \otimes_A A^{1/p^{ie}}\right) \cong R^{1/p^{ie}} \quad \text{for all } i\,.$$

From [Kun86, Appendix F], [Sch09, Lemma 3.9], or composition of Grothendieck trace, we have

$$\Big(\operatorname{Hom}_{R^{1/p^e} \otimes_{A^{1/p^e}} A^{1/p^{ne}}}\left(R^{1/p^{ne}}, R^{1/p^e} \otimes_{A^{1/p^e}} A^{1/p^{ne}}\right)$$

$$\otimes_{R^{1/p^e} \otimes_{A^{1/p^e}} A^{1/p^{ne}}} \operatorname{Hom}_{R_{A^{1/p^{ne}}}}\left(R^{1/p^e} \otimes_{A^{1/p^e}} A^{1/p^{ne}}, R_{A^{1/p^{ne}}}\right)\Big)$$

$$\cong \operatorname{Hom}_{R_{A^{1/p^{ne}}}}\left(R^{1/p^{ne}}, R_{A^{1/p^{ne}}}\right) \quad (2.6)$$

via the homomorphism induced by composition. Let θ_e be the R^{1/p^e}-module generator of

$$\operatorname{Hom}_{R \otimes_A A^{1/p^e}}\left(R^{1/p^e}, R \otimes_A A^{1/p^e}\right)$$

for each e, and let θ_e^n be defined as φ^n, with φ replaced by θ. Then by using (2.6) iteratively, $\theta_e^n = \theta_{ne}$ up to multiplication by a unit.

Further, let $r \in R$ be such that $\varphi(\_) = \theta_e\left(r^{1/p^e} \cdot \_\right)$. Then it is easy to verify that

$$\varphi^n(\_) = \theta_{ie}\left(\left(r^{\frac{p^{ne}-1}{p^e-1}}\right)^{1/p^{ne}} \cdot \_\right). \quad (2.7)$$

Then we see that if ϕ is generating at a point $P \in X$, or equivalently r is a unit at P, then so is ϕ^n. This shows that ϕ^n is relatively divisorial. Furthermore, (2.7) shows that $\Delta_{\phi^i} = \Delta_\phi$. $\qquad \square$

2.2 Base change of ϕ

Suppose that we are given a morphism of schemes $g\colon T \to V$ such that T is also excellent and integral and has a dualizing complex. For example, we could set T to be a closed point of V and let g be the inclusion. Alternately, we could let g be a regular alteration over some closed subscheme of V. We list the following maps:

$$
\begin{array}{rlll}
p_1\colon & X \times_V T & \to X & \text{(the projection)}\,, \\
(p_1)^{1/p^i}\colon & X^i \times_{V^i} T^i & \to X^i & \text{(the projection for any } i)\,, \\
q_i\colon & X \times_V T^i & \to X \times_V V^i & \text{(base change)}\,, \\
p_1 = q_0\colon & X \times_V T & \to X\,.
\end{array}
\quad (2.8)
$$

These are pictured below:

$$
\begin{array}{ccc}
X^i \times_{V^i} T^i & \xrightarrow{\;p_1^{1/p^i}\;} & X^i \\
{\scriptstyle F^i_{(X_T)^i/T^i}}\Big\downarrow & & \Big\downarrow{\scriptstyle F^i_{X^i/V^i}} \\
X \times_V T^i & \xrightarrow{\;q_i\;} & X \times_V V^i\,.
\end{array}
$$

Notice that given $\phi\colon L^{1/p^e} \to R_{A^{1/p^e}} = \mathscr{O}_{X_{V^e}}$, we can form $(q_e)^*\phi$, which we denote by

$$\phi_T\colon L_T^{1/p^e} \cong q_e^* L^{1/p^e} \to q_e^* \mathscr{O}_{X_{V^e}} = \mathscr{O}_{X_{T^e}}\,.$$

We explain the isomorphism $L_T^{1/p^e} \cong q_e^* L^{1/p^e}$ briefly. Working locally, let $V = \operatorname{Spec} A$, $T = \operatorname{Spec} B$, and $X = \operatorname{Spec} R$. Then the map ϕ_T is identified with the map

$$L^{1/p^e} \otimes_{R_{A^{1/p^e}}} R_{B^{1/p^e}} \cong L^{1/p^e} \otimes_{A^{1/p^e}} B^{1/p^e} \to \left(\left(R \otimes_A A^{1/p^e}\right) \otimes_{A^{1/p^e}} B^{1/p^e}\right) \cong R \otimes_A B^{1/p^e}\,.$$

The isomorphism in the definition of ϕ_T is now immediate.

The next lemma shows that base change of ϕ commutes with the self-composition defined in Section 2.1.

LEMMA 2.15. *Suppose that* $g \colon T \to V$ *is as above. Then* $(\phi^n)_T = (\phi_T)^n$.

Proof. It is sufficient to prove this in the affine case, assuming that our isomorphism is sufficiently canonical (which will be clear). We notice that $(\phi_T)^2$ is the composition

$$\left(\left(L^{p^e+1}\right)^{1/p^e} \otimes_{A^{1/p^e}} B^{1/p^e}\right)^{1/p^e} \xrightarrow{\left(\phi_T \otimes_{R_{B^{1/p^e}}} L_T\right)^{1/p^e}} L^{1/p^e} \otimes_{A^{1/p^e}} B^{1/p^e} \otimes_{B^{1/p^e}} B^{1/p^{2e}}$$

$$\xrightarrow{\phi_T \otimes_{B^{1/p^e}} B^{1/p^{2e}}} \left(\left(R \otimes_A A^{1/p^e}\right) \otimes_{A^{1/p^e}} B^{1/p^e}\right) \otimes_{B^{1/p^e}} B^{1/p^{2e}}.$$

Now, unraveling the definitions, we obtain

$$(\phi_T)^2 = \left(\left(\phi_T \otimes_{R_{B^{1/p^e}}} L_T\right)^{1/p^e}\right) \circ \left(\phi_T \otimes_{B^{1/p^e}} B^{1/p^{2e}}\right)$$

$$= \left(\left(\left(\phi \otimes_{A^{1/p^e}} B^{1/p^e}\right) \otimes_{R_{B^{1/p^e}}} \left(L \otimes_{A^{1/p^e}} B^{1/p^e}\right)\right)^{1/p^e}\right) \circ \left(\phi \otimes_{A^{1/p^e}} B^{1/p^e} \otimes_{B^{1/p^e}} B^{1/p^{2e}}\right)$$

$$= \left(\left(\phi \otimes_{R_{A^{1/p^e}}} L \otimes_{A^{1/p^e}} B^{1/p^e}\right)^{1/p^e}\right) \circ \left(\phi \otimes_{A^{1/p^e}} B^{1/p^{2e}}\right)$$

$$= \left(\left(\phi \otimes_{R_{A^{1/p^e}}} L\right)^{1/p^e} \otimes_{A^{1/p^{2e}}} B^{1/p^{2e}}\right) \circ \left(\left(\phi \otimes_{A^{1/p^e}} A^{1/p^{2e}}\right) \otimes_{A^{1/p^{2e}}} B^{1/p^{2e}}\right)$$

$$= \left(\left(\phi \otimes_{R_{A^{1/p^e}}} L\right)^{1/p^e} \circ \left(\phi \otimes_{A^{1/p^e}} A^{1/p^{2e}}\right)\right) \otimes_{A^{1/p^{2e}}} B^{1/p^{2e}}$$

$$= \left(\phi^2\right)_T,$$

as desired. The general nth self-composition is similar. $\square$

Our next goal is to describe how divisors, corresponding to maps φ, fare under base change. We thank Brian Conrad for pointing us in the right direction [Con00, Theorem 3.61].

LEMMA 2.16. *Suppose that* $f \colon X \to V$ *is a finite-type, Cohen–Macaulay morphism over an excellent scheme* V *of characteristic* $p > 0$ *and that* $g \colon T \to V$ *is as above. Then the Grothendieck trace map*

$$\left(F_{X^e/V^e}\right)_* \omega_{X^e/V^e} \cong \mathscr{H}om_{\mathscr{O}_{X_{V^e}}} \left(\left(F_{X^e/V^e}\right)_* \mathscr{O}_{X^e}, \omega_{X_{V^e}/V^e}\right) \to \omega_{X_{V^e}/V^e}$$

is compatible with base change.

Proof. We clearly have a commutative diagram:

$$\begin{array}{ccc}
q_e^* \mathscr{H}om_{\mathscr{O}_{X_{V^e}}} \left(\left(F_{X^e/V^e}\right)_* \mathscr{O}_{X^e}, \omega_{X_{V^e}/V^e}\right) & \longrightarrow & q_e^* \omega_{X_{V^e}/V^e} \\
\alpha \downarrow & & \downarrow \beta \\
\mathscr{H}om_{\mathscr{O}_{X_{T^e}}} \left(\left(F_{(X_T)^e/T^e}\right)_* \mathscr{O}_{(X_T)^e}, \omega_{X_{T^e}/T^e}\right) & \longrightarrow & \omega_{\omega_{X_{T^e}/T^e}},
\end{array}$$

and by [Con00, Theorem 3.6.1], the map β is an isomorphism. It is sufficient to verify that α is an isomorphism as well. We work locally on some affine chart on X and hence assume that $X_V \subseteq \mathbb{A}_V^N =: P_V$ embeds as a closed subscheme since f is of finite type. The map α can then be identified with

$$Q_e^* \mathscr{E}xt^j_{\mathscr{O}_{P_{V^e}}} \left(\left(F_{X^e/V^e}\right)_* \mathscr{O}_{X^e}, \omega_{P_{V^e}/V^e}\right) \to \mathscr{E}xt^j_{\mathscr{O}_{P_{T^e}}} \left(\left(F_{(X_T)^e/T^e}\right)_* \mathscr{O}_{(X_T)^e}, \omega_{P_{T^e}/T^e}\right),$$

where $Q_e \colon P_{T^e} \to P_{V^e}$ is the induced map and $j = N - \dim(X/V)$ (we leave off the pushforward for the inclusion $i \colon X_V \to P_V$ above). This in turn can be identified with

$$Q_e^* \mathscr{E}xt^j_{\mathscr{O}_{(P_V)^e}} \left(\mathscr{O}_{X^e}, \omega_{(P_V)^e/V^e}\right) \to \mathscr{E}xt^j_{\mathscr{O}_{(P_T)^e}} \left(\mathscr{O}_{(X_T)^e}, \omega_{(P_T)^e/T^e}\right).$$

This last map is exactly the bottom row of [Con00, Diagram (3.6.1) in Theorem 3.6.1], which is an isomorphism, and hence the proof is complete. □

Lemma 2.16 above allows us to show that if the divisor D_ϕ or Δ_ϕ is trivial, then it stays trivial after base change.

LEMMA 2.17 (Divisors which are zero stay zero). *Suppose that $f\colon X \to V$ is a G1 and S2 morphism and that $\phi\colon L^{1/p^e} \to R_{A^{1/p^e}}$ is as above. Additionally, suppose that $g\colon T \to V$ is any base change. Finally, suppose that the natural map*

$$\phi \cdot R^{1/p^e} \longrightarrow \mathscr{H}om_{R_{A^{1/p^e}}}\left(L^{1/p^e}, R_{A^{1/p^e}}\right) \cong \left(F^e_{X^e/V^e}\right)_*\left(L^{-1} \otimes \omega^{1-p^e}_{X^e/V^e}\right)^{**}$$

is an isomorphism of R^{1/p^e}-modules (here, we define $\phi \cdot R^{1/p^e}$ to be the R^{1/p^e}-submodule of the set $\mathscr{H}om_{R_{A^{1/p^e}}}\left(L^{1/p^e}, R_{A^{1/p^e}}\right)$ generated by ϕ). Then

$$\phi_T \cdot (R_T)^{1/p^e} \longrightarrow \mathscr{H}om_{R_{T^{1/p^e}}}\left(L_T^{1/p^e}, R_{T^{1/p^e}}\right) \cong \left(F^e_{(X_T)^e/T^e}\right)_*\left(L_T^{-1} \otimes \omega^{1-p^e}_{(X_T)^e/T^e}\right)^{**}$$

is also an isomorphism.

In particular, if ϕ is relatively divisorial (see Definition 2.9), then the following holds: if D_ϕ is zero, then so is D_{ϕ_T}.

Proof. The statement about divisors is trivial since it is easy to see that a divisor being zero corresponds to the map above being an isomorphism. Thus, we merely need to prove the assertion. However, since f and all the sheaves involved are relatively S2, it suffices to prove the statement off a set of relative codimension 2 by Corollary A.8. Therefore, by removing a set of codimension 2, we can assume that f is a Gorenstein morphism. Thus, working locally if needed, ϕ can be identified (up to multiplication by a unit in R^{1/p^e}) with the Grothendieck trace $\left(F^e_{X^e/V^e}\right)_*\omega_{X^e/V^e} \to \omega_{X_{V^e}/V^e}$. Hence, by Lemma 2.16, so can ϕ_T. The proof is complete. □

We now move on to a discussion of base change with respect to divisors. First, we observe that by Lemma 2.17, if ϕ is relatively divisorial, then so is ϕ_T for every base change $g\colon T \to V$.

DEFINITION 2.18. Suppose that for some relatively divisorial ϕ, we have Δ_ϕ as above and that $g\colon T \to V$ is a base change. Then, we write $\Delta_\phi|_{X_T}$ (respectively, $D_\phi|_{X_T}$) to denote the divisor Δ_{ϕ_T} (respectively, D_{ϕ_T}).

LEMMA 2.19 (Pulling back Δ_ϕ). *Suppose that $f\colon X \to V$ is G1 and S2 and that ϕ is relatively divisorial. Then for any $g\colon T \to V$, we have $\Delta_\phi|_{X_T} = \left(p_1^{1/p^e}\right)^*\Delta_\phi$ (recall the morphism $p_1^{1/p^e}\colon X^e \times_{V^i} T^i \to X_i$ from (2.8)).*

Here, even though Δ_ϕ is not necessarily $\mathbb{Q}$-Cartier, $\left(p_1^{1/p^e}\right)^*\Delta_\phi$ can be defined after removing a set of relative codimension 2 outside of which it is $\mathbb{Q}$-Cartier (since D_ϕ is Cartier on such a set).

Proof. The statement is local on X and can be checked after removing a relative codimension 2 set, and so we may assume that D_ϕ is a Cartier divisor. Thus, we assume that L is trivial on the affine scheme $X = \operatorname{Spec} R$ and that f is a Gorenstein morphism. Then, shrinking the neighborhood further if necessary, the map $\phi\colon R^{1/p^e} \to R_{A^{1/p^e}}$ can be identified with $s^{1/p^e} \cdot \operatorname{Tr}$, where $\operatorname{Tr}\colon \left(F^e_{X^e/V^e}\right) * \omega_{X^e/V^e} \to \omega_{X_{V^e}/V^e}$ is the Grothendieck trace. The divisor D_ϕ is then easily seen to be the divisor $\operatorname{div}_X(s)$. On the other hand, it is clear by Lemma 2.16 that ϕ_T is then the trace on X_T multiplied by s as well. In particular, it equals $\left(p_1^{1/p^e}\right)^*D_\phi$, as desired. □

COROLLARY 2.20. *Suppose that $g\colon z \to V$ is the inclusion of a point. Suppose that ϕ is relatively divisorial and corresponds to Δ_ϕ. Then $\Delta_\phi|_{X_z} = \Delta_{\phi_z}$.*

Proof. The divisor Δ_{ϕ_z} is determined in codimension 1, where $\Delta_\phi|_{X_z}$ and Δ_{ϕ_z} agree. $\qquad \square$

2.3 Passing to V^∞ and other perfect points

First, suppose that V is the spectrum of a perfect field A. Then, the map $L^{1/p^e} \to R_{A^{1/p^e}}$ is also a map $L^{1/p^e} \to R$ since $R_{A^{1/p^e}} \cong R$. In such a case, we often also typically write the map as ψ to help distinguish how we are thinking about it. We note that maps such as $\psi\colon L^{1/p^e} \to R$ can be composed with themselves as in [BS13, Section 4.1] and, furthermore, this composition ψ^n coincides with ϕ^n. For the rest of the subsection, we discuss base change to perfect points. We first consider the perfection of the generic point of V.

Set V^∞ to denote the not-necessarily-Noetherian scheme $\operatorname{Spec} \mathscr{O}_V^{1/p^\infty}$, and set η_∞ to be the generic point of V^∞ with perfect fraction field $k(V^\infty)$ (note that this is a perfect point). We obtain a map $f_{k(V^\infty)}\colon X_{k(V^\infty)} = X \times_V \operatorname{Spec} k(V^\infty) \to \operatorname{Spec} k(V^\infty)$; now, $X_{k(V^\infty)}$ is a scheme of finite type over a perfect field.

As in Section 2.2, the map ϕ then induces a map

$$\phi_\infty\colon L^{1/p^e} \otimes_{k(V^e)} k(V^\infty) \to R_{k(V^\infty)} := R \otimes_A k\left(A^{1/p^\infty}\right),$$

which can be identified with

$$\psi_\infty\colon \left(L_{k(V^\infty)}\right)^{1/p^e} \to R_{k(V^\infty)}$$

since $\left((F^e_{X^e/V^e})_* \mathscr{O}_{X^e}\right) \otimes_{V^e} k(V^\infty) \cong F^e_* \mathscr{O}_{X_{k(V^\infty)}}$.

More generally, if $s \in V$ is any perfect point, we can induce

$$\phi_s\colon L_s^{1/p^e} := L^{1/p^e} \otimes_{A^{1/p^e}} k(s)^{1/p^e} \to R \otimes_A k(s)^{1/p^e}.$$

Since $k(s) = k(s)^{1/p^e}$ is perfect, we can identify this with a p^{-e}-linear map

$$\psi_s\colon L_s^{1/p^e} \to R_s.$$

As above, we see that the composition of ψ_s in the sense of [BS13, Section 4.1] coincides with the composition ϕ as in Section 2.1. Finally, we study this process with respect to divisors.

LEMMA 2.21. *With notation as above, $\Delta_\phi|_{X_s} = \Delta_{\psi_s}$, where Δ_ϕ is as in Definition 2.10 and Δ_{ψ_s} is as in [BS13, Section 4].*

Proof. By Lemma 2.19, we simply must show that Δ_{ϕ_s} coincides with Δ_{ψ_s}. Working locally and removing a set of relative codimension 2, we may assume that $\Phi \in \operatorname{Hom}_{R_{s^{1/p^e}}}\left((R_s)^{1/p^e}, R_{s^{1/p^e}}\right)$ generates the group Hom as an $(R_s)^{1/p^e}$-module and that $\phi(\_) = \Phi(z^{1/p^e} \cdot \_)$. Thus, $\Delta_\phi = (1/(p^e - 1)) \operatorname{div}_X(z)$.

We then identify $R_{s^{1/p^e}}$ with R_s and hence ϕ with ψ. Likewise, we can identify Φ with Ψ, which now generates $\operatorname{Hom}_{R_s}\left((R_s)^{1/p^e}, R_s\right)$ as an $(R_s)^{1/p^e}$-module. Hence $\psi(\_) = \Psi(z^{1/p^e} \cdot \_)$ and so $\Delta_\psi = (1/(p^e - 1)) \operatorname{div}_X(z)$, as desired. $\qquad \square$

3. Relative non-F-pure ideals

With notation as above, by the Hartshorne–Speiser–Lyubeznik–Gabber theorem [Gab04, Lemma 13.1], cf. [HS77, Lyu97, Bli13], we know that the chain

$$R_{k(V^\infty)} \supseteq \psi_\infty\left(\left(L_{k(V^\infty)}\right)^{1/p^e}\right) \supseteq \psi_\infty^2\left(\left(L_{k(V^\infty)}^{p^e+1}\right)^{1/p^{2e}}\right) \supseteq \cdots \supseteq \psi_\infty^n\left(\left(L_{k(V^\infty)}^{\frac{p^{ne}-1}{p^e-1}}\right)^{1/p^{ne}}\right) \supseteq \cdots$$

eventually stabilizes. Say that it stabilizes at $n \geqslant n_0$. For the rest of this section, we fix this integer n_0 and make the following definition.

DEFINITION 3.1. With notation as above, we define the integer n_0 to be the *uniform integer for σ over the generic point of V*; in general, it will be denoted by $n_{\sigma(\phi),k(V)}$. We notice that for any point $\eta \in V$, we can base change $\operatorname{Spec} k(\eta) \to V$ and form a corresponding integer $n_{\sigma(\phi_\eta),k(\eta)}$.

On the other hand, without the passing to $k(V^\infty)$, we have the images

$$
\begin{aligned}
\mathfrak{a}_1 &:= \phi^1\big(L^{1/p^e}\big) &&\subseteq R_{A^{1/p^e}}\,, \\
\mathfrak{a}_2 &:= \phi^2\big((L^{(p^e+1)})^{1/p^{2e}}\big) &&\subseteq R_{A^{1/p^{2e}}}\,, \\
&\ \ \vdots \\
\mathfrak{a}_n &:= \phi^n\big((L^{\frac{p^{ne}-1}{p^e-1}})^{1/p^{ne}}\big) &&\subseteq R_{A^{1/p^{ne}}}\,, \\
&\ \ \vdots
\end{aligned}
$$

These are ideals of different rings. However, we do have the following relation for any $i > j$:

$$
\operatorname{im}\big(\mathfrak{a}_j \otimes_{A^{1/p^j}} A^{1/p^i} \to R_{A^{1/p^i}}\big) \supseteq \mathfrak{a}_i\,.
$$

This is straightforward, so we leave it to the reader to check (note that the image of (2.4) contains the image of ϕ^2). Additionally, observe that if A were regular, A^{1/p^e} would be flat over A by [Kun69], and so we could identify the tensor product $\mathfrak{a}_j \otimes_{A^{1/p^j}} A^{1/p^i}$ with its image in $R_{A^{1/p^i}}$. Therefore, for any integer $n \geqslant i$, we set

$$
\mathfrak{a}_{i,n} = \operatorname{im}\big(\mathfrak{a}_i \otimes_{A^{1/p^i}} A^{1/p^n} \to R_{A^{1/p^n}}\big)
$$

and consider the chain of ideals

$$
R_{A^{1/p^{ne}}} \supseteq \mathfrak{a}_{1,n} \supseteq \mathfrak{a}_{2,n} \supseteq \cdots \supseteq \mathfrak{a}_{n-1,n} \supseteq \mathfrak{a}_{n,n}\,.
$$

DEFINITION 3.2 (nth relative non-F-pure ideal). For every integer $n \geqslant n_0 = n_{\sigma(\phi),k(V)}$, we define the *$n$th limiting relative non-F-pure ideal* to be $\mathfrak{a}_{n,n} = \mathfrak{a}_n$. It is denoted by $\sigma_n(X/V,\phi)$.

We now obtain a relative version of the Hartshorne–Speiser–Lyubeznik–Gabber theorem.

PROPOSITION 3.3. *Fix the notation as above. Then there exists a non-empty open subset $U \subseteq V$ of the base scheme V satisfying the following for every integer $m \geqslant n \geqslant n_0$:*

$$
\sigma_m(X/V,\phi)|_{f^{-1}(U)} = \operatorname{im}\big(\sigma_n(X/V,\phi) \otimes_{A^{1/p^{ne}}} A^{1/p^{me}} \to R_{A^{1/p^{me}}}\big)\big|_{f^{-1}(U)}\,. \tag{3.1}
$$

Proof. For any $m \geqslant n$, we consider the containment $\mathfrak{a}_{n,m} \supseteq \mathfrak{a}_{m,m}$. Fix $k(V)$ to be the residue field of the generic point $\eta \in V$, and consider the induced containment

$$
\mathfrak{a}_{n,m} \otimes_A k(V) = \mathfrak{a}_{n,m} \otimes_{A^{1/p^m}} k(V)^{1/p^m} \supseteq \mathfrak{a}_{m,m} \otimes_{A^{1/p^m}} k(V)^{1/p^m} = \mathfrak{a}_{m,m} \otimes_A k(V)\,,
$$

since inverting an element is the same as inverting its pth power. We notice two identifications

$$
\mathfrak{a}_{n,m} \otimes_{A^{1/p^m}} k(V)^{1/p^\infty} = \mathfrak{a}_{n,m} \otimes_{A^{1/p^m}} k(V^\infty) = \psi_\infty^n\left(\left(L_{k(V^\infty)}^{\frac{p^{ne}-1}{p^e-1}}\right)^{1/p^{ne}}\right),
$$

$$
\mathfrak{a}_{m,m} \otimes_{A^{1/p^m}} k(V)^{1/p^\infty} = \mathfrak{a}_{m,m} \otimes_{A^{1/p^m}} k(V^\infty) = \psi_\infty^m\left(\left(L_{k(V^\infty)}^{\frac{p^{me}-1}{p^e-1}}\right)^{1/p^{me}}\right).
$$

Thus,

$$\mathfrak{a}_{n,m} \otimes_A k(V)^{1/p^\infty} = \mathfrak{a}_{m,m} \otimes_A k(V)^{1/p^\infty}$$

since $m > n \geqslant n_0$. It immediately follows that

$$\mathfrak{a}_{n,m} \otimes_{A^{1/p^m}} k(V)^{1/p^m} = \mathfrak{a}_{m,m} \otimes_{A^{1/p^m}} k(V)^{1/p^m}$$

since $k(V)^{1/p^m} \subseteq k(V)^{1/p^\infty}$ is a faithfully flat extension. But now by generic freeness [Eis95, Theorem 14.4], for a *fixed* m, (3.1) follows for some open set $U_{n,m}$ (in the case that V is affine, which we can certainly reduce to, we can invert a single element of A to form $U_{n,m}$).

We now vary m. Choose a $U = U_{n,n+1}$ that works for $m = n+1$ and consider the diagram
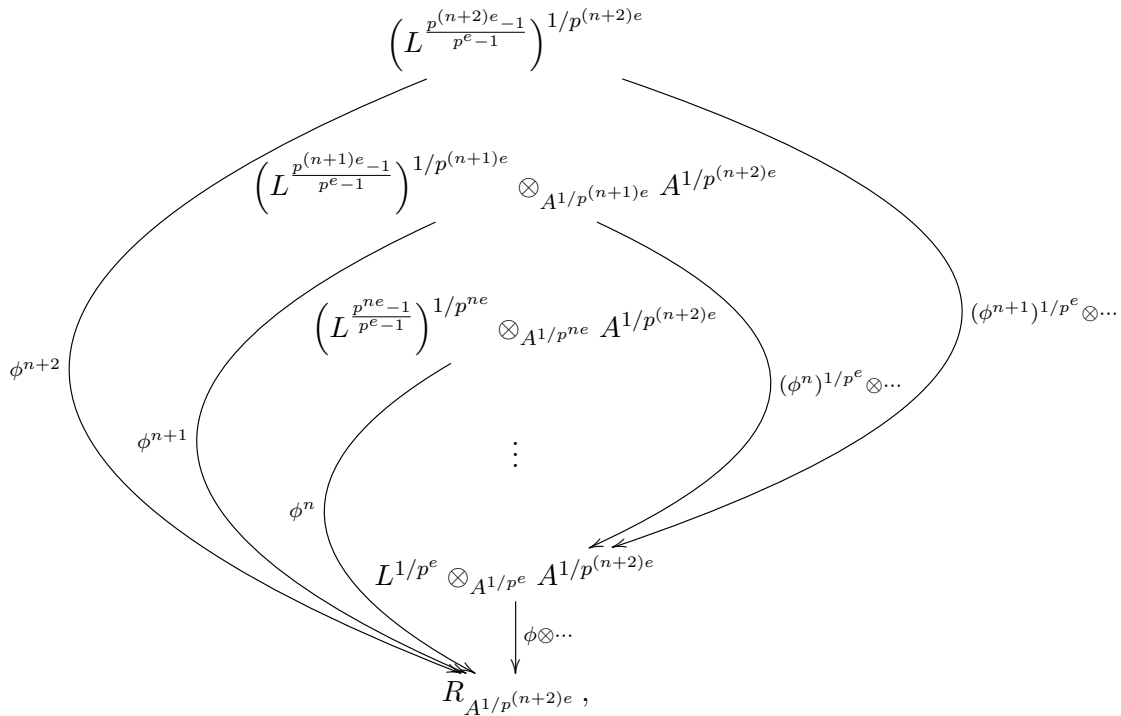
where the maps labeled $(\phi^n)^{1/p^e} \otimes \cdots$ and $(\phi^{n+1})^{1/p^e} \otimes \cdots$ are induced as in (2.3). We know that over U, the maps ϕ^{n+1} and ϕ^n have the same image. Therefore, so do $(\phi^n)^{1/p^e} \otimes \cdots$ and $(\phi^{n+1})^{1/p^e} \otimes \cdots$, again over U (since tensor is right exact). But then, composing with $(\phi \otimes \cdots)$ one more time, we know that $\phi^{n+1} = (\phi \otimes \cdots) \circ ((\phi^n)^{1/p^e} \otimes \cdots)$ and $\phi^{n+2} = (\phi \otimes \cdots) \circ ((\phi^{n+1})^{1/p^e} \otimes \cdots)$ also have the same image over U. Thus, they also share the image with ϕ^n over U. Hence, if $\mathfrak{a}_{n,n+1}|_U = \mathfrak{a}_{n+1,n+1}|_U$, then $\mathfrak{a}_{n,n+2}|_U = \mathfrak{a}_{n+2,n+2}|_U$. We iterate this to obtain that $\mathfrak{a}_{n,m}|_U = \mathfrak{a}_{m,m}|_U$, which is exactly the statement of the proposition. $\square$

We give three examples of these σ_n and U_n. In the first example, we show that it is possible that the images $\sigma_n(X/V, \phi)$ never stabilize in the sense of Proposition 3.3 on all of V but only over an open set. We do the same in the second example, but with respect to a more interesting choice of X and ϕ. Finally, we give an example where stabilization occurs at $n = 2$ (instead of at $n = 1$).

