Algebraic Geometry 2 (3) (2015) 298–314
doi:10.14231/AG-2015-013

On the *h*-principle and specialness for complex projective manifolds

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Abstract

We show that a complex projective manifold X which satisfies Gromov's h-principle is special in the sense of the first author's paper "Orbifolds, Special Varieties, and Classification Theory" (Annals of the Institut Fourier, 2004), and raise some questions about the reverse implication, the extension to the quasi-Kähler case, and the relationships of these properties to the Oka property. The guiding principle is that the existence of many Stein manifolds with degenerate Kobayashi pseudometric gives strong obstructions to the complex hyperbolicity of projective manifolds satisfying the h-principle.

1. Introduction

We start by recalling the notion of h-principle.

Definition 1.1 (Gromov). A complex space X is said to *satisfy the h-principle* (a property abbreviated as $hP(X)$) if for every Stein manifold S and every continuous map $f\colon S \to X$ there exists a holomorphic map $F\colon S \to X$ which is homotopic to f.

The origin of this notion lies in the work of Grauert and Oka. Grauert showed that for any continuous section s of a holomorphic principal bundle with fibre a complex Lie group G over a Stein manifold S, there exists a holomorphic section homotopic to s. The classification of continuous complex thus reduces to that of holomorphic vector bundles. This was established by Oka for complex line bundles. Considering products $G \times S$, Grauert's result also shows that complex Lie groups satisfy the h-principle. This has been extended by Gromov to elliptic (and later by Forstneric to subelliptic) manifolds. These include homogeneous complex manifolds (for example $\mathbb{P}_n$, Grassmannians, and tori) and complements $\mathbb{C}^n \setminus A$, where A is an algebraic subvariety of codimension at least two. Subelliptic manifolds contain as many entire curves as possible, and therefore are opposite to Brody hyperbolic complex manifolds. Since generic hyperbolicity is conjectured (and sometimes known) to coincide with general type in algebraic geometry, it is thus natural to assume that for projective varieties, satisfying the h-principle is related to being special as introduced in [Cam04], since specialness is conjectured there to be equivalent to $\mathbb{C}$-connectedness. In this article we investigate these relationships with particular emphasis on projective manifolds.

Received 27 January 2014, accepted in final form 9 October 2014.
2010 Mathematics Subject Classification 14E99, 14D99, 32E10, 32F45, 32J27, 55Q05.
Keywords: homotopy of complex manifolds, Gromov's homotopy principle, Kobayashi pseudometric, Special complex projective manifolds, quasi-albanese map.

The main result is as follows.

MAIN THEOREM. *Let X be a complex projective manifold satisfying the h-principle. Then*

(i) *the manifold X is* special*;*

(ii) *every holomorphic map from X to a Brody hyperbolic Kähler manifold is constant.*

For an arbitrary complex manifold we prove the statements below.

THEOREM 1.2. *Let X be a complex manifold satisfying the h-principle. Then*

(i) *the manifold X is* homotopically $\mathbb{C}$-connected*;*

(ii) *if X is an algebraic variety, its quasi-Albanese map is dominant.*

Let us now recall and introduce some notation.

DEFINITION 1.3. We say that a complex space X is

(i) $\mathbb{C}$-*connected* if any two points of X can be connected by a chain of entire curves, that is, holomorphic maps from $\mathbb{C}$ to X (this property is preserved by passing to unramified coverings and images by holomorphic maps; if X is smooth, this property is also preserved under proper modifications);

(ii) *Brody-hyperbolic* if any holomorphic map $h\colon \mathbb{C} \to X$ is constant;

(iii) *homotopically* $\mathbb{C}$-*connected* if every holomorphic map $f' \to Y$ from any unramified covering X' of X to a Brody-hyperbolic complex space Y induces maps $\pi_k(f)\colon \pi_k(X') \to \pi_k(Y)$ between the respective homotopy groups which are zero for every $k > 0$.

Observe that any holomorphic map $f\colon X \to Y$ between complex spaces is constant if X is $\mathbb{C}$-connected and Y is Brody-hyperbolic. Thus $\mathbb{C}$-connectedness implies weak $\mathbb{C}$-connectedness. Also, any contractible X is homotopically $\mathbb{C}$-connected.

There exist projective smooth threefolds which are homotopically $\mathbb{C}$-connected, but not $\mathbb{C}$-connected. An example can be found in [CW09].

It is easy to verify that every subelliptic manifold X is $\mathbb{C}$-connected. Conversely, all known examples of connected complex manifolds satisfying the h-principle admit a holomorphic homotopy equivalence to a subelliptic complex space.

This suggests the following question.

Question 1.4. Let X be a complex connected manifold. If X satisfies the h-principle, does this imply that there exists a holomorphic homotopy equivalence between X and a $\mathbb{C}$-connected complex space Z?

Since a compact manifold cannot be homotopic to a proper analytic subset for compact manifolds, this question may be reformulated as follows.

Question 1.5. Let X be a compact complex connected manifold. If X satisfies the h-principle, does this imply that X is $\mathbb{C}$-connected?

Combining Theorem 6.1 with the abelianity conjecture of [Cam04], we obtain the following purely topological conjectural obstruction to the h-principle.

CONJECTURE 1.6. Every projective manifold satisfying the h-principle has an almost abelian fundamental group.

Our proof of the implication "h-principle $\Rightarrow$ special" depends on Jouanoulou's trick, which is not available for non-algebraic manifolds.

Still, we believe that the statement should also hold in the Kähler case (for which specialness is defined as in Definition 2.1 below).

Conjecture 1.7. Every compact Kähler manifold satisfying the h-principle is special.

This implication might also hold for quasi-projective manifolds, provided their topology is sufficiently rich (non-contractibility being obviously a minimal requirement). Particular cases involving the quasi-Albanese map (dominance and connectedness) have been established, using [NWY07]. See Theorems 7.1 and 7.4 in Section 7.

The converse direction ("special $\Rightarrow$ h-principle") is almost completely open. Based on classification and known results (see [FL11] for a survey), the implication does hold for curves as well as surfaces which are rational, ruled over an elliptic curve, or blow-ups of either abelian or bielliptic surfaces. The question remains open for all other special surfaces, and thus in particular for K3, or even Kummer, surfaces. In higher dimensions even less is known; for example, the case of $\mathbb{P}^3$ blown-up in a smooth curve of degree three or more is far from being understood.

Still, with a sufficient amount of optimism one might hope for a positive answer to the following question.

Question 1.8. Let X be a smooth (or at least normal) quasi-projective variety. Assume that X is either special or $\mathbb{C}$-connected. Does it follow that X satisfies the h-principle? In that case, is it Oka (see Section 9)?

