



Counting invariant curves: A theory of Gopakumar–Vafa invariants for Calabi–Yau threefolds with an involution

Jim Bryan and Stephen Pietromonaco

ABSTRACT

We develop a theory of Gopakumar–Vafa (GV) invariants for a Calabi–Yau threefold (CY3) X which is equipped with an involution ι preserving the holomorphic volume form. We define integers $n_{g,h}(\beta)$ which give a virtual count of the number of genus g curves C on X , in the class $\beta \in H_2(X)$, which are invariant under ι , and whose quotient C/ι has genus h . We give two definitions of $n_{g,h}(\beta)$ which we conjecture to be equivalent: one in terms of a version of Pandharipande–Thomas theory and one in terms of a version of Maulik–Toda theory. We compute our invariants and give evidence for our conjecture in several cases. In particular, we compute our invariants when $X = S \times \mathbb{C}$, where S is an Abelian surface with $\iota(a) = -a$ or a $K3$ surface with a symplectic involution (a Nikulin $K3$ surface). For these cases, we give formulas for our invariants in terms of Jacobi modular forms.

1. Ordinary GV invariants

Let X be a Calabi–Yau threefold (CY3), by which we mean a smooth quasi-projective variety over $\mathbb{C}$ of dimension 3 with $K_X \cong \mathcal{O}_X$. In 1998 [GV98], Gopakumar and Vafa (GV) defined via physics integer invariants $n_g(\beta)$ which give a virtual count of curves $C \subset X$ of genus g and class $[C] \in H_2(X)$.

Mathematically, there are two conjecturally equivalent sheaf-theoretic approaches to defining $n_g(\beta)$, one by Pandharipande and Thomas (PT) via their stable pair invariants [PT10], and one more recently given by Maulik and Toda (MT) using perverse sheaves [MT018]. We begin by reviewing ordinary GV theory, and then we develop in a parallel fashion a theory of GV invariants for CY3s with an involution.

1.1 GV invariants via PT theory

Let $\text{PT}_{\beta,n}(X)$ be the moduli space of PT pairs [PT09]:

$$\text{PT}_{\beta,n}(X) = \{(F, s) : s \in H^0(X, F), [\text{supp}(F)] = \beta, \chi(F) = n\},$$

Received 28 March 2022, accepted in final form 17 April 2023.

2020 Mathematics Subject Classification 14J28, 14J32, 14N35, 11F03, 11F50.

Keywords: enumerative geometry, Gopakumar–Vafa invariants, Pandharipande–Thomas invariants, modular forms, Calabi–Yau threefolds, orbifolds, $K3$ surfaces.

This journal is © Foundation Compositio Mathematica 2024. This article is distributed with Open Access under the terms of the Creative Commons Attribution Non-Commercial License, which permits non-commercial reuse, distribution, and reproduction in any medium, provided that the original work is properly cited. For commercial re-use, please contact the Foundation Compositio Mathematica.

where F is a coherent sheaf on X with proper support of pure dimension 1 and $\text{coker}(s)$ has support of dimension zero.

For any scheme S over $\mathbb{C}$, Behrend [Beh09] defined a constructible function $\nu_S: S \rightarrow \mathbb{Z}$, and we define the *virtual Euler characteristic* to be the Behrend function weighted topological Euler characteristic:

$$e_{\text{vir}}(S) = e(S, \nu_S) = \sum_{k \in \mathbb{Z}} k \cdot e(\nu_S^{-1}(k)).$$

We then define the *PT invariants* by

$$N_{\beta, n}^{\text{PT}}(X) = e_{\text{vir}}(\text{PT}_{\beta, n}(X))$$

and the *PT partition function* by

$$Z^{\text{PT}}(X) = \sum_{\beta, n} N_{\beta, n}^{\text{PT}}(X) Q^\beta y^n.$$

DEFINITION 1.1. The Gopakumar–Vafa invariants (via PT theory) $n_g^{\text{PT}}(\beta)$ are defined via the following equation:

$$\log Z^{\text{PT}}(X) = \sum_{k > 0} \sum_{\beta, g} \frac{1}{k} \cdot Q^{k\beta} \cdot n_g^{\text{PT}}(\beta) \cdot \psi_{-(-y)^k}^{g-1}, \quad (1.1)$$

where

$$\psi_x = 2 + x + x^{-1}.$$

Remark 1.2. Writing $\log Z^{\text{PT}}(X)$ in the form given by the right-hand side of (1.1) uses the fact that the coefficient of Q^β in $Z^{\text{PT}}(X)$ is the Laurent expansion of a rational function in y which is invariant under $y \leftrightarrow y^{-1}$; see [PT09, Bri11, Tod10].¹ Although it is not clear from the definition, one expects that $n_g^{\text{PT}}(\beta) = 0$ if $g < 0$.

Remark 1.3. Gopakumar and Vafa gave a formula relating their invariants to the Gromov–Witten (GW) invariants. Equation (1.1) is equivalent to the Gopakumar–Vafa formula, where we use the expected relationship between PT and GW invariants.

1.2 GV invariants via MT theory

We define the moduli space of Maulik–Toda (MT) sheaves to be

$$\mathbf{M}_\beta(X) = \{F: [\text{supp}(F)] = \beta, \chi(F) = 1\},$$

where F is a coherent sheaf on X with proper sheaf-theoretic support of pure dimension 1 and where F is *Simpson stable*, which in this case is equivalent to the condition that if $F' \subsetneq F$, then $\chi(F') \leq 0$.

The MT moduli space is a quasi-projective scheme, and it has a proper morphism to the Chow variety given by the Hilbert–Chow morphism:

$$\begin{aligned} \pi: \mathbf{M}_\beta(X) &\rightarrow \text{Chow}_\beta(X) \\ [F] &\mapsto \text{supp}(F). \end{aligned}$$

There is a perverse sheaf $\phi^\bullet$ on $\mathbf{M}_\beta(X)$ which is locally given by the perverse sheaf of vanishing cycles associated with the local super-potential (the moduli space is locally the critical locus of

¹The proofs of the $y \leftrightarrow y^{-1}$ invariance in [Bri11, Tod10] require that X is compact, but this is widely expected to hold for non-compact geometries as well and holds in all known cases.

a holomorphic function on a smooth space, the so-called super-potential [JS12, Theorem 5.4]). The construction of $\phi^\bullet$ was done in [BBD⁺15] and requires the choice of “orientation data”: a square root of the virtual canonical line bundle on $M_\beta(X)$. Maulik and Toda conjecture the existence of a canonical choice of orientation data (one that is compatible with the morphism π). Using that choice, the *Maulik–Toda* polynomial is defined as follows:

$$\text{MT}_\beta(y) = \sum_{i \in \mathbb{Z}} \chi({}^p H^i(R^\bullet \pi_* \phi^\bullet)) y^i,$$

where ${}^p H^i(-)$ is the i th cohomology functor with respect to the perverse t -structure. By the self-duality of $\phi^\bullet$ and Verdier duality, $\text{MT}_\beta(y)$ is an integer-coefficient Laurent polynomial in y which is invariant under $y \leftrightarrow y^{-1}$. Noting that $\{\psi_y^g\}_{g \geq 0}$ forms an integral basis for such polynomials, we may write the MT polynomial as follows.

DEFINITION 1.4. The GV invariants (via MT theory) $n_g^{\text{MT}}(\beta)$ are defined by the equation

$$\text{MT}_\beta(y) = \sum_{g \geq 0} n_g^{\text{MT}}(\beta) \psi_y^g.$$

The main conjecture of Maulik and Toda is the following.

CONJECTURE 1.5. We have $n_g^{\text{MT}}(\beta) = n_g^{\text{PT}}(\beta)$.

Remark 1.6. Compared to the definition via PT theory, the above definition of GV invariants is more directly tied to the geometry of curves in the class β and more closely matches the original physics definition. In particular, the invariants $n_g^{\text{MT}}(\beta)$ only involve the single moduli space $M_\beta(X)$. In contrast, the invariants $n_g^{\text{PT}}(\beta)$ involve a subtle combination of the PT invariants associated with an infinite number of moduli spaces, namely the spaces $\text{PT}_{\beta', n}(X)$, where $\beta = k\beta'$ and n is unbounded from above. From this perspective, the conjectural formula $n_g^{\text{MT}}(\beta) = n_g^{\text{PT}}(\beta)$ can be viewed as a kind of multiple cover formula for PT invariants.

1.3 The local $K3$ surface: The Katz–Klemm–Vafa (KKV) formula

Suppose that S is a $K3$ surface and $\beta_d \in H_2(S)$ is a curve class with $\beta_d^2 = 2d$. The CY3 $X = S \times \mathbb{C}$ is sometimes called the local $K3$ surface. In this case, Conjecture 1.5 holds, and $n_g^{\text{MT}}(\beta_d) = n_g^{\text{PT}}(\beta_d)$ only depends on d and g (and not the divisibility of β_d). The invariants $n_g^{\text{PT}}(\beta_d)$ and $n_g^{\text{MT}}(\beta_d)$ were computed (in full generality) by Pandharipande and Thomas [PT16] and Shen and Yin [SY22, Theorem 0.5], respectively.² The MT polynomials are given by the famous KKV formula first conjectured by Katz, Klemm, and Vafa [KKV99]:

$$\sum_{d=-1}^{\infty} \text{MT}_{\beta_d}(y) q^d = -q^{-1} \prod_{n=1}^{\infty} (1 - q^n)^{-20} (1 + y^{-1} q^n)^{-2} (1 + y q^n)^{-2}. \quad (1.2)$$

The right-hand side can also be written as $\psi_y \cdot \phi_{10,1}(q, -y)^{-1}$, where $\phi_{10,1}(q, y)$ is the Fourier expansion of the unique Jacobi cusp form of weight 10 and index 1.

The fact that $n_g(\beta_d)$ is independent of the divisibility of β_d is a deep and unusual feature of the local $K3$ geometry.

²A subtlety here is that PT invariants for this non-compact geometry $S \times \mathbb{C}$ can be defined via $\mathbb{C}^*$ virtual localization (as in [PT16]) or by the Behrend function weighted Euler characteristic as we have done. The equivalence of these two definitions was shown in [MTh18].

2. GV invariants for CY3s with an involution

Let X be a CY3 equipped with an involution

$$\iota: X \rightarrow X$$

such that the induced action on K_X is trivial. The purpose of this article is to develop a theory of GV invariants which count ι -invariant curves on X . Namely, we seek to define integers $n_{g,h}(\beta)$ which give a virtual count of genus g , ι -invariant curves $C \subset X$ with $[C] = \beta \in H_2(X)$ and such that the genus of the quotient C/ι is h .

We develop this theory parallel to the presentation of the ordinary GV invariants given in Section 1. Namely, we define invariants $n_{g,h}^{\text{PT}}(\beta)$ and $n_{g,h}^{\text{MT}}(\beta)$ in terms of a version of PT and MT theory, respectively, and we conjecture that they are equal.

We do not currently know how to include curves that are *fixed* by ι (as opposed to merely invariant), and so we make the following general assumption.

ASSUMPTION 2.1. Throughout, we assume that the curve classes $\beta \in H_2(X)$ that we consider do not admit an effective decomposition $\beta = \sum_i d_i C_i$ containing an ι -fixed curve C_i .

2.1 ι -GV invariants via PT theory

We denote by R_+ and R_- the trivial and the non-trivial irreducible representation of the group of order 2, and we let $R_{\text{reg}} = R_+ \oplus R_-$ denote the regular representation.

Recall that a sheaf F on X is ι -invariant if $\iota^*F \cong F$ and that an ι -equivariant sheaf is an ι -invariant sheaf F along with a choice of a lift of ι to an isomorphism $\tilde{\iota}: \iota^*F \rightarrow F$ satisfying $\tilde{\iota} \circ \iota^*(\tilde{\iota}) = \text{id}$.

If F is an ι -equivariant sheaf, then

$$\chi(F) = \sum_k (-1)^k H^k(X, F)$$

is naturally a virtual representation and so can be written in the form

$$\chi(F) = nR_{\text{reg}} + \epsilon R_-$$

for some $n, \epsilon \in \mathbb{Z}$.

We define the space of ι -equivariant PT pairs

$$\text{PT}_{\beta,n,\epsilon}(X, \iota) = \{(F, s): s \in H^0(X, F), [\text{supp}(F)] = \beta, \chi(F) = nR_{\text{reg}} + \epsilon R_-\},$$

where (F, s) is a PT pair such that F is an ι -equivariant sheaf and s is an equivariant section. We note that

$$\text{PT}_{\beta,d}(X)^{\iota} = \bigsqcup_{2n+\epsilon=d} \text{PT}_{\beta,n,\epsilon}(X, \iota) \quad (2.1)$$

since the points of the ι -fixed locus $\text{PT}_{\beta,n}(X)^{\iota}$ corresponds to ι -invariant PT pairs, but each ι -invariant PT pair has a unique ι -equivariant structure making the section equivariant.

DEFINITION 2.2. We define the ι -PT invariants and the ι -PT partition function by

$$\begin{aligned} N_{\beta,n,\epsilon}^{\text{PT}}(X, \iota) &= e_{\text{vir}}(\text{PT}_{\beta,n,\epsilon}(X, \iota)), \\ Z^{\text{PT}}(X, \iota) &= \sum_{\beta,n,\epsilon} N_{\beta,n,\epsilon}^{\text{PT}}(X, \iota) Q^{\beta} y^n w^{\epsilon}, \end{aligned}$$

where in the sum, β is summed over the semi-group of classes satisfying Assumption 2.1.

These invariants are not new: they can be recovered from the orbifold PT invariants of the stack quotient $[X/\iota]$ studied for example in [BCR22]. It follows from [BCR22, Proposition 7.19] that the coefficient of Q^β in $Z^{\text{PT}}(X, \iota)$ is a Laurent expansion of a rational function in y and w which is invariant under $y \leftrightarrow y^{-1}$ and $w \leftrightarrow w^{-1}$. This allows us to make the following definition.

DEFINITION 2.3. The ι -GV invariants (via PT theory) $n_{g,h}^{\text{PT}}(\beta)$ (for classes β satisfying Assumption 2.1) are defined by the formula

$$\log(Z^{\text{PT}}(X, \iota)) = \sum_{k>0} \sum_{\beta, g, h} \frac{1}{k} Q^{k\beta} \cdot n_{g,h}^{\text{PT}}(\beta) \cdot \psi_{-(-y)^k}^{h-1} \cdot \psi_{w^k}^{g+1-2h},$$

where as before $\psi_x = 2 + x + x^{-1}$.

Remark 2.4. The number $g+1-2h$ is half the number of fixed points on a smooth genus g curve with an involution having a quotient of genus h .

Remark 2.5. Although it is not apparent from this definition, we expect $n_{g,h}^{\text{PT}}(\beta)$ to have good finiteness properties, namely that for fixed β , we expect $n_{g,h}^{\text{PT}}(\beta)$ to be non-zero for only a finite number of values of (g, h) and that those values should have $h \geq 0$ and $g \geq -1$. The possible occurrence of non-zero counts for $g = -1$ is due to (for example) the possibility of invariant curves $C = C_1 \cup C_2$ consisting of a disjoint ι -orbit of rational curves. Such a curve should be interpreted as having genus -1 via the formula $\chi(\mathcal{O}_C) = 1 - g$.