EXAMPLE 3.4. Fix k to be an algebraically closed field of characteristic $p > 2$, set $A = k[t]$, and set $R = k[x,t]$ with the obvious map $X \to V$. Let $\phi \colon R^{1/p} = k[x^{1/p}, t^{1/p}] \to R_{A^{1/p}} = k[x, t^{1/p}]$ be the composition of the local generator $\beta \in \operatorname{Hom}_{R_{A^{1/p}}}(R^{1/p}, R_{A^{1/p}})$ with pre-multiplication by

$t^{1/p}$ (note that since ϕ is $k[t^{1/p}]$-linear, this is also post-multiplication by $t^{1/p}$, and it corresponds to the divisor $\Delta_\phi = (1/(p-1))\{t = 0\}$). It easily follows that the image of ϕ is $\langle t^{1/p} \rangle \subseteq k[x, t^{1/p}]$. By tensoring with $\otimes_{k[t^{1/p}]} k[t^{1/p^2}]$, we obtain $\mathfrak{a}_{1,2} = \langle t^{1/p} \rangle = t^{1/p} \cdot k[x, t^{1/p^2}]$. More generally, we see that $\mathfrak{a}_{1,b} = t^{1/p} \cdot k[x, t^{1/p^n}]$. Now, we compute $\mathfrak{a}_{2,2}$ as the image of

$$k[x^{1/p^2}, t^{1/p^2}] \xrightarrow{\phi^{1/p}} k[x^{1/p}, t^{1/p^2}] \xrightarrow{\phi \otimes \cdots} k[x, t^{1/p^2}].$$

The image of $\phi^{1/p}$ is t^{1/p^2} and since the map $\phi \otimes \cdots$ is $k[t^{1/p^2}]$-linear, we see that the composition has image $\langle t^{1/p} \cdot t^{1/p^2} \rangle = \langle t^{(1+p)/p^2} \rangle = \mathfrak{a}_{2,2}$. In general, we see that

$$\mathfrak{a}_n = \mathfrak{a}_{n,n} = \langle t^{(1+p+\cdots+p^{n-1})/p^n} \rangle.$$

In particular, while we may take $U_i = \mathbb{A}^1 \setminus \{0\} = \operatorname{Spec} A \setminus \langle t \rangle$, we see that $\mathfrak{a}_n = \sigma_n(X/V, \phi)$ never stabilizes over all of V.

EXAMPLE 3.5. Let f be the morphism $X := \operatorname{Spec}\big(k[x, y, t]/(y^2 + x^3 + t)\big) \to V := \operatorname{Spec}(k[t])$. Consider the standard trace map $\phi \colon (F_{X^1/V^1})_* \omega_{X^1/V^1} \to \mathscr{O}_{X_{V^1}}$, for which $\Delta_\phi = 0$. This map can be identified with the descent of the following map to the quotient

$$k[x, y, t] \xrightarrow{\cdot (y^2 + x^3 + t)^{p-1}} k[x, y, t] \xrightarrow{\operatorname{Tr}} k[x, y, t],$$

where Tr is a k-linear map such that

$$\operatorname{Tr}(x^i y^j t^l) = \begin{cases} x^{\frac{i+1-p}{p}} y^{\frac{j+1-p}{p}} t^l & \text{if } p | i+1 \text{ and } p | j+1, \\ 0 & \text{otherwise}. \end{cases}$$

Then ϕ^n is given by

$$k[x, y, t] \xrightarrow{\cdot (y^2 + x^3 + t)^{p^n - 1}} k[x, y, t] \xrightarrow{\operatorname{Tr_n}} k[x, y, t],$$

where $\operatorname{Tr_n}$ is a k-linear map such that

$$\operatorname{Tr_n}\left(x^i y^j t^l\right) = \begin{cases} x^{\frac{i+1-p^n}{p^n}} y^{\frac{j+1-p^n}{p^n}} t^l & \text{if } p^n | i+1 \text{ and } p^n | j+1, \\ 0 & \text{otherwise}. \end{cases} \tag{3.2}$$

Assume that p is a prime such that $p \equiv 1 \pmod 6$, so that $2, 3 | p^n - 1$ for all $n > 0$. Let us now try to compute the following number:

$$d := \min\left\{ c \mid t^c \in \operatorname{im} \phi^n \right\}.$$

To have $t^c \in \operatorname{im} \phi^n$, it is necessary to have a polynomial in the ideal generated by $(y^2 + x^3 + t)^{p^n - 1}$, of which one of the non-zero monomials is $x^{p^n - 1} y^{p^n - 1} t^c$. Therefore, $(y^2 + x^3 + t)^{p^n - 1}$ itself has to have a non-zero monomial which divides $x^{p^n - 1} y^{p^n - 1} t^c$. That is, there have to be integers $0 \leqslant a \leqslant (p^n - 1)/2$ and $0 \leqslant b \leqslant (p^n - 1)/3$, such that $c = p^n - 1 - a - b$. So, we see that

$$d \geqslant p^n - 1 - \frac{p^n - 1}{2} - \frac{p^n - 1}{3} = \frac{p^n - 1}{6}.$$

On the other hand, we claim that $\operatorname{Tr_n}(y^2 + x^3 + t)^{p^n - 1} = t^{(p^n - 1)/6}$, which will show that $d = (p^n - 1)/6$ and also that $\sigma_n(X/V, \phi)$ does not stabilize. First, note that $x^{(p^n - 1)/2} y^{(p^n - 1)/3} t^{(p^n - 1)/6}$ is the only monomial in the expansion of $(y^2 + x^3 + t)^{p^n - 1}$ with non-zero image via $\operatorname{Tr_n}$. Indeed, the expansion can contain only monomials of the form $y^{2a} x^{3b} t^{p^n - 1 - a - b}$, where $a, b \geqslant 0$ are integers. Hence, higher powers ($2p^n - 1, 3p^n - 1$, etc.) of x and y that do not go to zero by $\operatorname{Tr_n}$ cannot be

obtained. Hence, as we stated, $x^{(p^n-1)/2}y^{(p^n-1)/3}t^{(p^n-1)/6}$ is the only interesting monomial that can show up in the expansion. Last, we have to verify that this monomial has non-zero coefficient in the expansion. That is,

$$\frac{(p^n-1)!}{\left(\frac{1}{2}(p^n-1)\right)!\left(\frac{1}{3}(p^n-1)\right)!\left(\frac{1}{6}(p^n-1)\right)!} \neq 0\,.$$

Equivalently, we have to show that the power of p in the prime factorization of the numerator is the same as in the denominator. For an arbitrary number m, this number for $m!$ is

$$\sum_{i=1}^{\infty} \left\lfloor \frac{m}{p^i} \right\rfloor\,.$$

Now, assume that $m = (p^n-1)/r$, where $r|p-1$. Then the number of times p divides $m!$ is

$$\sum_{i=1}^{\infty} \left\lfloor \frac{m}{p^i} \right\rfloor = \sum_{i=1}^{n-1} \left\lfloor \frac{p^n-1}{rp^i} \right\rfloor = \sum_{i=1}^{n-1} \left\lfloor \sum_{j=0}^{n-1} p^j \frac{p-1}{rp^i} \right\rfloor = \sum_{i=1}^{n-1} \left\lfloor \sum_{j=0}^{n-1} p^{j-i} \frac{p-1}{r} \right\rfloor$$

$$= \sum_{i=1}^{n-1} \sum_{j=i}^{n-1} p^{j-i} \frac{p-1}{r} = \sum_{i=1}^{n-1} \frac{p^{n-i}-1}{p-1} \frac{p-1}{r} = \sum_{i=1}^{n-1} \frac{p^{n-i}-1}{r} = \sum_{i=1}^{n-1} \frac{p^i-1}{r}$$

$$= \sum_{i=0}^{n-1} \frac{p^i-1}{r} = \frac{p^n-1}{r}\,.$$

Therefore, p divides p^n-1 times both $(p^n-1)!$ and $\left(\frac{1}{2}(p^n-1)\right)!\left(\frac{1}{3}(p^n-1)\right)!\left(\frac{1}{6}(p^n-1)\right)!$, as claimed earlier. Summarizing, $t^{(p^n-1)/6}$ is the smallest power of t that is in $\sigma_n(X/V,\phi)$. Hence, there is no stabilization over all of V.

Based on this example, one might ask the following.

QUESTION 3.6. *Is there a relation between the asymptotics of the σ_n over the non-stable locus and the singularities of those fibers (for example, the F-pure threshold)?*

We now give an example that stabilizes at the second step (over the entire base).

EXAMPLE 3.7 (cf. [MY09, Example 4.7]). Fix k to be an algebraically closed field of characteristic $p > 2$, set $A = k[t]$, and set $R = k[x,t]$ with the obvious map $X \to V$. Let $\phi\colon R^{1/p} = k[x^{1/p}, t^{1/p}] \to R_{A^{1/p}} = k[x, t^{1/p}]$ be the composition of the local generator $\beta \in \operatorname{Hom}_{R_{A^{1/p}}}\left(R^{1/p}, R_{A^{1/p}}\right)$ with pre-multiplication by $\left(x^{p^2}+t\right)^{1/p} = f^{1/p}$ (which corresponds to $\Delta_\phi = (1/(p-1))\{x^{p^2} = t\}$ restricting to $(p^2/(p-1))\{x = \lambda^{1/p^2}\}$ on the fiber over $t = \lambda$). Note that in $R_{A^{1/p}}$ or $R^{1/p}$, the map $f^{1/p}$ can be written as $x^p + t^{1/p}$. In particular, $f^{1/p}$ is already an element of $R_{A^{1/p}}$. Therefore, since ϕ is $R_{A^{1/p}}$-linear and β is clearly surjective, the image of ϕ is just $\langle f^{1/p} \rangle = \sigma_1(X/V,\phi) = \langle x^p + t^{1/p} \rangle$.

On the other hand, we now compose ϕ with itself as described above. In this case, $\phi^2\colon R^{1/p^2} \to R_{A^{1/p^2}}$ is induced by taking the generator $\beta^2\colon R^{1/p^2} \to R_{A^{1/p^2}}$ and pre-multiplying by $f^{(p+1)/p^2}$. Now, $f^{(p+1)/p^2} = \left(x^1 + t^{1/p^2}\right)^{p+1}$. This again is already an element of $R_{A^{1/p^2}}$, and so by the same argument as above, we see that

$$\sigma_2(X/V,\phi) = \langle x + t^{1/p^2} \rangle^{p+1} = \sigma_1(X/V,\phi) \cdot \langle x + t^{1/p^2} \rangle = \langle f^{(1+p)/p^2} \rangle\,.$$

Next, we form ϕ^3. In this case, it is obtained by taking a generator $R^{1/p^3} \to R_{A^{1/p^3}}$ and pre-multiplying by $f^{(1+p+p^2)/p^3}$. However, this time $f^{(1+p+p^2)/p^3}$ is *not* contained in $R_{A^{1/p^3}}$, and so we cannot argue as above. However, f still has a p^2rd root in $R_{A^{1/p^3}}$, and so

$$\phi^3\left(\left\langle f^{\frac{1+p+p^2)}{p^3}}\right\rangle\right) = \phi^3\left(f^{p(1+p)}\left\langle f^{1/p^3}\right\rangle\right) = f^{\frac{1+p}{p^2}}\phi^3\left(\left\langle f^{1/p^3}\right\rangle\right).$$

So we must only compute $\phi^3\left(\left\langle f^{1/p^3}\right\rangle\right)$. Now, we may take $\phi^3 \colon k\left[x^{1/p^3}, t^{1/p^3}\right] \to k\left[x, t^{1/p^3}\right]$ as the map which sends $x^{(p^3-1)/p^3}$ to 1 and other monomials in x to 0 (those monomials form a basis for R^{1/p^3} over $R_{A^{1/p^3}}$). It is thus obvious that $\phi^3\left(\left\langle f^{1/p^3}\right\rangle\right) = k\left[x, t^{1/p^3}\right] = R_{A^{1/p^3}}$. In particular,

$$\sigma_3(X/V, \phi) = \left\langle f^{\frac{1+p}{p^2}}\right\rangle = \sigma_2(X/V, \phi) \otimes_{A^{1/p^2}} A^{1/p^3}.$$

We now obtain a surprising base change statement.

PROPOSITION 3.8. *Suppose that $T \to V$ is a map from an excellent integral scheme with a dualizing complex; then, using the notation of Section 2.2,*

$$\mathrm{im}\left((q_{ne})^*\sigma_n(X/V, \phi) \to \mathscr{O}_{X_{Tne}}\right) = \sigma_n(X/V, \phi) \cdot \mathscr{O}_{X_{Tne}} = \sigma_n(X_T/T, \phi_T).$$

Furthermore, if U satisfies condition (3.1) from Proposition 3.3, then $W = g^{-1}(U) \subseteq T$ satisfies the same condition for $\sigma_n(X_T/T, \phi_T)$.

Proof. Indeed, images of maps are compatible with arbitrary base change by the right exactness of tensor. Thus, the first statement follows immediately. The second statement follows from the first since if the two images $\mathfrak{a}_{n,n}$ and $\mathfrak{a}_{n-1,n}$ are equal, they are also equal after base change. $\square$

THEOREM 3.9 (Base change for σ_n). *There exists an integer $N \geqslant 0$ such that for all points $s \in V$, we have $N \geqslant n_{\sigma(\phi_s),k(s)}$. In other words, we have both that $\sigma_n(X/V, \phi) \cdot \mathscr{O}_{X_{sne}} = \sigma_n(X_s/s, \phi_s)$ (which always holds) and also that for all $m \geqslant n \geqslant N$,*

$$\sigma_n(X/V, \phi) \otimes_{V^n} k(s)^{1/p^m} = \sigma_m(X_s/s, \phi_s).$$

Proof. Of course, the statement already holds on U with respect to some integer N_0. Set $V_1' := V \setminus U$ to be the complement with reduced scheme structure, and let

$$i_1 \colon V_1 = (V_1')_{\mathrm{reg}} \hookrightarrow V_1'$$

be the regular locus. We notice that V_1 has dimension strictly less than $\dim V$. We consider the base change $X_{V_1} \to V_1$. Each fiber of $X_{V_1} \to V_1$ is isomorphic to a fiber of $X \to V$. Choose an open set $U_1 \subseteq V_1$ for which the statement holds for some integer N_1.

Now, fix $V_2' = V_1' \setminus U_1$ and $i_2 \colon V_2 = (V_2')_{\mathrm{reg}} \hookrightarrow V_2'$ to be the regular locus and repeat. This process terminates by Noetherian induction.

Setting $N = \max\{N_0, N_1, N_2, \dots\}$ completes the proof. $\square$

Now, suppose that $s \in V$ is a perfect point; see Definition 2.2. It follows that X_s/s is a variety over a perfect field and so since $s^e \cong s$, we have $X_{s^e} \cong X_s$. Thus, we can identify $\phi_s \colon L_s^{1/p^e} \to R_{s^e}$ with a p^{-e}-linear map ψ_s as in Section 2.3. Thus, under these identifications, $\bigcap_{n \geqslant 0} \sigma_n(X_s/s, \phi) = \sigma(X_s, \psi_s)$. However, if we have the stabilization $\sigma_n(X_s/s, \phi) \cong \sigma_n(X_s/s, \phi) \otimes_{k(s)^{1/p^{ne}}} k(s)^{1/p^{me}} \cong \sigma_m(X_s/s, \phi)$ for every $m \geqslant n$, the equality $\sigma_n(X_s/s, \phi) = \sigma(X_s, \psi)$ holds. Combining this with our previous work, we obtain the following corollary.

COROLLARY 3.10. *With notation as above, there exists an integer $N \geqslant 0$ such that*

$$\sigma_n(X/V, \phi) \cdot \mathscr{O}_{X_s^{ne}} = \sigma(X_s, \psi_s)$$

for all perfect points $s \in V$ and $n \geqslant N$.

3.1 Iterated non-F-pure ideals versus the absolute non-F-pure ideal

We now compare the iterated non-F-pure ideal with the absolute non-F-pure ideal. First however, we make the following assumption.

CONVENTION 3.11. In this subsection, Section 3.1, we always assume that our base V is an F-pure, quasi-Gorenstein (that is, ω_V is a line bundle), F-finite, integral scheme.

Let $\Psi_V^e \colon F_*^e \mathscr{O}_V((1-p^e)K_V) \to \mathscr{O}_V$ be a map generating $\mathscr{H}om_V(F_*^e\mathscr{O}_V((1-p^e)K_V), \mathscr{O}_V)$ as an $F_*^e\mathscr{O}_V$-module. By taking p^{ne}th roots, applying f^{-1}, and then writing the A-module $M := f^{-1}(\mathscr{O}_V((1-p^e)K_V))$, we obtain

$$\left(\Phi_A^e\right)^{1/p^{ne}} \colon M^{1/p^{(n+1)e}} = f^{-1}F_*^{(n+1)e}\left(\mathscr{O}_V((1-p^e)K_V)\right) \to f^{-1}F_*^{ne}\mathscr{O}_V = A^{1/p^{ne}}.$$

We tensor with L^{1/p^e} over A^{1/p^e} and so obtain

$$\Phi' \colon L^{1/p^e} \otimes_{A^{1/p^e}} M^{1/p^{(n+1)e}} \to L^{1/p^e} \otimes_{A^{1/p^e}} A^{1/p^{ne}}. \tag{3.3}$$

Set $N = L \otimes_A M^{1/p^{ne}}$. We observe that N is a line bundle on $X_{V^{ne}}$. In fact, if $q_1 \colon X_{V^{ne}} = X \times_V V^{ne} \to X$ is the first projection and $q_2 \colon X_{V^{ne}} \to V^{ne}$ is the second projection, then

$$N = q_1^* L \otimes_{\mathscr{O}_{X_{V^{ne}}}} q_2^* \mathscr{O}_{V^{ne}}\left((1-p^e)K_{V^{ne}}\right)$$

and so it follows for any integer $l > 0$ that

$$N^l \cong q_1^* L^l \otimes_{\mathscr{O}_{X_{V^{ne}}}} q_2^* \mathscr{O}_{V^{ne}}\left(l(1-p^e)K_{V^{ne}}\right) = L^l \otimes_A \left(M^l\right)^{1/p^{ne}}. \tag{3.4}$$

Finally, we compose (3.3) with $\phi' := \phi \otimes_{A^{1/p^e}} A^{1/p^{ne}}$ to construct

$$\gamma \colon N^{1/p^e} = L^{1/p^e} \otimes_{A^{1/p^e}} M^{1/p^{(n+1)e}} \xrightarrow{\Phi'} L^{1/p^e} \otimes_{A^{1/p^e}} A^{1/p^{ne}}$$

$$\xrightarrow{\phi'} R_{A^{1/p^e}} \otimes_{A^{1/p^e}} A^{1/p^{ne}} = R \otimes_A A^{1/p^{ne}} = R_{A^{1/p^{ne}}}. \tag{3.5}$$

This is a map from an invertible sheaf on $X_{V^{(n+1)e}}^e = (X_{V^{ne}})^e$ to the structure sheaf on $X_{V^{ne}}$. In particular, it is a map such as one studied in [BS13, Section 4].

Let us point out an alternate way to construct γ. Indeed, take

$$\phi'' := \phi \otimes_{A^{1/p^e}} M^{1/p^{(n+1)e}} \colon L^{1/p^e} \otimes_{A^{1/p^e}} M^{1/p^{(n+1)e}} \to R_{A^{1/p^e}} \otimes_{A^{1/p^e}} M^{1/p^{(n+1)e}}$$

$$\cong R \otimes_A M^{1/p^{(n+1)e}}.$$

We can then compose this with $R_{A^{1/p^e}} \otimes_{A^{1/p^e}} \left(\Phi_A^e\right)^{1/p^{ne}} = \left(\Phi_A^e\right)'$ to obtain

$$N^{1/p^e} = L^{1/p^e} \otimes_{A^{1/p^e}} M^{1/p^{(n+1)e}} \xrightarrow{\phi''} R_{A^{1/p^e}} \otimes_{A^{1/p^e}} M^{1/p^{(n+1)e}}$$

$$\xrightarrow{(\Phi_A^e)'} R_{A^{1/p^e}} \otimes_{A^{1/p^e}} A^{1/p^{ne}} = R_{A^{1/p^{ne}}}.$$

This composition is easily seen to coincide with γ.

We can then compose γ with itself m times as in [BS13, Section 4] or [Sch09]. We recall this construction for the benefit of the reader. To construct γ^2, we tensor the map

$$\gamma \colon N^{1/p^e} \to R \otimes_A A^{1/p^{ne}} = R_{A^{1/p^{ne}}}$$

with the line bundle N over $R_{A^{1/p^{ne}}}$, take p^eth roots, and then compose it with γ to obtain

$$\gamma^2 \colon \left(N^{1+p^e}\right)^{1/p^{2e}} \to N^{1/p^e} \xrightarrow{\gamma} R \otimes_A A^{1/p^{ne}}.$$

Recursively, we can construct

$$\gamma^m \colon \left(N^{\frac{p^{me}-1}{p^e-1}}\right)^{1/p^{me}} = \left(L^{\frac{p^{me}-1}{p^e-1}}\right)^{1/p^{me}} \otimes_{A^{1/p^{me}}} \left(M^{\frac{p^{me}-1}{p^e-1}}\right)^{1/p^{(n+m)e}}$$
$$\to R_{A^{1/p^{ne}}}.$$

We are now in a position to relate the absolute $\sigma(X_{V^{1/p^{ne}}}, \gamma)$ with the relative $\sigma_n(X/V, \phi)$.

First, we observe that for each $m \leqslant n$, the map Φ_A^m induces maps

$$\mu_m \colon \left(N^{\frac{p^{me}-1}{p^e-1}}\right)^{1/p^{me}} = \left(L^{\frac{p^{me}-1}{p^e-1}}\right)^{1/p^{me}} \otimes_{A^{1/p^{me}}} \left(M^{\frac{p^{me}-1}{p^e-1}}\right)^{1/p^{(n+m)e}}$$
$$\xrightarrow{\cdots \otimes (\Phi_A^m)^{1/p^{ne}}} \left(L^{\frac{p^{me}-1}{p^e-1}}\right)^{1/p^{me}} \otimes_{A^{1/p^{me}}} A^{1/p^{ne}}.$$

Furthermore, these maps are surjective since Φ_A^m is.

LEMMA 3.12. *Assuming Convention* 3.11, *we have the factorization of γ^n and γ^{n-1} indicated by the diagram below:*

where the arrow γ' is induced by γ and $\phi^{n-1} \otimes \cdots$ is simply $\phi^{n-1} \otimes_{A^{1/p^{(n-1)e}}} A^{1/p^{ne}}$. Furthermore, since μ_{n-1} is surjective, we have

$$\mathrm{im}\left(\gamma^{n-1}\right) = \mathfrak{a}_{n-1,n} = \mathrm{im}\left(\phi^{n-1} \otimes_{A^{1/p^{(n-1)e}}} A^{1/p^{ne}}\right).$$

Likewise, since μ_n is surjective, we have

$$\mathrm{im}\left(\gamma^n\right) = \mathfrak{a}_{n,n} = \mathfrak{a}_n = \mathrm{im}\left(\phi^n\right).$$

Proof. The two equalities are immediate from the surjectivity of μ_{n-1} and μ_n and the commutativity of our diagram. The surjectivity of μ_{n-1} and μ_n is clear since $M^{1/p^{(n+1)e}} \to A^{1/p^{ne}}$ is surjective and tensor is right exact. It remains to prove the commutativity of our diagram. It suffices to prove the commutativity of the square

we will prove it by induction on m. It is clear that if the above square is commutative for m, then, by tensoring it with $L \otimes_R (-) \otimes_{A^{1/p^{ne}}} M^{1/p^{(n+1)e}}$ and taking p^eth roots, we will have the

commutativity for $m + 1$. Hence, it remains to prove the commutativity when $m = 2$. To this end, we will denote the map $M^{1/p^{(n+1)e}} \to A^{1/p^{ne}}$ by α. Then α induces a map

$$\alpha' = (\mathrm{id} \otimes \alpha)^{1/p^e} : \left(M^{\frac{p^{2e}-1}{p^e-1}} \right)^{1/p^{(n+2)e}} = \left(M^{1/p^{ne}} \otimes_{A^{1/p^{ne}}} M^{1/p^{(n+1)e}} \right)^{1/p^e}$$

$$\to M^{1/p^{ne}} \otimes_{A^{1/p^{ne}}} A^{1/p^{ne}} = M^{1/p^{(n+1)e}} .$$

It is clear that $\alpha \circ \alpha' = \alpha^2$. Likewise, the map $\varphi \colon L^{1/p^e} \to R \otimes_A A^{1/p^e}$ induces

$$\varphi' = (\mathrm{id} \otimes \varphi)^{1/p^e} : \left(L^{\frac{p^{2e}-1}{p^e-1}} \right)^{1/p^{2e}} = \left(L^{1/p^e} \otimes_{R^{1/p^e}} L^{1/p^{2e}} \right)^{1/p^e} \to L^{1/p^e} \otimes_{A^{1/p^e}} A^{1/p^{2e}} .$$

We will analyze the four maps involved in the square. The left vertical map

$$\mu_2 : \left(L^{\frac{p^{2e}-1}{p^e-1}} \right)^{1/p^{2e}} \otimes_{A^{1/p^{2e}}} \left(M^{\frac{p^{2e}-1}{p^e-1}} \right)^{1/p^{(n+2)e}} \to \left(L^{\frac{p^{2e}-1}{p^e-1}} \right)^{1/p^{2e}}$$

is given by $\mathrm{id} \otimes (\alpha \circ \alpha')$; similarly, the right vertical map μ_1 is $\mathrm{id} \otimes \alpha$. The top horizontal map is given by

$$\left(L^{\frac{p^{2e}-1}{p^e-1}} \right)^{1/p^{2e}} \otimes_{A^{1/p^{2e}}} \left(M^{\frac{p^{2e}-1}{p^e-1}} \right)^{1/p^{(n+2)e}} \xrightarrow{\varphi' \otimes \alpha'} L^{1/p^e} \otimes_{A^{1/p^e}} A^{1/p^{2e}} \otimes_{A^{1/p^{2e}}} M^{1/p^{(n+1)e}}$$

$$\xrightarrow{\sim} L^{1/p^e} \otimes_{A^{1/p^e}} M^{1/p^{(n+1)e}} .$$

The bottom horizontal map is given by

$$\left(L^{\frac{p^{2e}-1}{p^e-1}} \right)^{1/p^{2e}} \otimes_{A^{1/p^{2e}}} A^{1/p^{ne}} \xrightarrow{\varphi' \otimes \mathrm{id}} L^{1/p^e} \otimes_{A^{1/p^e}} A^{1/p^{2e}} \otimes_{A^{1/p^{2e}}} A^{1/p^{ne}}$$

$$\xrightarrow{\sim} L^{1/p^e} \otimes_{A^{1/p^e}} A^{1/p^{ne}} .$$

Now, given an arbitrary $x \otimes z \in \left(L^{\frac{p^{2e}-1}{p^e-1}} \right)^{1/p^{2e}} \otimes_{A^{1/p^{2e}}} \left(M^{\frac{p^{2e}-1}{p^e-1}} \right)^{1/p^{(n+2)e}}$, write $\varphi'(x) = \sum_i x_i \otimes y_i$ with $x_i \in L^{1/p^e}$ and $y_i \in A^{1/p^{2e}}$. Then, on one hand, if we follow the top horizontal map and then the right vertical map, we have

$$x \otimes z \mapsto \left(\sum_i x_i \otimes y_i \right) \otimes \alpha'(z) \mapsto \sum_i x_i \otimes y_i \alpha'(z) \mapsto \sum_i x_i \alpha(y_i \alpha'(z)) = \sum_i x_i \otimes y_i \alpha(\alpha'(z)),$$

where the last equality holds since α is $A^{1/p^{ne}}$-linear and hence $A^{1/p^{2e}}$-linear.

On the other hand, if we follow the other path (that is, the left vertical map first and then the bottom horizontal map), we have

$$x \otimes z \mapsto x \otimes \alpha'(z) \mapsto x \otimes \alpha(\alpha'(z)) \mapsto \left(\sum_i x_i \otimes y_i \right) \otimes \alpha(\alpha'(z)) \mapsto \sum_i x_i \otimes y_i \alpha(\alpha'(z)).$$

This proves that the square is indeed commutative when $m = 2$ and concludes the proof of our lemma. $\qquad \square$

THEOREM 3.13. *With notation as in Convention* 3.11 *and below it, choose* $n > n_0 = n_{\sigma(\phi),k(V)}$. *Then, there exists a dense open set* $U \subseteq V \cong V^e$ *with* $W = f^{-1}(U) \subseteq X$ *such that*

$$\sigma\big(X_{V^{1/p^{ne}}}, \gamma\big)\big|_W = \sigma_n\big(X/V, \phi\big)\big|_W .$$

Furthermore, shrinking U *further if necessary, we can require for all perfect points* $u \in U$ *that*

$$\sigma\big(X_{V^{1/p^{ne}}}, \gamma\big) \cdot \mathscr{O}_{X_u} = \sigma_n\big(X_u/u, \phi_u\big) = \sigma\big(X_u, \psi_u\big) ,$$

where ψ_u *is* ϕ_u *viewed as an absolute* p^{-e}-linear map.

