



Milnor and Tjurina numbers for smoothings of surface singularities

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To Eduard Looijenga on his 65th birthday

ABSTRACT

For an isolated hypersurface singularity $\{f = 0\}$, the Milnor number μ is greater than or equal to the Tjurina number τ (the dimension of the base of the semi-universal deformation), with equality if f is quasi-homogeneous. Saito proved the converse. The same result is true for complete intersections, but is much harder. For a Gorenstein surface singularity $(V, 0)$, the difference $\mu - \tau$ can be defined whether or not $(V, 0)$ is smoothable; the author has proved that it is non-negative, and equal to 0 if and only if $(V, 0)$ is quasi-homogeneous. We conjecture a similar result for non-Gorenstein surface singularities. Here, $\mu - \tau$ must be modified so that it is independent of any smoothing. This expression, involving cohomology of exterior powers of the bundle of logarithmic derivations on the minimal good resolution, is conjecturally non-negative, with equality if and only if $(V, 0)$ is quasi-homogeneous. We prove the “if” part; identify special cases where the conjecture is particularly interesting; verify it in some non-trivial cases; and prove it for a \mathbb{Q} -Gorenstein smoothing when the index one cover is a hypersurface. This conjecture arose regarding the classification of surface singularities with rational homology disk smoothings.

1. Introduction

Suppose that the n -dimensional hypersurface $V = \{f(z_1, \dots, z_{n+1}) = 0\} \subset \mathbb{C}^{n+1}$ has an isolated singularity at the origin. Then the Milnor fibre M of $(V, 0)$ has the homotopy type of a bouquet of a certain number μ of n -spheres, where the Milnor number μ is the length of the Jacobian algebra

$$J_f = \mathbb{C}[[z_1, \dots, z_{n+1}]] / (\partial f / \partial z_1, \dots, \partial f / \partial z_{n+1}).$$

The base space of the semi-universal deformation of $(V, 0)$ has dimension τ , which is the length of the Tjurina algebra

$$T_f = \mathbb{C}[[z_1, \dots, z_{n+1}]] / (f, \partial f / \partial z_1, \dots, \partial f / \partial z_{n+1}).$$

Received 11 September 2013, accepted in final form 7 November 2014.

2010 Mathematics Subject Classification 14J17, 14B07, 32S30, 32S25

Keywords: surface singularity, Milnor number, Tjurina number, smoothing, quasi-homogeneity, \mathbb{Q} -Gorenstein smoothing

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Clearly, $\mu \geq \tau$, with equality if and only if f belongs to the Jacobian ideal. This is the case when f is a weighted homogeneous polynomial. Saito [Sai71] proved the converse: equality implies that f is quasi-homogeneous, that is, analytically equivalent to such a polynomial.

A local complete intersection singularity $(V, 0)$ of dimension $n \geq 1$ also has a Milnor fibre M which is a bouquet of a certain number μ of n -spheres. The base space of its semi-universal deformation is again smooth of dimension $\tau = l(\text{Ext}_R^1(\Omega_R^1, R))$, where l denotes length and R is the local ring $\mathcal{O}_{V,0}$. In this case it is harder to relate the size of the two invariants, which are dimensions of very different-looking spaces. But it was eventually proved by work of (among others) Greuel [Gre80], Looijenga–Steenbrink [LS85], the author [Wah85], and Vosegaard [Vos02] that also in these cases one has $\mu \geq \tau$, with equality if and only if $(V, 0)$ is quasi-homogeneous.

Smoothings for $(V, 0)$ a general normal surface singularity were investigated in [Wah81]. A singularity (for example, the cone over the rational quartic curve in \mathbb{P}^4) can have several topologically distinct smoothings, occurring over “smoothing components” of different dimension in the base space of the semi-universal deformation of $(V, 0)$. Thus, for a smoothing $\pi: (\mathcal{V}, 0) \rightarrow (\mathbb{C}, 0)$, a Milnor fibre M may be defined, but the rank μ_π of $H^2(M)$ and the dimension τ_π of the corresponding smoothing component depend on the smoothing. (It is known that the first Betti number of the Milnor fibre is zero, by [GS83]). Define α_π to be the colength of the restriction map of the dual of (relative) dualizing sheaves

$$\omega_{\mathcal{V}/\mathbb{C}}^* \otimes \mathcal{O}_V \rightarrow \omega_V^*.$$

The following theorem was proved in special cases in [Wah81], and in general modulo several conjectures which were later established by Greuel, Looijenga, and Steenbrink ([GS83, GL85, Loo86]).

THEOREM 1.1 ([Wah81]). *Let $\pi: (\mathcal{V}, 0) \rightarrow (\mathbb{C}, 0)$ be a smoothing of a normal surface singularity $(V, 0)$, with $\mu_\pi, \tau_\pi, \alpha_\pi$ as above. Let $(X, E) \rightarrow (V, 0)$ be a good resolution. Then*

- (i) $1 + \mu_\pi = \alpha_\pi + 13h^1(\mathcal{O}_X) + \chi_T(E) - h^1(-K_X)$;
- (ii) $\tau_\pi = 2\alpha_\pi + 12h^1(\mathcal{O}_X) + h^1(\Theta_X) - 2h^1(-K_X)$.

If $(V, 0)$ is Gorenstein, then $\alpha_\pi = 0$, so μ and τ are independent of the smoothing.

In the Gorenstein case, one may use the expressions from the preceding theorem to define (possibly negative) singularity invariants μ and τ , independent of smoothability. With this definition, we have in general the following result.

THEOREM 1.2 ([Wah85]). *If $(V, 0)$ is a Gorenstein surface singularity, then $\mu - \tau \geq 0$, with equality if and only if $(V, 0)$ is weighted homogeneous.*

The point of this work is to study non-Gorenstein (for example, rational) surface singularities. Rather than $\mu - \tau$, consider the intrinsic invariant

$$(\mu_\pi - \tau_\pi) + \alpha_\pi,$$

depending only on $(V, 0)$. A more useful version of Theorem 1.1 is the following, given in Section 2

COROLLARY 2.5. *Let $(X, E) \rightarrow (V, 0)$ be the minimal good resolution of a normal surface singularity, not a rational double point. Denote by $S_X = (\Omega_X^1(\log(E)))^*$ the sheaf of derivations on X , logarithmic along E . Then for any smoothing π , one has*

$$1 + (\mu_\pi - \tau_\pi) + \alpha_\pi = h^1(\mathcal{O}_X) - h^1(S_X) + h^1(\wedge^2 S_X).$$

In the Gorenstein case, $\alpha_\pi = 0$ and the methods of [Wah85] allow one to compute the right-hand side, particularly the difficult term $h^1(S_X)$ (whose dimension can vary in an “equisingular family”). Note that $h^1(\wedge^2 S_X) = h^1(-(K_X + E))$ is the *second plurigenus* $\delta_2(V)$ of K. Watanabe ([Wat80]; see Lemma 2.1 below). It is not so easy to compute even for rational or quasi-homogeneous singularities.