We present some examples showing that there is no positive answer for arbitrary (that is, non-normal, non-Kähler, or non-algebraic varieties). There are examples of the following types which do *not* satisfy the h-principle despite being $\mathbb{C}$-connected, or being special in the sense of Definition 2.1:

(i) a non-normal projective curve which is special and $\mathbb{C}$-connected;

(ii) a non-compact and non-algebraic complex manifold which is $\mathbb{C}$-connected;

(iii) a compact non-Kähler surface which is special.

See Section 9 for more details.

Remark 1.9. (1) Any contractible complex space trivially satisfies the h-principle. The notion of h-principle is thus of interest only for non-contractible manifolds X. Since positive-dimensional compact manifolds are never contractible, this is not relevant for projective manifolds. However, there do exist examples of contractible affine varieties of log general type ([Ram71, Miy01]) indicating that for non-compact varieties an equivalence "h-principe $\Leftrightarrow$ special" can hold only if the topology of the variety is sufficiently non-trivial.

(2) Let $u'\colon X' \to X$ be an unramified covering, with X and X' smooth and connected. Then $hP(X)$ implies $hP(X')$ (see Lemma 6.6), but the converse is not true. To see this, consider a compact Brody-hyperbolic manifold X which is an Eilenberg–MacLane $K(\pi,1)$-space, but is not contractible (for example, a projective curve of genus $g \geqslant 2$ or a compact ball quotient). Then its universal cover $\tilde{X}$ is contractible and therefore satisfies the h-principle. On the other hand, being Brody-hyperbolic and non-contractible, X itself cannot satisfy the h-principle.

(3) For any given X and f, after possibly replacing the initial complex structure J_0 of S by another one, $J_1 = J_1(f)$, homotopic to J_0, the existence of F as in Definition 1.1 is always

true if $\dim_{\mathbb{C}}(S) \geqslant 3$. If $\dim_{\mathbb{C}}(S) = 2$, one must first compose with an orientation preserving homeomorphism of S; see [For11, Section 9.10].

2. Specialness

2.1 Specialness and the core map

We refer to [Cam04] for more details on this notion, to which the present section is an extremely sketchy introduction. Roughly speaking, special manifolds are higher-dimensional generalisations of rational and elliptic curves, thus opposite to manifolds of general type in the sense that they, and their finite étale covers, do not admit non-degenerate meromorphic maps to orbifolds of general type. Many qualitative properties of rational or elliptic curves extend or are expected to extend to special manifolds, although they are much more general (see Remark 2.2(7) below).

DEFINITION 2.1. Let X be a connected compact Kähler manifold. Let $p > 0$, and let $L \subset \Omega^p_X$ be a saturated rank one coherent subsheaf. We define

$$\kappa^{\mathrm{sat}}(X, L) := \limsup_{m>0} \left\{ \frac{\log(h^0(X, \overline{mL}))}{\log(m)} \right\},$$

where $H^0(X, \overline{mL}) \subset H^0(X, (\Omega^p_X)^{\otimes m})$ is the subspace of sections taking values in $L_x^{\otimes m} \subset (\Omega^p_X)_x^{\otimes m}$ at the generic point x of X.

By a generalisation of Castelnuovo–De Franchis due to Bogomolov (see [Bog79]), we have $\kappa^{\mathrm{sat}}(X, L) \leqslant p$, with equality if and only if $L = f^*(K_Y)$ at the generic point of X for some meromorphic dominant map $f\colon X \dashrightarrow Y$ with Y a compact p-dimensional manifold.

We say that L is a *Bogomolov sheaf* on X if $\kappa^{\mathrm{sat}}(X, L) = p > 0$, and that X is *special* if it has no Bogomolov sheaf.

Remark 2.2. (1) A special manifold has no dominant meromorphic map $f\colon X \dashrightarrow Y$ onto a positive-dimensional manifold Y of general type, since $L := f^*(K_Y)^{\mathrm{sat}}$ would provide a Bogomolov sheaf on X. In particular, X is not of general type (that is, $\kappa(X) := \kappa(X, K_X) < \dim(X)$).

(2) Specialness is a bimeromorphic property. If X is special, so is any Y dominated by X (that is, for which a dominant rational map $f\colon X \dashrightarrow Y$ exists).

(3) If X is special, and if $f\colon X' \to X$ is unramified finite, then X' is also special. The proof (see [Cam04]) is surprisingly difficult. It shows that specialness implies weak specialness, defined as follows: X is *weakly special* if none of its unramified covers has a dominant meromorphic map $f\colon X \dashrightarrow Y$ onto a positive-dimensional manifold Y of general type.

(4) The notion of weak specialness looks natural, and is easy to define. Unfortunately, it does not lead to any meaningful structure result such as the one given by the core map, stated below. On the other hand, it is also too weak to characterise the vanishing of the Kobayashi pseudometric (see item (10) below).

(5) Geometrically speaking, a manifold X is special if and only if it has no dominant rational map onto an orbifold pair (Y, Δ) of general type. We do not define these concepts here. See [Cam04, Cam11a] for details.

(6) Compact Kähler manifolds which are either rationally connected or have $\kappa = 0$ are special (see [Cam04]).

(7) For any $n > 0$ and any $\kappa \in \{-\infty, 0, 1, \ldots, (n-1)\}$, there exist special manifolds with $\dim(X) = n$ and $\kappa(X) = \kappa$. For details, see [Cam04, Section 6.5].

(8) For curves, special is equivalent to weakly special, and also to non-hyperbolic. For surfaces, special is equivalent to weakly special, and also to $\kappa < 2$ together with $\pi_1(X)$ almost abelian. Thus special surfaces are exactly the ones with

(i) $\kappa = -\infty$ and $q \leqslant 1$; or
(ii) $\kappa = 0$; or
(iii) $\kappa = 1$ and $q(X') \leqslant 1$ for any finite étale cover X' of X.

(9) Another quite different characterisation of compact Kähler special surfaces X is as follows: X is special if and only if it is $\mathbb{C}^2$-dominable (with the possible exception of non-elliptic K3 surfaces, which are special, but not known to be $\mathbb{C}^2$-dominable). One direction is essentially due to [BL00].

(10) For $n \geqslant 3$ there exist n-dimensional projective manifolds X which are weakly special, but not special (see [BT04]), and no simple characterisation of specialness depending only on κ and π_1 exists. Moreover, there are examples of weakly special varieties for which the Kobayashi pseudometric does not vanish identically (see [CP07, CW09]).

The central results concerning specialness, which motivated its introduction, are the following two structure theorems (see [Cam04, Cam11a] for definitions and details).