Proof. The second statement is immediate from the first by Proposition 3.8, so we need only prove the first statement. By the diagram in Lemma 3.12 and applying Proposition 3.3, we see that

$$\operatorname{im}\left(\gamma^n\right)\big|_W = \mathfrak{a}_{n,n}\big|_W = \mathfrak{a}_{n-1,n}\big|_W = \operatorname{im}\left(\gamma^{n-1}\right)\big|_W$$

for all $n > n_0$. Hence, $\operatorname{im}(\gamma^n)|_W = \sigma(X_{V^{1/p^{ne}}}, \gamma)|_W$, and the result follows. □

CONVENTION 3.14. Assuming everything in Convention 3.11, we additionally assume that ϕ is relatively divisorial.

Before we proceed, we prove a lemma about interpreting maps as divisors in the relative versus absolute setting.

LEMMA 3.15. *With notation as in Convention 3.14, let Δ_ϕ be the divisor corresponding to ϕ as in Definition 2.8. Then Δ_γ, the divisor on $X_{V^{ne}}$ corresponding to γ as in [BS13, Section 4], is equal to $h^*\Delta$, where $h\colon X_{V^{ne}} = X \times_V V^{ne} \to X$ is the projection.*

Proof. Working off a set of relative codimension 2, and then working locally on X and V, we can assume that L and M are trivial, and further assume that $\phi(\_) = \Phi\left(z^{1/p^e} \cdot \_\right)$, where Φ generates $\mathscr{H}om_{R_{A^{1/p^e}}}\left(R^{1/p^e}, R_{A^{1/p^e}}\right)$ as an R^{1/p^e}-module. Then Δ_ϕ is the divisor corresponding to $(1/(p^e - 1))\operatorname{div}_X(z)$. On the other hand, since the map Γ corresponding to Φ is a composition of generating maps, Γ generates $\mathscr{H}om_{R_{A^{1/p^{ne}}}}\left(R^{1/p^e}_{A^{1/p^{(n+1)e}}}, R_{A^{1/p^{ne}}}\right)$; see for example [Kun86, Appendix F], and so corresponds to the zero divisor. Furthermore, we see that $\gamma(\_) = \Gamma\left(\bar{z}^{1/p^e} \cdot \_\right)$ where $\bar{z}$ is the element corresponding to $z \otimes 1 \in R_{A^{1/p^{ne}}}$. Hence, $\Delta_\gamma = (1/(p^e - 1))\operatorname{div}_{X_{V^{ne}}}(\bar{z})$ and the claim follows. □

We now need a lemma which says roughly that in an F-pure, F-finite ring A with maximal ideal $\mathfrak{m}$, at least locally, one can always choose an $\alpha\colon A^{1/p^e} \to A$ such that the closed point $V(\mathfrak{m})$ is an F-pure center of Δ_α. To this end, we introduce the following sets:

$$M_{e,A} := \left\{\phi \in \operatorname{Hom}_A\left(A^{1/p^e}, A\right) \mid \phi\left(\mathfrak{m}^{1/p^e}\right) \subseteq \mathfrak{m}\right\}$$
$$N_{e,A} := \left\{\phi \in \operatorname{Hom}_A\left(A^{1/p^e}, A\right) \mid \phi\left(A^{1/p^e}\right) \subseteq \mathfrak{m}\right\}.$$

It is straightforward to check that both $M_{e,A}$ and $N_{e,A}$ are A^{1/p^e}-submodules of $\operatorname{Hom}_A\left(A^{1/p^e}, A\right)$ and that $N_{e,A} \subseteq M_{e,A}$. We now observe the following result.

LEMMA 3.16. *For any F-finite, reduced, G1 and S2 ring A with maximal ideal $\mathfrak{m}$, the formation of $M_{e,A}$ and $N_{e,A}$ commutes with localization and completion; that is, if W is a multiplicative system and $\hat{\_}$ denotes completion along $\mathfrak{m}$, then*

$$W^{-1}M_{e,A} \cong \left(W^{-1}M\right)_{e,W^{-1}A} \quad \text{and} \quad W^{-1}N_{e,A} \cong \left(W^{-1}N\right)_{e,W^{-1}A} \quad \text{and}$$
$$M_{e,A} \otimes_A \hat{A} \cong M_{e,\hat{A}} \quad \text{and} \quad N_{e,A} \otimes_A \hat{A} \cong N_{e,\hat{A}}.$$

The following proof was suggested to us by Manuel Blickle and Kevin Tucker. We believe that there are more general proofs that work without the G1 and S2 hypothesis (but this proof is short).

Proof. We have a natural injective map $\operatorname{Hom}_A\left(\mathfrak{m}^{1/p^e}, \mathfrak{m}\right) \hookrightarrow \operatorname{Hom}_A\left(\mathfrak{m}^{1/p^e}, A\right)$, but $\operatorname{Hom}_A\left(\mathfrak{m}^{1/p^e}, A\right) \cong \operatorname{Hom}_A\left(A^{1/p^e}, A\right)$ since both modules are S2 (since a reflexive module is S2 in a G1 and S2 ring [Har94]). Thus, we have $\operatorname{Hom}_A\left(\mathfrak{m}^{1/p^e}, \mathfrak{m}\right) \hookrightarrow \operatorname{Hom}_A\left(A^{1/p^e}, A\right)$, the image of which is $M_{e,A}$.

Therefore, $M_{e,A}$ clearly commutes with localization and completion since the formation of both Hom sets commutes with localization and completion. Similarly, the formation of $N_{e,A}$ commutes with localization and completion. $\square$

LEMMA 3.17. *Suppose that A is an F-pure, F-finite, G1 and S2 ring with maximal ideal $\mathfrak{m}$. Then there exists an A-linear map $\alpha_A\colon A^{1/p^e} \to A$ such that α_A is surjective and $\alpha_A\big(\mathfrak{m}^{1/p^e}\big) \subseteq \mathfrak{m}$.*

Proof. We first assume that A is a local ring. In this case, it suffices to show that $N_{e,A} \neq M_{e,A}$. By Lemma 3.16, we may assume that A is complete since $\hat{A}$ is faithfully flat over A. By the Cohen structure theorem, there is a ring of formal power series $S = k[\![x_1,\ldots,x_n]\!]$ over a coefficient field k of A with surjection $S \twoheadrightarrow A$. Write $A = S/I$. Recall that $\mathrm{Hom}_S\big(S^{1/p^e}, S\big)$ is generated (as an S^{1/p^e}-module) by $g\colon S^{1/p^e} \to S$, which sends $(x_1 \cdots x_n)^{(p^e-1)/p^e}$ to 1 and other basis monomials (including 1) to zero. By Fedder's lemma [Fed83, Lemma 1.6], each element of $\mathrm{Hom}_A\big(A^{1/p^e}, A\big)$ has the form $f\big(u^{1/p^e}\_\big)$ with $u \in \big(I^{[p^e]} : I\big)$. Since A is F-pure, $\big(I^{[p^e]} : I\big) \not\subset \mathfrak{n}^{[p^e]}$, where $\mathfrak{n} = (x_1,\ldots,x_n)$. Let u be an element of $\big(I^{[p^e]} : I\big)\backslash\mathfrak{n}^{[p^e]}$. Then u must have a monomial term $cx_1^{a_1}\cdots x_n^{a_n}$ with $c \in k$ and $a_i < p^e$ for each i. Choose such a monomial appearing in u with the least degree and still denote it by $cx_1^{a_1}\cdots x_n^{a_n}$. Then it is clear that $\big(x_1^{p^e-1-a_1}\cdots x_n^{p^e-1-a_n}\big)u - c(x_1\cdots x_n)^{p^e-1} \in \mathfrak{n}^{[p^e]}$. Now, define $\alpha\colon A^{1/p^e} \to A$ by $\alpha(\_) = g\big(\big(x_1^{p^e-1-a_1}\cdots x_n^{p^e-1-a_n}u\big)^{1/p^e}\_\big)$. Then α is in $M_{e,A}\backslash N_{e,A}$. This completes the local case by Lemma 3.16.

For the non-local case, we have the map $M_{e,A} \to A$ which is evaluation at 1. Since the formation of $M_{e,A}$ commutes with localization, this map is surjective if and only if it is surjective locally. The above work proves the result after localizing at $\mathfrak{m}$. However, clearly $(M_{e,A})_{\mathfrak{n}} = \big(\mathrm{Hom}_A\big(A^{1/p^e}, A\big)\big)_{\mathfrak{n}}$ for prime ideals not equal to $\mathfrak{m}$. The result follows. $\square$

For the moment, we work sufficiently locally so that $X = \mathrm{Spec}\,R$ and $V = \mathrm{Spec}\,A$ are affine and that L and M are isomorphic to R and A, respectively. Continue to assume that A is F-pure and quasi-Gorenstein, and fix a point $s \in \mathrm{Spec}\,A$. Shrinking V if necessary, choose a map $\alpha_s \in \mathrm{Hom}_A\big(A^{1/p^e}, A\big)$ which is surjective and which satisfies $\alpha\big(\mathfrak{m}_s^{1/p^e}\big) \subseteq \mathfrak{m}_s$ (whose existence is guaranteed by Lemma 3.17). We write $\alpha_s(\_) = \Phi_A^e\big(z^{1/p^e}\cdot\_\big)$ for some $z \in A$. Note we have an induced map $\overline{\alpha_s}\colon (A/\mathfrak{m}_s)^{1/p^e} \to A/\mathfrak{m}_s$.

We thus induce the following map: $\beta\colon \big(R_{A^{1/p^{ne}}}\big)^{1/p^e} \to R_{A^{1/p^{ne}}}$, defined by the rule $\beta(\_) = \gamma\big(z^{1/p^{(n+1)e}}\cdot\_\big)$. Certainly, using the notation of (3.5),

$$\beta\big(\mathfrak{m}_s^{1/p^{(n+1)e}}\cdot R_{A^{1/p^{(n+1)e}}}^{1/p^e}\big) = \gamma\big(z^{1/p^{(n+1)e}}\mathfrak{m}_s^{1/p^{(n+1)e}}\cdot R_{A^{1/p^{(n+1)e}}}^{1/p^e}\big)$$
$$= \big(\phi'\circ\big((\Phi_A^e)^{1/p^{ne}}\otimes_{A^{1/p^{ne}}} R^{1/p^e}\big)\big)\big(z^{1/p^{(n+1)e}}\mathfrak{m}_s^{1/p^{(n+1)e}}\cdot R_{A^{1/p^{(n+1)e}}}^{1/p^e}\big)$$
$$= \big(\phi'\circ\big((\alpha_s)^{1/p^{ne}}\otimes_{A^{1/p^{ne}}} R^{1/p^e}\big)\big)\big(\mathfrak{m}_s^{1/p^{(n+1)e}}\cdot R_{A^{1/p^{(n+1)e}}}^{1/p^e}\big)$$
$$\subseteq \phi'\big(\mathfrak{m}_s^{1/p^{ne}}\cdot R_{A^{1/p^{ne}}}^{1/p^e}\big)$$
$$\subseteq \mathfrak{m}_s^{1/p^{ne}} R_{A^{1/p^{ne}}}.$$

And, hence, we induce a map

$$\overline{\beta}\colon R_{(A/\mathfrak{m}_s)^{1/p^{(n+1)e}}}^{1/p^e} \to R_{(A/\mathfrak{m}_s)^{1/p^{ne}}}.$$

LEMMA 3.18. *With notation as above, $\Delta_{\overline{\beta}} = \Delta_\gamma|_{X_s^{ne}}$, where the divisors are induced as in [BS13, Section 4].*

Proof. Since ϕ is relatively divisorial, by working in relative codimension 1, it suffices to show the result in the case that Δ_γ (and so Δ_ϕ) is trivial (since locally, ϕ or ψ is a generator pre-multiplied by the element which determines Δ_ϕ or Δ_ψ). But now consider the composition constructing β,

$$R^{1/p^e}_{A^{1/p^{(n+1)e}}} \xrightarrow{\cdot z^{1/p^{(n+1)e}}} R^{1/p^e}_{A^{1/p^{(n+1)e}}} \xrightarrow{\Phi'} R^{1/p^e}_{A^{1/p^{ne}}} \xrightarrow{\phi'} R_{A^{1/p^{ne}}}.$$

The composition of the first two maps sends $\mathfrak{m}^{1/p^{(n+1)e}} \cdot R^{1/p^e}_{A^{1/p^{(n+1)e}}}$ to $\mathfrak{m}^{1/p^{ne}} \cdot R^{1/p^e}_{A^{1/p^{ne}}}$, and modding out by these ideals induces a generating map for

$$\mathrm{Hom}_{R^{1/p^e}_{(A/\mathfrak{m})^{1/p^{ne}}}} \left(R^{1/p^e}_{(A/\mathfrak{m})^{1/p^{(n+1)e}}}, R^{1/p^e}_{(A/\mathfrak{m})^{1/p^{ne}}} \right)$$

via the fact that α_s is surjective and so the induced map $(A/\mathfrak{m})^{1/p^{(n+1)e}} \to (A/\mathfrak{m})^{1/p^{ne}}$ is a generating map as well. On the other hand, if ϕ is a generating map, so is $\phi' \otimes_{A^{1/p^{ne}}} (A/\mathfrak{m})^{1/p^{ne}}$ by Lemma 2.17. Since the composition of two generating maps is a generating map, we are done. $\square$

We now explain how the absolute σ behaves when restricting to fibers. First, we need two lemmas.

LEMMA 3.19. *With notation as above and $v \in V$ a closed point,*

$$\sigma\big(X_{V^{ne}}, \Delta_\gamma\big) \cdot \mathscr{O}_{X_{v^{ne}}} = \sigma\big(X_{V^{ne}}, \gamma\big) \cdot \mathscr{O}_{X_{v^{ne}}} \supseteq \sigma\big(X_{v^{ne}}, \overline{\beta}\big) = \sigma\big(X_{v^{ne}}, \Delta_{\overline{\beta}}\big) = \sigma\big(X_{v^{ne}}, \Delta_\gamma|_{X_{v^{ne}}}\big).$$

Proof. Obviously, $\sigma(X_{V^{ne}}, \gamma) \supseteq \sigma(X_{V^{ne}}, \beta)$. On the other hand, it easily follows that $\sigma(X_{V^{ne}}, \beta)$ restricts to $\sigma(X_{v^{ne}}, \overline{\beta})$ by F-adjunction; see for instance [FST11]. $\square$

The next lemma is a generalization of the fact that for an ideal in a regular ring, $I \subseteq \big(I^{[1/p]}\big)^{[p]}$ using the $\_^{[1/p]}$ notation from [BMS08].

LEMMA 3.20. *Let B be an F-finite regular ring such that $B^{1/p}$ is a free B-module and that $\mathrm{Hom}_B\big(B^{1/p}, B\big) \cong B^{1/p}$ (for example, this happens if B is local). Further, suppose that Q is a B^{1/p^a}-submodule of $B^{1/p^a} \otimes_B P$ for some B-module P, so $Q \subseteq B^{1/p^a} \otimes_B P$. Then $Q \subseteq B^{1/p^a} \otimes_B \big((\theta \otimes \mathrm{id}_P)(Q)\big)$, where $\theta \in \mathrm{Hom}_B\big(B^{1/p^a}, B\big)$ is the generator.*

Proof. The strategy is similar to [BMS08, Proposition 2.5]. By [Kun69, Theorem 2.1], any element of Q can be written as a finite sum $\sum \lambda_i \otimes m_i$, where $\{\lambda_i\}$ is a basis for B^{1/p^a} over B and $m_i \in P$. Now, since θ is a local generating map of $\mathrm{Hom}_B(B^{1/p^a}, B)$, the projection maps onto the λ_i are multiples of θ. In other words, for each λ_i, there exists a $u_i \in B^{1/p^a}$ such that both $\theta(u_i \lambda_i) = 1 \in R$ and $\theta(u_i \lambda_j) = 0 \in R$ for $j \neq i$.

We observe that each m_j is then in $(\theta \otimes \mathrm{id}_P)(Q)$ since $\theta(u_j \sum \lambda_i \otimes m_i) = m_j$. It then follows that $\sum \lambda_i \otimes m_i \in B^{1/p^a} \otimes_B \big((\theta \otimes \mathrm{id}_P)(Q)\big)$. $\square$

LEMMA 3.21. *Suppose that $A = k$ is an F-finite field and $K \supseteq k$ is a field extension of k such that K is perfect. Choose a flat map $X = \mathrm{Spec}\, R \to V = \mathrm{Spec}\, A = \mathrm{Spec}\, k$ of finite type with $X \to V$ possessing geometrically reduced, G1 and S2 fibers. Additionally, suppose that $\gamma \colon R^{1/p^e} \to R$ is any R-linear map which is a local generator of $\mathrm{Hom}_R\big(R^{1/p^e}, R\big)$ at the generic points of the codimension 1 components of the non-smooth locus of X. Then, setting Δ_γ as in [BS13, Section 4],*

$$\sigma(X, \Delta_\gamma) \otimes_k K \supseteq \sigma(X_K, (\Delta_\gamma) \times_k K).$$

Proof. First, choose $n > 0$ such that $\sigma(X, \Delta_\gamma) = \gamma^n(R^{1/p^{ne}})$. Since $\operatorname{Hom}_k(k^{1/p^{ne}}, k)$ is a free $k^{1/p^{ne}}$-module of rank 1, by [Kun86, Appendix F] we can factor γ^n as

$$R^{1/p^{ne}} \xrightarrow{\beta} R_{k^{1/p^{ne}}} \xrightarrow{\theta} R_k$$
$$\gamma^n$$

with θ a local generator of $\mathscr{H}om_{R_k}\left(R_{k^{1/p^{ne}}}, R_k\right)$ and such that Δ_β coincides with Δ_γ. To see this last point, note that if γ is a local generator of $\operatorname{Hom}_R\left(R^{1/p^e}, R\right)$, then so is γ^n for $\operatorname{Hom}_R\left(R^{1/p^{ne}}, R\right)$ and hence β is also a local generator of $\operatorname{Hom}_{R_{A^{1/p^{ne}}}}\left(R^{1/p^{ne}}, R_{A^{1/p^{ne}}}\right)$. More generally then, if γ^n is a local generator pre-multiplied by some $g^{1/p^{ne}} \in R^{1/p^{ne}}$ (working in codimension 1), then β is also a generator times $g^{1/p^{ne}}$ (possibly times a unit depending on our choice of θ). This proves that $\Delta_\beta = \Delta_\gamma$, as asserted.

We now observe that $\gamma^n\left(R^{1/p^{ne}}\right) \otimes_k k^{1/p^{ne}} = \theta\left(\beta\left(R^{1/p^{ne}}\right) \otimes_k k^{1/p^{ne}}\right) \supseteq \beta\left(R^{1/p^{ne}}\right)$ by Lemma 3.20 (here $B = k$ and $P = \beta\left(R^{1/p^{ne}}\right)$). After embedding $k^{1/p^{ne}} \subseteq K$, we tensor β by K to obtain a map

$$\beta_K \colon R^{1/p^{ne}} \otimes_{k^{1/p^{ne}}} K \longrightarrow R_{k^{1/p^{ne}}} \otimes_{k^{1/p^{ne}}} K\,.$$

We immediately see that $\Delta_{\beta_K} = (\Delta_\gamma) \times_k K$ (again by an argument about local generators). Then, by Lemma 3.20,

$$\sigma(X, \gamma) \otimes_k K \cong \gamma^n\left(R^{1/p^{ne}}\right) \otimes_k k^{1/p^{ne}} \otimes_{k^{1/p^{ne}}} K \supseteq \beta\left(R^{1/p^{ne}}\right) \otimes_{k^{1/p^{ne}}} K \supseteq \sigma(X, \beta_K)\,.$$

This completes the proof. $\qquad\square$

THEOREM 3.22. *With notation as in Convention* 3.14, *there exists an* $N > 0$ *such that for all* $n > N$, *we have* $\sigma\left(X_{V^{1/p^{ne}}}, \Delta_\gamma\right) \cdot \mathscr{O}_{X_{s^{ne}}} = \sigma\left(X_{s^{ne}}, (\Delta_\gamma)|_{X_{s^{ne}}}\right)$ *for all perfect points* $s \in V$.

Proof. The statement is local, so we may assume that X and V are affine and that L and M are trivial just as above in Lemma 3.18. We know that

$$\sigma\left(X_{V^{1/p^{ne}}}, \gamma^n\right) \subseteq \operatorname{im}\left(\gamma^n\right) = \sigma_n(X/V, \phi)$$

by Lemma 3.12. On the other hand, by Theorem 3.9, we have $\sigma_n(X/V, \phi) \cdot \mathscr{O}_{X_s} = \sigma(X_s, \phi_s)$. Therefore, it is sufficient to show that $\sigma\left(X_{V^{1/p^{ne}}}, \gamma\right) \cdot \mathscr{O}_{X_{s^{ne}}} \supseteq \sigma\left(X_{s^{ne}}, \psi_{s^{ne}}\right)$.

For this we may assume that V is the spectrum of an F-finite, F-pure local ring and that $\operatorname{Spec} K = s \mapsto V$ has image the closed point $\operatorname{Spec} k = v \in V$. We first see that $\sigma\left(X_{V^{ne}}, \Delta_\gamma\right) \cdot \mathscr{O}_{X_{v^{ne}}} \supseteq \sigma\left(X_{v^{ne}}, \Delta_{\overline{\beta}}\right) = \sigma\left(X_{v^{ne}}, \Delta_\gamma|_{X_{v^{ne}}}\right)$ by Lemma 3.19 (and using its notation). On the other hand, by Lemma 3.21,

$$\sigma\left(X_{v^{ne}}, \Delta_\gamma|_{X_{v^{ne}}}\right) \cdot \mathscr{O}_{X_{s^{ne}}} = \sigma\left(X_{v^{ne}}, \Delta_{\overline{\beta}}\right) \cdot \mathscr{O}_{X_{s^{ne}}} \supseteq \sigma\left(X_{s^{ne}}, \Delta_{\overline{\beta}} \times_{v^{ne}} s^{ne}\right) = \sigma\left(X_{s^{ne}}, \Delta_\gamma|_{X_{s^{ne}}}\right)\,.$$

This completes the proof. $\qquad\square$

Using the same method as for Lemma 3.19, we also obtain the following result, which may also be of independent interest.

PROPOSITION 3.23. *Assume that* $f \colon X \to V$ *is a flat, finite-type, reduced, G1 and S2 morphism and that* V *is regular. Additionally, assume that* $\gamma \colon R^{1/p^e} \to R$ *is an* R-*linear map which is a local generator at the generic points of the codimension 1 components of the non-smooth locus of* f *for each fiber of* f. *Then, for any* $a > 0$, *we have*

$$\sigma\left(X, \Delta_\gamma\right) \otimes_A A^{1/p^a} \supseteq \sigma\left(X_{V^a}, \Delta_\gamma \times_V V^a\right)\,.$$

Proof. The statement is local over the base and so we may assume that $K_V \sim 0$ and that A^{1/p^e} is a free A-module since V is regular [Kun69]. Note, then, that $K_{X/V} \cong K_X$. Choose $ne \geqslant a > 0$ such that $\sigma(X, \Delta_\gamma) = \gamma^n(R^{1/p^{ne}})$. Since $\mathrm{Hom}_A(A^{1/p^a}, A)$ is a free $A^{1/p^{ne}}$-module of rank 1, by [Kun86, Appendix F] we can factor γ^n as

$$R^{1/p^{ne}} \xrightarrow{\beta} R_{A^{1/p^a}} \xrightarrow{\theta} R,$$

with γ^n being the composite.

where θ is a local generator of $\mathscr{H}\mathrm{om}_R\left(R_{A^{1/p^a}}, R\right)$. Note that $\beta \in \mathrm{Hom}_{R_{A^{1/p^a}}}\left(R^{1/p^{ne}}, R_{A^{1/p^a}}\right)$; the latter is identified with

$$\mathscr{H}\mathrm{om}_{R_{A^{1/p^a}}}\left(R^{1/p^{ne}} \otimes_{R_{A^{1/p^a}}} \left(A^{1/p^a} \otimes_A \omega_{R/A}\right), \left(A^{1/p^a} \otimes_A \omega_{R/A}\right)\right)$$

$$\cong \mathscr{H}\mathrm{om}_{R_{A^{1/p^a}}}\left(\left(\mathscr{O}_X\left(p^{ne}K_{X/V}\right)\right)^{1/p^{ne}}, \omega_{R_{A^{1/p^a}}/A^{1/p^a}}\right)$$

$$\cong \mathscr{H}\mathrm{om}_{R_{A^{1/p^a}}}\left(\left(\mathscr{O}_X\left(p^{ne}K_X\right)\right)^{1/p^{ne}}, \omega_{R_{A^{1/p^a}}}\right)$$

$$\cong \mathscr{H}\mathrm{om}_{R^{1/p^{ne}}}\left(\left(\mathscr{O}_X\left(p^{ne}K_X\right)\right)^{1/p^{ne}}, \omega_{R^{1/p^{ne}}}\right)$$

$$\cong \left(\mathscr{O}_X\left((1-p^{ne})K_X\right)\right)^{1/p^{ne}}.$$

Note that since the base is regular, all these sheaves are automatically reflexive and so there is no need to double dualize as we did before in Definition 2.8. By dividing by $(p^{ne} - 1)$, we see that β coincides with a divisor Δ_β such that $(1 - p^{ne})(K_X + \Delta_\beta) \sim 0$. Furthermore, it is easy to see that Δ_β coincides with Δ_γ just as in Lemma 3.19 since θ is a local generator.

By composing β with a local generating (and surjective) map

$$R^{1/p^{ne}} \otimes_{A^{1/p^{ne}}} A^{1/p^{ne+a}} \xrightarrow{\cdots \otimes (\Phi_A^a)^{1/p^{ne}}} R^{1/p^{ne}},$$

we then obtain a map

$$\gamma'\colon (R_{A^{1/p^a}})^{1/p^{ne}} \cong R^{1/p^{ne}} \otimes_{A^{1/p^{ne}}} A^{1/p^{ne+a}} \longrightarrow R_{A^{1/p^a}}$$

satisfying the condition $\Delta_{\gamma'} = \Delta_\gamma \times_V V^a$. We also notice that γ' has the same image as β.

Now, we observe that $\gamma^n\left(R^{1/p^{ne}}\right) \otimes_A A^{1/p^a} = \theta\left(\beta\left(R^{1/p^{ne}}\right)\right) \otimes_A A^{1/p^a} \supseteq \beta\left(R^{1/p^{ne}}\right)$ by Lemma 3.20, setting $B = A$ and $P = \beta\left(R^{1/p^{ne}}\right)$. The remainder of the proof is easy since

$$\sigma(X, \Delta_\gamma) \otimes_A A^{1/p^a} = \gamma^n\left(R^{1/p^{ne}}\right) \otimes_A A^{1/p^a}$$

$$\supseteq \beta\left(R^{1/p^{ne}}\right)$$

$$= \gamma'\left((R_{A^{1/p^a}})^{1/p^{ne}}\right)$$

$$\supseteq \sigma\left(X_{V^a}, \Delta_{\gamma'}\right)$$

$$= \sigma\left(X_{V^a}, \Delta_\gamma \times_V V^a\right). \qquad \square$$

3.2 Application to sharply *F*-pure singularities and HSL numbers in families

We observe that Theorem 3.10 has an immediate application. Recall that if $\gamma\colon L^{1/p^e} \to R$ is an R-linear map, then the *HSL number* is the first integer n such that $\mathrm{im}(\gamma^n) = \mathrm{im}(\gamma^{n+1}) = \sigma(R, \gamma)$; cf. [Sha07].

COROLLARY 3.24 (Uniform behavior of HSL numbers). *Given a flat family $f\colon X \to V$ over an excellent integral scheme V with a dualizing complex of characteristic $p > 0$ and $\phi\colon L^{1/p^e} \to$*

$\mathscr{O}_{X_{V^e}}$ as before, there exists an integer $N > 0$ that gives an upper bound on the HSL number of $\left(X_s, \phi_s \colon L_s^{1/p^e} \to \mathscr{O}_{X_s}\right)$ for every perfect point $s \in V$.

Proof. The statement immediately follows from Theorem 3.9 since, clearly, $\sigma_m(X_s/s, \phi_s)$ agrees with $\sigma(X_s, \gamma_s = \phi_s)$ for $m \gg n$. $\qquad\square$

Now, we study the deformation of sharp F-purity, a question which has been studied before in [SZ09, Has10]. We believe that the hypothesis that f is proper is necessary in Theorem 3.28.

DEFINITION 3.25. Recall that a pair $\left(X, \psi \colon L^{1/p^e} \to \mathscr{O}_X = R\right)$ is called *sharply F-pure* if $\sigma(X, \psi) = \mathscr{O}_X$. Given $\left(X/S, \phi \colon L^{1/p^e} \to R_{A^{1/p^e}}\right)$, we define $(X/S, \phi)$ to be *relatively sharply F-pure* if $\sigma_n(X/S, \phi) = R_{A^{1/p^{ne}}}$ for some $n > 0$; cf. [Has10].

LEMMA 3.26. *If $\sigma_n(X/S, \phi) = R_{A^{1/p^{ne}}}$ for some $n > 0$, then the same holds for all $n > 0$.*

Proof. Suppose $\sigma_n(X/S, \phi) = R_{A^{1/p^{ne}}}$ for some $n > 0$. Choose $m < n$ to start. Then we know $\mathfrak{a}_{m,n} \supseteq \mathfrak{a}_{n,n} = \sigma_n(X/S, \phi)$ and so $\mathfrak{a}_{m,n} = R_{A^{1/p^{ne}}}$. But $\mathfrak{a}_{m,n} = \mathrm{im}\left(\mathfrak{a}_m \otimes_{A^{1/p^m}} A^{1/p^n} \to R_{A^{1/p^n}}\right)$, which is just the extension of $\sigma_m(X/S, \phi) \subseteq R_{A^{1/p^{me}}}$ to $R_{A^{1/p^{ne}}}$. Additionally, $R_{A^{1/p^{me}}} \subseteq R_{A^{1/p^{ne}}}$ is an integral extension and hence we must have $\sigma_m(X/S, \phi) = R_{A^{1/p^{me}}}$ as well.