The main purpose of this paper is to offer the following conjecture.

CONJECTURE (Main Conjecture). Let $(X, E) \rightarrow (V, 0)$ be the minimal good resolution of a non-Gorenstein normal surface singularity. Denoting by S_X the sheaf of logarithmic derivations on X , one has

$$h^1(\mathcal{O}_X) - h^1(S_X) + h^1(\wedge^2 S_X) \geq 0,$$

with equality if and only if $(V, 0)$ is quasi-homogeneous.

The relevant expression $\sum_{i=0}^2 (-1)^i h^1(\wedge^i S_X)$ looks somewhat like a second Chern class, as explained in Proposition 2.7.

The Main Conjecture (given also in Conjecture 2.6 below) might be overly optimistic. One can check it in certain cases by computing the right or left side in Corollary 2.5.

One implication is proved in this paper.

THEOREM 3.3. *If $(V, 0)$ is quasi-homogeneous and not Gorenstein, then*

$$h^1(\mathcal{O}_X) - h^1(S_X) + h^1(\wedge^2 S_X) = 0.$$

Certain special cases are worth pursuing. In Section 4, we have the following conjecture.

CONJECTURE 4.1 (Rational Conjecture). Let $(X, E) \rightarrow (V, 0)$ be the minimal good resolution of a rational surface singularity, not an RDP (rational double point). Then

$$h^1(S_X) \leq h^1(-(K_X + E)),$$

with equality if and only if $(V, 0)$ is quasi-homogeneous.

Recall that $h^1(S_X)$ is the dimension of the smooth space of equisingular deformations of $(V, 0)$, obtained from the deformations of X which preserve every exceptional curve [Wah76]. As for $h^1(-(K_X + E))$, we prove in Corollary 4.5 that it can be computed from the resolution graph; thus the conjecture would give a topological upper bound for the dimension $h^1(\Theta_X)$ of the Artin component of $(V, 0)$. As evidence for the Rational Conjecture, we show in Section 4 validity in several cases:

- (i) rational singularities for which $h^1(S_X) = 0$ (Proposition 4.9);
- (ii) rational singularities for which the resolution graph of $(V, 0)$ is star-shaped (Proposition 4.8);
- (iii) an example with non-star-shaped graph and $h^1(S_X) = 1$ (Example 4.10).

In a forthcoming paper, we verify the conjecture (in a stronger form) for any rational graph which is “sufficiently negative at the nodes”; see Remark 4.11 for a precise statement.

Recall that $(V, 0)$ is called *Q-Gorenstein* if K_V is \mathbb{Q} -Cartier, that is, some rK_V is invertible; then $(V, 0)$ is an r -cyclic quotient of its index one (or canonical) cover $(W, 0)$, which is Gorenstein. A smoothing $\pi: (V, 0) \rightarrow (\mathbb{C}, 0)$ is called *Q-Gorenstein* if it is an r -cyclic quotient of a smoothing of $(W, 0)$. In this case, $\alpha_\pi = 0$ (see Lemma 5.1), so the conjecture is again about $\mu - \tau$. We discuss in Section 5 the following conjecture.

CONJECTURE (\mathbb{Q} -Gorenstein Conjecture). For a \mathbb{Q} -Gorenstein smoothing of a non-Gorenstein singularity $(V, 0)$, one has

$$\mu \geq \tau - 1,$$

with equality if and only if $(V, 0)$ is quasi-homogeneous.

We verify this last conjecture in the case that the canonical cover is a hypersurface singularity. In fact, a much more general result, in all dimensions, is proved.

THEOREM 5.2. *Let $(W, 0)$ be an isolated hypersurface singularity $(\{f(z_1, \dots, z_{n+1}) = 0\}, 0) \subset (\mathbb{C}^{n+1}, 0)$, and let $G \subset \mathrm{GL}(n+1, \mathbb{C})$ be a finite group acting freely off 0 and leaving f invariant. The map $f: (\mathbb{C}^{n+1}/G, 0) \rightarrow (\mathbb{C}, 0)$ is a \mathbb{Q} -Gorenstein smoothing of $(V, 0) \equiv (W/G, 0)$, with smoothing invariants $\bar{\mu}$ and $\bar{\tau}$.*

- (i) *If $G \subset \mathrm{SL}(n+1, \mathbb{C})$, then $(V, 0)$ is Gorenstein and $\bar{\mu} \geq \bar{\tau}$, with equality if and only if $(V, 0)$ is quasi-homogeneous.*
- (ii) *If $G \not\subset \mathrm{SL}(n+1, \mathbb{C})$, then $(V, 0)$ is not Gorenstein and $\bar{\mu} \geq \bar{\tau} - 1$, with equality if and only if $(V, 0)$ is quasi-homogeneous.*

(The Milnor fibre for the quotient smoothing again has rational homology only in dimension n , of rank $\bar{\mu}$). The key ingredients of the proof are the main results of [Wal80] and [GL85] plus the Lefschetz fixed-point theorem.

A normal surface singularity in $(\mathbb{C}^4, 0)$ is smoothable, with a smooth base space for the semi-universal deformation. Gorenstein examples are complete intersections. It is proved in [Wah81, (3.14.4)] that α is zero. The conjecture in Section 6 is as follows.

CONJECTURE 6.1 (Codimension Two Conjecture). Let $(V, 0)$ be a normal surface singularity in $(\mathbb{C}^4, 0)$, not a complete intersection. Then

$$\mu \geq \tau - 1,$$

with equality if and only if $(V, 0)$ is quasi-homogeneous.

Examples of $(V, 0) \subset (\mathbb{C}^4, 0)$ which are neither \mathbb{Q} -Gorenstein nor quasi-homogeneous, with $\mu = \tau$ (consistently with the conjecture), can be found in Example 6.2.

Our original motivation for the conjecture (even for rational singularities) concerns those $(V, 0)$ admitting a “rational homology disk smoothing,” that is, a smoothing with Milnor number zero. These are especially interesting to topologists because the link of such a singularity possesses a symplectic filling with no rational homology. A complete classification exists in case $(V, 0)$ is quasi-homogeneous ([SSW08, BS11]), and it was conjectured in [Wah11] that these are the only examples. If a $(\mu = 0)$ -smoothing were \mathbb{Q} -Gorenstein (as happens in the quasi-homogeneous case [Wah13]), one would have $\alpha = 0$ and $\tau \geq 1$, whence (assuming the Rational or \mathbb{Q} -Gorenstein Conjecture) $\tau = 1$ and $(V, 0)$ would be quasi-homogeneous. Thus, there would be no other examples with \mathbb{Q} -Gorenstein smoothings.

Note finally that there exist similar results and conjectures for a reduced curve singularity $(C, 0)$. Here, μ is defined by Buchweitz–Greuel [BG80], and τ (the dimension of a smoothing component) is (via Deligne [Del73]) an expression in terms of curve invariants. Work of Greuel and others ([Gre81, GMP85]) gives the following results:

- (i) For $(C, 0)$ Gorenstein, one has $\mu \geq \tau$, with equality if and only if $(C, 0)$ is quasi-homogeneous.