Theorem 2.3. *For any compact Kähler manifold X, there exists a unique almost holomorphic*[1] *meromorphic map with connected fibres $c_X\colon X \dashrightarrow C(X)$ such that:*

(i) *its general fibre is special; and*
(ii) *its orbifold base $(C(X), \Delta_{c_X})$ is of general type (and a point exactly when X is special).*

The map c_X is called the core map *of X. It functorially splits X into its parts of opposite geometries (special vs. general type).*

Conjecture 2.4. For any X as above, $c_X = (J \circ r)^n$, where $n := \dim(X)$. Here J and r are orbifold versions of the Moishezon fibration and of the rational quotient, respectively. In particular, special manifolds are then towers of fibrations with general fibres having either $\kappa = 0$ or $\kappa_+ = -\infty$.

Theorem 2.5. *The preceding conjecture holds if the orbifold version of Iitaka's $C_{n,m}$-conjecture is true.*

Remark 2.6. The two theorems above extend naturally to the full orbifold category.

The last (conditional) decomposition naturally leads (see [Cam11a]) to the following conjectures.

Conjecture 2.7. (i) If X is special, $\pi_1(X)$ is almost abelian.
(ii) The manifold X is special if and only if its Kobayashi pseudometric vanishes identically.
(iii) The manifold X is special if and only if X is $\mathbb{C}$-connected.

2.2 Orbifold Kobayashi–Ochiai and factorisation through the core map

The following orbifold version of the Kobayashi–Ochiai extension theorem will be crucial in the proof of our main result.

Theorem 2.8 ([Cam04, Theorem 8.2]). *Let X be a compact Kähler manifold, let $c_X\colon X \dashrightarrow C(X)$ be its core map*[2]*, let $M \subset \overline{M}$ be a non-empty Zariski open subset of the connected complex*

[1] This means that its generic fibre does not meet its indeterminacy locus.
[2] Or, more generally, any map $f\colon X \to Y$ of general type in the sense of [Cam04].

manifold $\overline{M}$, and let $\varphi\colon M \dashrightarrow X$ be a meromorphic map such that $g := c_X \circ \varphi\colon M \dashrightarrow C(X)$ is non-degenerate (that is, submersive at some point of M). Then g extends meromorphically to $\overline{M}$.

Corollary 2.9. *Let X be a compact Kähler manifold. If there exists a non-degenerate meromorphic map $\varphi\colon \mathbb{C}^n \dashrightarrow X$, then X is special.*

This is an indication in the direction of Conjecture 2.7(ii).

Proof. By Theorem 2.8 applied to $M := \mathbb{C}^n \subset \overline{M} := \mathbb{P}^n$, the existence of such a map φ implies the existence of a surjective meromorphic map $\bar{\varphi}\colon \mathbb{P}^n \dashrightarrow X$. This contradicts Theorem 2.3(ii) unless $C(X)$ is a point, that is, unless X is special. □

Theorem 2.10. *Let X, Z be complex projective manifolds, and let M be a smooth algebraic variety admitting a surjective algebraic morphism $\tau\colon M \to Z$ whose fibres are all affine spaces (biholomorphic to $\mathbb{C}^k$). Let $G\colon M \dashrightarrow X$ be a meromorphic map such that $g := c_X \circ G\colon M \to C(X)$ is non-degenerate. Then g also factors through τ and the core map $c_Z\colon Z \dashrightarrow C(Z)$; that is, $g = \varphi \circ c_Z \circ \tau$ for some $\varphi\colon C(Z) \dashrightarrow C(X)$.*

Proof. The variety M can be compactified to a compact smooth projective variety $\overline{M}$ by adding a hypersurface D with normal crossings. By Theorem 2.8, the map g extends algebraically to $\bar{g}\colon \overline{M} \to C(X)$. Denote by $\overline{\tau}\colon \overline{M} \to Z$ the extension of τ to $\overline{M}$.

The functoriality of the core gives two maps, $c_{\overline{\tau}}\colon C(\overline{M}) \to C(Z)$ and $c_G\colon C(\overline{M}) \to C(X)$. We have $\bar{g} = c_{\bar{g}} \circ c_{\overline{M}}$. The fibres of $c_{\overline{\tau}}$ are rationally connected, since those of $\overline{\tau}$ are. Thus $c_{\overline{\tau}}$ is an isomorphism by [Cam04, Theorem 3.26]. The composed map $\varphi := c_{\bar{g}} \circ c_{\overline{\tau}}^{-1}\colon C(Z) \to C(X)$ provides the desired factorisation, since $\bar{g} = c_{\bar{g}} \circ c_{\overline{M}} = c_{\bar{g}} \circ c_{\overline{\tau}}^{-1} \circ c_Z \circ \overline{\tau} = \varphi \circ c_Z \circ \overline{\tau}$. □

Remark 2.11. The conclusion still holds if we replace c_X by any fibration with general type orbifold base and only assume that the fibres of $\overline{\tau}$ are rationally connected manifolds and that all components of D are mapped surjectively onto Z by $\overline{\tau}$. This follows from [GHS03] and [Cam04, Theorem 3.26].

3. Jouanoulou's trick

Proposition 3.1. *Let X be a projective manifold. Then there exist a smooth affine complex variety M and a surjective morphism $\tau\colon M \to X$ such that*

(i) *the morphism $\tau\colon M \to X$ is a homotopy equivalence;*

(ii) *every fibre of τ is isomorphic to some $\mathbb{C}^n$; in particular, every fibre has vanishing Kobayashi pseudodistance;*

(iii) *the morphism τ is a locally holomorphically trivial fibre bundle;*

(iv) *the morphism τ admits a real-analytic section.*

Remark 3.2. This is known as Jouanoulou's trick (see [Jou73]). This construction was introduced in Oka's theory in [Lár05], where the class G of good manifolds is introduced. The latter are defined as manifolds with a Stein affine bundle over some quasi-projective manifold, with fibre $\mathbb{C}^n$ for some n. This class contains Stein manifolds, quasi-projective manifolds, and is stable by various usual geometric operations.

Proof. We first treat the case of $X := \mathbb{P}^N$, denoting by $\mathbb{P}^{N*}$ its dual projective space. Let $D \subset P := \mathbb{P}^N \times \mathbb{P}^{N*}$ be the divisor consisting of pairs (x, H) such that $x \in H$. In other words, it is the incidence graph of the universal family of hyperplanes of $\mathbb{P}^N$. This divisor D is ample, since it intersects positively the two family of lines contained in the fibres of both projections of P. Let V be its complement in P. The projection τ_P onto the first factor of P, restricted to V, satisfies the requirements for $X := \mathbb{P}^N$. A real-analytic section is obtained by choosing a hermitian metric on $\mathbb{C}^{n+1}$ and sending a complex line to its orthogonal hyperplane.