We finish the proof by showing $\sigma_{2n}(X/S, \phi) = R_{A^{1/p^{2ne}}}$. This follows quickly since $\sigma_{2n}(X/S, \phi)$ factors as a composition of two surjective maps (as tensor is right exact) as in (2.3) and (2.3). $\qquad\square$

Remark 3.27. If S is a point, then $(X/S, \phi)$ being relatively sharply F-pure is equivalent to $(X/S, \phi)$ being geometrically sharply F-pure (in other words, that (X_t, γ_t) is sharply F-pure for every geometric point $t \to S$, where γ_t is induced from ϕ_t as in Section 3.1). To see this, certainly observe that if $(X/S, \phi)$ is relatively sharply F-pure, then so is any base change $(X_t/t, \phi_t)$. But then, (X_t, γ) is sharply F-pure by Theorem 3.13. Conversely, if (X_t, γ_t) is sharply F-pure, then certainly $(X_t/t, \phi_t)$ is relatively sharply F-pure by Lemma 3.12. But then so is $(X/S, \phi)$ near that t by Nakayama's lemma.

THEOREM 3.28 (Openness of sharp F-purity). *With notation as before, assume that $f \colon X \to V$ is proper. Assume that $s \in V$ is a point and that $(X_s/s, \phi_s)$ is relatively sharply F-pure (in other words, geometrically F-pure). Then there exists a dense open set $U \subseteq V$ containing s such that $(X_u/u, \phi_u)$ is relatively sharply F-pure for all $u \in U$ (in particular, (X_u, ϕ_u) is sharply F-pure for all perfect points $u \in U$).*

Proof. Choose $n \gg 0$ such that $\sigma_n(X_s/s, \phi_s) = R_{\mathscr{O}_s^{1/p^{ne}}}$ since $(X_s/s, \phi_s)$ is sharply F-pure. By Theorem 3.9, we know that $\sigma_n(X/V, \phi) = R_{A^{1/p^{ne}}}$ in a neighborhood $W \subseteq X$ of X_s. Let $Z = X \setminus W \subseteq X$ be the complement of that neighborhood. Since f is proper, $f(Z)$ is closed and also does not contain s. Set $U = V \setminus f(Z)$. Then $\sigma_n(X_U/U, \phi_U) = R_{\mathscr{O}_U^{1/p^{ne}}}$. It follows from Theorem 3.9 that all the fibers $(X_u/u, \phi_u)$ are relatively sharply F-pure for perfect $u \in U$, as desired. $\qquad\square$

We now state the above theorem in the context of divisors.

COROLLARY 3.29 (Openness of sharp F-purity for divisor pairs). *Suppose that $f \colon X \to V$ is a proper, G1 and S2 morphism as before and now also suppose that Δ is a $\mathbb{Q}$-divisor satisfying conditions (a)–(d) from Remark 2.11. Additionally, suppose that $s \in V$ is a perfect point and that $(X_s, \Delta|_{X_s})$ is sharply F-pure (or, more generally, if s is not perfect, that $(X_{s^e}, \Delta|_{X_{s^e}})$ is sharply F-pure for some or all $e > 0$). Then, there exists an open set $U \subseteq V$ containing s such that for all $u \in U$, the pair $(X_u, \Delta|_{X_u})$ is sharply F-pure (respectively, the pair $(X_{u^e}, \Delta|_{X_{u^e}})$ is sharply F-pure for, respectively, some or all $e > 0$ and all $u \in U$).*

Proof. Associate to Δ a relatively divisorial $\phi\colon L^{1/p^e} \to R_{A^{1/p^e}}$ as in Remark 2.11. The result then follows immediately from Theorem 3.28 and Corollary 2.20. $\qquad\square$

4. Relative test ideals

In this section, we define a notion of a relative test ideal. We construct it using the following method. We first find some ideal I which when restricted to every geometric fiber is contained in the test ideal of that fiber. We then sum up the images of the $I^{1/p^{ne}}$ under ϕ^n (the same method is used to construct the test ideal in the absolute case). This sounds simple enough, but it actually leads to a somewhat disappointing construction. The problem is that the ideal we pick is in no sense canonical and that we do not see a way to make it both canonical and stable under base change. In the case that ϕ corresponds to a trivial divisor, we can make our choices canonical by setting I to be the Jacobian ideal (this is done later when we discuss relative F-rationality in Section 5), but for more general ϕ we do not know of an analogous choice.

We fix the notation of the previous sections (in particular, we fix $\phi\colon L^{1/p^e} \to R_{A^{1/p^e}}$). We begin by choosing an *arbitrary ideal* $I \subseteq \mathscr{O}_X$. After base change to $k(V^\infty)$ as before, setting $I_\infty = I \cdot R_\infty$, we can form the following sum:

$$\tau\big(R_{k(V^\infty)}, \psi_\infty I_\infty\big) := \sum_{i=0}^\infty \psi_\infty^i \left(\left(I_\infty \cdot L_{k(V^\infty)}^{\frac{p^{ie}-1}{p^e-1}}\right)^{1/p^{ie}}\right).$$

If it happens that I_∞ is contained in the test ideal of $(R_{k(V^\infty)}, \phi_\infty)$ and non-zero on any component of X_∞, then we see that the absolute test ideal $\tau(R_{k(V^\infty)}, \psi_\infty(I_\infty))$ is equal to $\tau(R_{k(V^\infty)}, \psi_\infty)$, for example by [ST12, Section 7].

We can also describe this as follows, as first essentially observed in [Kat08]. Fix ideals

$$\mathfrak{b}_0^\infty := I_\infty = \psi_\infty^0(I_\infty),$$
$$\mathfrak{b}_1^\infty := \mathfrak{b}_0 + \psi_\infty\big((\mathfrak{b}_0^\infty \cdot L_{k(V^\infty)})^{1/p^e}\big) = I_\infty + \psi_\infty\big((I_\infty \cdot L_{k(V^\infty)})^{1/p^e}\big),$$
$$\mathfrak{b}_2^\infty := \mathfrak{b}_1^\infty + \psi_\infty\big((\mathfrak{b}_1^\infty \cdot L_{k(V^\infty)})^{1/p^e}\big)$$
$$\quad = \mathfrak{b}_1^\infty + \psi_\infty\big((I_\infty + \psi_\infty((I_\infty \cdot L_{k(V^\infty)})^{1/p^e}) \cdot L_{k(V^\infty)})^{1/p^e}\big)$$
$$\quad = \sum_{i=0}^2 \psi_\infty^i\left(\left(I_\infty \cdot L_{k(V^\infty)}^{\frac{p^{ie}-1}{p^e-1}}\right)^{1/p^{ie}}\right),$$
$$\vdots$$
$$\mathfrak{b}_n^\infty := \mathfrak{b}_{n-1}^\infty + \psi_\infty\big((\mathfrak{b}_{n-1}^\infty \cdot L_{k(V^\infty)})^{1/p^e}\big) = \sum_{i=0}^n \psi_\infty^i\left(\left(I_\infty \cdot L_{k(V^\infty)}^{\frac{p^{ie}-1}{p^e-1}}\right)^{1/p^{ie}}\right). \tag{4.1}$$

We notice that this ascending chain stabilizes, say at t. Also note that the first time $\mathfrak{b}_t^\infty = \mathfrak{b}_{t+1}^\infty$, we then have $\mathfrak{b}_t^\infty = \mathfrak{b}_{t+1}^\infty = \mathfrak{b}_{t+2}^\infty = \cdots$. We fix this integer t.

DEFINITION 4.1. With notation as above, we define the integer t to be the *uniform integer for τ and I over the generic point of V*, and, in general, it will be denoted by $n_{\tau(\phi I),k(V)}$. We notice that for any point $\eta \in V$, by base change we can replace V by $\operatorname{Spec} k(\eta)$ and form a corresponding integer $n_{\tau(\phi_\eta I_\eta),k(\eta)}$.

On the other hand, without the passing to $k(V^\infty)$, we have the images

$$\mathfrak{b}_1 := \mathrm{im}\left(I \otimes_A A^{1/p^e} \to R_{A^{1/p^e}}\right) + \phi((I \cdot L)^{1/p^e}) \subseteq R_{A^{1/p^e}}\,,$$

$$\mathfrak{b}_2 := \sum_{i=0}^{2} \mathrm{im}\left(\phi^i\left(\left(I \cdot L^{\frac{p^{ie}-1}{p^e-1}}\right)^{1/p^{ie}}\right) \otimes_{A^{1/p^{ie}}} A^{1/p^{2e}} \to R_{A^{1/p^{2e}}}\right) \subseteq R_{A^{1/p^{2e}}}\,,$$

$$\vdots$$

$$\mathfrak{b}_n := \sum_{i=0}^{n} \mathrm{im}\left(\phi^i\left(\left(I \cdot L^{\frac{p^{ie}-1}{p^e-1}}\right)^{1/p^{ie}}\right) \otimes_{A^{1/p^{ie}}} A^{1/p^{ne}} \to R_{A^{1/p^{ne}}}\right) \subseteq R_{A^{1/p^{ne}}}\,,$$

$$\vdots$$

Notice that $\mathrm{im}\left(\mathfrak{b}_1 \otimes_{A^{1/p^e}} A^{1/p^{2e}} \to R_{A^{1/p^{2e}}}\right) \subseteq \mathfrak{b}_2$ and, more generally, that $\mathrm{im}\left(\mathfrak{b}_i \otimes_{A^{1/p^{ie}}} A^{1/p^{je}} \to R_{A^{1/p^{ne}}}\right) \subseteq \mathfrak{b}_j$ for $j > i$. Also observe that this is the *opposite* containment compared to what we had in Section 3.

By the same argument as in Section 3, we know that there exists an open set $U \subseteq V$ with $W = f^{-1}(U)$ such that

$$\mathrm{im}\left(\mathfrak{b}_t \otimes_{A^{1/p^{te}}} A^{1/p^{(t+1)e}} \to R_{A^{1/p^{(t+1)e}}}\right)\big|_W = \mathfrak{b}_{t+1}|_W\,. \tag{4.2}$$

As before in Proposition 3.3, we claim that the following holds.

LEMMA 4.2. *With notation as above,* $\mathrm{im}\left(\mathfrak{b}_n \otimes_{A^{1/p^{ne}}} A^{1/p^{(n+1)e}} \to R_{A^{1/p^{(n+1)e}}}\right)|_W = \mathfrak{b}_{n+1}|_W$ *for all* $n \geqslant t$.

Proof. Replacing V by U and X by W, we may assume

$$\mathfrak{b}_{t+1} = \mathrm{im}\left(\mathfrak{b}_t \otimes_{A^{1/p^{te}}} A^{1/p^{(t+1)e}} \to R_{A^{1/p^{(t+1)e}}}\right)\,.$$

By induction on n, it suffices to show that if $\mathfrak{b}_{n+1} = \mathrm{im}\left(\mathfrak{b}_n \otimes_{A^{1/p^{ne}}} A^{1/p^{(n+1)e}} \to R_{A^{1/p^{(n+1)e}}}\right)$, then $\mathfrak{b}_{n+2} = \mathrm{im}\left(\mathfrak{b}_{n+1} \otimes_{A^{1/p^{(n+1)e}}} A^{1/p^{(n+2)e}} \to R_{A^{1/p^{(n+2)e}}}\right)$. Note that $\mathfrak{b}_{n+1} = \mathrm{im}\left(\mathfrak{b}_n \otimes_{A^{1/p^{ne}}} A^{1/p^{(n+1)e}} \to R_{A^{1/p^{(n+1)e}}}\right)$ if and only if

$$\mathrm{im}\left(\phi^{n+1}\left(\left(I \cdot L^{\frac{p^{(n+1)e}-1}{p^e-1}}\right)^{1/p^{(n+1)e}}\right) \to R_{A^{1/p^{(n+1)e}}}\right)$$
$$\subseteq \sum_{i=0}^{n} \mathrm{im}\left(\phi^i\left(\left(I \cdot L^{\frac{p^{ie}-1}{p^e-1}}\right)^{1/p^{ie}}\right) \otimes_{A^{1/p^{ie}}} A^{1/p^{(n+1)e}} \to R_{A^{1/p^{(n+1)e}}}\right)\,.$$

Tensoring with L over R, taking p^eth roots, and applying ϕ yields

$$\mathrm{im}\left(\phi^{n+2}\left(\left(I \cdot L^{\frac{p^{(n+2)e}-1}{p^e-1}}\right)^{1/p^{(n+2)e}}\right) \to R_{A^{1/p^{(n+2)e}}}\right)$$
$$\subseteq \sum_{i=1}^{n+1} \mathrm{im}\left(\phi^i\left(\left(I \cdot L^{\frac{p^{ie}-1}{p^e-1}}\right)^{1/p^{ie}}\right) \otimes_{A^{1/p^{ie}}} A^{1/p^{(n+2)e}} \to R_{A^{1/p^{(n+2)e}}}\right)\,.$$

Hence,

$$\mathfrak{b}_{n+2} = \mathrm{im}\left(\mathfrak{b}_{n+1} \otimes_{A^{1/p^{(n+1)e}}} A^{1/p^{(n+2)e}} \to R_{A^{1/p^{(n+2)e}}}\right)\,.$$

This finishes the proof. $\qquad\square$

As before, we define relative test ideals as follows.

DEFINITION 4.3. The *nth limiting relative test ideal with respect to I (of the pair $(X/V, \phi)$) is* defined to be $\mathfrak{b}_n \subseteq R_{A^{1/p^{ne}}}$ and is denoted by $\tau_n(X/V, \phi I)$.

Before proceeding to various properties of relative test ideals, we give one example of relative test ideals.

EXAMPLE 4.4 (Example 3.7, revisited). Fix k to be an algebraically closed field of characteristic $p > 2$, set $A = k[t]$, and set $R = k[x, t]$ with the obvious map $X = \operatorname{Spec}(R) \to V = \operatorname{Spec}(A)$. Let $\phi \colon R^{1/p} = k[x^{1/p}, t^{1/p}] \to R_{A^{1/p}} = k[x, t^{1/p}]$ be the composition of the local generator $\beta \in \operatorname{Hom}_{R_{A^{1/p}}}(R^{1/p}, R_{A^{1/p}})$ with pre-multiplication by $(x^{p^2} + t)^{1/p} = f^{1/p}$. Set $I = \langle x^p + t \rangle$. Then

$$\mathfrak{b}_1 = \operatorname{im}(I \otimes_A A^{1/p} \to R_{A^{1/p}}) + \phi(I^{1/p}) = \langle x^p + t \rangle + \langle (x^p + t^{1/p})(x + t^{1/p}) \rangle$$
$$= \langle x^p + t, (x^p + t^{1/p})(x + t^{1/p}) \rangle.$$

To calculate $\mathfrak{b}_2$, it suffices to calculate

$$\operatorname{im}(\phi^2(I^{1/p^2})) = \operatorname{im}(\beta^2((x^{p^2} + t)^{\frac{1+p}{p^2}} I^{1/p^2})).$$

Since $(x^{p^2} + t)^{(1+p)/p^2} \in R_{A^{1/p^2}}$, we can see that $\operatorname{im}(\phi^2(I^{1/p^2})) \subseteq \langle (x^{p^2} + t)^{(1+p)/p^2} \rangle$. On the other hand, it is straightforward to check that

$$\beta^2\left((x^{p^2} + t)^{\frac{1+p}{p^2}} x^{\frac{p^2-p-1}{p^2}} (x^p + t)^{1/p^2}\right) = (x^{p^2} + t)^{\frac{1+p}{p^2}} = (x + t^{1/p^2})(x^p + t^{1/p}).$$

Therefore,

$$\mathfrak{b}_2 = \langle x^p + t, (x^p + t^{1/p})(x + t^{1/p}), (x + t^{1/p^2})(x^p + t^{1/p}) \rangle.$$

For each $i \geqslant 2$, to calculate $\mathfrak{b}_{i+1}$, it suffices to calculate $\operatorname{im}(\beta^{i+1}((x^{p^2} + t)^{\frac{1+p+\cdots+p^i}{p^{i+1}}} I^{1/p^{i+1}}))$. It is straightforward to check that it is contained in $\langle (x^p + t^{1/p})(x + t^{1/p^2}) \rangle$ and hence is contained in $\mathfrak{b}_2$. Therefore, $\mathfrak{b}_{i+1} = \mathfrak{b}_2$ as well. So we have

$$\mathfrak{b}_n = \begin{cases} \langle x^p + t, (x^p + t^{1/p})(x + t^{1/p}) \rangle & \text{when } n = 1, \\ \mathfrak{b}_2 = \langle x^p + t, (x^p + t^{1/p})(x + t^{1/p}), (x + t^{1/p^2})(x^p + t^{1/p}) \rangle & \text{when } n \geqslant 2. \end{cases}$$

We now prove a base change statement for relative test ideals. Compare with Proposition 3.8. As before, we fix $q_i \colon X \times_V T^i \to X \times_V V^i$ to be the natural map.

THEOREM 4.5 (Relative test ideals and base change). *Fix $g \colon T \to V$ to be any morphism with T excellent, integral, and admitting a dualizing complex. Then*

$$\operatorname{im}((q_{ne})^* \tau_n(X/V, \phi I) \hookrightarrow \mathscr{O}_{X_{T^{ne}}}) := \tau_n(X/V, \phi I) \cdot \mathscr{O}_{X_{T^{ne}}} = \tau_n(X_T/T, \phi_T I_T).$$

Furthermore, if $U = U_n$ satisfies the condition of Lemma 4.2, then $W = g^{-1}(U) \subseteq T$ satisfies the same condition for $\tau_n(X_T/T, \phi_T I_T)$.

Proof. We write $B = \mathscr{O}_T$ and work locally. By construction and right exactness of tensor,

$$\tau_n(X/V, \phi) \cdot \mathscr{O}_{X_{T^{ne}}} = \mathfrak{b}_n \otimes_{A^{1/p^{ne}}} B^{1/p^{ne}}$$
$$= \sum_{i=0}^{n} \left(\phi^i\left(\left(I \cdot L^{\frac{p^{ie}-1}{p^e-1}}\right)^{1/p^{ie}}\right) \otimes_{A^{1/p^{ie}}} B^{1/p^{ne}}\right) \subseteq R_{B^{1/p^{ne}}}.$$

But, by Lemma 2.15,

$$\sum_{i=0}^{n} \left(\phi^i \left(\left(I \cdot L^{\frac{p^{ie}-1}{p^e-1}} \right)^{1/p^{ie}} \right) \otimes_{A^{1/p^{ie}}} B^{1/p^{ne}} \right) = \sum_{i=0}^{n} \left(\phi_T^i \left(\left(I_T \cdot L_T^{\frac{p^{ie}-1}{p^e-1}} \right)^{1/p^{ie}} \right) \otimes_{B^{1/p^{ie}}} B^{1/p^{ne}} \right)$$
$$= \tau_n \left(X_T/T, \phi_T I_T \right),$$

which completes the proof. $\qquad \square$

THEOREM 4.6 (Restriction of τ_n to fibers). *With notation as above, there exists an integer $N \geqslant 0$ such that for all points $s \in V$, we have $N \geqslant n_{\tau(\phi_s I),k(s)}$. In other words, we have both that $\tau_n(X/V, \phi) \cdot \mathscr{O}_{X_s^{ne}} = \tau_n(X_s/s, \phi_s)$ (which always holds) and that for all $m \geqslant n \geqslant N$,*

$$\tau_n \left(X/V, \phi I \right) \otimes_{V^n} k(s)^{1/p^m} = \tau_m \left(X_s/s, \phi_s I_s \right).$$

Proof. The idea is the same as for Theorem 3.9 and we only sketch it here. We stratify V as follows. The result holds on a dense open subset of $U_0 \subseteq V$. Let V_1' be the complement, and let V_1 denote the regular locus of V_1'. Base change to V_1, and then repeat. This procedure stops after finitely many iterations by Noetherian induction, and we choose an N that works for them all. $\qquad \square$

We now construct $I \subseteq R$ whose restriction is contained in the test ideal of every geometric fiber.

PROPOSITION 4.7 (Existence of relative test elements). *With notation as above, suppose additionally that $f \colon X \to V$ is relatively G1 and S2. Then, there exists an ideal $I \subset \mathscr{O}_X$ such that for every point $s \in V$ and every perfect extension $K \supseteq k(s)$, we have $I_K \subseteq \tau(X_K, \psi_K)$, where again ψ_K is simply ϕ_K interpreted as in [BS13]. Additionally, we can assume $I = \mathscr{O}_X$ at all points of X such that both X/V is smooth and ϕ locally generates $\mathscr{H}om_{\mathscr{O}_X} \left(L^{1/p^e}, R_{A^{1/p^e}} \right)$. Furthermore, if V is strongly F-regular and quasi-Gorenstein, then I can additionally be chosen such that I_{V^e} is within the absolute test ideal of (X_{V^e}, γ) (where γ is as in Section 3.1).*

We caution the reader that it is possible that ψ_s (and thus ψ_K) could be the zero map, and hence I_K the zero ideal.

Proof. First, let $Z_1 \subseteq X$ denote the locus where $\phi R^{1/p^e} \subseteq \mathscr{H}om_{R_{A^{1/p^e}}} \left(L^{1/p^e}, R_{A^{1/p^e}} \right)$ is not an isomorphism. Additionally, let Z_2 denote the locus where X is not smooth over V. Set $Z = Z_1 \cup Z_2$. Set $W = X \setminus Z$ (and note that we can also view this as a subset of X_{V^e} for any e) and notice that on W, we know that ϕ can be identified with the trace map up to multiplication by a unit.

Since $f|_W$ is smooth, we observe that ϕ (which is identified with the trace map) is surjective when restricted to W by Corollary 2.5. Now, choose an exponent $m > 0$ such that

$$I_Z^m \cdot R_{A^{1/p^e}} \subseteq \mathfrak{a}_{1,1} = \sigma_1(X/S, \phi).$$

Now, choose an integer $l > 0$ such that I_Z^{ml} is locally generated by cubes of elements of I_Z^m (this only depends on the number of local generators of I_Z). Then the formation of I_Z^{ml} is obviously compatible with base change (in that the extension of I_Z^{ml} to the base change will also satisfy the same containment condition.). Thus, set $I = I_Z^{ml}$. The first result follows immediately from [Hoc07, Theorem on p. 90] or [Sch11, proof of Proposition 3.21].

For the second result, we notice that γ locally generates $\mathscr{H}om_{R_{A^{1/p^e}}} \left(N^{1/p^e}, R_{A^{1/p^e}} \right)$ and that I_Z^m is also in the image of γ. Therefore, I^{ml} works by the above references. $\qquad \square$

Therefore we obtain the following result.

COROLLARY 4.8. *Using the notation of Theorem 4.6, assume further that I satisfies the condition of Proposition 4.7. If s is a perfect point, then for all $n \geqslant N$ as in Theorem 3.9,*

$$\tau_n(X/V, \phi I) \otimes_{V^n} k(s)^{1/p^m} = \tau(X_s, \psi_s),$$

where ψ_s is ϕ_s interpreted as a p^{-e}-linear map.

Proof. Simply observe that, in general, the ascending chain $\tau_j(X_s/s, \phi_s I_s) \subseteq \tau_{j+1}(X_s/s, \phi_s I_s) \subseteq \cdots$ stabilizes to $\tau(X_s, \psi_s)$. Furthermore, by Theorem 4.6, this ascending chain stabilizes. □

COROLLARY 4.9. *Using the notation of Theorem 4.6, assume further that I satisfies the condition of Proposition 4.7. If the perfect closure of the generic fiber of $f \colon X \to V$ is strongly F-regular, that is, $\tau\big(R_{k(V^\infty)}, \psi_\infty I_\infty\big) = R_{k(V^\infty)}$, then there exists an open subset $U \subseteq V$ such that (X_s, ψ_s) is also strongly F-regular for each perfect point $s \in U$.*

Proof. It follows immediately from the proof of Proposition 3.3 and Corollary 4.8. □

4.1 Relative test ideals versus absolute test ideals

We make the following assumption.

CONVENTION 4.10. For the rest of Section 4.1, we assume that V is regular and F-finite.

It is natural to try to relate the relative test ideal $\tau_n(X/V, \phi I)$ to the absolute test ideal $\tau(X_{V^{ne}}, \gamma^n)$ as in Section 3.1. Indeed, if we construct I as we did in the proof of Proposition 4.7, then it follows easily that $I \cdot R_{A^{1/p^{ne}}} \subseteq \tau(X_{V^{ne}}, \gamma^n)$. Thus, fix such an I.

We now recall the following diagram from Lemma 3.12:

We let $I^{1/p^{je}}_{A^{1/p^{ke}}}$ denote the extension of $I^{1/p^{je}}$ to $R^{1/p^{je}}_{A^{1/p^{ke}}}$. Then notice that μ_j sends $I^{1/p^{je}}_{A^{1/p^{(j+n)e}}}$ (times the domain of μ_j) back *onto* $I^{1/p^{je}}$ since each μ_j is surjective.

We then observe that the following holds.

LEMMA 4.11. *With notation as above and assuming Convention 4.10, we have $\tau_n(X/V, \phi I) \subseteq \tau(X_{V^{ne}}, \gamma)$.*

Proof. We know that $\tau_n(X/V, \phi I)$ is the sum

$$\sum_{j=0}^{n} \Big(\phi^j \otimes_{A^{1/p^{je}}} A^{1/p^{ne}}\Big)\Big(I^{1/p^{je}} \cdot \Big(L^{\frac{p^{je}-1}{p^e-1}}\Big)^{1/p^{je}} \otimes_{A^{1/p^{je}}} A^{1/p^{ne}}\Big). \tag{4.3}$$

On the other hand, $\tau(X, \gamma)$ is the sum

$$\sum_{j=0}^{\infty} \gamma^j \Big(I^{1/p^{je}}_{A^{1/p^{(j+n)e}}} \cdot \Big(L^{\frac{p^{je}-1}{p^e-1}}\Big)^{1/p^{je}} \otimes_{A^{1/p^{je}}} \Big(M^{\frac{p^{je}-1}{p^e-1}}\Big)^{1/p^{(j+n)e}}\Big). \tag{4.4}$$

By our above observations about the surjectivity of the μ_j above, the sum of the 0th through jth terms of (4.4) is equal to the sum (4.3). The result follows. □

Additionally, at least over an open subset of the base, the relative test ideal and absolute test ideal actually agree; cf. Theorem 3.13.

THEOREM 4.12. *With notation as above and assuming Convention 4.10, choose $n > t = n_{\tau(\phi I),k(V)}$. Then there exists a dense open set $U \subseteq V \cong V^e$ with $W = f^{-1}(U) \subseteq X$ such that $\tau(X_{V^{1/p^{ne}}}, \gamma)|_W = \tau_n(X/V, \phi I)|_W$. Furthermore, shrinking U further if necessary and possibly increasing n, we can require for all perfect points $u \in U$ that*

$$\tau(X_{V^{1/p^{ne}}}, \gamma) \cdot \mathscr{O}_{X_u} = \tau_n(X_u/u, \phi_u I_u) = \tau(X_u, \psi_u),$$

where again ψ_u is ϕ_u viewed as a p^{-e}-linear map.

Proof. First, we observe that by taking U as in (4.2), we can assume $\mathfrak{b}_{n-1,n} = \mathfrak{b}_{n,n}$. But by the diagram above, we see that the $(n-1)$st partial sum defining $\tau(X_{V^{1/p^{ne}}}, \gamma)$ in (4.4) is equal to $\mathfrak{b}_{n-1,n}$. Likewise, the nth partial sum is equal to $\mathfrak{b}_{n,n}$. However, once two adjacent partial sums defining $\tau(X_{V^{1/p^{ne}}}, \gamma)$ coincide, the sum stabilizes for further powers by the computation we made when defining the $\mathfrak{b}_n^\infty$ in (4.1). The second statement follows from the first and from Theorem 4.5; cf. the argument of Corollary 4.8. □

When dealing with relatively non-F-pure ideals σ, we actually obtained the above restriction theorem without shrinking X to U in Corollary 3.22. The difference is that for σ, we have the easy containment $\sigma(X_{V^{1/p^{ne}}}, \gamma^n) \subseteq \sigma_n(X/V, \phi)$ by Lemma 3.12. For τ, however, the easy containment is reversed. This leads us to the following question.

QUESTION 4.13. *Is it true that $\tau(X_{V^{1/p^{ne}}}, \gamma) \cdot \mathscr{O}_{X_{s^{ne}}} = \tau(X_s, \psi_s)$ for all perfect points $s \in V$, at least when $n \gg 0$?*

Rephrasing Theorem 4.12 for divisors, we obtain the following statement.

COROLLARY 4.14. *Suppose that $f\colon X \to V$ is a flat finite map to an excellent regular scheme V. Additionally, suppose that f is relatively G1 and S2 and I is chosen as in Proposition 4.7+. Choose Δ satisfying conditions (a)–(d) of Remark 2.11. Then, there exists an open dense set $U \subseteq V$ such that*

$$\tau(X_{V^{1/p^{ne}}}, \Delta) \cdot \mathscr{O}_{X_{u^{1/p^{ne}}}} = \tau(X_u, \Delta|_{X_u})$$

for all perfect points $u \in U$.