- (ii) For $(C, 0)$ quasi-homogeneous, one has $\mu = \tau + 1 - t$, where t (the “type” of $(C, 0)$) is the minimal number of generators of the dualizing sheaf ω_C .
- (iii) (Conjecture) One has $\mu \geq \tau + 1 - t$, with equality if and only if $(C, 0)$ is quasi-homogeneous.

2. Formulas for μ and τ

Formulas for μ and τ for a smoothing $\pi: (\mathcal{V}, 0) \rightarrow (\mathbb{C}, 0)$ are given in Theorem 3.13 of [Wah81], modulo three conjectures, proved there only in certain cases, but later established in general. Specifically, the first Betti number of the Milnor fibre is zero ([GS83]); any smoothing can be appropriately globalized ([Loo86]); and the dimension of the smoothing component corresponding to π is the length of the cokernel of

$$\Theta_{\mathcal{V}/\mathbb{C}} \otimes \mathcal{O}_{\mathcal{V}} \rightarrow \Theta_{\mathcal{V}}$$

(see [GL85]). The invariant measuring the change in the dual of the dualizing sheaves is

$$\alpha = l(\text{Coker}(\omega_{\mathcal{V}/\mathbb{C}}^* \otimes \mathcal{O}_{\mathcal{V}} \rightarrow \omega_{\mathcal{V}}^*)).$$

Writing $R = \mathcal{O}_{(V,0)}$ for the local ring of V at 0, one has ([Wah81, Corollary A.2])

$$0 \leq \alpha \leq l(\text{Ext}_R^1(\omega, R)).$$

One deduces that $\alpha = 0$ if $(V, 0)$ is Gorenstein or a normal surface in \mathbb{C}^4 .

We rewrite the formulas relating μ, τ , and α not in terms of Θ_X and its second exterior power $-K_X$, but rather in terms of the sheaf $S = S_X$ of derivations logarithmic along E , and its second exterior power $-(K_X + E)$.

LEMMA 2.1 ([Oku97]). *Let $(X, E) \rightarrow (V, 0)$ be the minimal good resolution of a normal surface singularity, not an RDP. Then*

- (i) $H_E^1(X, -(K_X + E)) = 0$;
- (ii) *the dimension $h^1(X, -(K_X + E))$ equals the second plurigenus*

$$\delta_2(V) = \dim H^0(X - E, 2K_X + E) / H^0(X, 2K_X + E).$$

Proof. Corollary 1.9 of [Oku97] asserts that $h^1(2K_X + E) = 0$, whence by the long exact sequence in local cohomology $\delta_2(V) = h_E^1(2K_X + E)$. Local duality for a line bundle L on X yields $h_E^1(L) = h^1(K_X - L)$, from which both assertions follow. \square

PROPOSITION 2.2. *Let $\pi: (\mathcal{V}, 0) \rightarrow (\mathbb{C}, 0)$ be a smoothing of a normal surface singularity $(V, 0)$, not an RDP, with μ, τ , and α as before. Let $(X, E) \rightarrow (V, 0)$ be the minimal good resolution. Then*

- (i) $1 + \mu = \alpha + 13h^1(\mathcal{O}_X) + \chi_T(E) - (1/2)E \cdot (E + 3K) - h^1(-(K_X + E))$;
- (ii) $\tau = 2\alpha + 12h^1(\mathcal{O}_X) + \chi_T(E) - (1/2)E \cdot (E + 3K) + h^1(S) - 2h^1(-(K_X + E))$.

Proof. For each exceptional curve E_i , denote the genus by g_i , the degree by $-d_i$, and the number of intersections with other curves by t_i . One has the standard short exact sequence

$$0 \rightarrow S \rightarrow \Theta \rightarrow \oplus N_{E_i} \rightarrow 0,$$

where N_{E_i} is the normal bundle of E_i . Since $h^0(N_{E_i}) = 0$, one has $h^1(N_{E_i}) = g_i + d_i - 1$, so

$$h^1(\Theta) = h^1(S) + \sum (g_i + d_i - 1).$$

LEMMA 2.3. *Let $(X, E) \rightarrow (V, 0)$ be the minimal good resolution of a normal surface singularity, not an RDP. Then*

$$H^0(X, (-K_X) \otimes \mathcal{O}_E) = 0.$$

Proof. The group $H_E^1(-K_X + E)$ is the direct limit of the direct system given by the injective maps

$$H^0(-K_X + E) \otimes \mathcal{O}_Z(Z) \rightarrow H^0(-K_X + E) \otimes \mathcal{O}_{Z+Z'}(Z + Z'),$$

where Z and Z' are effective exceptional divisors. As $h_E^1(-K_X + E) = 0$, each of these spaces is 0, in particular for $Z = E$, as desired. \square

LEMMA 2.4. *Let $(X, E) \rightarrow (V, 0)$ be the minimal good resolution of a normal surface singularity, not an RDP. Then*

$$h^1(X, -K_X) = h^1(X, -(K_X + E)) + (1/2)E \cdot (E + 3K).$$

Proof. By Lemma 2.3 and the standard exact sequence, it suffices to compute $h^1((-K_X) \otimes \mathcal{O}_E) = -\chi((-K_X) \otimes \mathcal{O}_E)$. The Riemann–Roch theorem for a line bundle L and any exceptional divisor Z states

$$\chi(L \otimes \mathcal{O}_Z) = (-1/2)Z \cdot (Z + K) + Z \cdot L,$$

from which the result follows. \square

Proof of Proposition 2.2, continued. Lemma 2.4 provides the formula for $1 + \mu$. The formula for τ requires checking the easily verified relation

$$\chi_T(E) = \sum (g_i + d_i - 1) - E \cdot (E + 3K)/2. \quad \square$$

COROLLARY 2.5. *With notation as before, for a smoothing of a normal surface singularity (not an RDP), one has*

$$1 + (\mu - \tau) + \alpha = h^1(\mathcal{O}_X) - h^1(S_X) + h^1(-K_X + E).$$

This allows one to formulate the Main Conjecture of the paper, which is motivated by (but independent of) smoothing questions.

CONJECTURE 2.6. *Let $(X, E) \rightarrow (V, 0)$ be the minimal good resolution of a normal surface singularity, not an RDP.*

- (i) *If $(V, 0)$ is Gorenstein, then $h^1(\mathcal{O}_X) - h^1(S_X) + h^1(-K_X + E) \geq 1$, with equality if and only if $(V, 0)$ is quasi-homogeneous.*
- (ii) *If $(V, 0)$ is not Gorenstein, then $h^1(\mathcal{O}_X) - h^1(S_X) + h^1(-K_X + E) \geq 0$, with equality if and only if $(V, 0)$ is quasi-homogeneous.*

As previously mentioned, assertion (i) has been proved in [Wah85].