In the general case, first embed X in some $\mathbb{P}_N$. Then, let $M = \tau_P^{-1}(X)$ and denote by τ the restriction of τ_P to M. Now M is a closed algebraic subset of V and therefore likewise affine. Everything then restricts from $\mathbb{P}^N$ to X. □

Note that when $X = \mathbb{P}^1$, we recover the two-dimensional affine quadric as M (and indeed, $\mathbb{P}^1$ is diffeomorphic to S^2).

If X is a projective curve, we may also obtain a bundle $M \to X$ with the desired properties in a different way. Let $Q_2 = \mathbb{P}^1 \times \mathbb{P}^1 - D$, where D is the diagonal. Taking the first projection, we get an affine bundle $Q_2 \to \mathbb{P}^1$ with fibre $\mathbb{C}$ over $\mathbb{P}^1$, which is an affine variety. Now we choose a finite morphism f from X to $\mathbb{P}_1$ and define $M \to X$ via base change.

Question 3.3. Given a complex manifold Z, do there exist a Stein manifold S and a holomorphic map $f\colon S \to Z$ whose fibres are isomorphic to $\mathbb{C}^n$? Is this true at least when Z is compact Kähler?

4. Opposite complex structures and associated cohomological integrals

4.1 Inverse images of forms under meromorphic maps

Lemma 4.1. *Let $f\colon X \dashrightarrow Y$ be a dominant meromorphic map between compact complex manifolds, where $\dim(X) = n$, and let $I(f) \subset X$ be the indeterminacy set. For every $c \in H^{k,k}(Y)$, there exists a unique cohomology class $c' \in H^{k,k}(X)$ such that*

$$[\alpha] \cdot c' = \int_{X \setminus I(f)} \alpha \wedge f^*\beta \tag{4.1}$$

for every closed smooth $(n-k, n-k)$-form α on X and every closed smooth (k,k)-form β on Y with $[\beta] = c$.

We define the inverse image of the De Rham cohomology class $[c]$ with respect to the meromorphic map f by $f^*([c]) := c'$.

Proof. First, we observe that C' is unique, if it exists. Indeed, if two coholomology classes c and c' both satisfy (4.1), then $(c' - c'') \cdot a = 0$ for all $a \in H^{k,k}(X)$, implying that they are equal.

Next, let $\tau'\colon X' \to X$ be a blow-up such that f lifts to a holomorphic map $F\colon X' \to Y$. Using Poincaré duality, $F^*\beta$ may be identified with a linear form on $H^{n-k,n-k}(X')$. Restricting this linear form to $\tau^* H^{n-k,n-k}(X)$ and again using Poincaré duality, we see that there is a unique cohomology class c' such that

$$[\alpha] \cdot c' = \int_{X'} \tau^*(\alpha) \wedge F^*\beta\,.$$

Furthermore,

$$\int_{X'} \tau^*(\alpha) \wedge F^*\beta = \int_{X \setminus I(f)} \alpha \wedge f^*\beta$$

since $\alpha \wedge f^*\beta$ is a top-degree form and both the exceptional divisor of the blow-up and the indeterminacy set $I(f)$ of the meromorphic f are sets of measure zero. □

From the characterization of this inverse image, it is clear that it is compatible with composition of dominant meromorphic maps. It is also clear that it specializes to the usual pull-back if the meromorphic map under discussion happens to be holomorphic.

Caveat: This inverse image gives linear maps between the cohomology groups, but, as can be seen by easy examples, it does not define a ring homomorphism between the cohomology rings.

4.2 Opposite complex structures

Given a complex manifold X, we define the *opposite* or *conjugate* complex manifold (also called *opposite complex structure* on M) as follows. If X_0 is the underlying real manifold and J is the almost complex structure tensor of X, we define the opposite complex manifold to be X_0 equipped with $-J$ as complex structure tensor.

If $(z_i)_i$ are local holomorphic coordinates on (X, J), their complex conjugates $(\bar{z}_i)_i$ provide holomorphic coordinates for $(X, -J)$. Thus the opposite almost complex structure $-J$ is integrable if and only if J is integrable.

Now, consider the complex projective space $\mathbb{P}_n(\mathbb{C})$. The map

$$[z_0 : \ldots : z_n] \mapsto [\bar{z}_0 : \ldots : \bar{z}_n]$$

defines a biholomorphic map between $\mathbb{P}_n(\mathbb{C})$ and its opposite. As a consequence, we deduce that if a complex manifold X is projective, so is its opposite $\overline{X}$.

Now assume that X admits a Kähler form ω. Then the opposite complex manifold $\overline{X}$ is again a Kähler manifold. Indeed, since $\omega(v, w) = g(Jv, w)$ defines the Kähler form on a complex manifold admitting a Riemannian metric g for which J is an isometry, we see that $\overline{X}$ admits a Kähler metric with $-\omega$ as Kähler form. The same property applies if g is, more generally, a hermitian metric on X, and ω its associated Kähler form, defined from J and g by the formula above.

Orientation. On a Kähler manifold X with Kähler form ω, the orientation is defined by imposing that ω^n be positively oriented, where $n = \dim_{\mathbb{C}}(X)$. This implies that if X is a Kähler manifold and $\overline{X}$ is its opposite, the identity map of the underlying real manifold defines an orientation-preserving diffeomorphism if $n = \dim_{\mathbb{C}}(X)$ is even and an orientation-reversing one if n is odd.

4.3 Inverse image of forms and opposite complex structures

Lemma 4.2. *Let X be an n-dimensional compact complex manifold, let $\overline{X}$ be its conjugate, and let $\zeta\colon \overline{X} \to X$ be a smooth map homotopic to the identity map id_X of X. Let $c\colon X \dashrightarrow Y$ be a meromorphic map to a compact complex manifold Y. Let $c \circ \zeta =: \varphi\colon \overline{X} \to Y$. Let α be a d-closed smooth differential form of degree $2d$ on Y, and let ω_X be a smooth closed $(1,1)$-form on X. Then we have*

$$I' =: \int_{\overline{X}} \zeta^*(\omega_X^{n-d} \wedge c^*(\alpha)) = (-1)^d \cdot \int_X \omega_X^{n-d} \wedge c^*(\alpha) := (-1)^d \cdot I\,.$$

Proof. From the remarks above on the orientations of X and $\overline{X}$, and the fact that $\mathrm{id}_X^*(\omega_X) = -\omega_{\overline{X}}$, we get $I = (-1)^n \int_{\overline{X}} \omega_X^{n-d} \wedge c^*\alpha$.