2.2 ι -GV invariants via MT theory

We define the moduli space of ι -MT sheaves to be

$$\mathbf{M}_\beta^\epsilon(X, \iota) = \{F: [\text{supp}(F)] = \beta, \chi(F) = R_{\text{reg}} + \epsilon R_-\}, \quad (2.2)$$

where F is an ι -equivariant coherent sheaf on X with proper sheaf-theoretic support of pure dimension 1 and where F is ι -stable.

DEFINITION 2.6. We say that an ι -equivariant sheaf F on X of pure dimension 1 with $\chi(F) = R_{\text{reg}} + \epsilon R_-$ is ι -stable if all ι -equivariant subsheaves $F' \subsetneq F$ with $\chi(F') = kR_{\text{reg}} + \gamma R_-$ satisfy $k \leq 1$ and if when $k = 1$, we have $\gamma < \epsilon$ and $[\text{supp}(F')] = [\text{supp}(F)] \in H_2(X)$.

Remark 2.7. We will show in Proposition 5.5 that ι -stability can be reformulated in terms of Nironi stability for the corresponding sheaf on the stack quotient $[X/\iota]$. A consequence is that $\mathbf{M}_\beta^\epsilon(X, \iota)$ is a scheme and it is proper over $\text{Chow}_\beta(X)$.

Let

$$\pi^\epsilon: \mathbf{M}_\beta^\epsilon(X, \iota) \rightarrow \text{Chow}_\beta(X) \quad (2.3)$$

be the Hilbert–Chow morphism. Since $\mathbf{M}_\beta^\epsilon(X, \iota)$ parameterizes objects in the CY3 category of ι -equivariant coherent sheaves on X , there exists a perverse sheaf of vanishing cycles³ $\phi^\bullet$ on $\mathbf{M}_\beta^\epsilon(X, \iota)$, and we can define the ι -MT polynomial in a fashion analogous to the ordinary MT polynomial:

$$\text{MT}_\beta(y, w) = \sum_{i, \epsilon \in \mathbb{Z}} \chi({}^p H^i(R^\bullet \pi_*^\epsilon \phi^\bullet)) y^i w^\epsilon. \quad (2.4)$$

As before, self-duality and Verdier duality imply that $\text{MT}_\beta(y, w)$ is a Laurent polynomial in y invariant under $y \leftrightarrow y^{-1}$. We conjecture that in general $\text{MT}_\beta(y, w)$ is also a Laurent invariant

³As in [MT018, Definition 2.7], we assume that our orientation is *strictly Calabi–Yau*.

polynomial in w invariant under $w \leftrightarrow w^{-1}$. Assuming this conjecture, we can write $\text{MT}_\beta(y, w)$ as a polynomial in ψ_y and ψ_w and make the following definition.

DEFINITION 2.8. The ι -GV invariants (via MT theory) $n_{g,h}^{\text{MT}}(\beta)$ (for classes satisfying Assumption 2.1) are defined by the formula

$$\text{MT}_\beta(y, w) = \sum_{g,h} n_{g,h}^{\text{MT}}(\beta) \psi_y^h \psi_w^{g+1-2h}.$$

Our main conjecture is that our two definitions of ι -GV invariants are equivalent.

CONJECTURE 2.9. We have $n_{g,h}^{\text{PT}}(\beta) = n_{g,h}^{\text{MT}}(\beta)$.

2.3 Examples: Local Abelian surfaces and local Nikulin $K3$ surfaces

One of the main results of this paper are various ι -equivariant versions of the KKV formula. Namely, we compute our invariants and prove our conjecture for the case $X = S \times \mathbb{C}$, where S is either an Abelian or a $K3$ surface and where ι acts trivially on $\mathbb{C}$ and symplectically on S .

For the case of an Abelian surface, the involution is the natural one arising from the group structure: $\iota(a) = -a$. A $K3$ surface equipped with a symplectic involution is called a Nikulin surface, and there are two distinct deformation types, which we call Type (I) and Type (II) (see Definition 4.12).

As with the ordinary GV invariants of a local $K3$ surface, our invariants admit a surprising lack of dependency on the divisibility of the curve class, and they are given by formulas involving Jacobi modular forms.

THEOREM 2.10. Let $X = S \times \mathbb{C}$, where S is an Abelian surface or a Nikulin $K3$ surface, and let $\beta \in H_2(S)$ be an effective invariant curve class with $\beta^2 = 2d$. Then:

- (1) If S is a Type (II) Nikulin surface, $n_{g,h}^{\text{PT}}(\beta)$ only depends on (g, h, d) . In particular, it does not depend on the divisibility of β .
- (2) If S is an Abelian surface or a Type (I) Nikulin surface, $n_{g,h}^{\text{PT}}(\beta)$ only depends on (g, h, d) as well as the parity of the divisibility of β .

We denote these invariants by $n_{g,h}(d; \text{type})$, where $\text{type} \in \{\mathbf{A}^{\text{ev}}, \mathbf{A}^{\text{odd}}, \mathbf{N}_I^{\text{ev}}, \mathbf{N}_I^{\text{odd}}, \mathbf{N}_{II}\}$ distinguishes the cases in the obvious way. Then the invariants are determined from the formula

$$\sum_{g,h} n_{g,h}(d; \text{type}) \psi_y^{h-1} \psi_w^{g+1-2h} = \left[\frac{\Theta_T(q^2, w)}{\phi_{10,1}(q^2, -y)} \right]_{q^d},$$

where $[\cdots]_{q^d}$ denotes the coefficient of q^d in the expression $[\cdots]$.

Moreover, T is a lattice or shifted lattice depending on the type, $\Theta_T(q^2, w)$ is an explicitly determined Jacobi theta function (see Theorem 4.4), and $\phi_{10,1}(q, y)$ is the unique Jacobi cusp form of weight 10 and index 1. In particular, for types $\mathbf{A}^{\text{odd}}$ and $\mathbf{N}_I^{\text{odd}}$, we get infinite product formulas:

$$\begin{aligned} \sum_{g,h,d} n_{g,h}(d; \mathbf{A}^{\text{odd}}) \psi_y^h \psi_w^{g-1-2h} q^d &= -4 \prod_{n=1}^{\infty} \frac{(1+q^n)^8 (1+wq^n)^4 (1+w^{-1}q^n)^4}{(1-q^{2n})^4 (1+yq^{2n})^2 (1+y^{-1}q^{2n})^2}, \\ \sum_{g,h,d} n_{g,h}(d; \mathbf{N}_I^{\text{odd}}) \psi_y^h \psi_w^{g-2h} q^{d+1} &= - \prod_{n=1}^{\infty} \frac{(1+q^n)^4 (1+wq^n)^2 (1+w^{-1}q^n)^2}{(1-q^{2n})^{12} (1+yq^{2n})^2 (1+y^{-1}q^{2n})^2}. \end{aligned}$$

Remark 2.11. The specialization of the invariants $n_{g,h}(\beta)$ to $h = 0$ count ι -invariant hyperelliptic curves. The problem of counting the number of genus g hyperelliptic curves in a primitive class β_d on an Abelian surface A was first considered by Rose [Ros14] and then solved by Bryan–Oberdieck–Pandharipande–Yin [BOPY18]. We may specialize our invariants $n_{g,h}^{\text{PT}}(d; \mathbf{A}^{\text{odd}})$ to $h = 0$ by setting $y = -1$. The above formula then becomes

$$\sum_{d=0}^{\infty} \sum_{g>0} n_{g,0}(d; \mathbf{A}^{\text{odd}}) \psi_w^{g-1} q^d = -4 \prod_{n=1}^{\infty} \frac{(1 + wq^n)^4 (1 + w^{-1}q^n)^4}{(1 - q^n)^8}.$$

We note that the invariant $n_{g,0}(d; \mathbf{A}^{\text{odd}})$ is equal to $\mathbf{h}_{\mathbf{g}, \mathbf{A}, \beta_d}^{\text{Hilb}}$ in the notation of [BOPY18] and the above formula is equivalent to the equation in [BOPY18, Proposition 4].

Remark 2.12. For the case where β_d is the primitive class on an abelian surface, our invariants $n_{g,h}(d; \mathbf{A}^{\text{odd}})$ are refinements of the invariants $\mathbf{n}_d(h)$ considered in [Pie22]. The relationship is given by

$$\mathbf{n}_d(h) = - \sum_g 4^{g-2h} \cdot n_{g,h}(d, \mathbf{A}^{\text{odd}}).$$

Our main technique to prove Theorem 2.10/Theorem 4.4 is to use the Donaldson–Thomas crepant resolution conjecture (DT-CRC) [BCY12, BCR22] to compute orbifold PT invariants in terms of the crepant resolution which in this case is a local $K3$ surface. We can then apply the KKV formula, making crucial use of the description of the Picard lattice of Kummer $K3$ surfaces and Nikulin resolutions given by Garbagnati–Sarti [GS16, GS08]. This is carried out in Section 4.

In Section 5, we use the derived McKay correspondence to compute the MT versions of our invariants $n_{g,h}^{\text{MT}}(\beta)$ for $X = S \times \mathbb{C}$. The final result is as follows.

THEOREM 2.13. *The two definitions of ι -GV invariants coincide for $X = S \times \mathbb{C}$ for S an Abelian surface or a Nikulin $K3$ surface:*

$$n_{g,h}^{\text{PT}}(\beta) = n_{g,h}^{\text{MT}}(\beta).$$

2.4 Further examples

In Section 6, we give two other examples illustrating our theory and providing evidence of our conjecture. For the case where $X \rightarrow B$ is an elliptically fibered CY3 over a surface B with integral fibers and $\iota: X \rightarrow X$ is the composition of an involution on B and fiberwise multiplication by -1 , we compute both $n_{g,h}^{\text{PT}}(\beta)$ and $n_{g,h}^{\text{MT}}(\beta)$ completely for $\beta = [F]$, the class of the fiber. Let $C \subset B$ be the fixed locus of the involution on the base, and let $S = X|_C$ be the restriction of $X \rightarrow B$ to C . The result is the following.

THEOREM 2.14. *Let (X, ι) be an elliptically fibered CY3 with notation as above. Then for all g and h , we have $n_{g,h}^{\text{PT}}([F]) = n_{g,h}^{\text{MT}}([F])$, and these integers are given by*

$$n_{g,h}([F]) = \begin{cases} -e(C) & \text{if } (g, h) = (1, 0), \\ e(S) & \text{if } (g, h) = (0, 0), \\ 0 & \text{otherwise.} \end{cases}$$

We also consider the case where $X = \text{Tot}(\mathcal{O}(-1) \oplus \mathcal{O}(-1))$ is the conifold, $\iota: X \rightarrow X$ is the Calabi–Yau involution which acts non-trivially on the base, and $C \subset X$ is the zero section. We

use the orbifold topological vertex to compute $n_{g,h}^{\text{PT}}(d[C])$, and we use stability considerations to compute $n_{g,h}^{\text{MT}}(d[C])$. The result is as follows.

PROPOSITION 2.15. *For X the conifold with ι as above,*

$$n_{g,h}^{\text{PT}}(d[C]) = n_{g,h}^{\text{MT}}(d[C]) = \begin{cases} 1 & \text{if } (g, h, d) = (0, 0, 1), \\ 0 & \text{otherwise.} \end{cases}$$

We remark that despite the simplicity of the answer, the above proposition is the result of a rather involved orbifold topological vertex computation which was done in [BCY12], and it gives a non-trivial instance of our conjecture.

3. Motivating example of an isolated smooth invariant curve

The simplest evidence of our conjecture is the case of a rigid local curve in the primitive class of the curve. Let C be a non-singular curve of genus g . Suppose that there exists an involution $\iota: C \rightarrow C$ with fixed points $C^\iota = \{p_1, \dots, p_{2m}\}$. If h is the genus of the quotient C/ι , then by the Riemann–Hurwitz formula, we have

$$m = g + 1 - 2h.$$

Let N be an ι -equivariant line bundle on C such that $H^0(N) = H^1(N) = 0$. Then

$$X = \text{Tot}(N \oplus K_C N^{-1})$$

is a CY3 with an induced involution (also denoted by ι) acting trivially on K_X . Let $[C]$ be the class of the zero section $C \hookrightarrow X$, which we note is rigid.

The moduli spaces for the class $[C]$ can be determined explicitly.

PROPOSITION 3.1. *The ι -PT and ι -MT moduli spaces in the class $[C]$ are given by*

$$\begin{aligned} \text{PT}_{[C],n,\epsilon}(X, \iota) &= \coprod_{\substack{T \subseteq \{1, \dots, 2m\} \\ |T| = \epsilon + m}} \text{Sym}^{n+h-1}(C/\iota), \\ \text{M}_{[C]}^\epsilon(X, \iota) &= \coprod_{\substack{T \subseteq \{1, \dots, 2m\} \\ |T| = \epsilon + m}} \text{Pic}^h(C/\iota). \end{aligned}$$

As a consequence, we find that there is a single non-zero ι -GV invariant in the class $[C]$.

COROLLARY 3.2. *We have*

$$n_{g',h'}^{\text{PT}}([C]) = n_{g',h'}^{\text{MT}}([C]) = \begin{cases} 1 & \text{if } (g', h') = (g, h), \\ 0 & \text{if } (g', h') \neq (g, h). \end{cases}$$

Remark 3.3. We expect the invariants $n_{g',h'}(d[C])$ to be complicated for $d > 1$ and $g > 0$. For the related case of $X = \text{Tot}(K_C \oplus \mathcal{O})$, Conjecture 1.5 is equivalent to the well-known $P = W$ conjecture for the GL_d Hitchin system on C ; see [MT18, § 9.3]. We expect that for $\iota: X \rightarrow X$, the natural lift of an involution on C , our Conjecture 2.9 should be equivalent to an orbifold version of $P = W$ for the orbifold curve $[C/\iota]$.

To prove Proposition 3.1 and Corollary 3.2, we need the following lemma.

LEMMA 3.4. *Let $L = \mathcal{O}(D)$ be an ι -equivariant line bundle on C admitting an ι -invariant section $\mathcal{O}_C \rightarrow L$ vanishing on an ι -invariant effective divisor D . Then D can be written*

$$D = \sum_j d_j(x_j + \iota(x_j)) + \sum_{i \in T} p_i,$$

where $T \subset \{1, \dots, 2m\}$, $x_j \in C$, and

$$\chi(L) = nR_{\text{reg}} + \epsilon R_-$$

with $n = 1 - h + \sum_j d_j$ and $\epsilon = |T| - m$.

Moreover, the above formula still holds for $L = \mathcal{O}(D)$ with D ι -invariant but not necessarily effective.

Proof. Since the support of D is ι -invariant, it must consist of free orbits and fixed points. Absorbing multiplicities which are not zero or one on the p_i into the first term using $2p_i = p_i + \iota(p_i)$, we see that D may be written in the given form. Assuming that D is effective, we then get the sequence

$$0 \rightarrow \mathcal{O}_C \rightarrow L \rightarrow \mathcal{O}_D(D) \rightarrow 0,$$

and consequently we find

$$\chi(L) = \chi(\mathcal{O}_C) + \left(\sum_j d_j \right) R_{\text{reg}} + \sum_{i \in T} \chi(\mathcal{O}_{p_i}(p_i)).$$

Using a local coordinate about p_i , it is easy to determine that $\chi(\mathcal{O}_{p_i}(p_i)) = R_-$. We also observe that

$$\chi(\mathcal{O}_C) = (1 - h)R_+ + (h - g)R_- = (1 - h)R_{\text{reg}} - mR_-,$$

and the formula for $\chi(L)$ follows. The general case is then obtained by writing $D = D' - D''$ with D' and D'' effective and using the sequence

$$0 \rightarrow \mathcal{O}(D' - D'') \rightarrow \mathcal{O}(D') \rightarrow \mathcal{O}_{D''}(D') \rightarrow 0. \quad \square$$

Proof of Proposition 3.1. An ι -PT pair on X in the class $[C]$ must be supported on C and hence be an ι -equivariant line bundle with a non-zero invariant section $\mathcal{O}_C \rightarrow L$. Such ι -PT pairs are determined up to isomorphism by the associated invariant divisor D , which by Lemma 3.4 is determined by the subset $T \subset \{1, \dots, 2m\}$ and $\sum_j d_j = n + h - 1$ ι -orbits on C or, equivalently, $n + h - 1$ points on C/ι . The first equation in the proposition follows.