Proof. Using Lemmas 2.21 and 3.15, we see that Δ corresponds to some $\phi\colon L^{1/p^e} \to R_{A^{1/p^e}}$. Thus, we simply apply Corollary 2.20 and Theorem 4.12. □

4.2 Applications to strong F-regularity and divisor pairs

DEFINITION 4.15. Recall that a pair $(X, \psi\colon L^{1/p^e} \to \mathscr{O}_X = R)$ is called *strongly F-regular* if $\tau(X, \psi) = \mathscr{O}_X$. Given $(X/S, \phi\colon L^{1/p^e} \to R_{A^{1/p^e}})$ and some $I \subseteq R$ satisfying the condition of Proposition 4.7, we say that $(X/S, \phi I)$ is *relatively strongly F-regular* if we have $\tau_n(X/S, \phi I) = R_{A^{1/p^{ne}}}$ for some $n > 0$ (equivalently, all $n \gg 0$ since the τ_n ascend); cf. [Has10].

Remark 4.16. Suppose that S is a point and I is chosen as in Proposition 4.7; then $(X/S, \phi I)$ is relatively strongly F-regular if and only if it is geometrically strongly F-regular (that is, (X_t, γ_t) is strongly F-regular for every point $t \to S$). The argument is the same as in Remark 3.27.

The same ideas imply that the strongly F-regular locus of a proper map is open.

COROLLARY 4.17 (Openness of the strongly F-regular locus). *With notation as before, assume additionally that $f\colon X \longrightarrow V$ is proper and that I satisfies the condition of Proposition 4.7. Assume that $s \in V$ is a point and that $(X_s/s, \phi_s I_s)$ is relatively strongly F-regular (for example, if s is a perfect point, this just means that it is strongly F-regular and independent of I). Then, there exists a dense open set $U \subseteq V$ containing s such that $(X_u/u, \phi_u I_u)$ is relatively strongly F-regular for all $u \in U$ (in particular, (X_u, ϕ_u) is strongly F-regular for all perfect points $u \in U$).*

Proof. Choose $n \gg 0$ such that $\tau_n(X_s/s, \phi_s I_s) = R_{\mathscr{O}_s^{1/p^{ne}}}$, since $(X_s/s, \phi_s I_s)$ is strongly F-regular. By Theorem 4.6, we know that $\tau_n(X/V, \phi I) = R_{A^{1/p^{ne}}}$ in a neighborhood $W \subseteq X$ of X_s. Let $Z = X \setminus W \subseteq X$ be the complement of that neighborhood. Since f is proper, $f(Z)$ is closed, and also does not contain s. Set $U = V \setminus f(Z)$. Then $\tau_n(X_U/U, \phi_U I|_U) = R_{\mathscr{O}_U^{1/p^{ne}}}$. It follows from Theorem 4.6 that all the fibers $(X_u/u, \phi_u I_u)$ are relatively strongly F-regular, as desired. $\square$

At least for proper maps, we also obtain that the definition of relative strongly F-regularity (Definition 4.15) is independent of the choice of I.

LEMMA 4.18. *Suppose that $f\colon X \longrightarrow V$ is proper and that $(X/V, \phi I)$ is relatively strongly F-regular for some I satisfying the condition of Proposition 4.7. Then for all $J \subseteq R$ such that J_s is non-zero on every component of every fiber X_s, we have that $\tau_n(X/S, \phi J) = R_{A^{1/p^{ne}}}$ for some $n > 0$ (equivalently, all $n \gg 0$).*

In particular, in Definition 4.15, it would be equivalent to require $\tau_n(X/S, \phi J) = R_{A^{1/p^{ne}}}$ for all such J and some n.

Proof. Choose a closed point $s \in V$ and a perfect extension $K \supseteq k(s)$. Then $\tau_m(X_K/K, \phi_K I_K) = R_{K^{1/p^{ne}}}$ for some $m > 0$ by Theorem 4.6. But, since K is a perfect field extension, this implies that $\tau_m(X_K, \psi_K I_K) = R_{K^{1/p^{me}}}$ as well. On the other hand, since (X_K, ψ_K) is strongly F-regular, we see that $\tau_n(X_K/K, \phi_K J_K) = R_{K^{1/p^{ne}}}$ for some $n > 0$. Hence, since $k(s) \subseteq K$ is faithfully flat, we see that $\tau_n(X_s/s, \phi_s J_s) = R_{k(s)^{1/p^{ne}}}$. But then, since $\tau_n(X/V, \phi J)$ restricts to $\tau_n(X_s/s, \phi_s J_s)$, we observe that $\tau_n(X/V, \phi J) = R_{A^{1/p^{ne}}}$ at least in a neighborhood of s, using that f is proper and the same argument we made in Corollary 4.17. But we can find such an n for every $s \in V$, and so the lemma holds by the quasi-compactness of V. $\square$

We now state our result in the divisorial case.

COROLLARY 4.19. *With notation as above, suppose that $f\colon X \longrightarrow V$ is a proper map and that Δ is a $\mathbb{Q}$-divisor satisfying conditions (a)–(d)[8] of Remark 2.11. Additionally, suppose that for some perfect point $s \in V$, the fiber $(X_s, \Delta|_{X_s})$ is strongly F-regular. Then there exists a dense open set $U \subseteq V$ containing s such that $(X_u, \Delta|_{X_u})$ is strongly F-regular for all perfect $u \in U$.*

Proof. Using Remark 2.11, we construct a relatively divisorial $\phi\colon L^{1/p^e} \longrightarrow R_{A^{1/p^e}}$ corresponding to Δ. Now, choose I satisfying the condition of Proposition 4.7. It follows that (X_s, ψ_s) is strongly F-regular and hence that $(X_s/s, \phi_s I_s)$ is relatively strongly F-regular since s is a perfect point. Then Corollaries 4.17 and 2.20 complete the proof. $\square$

By a perturbation trick we can also handle the case that the index of $K_X + \Delta$ is divisible by $p > 0$, at least over curves.

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[8]For example, these conditions hold if V is regular, $f\colon X \longrightarrow V$ is geometrically normal, $K_X + \Delta$ is $\mathbb{Q}$-Cartier with index not divisible by p, and Δ does not contain any fiber of f in its support.

COROLLARY 4.20. *With notation as above, suppose that $f \colon X \to V$ is a projective map to a regular 1-dimensional base and that Δ is a $\mathbb{Q}$-divisor satisfying the following four conditions:*

(a') *We have $\Delta' = \frac{1}{m}D$ for some Weil divisor D.*

(b) *The divisor D is a Weil divisor on X which is Cartier in codimension 1 and Cartier at every codimension 1 point of every fiber.*

(c) *The divisor D is trivial along the codimension 1 components of the non-smooth locus of $X \to V$ and the codimension 1 components of the non-smooth locus of every fiber.*

(d') *We have $l/m \in \mathbb{Z}$, and $\left(\omega_{X/V}^l \otimes \mathscr{O}_X(l\Delta')\right)^{**}$ is a line bundle.*

Additionally, suppose that for some perfect point $s \in V$, the fiber $(X_s, \Delta|_{X_s})$ is strongly F-regular. Then there exists a dense open set $U \subseteq V$ containing s such that $(X_u, \Delta|_{X_u})$ is strongly F-regular for all perfect $u \in U$.

Proof. We may certainly suppose that V is affine. Since the fiber X_s is normal, and hence geometrically normal, we may also assume that the nearby fibers satisfy the same condition (using that f is proper). Thus, we may assume that all the fibers are geometrically normal and in fact that the map $f \colon X \to V$ is geometrically integral. Without loss of generality, by base change we can assume that V is normal and hence X is normal itself. Thus, we may assume that X is normal.

Now, since the base is 1-dimensional, we claim that we can assume that $K_{X/V}$ does not contain any fiber. The only fiber we must worry about is X_s (as the others can be handled by shrinking V). We argue as follows: First, note that $K_{X/V}$ can be viewed as an honest Weil divisor since we already assumed that X is normal. On the other hand, each fiber is a Cartier divisor. Hence, if $K_{X/V}$ is non-trivial along the generic point of X_s, by twisting by the pullback of s, we can assume that $K_{X/V}$ does not contain X_s.

Write $l\Delta' = cp^{e_0}\Delta'$, where p does not divide c. Let E be an effective Cartier divisor on X not containing any fiber such that $K_{X/V} + E$ is effective. We also observe that $K_{X/V} + E$ satisfies conditions (b) and (c) above. Let $\Gamma = (1/(p^e - 1))(E + K_{X/V} + \Delta')$ for some $e \gg e_0$. Then consider $\Delta := \Delta' + \Gamma$. Note that

$$\begin{aligned}
c(p^e - 1)(K_{X/V} + \Delta) &= c(p^e - 1)(K_{X/V} + \Delta' + \Gamma) \\
&= c(p^e - 1)(K_{X/V} + \Delta') + c(E + K_{X/V} + \Delta') \\
&= cp^e(K_{X/V} + \Delta') + cE \,,
\end{aligned}$$

which is certainly Cartier. Hence $K_{X/V} + \Delta$ satisfies conditions (a)–(d) from Remark 2.11. On the other hand, since $e \gg 0$, we know that $(X_s, \Delta|_{X_s}) = (X_s, \Delta'|_{X_s} + \Gamma|_{X_s})$ is still strongly F-regular, and so by Corollary 4.19, there exists an open set U such that $(X_u, \Delta|_{X_u})$ is strongly F-regular for all $u \in U$. But $\Delta \geqslant \Delta'$ and so the result follows for Δ' as well. $\qquad\square$

Remark 4.21. In the case of a normal X and in the non-relative case, we know that $\tau(X; \Delta) = \tau(\omega_X, K_X + \Delta)$. Furthermore, we then know that

$$\mathrm{Tr}^e\left(F_*^e \tau(\omega_X, K_X + \Delta)\right) = \tau\left(\omega_X, \frac{1}{p^e}(K_X + \Delta)\right) = \tau\left(X, \frac{1}{p^e}(K_X + \Delta) - K_X\right).$$

Reversing this process gives us a nice means to compute $\tau(X, \Delta)$ when the index of $K_X + \Delta$ is divisible by p. It would be natural to try to prove a relative version of this, which may yield a suitable definition of relative test ideals for $K_X + \Delta$ of any index. We will not attempt this here.

5. Relative test submodules, F-rationality, and F-injectivity

Our goal in this (somewhat shorter) section is to explore relative test submodules and non-F-injective modules. Throughout this section, we assume that $f \colon X \to V$ is a Cohen–Macaulay morphism. This provides us with base change for relative canonical sheaves $\omega_{X/V}$ [Con00].

We first define relative non-F-injective modules $\sigma_n(X/V, \omega_{X/V})$ and relative test submodules $\tau_n(X/V, \omega_{X/V})$. Recall the trace map, Lemma 2.16:

$$\Phi_{X/V,n} \colon \omega_{X/V}^{1/p^n} = \omega_{X^n/V^n} \longrightarrow \omega_{X_{V^n}/V^n} \cong \omega_{X/V} \otimes_A A^{1/p^{ne}} \,.$$

Here, the final isomorphism follows from [Con00, Theorem 3.6.1] since f is a Cohen–Macaulay morphism. We will form σ and τ relative to these maps, instead of relative to the map ϕ discussed previously. One key point to remember is that by Lemma 2.16, the map $\Phi_{X/V,n}$ is compatible with arbitrary base change (since in this section, f is a Cohen–Macaulay morphism). We also observe that the composition of trace maps

$$\omega_{X^m/V^m} \xrightarrow{\Phi_{X/V,m-n}^{1/p^{m-n}}} \omega_{X_{V^m}^n/V^m} \xrightarrow{\Phi_{X/V,n} \otimes_{A^{1/p^n}} A^{1/p^m}} \omega_{X/V} \otimes_A A^{1/p^m} \cong \omega_{X_{V^m}/V^m} \tag{5.1}$$

can be identified with $\Phi_{X/V,m}$.

5.1 The definition and basic properties of $\sigma_n(X/V, \omega_{X/V})$

For each integer $n > 0$, define $\mathfrak{c}_n := \operatorname{im}(\Phi_{X/V,n}) \subseteq \omega_{X_{V^n}/V^n} \cong \omega_{X/V} \otimes_A A^{1/p^n}$. Furthermore, for each $m \geqslant n$, using the factorization in (5.1) it is easy to see that

$$\mathfrak{c}_m \subseteq \operatorname{im}\left(\mathfrak{c}_n \otimes_{A^{1/p^n}} A^{1/p^m} \longrightarrow \omega_{R_{A^{1/p^m}}}\right), \tag{5.2}$$

just as we observed in Section 3.

DEFINITION 5.1. With notation as above, we define the *nth relative non-F-injective submodule*, denoted by $\sigma_n(X/V, \omega_{X/V})$, to be $\mathfrak{c}_n \subseteq \omega_{X_{V^n}/V^n}$.

As before, we will prove a stabilization statement for $\mathfrak{c}_n$ and $\mathfrak{c}_m$, in particular that the containment (5.2) is an equality for all $m > n \gg 0$. By base changing with $k(V^\infty)$, the perfection of the residue field of the generic point of V, we can again find an integer $n_0 > 0$ and an open set $U \subseteq V$ such that

$$\mathfrak{c}_{n_0+1}\big|_{f^{-1}U} = \operatorname{im}\left(\mathfrak{c}_{n_0} \otimes_{A^{1/p^{n_0}}} A^{1/p^{n_0+1}} \longrightarrow \omega_{R_{A^{1/p^{n_0+1}}}}\right)\big|_{f^{-1}U} \,.$$

Again, observe that for U inside the regular locus of V, we can identify the image above with $\left(\mathfrak{c}_{n_0} \otimes_{A^{1/p^{n_0}}} A^{1/p^{n_0+1}}\right)\big|_U$ by the flatness of A^{1/p^m} over A^{1/p^n}. We then obtain the following result.

PROPOSITION 5.2. *Fix the notation as above. For every integer $n \geqslant n_0$, there exists a nonempty open subset $U_n \subseteq V$ of the base scheme V satisfying the following condition. If one sets $X_n = f^{-1}(U_n)$, then for every $m \geqslant n$,*

$$\sigma_m\big(X/V, \omega_{X/V}\big)\big|_{X_n} = \operatorname{im}\left(\sigma_n\big(X/V, \omega_{R/A}\big) \cdot \mathscr{O}_{X_n} \otimes_{A^{1/p^n}} A^{1/p^m} \longrightarrow \omega_{R_{A^{1/p^m}}}\right). \tag{5.3}$$

Furthermore, we may assume that $U_{n_0} \subseteq U_{n_0+1} \subseteq \cdots \subseteq U_n \subseteq U_{n+1} \subseteq \cdots$ form an ascending chain of open sets.

Proof. The proof is identical to that of Proposition 3.3 and so we omit it. $\square$

We now point out that relative non-F-injective modules behave well with respect to base change. Recall that if $g \colon T \to V$ is a map, then $q_n \colon X_{T^n} \to X_{V^n}$ is the induced map.

PROPOSITION 5.3 (Base change for $\sigma_n(X/V, \omega_{X/V})$). *Suppose that* $g\colon T \to V$ *is a map from an excellent scheme with a dualizing complex; then using the notation of Section* 2.2,

$$\mathrm{im}\left((q_n)^*\sigma_n\big(X/V, \omega_{X/V}\big) \to \omega_{X_{T^n}/T^n}\right) = \sigma_n\big(X_T/T, \omega_{X_T/T}\big).$$

Furthermore, if $U = U_n$ *satisfies condition* (5.3) *from Proposition* 5.2, *then* $W = g^{-1}(U) \subseteq T$ *satisfies the same condition for* $\sigma_n(X_T/T, \omega_{X_T/T})$.

Proof. The first statement is an immediate consequence of base change relative to canonical sheaves and the trace map (Lemma 2.16). For the second, if $\Phi_{X/V,n} \otimes_{A^{1/p^n}} A^{1/p^{n+1}}$ and $\Phi_{X/V,n+1}$ have the same image in $\omega_{X_{V^n}/V^n}$, then it is easy to see that the base changed maps also have the same images. $\qquad\square$

Recall that for any F-finite scheme Y with canonical module ω_Y, the image $\sigma(\omega_Y)$ is equal to $\mathrm{im}(F_*^e\omega_Y \xrightarrow{\mathrm{Tr}^e} \omega_Y)$ for any $e \gg 0$.

COROLLARY 5.4 (Restriction theorem for $\sigma_n(X/V, \omega_{X/V})$). *With notation as above, there exists an integer* $N > 0$ *such that for every perfect point* $s \in V$,

$$\sigma(\omega_{X_s}) = \mathrm{im}\left(\sigma_n\big(X/V, \omega_{X/V}\big) \otimes_{A^{1/p^{ne}}} k(s)^{1/p^{ne}} \to \omega_{X_s/s}\right)$$

for all $n > N$.

Proof. Taking $g\colon s \to V$ in Proposition 5.3, we obtain that the image in the statement of the proposition equals $\sigma_n(X_s/s, \omega_{X_s/s})$. Furthermore, since $k(s)$ is perfect, we have containments $\cdots \supseteq \sigma_n(X_s/s, \omega_{X_s/s}) \supseteq \sigma_{n+1}(X_s/s, \omega_{X_s/s}) \supseteq \cdots$ with a descending intersection that coincides with $\sigma(\omega_{X_s})$. Furthermore, by Proposition 5.3, over a dense open set $U \subseteq V$ and some $N > 0$, we have $\sigma_n(X_s/s, \omega_{X_s/s}) = \sigma_{n+1}(X_s/s, \omega_{X_s/s})$ and hence $\sigma_n(X_s/s, \omega_{X_s/s}) = \sigma(\omega_{X_s})$ by the construction of $\sigma(\omega_{X_s})$ for all $n \geqslant N_0$. Let $V_1 = V \setminus U$, base change with V_1, and obtain the result over a dense open subset U_1 of V_1 (for some N_1). By Noetherian induction, this process terminates. $\qquad\square$

5.2 The definition and basic properties of $\tau_n(X/V, \omega_{X/V})$

The goal of this section is to develop the basics of a relative theory of test submodules. We fix the notation of the previous sections and additionally assume that X is geometrically normal over V, which we now also assume is regular. We let $J = J_{X/V} \subseteq R$ be the Jacobian ideal sheaf of X over V. We observe that the formation of $J_{X/V}$ commutes with base change in the following sense: for any $T \to V$, we have $J_{X/V} \cdot \mathscr{O}_{X_T} = J_{X_T/T}$. To see this, just note that $J_{X/V}$ can be defined as a Fitting ideal of $\Omega_{X/V}$ ([HS06, Discussion 4.4.7]) and that the formation of $\Omega_{X/V}$ ([Eis95, Proposition 16.4]) and Fitting ideals ([HS06, Discussion 4.4.7]) commutes with arbitrary base change. Note that J is non-zero at any generic point of X since X is geometrically reduced. Furthermore, on every perfect fiber, the Jacobian ideal is contained in the (big) test ideal [Hoc07, Theorem on p. 213].

We have the images

$$\mathfrak{d}_1 := \mathrm{im}\left(J \cdot \omega_{R_{A^{1/p}}/A^{1/p}} \to \omega_{R_{A^{1/p}}/A^{1/p}}\right) + \Phi_{X/V,1}\big((J \cdot \omega_R)^{1/p}\big) \subseteq \omega_{R_{A^{1/p}}},$$

$$\mathfrak{d}_2 := \sum_{i=0}^{2} \mathrm{im}\left(\big(\Phi_{X/V,i}(J \cdot \omega_{R/A})^{1/p^i}\big) \otimes_{A^{1/p^i}} A^{1/p^2} \to \omega_{R_{A^{1/p^2}}}\right) \subseteq \omega_{R_{A^{1/p^2}}},$$

$$\vdots$$

$$\mathfrak{d}_n := \sum_{i=0}^{n} \operatorname{im}\left(\left(\Phi_{X/V,i}(J \cdot \omega_{R/A})^{1/p^i}\right) \otimes_{A^{1/p^i}} A^{1/p^n} \longrightarrow \omega_{R_{A^{1/p^n}}}\right) \subseteq \omega_{R_{A^{1/p^n}}},$$

$$\vdots$$

Notice that $\mathfrak{d}_{1,2} := \operatorname{im}\left(\mathfrak{d}_1 \otimes_{A^{1/p}} A^{1/p^2} \longrightarrow \omega_{R_{A^{1/p^2}}}\right) \subseteq \mathfrak{d}_2$ and more generally for $j > i$ that $\mathfrak{d}_{i,j} := \operatorname{im}\left(\mathfrak{d}_i \otimes_{A^{1/p^i}} A^{1/p^j} \longrightarrow \omega_{R_{A^{1/p^n}}}\right) \subseteq \mathfrak{d}_j$.

By the same argument as in Section 3, we know that there exists an open set $U \subseteq V$ with $W = f^{-1}(U)$ such that $\operatorname{im}\left(\mathfrak{d}_t \otimes_{A^{1/p^{te}}} A^{1/p^{(t+1)e}} \longrightarrow \omega_{R_{A^{1/p^{t+1}}}}\right)\big|_W = \mathfrak{d}_{t+1}|_W$.

LEMMA 5.5. *With notation as above,* $\operatorname{im}\left(\mathfrak{d}_n \otimes_{A^{1/p^{ne}}} A^{1/p^{(n+1)e}} \longrightarrow \omega_{R_{A^{1/p^{(n+1)e}}}}\right)\big|_W = \mathfrak{d}_{n+1}|_W$ *for all* $n > t$.

Proof. The proof is the same as the proof of Lemma 4.2 and so we omit it. $\square$

DEFINITION 5.6 (Relative test submodules). With notation as above (in particular, $\omega_{X/V}$ is still compatible with base change), we define the *nth iterated relative test submodule* to be $\mathfrak{d}_n$ and denote it by $\tau_n(X/V, \omega_{X/V})$.

We now discuss base change for relative test ideals.

PROPOSITION 5.7 (Base change for $\tau_n(X/V, \omega_{X/V})$). *Suppose that* $g \colon T \to V$ *is a map from a excellent scheme with a dualizing complex; then using the notation of Section 2.2,*

$$\operatorname{im}\left((q_n)^* \tau_n\left(X/V, \omega_{X/V}\right) \to \omega_{X_{T^n}/T^n}\right) = \tau_n\left(X_T/T, \omega_{X/T}\right).$$

Furthermore, if $U = U_n$ *satisfies condition from Lemma 5.5, then* $Y = g^{-1}(U) \subseteq T$ *satisfies the same condition for* $\tau_n(X_T/T, \omega_{X_T/T})$.

Proof. It is just as before since $\omega_{X/V}$ and J are compatible with arbitrary base change. $\square$

COROLLARY 5.8 (Restriction theorem for $\tau_n(X/V, \omega_{X/V})$). *With notation as above, there exists an integer* $N > 0$ *such that for every perfect point* $s \in V$, *we have*

$$\operatorname{im}\left(\tau_n\left(X/V, \omega_{X/V}\right) \otimes_{A^{1/p^{ne}}} k(s)^{1/p^{ne}} \longrightarrow \omega_{X_s/s}\right) = \tau(\omega_{X_s}).$$

for all $n > N$.

Proof. Taking $g \colon s \to V$ in Proposition 5.7, we obtain that the image is equal to $\tau_n(X_s/s, \omega_{X_s/s})$. Furthermore, since $k(s)$ is perfect, we can make identifications so as to have containments $\cdots \subseteq \tau_n(X_s/s, \omega_{X_s/s}) \subseteq \tau_{n+1}(X_s/s, \omega_{X_s/s}) \subseteq \cdots$ with an ascending union that coincides with $\tau(\omega_{X_s})$. Furthermore, by Proposition 5.3, over a dense open set $U \subseteq V$ and some $N > 0$, we have $\tau_n(X_s/s, \omega_{X_s/s}) = \tau_{n+1}(X_s/s, \omega_{X_s/s})$ for all $n \geqslant N_0$ and hence $\tau_n(X_s/s, \omega_{X_s/s}) = \tau(\omega_{X_s})$ by the computation of (4.1). Let $V_1 = V \setminus U$, base change with V_1, and obtain the result over a dense open subset U_1 of V_1 (for some N_1). By Noetherian induction, this process must terminate. $\square$

Remark 5.9 (Relative versus absolute $\tau(\omega)$). It would be natural to relate the relative $\sigma(\omega)$ and $\tau(\omega)$ to the absolute $\sigma(\omega)$ and $\tau(\omega)$. While the authors believe that this is possible along the lines of Section 3.1 or Section 4.1, we will not work out the details here.

5.3 Applications to families of F-injective and F-rational singularities

In this section, we obtain new proofs of results of Hashimoto [Has01], deformation of F-injectivity, and F-rationality in proper flat families. Note that Hashimoto proved the local version of these results (and in the F-rationality case, over a variety). However, the local results generalize to the non-local case via straightforward computations. We begin with a definition essentially first made by Hashimoto.

DEFINITION 5.10 (cf. [Has01, Has10]). We say that X/V (which is still assumed to be relatively Cohen–Macaulay) is *relatively F-injective* if for some $n > 0$, we have $\sigma_n(X/V, \omega_{X/V}) = \omega_{X_{V^n}/V^n}$. Likewise, X/V is *relatively F-rational* if for some $n > 0$, we have $\tau_n(X/V, \omega_{X/V}) = \omega_{X_{V^n}/V^n}$.

Remark 5.11. If V is the spectrum of a perfect field k, it is easy to see that $f : X \longrightarrow V$ is relatively F-injective (respectively. relatively F-rational) if and only if X is F-injective (respectively, F-rational) in the usual sense [ST12, Section 8] via an identification $k \cong k^{1/p}$. Note that we are implicitly assuming that X is Cohen–Macaulay in this case.

LEMMA 5.12 (cf. [Has01, Proposition 5.5]). *With notation as above, if $\sigma_n(X/V, \omega_{X/V}) = \omega_{X_{V^n}/V^n}$ for some $n > 0$, then $\sigma_m(X/V, \omega_{X/V}) = \omega_{X_{V^m}/V^m}$ for all $m > 0$ divisible by n. Furthermore, if V is regular, then the result holds for all $m \geqslant n$. Additionally, if $\tau_n(X/V, \omega_{X/V}) = \omega_{X_{V^n}/V^n}$ for some $n > 0$, then $\tau_m(X/V, \omega_{X/V}) = \omega_{X_{V^m}/V^m}$ for all $m \gg 0$.*

Proof. We begin with σ. We notice that

$$\sigma_{2n}\big(X/V, \omega_{X/V}\big) = \mathrm{im}\,\big(\,\mathrm{im}\,\big(\omega_{X^n/V^n} \longrightarrow \omega_{X_V^n/V^n}\big) \otimes_{A^{1/p^n}} A^{1/p^{2n}} \longrightarrow \omega_{X_{V^{2n}}/V^{2n}}\big)$$

and our hypothesis $\sigma_n(X/V, \omega_{X/V}) = \omega_{X_{V^n}/V^n}$ implies that the inner map is surjective. But then, the outer map is surjective too by right exactness of the tensor product and so $\sigma_{2n}(X/V, \omega_{X/V}) = \omega_{X_{V^{2n}}/V^{2n}}$. The general case repeats this process and so follows similarly.

Now we assume that V is regular for the second statement about σ_n. We fix an $n > 0$ such that $\sigma_n(X/V, \omega_{X/V}) = \omega_{X_{V^n}/V^n}$. Then for all $m \leqslant n$, we have

$$\sigma_m\big(X/V, \omega_{X/V}\big) \otimes_{A^{1/p^m}} A^{1/p^n} \supseteq \sigma_n\big(X/V, \omega_{X/V}\big) = \omega_{X_{V^n}/V^n} \cong \omega_{X_{V^m}/V^m} \otimes_{A^{1/p^m}} A^{1/p^n}$$

and so $\sigma_m(X/V, \omega_{X/V}) \otimes_{A^{1/p^m}} A^{1/p^n} = \omega_{X_{V^m}/V^m} \otimes_{A^{1/p^m}} A^{1/p^n}$. On the other hand, if we know $\sigma_m(X/V, \omega_{X/V}) \subsetneq \omega_{X_{V^m}/V^m}$, then the previous equality is impossible since A^{1/p^n} is faithfully flat over A [Kun69] (since V is regular).

Handling τ is easy. Again use that $\omega_{X/V}$ is compatible with base change, since

$$\begin{aligned}
\omega_{X_V^m/V^m} &= \mathrm{im}\,\big(\omega_{X/V} \otimes_{A^{1/p^n}} A^{1/p^m} \longrightarrow \omega_{X_V^m/V^m}\big) \\
&= \mathrm{im}\,\big(\tau_n\big(X/V, \omega_{X/V}\big) \otimes_{A^{1/p^n}} A^{1/p^m} \longrightarrow \omega_{X_V^m/V^m}\big) \\
&= \mathfrak{d}_{n,m} \subseteq \mathfrak{d}_m = \tau_n(X/V, \omega_{X/V}) \subseteq \omega_{X_V^m/V^m}. \qquad \square
\end{aligned}$$

THEOREM 5.13 (Deformation of F-rationality and F-injectivity; cf. [Has01, Theorem 5.8, Remark 6.7]). *Suppose that $f : X \longrightarrow V$ is a proper, flat, finite-type, equidimensional, reduced Cohen–Macaulay morphism to an excellent integral scheme V with a dualizing complex. Suppose that for some point $s \in V$, the fiber X_s/s is relatively F-injective (respectively, F-rational). Then there exists an open neighborhood $U \subseteq V$ containing s such that $X_u \longrightarrow u$ is relatively F-injective (respectively, F-rational) for all $u \in U$.*

Proof. We first show that $\sigma_n(\omega_{X/V}) = \omega_{X_{V^{ne}}/V^{ne}}$ at each point of the fiber X_s. Indeed, let $z \in X$ be a point on X_s, and let I denote the ideal sheaf of X_s. We observe that for some n, the natural

map $\sigma_n(\omega_{X/V})/(I \cdot \sigma_n(\omega_{X/V})) \longrightarrow \omega_{X/V}/(I \cdot \omega_{X/V}) = \omega_{X_s/V_s}$ is surjective. This is preserved after localizing at z, and so since $I \subseteq \mathfrak{m}_z$, we see that the generators of the stalk $(\sigma_n(\omega_{X/V}))_z$ generate $(\omega_{X/V})_z$ by Nakayama's lemma. Hence $(\sigma_n(\omega_{X/V}))_z = (\omega_{X/V})_z$. Since this holds at every point $z \in X_s \subseteq X$, it holds in a neighborhood of X_s.