Since ζ is homotopic to id_X, and $c \circ \zeta = \varphi$, we have

$$\begin{aligned} I &= (-1)^n \int_{\overline{X}} \zeta^*(\omega_X^{n-d} \wedge c^*(\alpha)) \\ &= (-1)^n \int_{\overline{X}} \zeta^*(\omega_X^{n-d}) \wedge \varphi^*(\alpha) = (-1)^n \int_{\overline{X}} (-1)^{n-d} \omega_{\overline{X}}^{n-d} \wedge \varphi^*(\alpha) \\ &= (-1)^d \int_{\overline{X}} \omega_{\overline{X}}^{n-d} \wedge \varphi^*(\alpha) = (-1)^d \cdot I'. \end{aligned}$$ □

Corollary 4.3. *In the situation of Lemma 4.2, assume that X is compact Kähler,* $\dim(Y) > 0$, *and that $c\colon X \dashrightarrow Y$ is non-degenerate (that is, dominant). Then $\varphi := c \circ \zeta\colon \overline{X} \dashrightarrow Y$ is not meromorphic.*

Proof. Assume that φ is meromorphic. After suitable modifications, we may assume that Y is Kähler. Let $\alpha := \omega_Y$ be a Kähler form on Y. Choose $d = 1$ in Lemma 4.2. Then $I := \int_X \omega_X^{n-1} \wedge c^*(\omega_Y) > 0$. On the other hand, $I' := \int_{\overline{X}} \omega_{\overline{X}}^{n-1} \wedge \varphi^*(\omega_Y) > 0$. From Lemma 4.2 we deduce that $I' = -I$, giving a contradiction. □

5. The h-principle and Brody-hyperbolicity

5.1 The h-principle and weak $\mathbb{C}$-connectedness

Proposition 5.1. *For any $n > 0$, the n-dimensional sphere S^n is homotopic to the (complex) n-dimensional affine quadric Q_n defined as follows:*

$$Q_n = \left\{ z = (z_0, \ldots, z_n) \in \mathbb{C}^{n+1} : \sum_k z_k^2 = 1 \right\}.$$

Any two points of Q_n are connected by an algebraic $\mathbb{C}^*$, and so its Kobayashi pseudometric vanishes identically.

Proof. Let q be the standard non-degenerate quadratic form in $\mathbb{R}^{n+1}$. The set $Q_n(\mathbb{R})$ of real points of Q_n obviously coincides with S^n. An explicit real analytic isomorphism $\rho\colon Q_n \to N_n$ with the real normal (that is, orthogonal) bundle $N_n := \{(x, y) \in S^n \times \mathbb{R}^{n+1} : q(x, y) = 0\}$ of S^n in $\mathbb{R}^{n+1}$ is given by $\rho(z = x + i \cdot y) := (\lambda(z) \cdot x, \lambda \cdot y)$, where $\lambda(z)^{-1} := \sqrt{1 + q(y, y)}$. The map ρ is in particular a homotopy equivalence.

The last assertion is obvious, since any complex affine plane in $\mathbb{C}^{n+1}$ intersects Q_n either in a conic with one or two points deleted, or in a two-dimensional complex affine space. □

Question 5.2. Let Z be a connected differentiable manifold or a finite-dimensional CW-complex. Do there exist topological obstructions to the existence of a Stein manifold S homotopic to Z with vanishing Kobayashi pseudodistance?

In particular, does there exist a Stein manifold with vanishing Kobayashi pseudodistance (for example, $\mathbb{C}$-connected) and homotopic to a smooth connected projective curve of genus $g \geqslant 2$?

The main difficulty here is the condition on the Kobayashi pseudodistance. In fact, it is not too hard to give an always positive answer if one drops the condition on the Kobayashi pseudodistance.

Proposition 5.3. *Let Z be a connected differentiable manifold or a finite-dimensional CW-complex* (*as always with countable base of topology*)*. Then there exists a Stein manifold M homotopic to Z.*

Proof. This is a known consequence of the classical characterisation of Stein spaces by Grauert (see [For11, Corollary 3.5.3] and the references there, for example). We give here a short proof, using a deep theorem of Eliashberg.

If Z is a CW-complex, we embed Z into some $\mathbb{R}^n$. Then Z is homotopic to some open neighbourhood of Z in $\mathbb{R}^n$. Since open subsets of $\mathbb{R}^n$ are manifolds, it thus suffices to deal with the case where Z is a differentiable manifold. By taking a direct product with some $\mathbb{R}^k$, we may furthermore assume $\dim_{\mathbb{R}}(Z) > 2$. Let $M = T^*Z \overset{\tau}{\mapsto} Z$ denote the cotangent bundle. Then M carries a symplectic structure in a natural way and therefore admits an almost complex structure. Fixing a metric h on $M = T^*Z$ and choosing an exhaustive Morse function ρ on Z, we can use $p(v) = \rho(\tau(v)) + h(v)$ as an exhaustive Morse function on M. By construction, the critical points of p are all in the zero-section of the cotangent bundle of Z and coincide with the critical points of ρ. Therefore, there is no critical point of index greater than $\dim(Z) = \frac{1}{2}\dim(M)$. By a result of Eliashberg ([Eli90, Theorem 1.3.1]) it follows from the existence of such a Morse function and the existence of an almost complex structure that M can be endowed with a structure of Stein complex manifold. This completes the proof since M is obviously homotopy equivalent to Z. □

Theorem 5.4. *Let X be a complex space which satisfies the h-principle. Then X is homotopically $\mathbb{C}$-connected.*

Proof. Assume that it is not. Since $hP(X)$ is preserved by passing to unramified coverings (see Lemma 6.6), we may assume that $X' = X$ in Definition 1.3(iii). Then there exists a holomorphic map $g \to Y$ with Y Brody-hyperbolic and such that there exists a non-zero induced homotopy map $\pi_k(g)\colon \pi_k(X) \to \pi_k(Y)$ with $k > 0$. Let $f\colon S^k \to X$ be a continuous map defining a non-trivial element $g \circ f\colon S^k \to Y$ in $\pi_k(Y)$, where S^k denotes the k-dimensional sphere. Let Q_k be the k-dimensional affine quadric, and let $\varphi\colon Q_k \to S^k$ be a continuous map which is a homotopy equivalence (its existence is due to Proposition 5.1). Then $f \circ \varphi\colon Q_k \to Y$ is a continuous map which is not homotopic to a constant map. But due to the Brody-hyperbolicity of Y, every holomorphic map from Q_k to Y must be constant, contradicting our initial assumption. □

Corollary 5.5. *Let X be a Brody-hyperbolic complex manifold. Then X satisfies the h-principle if and only if it is contractible.*

Proof. The h-principle is trivially true if X is contractible. Conversely, assume that $hP(X)$ holds. Then X is homotopically $\mathbb{C}$-connected by Theorem 5.4. Since in addition X is Brody-hyperbolic, the identity map $\mathrm{id}\colon X \to X$ must induce the zero morphism on every homotopy group $\pi_k(X)$. Hence $\pi_k(X) = 0$ for every k and consequently X is contractible. □

Corollary 5.6. *Let X be a positive-dimensional compact complex Brody-hyperbolic manifold. Then X does not satisfy the h-principle.*

Proof. Positive-dimensional compact manifolds are not contractible. □

In particular, compact Riemann surfaces of genus $g \geqslant 2$ do not satisfy the h-principle.