For the same reasons, an ι -MT sheaf on X in the class $[C]$ must be an ι -equivariant line bundle $L \rightarrow C$ or, equivalently, a line bundle on the stack quotient $[C/\iota]$. Since $\chi(L) = R_{\text{reg}} + \epsilon R_-$, the line bundle L admits a non-zero ι -invariant section and hence is of the form $L \cong \mathcal{O}_C(D)$, where D is an invariant divisor given as in Lemma 3.4 with $\sum_j d_j = h$.

The Picard group of an orbifold is given in general in [Pop17, §B]. In particular, for $[C/\iota]$, we have

$$\begin{aligned} \text{Pic}([C/\iota]) &\cong \text{Pic}^0(C/\iota) \oplus H_{\text{orb}}^2([C/\iota]) \\ &\cong \text{Pic}^0(C/\iota) \oplus \mathbb{Z} \oplus (\mathbb{Z}/2)^{2m} \cong \text{Pic}(C/\iota) \oplus (\mathbb{Z}/2)^{2m}. \end{aligned}$$

Under the above isomorphism, the line bundle $\mathcal{O}_C(D)$ with D as in the lemma goes to $(\mathcal{O}_{C/\iota}(\sum_j d_j \bar{x}_j), 1_T)$, where $\bar{x}_j \in C/\iota$ is the point corresponding to the orbit $x_j + \iota(x_j)$ and $1_T = (t_1, \dots, t_{2m})$, where $t_i = 1$ if $i \in T$ and 0 otherwise. The second equation of Proposition 3.1 follows.

We remark that for clarity of exposition, we have given the above argument on the level of points in the moduli spaces, but we note that our argument works for flat families of PT pairs and MT sheaves, and hence the above bijections induce the isomorphisms in Proposition 3.1. $\square$

Proof of Corollary 3.2. Since $[C]$ is a primitive class, we have

$$[Z^{\text{PT}}(X, \iota)]_{q[C]} = [\log Z^{\text{PT}}(X, \iota)]_{q[C]} = \sum_{g', h'} n_{g', h'}^{\text{PT}}([C]) \psi_y^{h'-1} \psi_w^{g'+1-2h'}.$$

On the other hand, by Proposition 3.1 and using the fact that the Behrend function is $(-1)^d$ on a smooth scheme of dimension d , we get

$$\begin{aligned} [Z^{\text{PT}}(X, \iota)]_{q[C]} &= \sum_{T \subset \{1, \dots, 2m\}} \sum_n (-1)^{n+h-1} e(\text{Sym}^{n+h-1}(C/\iota)) y^n w^{|T|-m} \\ &= \left(\sum_{k=0}^{2m} \binom{2m}{k} w^{k-m} \right) y^{1-h} \sum_{d=0}^{\infty} e(\text{Sym}^d(C/\iota)) (-y)^d \\ &= \psi_w^m y^{1-h} (1+y)^{2h-2} = \psi_y^{h-1} \psi_w^{g+1-2h}, \end{aligned}$$

where we used Macdonald's formula [Mac62] for the penultimate equality. The formula for $n_{g', h'}^{\text{PT}}([C])$ then follows.

To compute $n_{g', h'}^{\text{MT}}([C])$, we observe that $\text{Chow}_{[C]}(X)$ is a point and that $\mathbf{M}_{[C]}^\epsilon(X, \iota)$ is smooth. Consequently, the Maulik–Toda polynomial is given by the (symmetrized) Poincaré polynomial:

$$\text{MT}_{[C]}(y, w) = \sum_{T \subset \{1, \dots, 2m\}} w^{|T|-m} \tilde{P}_y(\text{Pic}^h(C/\iota)).$$

Since $\text{Pic}^h(C/\iota)$ is an Abelian variety of dimension h , its symmetrized Poincaré polynomial is given by $y^{-h}(1+y)^{2h} = \psi_y^h$. Thus $\text{MT}_{[C]}(y, w) = \psi_y^h \psi_w^{g+1-2h}$, and the formula for $n_{g', h'}^{\text{MT}}([C])$ is proved. $\square$

4. Local Abelian and Nikulin surfaces (PT theory)

4.1 Overview

In this section, we compute $Z^{\text{PT}}(X, \iota)$ and thus determine all the invariants $n_{g, h}^{\text{PT}}(\beta)$ for the case of $X = S \times \mathbb{C}$, where S is an Abelian surface with its symplectic involution $\iota(a) = -a$ or a Nikulin $K3$ surface which by definition comes with a symplectic involution (in both cases, ι acts trivially on the second factor). The main theorem is given by Theorem 4.4.

Our basic tool for computing the ι -PT invariants of $S \times \mathbb{C}$ is the Donaldson–Thomas crepant resolution conjecture (DT-CRC), which was conjectured in [BCY12] and recently proven by Beentjes, Calabrese, and Rennemo in [BCR22]. The idea is the following. Our ι -PT partition function $Z^{\text{PT}}(X, \iota)$ can be written in terms of the orbifold PT partition function $Z^{\text{PT}}([X/\iota])$, and then the DT-CRC asserts that

$$Z^{\text{PT}}([X/\iota]) = \frac{Z^{\text{PT}}(Y)}{Z_{\text{exc}}^{\text{PT}}(Y)}, \quad (4.1)$$

where $Y \rightarrow X/\iota$ is the crepant resolution, $Z^{\text{PT}}(Y)$ is the ordinary PT partition function of Y , and $Z_{\text{exc}}^{\text{PT}}(Y)$ is the partition function for curve classes supported on the exceptional fibers. The variables in the above equality are identified via the Fourier–Mukai isomorphism in numerical

K -theory. In the case of $X = S \times \mathbb{C}$, we let

$$Y = \widehat{S} \times \mathbb{C},$$

where $\widehat{S} \rightarrow S/\iota$ is the minimal resolution. In the case where S is an Abelian surface, $\widehat{S}$ is the associated Kummer $K3$ surface, and in the case where S is a Nikulin $K3$ surface, $\widehat{S}$ is a special kind of $K3$ surface which we call a Nikulin resolution.

We then can compute the right-hand side of (4.1) using the KKV formula (see Section 1.3). Doing this requires an explicit description of the Picard lattice of $\widehat{S}$, which was given by Garbagnati–Sarti [GS16, GS08]. Finally, to complete the computation, we will need some theta function identities which we prove in Section 4.4.

4.2 Using the DT-CRC

To use (4.1), we will need to be explicit with our choice of variables. Let $X = S \times \mathbb{C}$, and let β_d be an effective, ι -invariant, primitive curve class on S with $\beta_d^2 = 2d$ which we identify with the corresponding class on X . To determine our invariants $n_{g,h}^{\text{PT}}(m\beta_d)$, we need to compute the partition function

$$Z^{\text{PT}}(X, \iota) = \sum_{m \geq 0} \sum_{n, \epsilon} N_{m\beta_d, n, \epsilon}^{\text{PT}}(X, \iota) Q^m y^n w^\epsilon.$$

Remark 4.1. Strictly speaking, when the rank of the invariant Picard group $\text{Pic}(S)^\iota$ is greater than 1, the above is a *restricted* partition function: we do not sum over all invariant curve classes but only over the semi-group generated by β_d . This suffices for determining the invariants $n_{g,h}^{\text{PT}}(m\beta_d)$. A few of the statements made in this section require minor adjustments in the case where the invariant Picard rank is greater than 1. Note that for $d \leq 0$, the invariant Picard rank is necessarily greater than 1.

Our partition function $Z^{\text{PT}}(X, \iota)$ can be determined from the PT partition function of the orbifold $[X/\iota]$. By definition (see [BCR22]),

$$Z^{\text{PT}}([X/\iota]) = \sum_{\alpha \in N_{\leq 1}([X/\iota])} N_{\alpha}^{\text{PT}}([X/\iota]) \mathbf{Q}^{\alpha},$$

where the sum ranges over the numerical K -theory of $\text{Coh}_{\leq 1}([X/\iota])$, the category of coherent sheaves on $[X/\iota]$ having proper support of dimension less than or equal to 1.

We need to choose generators for the free $\mathbb{Z}$ -modules

$$N_{\leq 1}([X/\iota]) \cong N_{\leq 1}([S/\iota]) \quad \text{and} \quad N_{\leq 1}(Y) \cong N_{\leq 1}(\widehat{S}) \cong H_0(\widehat{S}) \oplus \text{Pic}(\widehat{S})$$

in a way that is compatible with the Fourier–Mukai isomorphism

$$N_{\leq 1}([S/\iota]) \cong N_{\leq 1}(\widehat{S}).$$

It will be convenient to choose generators over $\mathbb{Q}$. Determining which linear combinations of our generators are integral classes is somewhat subtle and is addressed in Section 4.3. The generators, and their corresponding variables in the partition functions, are given in the following table:

Here $\mathcal{O}_{\text{pt}}$ is the structure sheaf of a generic point on $[S/\iota]$, the sheaf $\mathcal{O}_{x_i} \otimes R_-$ is the structure sheaf of the i th orbifold point x_i equipped with the non-trivial action of its stabilizer group, and

$$\alpha_d = \frac{1}{2} t_* (\text{ch}^{-1}(\beta_d)),$$

Class in $N_{\leq 1}([S/\iota])$	Class in $H_0(\widehat{S}) \oplus \text{Pic}(\widehat{S})$	Variable
$[\mathcal{O}_{\text{pt}}]$	$[\text{pt}]$	y
$[\mathcal{O}_{x_i} \otimes R_-]$	$[E_i]$	w_i
α_d	γ_d	Q

where $t: S \rightarrow [S/\iota]$ and $\text{ch}^{-1}(\beta_d)$ is the class in $N_{\leq 1}(S)$ corresponding to $\beta_d \in \text{Pic}(S)$ under the Chern character isomorphism. The generators of $H_0(\widehat{S})$ and $\text{Pic}(\widehat{S})$ are given by the point class $[\text{pt}]$, the classes of the exceptional divisors E_i , and

$$\gamma_d = c_1(FM(\alpha_d)),$$

the divisor class associated with the image of α_d under the Fourier–Mukai isomorphism.

The fact that the above choices are compatible with the Fourier–Mukai isomorphism uses the well-known fact that the isomorphism takes $\mathcal{O}_{x_i} \otimes R_-$ to $\mathcal{O}_{E_i}(-1)$.

With these variables, (4.1) can be viewed as an equality of formal series⁴ in the variables y, w_i, Q .

LEMMA 4.2. *We have $Z^{\text{PT}}(X, \iota) = Z^{\text{PT}}([X/\iota])|_{w_i=w}$.*

Proof. Our ι -PT pairs on X are equivalent to PT pairs on the stack quotient $[X/\iota]$. However, keeping track of the K -theory class of the sheaf on $[X/\iota]$ is a refinement of the discrete data used for ι -PT pairs. The lemma follows from observing that a sheaf on $[X/\iota]$ in the K -theory class

$$m\alpha_d + n[\mathcal{O}_{\text{pt}}] + \sum_i v_i[\mathcal{O}_{x_i} \otimes R_-]$$

corresponds to an ι -equivariant sheaf F with $[\text{supp}(F)] = m\beta_d$ and

$$\chi(F) = nR_{\text{reg}} + \left(\sum_i v_i \right) R_- . \quad \square$$

Next we define the *exceptional lattice*

$$\Lambda = \oplus_i \mathbb{Z} \langle E_i \rangle \subset \text{Pic}(\widehat{S}),$$

and we define

$$\Gamma_{m,d} = \{v \in \Lambda \otimes \mathbb{Q} : m\gamma_d + v \text{ is a non-zero integral class in } \text{Pic}(\widehat{S})\}.$$

For $v = \sum_i v_i E_i$, we will use the following notation:

$$l(v) = \sum_i v_i, \quad v^2 = -2 \sum_i v_i^2. \quad (4.2)$$

We can write the log of the partition function on Y in terms of the Gopakumar–Vafa invariants of Y using (1.1). We then specialize w_i to w to get

$$\log Z^{\text{PT}}(Y) \Big|_{w_i=w} = \sum_{k>0} \sum_{m \geq 0} \sum_{v \in \Gamma_{m,d}} \sum_{h \geq 0} \frac{1}{k} n_h^{\text{PT}}(m\gamma_d + v) Q^{km} w^{kl(v)} \psi_{-(-y)^k}^{h-1}.$$

⁴In general, the statement of the DT-CRC requires viewing the partition functions as rational functions in certain variables and the equality as an equality of rational functions. This issue does not arise in this case.

On the other hand, the invariants $n_{g,h}^{\text{PT}}(m\beta_d)$ on X are by definition given by

$$\log Z^{\text{PT}}(X, \iota) = \sum_{k,m>0} \sum_{g,h} \frac{1}{k} Q^{km} n_{g,h}^{\text{PT}}(m\beta_d) \psi_{-(-y)^k}^{h-1} \psi_w^{g+1-2h}.$$

Taking the log of (4.1), observing that $Z_{\text{exc}}^{\text{PT}}(Y) = Z^{\text{PT}}(Y)|_{Q=0}$, and applying Lemma 4.2, we get

$$\log Z^{\text{PT}}(X, \iota) = \log Z^{\text{PT}}(Y) \Big|_{w_i=w} - \log Z^{\text{PT}}(Y) \Big|_{w_i=w, Q=0}.$$

Combining this with the previous two equations, we arrive at

$$\begin{aligned} \sum_{k,m>0} \frac{Q^{km}}{k} \left(\sum_{g,h} n_{g,h}^{\text{PT}}(m\beta_d) \psi_{-(-y)^k}^{h-1} \psi_w^{g+1-2h} \right) \\ = \sum_{k,m>0} \frac{Q^{km}}{k} \left(\sum_{h \geq 0} \sum_{v \in \Gamma_{m,d}} n_h^{\text{PT}}(m\gamma_d + v) w^{kl(v)} \psi_{-(-y)^k}^{h-1} \right). \end{aligned}$$

By Möbius inversion (or a simple induction argument), the quantities in the parentheses in the above equation must be equal for all k and m . In particular, by setting $k = 1$, we have proved

$$\sum_{g,h} n_{g,h}^{\text{PT}}(m\beta_d) \psi_y^{h-1} \psi_w^{g+1-2h} = \sum_{h \geq 0} \sum_{v \in \Gamma_{m,d}} n_h^{\text{PT}}(m\gamma_d + v) w^{l(v)} \psi_y^{h-1}. \quad (4.3)$$

The invariants $n_h^{\text{PT}}(m\gamma_d + v)$ are determined by the KKV formula since $Y = \widehat{S} \times \mathbb{C}$ is a local $K3$ surface. One formulation of the KKV formula from Section 1.3 is the following. For any effective curve class C , the invariant $n_h(C)$ is given by

$$\sum_{h \geq 0} n_h(C) \psi_y^{h-1} = \left[\frac{1}{\phi_{10,1}(q, -y)} \right]_{q^{\frac{1}{2}C^2}},$$

where

$$\phi_{10,1}(q, -y) = -\psi_y \cdot q \prod_{n=1}^{\infty} (1 + yq^n)^2 (1 + y^{-1}q^n)^2 (1 - q^n)^{20}$$

and where $[\cdots]_{q^a}$ denotes the coefficient of q^a in the expression $[\cdots]$. Applying this to (4.3) and using the facts that $\gamma_d^2 = d$ and $\gamma_d \in \Lambda^\perp$, we get

$$(m\gamma_d + v)^2 = m^2d + v^2$$

and so

$$\sum_{g,h} n_{g,h}^{\text{PT}}(m\beta_d) \psi_y^{h-1} \psi_w^{g+1-2h} = \sum_{v \in \Gamma_{m,d}} \left[\frac{w^{l(v)}}{\phi_{10,1}(q, -y)} \right]_{q^{\frac{1}{2}(m^2d+v^2)}} = \left[\frac{\Theta_{\Gamma_{m,d}}(q, w)}{\phi_{10,1}(q, -y)} \right]_{q^{\frac{1}{2}m^2d}},$$

where for any subset $T \subset \Lambda \otimes \mathbb{Q}$, we have defined

$$\Theta_T(q, w) = \sum_{v \in T} q^{-\frac{1}{2}v^2} w^{l(v)}.$$

In Section 4.3, we compute $\Gamma_{m,d}$.