As before, let Z denote the locus where $\sigma_n(\omega_{X/V}) \neq \omega_{X_{V^{ne}}/V^{ne}}$. This is closed and since f is proper, its image $f(Z)$ is closed too. But $f(Z)$ is then a closed set not containing s. The result follows for F-injectivity. The proof for F-rationality is the same. $\qquad\square$

6. Global applications

The purpose of this section is to develop a global theory of the previous sections for a projective family $f\colon X \longrightarrow V$. In the last few years, there has been a new push to use the Frobenius morphism and the trace map to replace the Kodaira vanishing theorem. In this section, we extend some of these ideas to families. We study how the canonical linear subsystems $S^0(X_s, \sigma(X_s, \Delta_s) \otimes M_s) \subseteq H^0(X_s, M_s)$, introduced in [Sch14], behave as we vary $s \in V$. Furthermore, as mentioned in the introduction, we also obtain some global generation and semi-positivity statements.

6.1 Basic definitions

We use the following setup throughout Section 6.

NOTATION 6.1. In the situation of Notation 2.1, also assume that $f\colon X \longrightarrow V$ is projective and that V is regular (which implies that $F_V^e\colon V^e \longrightarrow V$ is flat). Furthermore, fix a line bundle M on X. Sometimes we also assume the following (in which case we write Notation 6.1*):

(∗): There is an integer $N \geqslant 0$ such that for every integer $m \geqslant N$,
$$\sigma_m(X/V, \phi) = \sigma_N(X/V, \phi) \otimes_{A^{1/p^{Ne}}} A^{1/p^{me}}.$$

In this situation, we denote $\sigma_N(X/V, \phi)$ by σ_N. We notice that this condition (∗) always holds over a dense open set of the base by Proposition 3.3.

Remark 6.2. Note that since X (respectively, V) is topologically isomorphic to $X_{V^{me}}^{ne}$ (respectively, V^{me}) for every $m \geqslant n$, the push-forward f_* can be identified with g_*, where g is any of the induced morphisms $X_{V^{me}}^{ne} \longrightarrow V^{me}$. Hence, we use only f_* for all purposes, even when g_* for one of the above maps $g\colon X_{V^{me}}^{ne} \longrightarrow V^{me}$ would be more natural. The downside of this notation is that it does not show if a sheaf has an $A^{1/p^{me}}$-structure and consequently its pushforward by f an $\mathscr{O}_{V^{me}}$-structure. We decided to still use it because it greatly simplifies the notation.

Summarizing: When reading the following arguments, it is important to trace through the space $X_{V^{me}}^{ne}$ on which the adequate sheaves live. Then f_* of these sheaves will live on V^{me}.

NOTATION 6.3. When dealing with sheaves G on V^{ne}, we will frequently pull them back to V^{me} for $m \geqslant n$. When doing this, instead of writing $\mathscr{O}_{V^{me}} \otimes_{\mathscr{O}_{V^{ne}}} G$ or $\mathscr{O}_V^{1/p^{me}} \otimes_{\mathscr{O}_V^{1/p^{ne}}} G$, we will write simply $V^{me} \times_{V^{ne}} G$ or $G_{V^{me}}$. We trust that this abuse of notation will cause no confusion, as it helps compactify the notation substantially.

Our approach to understanding how the canonical linear systems of [Sch14] behave in families is to define a relative version of them and then show certain base-change properties. These objects will be relative versions of σ for global sections, and they play the same role for σ_n that T^0 and S^0 play for τ and σ in [BST15] and [Sch14], respectively.

DEFINITION 6.4. In the situation of Notation 6.1, define

$$S^0_{\varphi^n} f_*(M) := \mathrm{im}\left(f_*\left(\left(L^{\frac{p^{ne}-1}{p^e-1}}\right)^{1/p^{ne}} \otimes_R M\right) \xrightarrow{f_*(\varphi^n \otimes_R \mathrm{id}_M)} f_*\left(A^{1/p^{ne}} \otimes_A M\right)\right).$$

Note that $S^0_{\varphi^n} f_*(M)$ is a sheaf on V^{ne}; it is a subsheaf of $(f_*(M))_{V^{ne}}$ by flat base change. In case (X, Δ) is a pair, we define $S^0_{\Delta,ne} f_*(M) := S^0_{\varphi^n_\Delta} f_*(M)$ (assuming that $(p^{ne} - 1)(K_X + \Delta)$ is Cartier and ϕ is the corresponding map). If $\Delta = 0$, then we write $S^0_{ne} f_*(M)$ for $S^0_{\Delta,ne} f_*(M)$.

Since the image of $\varphi^n \otimes_R \mathrm{id}_M$ in the above definition is $\sigma_n(X/V, \phi) \otimes_R M$, the following proposition is immediate.

PROPOSITION 6.5. *In the situation of Notation 6.1, we have $S^0_{\varphi^n} f_*(M) \subseteq f_*(\sigma_n(X/V, \phi) \otimes_R M)$.*

Our first goal is to show that the images in Definition 6.4 descend (up to appropriate base change by the Frobenius morphism). Compare with the containments $\mathfrak{a}_{i,n} \supseteq \mathfrak{a}_{i+1,n} \supseteq \cdots \supseteq \mathfrak{a}_{n,n}$ of Section 3.

PROPOSITION 6.6. *For all integers $m \geqslant n \geqslant 0$,*

$$V^{me} \times_{V^{ne}} S^0_{\varphi^n} f_*(M) \supseteq S^0_{\varphi^m} f_*(M) \tag{6.1}$$

as subsheaves of $V^{me} \times_V f_*(M)$.

Remark 6.7. To be precise, the left- and right-hand sides of (6.1) are subsheaves of

$$V^{me} \times_{V^{ne}} f_*\left(A^{1/p^{ne}} \otimes_A M\right) \quad \text{and} \quad f_*\left(A^{1/p^{me}} \otimes_A M\right),$$

respectively. However, both $V^{me} \times_{V^{ne}} f_*\left(A^{1/p^{ne}} \otimes_A M\right)$ and $f_*\left(A^{1/p^{me}} \otimes_A M\right)$ are canonically isomorphic to $V^{me} \times_V f_*(M)$ via flat base change (since V is regular).

Proof of Proposition 6.6. The following commutative diagram shows that $f_*\left(\varphi^m \otimes_R \mathrm{id}_M\right)$ factors through $V^{me} \times_{V^{ne}} f_*\left(\varphi^n \otimes_R \mathrm{id}_M\right)$:

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Hence, the statement of the proposition holds by the following computation:

$$\begin{aligned}
S^0_{\phi^m} f_*(M) &= \mathrm{im}\left(f_*\left(\varphi^m \otimes_R \mathrm{id}_M\right)\right) \\
&\subseteq \mathrm{im}\left(V^{me} \times_{V^{ne}} f_*\left(\varphi^n \otimes_R \mathrm{id}_M\right)\right) \\
&= \underbrace{V^{me} \times_{V^{ne}} \mathrm{im}\left(f_*\left(\varphi^n \otimes_R \mathrm{id}_M\right)\right)}_{V^{ne} \to V^{me} \text{ is flat}} \\
&= V^{me} \times_{V^{ne}} S^0_{\phi^n} f_*(M). \qquad\qquad \square
\end{aligned}$$

6.2 Auxiliary definition and stabilization

We would now like to obtain a global result similar to Proposition 3.3. In particular, we would like the containments of Proposition 6.6 to be equalities over a dense open subset of the base V. There is a complicating factor, however: while we can still find an open set U of V such that

$$\left(V^{(n+1)e} \times_{V^{ne}} S^0_{\varphi^n} f_*(M)\right)\big|_U = \left(S^0_{\varphi^{n+1}} f_*(M)\right)\big|_U,$$

we do not see how to use this to show that we have the $n+1$ to $n+2$ equality without additional assumptions. The issue is that in the proof of Proposition 3.3, twisting by line bundles is exact. For $S^0_{\phi^n}$, however, we also push forward. Therefore, in order to obtain our stabilization over a dense open set of the base, we need additional positivity assumptions on M and L.

Furthermore, we need an auxiliary version of S^0_ϕ which involves the $\sigma_N := \sigma_N(X/V, \phi)$ from Notation 6.1*. To do this, first observe that since f is flat, the tensor product $L^{\frac{p^{me}-1}{p^e-1}} \otimes_R \sigma_N$ is naturally identified with a subsheaf of $L^{\frac{p^{me}-1}{p^e-1}} \otimes_R R_{A^{1/p^{Ne}}} \cong L^{\frac{p^{me}-1}{p^e-1}} \otimes_A A^{1/p^{Ne}}$. In order to motivate this auxiliary definition, we make the following observation.

LEMMA 6.8. *In the situation of Notation 6.1*[*], the image of the natural map*

$$\alpha_{N+m} \colon \left(L^{\frac{p^{me}-1}{p^e-1}} \otimes_R \sigma_N\right)^{1/p^{me}} \hookrightarrow \left(L^{\frac{p^{me}-1}{p^e-1}} \otimes_R R_{A^{1/p^{Ne}}}\right)^{1/p^{me}} \xrightarrow{\phi^m \otimes_{A^{1/p^{me}}} A^{1/p^{(N+m)e}}} R_{A^{1/p^{(N+m)e}}}$$

is equal to $\sigma_N \otimes_{A^{1/p^{Ne}}} A^{1/p^{(N+m)e}} \cong \sigma_{m+N}(X/V, \phi)$.

Proof. The first term is the image of the p^{me}th root of $L^{\frac{p^{me}-1}{p^e-1}} \otimes_R \phi^N$. Therefore, the image of the composition also equals $\sigma_{m+N}(X/V, \phi)$, by our construction of ϕ^j in Section 2.1. $\square$

This stabilization suggests that it is reasonable to make the following definition.

DEFINITION 6.9. In the situation of Notation 6.1*,

$$S^0_{\varphi^n, \sigma_N} f_*(M) := \mathrm{im}\left(f_*\left(\left(L^{\frac{p^{me}-1}{p^e-1}} \otimes_R \sigma_N\right)^{1/p^{me}} \otimes_R M\right) \xrightarrow{f_*(\alpha_{N+m} \otimes_R M)} f_*\left(A^{1/p^{(n+N)e}} \otimes_A M\right)\right),$$

where α_{N+m} is as in Lemma 6.8.

Consider the following proposition relating the various $S^0 f_*$-objects described so far.

PROPOSITION 6.10. *In the situation of Notation 6.1*[*], for every integer $m \geqslant 0$,*

$$V^{(N+m)e} \times_{V^{me}} S^0_{\varphi^m} f_*(M) \supseteq S^0_{\varphi^m, \sigma_N} f_*(M) \supseteq S^0_{\varphi^{m+N}} f_*(M).$$

(Here, all sheaves are regarded as subsheaves of $V^{(N+m)e} \times_V f_*(M)$ via flat base change.)

Proof. This follows directly from the definitions. The first containment is trivial, and the second follows from a factorization similar to the one from Proposition 6.6. $\square$

PROPOSITION 6.11. *In the situation of Notation* 6.1*,

(a) *for all integers $m \geqslant n \geqslant 0$,*

$$V^{(N+m)e} \times_{V^{(N+n)e}} S^0_{\varphi^n, \sigma_N} f_*(M) \supseteq S^0_{\varphi^m, \sigma_N} f_*(M),$$

as subsheaves of $V^{(N+m)e} \times_V f_*(M)$;

(b) *if, furthermore, $L \otimes M^{p^e-1}$ is f-ample, then there is an integer $n > 0$ such that for every integer $m \geqslant n$, the above inclusion is an equality;*

(c) *if, furthermore, $M = Q^l \otimes P$, where Q is f-ample, then there is an integer $l_0 > 0$ such that for every integer $l \geqslant l_0$, $m \geqslant n \geqslant 0$, and nef line bundle P, the above inclusion is an equality.*

Proof. Define the coherent sheaf B_ϕ on $X_{V^{(N+1)e}}$ as the kernel of the top horizontal map in the following commutative diagram:

$$
\begin{array}{ccc}
B_\phi \hookrightarrow \left(L \otimes_R \sigma_N\right)^{1/p^e} \xrightarrow{\phi\otimes} A^{1/p^{(N+1)e}} \otimes_{A^{1/p^{Ne}}} \sigma_N \\
\left(\mathrm{id}_L \otimes_R \phi^N\right)^{1/p^e} \Big\uparrow \qquad\qquad \Big\uparrow \phi^{N+1} \\
\left(L \otimes \left(L^{\frac{p^{Ne}-1}{p^e-1}}\right)^{1/p^{Ne}}\right)^{1/p^e} \cong \left(L^{\frac{p^{(N+1)e}-1}{p^e-1}}\right)^{1/p^{(N+1)e}}
\end{array}
\tag{6.2}
$$

Note that here, we used that the image of ϕ^{N+1} is $A^{1/p^{(N+1)e}} \otimes_{A^{1/p^{Ne}}} \sigma_N$ by the assumptions made in Notation 6.1*. Also, the horizontal arrow is surjective by either Lemma 6.8 or from the diagram (which is how we proved Lemma 6.8). Then one can apply $f_*\left(\left(L^{\frac{p^{me}-1}{p^e-1}} \otimes_R \_\right)^{1/p^{me}} \otimes_R M\right)$ to the top row of (6.2), which is shown in the following commutative diagram. We have also included important isomorphisms to the different terms of the exact sequence

$$f_*\left(\left(L^{\frac{p^{me}-1}{p^e-1}} \otimes_R B_\phi\right)^{1/p^{me}} \otimes_R M\right)$$

$$\downarrow$$

$$f_*\left(\left(L^{\frac{p^{me}-1}{p^e-1}} \otimes_R (L \otimes \sigma_N)^{1/p^e}\right)^{1/p^{me}} \otimes_R M\right) \cong f_*\left(\left(L^{\frac{p^{(m+1)e}-1}{p^e-1}} \otimes_R \sigma_N\right)^{1/p^{(m+1)e}} \otimes_R M\right)$$

$$\nu \downarrow$$

$$f_*\left(\left(L^{\frac{p^{me}-1}{p^e-1}} \otimes_R \left(\sigma_N \otimes_{A^{1/p^{Ne}}} A^{1/p^{(N+1)e}}\right)\right)^{1/p^{me}} \otimes_R M\right)$$

$$\downarrow$$

$$R^1 f_*\left(\left(L^{\frac{p^{me}-1}{p^e-1}} \otimes_R B_\phi\right)^{1/p^{me}} \otimes_R M\right) \qquad \cong$$

$$f_*\left(\left(\left(L^{\frac{p^{me}-1}{p^e-1}} \otimes_R \sigma_N\right)^{1/p^{me}} \otimes_{A^{1/p^{(N+m)e}}} A^{1/p^{(N+m+1)e}}\right) \otimes_R M\right)$$

$$\| \wr \text{ by flat base change}$$

$$V^{(N+m+1)e} \times_{V^{(N+m)e}} f_*\left(\left(L^{\frac{p^{me}-1}{p^e-1}} \otimes_R \sigma_N\right)^{1/p^{me}} \otimes_R M\right)$$

$$\downarrow$$

$$V^{(N+m+1)e} \times_{V^{(N+m)e}} S^0_{\phi^m, \sigma_N} f_*(M) \longleftarrow S^0_{\phi^{m+1}, \sigma_N} f_*(M).$$

Both $V^{(N+m+1)e} \times_{V^{(N+m)e}} S^0_{\phi^m, \sigma_N} f_*(M)$ and $S^0_{\phi^{m+1}, \sigma_N} f_*(M)$ can be regarded as subsheaves of $V^{(N+m+1)e} \times_V f_*(M)$ via flat base change. Furthermore, the bottom horizontal arrow becomes a map of subsheaves, that is, an injection, this way. This shows point (a).

To prove point (b), we are supposed to prove that whenever $L \otimes M^{p^e-1}$ is f-ample, this arrow is also surjective for $m \gg 0$. This would follow if the third vertical arrow (from above), labeled ν, is surjective. Therefore, it is sufficient to show that for $m \gg 0$,

$$0 = R^1 f_* \left(\left(L^{\frac{p^{me}-1}{p^e-1}} \otimes_R B_\phi \right)^{1/p^{me}} \otimes_R M \right) = R^1 f_* \left(\left(L^{\frac{p^{me}-1}{p^e-1}} \otimes_R B_\phi \otimes_R M^{p^{me}} \right)^{1/p^{me}} \right)$$

$$= R^1 f_* \left(\left((L \otimes M^{p^e-1})^{\frac{p^{me}-1}{p^e-1}} \otimes_R B_\phi \otimes_R M \right)^{1/p^{me}} \right).$$

Note that since applying $(\_)^{1/p^{me}}$ does not change the sheaf of abelian groups structure, this is equivalent to showing

$$0 = R^1 f_* \left((L \otimes M^{p^e-1})^{\frac{p^{me}-1}{p^e-1}} \otimes_R B_\phi \otimes_R M \right)$$

$$= R^1 f_* \left((L \otimes M^{p^e-1} \otimes_A A^{1/p^{N+1}})^{\frac{p^{me}-1}{p^e-1}} \otimes_{R_{A^{1/p^{(N+1)e}}}} (B_\phi \otimes_R M) \right).$$

Furthermore, $L \otimes M^{p^e-1} \otimes_A A^{1/p^{N+1}}$ is a relatively ample line bundle by the assumption of point (b) and $B_\phi \otimes_R M$ is a coherent sheaf on $X_{V^{(N+1)e}}$. Therefore, the relative Serre vanishing theorem concludes our proof.

Point (c) follows immediately from the above argument. Indeed, if $M = Q^l \otimes P$, then by the relative Fujita vanishing theorem [Kee03, Theorem 1.5], there is an integer $l_0 > 0$ such that the above vanishing holds for every $m \geqslant 0$, $l \geqslant l_0$, and nef line bundle P. $\qquad \square$

COROLLARY 6.12. *In the situation of Notation* 6.1[*], *if* $L \otimes M^{p^e-1}$ *is* f-ample, *then there is an integer* $n > 0$ *such that for every integer* $m \geqslant n$,

$$V^{me} \times_{V^{ne}} S^0_{\varphi^n} f_*(M) = S^0_{\varphi^m} f_*(M) \quad \left(= S^0_{\varphi^{m-N}, \sigma_N} f_*(M) \right)$$

as subsheaves of $V^{me} \times_V f_*(M)$. *Furthermore, in the situation of point* (c) *of Proposition* 6.11, *where* $M = Q^l \otimes P = (\text{ample})^l \otimes (\text{nef})$ *with* $l \gg 0$, *we can pick* $n = N$.

Proof. By Proposition 6.10,

$$V^{(m+2N)e} \times_{V^{(m+N)e}} S^0_{\varphi^m, \sigma_N} f_*(M) \supseteq V^{(m+2N)e} \times_{V^{(m+N)e}} S^0_{\varphi^{m+N}} f_*(M) \supseteq S^0_{\varphi^{m+N}, \sigma_N} f_*(M). \quad (6.3)$$

Furthermore, by Proposition 6.11(b), we know that the statement holds when $S^0_{\varphi^m} f_*(M)$ is replaced by $S^0_{\varphi^m, \sigma_N} f_*(M)$. Hence, the inclusion of the right end of (6.3) in the left end is an equality for $m \gg 0$. However, then all inclusions in (6.3) are equalities. In particular, from the first equality of (6.3), using that $V^{(m+2N)e} \to V^{(m+N)e}$ is faithfully flat, we obtain that $S^0_{\varphi^m, \sigma_N} f_*(M) = S_{\varphi^{m+N}} f_*(M)$ for $m \gg 0$. Using Proposition 6.11(b) once more concludes our proof of the main statement. The final statement is similar. $\qquad \square$

We can now obtain our promised analog of Proposition 3.3.

THEOREM 6.13. *In the situation of Notation* 6.1, *if* $L \otimes M^{p^e-1}$ *is ample, there exist an integer* n_0 *and a dense open set* $U \subseteq V$ *such that for all* $m \geqslant n \geqslant n_0$, *we have*

$$\left(V^{me} \times_{V^{ne}} S^0_{\varphi^n} f_*(M) \right)\big|_U = \left(S^0_{\varphi^m} f_*(M) \right)\big|_U.$$

Proof. By shrinking V and applying Proposition 3.3, we can reduce to the case of Notation 6.1*. The result follows directly from Corollary 6.12. □

COROLLARY 6.14. *In the situation of Notation* 6.1, *there is a natural inclusion*

$$S^0_{\phi^n} f_*(M) \subseteq f_* \big(\sigma_n(X/V, \phi) \otimes_R M \big)$$

as subsheaves of $V^{ne} \times_V f_*(M)$.

Further, *in the situation of Notation* 6.1*, *if* $M = Q^l \otimes P$, *where* Q *is* f-*ample, then there is an integer* $l_0 > 0$ *such that for every integer* $l \geqslant l_0$, $n \geqslant N$, *and* f-*nef line bundle* P, *the above inclusion is an equality.*

Proof. Consider the surjection

$$\xi \colon \left(L^{\frac{p^{ne}-1}{p^e-1}} \right)^{1/p^{ne}} \xrightarrow{\quad \phi^n \quad} \sigma_n(X/V, \phi) \,.$$

Then $S^0_{\phi^n} f_*(M) = \operatorname{im} f_*(\xi \otimes_R \operatorname{id}_M)$, which is a subsheaf of $f_*(\sigma_n(X/V, \phi) \otimes_R M)$.

We prove the addendum by induction on n. If $n = N$,

$$S^0_{\phi^N} f_*(M) = \underbrace{S^0_{\phi^0, \sigma_N} f_*(M)}_{\text{by Corollary } 6.12} = \underbrace{f_*(\sigma_N \otimes_R M)}_{\text{by Definition } 6.9} \,.$$

Let us then assume $n > N$. By flat base change and the assumption of Notation 6.1*,

$$f_* \big(\sigma_n(X/V) \otimes_R M \big) = f_* \big(\sigma_N(X/V) \otimes_R M \big) \times_{V^{Ne}} V^{ne} \,.$$

Hence, using the induction hypothesis, it is enough to see that for every $n \geqslant N$,

$$S^0_{\phi^n} f_*(M) = S^0_{\phi^{n-1}} f_*(M) \,.$$

However, this follows from point (c) of Proposition 6.11 and Corollary 6.12. □

6.3 Base change

We now prove base change for $S^0_{\phi^n} f_*$.

NOTATION 6.15. In the situation of Notation 6.1, choose

- ○ a regular, integral, excellent scheme T with a dualizing complex,
- ○ a morphism $T \to V$.

We indicate base change via this morphism by T in subscript. Define the following sheaves on X_T:

(a) $B := (f_T)^{-1} \mathcal{O}_T$,

(b) $Q := \mathcal{O}_{X_T}$.

Denote by $(p_1)^{1/p^i}$ the natural projection morphism $X^i \times_{V^i} T^i \to X^i$.

PROPOSITION 6.16. *In the situation of Notation* 6.15, *if* $L \otimes M^{p^e-1}$ *is* f-*ample, then for* $n \gg 0$, *the natural base change morphism*

$$f_* \big(A^{1/p^{ne}} \otimes_A M \big) \times_{V^{ne}} T^{ne} \to (f_T)_* \big(B^{1/p^{ne}} \otimes_B M_T \big)$$

induces a surjective morphism on subsheaves

$$S^0_{\phi^n} f_*(M) \times_{V^{ne}} T^{ne} \twoheadrightarrow S^0_{\phi_T^n} (f_T)_*(M_T) \,. \tag{6.4}$$

Furthermore, the following hold:

(a) *The lower bound on the n for which the above statement holds depends only on f and M. In particular, it is independent of T.*

(b) *If $M = Q^l \otimes P$ for some f-ample line bundle Q, nef line bundle P, and integer $l > 0$, then there is a uniform lower bound on n independent of l and P.*

(c) *Even if $L \otimes M^{p^e - 1}$ is not assumed to be f-ample, (6.4) is an isomorphism if $T \to V$ is flat.*

(d) *There is a dense open set $U \subseteq V$ such that if the image of $T \to V$ is contained in U, then (6.4) is an isomorphism.*

Proof of Proposition 6.16. First, note that

$$\left((p_1)^{1/p^{ne}}\right)^* \left(\left(L^{\frac{p^{ne}-1}{p^e-1}} \otimes_R M^{p^{ne}}\right)^{1/p^{ne}}\right) \cong \left((p_1)^{1/p^{ne}}\right)^* \left(\left(L^{\frac{p^{ne}-1}{p^e-1}}\right)^{1/p^{ne}} \otimes_R M\right)$$

$$\cong \left(L_T^{\frac{p^{ne}-1}{p^e-1}}\right)^{1/p^{ne}} \otimes_Q M_T.$$

Hence, there is a natural base-change morphism below in (6.5). We will show that it is an isomorphism for $n \gg 0$. Furthermore, this n can be chosen independently of T.

$$\left(f_*\left(\left(L^{\frac{p^{ne}-1}{p^e-1}}\right)^{1/p^{ne}} \otimes_R M\right)\right) \times_{V^{ne}} T^{ne} \to (f_T)_*\left(\left(L_T^{\frac{p^{ne}-1}{p^e-1}}\right)^{1/p^{ne}} \otimes_Q M_T\right). \qquad (6.5)$$

Indeed, by cohomology and base change [Sta17, Lemma 25.20.1], it is enough to show that for every integer $i > 0$,

$$R^i f_*\left(\left(L^{\frac{p^{ne}-1}{p^e-1}}\right)^{1/p^{ne}} \otimes_R M\right) = 0.$$

Since $V^{ne} \to V$ is affine, this is equivalent to showing that for $i > 0$,

$$0 = R^i(f_{V^{ne}})_*\left(\left(L^{\frac{p^{ne}-1}{p^e-1}}\right)^{1/p^{ne}} \otimes_R M\right) = R^i(f_{V^{ne}})_*\left(\left(L^{\frac{p^{ne}-1}{p^e-1}} \otimes_R M^{p^{ne}}\right)^{1/p^{ne}}\right),$$

which is further equivalent to showing that for $i > 0$,

$$0 = R^i f_*\left(L^{\frac{p^{ne}-1}{p^e-1}} \otimes_R M^{p^{ne}}\right). \qquad (6.6)$$

However, the last vanishing holds for $n \gg 0$ by the relative Serre vanishing theorem, independently of T. Hence, the base-change homomorphism of (6.5) is indeed an isomorphism for $n \gg 0$, which can be chosen independently of T. In particular, for $n \gg 0$, there is a commutative base-change diagram as follows, which implies the statement of the proposition together with addendum (a):

$$\begin{array}{ccc}
f_*\left(\left(L^{\frac{p^{ne}-1}{p^e-1}}\right)^{1/p^{ne}} \otimes_R M\right) \times_{V^{ne}} T^{ne} & \longrightarrow & f_*\left(A^{1/p^{ne}} \otimes_A M\right) \times_{V^{ne}} T^{ne} \\
\| \wr & & \downarrow \\
(f_T)_*\left(\left(L_T^{\frac{p^{ne}-1}{p^e-1}}\right)^{1/p^{ne}} \otimes_Q M_T\right) & \longrightarrow & (f_T)_*\left(B^{1/p^{ne}} \otimes_B M_T\right).
\end{array} \qquad (6.7)$$

For addendum (c), note that if $T \to V$ is flat, then both vertical morphisms in (6.7) are isomorphisms even if $L \otimes M^{p^e - 1}$ is not assumed to be f-ample. Furthermore, images via flat pullbacks are pullbacks of images, which concludes the proof of addendum (c).

For addendum (b), note that if $M = Q^l \otimes P$ for an f-ample line bundle Q and a nef line

bundle P, then (6.6) holds for every $l > 0$ uniformly, by the relative Fujita vanishing theorem [Kee03].

For addendum (d), note that by [Har77, Theorem 12.8 and Corollary 12.9], there is a dense open set $U \subseteq V$ (that is, the open set where $h^0(X_s, M_s)$ is constant) such that if T maps into U, then the right vertical arrow in (6.7) is an isomorphism. In particular, the homomorphism (6.4) is then injective, because it is a homomorphism between subsheaves of the two sheaves involved in the above vertical map. Therefore, by the already proven surjectivity, (6.4) is an isomorphism. This concludes addendum (d). $\qquad\square$

Remark 6.17. In particular, if in Notation 6.15, $T = s$ is a perfect point of V (that is, $T = \mathrm{Spec}(K)$ for some perfect field K), then $S^0_{\phi^n_T}(f_T)_*(M_T)$ can be interpreted as follows. Since the natural map $K \to (K)^{1/p^i}$ is an isomorphism, the natural projection $X_{T^i} \to X_T$ is an isomorphism. Furthermore, $(f_T)_*(M_{T^i})$ can be identified with $(f_T)_*(M_T) = H^0(X_T, M_T)$ via $K \to K^{1/p^i}$. Thus, $S^0_{\phi^n_T}(f_T)_*(M_T)$ is identified with

$$S^0_{s,m} := \mathrm{im}\left(H^0\left(X_s, \left(L_s^{\frac{p^{me}-1}{p^e-1}}\right)^{1/p^{me}} \otimes M_s\right) \to H^0(X_s, M_s)\right) \tag{6.8}$$

In particular, for $m \gg 0$, the sheaf (6.8) can be identified with $S^0(X_s, \sigma(X_s, \phi_s) \otimes M_s)$.