Remark 5.7. There exist holomorphic maps $f\colon X \to Y$ with X and Y both smooth and projective which are not homotopic to a constant map, since the homological degree is non-zero, although $\pi_k(f) = 0$ for all $k > 0$. For example, take a compact Riemann surface X of genus $g \geqslant 2$ and let f be any non-constant map to $\mathbb{P}^1$ (example suggested by Bogomolov).

Therefore it is not clear whether being homotopically $\mathbb{C}$-connected implies that every holomorphic map to a Brody-hyperbolic complex space must be homotopic to a constant map.

Theorem 5.8 solves this issue in the projective case, assuming the h-principle.

5.2 Projective Brody-hyperbolic targets

Theorem 5.8. *Let X be an irreducible projective complex space satisfying the h-principle. Let $f\colon X \dashrightarrow Y$ be a meromorphic map to a Brody hyperbolic Kähler manifold Y. Assume either that f is holomorphic, or that X is smooth. Then f is constant.*

Proof. For every meromorphic map $f\colon X \dashrightarrow Y$ there exists a proper modification $\hat{X} \to X$ such that f can be lifted to a holomorphic map defined on $\hat{X}$. If X is smooth, this modification can be obtained by blowing up smooth centers, implying that the fibres of $\hat{X} \to X$ are rational. Since Y is Brody-hyperbolic, holomorphic maps from rational varieties to Y are constant. Hence X being smooth implies that f is already holomorphic.

Thus in any case, we may assume that f is holomorphic. Assume by contradiction that f is not constant. Because X is projective, we can find a compact complex curve C on X such that $f|_C$ is non-constant.

Let $\overline{C}$ be C equipped with its conjugate (that is, opposite) complex structure, and let $j\colon \overline{C} \to C$ be the set-theoretic identity map. Let $\tau\colon E \to \overline{C}$ be an holomorphic affine $\mathbb{C}$-bundle as given by Proposition 3.1.

Since X is assumed to satisfy the h-principle, the continuous map $j \circ \tau\colon E \to X$ is homotopic to a holomorphic map $h\colon E \to X$. Because Y is Brody-hyperbolic, the map $f \circ h\colon E \to Y$ is constant along the fibres of τ. Hence $f \circ h$ is equal to $\varphi \circ \tau$ for a holomorphic map $\varphi\colon \overline{C} \to Y$. Observe that φ and $f \circ j\colon \overline{C} \to Y$ are homotopic to each other, but the first map is holomorphic while the latter is antiholomorphic. This is a contradiction, because now

$$0 < \int_{\overline{C}} \varphi^* \omega = \int_{\overline{C}} (f \circ j)^* \omega < 0$$

for any Kähler form ω on Y. □

6. The h-principle implies specialness for projective manifolds

Theorem 6.1. *Let X be a complex projective manifold. If X satisfies the h-principle, then X is special in the sense of* [Cam04].

Proof. Let $\overline{X}$ denote the underlying real manifold equipped with the opposite complex structure and let $\iota\colon \overline{X} \to X$ denote the antiholomorphic diffeomorphism induced by the identity map of this underlying real manifold. Recall that $\overline{X}$ is also projective. Hence we can find a Stein manifold M together with a holomorphic fibre bundle $\tau\colon M \to \overline{X}$ with some $\mathbb{C}^k$ as fibre (Proposition 3.1).

Let $\sigma\colon \overline{X} \to M$ denote a smooth (real-analytic, for example) section (whose existence is guaranteed by Proposition 3.1). Since we have assumed that X satisfies the h-principle, there must exist a holomorphic map $h\colon M \to X$ homotopic to $\iota \circ \tau$. Define $\zeta := h \circ \sigma\colon \overline{X} \to X$. Thus ζ is homotopic to id_X.

Let $c\colon X \dashrightarrow C$ be the core map of X. We assume that X is not special, that is, that $d := \dim(C) > 0$. Let $n = \dim(X)$. We claim that $c \circ \zeta\colon \overline{X} \dashrightarrow C$ is non-degenerate, and thus, that so is $g := c \circ h\colon M \to C(X)$.

Let indeed ω_C and ω_X be Kähler forms on C and on X, respectively, and let $d := \dim(C)$. Then $I := \int_X \omega_X^{n-d} \wedge c^*(\omega_C^d) > 0$. By Lemma 4.2, we have $I' := \int_{\overline{X}} \zeta^*(\omega_X^{n-d} \wedge c^*(\omega_C^d)) = (-1)^d \cdot I \neq 0$. This implies $(c \circ \zeta)^*(\omega_C^d) \neq 0$, and thus that $c \circ \zeta$ is not of measure zero. By Sard's theorem, this implies that $c \circ \zeta$ is non-degenerate, and therefore so is $c \circ h$.

We consider the meromorphic map $c \circ h := g\colon M \dashrightarrow C$. By Theorem 2.10, it follows that we obtain an induced meromorphic map $\varphi\colon \overline{X} \dashrightarrow C$ such that $\varphi \circ \tau = g$, and thus such that $\varphi = \varphi \circ \tau \circ \sigma = c \circ h \circ \sigma = c \circ \zeta$.

We now consider the integral $J = \int_X \omega_X^{n-1} \wedge c_X^*(\omega_C)$. We have $J > 0$, giving a contradiction by Corollary 4.3. Hence X cannot satisfy the h-principle unless $\dim(C) = 0$, that is, unless X is special. □

A consequence of Theorem 6.1 and Conjecture 2.7 is the following homotopy restriction for the h-principle to hold.

Conjecture 6.2. If X is complex projective manifold satisfying the h-principle, then $\pi_1(X)$ is almost abelian.

Notice that this conjecture is true if $\pi_1(X)$ has a faithful linear representation in some $\mathrm{GL}(N, \mathbb{C})$ or is solvable, by [Cam11a] and [Cam11b], respectively.

The result above on projective manifolds raises the following questions.

Question 6.3. (1) Are compact Kähler manifolds satisfying the h-principle special? This is true, at least, for compact Kähler surfaces (see Theorem 6.4 and its corollary below).

(2) Let X be a quasi-projective manifold satisfying the h-principle. Assume that X is not homotopy-equivalent to any proper subvariety $Z \subset X$. Does it follow that X is special?

We have some partial results towards answering these questions.