PROPOSITION 4.3. *The subset $\Gamma_{m,d} \subset \Lambda \otimes \mathbb{Q}$ is given as follows:*

- If S is an Abelian surface or a Type (I) Nikulin surface, then

$$\Gamma_{m,d} = \begin{cases} L & \text{if } m \text{ is even,} \\ L + r_0 & \text{if } m \text{ is odd and } d \text{ is even,} \\ L + r_1 & \text{if } m \text{ is odd and } d \text{ is odd,} \end{cases}$$

where in the abelian surface case, $L = K$ is the so-called Kummer lattice, an even, negative-definite rank 16 lattice, and in the Type (I) Nikulin case, $L = N$ is the so-called Nikulin lattice, an even, negative-definite rank 8 lattice. The vectors r_0 and r_1 are particular vectors we will define in Section 4.3. See Section 4.3 for the definitions of K and N .

- If S is a Type (II) Nikulin surface, then d is even and $\Gamma_{m,d} = N$, where N is the Nikulin lattice.

The shifted lattices $L + r_i$ have the property that all their vectors have squares which are congruent to i modulo 2. It follows that we may write

$$\Theta_{L+r_0}(q, w) + \Theta_{L+r_1}(q, w) = \Theta_{L_{\text{sh}}}(q, w),$$

where $L_{\text{sh}} = (L + r_0) \cup (L + r_1)$.

In summary, we have shown that

$$\sum_{g,h} n_{g,h}^{\text{PT}}(m\beta_d) \psi_y^{h-1} \psi_w^{g+1-2h} = \left[\frac{\Theta_T(q, w)}{\phi_{10,1}(q, -y)} \right]_{q^{\frac{1}{2}m^2d}}, \quad (4.4)$$

where

$$T = \begin{cases} N & \text{if } S \text{ is Type (II) Nikulin } (N_{\text{II}}), \\ N & \text{if } S \text{ is Type (I) Nikulin and } m \text{ is even } (N_{\text{I}}^{\text{ev}}), \\ N_{\text{sh}} & \text{if } S \text{ is Type (I) Nikulin and } m \text{ is odd } (N_{\text{I}}^{\text{odd}}), \\ K & \text{if } S \text{ is Abelian and } m \text{ is even } (A^{\text{ev}}), \\ K_{\text{sh}} & \text{if } S \text{ is Abelian and } m \text{ is odd } (A^{\text{odd}}). \end{cases} \quad (4.5)$$

We now make the crucial observation that the right-hand side of (4.4) only depends on the curve class $m\beta_d$ through its square $(m\beta_d)^2 = 2m^2d$ and (possibly) its divisibility modulo 2 (that is, $m \bmod 2$). This leads to the main theorem of this section.

THEOREM 4.4. *Let β be any effective ι -invariant curve class (not necessarily primitive) on an Abelian or Nikulin surface S with $\beta^2 = 2d$. Then the invariants $n_{g,h}^{\text{PT}}(\beta)$ of $S \times \mathbb{C}$ only depend on (g, h, d) in the case, where S is a Type (II) Nikulin surface and on (g, h, d) and whether the divisibility of β is odd or even in the other cases. Denoting these invariants as $n_{g,h}^{\text{PT}}(d; \text{type})$, where $\text{type} \in \{A^{\text{odd}}, A^{\text{ev}}, N_{\text{I}}^{\text{odd}}, N_{\text{I}}^{\text{ev}}, N_{\text{II}}\}$, we have*

$$\sum_{g,h} n_{g,h}^{\text{PT}}(d; \text{type}) \psi_y^{h-1} \psi_w^{g+1-2h} = \left[\frac{\Theta_T(q^2, w)}{\phi_{10,1}(q^2, -y)} \right]_{q^d}.$$

Moreover, $\Theta_T(q^2, w)$ is given explicitly by

$$\Theta_T(q^2, w) = \begin{cases} \theta_0^{16} + 30\theta_0^8\theta_1^8 + \theta_1^{16} & \text{if } T = K \text{ (type } \mathbf{A}^{\text{ev}}), \\ \theta_0^8 + \theta_1^8 & \text{if } T = N \text{ (types } \mathbf{N}_I^{\text{ev}} \text{ and } \mathbf{N}_{II}), \\ 4 \cdot \frac{\Delta(q^2)}{\Delta(q)^2} \cdot \phi_{10,1}^2(q, -w) & \text{if } T = K_{\text{sh}} \text{ (type } \mathbf{A}^{\text{odd}}), \\ -\frac{\Delta(q^2)^{\frac{1}{2}}}{\Delta(q)} \cdot \phi_{10,1}(q, -w) & \text{if } T = N_{\text{sh}} \text{ (type } \mathbf{N}_I^{\text{odd}}), \end{cases} \quad (4.6)$$

where

$$\Delta(q) = q \prod_{n=1}^{\infty} (1 - q^n)^{24}$$

is the unique modular cusp form of weight 12,

$$\phi_{10,1}(q, y) = -\psi_{-y} \cdot q \prod_{n=1}^{\infty} (1 - yq^n)^2 (1 - y^{-1}q^n)^2 (1 - q^n)^{20}$$

is the unique Jacobi cusp form of weight 10 and index 1, and

$$\theta_i = \theta_i(q^2, w) = \sum_{k \in \mathbb{Z} + \frac{i}{2}} q^{2k^2} w^k$$

are the standard rank 1 theta functions.

We note that for $n_{g,h}^{\text{PT}}(d; \mathbf{A}^{\text{ev}})$ and $n_{g,h}^{\text{PT}}(d; \mathbf{N}_I^{\text{ev}})$, the integer d is necessarily divisible by 4, and for $n_{g,h}^{\text{PT}}(d; \mathbf{N}_{II})$, the integer d is necessarily even.

Remark 4.5. It is straightforward to see that the above formulas for the case of $\mathbf{A}^{\text{odd}}$ and $\mathbf{N}_I^{\text{odd}}$ lead to the product formulation given in Theorem 2.10 in the introduction.

Remark 4.6. The theta functions $\Theta_K(q^2, w)$ and $\Theta_{K_{\text{sh}}}(q^2, -w)$ are Jacobi forms of weight 8 and index 2 (for some congruence subgroup), while the theta functions $\Theta_N(q^2, w)$ and $\Theta_{N_{\text{sh}}}(q^2, -w)$ are Jacobi forms of weight 4 and index 1 (for some congruence subgroup).

Remark 4.7. In Appendix B, we will give a direct lattice-theoretic explanation of the identity $\Theta_{K_{\text{sh}}} = 4\Theta_{N_{\text{sh}}}^2$.

To complete the proof of Theorem 4.4, we must prove Proposition 4.3, and we must prove the formulas for $\Theta_T(q^2, w)$ given by (4.6). This is carried out in the next two subsections.

4.3 The Picard lattice of $\widehat{S}$

Recall that S is an Abelian or Nikulin surface and $\widehat{S} \rightarrow S/\iota$ is the associated Kummer $K3$ or Nikulin resolution, respectively. In this subsection, we describe $\text{Pic}(\widehat{S})$ and in particular prove Proposition 4.3. Also recall that we defined the exceptional lattice as follows:

$$\Lambda = \oplus_i \mathbb{Z} \langle E_i \rangle \subset \text{Pic}(\widehat{S}).$$

DEFINITION 4.8. Let $L \subset \text{Pic}(\widehat{S})$ be the saturation of Λ in $\text{Pic}(\widehat{S})$, that is, the smallest primitive sublattice containing Λ such that Λ generates L over $\mathbb{Q}$. If $\widehat{S}$ is a Kummer surface, then L is by definition K , the Kummer lattice. If $\widehat{S}$ is a Nikulin resolution, then L is by definition N , the Nikulin lattice.

We note that by construction, we have the inclusions

$$\Lambda \subset L \subset L^\vee \subset \Lambda^\vee = \tfrac{1}{2}\Lambda.$$

Explicit descriptions of K and N are given in the following lemmas. The first is due to Nikulin; see for example [Mor84, Lemma 5.2].

LEMMA 4.9. *The Nikulin lattice N is the overlattice of Λ generated by Λ and $\widehat{E} = \tfrac{1}{2} \sum_i E_i$.*

While the above shows that the Nikulin lattice is obtained from the exceptional lattice by adding a single vector, the situation for the Kummer lattice is more complicated. The pithiest way to state the result is as follows (see [BHPVdV04, § VIII.5]).

LEMMA 4.10. *Under the natural identification $\Lambda^\vee/\Lambda \cong \text{Maps}(\mathbb{F}_2^4, \mathbb{F}_2)$, the Kummer lattice K is the overlattice of Λ such that the inclusion $K/\Lambda \subset \Lambda^\vee/\Lambda$ corresponds to the inclusion $\text{Aff}(\mathbb{F}_2^4, \mathbb{F}_2) \subset \text{Maps}(\mathbb{F}_2^4, \mathbb{F}_2)$ of affine linear maps (including the two constant maps) into all maps.*

We next describe $\text{Pic}(\widehat{S})$, which will allow us to prove Proposition 4.3. The class γ_d might not be an integral class, but it turns out that the embedding

$$\mathbb{Z}\langle 2\gamma_d \rangle \oplus L \subset \text{Pic}(\widehat{S})$$

is always of index 2. Depending on the parity of d and the type of the surface S , the order 2 quotient group is generated either by γ_d or by $\gamma_d + r_d$, where r_d is a certain vector only depending on $d \bmod 2$.

In the case where S is Abelian and $\widehat{S}$ is a Kummer $K3$, we quote Garbagnati–Sarti [GS16, Theorem 2.7] adapted to our notation.

PROPOSITION 4.11. *The Picard lattice of the Kummer surface $\widehat{S}$ is the index 2 overlattice of $\mathbb{Z}\langle 2\gamma_d \rangle \oplus K$ generated by $\mathbb{Z}\langle 2\gamma_d \rangle \oplus K$ and $\gamma_d + r_d$, where*

$$r_d \in K^\vee - K, \quad 2r_d \in K \quad \text{and} \quad r_d^2 = d \bmod 2.$$

The class $[r_d] \in \Lambda^\vee/\Lambda \cong \text{Maps}(\mathbb{F}_2^4, \mathbb{F}_2)$ only depends on $d \bmod 2$ and is unique up to isometries of K . For d even, the corresponding map $\mathbb{F}_2^4 \rightarrow \mathbb{F}_2$ is the characteristic function of a fixed linear 2-plane $P_1 \subset \mathbb{F}_2^4$. For d odd, the corresponding map is the characteristic function of $P_1 \Delta P_2 \subset \mathbb{F}_2^4$, where P_1 and P_2 are transversely intersecting 2-planes and Δ denotes symmetric difference.

There are two families of Nikulin $K3$ surfaces determined as follows. Let S be a Nikulin surface, and recall that $\beta_d \in \text{Pic}(S)$ is a primitive ι -invariant effective class with $\beta_d^2 = 2d$. The existence of the Nikulin involution implies that there is an inclusion

$$\mathbb{Z}\langle \beta_d \rangle \oplus E_8(-2) \subset \text{Pic}(S).$$

The above is either (I) an isomorphism or (II) an index 2 sublattice. (This statement must be modified if the invariant Picard rank of S is greater than 1; cf. Remark 4.1.)

DEFINITION 4.12. We say that S is Nikulin of Type (I) in the first case and of Type (II) in the second case. The latter can occur only when d is even.

PROPOSITION 4.13. *The Picard lattice of a Type (II) Nikulin resolution $\widehat{S}$ is $\mathbb{Z}\langle \gamma_d \rangle \oplus N$. The Picard lattice of a Type (I) Nikulin resolution $\widehat{S}$ is the index 2 overlattice of $\mathbb{Z}\langle 2\gamma_d \rangle \oplus N$ generated by $\mathbb{Z}\langle 2\gamma_d \rangle \oplus N$ and $\gamma_d + r_d$, where*

$$r_d \in N^\vee - N, \quad 2r_d \in N \quad \text{and} \quad r_d^2 = d \bmod 2.$$

The class $[r_d] \in \Lambda^\vee / \Lambda$ only depends on $d \bmod 2$, is unique up to isometries of N , and is given by

$$r_d = \begin{cases} \frac{1}{2}(E_1 + E_2) & \text{if } d \text{ is odd,} \\ \frac{1}{2}(E_1 + E_2 + E_3 + E_4) & \text{if } d \text{ is even} \end{cases}$$

for a suitable numbering of the exceptional divisors $E_1, \dots, E_8$.

Proof. See [GS08, Proposition 2.1 and Corollary 2.2]. $\square$

Propositions 4.11 and 4.13 then imply Proposition 4.3.