The relevant expression in the conjecture can be written in an alternative and suggestive way. The standard Euler characteristic of a locally free sheaf \mathcal{F} on X is

$$\chi(\mathcal{F}) = \dim H^0(X - E, \mathcal{F})/H^0(X, \mathcal{F}) - \dim H^1(\mathcal{F}).$$

If $H_E^1(\mathcal{F}) = 0$, one has simply $\chi(\mathcal{F}) = -h^1(\mathcal{F})$. Now, in the case at hand, one has

- (i) $h_E^1(\wedge^0 S_X) = h_E^1(\mathcal{O}_X) = 0$, by Grauert–Riemenschneider;
- (ii) $h_E^1(S_X) = 0$, by the main theorem of [Wah75];

(iii) $h_E^1(\wedge^2(S_X)) = 0$, by [Oku97], as in Lemma 2.1.

PROPOSITION 2.7. *On the minimal good resolution X of a normal surface singularity, one has*

$$h^1(\mathcal{O}_X) - h^1(S_X) + h^1(-(K_X + E)) = -\sum_{i=0}^2 (-1)^i \chi(\wedge^i S_X).$$

It is interesting to compare with the Riemann–Roch theorem for a rank two vector bundle \mathcal{F} on a smooth projective surface Y , which yields

$$\sum_{i=0}^2 (-1)^i \chi(Y, \wedge^i \mathcal{F}) = c_2(\mathcal{F}).$$

3. The quasi-homogeneous case

The following result was asserted in [Wah81, (4.10.2)], but not carefully proved there.

PROPOSITION 3.1. *Let $(X, E) \rightarrow (V, 0)$ be the minimal good resolution of a quasi-homogeneous surface singularity, not a cyclic quotient. Then the Euler derivation D of $(V, 0)$ induces a nowhere-zero section of the vector bundle S_X , hence gives a short exact sequence*

$$0 \rightarrow \mathcal{O}_X \rightarrow S_X \rightarrow \wedge^2(S_X) \rightarrow 0.$$

We may assume that $V = \text{Spec } A$ is an affine variety, where $A = \bigoplus A_i$ is a positively graded normal domain. Write $A = \mathbb{C}[z_1, \dots, z_s]/(g_\alpha(z_i))$, a quotient of a graded polynomial ring, where $\deg z_i = m_i$. The Euler derivation on A_i is multiplication by m_i , and $D = \sum_{i=1}^s m_i z_i \partial / \partial z_i$.

Let $W \rightarrow V$ be the partial resolution obtained by blowing up the weight filtration of A . The variety W has cyclic quotient singularities along the (smooth) exceptional curve C , which is isomorphic to $\text{Proj } A$. Minimally resolving these cyclic quotients gives X and E , which provides the minimal good resolution. The derivation D lifts to a section of both Θ_W and Θ_X ; as $H^0(S_X) = H^0(\Theta_X)$, it is also a section of S_X . We must show that it is nowhere zero. This is clear on $X - E = V - \{0\}$.

The variety W is the union of s affines W_i , each the quotient of a smooth affine U_i by a cyclic group of order m_i . To define U_1 , write

$$z_1 = x^{m_1}, z_2 = x^{m_2} y_2, \dots, z_s = x^{m_s} y_s.$$

Then U_1 is the affine variety with coordinate ring

$$\mathbb{C}[x, y_2, \dots, y_s]/(g_\alpha(1, y_2, \dots, y_s)),$$

hence is polynomial in x . The derivation D lifts to the derivation $x\partial/\partial x$, and is a nowhere-zero section of the bundle of derivations on U_1 logarithmic along $x = 0$. The variety W_1 is the quotient of U_1 by the cyclic group of order m_1 generated by

$$T = (1/m_1)[-1, m_2, \dots, m_s].$$

If the action is free at a point of $x = 0$, the quotient map is a local analytic isomorphism; so at the corresponding point of $C \subset W_1$, the derivation D is still a nowhere-zero section of the corresponding bundle of logarithmic derivations. At points of $x = 0$ where there is isotropy (that is, above the cyclic quotient singularities of W_1), consider the corresponding local analytic model on U_1 .

Changing notation slightly, for local analytic coordinates x, y on \mathbb{C}^2 , consider the action of \mathbb{Z}/r of type $1/r[1, a]$ and the cyclic quotient singularity $W = \mathbb{C}^2/(\mathbb{Z}/r)$. Let $X \rightarrow W$ be the minimal equivariant resolution of W , with exceptional divisor $E = \sum_{i=1}^l E_i$. Denote by C the Weil divisor on W given by the image of $x = 0$, and by C' its proper transform on Y . Then C' intersects E transversally along one end, say E_l . The following lemma will complete the proof of the proposition.

LEMMA 3.2. *The derivation $D = x\partial/\partial x$ on \mathbb{C}^2 induces a derivation of W and lifts to a nowhere-zero section of the rank two vector bundle $\Omega_X^1(\log(E + C'))^*$ on X .*

Proof. Use the familiar description (cf. Reid's Warwick notes [Rei, page 10]) of the minimal resolution as a union $X_0 \cup X_1 \cup \dots \cup X_l$ of copies of \mathbb{C}^2 . A copy X_i has coordinates u_i, v_i , with the exceptional curve given by $v_0 = 0$ on X_0 ; $u_l = 0$ on X_l ; and $u_i v_i = 0$ on the intermediate X_i . The curve C' is given by $u_0 = 0$ on X_0 . On X_i , we have $u_i = x^{a_i} y^{b_i}, v_i = x^{c_i} y^{d_i}$, for appropriate integer exponents. Thus on this affine, one has $D = a_i u_i \partial/\partial u_i + c_i v_i \partial/\partial v_i$. The construction shows that all the a_i and c_i are non-zero, except that $c_l = 0$. Thus, D has the desired property on every X_i . \square

THEOREM 3.3. *Let $(X, E) \rightarrow (V, 0)$ be the minimal good resolution of a quasi-homogeneous singularity, not an RDP.*

(i) *If $(V, 0)$ is not Gorenstein, then*

$$h^1(\mathcal{O}_X) - h^1(S) + h^1(-(K_X + E)) = 0.$$

(ii) *If $(V, 0)$ is Gorenstein, then*

$$h^1(\mathcal{O}_X) - h^1(S) + h^1(-(K_X + E)) = 1.$$

Proof. As previously noted, the theorem has already been proved for Gorenstein singularities. It is also true for non-RDP cyclic quotients (since each individual term in (i) vanishes, for instance using Corollary 4.5 below).