Theorem 6.4. *Let X be a compact Kähler manifold satisfying the h-principle. Then the Albanese map of X is surjective.*

Proof. The proof of Theorem 7.1 applies. □

Corollary 6.5. *Let X be a compact Kähler surface satisfying the h-principle. Then X is special.*

Proof. Assume that it is not. Then X is in particular not weakly special. Since X is not of general type, by Theorem 6.1, there exist a finite étale cover $\pi'\colon X' \to X$ and a surjective holomorphic map $f'\colon X' \to C$ onto a curve C of general type. Because X' also satisfies the h-principle, by Lemma 6.6 below, this contradicts Theorem 6.4 □

Lemma 6.6. *Let $\pi'\colon X' \to X$ be an unramified covering between complex spaces. If X satisfies the h-principle, so does X'.*

Proof. Let $f\colon S \to X'$ be a continuous map from a Stein space S. By assumption, there is a holomorphic map $g\colon S \to X$ homotopic to $\pi \circ f$. The homotopy lifting property for coverings implies that g can be lifted to a holomorphic map $G\colon S \to X'$ which is homotopic to f. □

7. Necessary conditions on the quasi-Albanese map

We give two necessary conditions bearing on the quasi-Albanese map for a quasi-projective manifold X to satisfy the h-principle. These conditions are necessary for X to be special.

Theorem 7.1. *Let X be a complex quasi projective manifold for which the quasi-Albanese map is not dominant. Then X does not satisfy the h-principle.*

Proof. Let A be the quasi-Albanese variety of X, and let Z denote the closure of the image of X under the quasi-Albanese map $a\colon X \to A$. We may assume $e_A \in Z$. By a generalization of a theorem of Kawamata (see [Kaw80] and [NW14, Theorem 5.6.19]), there are finitely many sub-semitori $T_i \subset A$ and T_i-orbits $S_i \subset A$ such that $S_i \subset Z$ and such that every translated sub-semitorus of A which is contained in Z must already be contained in one of the S_i. By Lemma 7.2 below, there is an element $\gamma_0 \in \pi_1(A)$ which is not contained in any of the $\pi_1(S_i)$. By the functoriality properties of the quasi-Albanese map, the group homomorphism $\pi_1(X) \to \pi_1(A)$ is surjective. Thus we can lift γ_0 to an element $\gamma \in \pi_1(X)$. Let us now assume that the h-principle holds. In this case there must exist a holomorphic map f from $\mathbb{C}^*$ to X inducing γ. By composition we obtain a holomorphic map

$$F = a \circ f \circ \exp\colon \mathbb{C} \longrightarrow Z \subset A\,.$$

Now Noguchi's logarithmic version of the theorem of Bloch–Ochiai ([NW14, Theorem 4.8.17]) implies that the analytic Zariski closure of $F(\mathbb{C})$ in Z is a translated sub-semitorus of A. Therefore $F(\mathbb{C})$ must be contained in one of the S_i. But this implies

$$(a \circ f)_* \left(\pi_1(\mathbb{C}^*)\right) \subset \pi_1(S_i)\,,$$

which contradicts our choice of γ. □

Lemma 7.2. *Let $\Gamma_1, \dots, \Gamma_k$ be a family of subgroups of $G = \mathbb{Z}^n$ with $\mathrm{rank}_{\mathbb{Z}}\Gamma_i < n$. Then $\cup_i \Gamma_i \neq G$.*

Proof. For a subgroup $H \subset G \subset \mathbb{R}^n$, we denote by $N(H,r)$ the number of elements $x \in H$ with $||x|| \leqslant r$. Then $N(H,r) = O(r^d)$ if d is the rank of the $\mathbb{Z}$-module H. Now $N(\Gamma_i, r) = O(r^{n-1})$, but $N(G,r) = O(r^n)$. This implies the statement. □

Theorem 7.1 has the following consequence.

Corollary 7.3. *Let X be an algebraic variety which admits a surjective morphism F onto an algebraic curve C. If C is hyperbolic, then X does not satisfy the h-principle.*

Proof. Let A and J denote the quasi-Albanese varieties of X and C, respectively. By the functoriality of the quasi-Albanese, we have an induced map $F_1\colon A \to J$. Now $F_1(A)$ is a semi-abelian subvariety of J, the variety J is generated as an algebraic group by the image of C in J, and $F_1\colon X \to C$ is surjective. It follows that $F_1\colon A \to J$ is surjective.

Next, observe that $\dim(J) > \dim(C)$ due to the hyperbolicity of C. Therefore the induced map from X to J is not dominant. As a consequence, the quasi-Albanese map $X \to A$ is likewise not dominant. □

By similar reasoning, using [NWY07], we obtain the following result.

Proposition 7.4. *Let X be a quasi-projective manifold which admits a finite map onto a semi-abelian variety. Then X satisfies the h-principle only if X is a semi-abelian variety.*

8. (Counter-)examples

We now present examples showing that the desired implications "special $\implies$ h-principle" and "$\mathbb{C}$-connected $\implies$ special" certainly do not hold without imposing some normality and algebraicity, Kählerness, or completeness conditions on the manifold in question.

Example 8.1. There is a non-normal projective curve X which is rational and $\mathbb{C}$-connected, but does not satisfy the h-principle.

We start with $\hat{X} = \mathbb{P}_1$ and define X by identifying 0 and ∞ in $\hat{X} = \mathbb{C} \cup \{\infty\}$. Via the map $[x_0 : x_1] \mapsto [x_0^3 + x_1^3 : x_0^2 x_1 : x_0 x_1^2]$ the quotient space X can be realized as

$$X \simeq \{[z_0 : z_1 : z_2] : z_0 z_1 z_2 = z_1^3 + z_2^3\}.$$

Let $\tilde{X}$ denote the universal covering of X. Then $\tilde{X}$ consists of countably infinitely many 2-spheres glued together. By Hurewitz's theorem, $\pi_2(\tilde{X}) \simeq H_2(\tilde{X}, \mathbb{Z}) \simeq \mathbb{Z}^\infty$. The long homotopy sequence associated with the covering map implies $\pi_2(X) \simeq \mathbb{Z}^\infty$. As a consequence, the group homomorphism

$$\mathbb{Z} \simeq \pi_2(\hat{X}) \longrightarrow \pi_2(X) \simeq \mathbb{Z}^\infty$$

induced by the natural projection $\pi\colon \hat{X} \to X$ is not surjective. Now, let Q denote the two-dimensional affine quadric. Note that Q is a Stein manifold which is homotopic to the 2-sphere. Because $\pi_2(\hat{X}) \to \pi_2(X)$ is not surjective, there exists a continuous map $f\colon Q \to X$ which cannot be lifted to a continuous map from Q to $\hat{X}$. On the other hand, every holomorphic map from the complex manifold Q to X can be lifted to $\hat{X}$, because $\hat{X}$ is the normalization of X. Therefore, there exists a continuous map from Q to X which is not homotopic to any holomorphic map. Thus X does not satisfy the h-principle.