4.4 Theta function identities

To finish the proof of Theorem 4.4, we must prove the formulas given in (4.6). Recall that

$$\Lambda \subset L \subset L^\vee \subset \Lambda^\vee = \frac{1}{2}\Lambda,$$

where $\Lambda = \oplus_i \mathbb{Z}\langle E_i \rangle$ is the exceptional lattice and L is either K or N . Since any element $\rho \in \Lambda^\vee / \Lambda$ may be uniquely written as

$$\rho = \frac{1}{2} \sum_i \rho_i E_i, \quad \rho_i \in \{0, 1\},$$

we may define

$$c_1(\rho) = \sum_i \rho_i, \quad c_0(\rho) = \text{rk}(\Lambda) - c_1(\rho),$$

that is, the number of ρ_i which are 1 or 0, respectively. The following lemma is our basic tool for computing theta functions.⁵

LEMMA 4.14. *Let π be the projection $\Lambda^\vee \rightarrow \Lambda^\vee / \Lambda$, and suppose that $T \subset \Lambda^\vee$ is a union of cosets: $T = \cup_{\rho \in \pi(T)} (\Lambda + \rho)$. Then*

$$\Theta_T(q^2, w) = \sum_{\rho \in \pi(T)} \theta_0^{c_0(\rho)} \theta_1^{c_1(\rho)},$$

where

$$\theta_i = \theta_i(q^2, w) = \sum_{k \in \mathbb{Z} + \frac{i}{2}} q^{2k^2} w^k.$$

Proof. Since the cosets $\Lambda + \rho$ are disjoint, we have $\Theta_T(q^2, w) = \sum_{\rho \in \pi(T)} \Theta_{\Lambda + \rho}(q^2, w)$, and then we observe that

$$\Lambda + \rho \cong \mathbb{Z}\langle E \rangle^{\oplus c_0(\rho)} \oplus \left(\mathbb{Z}\langle E \rangle + \frac{1}{2}E \right)^{\oplus c_1(\rho)},$$

from which it follows that $\Theta_{\Lambda + \rho} = \theta_0^{c_0(\rho)} \theta_1^{c_1(\rho)}$. $\square$

PROPOSITION 4.15. *The theta functions of the Nikulin lattice N and the shifted Nikulin lattice N_{sh} are given by*

$$\Theta_N(q^2, w) = \theta_0^8 + \theta_1^8, \quad \Theta_{N_{\text{sh}}}(q^2, w) = \theta_0^2 \theta_1^6 + 2\theta_0^4 \theta_1^4 + \theta_0^6 \theta_1^2.$$

Proof. It follows from Lemma 4.9 and Proposition 4.13 that $\pi(N) = \{0, \frac{1}{2}(E_1 + \dots + E_8)\}$ and that

$$\begin{aligned} \pi(N_{\text{sh}}) &= \pi\left(N + \frac{1}{2}(E_1 + E_2)\right) \cup \pi\left(N + \frac{1}{2}(E_1 + \dots + E_4)\right) \\ &= \left\{ \frac{1}{2}(E_1 + E_2), \frac{1}{2}(E_3 + \dots + E_8), \frac{1}{2}(E_1 + \dots + E_4), \frac{1}{2}(E_5 + \dots + E_8) \right\}. \end{aligned}$$

⁵The authors are very grateful to John Duncan who explained this to us.

The value of c_1 on the above four elements is 2, 6, 4, and 4 respectively. The proposition then follows from Lemma 4.14. $\square$

PROPOSITION 4.16. *The theta functions of the Kummer lattice K and the shifted Kummer lattice K_{sh} are given by*

$$\begin{aligned}\Theta_K(q^2, w) &= \theta_0^{16} + 30\theta_0^8\theta_1^8 + \theta_1^{16}, \\ \Theta_{K_{\text{sh}}}(q^2, w) &= 4\theta_0^4\theta_1^{12} + 16\theta_0^6\theta_1^{10} + 24\theta_0^8\theta_1^8 + 16\theta_0^{10}\theta_1^6 + 4\theta_0^{12}\theta_1^4.\end{aligned}$$

Proof. As in the Nikulin case, we must determine the value of c_1 on all the elements of $\pi(K)$ and $\pi(K_{\text{sh}})$. By Lemma 4.10, the image $\pi(K)$ is given by the 32 elements $\frac{1}{2}\sum_i \rho_i E_i$, where the corresponding map $\mathbb{F}_2^4 \rightarrow \mathbb{F}_2$ given by $i \mapsto \rho_i$ is an affine linear function. The values of c_1 on the two constant functions are 0 and 16, respectively, while the value of c_1 on the remaining 30 non-constant affine linear functions is 8. The formula for Θ_K then follows from Lemma 4.14. A simple but tedious way to determine the elements of $\pi(K_{\text{sh}}) = \pi(K + r_0) \cup \pi(K + r_1)$ is to choose coordinates for $\mathbb{F}_2^4$ and write down all 64 elements explicitly. Doing so and reading off the value of c_1 on each element, we find that $\pi(K + r_0)$ has 4 elements with $c_1 = 4$, 24 elements with $c_1 = 8$, and 4 elements with $c_1 = 12$ and that $\pi(K + r_1)$ has 16 elements with $c_1 = 6$ and 16 elements with $c_1 = 10$. The formula for $\Theta_{K_{\text{sh}}}$ then follows from Lemma 4.14. There is a more coordinate-free approach to the same calculation using the affine geometry of $\mathbb{F}_2^4$. It requires analyzing the various symmetric differences of the affine hyperplanes and the 2-dimensional planes P_1 and P_2 appearing in Proposition 4.11. Unfortunately, the case-by-case analysis in this approach is not particularly less tedious than the direct enumeration. $\square$

To complete the proof of Theorem 4.4, it only remains to prove the identities

$$\begin{aligned}\Theta_{N_{\text{sh}}}(q^2, w) &= -\frac{\Delta(q^2)^{\frac{1}{2}}}{\Delta(q)} \cdot \phi_{10,1}(q, -w) \\ &= \psi_w \cdot q \cdot \prod_{n=1}^{\infty} (1 + q^n)^{12} (1 - q^n)^8 (1 + wq^n)^2 (1 + w^{-1}q^n)^2\end{aligned}$$

and

$$\begin{aligned}\Theta_{K_{\text{sh}}}(q^2, w) &= 4 \frac{\Delta(q^2)}{\Delta(q)^2} \cdot \phi_{10,1}^2(q, -w) \\ &= 4\psi_w^2 \cdot q^2 \cdot \prod_{n=1}^{\infty} (1 + q^n)^{24} (1 - q^n)^{16} (1 + wq^n)^4 (1 + w^{-1}q^n)^4.\end{aligned}$$

Since the equations for $\Theta_{N_{\text{sh}}}(q^2, w)$ and $\Theta_{K_{\text{sh}}}(q^2, w)$ given in Propositions 4.15 and 4.16 can be factored as

$$\Theta_{N_{\text{sh}}}(q^2, w) = \theta_0^2 \theta_1^2 (\theta_0^2 + \theta_1^2)^2, \quad \Theta_{K_{\text{sh}}}(q^2, w) = 4\theta_0^4 \theta_1^4 (\theta_0^2 + \theta_1^2)^4, \quad (4.7)$$

we see that it suffices to prove the identity

$$\theta_0^2 \theta_1^2 (\theta_0^2 + \theta_1^2)^2 = \psi_w \cdot q \cdot \prod_{n=1}^{\infty} (1 + q^n)^{12} (1 - q^n)^8 (1 + wq^n)^2 (1 + w^{-1}q^n)^2. \quad (4.8)$$

By the Jacobi triple product identity, we may write

$$\begin{aligned}\theta_0(q^2, w) &= \prod_{n=1}^{\infty} (1 - q^{4n})(1 + wq^{4n-2})(1 + w^{-1}q^{4n-2}), \\ \theta_1(q^2, w) &= q^{\frac{1}{2}} \cdot (w^{\frac{1}{2}} + w^{-\frac{1}{2}}) \cdot \prod_{n=1}^{\infty} (1 - q^{4n})(1 + wq^{4n})(1 + w^{-1}q^{4n}).\end{aligned}\tag{4.9}$$

LEMMA 4.17. *We have*

$$\theta_0(q^2, w)^2 + \theta_1(q^2, w)^2 = \prod_{n=1}^{\infty} (1 - q^{2n})^2 (1 + q^{2n-1})^2 (1 + wq^{2n-1})(1 + w^{-1}q^{2n-1}).$$

Proof. The left-hand side of the above equation is given by

$$\sum_{n, m \in \mathbb{Z}} q^{2m^2 + 2n^2} w^{n+m} + q^{2\left(m+\frac{1}{2}\right)^2 + 2\left(n+\frac{1}{2}\right)^2} w^{n+m+1}.$$

Set $n = \frac{1}{2}(a - b)$ and $m = \frac{1}{2}(a + b)$; the sum rearranges to

$$\sum_{\substack{a, b \in \mathbb{Z} \\ a \equiv b \pmod{2}}} q^{b^2} (q^{a^2} w^a + q^{(a+1)^2} w^{a+1}) = \sum_{a, b \in \mathbb{Z}} q^{b^2} q^{a^2} w^a = \theta_0(q, 1) \theta_0(q, w).$$

Then applying the Jacobi triple product identity to the right-hand side of the above proves the lemma. $\square$

Now applying Lemma 4.17 and (4.9) to the left-hand side of (4.8), we get

$$\begin{aligned}\theta_0^2 \theta_1^2 (\theta_0^2 + \theta_1^2)^2 &= q \cdot \psi_w \cdot \prod_{n=1}^{\infty} \left((1 - q^{4n})^2 (1 + wq^{4n})^2 (1 + w^{-1}q^{4n})^2 \right. \\ &\quad \cdot (1 - q^{4n})^2 (1 + wq^{4n-2})^2 (1 + w^{-1}q^{4n-2})^2 \\ &\quad \cdot (1 - q^{2n})^4 (1 + q^{2n-1})^4 (1 + wq^{2n-1})^2 (1 + w^{-1}q^{2n-1})^2 \Big) \\ &= q \cdot \psi_w \cdot \prod_{n=1}^{\infty} (1 + q^n)^{12} (1 - q^n)^8 (1 + wq^n)^2 (1 + w^{-1}q^n)^2,\end{aligned}$$

where in the last equality, we have used the fact that

$$\prod_{n=1}^{\infty} (1 + wq^{4n})(1 + wq^{4n-2})(1 + wq^{2n-1}) = \prod_{n=1}^{\infty} (1 + wq^n)$$

and similar considerations.

This completes the proof of Theorem 4.4.

5. Local Abelian and Nikulin surfaces (MT theory)

5.1 Overview

In this section, we prove some basic results about ι -stability, and we prove Conjecture 2.9 for local Abelian and Nikulin surfaces.

In Section 5.2, we show that ι -stability is equivalent to a certain kind of Nironi stability on the stack quotient $[X/\iota]$. In Section 5.3, we prove Conjecture 2.9 for $X = S \times \mathbb{C}$, where S is

an Abelian or Nikulin surface. The basic idea is the following. Using the results of Section 5.2 and the projection $X \rightarrow \mathbb{C}$, we show that all ι -stable MT sheaves on X are given by Nironi δ -stable sheaves on $[S/\iota] \times \{t\}$ for some $t \in \mathbb{C}$. We then show that Nironi δ -stability on $[S/\iota]$ is the large volume limit of a certain Bridgeland stability condition on $[S/\iota]$ constructed by Lim and Rota [LR22]. We then apply the derived Fourier–Mukai correspondence to show that our moduli spaces are given by, up to a factor of $\mathbb{C}$, moduli spaces of objects in the derived category of $\widehat{S}$ which are stable with respect to the large volume limit of one of the stability conditions on $K3$ surfaces constructed by Bridgeland [Bri08]. Finally, we use the results of Bayer and Macri [BM14] to show that these moduli spaces are deformation equivalent to moduli spaces of MT sheaves on $\widehat{S}$. This then allows us to apply Conjecture 1.5, which is known to hold for $\widehat{S} \times \mathbb{C}$ by [PT16, SY22]. The upshot is that we prove that (4.3) holds with MT GV invariants replacing PT GV invariants on both sides, and then the subsequent arguments of Section 4 apply word for word.

5.2 Nironi stability

The category of ι -equivariant sheaves on X and the category of sheaves on the stack $[X/\iota]$ are canonically equivalent, and in this subsection we will not notationally differentiate between a sheaf on the stack and the corresponding ι -equivariant sheaf.

In [Nir08], Nironi developed a theory of slope stability for Deligne–Mumford stacks analogous to Simpson stability for schemes. Nironi stability for the stack $[X/\iota]$ involves a choice of an ample divisor H on the coarse space X/ι and the choice of a “generating bundle” V which we may take to be (see [Nir08, Definition 2.2, Proposition 2.7])

$$V = (\mathcal{O}_X \otimes R_+)^{\oplus a} \oplus (\mathcal{O}_X \otimes R_-)^{\oplus b}$$

for any $a, b \in \mathbb{N}$.

Nironi’s slope function is obtained from the generalized Hilbert polynomial of a sheaf F (that is, the ι -invariant part of $\chi(F \otimes V(mH))$) by dividing the second coefficient by the leading coefficient. For 1-dimensional sheaves F with $[\text{supp}(F)] = \beta$, $\chi(F) = nR_{\text{reg}} + \epsilon R_-$, and our choice of V , Nironi’s slope function is given by

$$\mu(F) = \frac{(a+b)n + b\epsilon}{(a+b)H \cdot \beta}.$$

The slope function only depends on a and b through the number

$$\delta = \frac{b}{a+b} \in \mathbb{Q} \cap (0, 1),$$

so we write

$$\mu_\delta(F) = \frac{n + \delta\epsilon}{H \cdot \beta}.$$

DEFINITION 5.1. Let F be an ι -equivariant sheaf on the threefold X with pure 1-dimensional support, $[\text{supp}(F)] = \beta$, and $\chi(F) = nR_{\text{reg}} + \epsilon R_-$. Then F is *Nironi δ -(semi)stable* if for all ι -equivariant subsheaves $F' \subsetneq F$, we have $\mu_\delta(F') < \mu_\delta(F)$ (respectively, $\mu_\delta(F') \leq \mu_\delta(F)$).

Let $\mathbf{M}_{\beta, n, \epsilon}^{\delta\text{-s}}([X/\iota])$ (respectively, $\mathbf{M}_{\beta, n, \epsilon}^{\delta\text{-ss}}([X/\iota])$) be the moduli stack of Nironi δ -(semi)stable sheaves with β, n, ϵ as above. Nironi proves that the usual properties enjoyed by moduli stacks of Simpson (semi)stable sheaves hold for moduli stacks of Nironi (semi)stable sheaves [Nir08, Theorems 6.21 and 6.22]. In particular, we have the following.

THEOREM 5.2. *The stack $\mathbf{M}_{\beta,n,\epsilon}^{\delta\text{-s}}([X/\iota])$ is a $\mathbb{C}^*$ -gerbe over its coarse moduli space. In particular, any Nironi δ -stable sheaf is simple.*

THEOREM 5.3. *Assume that X is projective; then $\mathbf{M}_{\beta,n,\epsilon}^{\delta\text{-ss}}([X/\iota])$ has a projective coarse moduli space.*

The following corollary is then standard.

COROLLARY 5.4. *For X quasi-projective, the Hilbert–Chow morphism*

$$\mathbf{M}_{\beta,n,\epsilon}^{\delta\text{-ss}}([X/\iota]) \rightarrow \text{Chow}_{\beta}(X)^{\iota}$$

given by $F \mapsto [\text{supp}(F)]$ is proper.