6.4 Uniform stabilization

THEOREM 6.18. *In the situation of Notation 6.1, if $L \otimes M^{p^e-1}$ is f-ample, then there is an integer $n > 0$ such that for all integers $m \geqslant n$ and perfect points $s \in V$,*

$$\mathrm{im}\left(S^0_{\phi^m} f_*(M) \otimes_{\mathscr{O}_{V^{me}}} k(s)^{1/p^{me}} \to H^0(X_s, M_s)\right) = S^0_{s,m} = S^0\left(X_s, \sigma(X_s, \psi_s) \otimes M_s\right).$$

(Note that ψ_s is defined in Section 2.3 and $S^0_{s,m}$ in Remark 6.17.)

Proof. We show the statement by induction on the dimension of V. There is nothing to prove if $\dim V = 0$. Hence, we may proceed to the induction step and assume that $\dim V > 0$. By Proposition 6.16, there is an $n > 0$ such that for every perfect point $s \in V$ and every $m \geqslant n$, we have

$$\mathrm{im}\left(S^0_{\phi^m} f_*(M) \otimes_{\mathscr{O}_{V^{me}}} k(s)^{1/p^{me}} \to H^0(X_s, M_s)\right) = S^0_{s,m}.$$

According to Lemma 2.15, by possibly increasing n, we may find a non-empty, dense open set U such that for all $m \geqslant n$,

$$\sigma_m(X/V, \phi)\big|_U = \sigma_n(X/V, \phi) \times_{V^{ne}} V^{me}\big|_U.$$

Therefore, applying Corollary 6.12 yields (also by possibly increasing n) that for all $m \geqslant n$,

$$V^{me} \times_{V^{ne}} S^0_{\varphi^n} f_*(M)\big|_U = S^0_{\varphi^m} f_*(M)\big|_U.$$

It then follows that for every perfect point $s \in U$,

$$\mathrm{im}\left(S^0_{\phi^m} f_*(M) \otimes_{\mathscr{O}_{V^{me}}} k(s)^{1/p^{me}} \to H^0(X_s, M)\right)$$

is the same for all $m \geqslant n$. Therefore, $S^0_{s,m}$ is stabilized for all perfect $s \in U$ for values of $m \geqslant n$. Hence, it is equal to $S^0(X_s, \sigma(X_s, \psi_s) \otimes M_s)$. This shows the statement of the proposition for all perfect $s \in U$. We fix the $n_0 = n$ used above for future use.

On the other hand, consider the reduced scheme $V_1 = V \setminus U$. Write $V_1 = \coprod V_{1,i}$ as a disjoint union of regular, locally closed, integral subschemes. Each $V_{1,i}$ has smaller dimension than V. In

particular, there exists an n_i such that the statement holds for $f_i\colon X_{V_i} \to V_i$. Letting n be the maximum of n_0 and the n_i, we obtain our result since every perfect point of X factors through a point of U or of one of the V_i. □

COROLLARY 6.19. *In the situation of Notation* 6.1, *if* $L \otimes M^{p^e-1}$ *is* f-*ample, then there is a dense open set* $U \subseteq V$ *such that for every perfect point* $s \in U$,

$$S^0_{\phi^m} f_*(M) \otimes_{\mathscr{O}_{V^{me}}} k(s)^{1/p^{me}} = S^0\big(X_s, \sigma(X_s, \psi_s) \otimes M_s\big).$$

In particular, $\dim_{k(s)} S^0(X_s, \sigma(X_s, \psi_s) \otimes M_s)$ *is constant on an open set, and the rank of* $S^0_{\phi^m} f_*(m)$ *equals this dimension.*

Proof. The main statement follows immediately from Theorem 6.18 and point (d) of Proposition 6.16. The corollary follows from basic properties of coherent sheaves [Har77, Exercise II.5.8]. □

If there is a point $s_0 \in V$ such that in X_{s_0} we have $H^0 = S^0$, then we can say more than the above corollary: it turns out that it can be assumed that $s_0 \in U$, and further that U has other useful properties. This is proved in the following theorem.

THEOREM 6.20. *In the situation of Notation* 6.1, *if* $L \otimes M^{p^e-1}$ *is* f-*ample and there is a perfect point* $s_0 \in V$ *such that* $H^0(X_{s_0}, M_{s_0}) = S^0(X_{s_0}, \sigma(X_{s_0}, \psi_{s_0}) \otimes M_{s_0})$, *then there is an open neighborhood* U *of* s_0, *such that*

(a) $f_*M|_U$ *is locally free and compatible with base change and*

(b) $H^0(X_v, M_v) = S^0(X_v, \sigma(X_v, \psi_v) \otimes M_v)$ *for every perfect point* $v \in U$.

In particular, $\dim S^0(X_v, \sigma(X_v, \psi_v) \otimes M_v)$ *is constant for* $v \in U$.

Proof. By Theorem 6.18, there is an n such that for every $v \in V$ and integer $m \geqslant n$, the natural base-change homomorphism induces a surjection as follows:

$$\begin{array}{ccc} S^0_{\phi^m} f_*(M) \otimes_{\mathscr{O}_{V^{me}}} k(v)^1/p^{me} & \longrightarrow & S^0(X_v, \sigma(X_v, \psi_v) \otimes M_v) \\ \downarrow & & \uparrow \\ f_*\big(M \otimes_A A^{1/p^{me}}\big) \otimes_{\mathscr{O}_{V^{me}}} k(v)^{1/p^{me}} & \longrightarrow & H^0(X_v, M_v). \end{array} \qquad (6.9)$$

Furthermore, the right vertical arrow is surjective for $v = s_0$. It follows that the bottom horizontal arrow then also is. Using [Har77, Corollary III.12.11] concludes the proof of point (a).

To prove point (b), let us again consider (6.9) for $v = s_0$. Now, we know that the bottom horizontal arrow is an isomorphism. However, the left vertical arrow is then also surjective. By Nakayama's lemma, there is an equality $(S^0_{\phi^n} f_*(M))_v = (f_*(M))_v$ of stalks, which means that by possibly restricting U, for every $v \in U$, the left vertical arrow of (6.9) is surjective. However, then, since the bottom horizontal arrow is an isomorphism for every $v \in U$, point (b) follows. □

The surjection in Proposition 6.16 is not an isomorphism in general, by the following two examples. In the first example, the global geometry of the smooth fibers causes the anomaly, while in the second, the degeneration of F-pure singularities to non-F-pure singularities is the main culprit. However, we first show a lemma.

LEMMA 6.21. *Let* Y *be a smooth curve over* k *and* G *an effective divisor on* Y. *Define* $\mathscr{B}^1_Y$ *as the cokernel of* $\mathscr{O}_X \to F_*\mathscr{O}_X$. *Then*

$$S^0\big(Y, \sigma(X, 0) \otimes \omega_Y(G)\big) = H^0(Y, \omega_Y(G)) \Leftrightarrow H^0\big(Y, \mathscr{B}^1_Y(-p^nG)\big) = 0 \quad \forall\, n > 0.$$

Proof. Consider the exact sequence

$$0 \longrightarrow \mathscr{O}_Y \longrightarrow F_*\mathscr{O}_Y \longrightarrow \mathscr{B}_Y^1 \longrightarrow 0\,.$$

Twist this sequence by $-G$, apply cohomology, and use that since G is effective,

(a) either $G > 0$ and hence $H^0(Y, (F_*\mathscr{O}_Y)(-G)) \cong H^0(\mathscr{O}_Y(-F^*G)) = 0$,

(b) or $G = 0$ and then $H^0(Y, \mathscr{O}_Y) \to H^0(Y, F_*\mathscr{O}_Y)$ is an isomorphism.

In either case, we have another exact sequence

$$0 \longrightarrow H^0\big(Y, \mathscr{B}_Y^1(-G)\big) \longrightarrow H^1(Y, \mathscr{O}_Y(-G)) \longrightarrow H^1\big(Y, F_*\mathscr{O}_Y(-F^*G)\big)\,,$$

where the last map is the Serre dual to $H^0(Y, (F_*\omega_Y)(G)) \to H^0(Y, \omega_Y(G))$. Furthermore, observe that the map $H^0(Y, (F_*^{n+1}\omega_Y)(G)) \to H^0(Y, (F_*^n\omega_Y)(G))$ is harmlessly identified with the map $H^0(Y, (F_*\omega_Y)(p^nG)) \to H^0(Y, (\omega_Y(p^nG)))$. Therefore, combining the previous statements, we see that $H^0(Y, (F_*^{n+1}\omega_Y)(G)) \to H^0(Y, (F_*^n\omega_Y)(G))$ surjects if and only if $H^0(Y, \mathscr{B}_Y^1(-p^nG)) = 0$. $\qquad\square$

EXAMPLE 6.22. We choose two smooth projective curves C and D of genus 3 over an algebraically closed field k of prime characteristic with two ample line bundles N_C and N_D (of degree 1) such that $S^0(C, \omega_C \otimes N_C) \neq H^0(C, \omega_C \otimes N_C)$ but $S^0(D, \omega_D \otimes N_D) = H^0(D, \omega_D \otimes N_D)$. Since the relative Picard scheme over the moduli stack of smooth curves of genus g is smooth and irreducible, there is a (possibly reducible) curve connecting (C, N_C) and (D, N_D) in the above moduli space. However, by possibly replacing (C, N_C) and (D, N_D), we may then also find an irreducible curve connecting them. Therefore, by passing to the normalization of this curve, we may assume that there are a family $f\colon X \to V$, a line bundle N on X, and two points $c, d \in C$ such that if we set $(C, N_C) := (X_c, N|_{X_c})$ and $(C, N_C) := (X_c, N|_{X_c})$, then

(a) V is a smooth curve;

(b) X is a family of smooth curves of genus 3;

(c) $\deg_{X/V} N = 1$;

(d) $S^0(C, \omega_C \otimes N_C) \neq H^0(C, \omega_C \otimes N_C)$; and

(e) $S^0(D, \omega_D \otimes N_D) = H^0(D, \omega_D \otimes N_D)$.

Furthermore, fix $e = 1$ and $L := \omega_{X/V}^{1-p}$, and let $\phi\colon \omega_{X/V}^{1/p} \to \mathscr{O}_{X_{V^1}}$ be the map with $D_\phi = 0$ (see Definition 2.8 for the definition of D_ϕ). Note that by Lemma 2.21, we have $D_{\phi_s} = 0$ as well. Set $M := \omega_{X/V} \otimes N$.

By assumption (c), for every $v \in V$ and $i > 0$, we have $H^i(X_v, N_v) = 0$. Therefore, f_*N is a vector bundle of rank 3, and its formation is compatible with arbitrary base change. In particular, by Proposition 6.20 and assumption (e), the subsheaf $S_\phi^n f_*(\omega_{X/V} \otimes M) \subseteq f_*(\omega_{X/V} \otimes M)$ is generically isomorphic to $f_*(\omega_{X/V} \otimes M)$. Furthermore, since it is a subsheaf of $f_*(\omega_{X/V} \otimes M)$, it is torsion free, and since V is a curve, it is locally free of rank 3. Therefore, we have

$$\dim_k S_\phi^n f_*(\omega_{X/V} \otimes M) \otimes k(c) = 3\,.$$

However,

$$\dim_k S^0(C, \omega_C \otimes M_C) < \dim_k H^0(C, \omega_C \otimes M_C) = 3\,.$$

This shows that the base-change morphism

$$S_{\phi^n}^0 f_*(\omega_{X/V} \otimes M) \otimes k(c) \longrightarrow S^0(C, \omega_C \otimes M_C)$$

cannot be isomorphism.

We are left with giving the polarized curves (C, M_C) and (D, M_D). For that, consider the situation of Lemma 6.21. By [Tan72, Lemma 12], we have $H^0(Y, \mathscr{B}^1(-p^n G)) = 0$ if $p^n G \geqslant n(Y)$, where $n(Y)$ is a numerical invariant of the curve, which is at most 1 for genus 3 curves [Tan72, Lemma 10]. Hence $H^0(Y, \mathscr{B}^1(-p^n G)) = 0$ if $\deg G \geqslant 1$ and $n > 1$. In particular, in our special case (that is, if Y is of genus 3 and $\deg G \geqslant 1$), Lemma 6.21 states that

$$S^0\big(Y, \sigma(Y, 0) \otimes \omega_Y(G)\big) = H^0(Y, \omega_Y(G)) \Leftrightarrow H^0\big(Y, \mathscr{B}_Y^1(-G)\big) = 0.$$

Finding a curve where $H^0(Y, \mathscr{B}_Y^1(-G)) = 0$ is quite easy, because, again, using [Tan72, Lemma 12] yields that if $n(Y) = 1$, then there is a degree 1 divisor G for which $H^0(Y, \mathscr{B}_Y^1(-G)) \neq 0$, and then by passing to a linearly equivalent divisor, we may also assume that G is effective. Further, [Tan72, Example 1] gives a curve ($x^3 y + y^3 z + z^3 x = 0$) for which $n(Y) = 1$. Therefore, we have found C and N_C.

To find D and N_D, assume further that $k = \overline{\mathbb{F}}_3$. Then by [Mil72, Kob75], a general genus 3 curve curve is ordinary or, equivalently, $S^0(Y, \sigma(Y, 0) \otimes \omega_Y) = H^0(Y, \omega_Y)$. Therefore, we have $H^0(Y, \mathscr{B}_Y^1) = 0$ for such a Y. Now, take an arbitrary effective degree 1 divisor G. Then $\mathscr{B}_Y^1(-G)$ embeds into $\mathscr{B}_Y^1$ and hence $H^0(Y, \mathscr{B}_Y^1(-G)) = 0$ as well. In particular, we can choose D to be a generic curve and N_D to be an arbitrary degree 1 line bundle on D.

EXAMPLE 6.23. In the following example, for an f-ample line bundle Q, the surjection

$$S_{\varphi^n}^0 f_*\big(Q^l\big) \otimes_{\mathscr{O}_{V^{ne}}} \big(k(s)^{1/p^{ne}}\big) \twoheadrightarrow S^0\big(X_s, \sigma(X_s, \phi_s) \otimes Q_s^l\big) \qquad (6.10)$$

is not an isomorphism for any integer $n \geqslant n_0$ and $l > 0$. Therefore, the isomorphism in Proposition 6.16 cannot be obtained by stronger positivity assumptions.

Let C be the projective cone over a supersingular elliptic curve, and let D be a non-singular cubic surface. Then these can be put into a family $f \colon X \to V$ as above. More precisely, we may find a family $f \colon X \to V$ and a point $c \in V$, such that we have $C := X_c$, and hence the following hold:

(a) V is a smooth curve;

(b) X is a flat family of normal surfaces;

(c) X_c is not sharply F-pure at one point $P \in X_c$; and

(d) X_s is sharply F-pure for every $s \in V \setminus \{c\}$.

Let $L := \omega_{X/V}^{1-p}$ and $\phi := \phi_0$ as in the previous example. Then by Theorem 3.10 and Nakayama's lemma, $\sigma_n(X/V, \phi)|_{V \setminus \{c\}} \cong \mathscr{O}_{f^{-1}(V \setminus \{c\})}$ for every $n \gg 0$. Now, choose an arbitrary sufficiently f-ample line bundle M. By Corollary 6.14, for every $n \gg 0$, we have $S_{\phi^n}^0 f_*(M^l)|_{V \setminus \{c\}} \cong (f_*(M))_{V^{ne} \setminus \{c\}}$ for every integer $l > 0$. In particular, we have $\operatorname{rk} S_{\phi^n}^0 f_*(M^l) = \operatorname{rk} f_*(M^l)$ for every $n \gg 0$ and $l > 0$ (where n does not depend on l). Consequently,

$$\dim S_{\phi^n}^0 f_*\big(M^l\big) \otimes k(c) \geqslant \operatorname{rk} f_*\big(M^l\big) = \underbrace{\dim_k H^0\big(X_c, M_c^l\big)}_{M \text{ is relatively ample enough}}. \qquad (6.11)$$

On the other hand for all $n \gg 0$ (independently of l),

$$S_{\phi^n}^0 f_*\big(M^l\big) \otimes k(c) \twoheadrightarrow \underbrace{S_{\phi_c^n}^0 (f_c)_*\big(M_c^l\big)}_{\text{by point (b) of Proposition 6.16}} = \underbrace{H^0\big(X_c, \sigma(X_c, 0) \otimes M_c^l\big)}_{\text{since } M_c \text{ is ample enough [Pat14] and Lemma 2.21}}. \qquad (6.12)$$

However, by the relative ampleness assumption, M_c is globally generated; hence

$$\dim_k H^0\big(X_c, \sigma(X_c, 0) \otimes M_c^l\big) < \dim_k H^0\big(X_c, M_c^l\big)$$

(because $\sigma(X_c, 0) \subsetneq \mathscr{O}_{X_c}$). Therefore, (6.11) and (6.12) imply that, indeed, the surjection (6.10) is not an isomorphism for any $l > 0$ and any $n \gg 0$ (bounded independently of l).

Remark 6.24. The fundamental reason for the above example is that $\sigma_n(X/V, \phi)$ is *not* flat in general.

The above two examples show that $\dim_k(S^0(X_v, \sigma(X_v, \phi_v) \otimes M_v)$ is not upper semicontinuous. One might then guess that it is lower semicontinuous. The next example shows that this is also not the case (this can also be deduced from [Har98, Example 5.5] and [Tan15, Theorem 8.3] as pointed out to us by Tanaka). So, $\dim_k S^0(X_v, \sigma(X_v, \phi_v) \otimes M_v)$ is not semicontinuous in either direction.

EXAMPLE 6.25. Take a flat family $f \colon X \to V$ of ordinary elliptic curves and a line bundle M on X of relative degree 0, such that $M_{X_{v_0}} \cong \mathscr{O}_{X_{v_0}}$ for a special $v_0 \in V$ and $M_{X_v} \not\cong \mathscr{O}_{X_v}$ for generic $v \in V$. Then, by the ordinarity of the fibers, for all $v \in V$,

$$H^0(X_v, M_v) = S^0\big(X_v, \sigma(X_v, 0) \otimes M_v\big).$$

However, the dimension $\dim H^0(X_v, M_v)$ is 1 for the special fiber and 0 for the generic one. So,

$$\dim S^0(X_v, \sigma(X_v, 0) \otimes M_v)$$

is not lower semicontinuous in this example. One can also easily modify this example by taking an M with higher relative degree to obtain higher values of dimension for the generic fiber.

6.5 Global generation and semi-positivity

Now, suppose that V is a projective variety over a perfect field k. In this section, we explore global generation results if $L \otimes M^{p^e-1}$ is ample (instead of just relatively ample). In particular, $S^0_{\phi^n} f_*(M)$ is globally generated for all large n. This should not be surprising since $S^0_{\phi^n} f_*(M)$ lives on V^{ne}, where ampleness is amplified.

PROPOSITION 6.26. *In the situation of Notation* 6.1, *if V is projective over a perfect field k and $L \otimes M^{p^e-1}$ is ample, then $S^0_{\phi^n} f_*(M)$ is globally generated for every $n \gg 0$.*

Proof. Choose a globally generated ample divisor H on V, and let $d := \dim V$. Since $S^0_{\phi^n} f_*(M)$ is defined as the image of

$$f_*\left(\left(L^{\frac{p^{ne}-1}{p^e-1}}\right)^{1/p^{ne}} \otimes_R M\right),$$

it is enough to show that this sheaf is globally generated as an $\mathscr{O}_{V^{ne}}$-module. By Mumford's criterion [Laz04a, Theorem 1.8.5], it is enough to show that for every $i > 0$ and for the divisor H_{ne} on V^{ne} identified with H via the isomorphism $V^{ne} \cong V$,

$$H^i\left(V^{ne}, \mathscr{O}_{V^{ne}}(-iH_{ne}) \otimes_{\mathscr{O}_{V^{ne}}} f_*\left(\left(L^{\frac{p^{ne}-1}{p^e-1}}\right)^{1/p^{ne}} \otimes_R M\right)\right) = 0. \tag{6.13}$$

By the relative Serre vanishing theorem for $i > 0$,

$$R^i f_*\left(\left(L^{\frac{p^{ne}-1}{p^e-1}}\right)^{1/p^{ne}} \otimes_R M\right) = 0.$$

In particular, then to prove (6.13), it is enough to show

$$H^i\left(X, L^{\frac{p^{ne}-1}{p^e-1}} \otimes M^{p^{ne}}(-if^*H)\right) = 0.$$

However, this is equivalent to showing

$$H^i\left(X, \left(L \otimes M^{(p^e-1)}\right)^{\frac{p^{ne}-1}{p^e-1}} \otimes M(-if^*H)\right) = 0,$$

which holds by the Serre vanishing theorem for $n \gg 0$ and $i > 0$. $\qquad \square$

We recall the following definition.

DEFINITION 6.27 ([Vie95, Definition 2.11]). Let $\mathscr{F}$ be a sheaf on a normal, quasi-projective (over a perfect field k) variety V and $U \subseteq V$ a dense open set. Let U_{lf} be the open locus of V where $\mathscr{F}' := \mathscr{F}/(\text{torsion})$ is locally free. Then $\mathscr{F}$ is *weakly positive over U* if for a fixed (or, equivalently, every [Vie95, Lemma 2.14(a)]) ample line bundle $\mathscr{H}$ for every $a > 0$, there is a $b > 0$ such that $S^{\langle ab \rangle}(\mathscr{F}) \otimes \mathscr{H}^b$ is globally generated over $U \cap U_{\mathrm{lf}}$ (here $S^{\langle ab \rangle}(\mathscr{F})$ denotes the abth symmetric reflexive power of $\mathscr{F}$). We say that $\mathscr{F}$ is *weakly positive* if it is weakly positive over some dense open set.

LEMMA 6.28. *If $g \colon Y \to Z$ is a finite morphism of normal varieties, quasi-projective over k, $U \subseteq Z$ a dense open set, and $\mathscr{F}$ a sheaf on Z, then $\mathscr{F}$ is weakly positive over U if and only if $g^*\mathscr{F}$ is weakly positive over $g^{-1}(U)$.*

Proof. The "only if" direction is shown in [Vie95, Lemma 2.15.1]. For the other direction, according to Definition 6.27, by throwing out a codimension 2 subset, we may assume that $\mathscr{F}$ is locally free and Y is flat over Z. In particular, $g^*\mathscr{F}$ is then also locally free. Choose a very ample divisor $\mathscr{H}$ on Z such that

(a) $\mathscr{H}om(g_*\mathscr{O}_Y, \mathscr{H})$ is globally generated and

(b) $g_*\mathscr{O}_Y \otimes \mathscr{H}$ is globally generated.

Then, fix an $a > 0$. By the weak positivity of $g^*\mathscr{F}$, there is a $b > 0$ such that there is a homomorphism $\alpha \colon \mathscr{O}_Y^{\oplus N} \to S^{\langle ab \rangle}(g^*\mathscr{F}) \otimes g^*\mathscr{H}^b$ surjective over g^*U. Then, consider for every choice of $s \in \mathrm{Hom}(g_*\mathscr{O}_Y, \mathscr{H})$ and $s' \in H^0(\mathscr{H}^{b-1})$ the following composition:

$$(g_*\mathscr{O}_Y)^{\oplus N} \xrightarrow{\gamma := g_*\alpha} g_*(S^{\langle ab \rangle}(g^*\mathscr{F}) \otimes g^*\mathscr{H}^b) \cong S^{\langle ab \rangle}(\mathscr{F}) \otimes \mathscr{H}^b \otimes g_*\mathscr{O}_Y$$

$$\xdownarrow{\beta_s} \qquad \downarrow{\delta_{s,s'} := \mathrm{id}_{S^{\langle ab \rangle}(\mathscr{F}) \otimes \mathscr{H}^b} \otimes s \otimes s'}$$

$$S^{\langle ab \rangle}(\mathscr{F}) \otimes \mathscr{H}^{2b}.$$

Choose a point $P \in U$ and an element f of the fiber $S^{\langle ab \rangle}(\mathscr{F}) \otimes k(P)$ over P. Then, by assumption (b), for any preimage Q of P, there is a section $t' \in \mathscr{O}_Y^{\oplus N}$ such that $\alpha(t')_Q = g^*f \times$ "generator". The section t' descends to a section $t := g_*(t') \in (g_*\mathscr{O}_Y)^{\oplus N}$ such that $\gamma(t)_P = f \times$ "generator" $\times h$ for some $h \in (g_*\mathscr{O}_Y)_P$. However, then by assumption (a) and the very ampleness of $\mathscr{H}$ for a suitable choice of s and s', we have that $\delta_{s,s'}(\gamma(t))_P = f \times$ "generator" is not zero at P. Therefore, for every point P in U and every element f in the fiber of $S^{\langle ab \rangle}(\mathscr{F}) \otimes \mathscr{H}^{2b}$ at that point, we find a section of $S^{\langle ab \rangle}(\mathscr{F}) \otimes \mathscr{H}^{2b}$ whose image in the fiber is f up to multiplication by unit. This finishes our proof. $\qquad \square$

COROLLARY 6.29. *In the situation of Notation 6.1, let V be projective over a prefect field k, $L \otimes M^{p^e-1}$ ample, and let $n > 0$ be an integer such that there is an open set $U \subseteq V$ for which*

$$S^0_{\phi^n} f_*(M) \times_{U^{ne}} U^{me} = S^0_{\phi^m} f_*(M)|_{U^{me}}$$

for every integer $m \geqslant n$. Then $S^0_{\phi^n} f_*(M)$ is weakly positive. Note that an integer as above always exists by Corollary 6.12 and Proposition 5.2.

Proof. By the assumption, there is an embedding

$$S^0_{\phi^m} f_*(M) \hookrightarrow S^0_{\phi^n} f_*(M) \times_{V^{ne}} V^{me}$$

which is generically an isomorphism. Since the left sheaf is globally generated and hence weakly positive, so is the right one. However, then by Lemma 6.28, so is $S^0_{\phi^n} f_*(M)$. $\square$

LEMMA 6.30. *If $\mathscr{F}$ is a coherent sheaf on a normal variety V over a perfect field k, then $\mathscr{F}$ is weakly positive if and only if for every ample line bundle $\mathscr{H}$ and every integer $p \nmid r > 0$, there is a finite morphism $\tau \colon T \to V$ such that $\tau^* \mathscr{H} \cong (\mathscr{H}')^r$ for some line bundle $\mathscr{H}'$ and $\tau^* \mathscr{F} \otimes \mathscr{H}'$ is weakly positive.*

Proof. The proof is identical to the "$(b) \Rightarrow (a)$" part of [Vie95, Lemma 2.15.1]. $\square$

THEOREM 6.31. *In the situation of Notation 6.1, if V is projective and $L \otimes M^{p^e-1}$ is nef and f-ample, then $S^0_{\phi^n} f_*(M)$ is weakly positive for $n \gg 0$.*

Proof. Choose an integer $n > 0$ as in Corollary 6.29. Fix an ample line bundle $\mathscr{H}$ on V, an integer $p \nmid r > 0$, and a finite morphism $\tau \colon T \to V$, such that $\tau^* \mathscr{H} \cong (\mathscr{H}')^r$ for some line bundle $\mathscr{H}'$. Such a morphism exits by [Vie95, Lemma 2.1]. By Lemma 6.30, we are supposed to prove that $\left(\tau^{1/p^{ne}}\right)^* S^n_\phi f_*(M) \otimes_{\mathscr{O}_{V^{ne}}} (\mathscr{H}')^{1/p^{ne}}$ is weakly positive. By disregarding codimension 2 closed sets, we may assume that T is regular as well. Then by point (c) of Proposition 6.16, we see that n also satisfies the assumptions of Corollary 6.29, but for f and M replaced by f_T and $M_T \otimes f_T^* \mathscr{H}'_T$, respectively. In particular, since $M_T \otimes f_T^* \mathscr{H}'_T$ is ample,

$$S^0_{\phi_T^n}(f_T)_*(M_T \otimes f_T^* \mathscr{H}'_T) \cong \mathscr{H}'_T \otimes S^0_{\phi_T^n}(f_T)_*(M_T)$$

is weakly positive over T^{ne}. $\square$

6.6 Relation to global canonical systems

Another subsheaf $S^0 f_*(\sigma(X, \Delta) \otimes M)$ of $f_*(M)$ has been introduced in [HX15, Definition 2.14] with a definition similar to that of $S^0_{\phi^n} f_*(M)$. In this section, we show some of the similarities and differences between the two sheaves. The advantage of $S^0 f_*(\sigma(X, \Delta) \otimes M)$ over $S^0_{\phi^n} f_*(M)$ is that it lives on one V, there is no involvement of V^{ne} at all. On the other hand, we show that contrary to $S^0_{\phi^n} f_*(M)$, it does not restrict even generically to $S^0(X_s, \sigma(X_s, \Delta_s) \otimes M_s)$.

Throughout the section, we use the divisorial language since the presentation seems to be more straightforward this way; hence Δ satisfies the conditions of Remark 2.11. Recall that we defined the notion of pair in our setting in Definition 2.12. Whenever we say pair, we mean everything assumed there. In particular, this includes all the assumptions on $f \colon X \to V$ from Notation 2.1: flatness, equidimensionality, etc.

First, we recall the definition of $S^0 f_*(M)$.