Example 8.2. There are non-Kähler compact surfaces, namely Inoue surfaces, which do not satisfy the h-principle, although they are special.

These Inoue surfaces are compact complex surfaces of algebraic dimension zero with $\Delta \times \mathbb{C}$ as universal covering and foliated by complex lines. They are special in the sense of Definition 2.1, because since their algebraic dimension is zero, there are no Bogomolov sheaves. On the other hand, the image of any holomorphic map from $\mathbb{C}^*$ to such a surface is contained in one of those leaves. This implies that there are many group homomorphisms from $\mathbb{Z}$ to the fundamental group of the surface which are not induced by holomorphic maps from $\mathbb{C}^*$. For this reason Inoue surfaces do not satisfy the h-principle.

Example 8.3. There is a non-compact complex manifold which is $\mathbb{C}$-connected, but does not satisfy the h-principle.

Due to Rosay and Rudin ([RR88, Theorem 4.5]) there exists a discrete subset $S \subset \mathbb{C}^2$ such that $F(\mathbb{C}^2) \cap S \neq \emptyset$ for any non-degenerate holomorphic map $F\colon \mathbb{C}^2 \to \mathbb{C}^2$. (Here F is called non-degenerate if and only if there is a point p with $\operatorname{rank}(DF)_p = 2$.) Let $X = \mathbb{C}^2 \setminus S$. Due to the discreteness of S, it is easy to show that X is $\mathbb{C}$-connected. Now let $G = \mathrm{SL}_2(\mathbb{C})$. Then G is a Stein manifold which is homotopic to S^3. Let $p \in \mathrm{SL}_2(\mathbb{C})$ and let $v, w \in T_pG$. Using the exponential map, there is a holomorphic map from $\mathbb{C}^2$ to G for which v and w are in the image. From this it follows easily that for every holomorphic map $F\colon G \to X$ and every $p \in G$, we have $\operatorname{rank}(DF)_p \leqslant 1$. Hence $F^*\omega \equiv 0$ for every threeform ω on X and every holomorphic map $F\colon G \to X$. This implies that for every holomorphic map $F\colon G \to X$, the induced map $F^*\colon H^3(X, \mathbb{R}) \to H^3(G, \mathbb{R})$ is trivial. On the other hand, there are continuous maps $f\colon S^3 \to X$ for which $f^*\colon H^3(X, \mathbb{C}) \to H^3(S^3, \mathbb{C})$ is non-zero. Namely, choose $p \in S$. Since S is countable, there is a number $r > 0$ such that $||p - q|| \neq r$ for all $q \in S$. Then $f\colon v \mapsto p + rv$ defines a continuous map from $S^3 = \{v \in \mathbb{C}^2 |v|| = 1\}$ to X which induces a non-zero homomorphism $f^*\colon H^3(X, \mathbb{C}) \to H^3(S^3, \mathbb{C})$.

As a consequence, X does not satisfy the h-principle.

9. Does special imply the h-principle?

We consider the question: if X is projective, smooth, and special, does it satisfy the h-principle? The question is very much open, even in dimension two.

For projective curves, we know that the h-principle is satisfied if and only if X is special.

The projective surfaces known to satisfy the h-principle are the following ones: rational surfaces, minimal surfaces ruled over an elliptic curve, blown-up abelian surfaces and their étale undercovers, termed bielliptic.

This means that the special projective surfaces not known to satisfy the h-principle are, on the one hand, blown-up K3 and Enriques surfaces, and, on the other hand, the blown-ups of surfaces with $\kappa = 1$, which are either

(i) elliptic fibrations over an elliptic base without multiple fibre, or

(ii) elliptic fibrations over a rational base with at most four multiple fibres, where the sum of the inverses of the multiplicities is at least two (respectively, one) if there are four (respectively, three) multiple fibres.

In higher dimension (even three), essentially nothing is known. In particular, the cases of Fano, rationally connected, and even rational manifolds (for example, $\mathbb{P}^3$ blown up along a smooth curve of degree three or more) are open.

For n-dimensional Fano or rationally connected manifolds with $n \geqslant 3$, even the existence of a non-degenerate meromorphic map from $\mathbb{C}^n$ to X is open. Non-existence would contradict the Oka property (see the definition below). If such a map exists, nothing is known about the unirationality of X (see [Uen75, Cam04], for example).

Let us first remark that the h-principle is not known to be preserved by many standard geometric operations preserving specialness. In particular, this concerns

(i) smooth blow-ups and blow-downs,

(ii) products,

(iii) (finite) étale coverings, for which only one direction is known (cf. [For11]).

Except for trivial cases it is very hard to verify the h-principle directly. The most important method for verifying the h-principle is Gromov's theorem that the h-principle is satisfied by elliptic manifolds. In the terminology of Gromov ellipticity means the existence of a holomorphic vector bundle $p\colon E \to X$ with zero section $z\colon X \to E$ and a holomorphic map $s\colon E \to X$ such that $s \circ z\colon X \to X$ is the identity map, and the derivative $ds\colon E \to TX$ is surjective along $z(E)$, where $E \subset TE$ is the kernel of the derivative $dp\colon TE \to TX$ along $z(X) \subset E$.

Homogeneous complex manifolds (for example $\mathbb{P}_n$, Grassmannians, tori) are examples of elliptic manifolds. Complements $\mathbb{C}^n \setminus A$ of algebraic subvarieties A of codimension at least two are also known to be elliptic.

For a complex manifold X, being elliptic also implies that X is Oka; that is, every holomorphic map $h\colon K \to X$ on a compact convex subset K of $\mathbb{C}^n$ can be uniformly approximated to any precision by holomorphic maps $H\colon \mathbb{C}^n \to X$. Forstneric's theorems ([For11]) show that Oka manifolds satisfy stronger approximation properties. All known examples of Oka manifolds are subelliptic, a slight weakening of ellipticity. We refer to [Gro89, For11, FL11] for more details and generalisations of these statements. See also [Lár04] for an interpretation of the Oka property in terms of model structures.

We thus have the following sequence of implications (the first two are always valid, the last for projective manifolds; see [For11]):

$$\text{elliptic} \Rightarrow \text{Oka} \Rightarrow h\text{-principle} \Rightarrow \text{special}$$

Although the notions of Oka and of satisfying the h-principle differ in general (for example, the unit disc is evidently not Oka, but satisfies the h-principle because it is contractible), one may ask the following question.

Question 9.1. Is any projective manifold satisfying the h-principle Oka?

Acknowledgements

We thank Finnur Làrusson for useful comments on an initial version of the present text and the referee for his suggestions to improve and restructure our first version.

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