PROPOSITION 5.5. *Let F be an ι -equivariant sheaf on X having proper pure 1-dimensional support and with $[\text{supp}(F)] = \beta$ and $\chi(F) = R_{\text{reg}} + \epsilon R_-$. Let $\delta > 0$ be a sufficiently small rational number. Then the following conditions are equivalent:*

- (1) *The sheaf F is Nironi δ semistable.*
- (2) *The sheaf F is Nironi δ stable.*
- (3) *The sheaf F is ι -stable (see Definition 2.6).*

Proof. Let $F' \subsetneq F$ be an ι -equivariant subsheaf with

$$\chi(F') = kR_{\text{reg}} + \gamma R_- \quad \text{and} \quad [\text{supp}(F')] = \beta'.$$

First we claim that the quantities $H \cdot \beta'$ and γ can be uniformly bounded over all F' for fixed F . The positive number $H \cdot \beta'$ is uniformly bounded by $H \cdot \beta$ since $\text{supp}(F) = \sum_i d_i C_i$ for some set of effective reduced curves C_i and $\text{supp}(F') = \sum_i d'_i C_i$ with $0 \leq d'_i \leq d_i$. Moreover, γ is bounded by the length of the zero-dimensional sheaf $F'|_{X^{\iota}}$ which is in turn uniformly bounded by the length of $F|_{X^{\iota}}$ (we note that $F|_{X^{\iota}}$ is zero-dimensional by Assumption 2.1). This claim follows from general properties of Toen’s GRR theorem for stacks [Toe99], [BCY12, Appendix A] applied to the morphism $f: [X/\iota] \rightarrow BG$, where $G = \{\pm 1\}$. Briefly, there is a Toen operator $\tau_{\mathcal{X}}: K(\mathcal{X}) \rightarrow A(I\mathcal{X})$ which is functorial with respect to proper pushforwards, so that in particular

$$\begin{aligned} f_*(\tau_{[X/\iota]} F') &= \tau_{BG}(f_* F') = (\text{tr}_{+1}(f_* F'), \text{tr}_{-1}(f_* F')) \\ &= (\text{tr}_{+1}(kR_{\text{reg}} + \gamma R_-), \text{tr}_{-1}(kR_{\text{reg}} + \gamma R_-)) \\ &= (2k + \gamma, -\gamma), \end{aligned}$$

where the above equalities are taking place in $A(IBG) \cong \mathbb{Z} \oplus \mathbb{Z}$. Thus we see that

$$\gamma = -f_*(\tau_{[X/\iota]} F'|_{X^{\iota}}).$$

A cursory examination of the Toen operator shows that the absolute value of the right-hand side is bounded by the length of $F'|_{X^{\iota}}$. We remark that Toen’s GRR in this context is essentially the holomorphic Lefschetz trace formula (also known as the Woods Hole theorem) but is general enough to include our situation (torsion sheaves and non-isolated fixed points).

Now we suppose that F is Nironi δ -semistable. Then the inequality

$$\frac{k + \delta\gamma}{H \cdot \beta'} \leq \frac{1 + \delta\epsilon}{H \cdot \beta}$$

holds and is equivalent to

$$k \leq \left(\frac{H \cdot \beta'}{H \cdot \beta} \right) (1 + \delta\epsilon) - \delta\gamma. \tag{5.1}$$

Since $\text{supp}(F') \subseteq \text{supp}(F)$ and $\dim \text{supp}(F') \neq 0$, we have

$$0 < \frac{H \cdot \beta'}{H \cdot \beta} \leq 1.$$

If $(H \cdot \beta')/(H \cdot \beta) < 1$, then for sufficiently small δ , we have $k < 1$.

If $(H \cdot \beta')/(H \cdot \beta) = 1$, then $\beta' = \beta$ and $k \leq 1 + \delta(\epsilon - \gamma)$. Thus we see that either $k < 1$, or $k = 1$ and $\beta' = \beta$ and $\gamma \leq \epsilon$. Finally, if $k = 1$, $\beta' = \beta$, and $\gamma = \epsilon$, then $\chi(Q) = 0$, where

$$0 \rightarrow F' \rightarrow F \rightarrow Q \rightarrow 0.$$

But since $\beta' = \beta$, we have $\dim Q = 0$, and so $\chi(Q) = 0$ implies that $Q = 0$, which implies that $F' = F$. Thus we have shown that condition (1) implies that either $k < 1$, or $k = 1$ and $\beta' = \beta$ and $\gamma < \epsilon$, which by Definition 2.6 means that F is ι -stable. Thus condition (1) implies condition (3). Moreover, we have shown that inequality (5.1) must be strict, so that condition (1) implies condition (2). And of course, condition (2) implies condition (1), so it remains to show that condition (3) implies condition (2).

To that end, suppose that F is ι -stable, so that the integers k, γ, ϵ satisfy $k \leq 1$ and that if $k = 1$, then $\gamma < \epsilon$ and $\beta' = \beta$. We need to prove that inequality (5.1) holds. Since we have $\dim \text{supp}(F') > 0$, we have $(H \cdot \beta')/(H \cdot \beta) > 0$, which means the right-hand side of (5.1) is positive for sufficiently small δ , and so if $k < 1$, then (5.1) is true. If $k = 1$, then by hypothesis, $\gamma < \epsilon$ and $\beta' = \beta$, so (5.1) becomes $1 \leq 1 + \delta\epsilon - \delta\gamma$, which is true. $\square$

5.3 Proof of Conjecture 2.9 for local Abelian and Nikulin surfaces

We now consider $X = S \times \mathbb{C}$ with S an Abelian or Nikulin surface, and we adopt the notation of Section 4.

By Proposition 5.5, we may identify the moduli space of ι -stable MT sheaves with the moduli space of Nironi δ -stable sheaves:

$$\mathbf{M}_{m\beta_d}^\epsilon(X, \iota) = \mathbf{M}_{m\alpha_d, 1, \epsilon}^{\delta\text{-ss}}([X/\iota]) = \mathbf{M}_{m\alpha_d, 1, \epsilon}^{\delta\text{-s}}([X/\iota]).$$

LEMMA 5.6. *Let F be an ι -stable MT sheaf on X . Then F is scheme-theoretically supported on $S \times \{t\}$ for some $t \in \mathbb{C}$.*

Proof. First suppose that the image of the support of F under the map $S \times \mathbb{C} \rightarrow \mathbb{C}$ is not a single point. Then F can be written as $F_1 \oplus F_2$, which violates stability since both factors are equivariant subsheaves. Thus we may suppose that F is set-theoretically supported on some $S_t = S \times \{t\}$. Consider the short exact sequence of ι -equivariant sheaves

$$0 \rightarrow \mathcal{O}_X(-S_t) \rightarrow \mathcal{O}_X \rightarrow \mathcal{O}_{S_t} \rightarrow 0.$$

Noting that $\mathcal{O}_X(-S_t) \cong \mathcal{O}_X$ and tensoring with F , we get the right-exact sequence

$$F \rightarrow F \rightarrow F \otimes \mathcal{O}_{S_t} \rightarrow 0.$$

By Theorem 5.2, the sheaf F is simple, and hence the first map is either 0 or an isomorphism. Since $F \otimes \mathcal{O}_{S_t}$ is non-zero by construction, the first map must be zero, and thus the second map induces an isomorphism $F \cong F \otimes \mathcal{O}_{S_t}$ by exactness. $\square$

By Lemma 5.6, every ι -stable MT sheaf on X can be identified with a Nironi δ -stable sheaf on $[S/\iota] \times \{t\}$ for some $t \in \mathbb{C}$. We let $\mathbf{M}_\eta^{\delta\text{-s}}([S/\iota])$ denote the moduli space of Nironi δ -stable sheaves on $[S/\iota]$ in the K -theory class η . Then applying the above discussion and the analysis of

K -theory classes done in Section 4 (and using the same notation), we get

$$\mathbf{M}_{m\beta_d}^\epsilon(X, \iota) = \bigsqcup_{\substack{v \in \Gamma_{m,d} \\ l(v) = \epsilon}} \mathbf{M}_{\eta(m\alpha_d, v)}^{\delta-s}([S/\iota]) \times \mathbb{C}, \quad (5.2)$$

where

$$\eta(m\alpha_d, v) = m\alpha_d + [\mathcal{O}_{\text{pt}}] + \sum_i v_i [\mathcal{O}_{x_i} \otimes R_-].$$

We again note that for clarity, we have proven (5.2) as a bijection of sets. However, the argument in Lemma 5.6 works for flat families of MT sheaves, and so (5.2) is an isomorphism of schemes.)

The following proposition is key.

PROPOSITION 5.7. *The moduli space $\mathbf{M}_{\eta(m\alpha_d, v)}^{\delta-s}([S/\iota])$ is deformation equivalent to the moduli space $\mathbf{M}_{(0, m\gamma_d + v, 1)}^s(\widehat{S})$ of Simpson sheaves on $\widehat{S}$ with Mukai vector $(0, m\gamma_d + v, 1)$. Moreover, the deformation equivalence is compatible with the Hilbert–Chow morphism.*

Assuming Proposition 5.7 and noting that

$$\mathbf{M}_{m\gamma_d + v}(\widehat{S} \times \mathbb{C}) = \mathbf{M}_{(0, m\gamma_d + v, 1)}^s(\widehat{S}) \times \mathbb{C},$$

we may compute the Maulik–Toda polynomial of (X, ι) as follows. We use the definition, (5.2), and Proposition 5.7 to get

$$\mathbf{MT}_{m\beta_d}(y, w) = \sum_{v \in \Gamma_{m,d}} \mathbf{MT}_{m\gamma_d + v}(y) w^{l(v)},$$

where $\mathbf{MT}_{m\beta_d}(y, w)$ is the Maulik–Toda polynomial of (X, ι) in the class $m\beta_d$ and $\mathbf{MT}_{m\gamma_d + v}(y)$ is the Maulik–Toda polynomial of $\widehat{S} \times \mathbb{C}$ in the class $m\gamma_d + v$. We note that the perverse cohomologies appearing in the definition of the MT polynomial only depend on the topology of the Hilbert–Chow morphism and not the algebraic structure. Consequently, they are invariant under the deformation equivalence in Proposition 5.7.

It then follows immediately from the above equation that the MT analog of (4.3) holds:

$$\sum_{g,h} n_{g,h}^{\text{MT}}(m\beta_d) \psi_y^{h-1} \psi_w^{g+1-2h} = \sum_{h \geq 0} \sum_{v \in \Gamma_{m,d}} n_h^{\text{MT}}(m\gamma_d + v) w^{l(v)} \psi_y^{h-1}.$$

All the analysis in Section 4 subsequent to equation (4.3) then goes through word for word with the MT versions of the invariants, and we see that they are given by the same formulas (in Theorem 4.4) as the PT versions of the invariants. We thus conclude that

$$n_{g,h}^{\text{PT}}(m\beta_d) = n_{g,h}^{\text{MT}}(m\beta_d)$$

holds and thus Conjecture 2.9 holds for (X, ι) .

It remains only to prove Proposition 5.7.

Proof of Proposition 5.7. We remark that although we do not directly use it, these moduli spaces are all hyperkähler manifolds of $K3[n]$ type, and the map to Chow is a Lagrangian fibration.

In [LR22], Lim and Rota construct Bridgeland stability conditions on orbifold surfaces with Kleinian orbifold points. For notational simplicity, they assume that their orbifold surface has

a single orbifold point, but their method easily applies to orbifold surfaces with multiple Kleinian orbifold points such as $[S/\iota]$.

Their stability condition depends (in the case of $[S/\iota]$) on parameters $\gamma \in (0, \frac{1}{2})$ and $w \in \mathbb{C}$ and has a central charge $Z_{w,\gamma}$ which in our situation takes values

$$Z_{w,\gamma} \left(m\alpha_d + n[\mathcal{O}_{\text{pt}}] + \sum_i v_i [\mathcal{O}_{x_i} \otimes R_-] \right) = -n - \frac{1}{2}\gamma \sum_i v_i + iH \cdot (m\alpha_d).$$

The parameter w must be chosen satisfying two inequalities which in our situation read

$$\Re(w) > -\frac{(\Im(w))^2}{H^2} + 3 - \gamma^2, \quad 2\Re(w) > \frac{\Im(w)}{H^2} - 3\gamma > 0,$$

and the corresponding heart of the derived category is given by a certain tilt $\text{Coh}^{-\Im(w)}([S/\iota])$ (see [LR22, § 4.2]).

Recalling that $\epsilon = \sum_i v_i$, we see that the slope function associated with the central charge is exactly the Nironi slope function μ_δ with $\gamma = 2\delta$. Consequently, in any limit with $\Im(w) \rightarrow \infty$ (and satisfying the necessary inequalities), the Lim–Rota stable objects become Nironi stable sheaves. Thus we may identify $\mathbf{M}_{\eta(m\alpha_d, v)}^{\delta-s}([S/\iota])$ with the moduli space of Lim–Rota stable objects for $\gamma = 2\delta$ and a choice of w with appropriately large $\Im(w)$.

We may now consider the derived Fourier–Mukai equivalence

$$FM: D^b([S/\iota]) \rightarrow D^b(\widehat{S}).$$

This equivalence will take the Lim and Rota stability conditions on $[S/\iota]$ to some Bridgeland stability condition on the $K3$ surface $\widehat{S}$. We claim that these stability conditions are in fact the stability conditions on $K3$ constructed by Bridgeland in [Bri08]. These stability conditions are characterized by lying in the connected component of the stability conditions on $\widehat{S}$ where $\mathcal{O}_{\text{pt}}$ is stable. In [LR22, § 5], Lim and Rota analyze the stability of $\mathcal{O}_{\text{pt}}$ and show that for generic deformations of their stability conditions, it is stable. Our claim follows.

We thus can make the identification

$$\mathbf{M}_{\eta(m\alpha_d, v)}^{\delta-s}([S/\iota]) = \mathbf{M}_{(0, m\gamma_d + v, 1)}^{\text{Bridgeland}}(\widehat{S}),$$

where the moduli space on the right is the moduli space of objects on $\widehat{S}$ with Mukai vector $(0, m\gamma_d + v, 1)$ which are stable with respect to the Bridgeland stability condition, which is Fourier–Mukai equivalent to our chosen Lim–Rota condition on $[S/\iota]$.

We may now apply the results of Bayer and Macri [BM14], who analyze in great detail all of the moduli spaces of Bridgeland semistable objects on a $K3$ surface. They show that all moduli spaces of objects semistable with respect to one of Bridgeland’s constructed stability conditions are deformation equivalent hyperkähler manifolds, provided that the moduli space has no strictly semistable objects. In particular, the moduli space $\mathbf{M}_{(0, m\gamma_d + v, 1)}^{\text{Bridgeland}}(\widehat{S})$ is deformation equivalent to $\mathbf{M}_{(0, m\gamma_d + v, 1)}^s(\widehat{S})$, which proves the main assertion of the proposition. Moreover, Bayer–Macri analyze the Hilbert–Chow morphism, which they show to be a Lagrangian fibration (whenever it is well defined) and compatible with the deformation equivalence. $\square$

6. Additional examples

6.1 Elliptically fibered Calabi–Yau threefolds

Let $\pi: X \rightarrow B$ be an elliptically fibered Calabi–Yau threefold with section $B \hookrightarrow X$ and with integral fibers. We additionally require the following conditions:

- (1) There is an involution $\bar{\iota}: B \rightarrow B$ whose fixed point locus is a smooth curve $C \subset B$.
- (2) The restriction $\pi|_S: S \rightarrow C$ is a smooth elliptic surface, where $S = \pi^{-1}(C)$.
- (3) The involution $\bar{\iota}$ lifts to a Calabi–Yau involution $\iota: X \rightarrow X$ which restricts to a fiberwise action by -1 on S over C .

A simple example of such an X is given by the Schoen Calabi–Yau threefold: the intersection of two generic hypersurfaces of degrees $(1, 0, 3)$ and $(1, 3, 0)$ in $\mathbb{P}^1 \times \mathbb{P}^2 \times \mathbb{P}^2$. This is an elliptic fibration over a base B which is itself a rational elliptic surface. The involution $\bar{\iota}: B \rightarrow B$ is given by -1 acting on the elliptic fibers of the base.

Let $[F] \in H_2(X)$ be the class of a fiber F of $\pi: X \rightarrow B$. The following is our main result for this example.