For $(V, 0)$ non-Gorenstein, by Proposition 3.1 one must show that the (graded) coboundary map

$$\delta: H^0(X, -(K_X + E)) \rightarrow H^1(X, \mathcal{O}_X)$$

is zero. We examine the graded pieces of the map, using the notation and results of Demazure–Watanabe in [Wat81]. The analytic data of $(V, 0)$ is given by the central curve $C = \text{Proj } A$, its normal divisor D on X , and cyclic quotient singularities of type n_i/q_i at points P_i for $i = 1, \dots, t$. Consider the \mathbb{Q} -divisors

$$F = D - \sum (q_i/n_i)P_i, \quad \Xi = K + \sum (1 - 1/n_i)P_i.$$

Recall that for these types of \mathbb{Q} -divisors G , one defines an invertible sheaf

$$\mathcal{O}(G) \equiv \mathcal{O}(\lfloor G \rfloor) \subset k(C).$$

Pinkham's basic result is

$$A = \oplus A_k = H^0(X, \mathcal{O}_X) = \oplus_{k=-\infty}^{\infty} H^0(C, kF)T^k.$$

Watanabe proved (see [Wat81])

$$\omega_A = H^0(X - E, K_X) = \oplus_{k=-\infty}^{\infty} H^0(C, \Xi + kF)T^k.$$

It follows from the general constructions that

$$\begin{aligned} H^1(X, \mathcal{O}_X) &= \bigoplus_{k=0}^{\infty} H^1(C, kF)T^k, \\ \omega_A^* &= H^0(X - E, -K_X) = \bigoplus_{k=-\infty}^{\infty} H^0(C, -\Xi + kF)T^k. \end{aligned}$$

Since $H^0(X, -(K_X + E)) \subset H^0(X - E, -(K_X + E)) = \omega_A^*$, it follows that the k th graded piece of the coboundary map factors through

$$H^0(C, -\Xi + kF) \rightarrow H^1(C, kF).$$

The second space is dual to $H^0(C, K - [kF])$; in order for the map to be non-zero, both this space and $H^0(C, [-\Xi + kF])$ must be non-zero. Therefore, the sum of the corresponding integral divisors is effective. One checks that the sum is supported at the points P_i , with coefficients

$$\lfloor -((kq_i - 1)/n_i + 1) \rfloor - \lfloor -kq_i/n_i \rfloor.$$

But this expression equals -1 , unless $kq_i \equiv 1 \pmod{n_i}$, in which case it is zero. Thus, a non-zero coboundary map requires $kq_i \equiv 1 \pmod{n_i}$ for all i . Since the sum of the two divisors is effective and of degree zero, and each divisor has a section, the divisors themselves would have to be trivial. This implies, in particular, $K = [kF]$. By Watanabe's criterion, $(V, 0)$ is Gorenstein.

It is not necessary to prove here the delicate fact that the coboundary map has rank one in the Gorenstein case [Hoc77]. \square

4. Rational surface singularities

CONJECTURE 4.1 (Rational Conjecture). Let $(X, E) \rightarrow (V, 0)$ be the minimal resolution of a rational surface singularity, not an RDP. Then

$$h^1(S_X) \leq h^1(-(K_X + E)),$$

with equality if and only if $(V, 0)$ is quasi-homogeneous.

The dimension $h^1(S_X)$ of the space of equisingular deformations is very difficult to compute (but see Example 4.10 below). The term $h^1(-K_X)$ has been encountered in [Wah80, (1.12)], where it is proved that for R the local ring of $(V, 0)$,

- (i) $h^1(-K_X) = l(\text{Ext}_R^1(\omega, R))$;
- (ii) $h^1(-K_X) \geq \text{mult}(R) - 3$.

As $E \cdot (E + K) = -2$, Lemma 2.3 implies

$$h^1(-K_X) = h^1(-(K_X + E)) + ((-E \cdot E) - 3).$$

If E is the fundamental cycle, $\text{mult}(R) = -E \cdot E$, and $h^1(-(K_X + E))$ calculates the difference in the second inequality above.

We show that $h^1(-(K_X + E))$ can be calculated from the graph of $(V, 0)$. Recall the notation and results of Giraud [Gir82]. For a line bundle L on a good resolution $(X, 0) \rightarrow (V, 0)$ of a normal surface singularity, define $[L]$ to be the smallest (in the usual ordering) integral divisor D such that

$$D \cdot E_i \leq L \cdot E_i \quad \text{for all } i.$$

Suppose that L is numerically equivalent to $\sum a_i E_i$ with $a_i \in \mathbb{Q}$. One forms $[L]$ by first rounding up all a_i to form an integral D_1 , and inductively defining $D_{n+1} = D_n + E_j$ if one finds that $D_n \cdot E_j > L \cdot E_j$.

THEOREM 4.2 (Giraud [Gir82]). *Let L be a line bundle on X such that*

$$[L] \cdot E_i \leq 0 \quad \text{for all } E_i.$$

Then one has $H_E^1(L) = 0$.

For L a line bundle and D an integral divisor, one has $[L - D] = [L] - D$; in particular

$$[(L - [L])] = 0.$$

COROLLARY 4.3. *For any line bundle L on X , one has $H_E^1(L - [L]) = 0$.*

Note that $[L]$ is an effective divisor if $L \cdot E_i \leq 0$ for all i , or more generally if the coefficients a_i as above are all greater than -1 .

PROPOSITION 4.4. *On the minimal resolution X of a rational singularity, let L be a line bundle for which $[L] \equiv Z$ is an effective divisor. Then*

$$\dim H^1(X, L) = Z \cdot (Z + K)/2 - Z \cdot L.$$

Proof. The result is easy if $Z = 0$, so assume $Z > 0$. By duality and Corollary 5.2, one has

$$0 = h_E^1(\mathcal{O}(L - Z)) = h^1(\mathcal{O}(K - L + Z)),$$

so that

$$0 = h^1(\mathcal{O}(K - L + Z) \otimes \mathcal{O}_Z) = h_E^1(\mathcal{O}(K - L + Z) \otimes \mathcal{O}_Z).$$

Consider the short exact sequence of sheaves on X ,

$$0 \rightarrow \mathcal{O}(K - L) \rightarrow \mathcal{O}(K - L + Z) \rightarrow \mathcal{O}(K - L + Z) \otimes \mathcal{O}_Z \rightarrow 0.$$

Taking local cohomology yields

$$0 \rightarrow H_E^0(\mathcal{O}(K - L + Z) \otimes \mathcal{O}_Z) \rightarrow H_E^1(\mathcal{O}(K - L)) \rightarrow H_E^1(\mathcal{O}(K - L + Z)) \rightarrow 0.$$

The first term (because of the vanishing of the corresponding H_E^1) is the Euler characteristic of a line bundle on Z , so by the Riemann–Roch theorem its dimension is $Z \cdot (Z + K)/2 - Z \cdot L$. The middle term has dimension $h^1(L)$, and one must show $h_E^1(K - L + Z) = 0$. As usual, it suffices to prove that for any effective cycle Y , there is an E_i in the support of Y such that

$$(K - L + Z + Y) \cdot E_i < 0.$$

By definition $(Z - L) \cdot E_i \leq 0$ for all i ; so it suffices to find $(K + Y) \cdot E_i < 0$. This is standard, because by rationality one always has $Y \cdot (Y + K) \leq -2$. \square