DEFINITION 6.32 ([HX15, Definition 2.14]). Let Δ be an effective $\mathbb{Q}$-divisor such that (X, Δ) is a pair and assume that $f \colon X \to V$ is projective. Let e be the least[9] positive integer such that $(p^e - 1)(K_X + \Delta)$ is Cartier. Then given a line bundle M on X, we define

$$S^0 f_*(\sigma(X, \Delta) \otimes M) := \bigcap_{n > 0} \operatorname{im}\left(f_*(F^{ne}_* \mathscr{O}_X((1 - p^{ne})(K_X + \Delta)) \otimes M) \to f_*(M)\right).$$

---

[9]The intersection in the definition is easily seen to be descending; hence if one chooses another e, the same object results.

We say that $S^0 f_*(\sigma(X, \Delta) \otimes M)$ *stabilizes* if the above intersection stabilizes. Note that in that case, $S^0 f_*(\sigma(X, \Delta) \otimes M)$ is a coherent sheaf.

Recall that $S^0 f_*(\sigma(X, \Delta) \otimes M)$ stabilizes if $M - K_X - \Delta$ is relatively ample; see [HX15, Proposition 2.15] for a proof.

PROPOSITION 6.33. *Let Δ be an effective $\mathbb{Q}$-divisor such that (X, Δ) is a pair and assume that $f : X \to V$ is projective. Let M be a line bundle on X such that $S^0 f_*(\sigma(X, \Delta) \otimes M)$ stabilizes. Further assume that V is regular. Then, for every $n \gg 0$,*

$$S^0_{\Delta, ne} f_*(M) \subseteq \left(S^0 f_*(\sigma(X, \Delta) \otimes M) \right)_{V^{ne}}$$

as subsheaves of $(f_*(M))_{V^{ne}}$, where $S^0_{\Delta, ne} f_*(M)$ is defined in Definition 6.4.

Proof. The statement is local over the base; hence, we can assume that V is affine, $K_V \sim 0$, and that $\mathscr{H}om_{\mathscr{O}_V}(F^i_* \mathscr{O}_V, \mathscr{O}_V) \cong F^i_* \mathscr{O}_V$ is a free $\mathscr{O}_V$-module for all $i \geqslant 0$. Note, then, that we can also identify $K_{X/V}$ with K_X. We introduce the notation

$$L := \mathscr{O}_X \left((1 - p^{ne})(K_X + \Delta) \right).$$

We have the evaluation-at-1 map $F^{ne}_* \mathscr{O}_V \cong \mathscr{H}om_{\mathscr{O}_V}(F^{ne}_* \mathscr{O}_V, \mathscr{O}_V) \to \mathscr{O}_V$, which we identify with the trace of V based on our previous assumption $\omega_V \cong \mathscr{O}_V$. This map pulls back to X to provide us with a map which we also denote by $\operatorname{Tr}^{ne}_{F_V} : \mathscr{O}_{X_{V^{ne}}} \cong \mathscr{O}_X \otimes f^* F^{ne}_* \mathscr{O}_V \to \mathscr{O}_X \otimes_{\mathscr{O}_X} f^* \mathscr{O}_V \cong \mathscr{O}_X$. Then there is a diagram as follows, which is commutative up to multiplication by a unit; cf. [Sch09, Lemma 3.9]:

$$
\begin{array}{ccccc}
F^{ne}_* L & \hookrightarrow & F^{ne}_* \mathscr{O}_X((1 - p^{ne})K_X) & \xrightarrow{\operatorname{Tr}_{F_X}} & \mathscr{O}_X \\
\| & & \| & & \uparrow{\scriptstyle \operatorname{Tr}^{ne}_{F_V}} \\
F^{ne}_* \mathscr{O}_X((1 - p^{ne})(K_{X/V} + \Delta)) & \hookrightarrow & F^{ne}_* \mathscr{O}_X((1 - p^{ne})K_{X/V}) & \xrightarrow{\operatorname{Tr}_{F^{ne}_{X/V}}} & \mathscr{O}_X \otimes_{\mathscr{O}_X} f^* F^{ne}_* \mathscr{O}_V .
\end{array}
$$

Applying the functor $f_*(\_ \otimes_{\mathscr{O}_X} M)$ to this diagram, we obtain the following:

$$
\begin{array}{ccc}
f_*((F^{ne}_* L) \otimes_{\mathscr{O}_X} M) & \longrightarrow & f_*(M) \\
\| & & \uparrow{\scriptstyle \kappa} \\
f_*((F^{ne}_* \mathscr{O}_X((1 - p^{ne})(K_{X/V} + \Delta))) \otimes_{\mathscr{O}_X} M) & \longrightarrow & f_*(M \otimes_{\mathscr{O}_X} f^* F^{ne}_* \mathscr{O}_V) .
\end{array}
$$

Note that for every $n \gg 0$, the image of the top horizontal row is $S^0 f_*(\sigma(X, \Delta) \otimes M)$ and for every $n > 0$, the image of the bottom horizontal row is $S^0_{\Delta, n} f_*(M)$. Hence for every $n \gg 0$,

$$S^0 f_*(\sigma(X, \Delta) \otimes M) = \operatorname{im} \left(S^0_{\Delta, n} f_*(M) \to f_*(M) \right),$$

where the map is induced by Tr_{F_V}.

Since we want containment for subsheaves of $(f_*(M))_{V^{ne}}$, we localize at a point of V. By setting $B := \Gamma(V, \mathscr{O}_V)$, $P := f_*(M)$, and $Q := F^{ne}_{V,*}(S^0_{\Delta, n} f_*(M))$, we are in the following situation: if Q is a $B^{1/p^{ne}}$-submodule of $B^{1/p^{ne}} \otimes_B P$ for some B-module P, we would like to show that $Q \subseteq \theta(Q) \otimes_B B^{1/p^{ne}}$, where $\theta \in \operatorname{Hom}_B(B^{1/p^{ne}}, B)$ is the generator. This is simply Lemma 3.20. $\square$

EXAMPLE 6.34. We provide an example where the two sheaves of Proposition 6.33 have different ranks for every $n \gg 0$. Fix an algebraically closed field k of characteristic $p > 0$. Let $X := \mathbb{P}^1 \times \mathbb{A}^1$, $U = \mathbb{A}^1 \times \mathbb{A}^1 = \operatorname{Spec} k[y, x] \subseteq X$, and $V := \mathbb{A}^1 := \operatorname{Spec} k[x]$. Define the divisor $\Delta := (1/(p-1)) V(y^p - x)$, which is a priori a divisor in U, but it happens to have a support

which is a closed subvariety of X as well. Note that $V(y^p - x)$ is a subvariety of X isomorphic to $\mathbb{A}^1$. In particular, (X, Δ) is sharply F-pure. Now, choose a line bundle N on X which is relatively ample over V, and let $M := N^l$ for some $l \gg 0$. Then by [HX15, Lemma 2.19], we have $S^0 f_*(\sigma(X, \Delta) \otimes M) = f_*(M)$. Therefore,

$$\left(S^0 f_*(\sigma(X, \Delta) \otimes M)\right)_{V^{ne}} = (f_*(M))_{V^{ne}}. \tag{6.14}$$

On the other hand, by Proposition 6.5, we have $S^0_{\Delta,n} f_*(M) \subseteq f_*(\sigma_n(X/V, \Delta) \otimes M)$. By Theorem 3.10 and Lemma 2.21, for every $n \gg 0$ and $s \in V$, we have $\sigma_n(X/V, \Delta)|_{X_s} = \sigma(X_s, \Delta)$. However, $\Delta|_{X_s}$ is one point with multiplicity $p/(p-1)$ and hence not sharply F-pure. In particular, $\sigma(X_s, \Delta) \neq \mathscr{O}_{X_s}$ for every $s \in S$. Hence, for every $n \gg 0$, we have $\sigma_n(X/V, \Delta)|_{X_s} \subsetneq \mathscr{O}_{X_s}$. It follows by the relative ampleness of M that $f_*(\sigma_n(X/V, \Delta) \otimes M)$ has smaller rank than $f_*(M)$. Then $S^0_{\Delta,n} f_*(M)$ has also smaller rank than $(S^0 f_*(\sigma(X, \Delta) \otimes M))_{V^{ne}}$ by (6.14).

Example 6.34 has an important corollary, which follows immediately from point (d) of Proposition 6.16 and Theorem 6.18.

COROLLARY 6.35. *In general, $S^0 f_*(\sigma(X, \Delta) \otimes M) \otimes k(s)$ is not isomorphic to $S^0(X_s, \sigma(X_s, \Delta_s) \otimes M_s)$. In fact, there are examples when the former has strictly bigger dimension than the latter for every closed point $s \in V$.*

Compare the following proposition with Proposition 3.23.

PROPOSITION 6.36. *Let Δ be an effective $\mathbb{Q}$-divisor such that (X, Δ) is a pair, and assume that $f : X \to V$ is projective. Let M be a line bundle on X, and assume that V is regular. Then for every $n \geqslant m \geqslant 0$,*

$$S^0 f_{V^{ne},*}\left(\sigma(X_{V^{ne}}, \Delta_{V^{ne}}) \otimes M_{V^{ne}}\right) \subseteq \left(S^0 f_{V^{me},*}(\sigma(X_{V^{me}}, \Delta_{V^{me}}) \otimes M_{V^{me}})\right)_{V^{ne}}$$

as subsheaves of $(f_*(M))_{V^{ne}}$. Furthermore, if these two sheaves stabilize, then

$$\kappa\left(F^{(n-m)e}_{V,*} S^0 f_{V^{ne},*}\left(\sigma(X_{V^{ne}}, \Delta_{V^{ne}}) \otimes f^* \mathscr{O}_V\left((1 - p^{(n-m)e}) K_{V^{ne}}\right) \otimes M_{V^{ne}}\right)\right)$$
$$= S^0 f_{V^{me},*}\left(\sigma(X_{V^{me}}, \Delta_{V^{me}}) \otimes M_{V^{me}}\right),$$

where κ is induced by $\mathrm{Tr}^{(n-m)e}_V$.

Proof. Note that by replacing $f : X \to V$ by $f_{V^{me}} : X_{V^{me}} \to V^{me}$, we may assume that $m = 0$, and then by replacing e by $(n-m)e$ that $n = 1$. As in the previous proof, let us assume that V is affine, $K_V \sim 0$, and that $\mathscr{H}om_{\mathscr{O}_V}(F^e_* \mathscr{O}_V, \mathscr{O}_V) \cong F^e_* \mathscr{O}_V$ is a free $\mathscr{O}_V$-module. Consider the following commutative diagram:

$$
\begin{array}{ccc}
(X_{V^e})^e \xrightarrow{\ \nu := (F^e_V)_{X^e}\ } & X^e & \\
& & \searrow^{F^e_X} \\
\searrow^{F^e_{X_{V^e}}} & X_{V^e} \xrightarrow{\ \eta := (F^e_V)_X\ } & X \\
& \ \ \downarrow^{f_{V^e}} & \ \ \downarrow^{f} \\
& V^e \xrightarrow{\ F^e_V\ } & V.
\end{array}
$$

We fix the notation

$$\mathscr{L}'_\Delta := \mathscr{O}_{X_{V^e}}\left((1 - p^e)(K_{X_{V^e}} + \eta^* \Delta)\right), \quad \mathscr{L}_\Delta := \mathscr{O}_X\left((1 - p^e)(K_X + \Delta)\right).$$

Then we have

$$(1 - p^e) K_{X_{V^e}} = (1 - p^e)(\eta^* K_{X/V} + f^*_{V^e} K_{V^e}), \quad (1 - p^e) K_X = (1 - p^e)(K_{X/V} + f^* K_V).$$

Therefore,

- ○ applying pullback of $\mathrm{Tr}_{F_V^e}$ yields a homomorphism $\nu_* \mathscr{L}'_\Delta \to \mathscr{L}_\Delta$, which then induces a homomorphism $\eta_* F^e_{X_{V^e},*} \mathscr{L}'_\Delta \cong F^e_{X,*} \nu_* \mathscr{L}'_\Delta \to F^e_{X,*} \mathscr{L}_\Delta$;
- ○ applying $\mathrm{Tr}_{F^e_{X_{V^e}}}$ yields a homomorphism $F^e_{X_{V^e},*} \mathscr{L}'_\Delta \to \mathscr{O}_{X_{V^e}}$;
- ○ applying $\mathrm{Tr}_{F^e_X}$ yields a homomorphism $F^e_{X,*} \mathscr{L}_\Delta \to \mathscr{O}_X$;
- ○ by the assumption $K_V \sim 0$, the trace map $\mathrm{Tr}_{F^e_V}$ corresponds to a homomorphism $F^e_{V,*} \mathscr{O}_V \to \mathscr{O}_V$ generating $\mathrm{Hom}_{\mathscr{O}_V}(F^e_{V,*} \mathscr{O}_V, \mathscr{O}_V)$; and
- ○ the previous homomorphism also induces a pullback homomorphism $\eta_* \mathscr{O}_{X_{V^e}} \to \mathscr{O}_X$.

Furthermore, the above homomorphisms fit into the following diagram:

$$
\begin{array}{ccc}
\eta_* F^e_{X_{V^e},*} \mathscr{L}'_\Delta & \longrightarrow & F^e_{X,*} \mathscr{L}_\Delta \\
& \searrow & \downarrow \\
& \eta_* \mathscr{O}_{X_{V^e}} \longrightarrow & \mathscr{O}_X .
\end{array}
$$

The diagram is a composition of trace maps (restricted to smaller domains determined by Δ); hence it is commutative. Now, applying $f_*(\_ \otimes M)$ to the above diagram and using the projection formula, we obtain the commutative diagram

$$
\begin{array}{ccc}
f_* \eta_* \big(F^e_{X_{V^e},*} \mathscr{L}'_\Delta \otimes M_{V^e} \big) & \longrightarrow & f_* \big(F^e_{X,*} \mathscr{L}_\Delta \otimes M \big) \\
& \searrow & \downarrow \\
& f_* \eta_* M_{V^e} \longrightarrow & f_*(M) .
\end{array}
$$

Observing that $f_* \eta_* = F^e_{V,*} f_{V^e,*}$ yields another commutative diagram:

$$
\begin{array}{ccc}
F^e_{V,*} f_{V^e,*} \big(F^e_{X_{V^e},*} \mathscr{L}'_\Delta \otimes M_{V^e} \big) & \longrightarrow & f_* \big(F^e_{X,*} \mathscr{L}_\Delta \otimes M \big) \\
& \searrow & \downarrow \\
& F^e_{V,*} f_{V^e,*} M_{V^e} \longrightarrow & f_*(M) .
\end{array}
$$

Now, note that the top horizontal arrow is split, because it is induced from $\nu_* \mathscr{L}'_\Delta \to \mathscr{L}_\Delta$, which is split as well. Therefore, if we define $P := f_*(M)$ and

$$
S := \mathrm{im}\big(f_*(F^e_{X,*} \mathscr{L}_\Delta \otimes M) \to f_*(M) \big), \quad Q := \mathrm{im}\big(f_{V^e,*}(F^e_{X_{V^e},*} \mathscr{L}'_\Delta \otimes M_{V^e}) \to f_{V^e,*} M_{V^e} \big),
$$

then we have $Q \subseteq P \otimes_A A^{1/p^e}$, $S \subseteq P$, and $(\mathrm{id}_P \otimes \mathrm{Tr}_{F^e_V})(Q) = S$. Therefore, by Lemma 3.20, we have $Q \subseteq S \otimes_A A^{1/p^e}$. Since this holds for any n, the statement of the proposition follows. $\square$

PROPOSITION 6.37. *Let Δ be an effective $\mathbb{Q}$-divisor such that (X, Δ) is a pair, and assume that $f \colon X \to V$ is projective and V is regular. Let M be a line bundle on X. Then for every $n \gg 0$,*

$$
S^0 f_{V^{ne},*}\big(\sigma(X_{V^{ne}}, \Delta_{V^{ne}}) \otimes M_{V^{ne}} \big) \subseteq S^0_{\Delta,ne} f_*(M)
$$

as subsheaves of $(f_*(M))_{V^{ne}}$.

Proof. As before, let us assume that V is affine, $K_V \sim 0$, and that $\mathscr{H}om_{\mathscr{O}_V}(F^i_* \mathscr{O}_V, \mathscr{O}_V) \cong F^i_* \mathscr{O}_V$ is a free $\mathscr{O}_V$-module for all $i \geqslant 0$. By Lemmas 3.12 and 3.15, there is a commutative diagram

$$
\begin{array}{ccc}
F^{ne}_{X_{V^{ne}},*} \mathscr{O}_{X_{V^{ne}}} \big((1 - p^{ne})(K_{X_{V^{ne}}} + \Delta_{V^{ne}}) \big) & \longrightarrow & F^{ne}_{X^{ne}/V^{ne},*} \mathscr{O}_X \big((1 - p^{ne})(K_{X/V} + \Delta) \big) \\
& \searrow & \downarrow \\
& & \mathscr{O}_{X_{V^{ne}}} .
\end{array}
$$

Furthermore, the horizontal arrow in this diagram is split surjective. Therefore, after applying $f_{V^{ne},*}(\_\_ \otimes M_{V^{ne}})$, the image of the vertical map still agrees with the image of the diagonal map. The former is exactly $S^0_{\Delta,n} f_*(M)$, while the latter contains $S^0 f_{V^{ne},*}(\sigma(X,(\Delta)_{V^{ne}}) \otimes M)$, since it is one of the terms in the intersection defining $S^0 f_{V^{ne},*}(\sigma(X,(\Delta)_{V^{ne}}) \otimes M)$. $\qquad\square$

Remark 6.38. Assuming that $f\colon X \longrightarrow V$ is projective and $M - K_X - \Delta$ is ample, it would be natural to ask whether $\operatorname{im}\big(S^0 f_{V^{ne},*}(\sigma(X_{V^{ne},\Delta_{V^{ne}}} \otimes M_{V^{ne}}) \otimes M_{V^{ne}}) \longrightarrow f_{V^{ne},*} M_s\big)$ equals $S^0(X_s, \sigma(X_s,\Delta_s) \otimes M_s)$ for all perfect points $s \in V$, in analogy with Corollary 3.22. We suspect that this is true but will not try to prove it here.

COROLLARY 6.39. *Let* $f\colon (X,\Delta) \longrightarrow V$ *be a projective morphism from a pair with* V *regular and projective over a perfect field* k. *Further, suppose that* M *is a line bundle on* X *such that* $M - K_{X/V} - \Delta$ *is nef and* f-*ample (here* M *denotes a Cartier divisor corresponding to* M*) and that* $\operatorname{rk} S^0 f_*(\sigma(X,\Delta) \otimes M)$ *equals the general value of* $H^0(X_s, \sigma(X_s,\Delta_s) \otimes M_s)$. *Then the sheaf* $S^0 f_*(\sigma(X,\Delta) \otimes M)$ *is weakly positive. In particular, if* V *is a smooth curve, then it is a nef vector bundle.*

Proof. From the assumption that the rank $\operatorname{rk} S^0 f_*(\sigma(X,\Delta) \otimes M)$ equals the general value of $H^0(X_s, \sigma(X_s,\Delta_s) \otimes M_s)$ follows that the inclusion of Proposition 6.33 is generically an isomorphism for every $n \gg 0$. However, since $S^0_{\phi^n} f_*(M)$ is weakly positive for every $n \gg 0$ by Theorem 6.31, we obtain by the above generically isomorphic inclusion that $(S^0 f_*(\sigma(X,\Delta) \otimes M))_{V^{ne}}$ is also weakly positive. Then the weak positivity of $S^0 f_*(\sigma(X,\Delta) \otimes M)$ follows from Lemma 6.28. $\qquad\square$

Appendix. Relative Serre's condition

Here we collect the statements of [HK04] and other sources that are important for the current paper for ease of reference. In some cases, we also state them in the greater generality that we need. All schemes are Noetherian and excellent and possess dualizing complexes, and all maps are separable.

DEFINITION A.1. Let $r > 0$ be an integer. A coherent sheaf $\mathscr{E}$ on a Noetherian scheme X is S_r if for every $x \in X$,
$$\operatorname{depth}_{\mathscr{O}_{X,x}} \mathscr{E}_x \geqslant \min\big\{r, \dim_{\mathscr{O}_{X,x}} \mathscr{E}_x\big\}.$$
The sheaf $\mathscr{E}$ is said to have *full support* if $\operatorname{Supp}\mathscr{E} = X$. It is *reflexive* if the natural map $\mathscr{E} \longrightarrow \mathscr{E}^{**} := \mathscr{H}om_{\mathscr{O}_X}(\mathscr{H}om_{\mathscr{O}_X}(\mathscr{E}, \mathscr{O}_X), \mathscr{O}_X)$ is an isomorphism.

DEFINITION A.2. If $f\colon X \longrightarrow V$ is a morphism of Noetherian schemes and $\mathscr{E}$ is a coherent sheaf on X, flat over V, then $\mathscr{E}$ is S_r over V if $\mathscr{E}|_{X_v}$ is S_r for every $v \in V$. That is, for every $x \in X$,
$$\operatorname{depth}_{\mathscr{O}_{X_{f(x)}}} \big(\mathscr{E}|_{X_{f(x)}}\big)_x \geqslant \min\big\{r, \dim_{\mathscr{O}_{X_{f(x)}}} \big(\mathscr{E}|_{X_{f(x)}}\big)_x\big\}.$$

We now recall two results from [Gro67] about how depth and dimension behave in families.

PROPOSITION A.3 ([Gro67, Proposition 6.3.1]). *Let* $\phi\colon A \longrightarrow B$ *be a local homomorphism of Noetherian local rings,* k *the residue field of* A, *and* M *and* N *finite* A- *and* B-*modules, respectively. If* N *is a flat non-zero* A-*module, then*
$$\operatorname{depth}_B(M \otimes_A N) = \operatorname{depth}_A(M) + \operatorname{depth}_{B \otimes_A k}(N \otimes_A k).$$

PROPOSITION A.4 ([Gro67, Corollaire 6.1.2]). *Let* $\phi\colon A \to B$ *be a local homomorphism of Noetherian local rings, k the residue field of A, and M and N non-zero finite A- and B-modules, respectively. If N is a flat A-module, then*

$$\dim_B(M \otimes_A N) = \dim_A(M) + \dim_{B \otimes_A k}(N \otimes_A k).$$

From these we obtain the following corollary.

COROLLARY A.5. *Let $f\colon X \to V$ be a morphism of Noetherian schemes and $\mathscr{E}$ a coherent sheaf on X flat over V; then $\mathscr{E}$ is S_r over V if and only if for every $x \in X$,*

$$\min\left\{i \mid H_x^i(\mathscr{E}) \neq 0\right\} \geqslant \operatorname{depth}_{\mathscr{O}_{V,v}} \mathscr{O}_{V,v} + \min\left\{r, \dim \mathscr{E}_x - \dim \mathscr{O}_{V,v}\right\}$$
$$\left(= \operatorname{depth}_{\mathscr{O}_{V,v}} \mathscr{O}_{V,v} + \min\left\{r, \dim \left(\mathscr{E}|_{X_v}\right)_x\right\}\right),$$

where $v := f(x)$.

Proof. To say that $\mathscr{E}|_{X_v}$ has depth equal to t at x is equivalent to asserting that $\operatorname{depth}_x \mathscr{E} = (\operatorname{depth}_{\mathscr{O}_{V,v}} \mathscr{O}_{V,v}) + t$, by Proposition A.3. The result follows. $\square$

The following vanishing allows us to extend sections over sets of relative codimension 2.

PROPOSITION A.6 ([HK04, Proposition 3.3]). *Let $f\colon X \to V$ be a morphism of Noetherian schemes and $\mathscr{E}$ a coherent sheaf on X, flat and S_r over V. Let $Z \subseteq X$ be a closed subscheme of X such that we have $\operatorname{codim}_{\operatorname{Supp} \mathscr{E}_v}(\operatorname{Supp} \mathscr{E}_v \cap Z_v) \geqslant r$ for every $v \in V$. Then we have $\mathscr{H}_Z^i(\mathscr{E}) = 0$ for each $0 \leqslant i < r$.*

Proof. We follow the proof in [HK04, Proposition 3.3]. First, we certainly have a spectral sequence $E_2^{p,q} := H_x^p \circ \mathscr{H}_Z^q \Rightarrow H_x^{p+q}$. We then induct on r, the base case of $r = 0$ being trivial. Now, assume $\mathscr{H}_Z^0(\mathscr{E}) = \ldots = \mathscr{H}_Z^{r-2}(\mathscr{E}) = 0$ but $\mathscr{H}_Z^{r-1}(\mathscr{E}) \neq 0$. Then, for any $x \in Z$ which is a generic point[10] of the support of $\mathscr{H}_Z^{r-1}(\mathscr{E})$, we have $H_x^0(\mathscr{H}_Z^{r-1}(\mathscr{E})) \neq 0$ (note here that x is probably not a closed point). Additionally, since x is a point of $Z \cap \operatorname{Supp}(\mathscr{E})$, we have $\dim(\mathscr{E}|_{X_v})_x \geqslant r$.

On the other hand, for all $i > 0$, we have $H_x^i(\mathscr{H}_Z^{r-1-i}(\mathscr{E})) = 0$, and so by the spectral sequence, $H_x^{r-1}(\mathscr{E}) \cong H_x^0(\mathscr{H}_Z^{r-1}(\mathscr{E})) \neq 0$. This contradicts Corollary A.5. $\square$

We now obtain a relative version of Hartog's phenomena, just as in [HK04].

PROPOSITION A.7 ([HK04, Proposition 3.5, 3.6.1]). *Let $f\colon X \to V$ be a morphism of Noetherian schemes and $\mathscr{E}$ a coherent sheaf on X which satisfies one of the following two conditions:*

(a) *either $\mathscr{E}$ is reflexive and f is flat and relatively S_2,*

(b) *or $\mathscr{E}$ is of full support, flat and S_2 over V.*

Let $j\colon U \hookrightarrow X$ be an open set such that for $Z := X \setminus U$, we have $\operatorname{codim}_{X_v} Z_v \geqslant 2$ for every $v \in V$. Then the following natural map is an isomorphism:

$$\mathscr{E} \to j_* \mathscr{E}|_U.$$

Proof. First, assume that $\mathscr{E}$ is reflexive. Consider a presentation of $\mathscr{E}^*$ by locally free sheaves

$$\mathscr{E}_2 \longrightarrow \mathscr{E}_1 \longrightarrow \mathscr{E}^* \longrightarrow 0 \quad \text{and the dual} \quad 0 \longrightarrow \mathscr{E} \longrightarrow \mathscr{E}_1^* \longrightarrow \mathscr{E}_2^*.$$

---

[10]A generic point is a minimal prime in the language of commutative algebra.

Then, applying Proposition A.6 to $\mathscr{O}_X$ and then applying local cohomology to the above exact sequence yields

$$\mathscr{H}_Z^1(\mathscr{E}) = \mathscr{H}_Z^0(\mathscr{E}) = 0. \tag{A.1}$$

If, instead, $\mathscr{E}$ is flat and S2 and has full support, then we still have the vanishing (A.1) by Proposition A.6. Consider then the exact sequence

$$0 \longrightarrow \mathscr{H}_Z^0(\mathscr{E}) \longrightarrow \mathscr{E} \longrightarrow j_*\mathscr{E} \longrightarrow \mathscr{H}_Z^1(\mathscr{E}).$$

Applying (A.1) concludes our proof. $\qquad\square$

COROLLARY A.8 ([HK04, Proposition 3.6.2]). *Let $f\colon X \to V$ be a flat, S_2 morphism of Noetherian schemes, $\mathscr{E}$ a coherent sheaf on X satisfying either part* (a) *or part* (b) *of Proposition A.7. Let $\mathscr{F}$ be another coherent sheaf satisfying part* (a) *or* (b) *of that proposition. Let $j\colon U \hookrightarrow X$ be an open set such that for $Z := X \setminus U$, we have $\operatorname{codim}_{X_v} Z_v \geqslant 2$ for every $v \in V$. Also assume that $\mathscr{E}|_U \cong \mathscr{F}|_U$. Then $\mathscr{E} \cong \mathscr{F}$.*

Proof. In any case, $j_*\mathscr{E}|_U = \mathscr{E}$ and $j_*\mathscr{F}|_U = \mathscr{F}$, and so the result follows. $\qquad\square$

PROPOSITION A.9 ([HK04, Corollary 3.7]). *Let $f\colon X \to V$ be a flat, S_2 morphism of Noetherian schemes and $j\colon U \hookrightarrow X$ an open set such that for $Z := X \setminus U$, we have $\operatorname{codim}_{X_v} Z_v \geqslant 2$ for every $v \in V$. Also assume that $\mathscr{E}$ is a reflexive, coherent sheaf on U. Then $j_*\mathscr{E}$ is reflexive.*

Proof. Choose a coherent subsheaf $\mathscr{F} \subseteq j_*\mathscr{E}$ such that $\mathscr{F}|_U = \mathscr{E}$ [Har77, Exercise II.5.15]. Then $j_*\mathscr{E}$ is reflexive by the following computation:

$$\mathscr{F}^{**} \cong \underbrace{j_*(\mathscr{F}^{**}|_U)}_{\text{by Proposition A.7}} \cong \underbrace{j_*\mathscr{E}^{**}}_{\mathscr{E}\cong\mathscr{F}|_U} \cong \underbrace{j_*\mathscr{E}}_{\mathscr{E} \text{ is reflexive}}.$$

This completes the proof. $\qquad\square$

PROPOSITION A.10. *If $f\colon X \to Y$ is a projective, flat, relatively S2 and G1, equidimensional morphism, then $\omega_{X/Y}$ is reflexive.*

Proof. The proof given in [PS14, Lemma 4.8] works. $\qquad\square$

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