THEOREM 6.1. *Conjecture 2.9 holds for the class $[F]$, and the invariants are given by*

$$n_{g,h}^{\text{PT}}([F]) = n_{g,h}^{\text{MT}}([F]) = \begin{cases} -e(C) & \text{if } (g, h) = (1, 0), \\ e(S) & \text{if } (g, h) = (0, 0), \\ 0 & \text{otherwise.} \end{cases}$$

We prove this theorem below by directly computing both the ι -PT and ι -MT invariants in the class $[F]$.

The ι -fixed locus X^ι is given by $S[2]$, the 2-torsion points of the elliptic fibration $S \rightarrow C$. It admits a decomposition

$$X^\iota = C_0 \sqcup C_1,$$

where $C_0 \cong C$ is the zero section and C_1 is the locus of non-trivial 2-torsion points. It is a degree 3 cover $C_1 \rightarrow C$ which is simply ramified at the nodes of the nodal fibers and doubly ramified at the cusps of the cuspidal fibers.

The compactified Jacobian $\text{Jac}^d(X/B)$ of the family of elliptic curves $X \rightarrow B$ can be identified with the moduli space of pure dimension 1 stable sheaves $\mathcal{E}$ with $ch(\mathcal{E}) = (0, 0, [F], d)$.

The following facts are standard for this situation:⁶

- $\text{Jac}^d(X/B) \cong X$.
- There is a rank d bundle $V_d \rightarrow \text{Jac}^d(X/B)$ whose fiber over $\mathcal{E}$ is $H^0(\mathcal{E})$.
- The map $\text{PT}_{[F],d}(X) \rightarrow \text{Jac}^d(X/B)$ given by $[\mathcal{O}_X \xrightarrow{s} \mathcal{E}] \mapsto \mathcal{E}$ induces an isomorphism $\text{PT}_{[F],d}(X) \cong \mathbb{P}(V_d)$.

The above constructions are ι -equivariant and exhibit $\text{PT}_{[F],d}(X)$ as a projective bundle over X , so that we may identify the fixed locus as follows:

$$\text{PT}_{[F],d}(X)^\iota = \mathbb{P}(V_d^+) \big|_{X^\iota} \sqcup \mathbb{P}(V_d^-) \big|_{X^\iota},$$

⁶See for example [MS13, §1, bottom of page 2] combined with the fact that the moduli space of stable pairs supported on a family of integral plane curves is isomorphic to the relative Hilbert scheme of points [MS13, §1, top of page 3] and [PT09, Proposition 1.8].

where

$$V_d|_{X^\iota} \cong V_d^+|_{X^\iota} \oplus V_d^-|_{X^\iota}$$

is the eigenbundle decomposition for the ι action. Since $X^\iota = C_0 \sqcup C_1$, we see that

$$\mathrm{PT}_{[F],d}(X)^\iota = \mathbb{P}_{d,0}^+ \sqcup \mathbb{P}_{d,1}^+ \sqcup \mathbb{P}_{d,0}^- \sqcup \mathbb{P}_{d,1}^-,$$

where we have abbreviated $\mathbb{P}(V_d^\pm)|_{C_i}$ as $\mathbb{P}_{d,i}^\pm$.

Then since $\mathbb{P}_{d,i}^\pm$ is a smooth projective bundle over a curve, we have

$$e_{\mathrm{vir}}(\mathbb{P}_{d,i}^\pm) = (-1)^{\dim V_d^\pm} \cdot \dim V_d^\pm \cdot e(C_i).$$

Using the decomposition given in (2.1),

$$\mathrm{PT}_{[F],d}(X)^\iota = \bigsqcup_{d=2n+\epsilon} \mathrm{PT}_{[F],n,\epsilon}(X, \iota),$$

we see that we need only compute $\dim V_d^\pm$ and ϵ for each component. To do so, we pick $[s: \mathcal{O}_F \rightarrow \mathcal{E}] \in \mathbb{P}_{d,i}^\pm$, where $\mathcal{E}$ is an ι -invariant sheaf supported on a smooth fiber F and $s \in H^0(\mathcal{E})^\pm$. Let $p_0 = F \cap C_0$ and $\{p_1, p_2, p_3\} = F \cap C_1$ be the origin and non-trivial 2-torsion points of F , respectively. Up to renumbering, we may assume $\mathcal{E} \mapsto p_i$ under the map $\mathbb{P}_{d,i}^\pm \rightarrow C_i$. The corresponding ι -PT pair is $[s: \mathcal{O}_F \rightarrow \mathcal{E} \otimes R_\pm]$, and so by definition

$$H^0(\mathcal{E} \otimes R_\pm) = nR_{\mathrm{reg}} + \epsilon R_- = nR_+ + (n + \epsilon)R_-.$$

On the other hand,

$$H^0(\mathcal{E} \otimes R_\pm) = H^0(\mathcal{E}) \otimes R_\pm = (V_d^+ \oplus V_d^-) \otimes R_\pm,$$

and so

$$\dim V_d^\pm = n = \frac{1}{2}(d - \epsilon).$$

Moreover, since s is an invariant section of $\mathcal{E} \otimes R_\pm$, it determines an isomorphism $\mathcal{E} \otimes R_\pm \cong \mathcal{O}_F(D)$, where D is an ι -invariant effective divisor. Such divisors are in the form given by Lemma 3.4, which by the same lemma shows $\epsilon = |T| - 2$, where $T \subset \{0, 1, 2, 3\}$. Finally, the map $\mathbb{P}_{d,i}^\pm \rightarrow C_i$ which sends $\mathcal{E} \mapsto p_i$ is given by summing the points in the divisor D in the group law of F . In particular, we must have $\sum_{j \in T} p_j = p_i$ in the group law. For $p_i = p_0$, the only possible subsets are

$$T = \emptyset, \{0\}, \{1, 2, 3\}, \{0, 1, 2, 3\} \quad \text{with } \epsilon = -2, -1, 1, 2, \text{ respectively,}$$

and for $p_i = p_1$, the only possible subsets are

$$T = \{1\}, \{0, 1\}, \{2, 3\}, \{0, 2, 3\} \quad \text{with } \epsilon = -1, 0, 0, 1, \text{ respectively.}$$

We then can compute the $Q^{[F]}$ coefficient of $Z^{\mathrm{PT}}(X, \iota)$ by taking e_{vir} of the components of $\mathrm{PT}_{[F],d}(X)^\iota$, multiplying by the appropriate $y^n w^\epsilon$, and then summing over n and ϵ :

$$\begin{aligned} [Z^{\mathrm{PT}}(X, \iota)]_{Q^{[F]}} &= \sum_{n=1}^{\infty} (-1)^n n y^n ((w^{-2} + w^{-1} + w + w^2)e(C_0) + (w^{-1} + 2 + w^1)e(C_1)) \\ &= \frac{-y}{(1+y)^2} ((\psi_w^2 - 3\psi_w)e(C_0) + \psi_w e(C_1)) \\ &= -e(C_0)\psi_y^{-1}\psi_w^2 + (3e(C_0) - e(C_1))\psi_y^{-1}\psi_w. \end{aligned}$$

Note that $[\log Z^{\mathrm{PT}}(X, \iota)]_{Q^{[F]}} = [Z^{\mathrm{PT}}(X, \iota)]_{Q^{[F]}}$ because $[F]$ is primitive; the formula for

$n_{g,h}^{\text{PT}}([F])$ in Theorem 6.1 then follows from the fact that

$$e(S) = 3e(C_0) - e(C_1).$$

The above formula is easily proved by observing that the equality holds when restricted to any fiber of $S \rightarrow C$ (smooth or singular).

To compute the ι -MT invariants, we use the following.

LEMMA 6.2. *We have $M_{[F]}^\epsilon(X, \iota) \cong \text{PT}_{[F], 1, \epsilon}(X, \iota)$.*

Proof. It suffices to show that for all $\mathcal{E} \in M_{[F]}^\epsilon(X, \iota)$, there is a unique (up to scale) ι -invariant section s . Then the isomorphism in the lemma is given by

$$\mathcal{E} \mapsto [s: \mathcal{O}_X \rightarrow \mathcal{E}].$$

Any ι -MT sheaf $\mathcal{E}$ admits some ι -invariant section s since $\chi(\mathcal{E}) = R_{\text{reg}} + \epsilon R_-$. We then get an ι -equivariant short exact sequence

$$0 \rightarrow \mathcal{O}_F \xrightarrow{s} \mathcal{E} \rightarrow Q \rightarrow 0,$$

where $F = \text{supp}(\mathcal{E})$ is some fiber of $S \rightarrow C$ and Q is necessarily zero-dimensional since F is integral. Moreover, $\chi(\mathcal{O}_F) = R_+ - R_-$ for any fiber F and so $\chi(Q) = H^0(Q) = (2 + \epsilon)R_-$. Taking the long exact sequence associated with the above short exact sequence and then restricting to the ι -invariant part yields an isomorphism $H^0(\mathcal{O}_F) \cong H^0(\mathcal{E})^\iota$, so that $H^0(\mathcal{E})^\iota$ is 1-dimensional and hence generated by s . $\square$

The lemma then allows us to apply our previous analysis of the components of $\text{PT}_{[F], d, \epsilon}(X, \iota)$ specialized to the case $d = 1$ where $\mathbb{P}_{1, i}^\pm \cong C_i$. As before, the components and the corresponding ϵ is determined by the subset $T \subset \{0, 1, 2, 3\}$. The result is

$$M_{[F]}^\epsilon(X, \iota) \cong \begin{cases} C_0 & \text{if } \epsilon = \pm 2, \\ C_0 \sqcup C_1 & \text{if } \epsilon = \pm 1, \\ C_1 \sqcup C_1 & \text{if } \epsilon = 0. \end{cases}$$

Since $M_{[F]}^\epsilon(X, \iota)$ is a smooth curve, the perverse sheaf of vanishing cycles is the shifted constant sheaf, $\phi^\bullet = \mathbb{Q}[1]$. Furthermore, $\text{Chow}_{[F]}(X)^\iota$ here is C with $\pi^\epsilon: M_{[F]}^\epsilon(X, \iota) \rightarrow C$ given by projection $C_i \rightarrow C$ on each component.

In general, if $\pi: C'' \rightarrow C'$ is a proper surjective morphism of smooth curves, then π is semismall, and consequently $R^\bullet \pi_* \mathbb{Q}[1]$ is a perverse sheaf on C' [dCM09, Theorem 4.2.7], and so

$${}^p H^i(R^\bullet \pi_* \mathbb{Q}[1]) = \begin{cases} R^\bullet \pi_* \mathbb{Q}[1] & \text{if } i = 0, \\ 0 & \text{if } i \neq 0. \end{cases}$$

Consequently, we have

$$\sum_i \chi({}^p H^i(R^\bullet \pi_* \mathbb{Q}[1])) y^i = \chi(R^\bullet \pi_* \mathbb{Q}[1]) = -e(C''),$$

where the last equality follows from the perverse Leray spectral sequence.

Applying this to $\pi^\epsilon: M_{[F]}^\epsilon(X, \iota) \rightarrow C$, we can then compute the Maulik–Toda polynomial:

$$\begin{aligned} \text{MT}_{[F]}(y, w) &= -e(C_0)(w^{-2} + w^{-1} + w^1 + w^2) - e(C_1)(w^{-1} + 2 + w^1) \\ &= -e(C)\psi_w^2 + e(S)\psi_w, \end{aligned}$$

and the formula for $n_{g,h}^{\text{MT}}([F])$ follows. The proof of Theorem 6.1 is then complete.

6.2 The local football

Let ι be an involution of $C \cong \mathbb{P}^1$ fixing two points z_0 and z_∞ . The line bundles $\mathcal{O}_C(-z_0)$ and $\mathcal{O}_C(-z_\infty)$ are naturally ι -equivariant, and consequently the CY3

$$X = \text{Tot}(\mathcal{O}_C(-z_0) \oplus \mathcal{O}_C(-z_\infty))$$

has a natural involution which we also call ι .

The global stack quotient $[X/\iota]$ is called a *local football*, though here we use this term to mean the pair (X, ι) . The purpose of this subsection is to give a proof of Proposition 2.15 which we restate here.

PROPOSITION 6.3. *For all $d > 0$, we have*

$$n_{g,h}^{\text{PT}}(d[C]) = n_{g,h}^{\text{MT}}(d[C]) = \begin{cases} 1 & \text{if } (d, g, h) = (1, 0, 0), \\ 0 & \text{otherwise.} \end{cases}$$

We start with ι -PT theory. As we did in Lemma 4.2, we may compute the ι -PT invariants of (X, ι) by computing orbifold PT invariants of the stack quotient $[X/\iota]$. Since $[X/\iota]$ is toric, we may use the orbifold topological vertex [BCY12]. The partition function $Z^{\text{PT}}([X/\iota])$ is computed in [BCY12, § 4.2], where it is given by $DT'(\mathcal{X}_{2,2})$ in the notation of [BCY12]. Here we are using [BCR22, Theorem 6.12], which states that the reduced Donaldson–Thomas partition function is equal to the Pandharipande–Thomas partition function. By the formula just below [BCY12, Proposition 3], we have

$$Z^{\text{PT}}([X/\iota]) = \prod_{u \in \{v, vp_0, vr_0, vp_0r_0\}} M(u, -q)^{-1}, \quad (6.1)$$

where

$$M(x, q) = \prod_{n=1}^{\infty} (1 - xq^n)^{-n}.$$

The variables v , p_0 , r_0 , and q track classes in the K -theory of sheaves on $[X/\iota]$ which we can equivalently regard as ι -equivariant sheaves on X . Recall that the variables in $Z^{\text{PT}}(X, \iota)$ track the curve class and χ of the sheaf. Thus to specialize $Z^{\text{PT}}([X/\iota])$ to $Z^{\text{PT}}(X, \iota)$, we must compute the curve class and χ of each of these classes. The results are given in the following table:

$Z^{\text{PT}}([X/\iota])$ variable	Class in $[X/\iota]$ (see [BCY12, § 3.3])	Equivariant class on X	(χ, β) of class	$Z^{\text{PT}}(X, \iota)$ variable
p_0	$[\mathcal{O}_{z_0} \otimes R_-]$	$[\mathcal{O}_{z_0} \otimes R_-]$	$(R_-, 0)$	w
r_0	$[\mathcal{O}_{z_\infty} \otimes R_-]$	$[\mathcal{O}_{z_\infty} \otimes R_-]$	$(R_-, 0)$	w
q	$[\mathcal{O}_{\bar{p}}]$	$[\mathcal{O}_p \oplus \mathcal{O}_{\iota(p)}]$	$(R_{\text{reg}}, 0)$	y
v	$[\mathcal{O}_{[C/\iota]}(-\bar{p})]$	$[\mathcal{O}_C(-p - \iota(p))]$	$(-R_-, [C])$	Qw^{-1}

Here $\bar{p} \in [C/\iota]$ is a generic point corresponding to the ι -orbit $\{p, \iota(p)\} \subset C$, and the formula $\chi(\mathcal{O}_C(-p - \iota(p))) = -R_-$ is obtained by applying Lemma 3.4.

Equation (6.1) then specializes to

$$Z^{\text{PT}}(X, \iota) = M(Qw^{-1}, -y)^{-1} M(Q, -y)^{-2} M(Qw, -y)^{-1}.$$

It is straightforward to show that

$$\log M(x, y)^{-1} = \sum_{k=1}^{\infty} \frac{x^k}{k} \psi_{-(-y)^k}^{-1},$$

and so

$$\log Z^{\text{PT}}(X, \iota) = \sum_{k=1}^{\infty} \frac{1}{k} Q^k \cdot (w^{-k} + 2 + w^k) \cdot \psi_{-(-y)^k}^{-1} = \sum_{k=1}^{\infty} \frac{1}{k} Q^k \cdot \psi_{-(-y)^k}^{-1} \cdot \psi_{w^k},$$

and the formula for $n_{g,h}^{\text{PT}}(d[C])$ then follows.