COROLLARY 4.5. *Let $(X, E) \rightarrow (V, 0)$ be the minimal resolution of a rational surface singularity, not an RDP. Then*

(i) *for $Y = [-K_X]$, one has*

$$h^1(X, -K_X) = Y \cdot (Y + 3K)/2;$$

(ii) *for $Z = [-(K_X + E)] = Y - E$, one has*

$$h^1(X, -(K_X + E)) = Z \cdot (Z + 3K)/2 + Z \cdot E;$$

(iii) *in particular, these dimensions are topological and depend only on the graph of E .*

Proof. Since $K \cdot E_i \geq 0$ for all i and $K \neq 0$, one has $-K \equiv \sum a_i E_i$ with all $a_i > 0$. Thus, both $[-K]$ and $[-(K + E)]$ are effective, so Proposition 4.4 applies. \square

Remark 4.6. For a rational singularity, $[-K] \cdot E_i \leq 2 - d_i \leq 0$ for all i , so that $[-K]$ is at least as big as the fundamental cycle Z_0 . The easiest way to compute $[-K]$ (and hence $[-K] - E$) is to add curves to Z_0 until one reaches a cycle Y satisfying $Y \cdot E_i \leq 2 - d_i$ for all i .

Remark 4.7. It follows from [LW86, (4.11.4)] that if a normal surface singularity admits a \mathbb{Q} -Gorenstein smoothing, then $h^1(-K_X) = -K \cdot K + h^1(\mathcal{O}_X)$.

PROPOSITION 4.8. Consider a rational surface singularity $(V, 0)$ with a star-shaped graph (not a cyclic quotient singularity), with $(X, E) \rightarrow (V, 0)$ the minimal resolution. Then the Main Conjecture holds for $(V, 0)$; that is,

$$h^1(X, S_X) \leq h^1(X, -(K_X + E)),$$

with equality if and only if $(V, 0)$ is quasi-homogeneous.

Proof. In the local ring of V , consider the filtration given by the order of vanishing along the central curve in the resolution. The associated graded ring A for this filtration is normal, and is a quasi-homogeneous rational singularity with isomorphic resolution graph. The corresponding degeneration gives a deformation of $\text{Spec } A$ whose general fibre is isomorphic to $(V, 0)$, and this family admits a simultaneous equisingular resolution (and, in fact, the reduced exceptional curve is analytically isomorphic). By Corollary 4.5, the dimension $h^1(-(K + E))$ is constant in the family, and it equals $h^1(S)$ on the special fibre. It suffices to show that if $h^1(S)$ is constant in the family, then the original singularity was already quasi-homogeneous. Let C denote the central curve, and consider on any of the resolutions the exact sequence

$$H^0(S) \rightarrow H^0(S \otimes \mathcal{O}_C) \rightarrow H^1(S(-C)) \rightarrow H^1(S) \rightarrow H^1(S \otimes \mathcal{O}_C) \rightarrow 0.$$

According to [Wah85, (3.11) and (3.2)], the second space has dimension one and the first map is surjective if and only if the singularity is quasi-homogeneous. Since the special fibre has

$$h^1(S) = h^1(S(-C)) + h^1(S \otimes \mathcal{O}_C),$$

if this quantity remained constant in a deformation then the same would be true for each summand, and the first map in the sequence would have to remain surjective. \square

PROPOSITION 4.9. The Rational Conjecture is true for rational singularities for which $h^1(S) = 0$ (for example, for the taut singularities classified by Laufer [Lau73]).

Proof. One must show that a non-quasi-homogeneous singularity $(V, 0)$ with $h^1(S_X) = 0$ must have $h^1(-(K_X + E)) > 0$. By Proposition 4.8, it suffices to assume that the graph is not star-shaped. Since $H^1(S_X) \rightarrow H^1(\Theta_E)$ is surjective, $h^1(S_X) = 0$ implies that all vertices in the graph have valency at most three. So, the graph contains a subgraph



with all negative self-intersections at least two. (Some of these can be taut.) Let F denote the reduced curve connecting the nodes, running from the $(-e)$ -curve to the $(-f)$ -curve; it suffices to show that $h^1(-(K_X + E) \otimes \mathcal{O}_F) \neq 0$. Since F is supported on a cyclic quotient singularity, the sheaf $-(K_X + E) \otimes \mathcal{O}_F$ depends only on the numerics of the line bundle $-(K_X + E)$ restricted to F . But this line bundle does to 0 with the interior curves, and to -1 with the nodes; thus, it

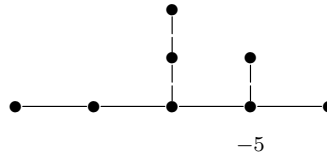
equals $(K_X + F) \otimes \mathcal{O}_F$, whence

$$h^1(-(K_X + E) \otimes \mathcal{O}_F) = h^1((K_X + F) \otimes \mathcal{O}_F).$$

Computing now on a resolution of the cyclic quotient, one has $h^1(K_X) = 0$, and standard dualities and vanishing give

$$h^1((K_X + F) \otimes \mathcal{O}_F) = h^1(K_X + F) = h^1_F(\mathcal{O}(-F)) = h^0(\mathcal{O}_F) = 1. \quad \square$$

Example 4.10. Consider a rational singularity with the following graph, where as usual the unmarked bullets are (-2) -curves:



First, we have $h^1(-(K_X + E)) = 2$. Starting from the fundamental cycle as in Remark 4.6, one finds $[-K_X] \equiv Y$, which has multiplicity three at the left hand node, one on the four outer vertices, and two on the others. Calculating with $Z \equiv Y - E$, one deduces from Corollary 4.5 that $h^1(-(K_X + E)) = 2$. Next, consistently with the conjecture, there is a singularity with this graph and with $h^1(S_X) = 1$. We can write it as a splice quotient [NW05], that is, the quotient of $x^3 + y^3 + zw^7 = z^2 + w^2 + xy = 0$ by the discriminant group G (of order 60). Calculating the semi-universal deformation via the computer algebra system SINGULAR, one finds that the only deformation on which G acts equivariantly is obtained by adding tz^3w^5 to the first equation. By general theory, this represents exactly the only equisingular deformation of the original singularity.

Remark 4.11. In a forthcoming paper, we show that the conjecture is true for any graph satisfying $d_i \geq 2t_i - 2$ for all i ; this is a condition only at the nodes. More precisely, if such a graph is not star-shaped, then $h^1(S_X)$ is equal to the number of ends of the graph minus four, which in turn is one less than $h^1(-(K_X + E))$.

5. \mathbb{Q} -Gorenstein smoothings

Suppose that $(V, 0)$ is an isolated Cohen–Macaulay singularity, whose dualizing sheaf ω has order r . If we write $(\omega^{\otimes i})^{**} \equiv \omega^{[i]}$ and choose an isomorphism $\omega^{[r]} \simeq \mathcal{O}$, the index one cover is the analytic spectrum of $\mathcal{O} \oplus \omega \oplus \omega^{[2]} \oplus \dots \oplus \omega^{[r-1]}$. The germ $(V, 0)$ is said to be \mathbb{Q} -Gorenstein if the index one cover is also Cohen–Macaulay (in which case it is Gorenstein). (Alternatively, $(V, 0)$ is the quotient of a Gorenstein singularity by a finite group acting freely off the singular point.) This is equivalent to requiring that all $\omega^{[i]}$ have maximum depth. In particular, $\omega^* \simeq \omega^{[r-1]}$ is Cohen–Macaulay. Recall that a rational surface singularity is \mathbb{Q} -Gorenstein.

A \mathbb{Q} -Gorenstein smoothing $\pi: (\mathcal{V}, 0) \rightarrow (\mathbb{C}, 0)$ of $(V, 0)$ is one which is the quotient of a smoothing of the index one cover.

LEMMA 5.1. *Let $\pi: (\mathcal{V}, 0) \rightarrow (\mathbb{C}, 0)$ be a \mathbb{Q} -Gorenstein smoothing of the normal surface singularity $(V, 0)$. Then $\alpha_\pi = 0$.*

Proof. As indicated above, the dualizing sheaf $\omega_{\mathcal{V}}^* \simeq \omega_{\mathcal{V}/\mathbb{C}}^*$ has depth three. Tensoring with $\mathcal{O}_{\mathcal{V}}$

gives a depth two subsheaf of ω_V which is equal off the singular point, hence equal. Therefore, $\alpha_\pi = 0$. \square

Suppose that $G \subset \mathrm{GL}(n + 1, \mathbb{C})$ is a finite group acting freely off the origin, with $n \geq 2$. Then $(\mathbb{C}^{n+1}/G, 0)$ is Cohen–Macaulay, and is Gorenstein if and only if $G \subset \mathrm{SL}(n + 1, \mathbb{C})$ (since G contains no pseudo-reflections). Next let $f: (\mathbb{C}^{n+1}, 0) \rightarrow (\mathbb{C}, 0)$ be a G -invariant function having an isolated singularity $(W, 0)$ at the origin, with invariants μ and τ . The induced map $\bar{f}: (\mathbb{C}^{n+1}/G, 0) \rightarrow (\mathbb{C}, 0)$ is a \mathbb{Q} -Gorenstein smoothing of the quotient singularity $(W/G, 0) \equiv (V, 0)$. The group G acts freely on the Milnor fibre M of W , so the new smoothing has Milnor fibre M/G , which continues to have (rational) homology only in dimensions zero and n . Euler characteristics (and hence Milnor numbers) are related by

$$1 + (-1)^n \mu = |G|(1 + (-1)^n \bar{\mu}).$$

Denote by $\bar{\tau}$ the dimension of the corresponding smoothing component of V .

THEOREM 5.2. *Consider the \mathbb{Q} -Gorenstein smoothing above of $(V, 0)$.*

- (i) *If $G \subset \mathrm{SL}(n + 1, \mathbb{C})$, then $\bar{\mu} \geq \bar{\tau}$, with equality if and only if $(V, 0)$ is quasi-homogeneous.*
- (ii) *If $G \not\subset \mathrm{SL}(n + 1, \mathbb{C})$, then $\bar{\mu} \geq \bar{\tau} - 1$, with equality if and only if $(V, 0)$ is quasi-homogeneous.*

We will prove this result via a series of lemmas. Denote by J and T the Jacobian and Tjurina algebras of f , respectively. The group G acts on both these spaces.

LEMMA 5.3. *For the G -invariant subspaces, one has*

$$\dim J^G \geq \dim T^G,$$

with equality if and only if f and hence V are quasi-homogeneous.

Proof. The inequality is clear. The function f is G -invariant, so is in J^G , and is 0 if and only if it is 0 in J . If f is quasi-homogeneous, one may choose weights so that G commutes with the \mathbb{C}^* action. Thus, V is also quasi-homogeneous. \square

Let $\tau' = \dim T^G$, and choose G -invariant polynomials $h_1, \dots, h_{\tau'}$ inducing a basis of T^G . The semi-universal deformation of $(W, 0)$ restricts to a family

$$\mathcal{W} = \left\{ f + \sum_{i=1}^{\tau'} t_i h_i = 0 \right\} \subset \mathbb{C}^{n+1} \times \mathbb{C}^{\tau'} \rightarrow \mathbb{C}^{\tau'}$$

on which G acts, trivially on the base and hence on the fibres. This gives a smooth family

$$(\mathcal{W}/G, 0) \equiv (\mathcal{V}, 0) \rightarrow (\mathbb{C}^{\tau'}, 0)$$

of deformations of $(V, 0)$.

LEMMA 5.4. *The family $(\mathcal{V}, 0) \rightarrow (\mathbb{C}^{\tau'}, 0)$ is a full smoothing component of the base space of the semi-universal deformation of $(V, 0)$. In particular, one has $\tau' = \bar{\tau}$.*

Proof. Denote the relative Kähler differentials of $f: (\mathbb{C}^{n+1}, 0) \rightarrow (\mathbb{C}, 0)$ by Ω_f^1 , and its dual by Θ_f ; similarly, one has $\Omega_{\bar{f}}^1$ and $\Theta_{\bar{f}}$. By [GL85], the dimension $\bar{\tau}$ is the length of the cokernel of the inclusion

$$\Theta_{\bar{f}} \otimes \mathcal{O}_V \subset \Theta_V.$$

But $\Theta_V = (\Theta_W)^G$, since W has an isolated normal singularity, G acts freely off that point, and Θ_W is reflexive and locally free off the singular point. Similarly, $\Theta_{\bar{f}} = (\Theta_f)^G$. Thus, $\bar{\tau}$ is the colength of the inclusion

$$\Theta_f^G \otimes \mathcal{O}_W \subset (\Theta_W)^G.$$

Writing $R = \mathbb{C}\{z_1, \dots, z_{n+1}\}$ and using the sequence

$$0 \rightarrow f^*(\Omega_{\mathbb{C}\{t\}}^1) \rightarrow \Omega_R^1 \rightarrow \Omega_f^1 \rightarrow 0$$

and basic homological algebra (for example, [Wah81, (A.1)] with $M = \Omega_f^1$ and $t = f$) gives the exact sequence

$$0 \rightarrow \Theta_f \rightarrow \Theta_f \rightarrow \Theta_W \rightarrow \text{Ext}_R^1(\Omega_f^1, R) \rightarrow \text{Ext}_R^1(\Omega_f^1, R) \rightarrow T_W^1 \rightarrow \text{Ext}_R^2(\Omega_f^1, R) = 0.$$

One has a natural G -isomorphism $\text{Ext}_R^1(\Omega_f^1, R) \cong J_f$, so taking G -invariants of the G -equivariant exact sequence

$$0 \rightarrow \Theta_f \otimes \mathcal{O}_W \rightarrow \Theta_W \rightarrow J_f \rightarrow J_f \rightarrow T_f \rightarrow 0$$

yields

$$\bar{\tau} = \dim T_f^G = \tau'. \quad \square$$

To compare μ to $\dim J_f^G$, we relate the action of G on J_f with the action on the cohomology of the Milnor fibre (in particular, on $H \equiv H^n(M; \mathbb{C})$.) The n -form $dx_1 \wedge dx_2 \wedge \dots \wedge dx_{n+1}/df$ is holomorphic on the Milnor fibre, and G acts on it via the determinant character. Recall the following result.

THEOREM 5.5 (Wall, [Wal80]). *The G -modules H and $J_f \otimes (\det)$ are isomorphic, where \det is the determinant character.*

(This result does not require that G act freely off the origin, only that it leave f invariant.) We conclude as follows.

LEMMA 5.6. *One has $\dim J_f^G = \dim H^\nu$, where H^ν is the isotypic component of H corresponding to the character $\nu = (\det)^{-1}$.*

The action of G on the full cohomology ring $H^*(M)$ can be deduced from the following general result.

LEMMA 5.7. *Let X be a finite connected CW complex with cohomology $H^*(X)$ only in even degrees (respectively, only in odd degrees and degree zero). Suppose that a finite group G acts freely on X . Then as a G -module, $H^*(X)$ is exactly $\chi(X)/|G|$ copies of the regular representation (respectively, the direct sum of two trivial representations and the direct sum of $-\chi(X)/|G|$ copies of the regular representation).*

Proof. Let $\phi_g: X \rightarrow X$ be the homeomorphism corresponding to $g \in G$. Consider the Lefschetz number

$$L(\phi_g) = \sum_i (-1)^i \text{Tr}(\phi_g^* : H^i(X)).$$

The group G acts freely, so by the Lefschetz fixed point theorem, for $g \neq e$ one has

$$L(\phi_g) = 0.$$

Clearly,

$$L(\phi_e) = L(\text{Id}) = \chi(X).$$

When $H^*(X)$ is non-zero only in even degrees, one has by definition

$$L(\phi_g) = \chi(g),$$

where χ is the character corresponding to the representation of G on $H^*(X)$. Note that the Euler characteristic satisfies $\chi(X) = \dim H^*(X)$. Thus, one has $\chi(g) = 0$ for $g \neq e$ and $\chi(e) = \dim H^*(X)$. Standard character theory implies that the representation is a direct sum of copies of the regular representation.

When $H^*(X)$ has cohomology only in odd degrees and degree zero, write $H^*(X) = H^0 \oplus H'$, where H' is a direct sum of the odd cohomology (which we assume is non-zero). Let η be the character corresponding to the representation of G on H' . Then the argument above yields

$$\eta(e) = \dim H', \quad \eta(g) = 1 \quad \text{if } g \neq e.$$

This time, character theory implies that the representation of G on H' is a direct sum of a trivial representation and $(\dim H' - 1)/|G|$ copies of the regular representation. As $\chi(X) = 1 - \dim H'$, the claim easily follows. \square

Back to the Milnor fibre M , we conclude that

- (i) if n is even, then $H^*(M)$ is a direct sum of $(1 + \mu)/|G|$ copies of the regular representation;
- (ii) if n is odd, then $H^*(M)$ is a direct sum of two trivial representations and $(\mu - 1)/|G|$ copies of the regular representation.

In particular, for n even, every multiplicative character χ of G occurs in $H^*(M)$ with the same multiplicity $(1 + \mu)/|G|$, which as mentioned before equals $1 + \bar{\mu}$. Thus, the dimension of H^\times is $(1 + \mu)/|G|$ unless χ is the trivial character, in which case the dimension is one less. But the determinant character of G is trivial if and only if $G \subset \text{SL}(n + 1, \mathbb{C})$. One now has all the ingredients to verify Theorem 5.2 in this case.

When n is odd, Lemma 5.7 implies that as a G -module, H consists of one copy of the trivial representation plus $(\mu - 1)/|G| = \bar{\mu} - 1$ copies of the regular representation. Thus, one knows the dimension of each isotypic component, and the same argument as in the even case completes the proof of Theorem 5.2.

6. Normal surfaces in \mathbb{C}^4

Let $(V, 0) \subset (\mathbb{C}^4, 0)$ be a germ of a normal surface singularity. If $(V, 0)$ is Gorenstein, it is a complete intersection; otherwise, it is maximal-minor determinantal. The base space of the semi-universal deformation is smooth, as $T_{(V,0)}^2 = 0$; computer programs can be used to calculate it. It is unknown whether the Milnor fibre is always simply-connected; this is true for the rational triple points, since one has a unique Milnor fibre, diffeomorphic to the minimal resolution. The Milnor number is difficult to compute. Since $\alpha = 0$ in these cases, one wants to prove the following conjecture.

CONJECTURE 6.1 (Codimension Two Conjecture). Let $(V, 0) \subset (\mathbb{C}^4, 0)$ be a non-Gorenstein singularity. Then

$$\mu \geq \tau - 1,$$

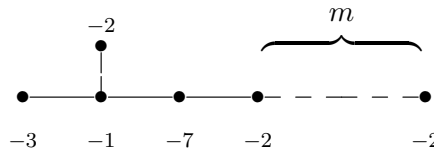
with equality if and only if $(V, 0)$ is quasi-homogeneous.

Example 6.2 ([Oku05, (6.3)]). There is a $(V, 0) \subset (\mathbb{C}^4, 0)$ which is not \mathbb{Q} -Gorenstein and not quasi-homogeneous (though it has a star-shaped graph), with $\mu = \tau$.

Consider the singularity $(V, 0)$ defined by the vanishing of the 2×2 minors of the matrix

$$\begin{pmatrix} x & y & z \\ y - 3w^2 & z + w^{3m} & x^2 + 6wy - 2w^3 \end{pmatrix} \quad \text{for } m \geq 1.$$

The singularity $(V, 0)$ has multiplicity three with $p_g = m$, and the same (integral homology sphere) resolution graph as the hypersurface singularity $x^2 + y^3 + z^{6m+7} = 0$ (which has $p_g = m + 1$):



Corollary 6.6 of [Oku05] yields $\mu = 12m + 1$. Van Straten has used SINGULAR to prove $\tau = 12m + 1$ for $m = 1$, and to indicate its likelihood for general m . The semi-universal deformation should be obtained by perturbing the matrix by adding

$$\begin{pmatrix} 0 & 0 & 0 \\ f & g & h \end{pmatrix},$$

where $f = \sum_{i=0}^{3m-1} a_i w^i$, $g = \sum_{i=0}^{3m-1} b_i w^i$, $h = \sum_{i=0}^{3m} c_i w^i + x \sum_{i=0}^{3m-1} d_i w^i$. Thus, the a_i, b_i, c_i, d_i are parameters for the semi-universal deformation.

ACKNOWLEDGEMENTS

We thank Shrawan Kumar for the proof of Lemma 5.7 and Duco van Straten and Jacob Fowler for help with computer calculations.

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