Now turning to the ι -MT theory, we begin with the following key result.

LEMMA 6.4. *The moduli space $M_{d[C]}^{\epsilon}(X, \iota)$ is empty for $d > 1$.*

Proof. Since $M_{d[C]}^{\epsilon}(X, \iota)$ is proper over $\text{Chow}_{d[C]}(X)$, which is a point, if it is non-empty, it admits a fixed point for the torus action induced from the action on X . In order to obtain a contradiction, we suppose that there exists a torus-invariant ι -MT sheaf $\mathcal{E}$ in the class $d[C]$ with $d > 1$.

Then the scheme-theoretic support of $\mathcal{E}$ is the thickened curve C_{λ} determined from a 2-dimensional partition λ . Since $\chi(\mathcal{E}) = R_{\text{reg}} + \epsilon R_{-}$, there is a ι -invariant section

$$0 \rightarrow \mathcal{O}_{C_{\lambda}} \xrightarrow{s} \mathcal{E} \rightarrow Q \rightarrow 0,$$

and then since $\mathcal{E}$ is ι -stable, $\chi(\mathcal{O}_{C_{\lambda}}) = kR_{\text{reg}} + mR_{-}$ with $k \leq 1$.

Let $\pi: X \rightarrow C$ be the projection; then $\chi(\mathcal{O}_{C_{\lambda}}) = \chi(\pi_* \mathcal{O}_{C_{\lambda}})$. We have

$$\pi_* \mathcal{O}_{C_{\lambda}} = \bigoplus_{(i,j) \in \lambda} \mathcal{O}_C(z_0)^i \otimes \mathcal{O}_C(z_{\infty})^j = \bigoplus_{(i,j) \in \lambda} \mathcal{O}_C(iz_0 + jz_{\infty}).$$

Writing $i = 2a + i'$ and $j = 2b + j'$ with $a, b \geq 0$ and $(i', j') \in \{(0, 0), (0, 1), (1, 0), (1, 1)\}$ and applying Lemma 3.4, we get

$$\chi(\mathcal{O}_C(iz_0 + jz_{\infty})) = (1 + a + b)R_{\text{reg}} + (i' + j' - 1)R_{-}.$$

Since we are taking sums of these terms, the only way to guarantee $k \leq 1$ as above is if $\lambda = \square$ is the unique partition of length 1. Thus $\mathcal{E}$ is scheme-theoretically supported on C , and hence it must be a torus-invariant, ι -equivariant, rank d vector bundle on C . Such vector bundles split as a direct sum of ι -equivariant line bundles, which then contradicts the ι -stability of $\mathcal{E}$. $\square$

It follows from Lemma 6.4 that $n_{g,h}^{\text{MT}}(d[C]) = 0$ for $d > 1$, and for $d = 1$, we apply Corollary 3.2, which then finishes the proof of Proposition 2.15.

Appendix A. Tables of values for $n_{g,h}(d, \mathbf{A}^{\text{odd}})$

In this appendix, we list explicitly the values of $n_{g,h}(d, \mathbf{A}^{\text{odd}})$ for $d \leq 7$. This case includes the primitive class β_d on a Picard rank 1 Abelian surface S where these numbers have some enumerative significance. The $h = 0$ numbers are (up to the minus sign due to the second factor in $X = S \times \mathbb{C}$) actual counts of ι -invariant hyperelliptic curves on S ; they coincide with the counts computed in [BOPY18]. For each d , the highest genus occurring is $g = d + 1$, the arithmetic genus of the class β_d . Let

$$\text{Chow}_{\beta_d}(X)_h \subset \text{Chow}_{\beta_d}(X)^{\iota}$$

be the dimension h components of the ι -fixed point locus. Then one can show that

$$n_{d+1,h}(d, A^{\text{odd}}) = e_{\text{vir}}(\text{Chow}_{\beta_d}(X)_h).$$

The right-hand side can be computed directly since $\text{Chow}_{\beta_d}(X)^{\iota} = \text{Chow}_{\beta_d}(S)^{\iota} \times \mathbb{C}$ and because the first factor here is the disjoint union of the ι -invariant linear systems of the ι -invariant line bundles in the class β_d .

For $d \leq 1$, the only non-zero values of $n_{g,h}(d, A^{\text{odd}})$ are given by

$$n_{1,0}(0, A^{\text{odd}}) = -4, \quad n_{2,0}(1, A^{\text{odd}}) = -16.$$

For $2 \leq d \leq 7$, see the table below:

Non-zero values of $n_{g,h}(d, A^{\text{odd}})$ for $2 \leq d \leq 7$.

	d = 2		d = 3	
	$h = 0$	$h = 1$	$h = 0$	$h = 1$
$g = 2$	-48		-64	
$g = 3$	-24	8	-160	
$g = 4$			-16	32

	d = 4			d = 5		
	$h = 0$	$h = 1$	$h = 2$	$h = 0$	$h = 1$	$h = 2$
$g = 2$	-112			-96		
$g = 3$	-456	24		-1056		
$g = 4$	-192	96		-912	224	
$g = 5$	-4	48	-12	-96	320	
$g = 6$					32	-48

	d = 6				d = 7			
	$h = 0$	$h = 1$	$h = 2$	$h = 3$	$h = 0$	$h = 1$	$h = 2$	$h = 3$
$g = 2$	-192				-128			
$g = 3$	-1920	32			-3264			
$g = 4$	-2992	512			-7776	704		
$g = 5$	-736	1056	-64		-3424	3072		
$g = 6$	-16	384	-144		-240	1920	-448	
$g = 7$		8	-72	16		192	-480	
$g = 8$							-48	64

Appendix B. Shifted Nikulin and Kummer lattices in terms of a shifted D_4 lattice

Let $\Lambda = \oplus_{i=1}^n \mathbb{Z}\langle e_i \rangle$ be the trivial lattice with inner-product $e_i \cdot e_j = \delta_{ij}$. We define the D_n lattice by

$$D_n = \{v \in \Lambda \mid l(v) \equiv 0 \pmod{2}\},$$

recalling the definition of $l(v)$ in (4.2). It is straightforward to show that the dual lattice is given by

$$D_n^\vee = \Lambda \cup \left(\Lambda + \frac{1}{2} \sum_{i=1}^n e_i \right).$$

In the case of $n = 8$, the following observation about the Nikulin lattice N follows immediately from Lemma 4.9.

PROPOSITION B.1. *We have $N \cong D_8^\vee(-2)$.*

Let λ be any root in the D_4 root system. Define the shifted D_4 lattice $D_{4,\text{sh}} = D_4 + \frac{1}{2}\lambda$ along with the corresponding Jacobi theta function

$$\Theta_{D_{4,\text{sh}}}(q, w) = \sum_{v \in D_{4,\text{sh}}} q^{v^2} w^{l(v)},$$

which we can explicitly evaluate.

PROPOSITION B.2. *We have*

$$\Theta_{D_{4,\text{sh}}}(q^2, w) = (q \psi_w)^{\frac{1}{2}} \cdot \prod_{n=1}^{\infty} (1 + q^n)^6 (1 - q^n)^4 (1 + wq^n)(1 + w^{-1}q^n).$$

We note that this formula also essentially appeared in [Obe18, Proposition 24].

Proposition B.2 explains an interesting connection we observed between the shifted Nikulin lattice N_{sh} and shifted Kummer lattice K_{sh} .

COROLLARY B.3. *We have*

$$\Theta_{N_{\text{sh}}}(q^2, w) = \Theta_{D_{4,\text{sh}}}(q^2, w)^2, \quad \Theta_{K_{\text{sh}}}(q^2, w) = 4\Theta_{D_{4,\text{sh}}}(q^2, w)^4.$$

Proof. Equations (4.7) and (4.8) reduce the above claims to Proposition B.2. □

Proof of Proposition B.2. Without loss of generality, suppose that the root of D_4 we specify is $\lambda = e_1 + e_2$. We define the lattice isomorphism $f: D_4^\vee(2) \rightarrow D_4$ by

$$f(e_1) = \lambda, \quad f(e_2) = e_1 - e_2, \quad f(e_3) = e_3 + e_4, \quad f(e_4) = e_3 - e_4$$

and observe that f identifies the two shifted lattices $D_{4,\text{sh}}$ and $D_{4,\text{sh}}^\vee = D_4^\vee(2) + \frac{1}{2}e_1$. Therefore, defining the theta function

$$\Theta_{D_{4,\text{sh}}^\vee}(q, w) = \sum_{v \in D_{4,\text{sh}}^\vee} q^{\frac{1}{2}v^2} w^{l(v)},$$

we have the formula

$$\Theta_{D_{4,\text{sh}}}(q, w) = \Theta_{D_{4,\text{sh}}^\vee}(q, w).$$

By the same methods as in Section 4.4, the description

$$D_{4,\text{sh}}^\vee = (\Lambda + \frac{1}{2}e_1) \cup (\Lambda + \frac{1}{2}(e_2 + e_3 + e_4))$$

allows us to compute the theta function explicitly to be

$$\Theta_{D_{4,\text{sh}}}(q^2, w) = \theta_0^3 \theta_1 + \theta_0 \theta_1^3 = \theta_0 \theta_1 (\theta_0^2 + \theta_1^2).$$

By (4.9) and Lemma 4.17, this completes the proof. $\square$

Remark B.4. Alternatively, one can give formulas for N_{sh} and K_{sh} in terms of shifts of the D_4 lattice, from which Corollary B.3 follows. Let $\lambda_k = f(e_k)$ denote the four roots from the proof of Proposition B.2, and define the shifted lattices $D_{4,k} = D_4 + \frac{1}{2}\lambda_k$. One can show that via the isomorphism f ,

$$\begin{aligned} N_{\text{sh}} &\cong D_{4,1}(-1) \oplus D'_{4,1}(-1), \\ K_{\text{sh}} &\cong \prod_{k=1}^4 D_{4,k}(-1) \oplus D'_{4,k}(-1) \oplus D''_{4,k}(-1) \oplus D'''_{4,k}(-1), \end{aligned}$$

where the primes signify that the different D_4 summands have different generators, for example $D'_4 \subset \oplus_i \mathbb{Z}\langle e'_i \rangle$ and $D''_4 \subset \oplus_i \mathbb{Z}\langle e''_i \rangle$, etc. In terms of the two-variable theta functions, we have

$$\Theta_{D_{4,k}}(q, w) = \Theta_{D_{4,\text{sh}}}(q, w)$$

for all $1 \leq k \leq 4$. One can then show that the above lattice formulas imply Corollary B.3.

ACKNOWLEDGEMENTS

The authors warmly acknowledge and thank the following people for helpful discussions: Arend Bayer, John Duncan, Bronson Lim, Daves Maulik, Georg Oberdieck, Franco Rota, and Junliang Shen.

REFERENCES

- BHPVdV04 W. P. Barth, K. Hulek, C. A. M. Peters, and A. Van de Ven, *Compact complex surfaces*, 2nd ed., *Ergeb. Math. Grenzgeb.* (3), vol. 4 (Springer, Berlin, 2004); [doi:10.1007/978-3-642-57739-0](#).
- BM14 A. Bayer and E. Macrì, *MMP for moduli of sheaves on K3s via wall-crossing: nef and movable cones, Lagrangian fibrations*, *Invent. Math.* **198** (2014), no. 3, 505–590; [doi:10.1007/s00222-014-0501-8](#).
- BCR22 S. V. Beentjes, J. Calabrese, and J. V. Rennemo, *A proof of the Donaldson–Thomas crepant resolution conjecture*, *Invent. Math.* **229** (2022), no. 2, 451–562; [doi:10.1007/s00222-022-01109-w](#).
- Beh09 K. Behrend, *Donaldson–Thomas type invariants via microlocal geometry*, *Ann. of Math.* (2) **170** (2009), no. 3, 1307–1338; [doi:10.4007/annals.2009.170.1307](#).
- BBD⁺15 C. Brav, V. Bussi, D. Dupont, D. Joyce, and B. Szendrői, *Symmetries and stabilization for sheaves of vanishing cycles* (with an appendix by Jörg Schürmann), *J. Singul.* **11** (2015), 85–151; [doi:10.5427/jsing.2015.11e](#).
- Bri08 T. Bridgeland, *Stability conditions on K3 surfaces*, *Duke Math. J.* **141** (2008), no. 2, 241–291; [doi:10.1215/S0012-7094-08-14122-5](#).
- Bri11 ———, *Hall algebras and curve-counting invariants*, *J. Amer. Math. Soc.* **24** (2011), no. 4, 969–998; [doi:10.1090/S0894-0347-2011-00701-7](#).
- BCY12 J. Bryan, C. Cadman, and B. Young, *The orbifold topological vertex*, *Adv. Math.* **229** (2012), no. 1, 531–595; [doi:10.1016/j.aim.2011.09.008](#).

- BOPY18 J. Bryan, G. Oberdieck, R. Pandharipande, and Q. Yin, *Curve counting on abelian surfaces and threefolds*, *Algebr. Geom.* **5** (2018), no. 4, 398–463; doi:10.14231/ag-2018-012.
- dCM09 M. A. A. de Cataldo and L. Migliorini, *The decomposition theorem, perverse sheaves and the topology of algebraic maps*, *Bull. Amer. Math. Soc. (N.S.)* **46** (2009), no. 4, 535–633; doi:10.1090/S0273-0979-09-01260-9.
- GS08 A. Garbagnati and A. Sarti, *Projective models of K3 surfaces with an even set*, *Adv. Geom.* **8** (2008), no. 3, 413–440; doi:10.1515/ADVGEOM.2008.027.
- GS16 ———, *Kummer surfaces and K3 surfaces with $(\mathbb{Z}/2\mathbb{Z})^4$ symplectic action*, *Rocky Mountain J. Math.* **46** (2016), no. 4, 1141–1205; doi:10.1216/RMJ-2016-46-4-1141.
- GV98 R. Gopakumar and C. Vafa, *M-theory and topological strings-II*, 1998, arXiv:hep-th/9812127.
- JS12 D. Joyce and Y. Song, *A theory of generalized Donaldson–Thomas invariants*, *Mem. Amer. Math. Soc.* **217** (2012), no. 1020; doi:10.1090/S0065-9266-2011-00630-1.
- KKV99 S. Katz, A. Klemm, and C. Vafa, *M-theory, topological strings and spinning black holes*, *Adv. Theor. Math. Phys.* **3** (1999), no. 5, 1445–1537; doi:10.4310/ATMP.1999.v3.n5.a6.
- LR22 B. Lim and F. Rota, *Characteristic classes and stability conditions for projective Kleinian orbisurfaces*, *Math. Z.* **300** (2022), no. 1, 827–849; doi:10.1007/s00209-021-02805-8.
- Mac62 I. G. Macdonald, *The Poincaré polynomial of a symmetric product*, *Proc. Cambridge Philos. Soc.* **58** (1962), 563–568; doi:10.1017/S0305004100040573.
- MTh18 D. Maulik and R. P. Thomas, *Sheaf counting on local K3 surfaces*, *Pure Appl. Math. Q.* **14** (2018), no. 3–4, 419–441; doi:10.4310/PAMQ.2018.v14.n3.a1.
- MTo18 D. Maulik and Y. Toda, *Gopakumar–Vafa invariants via vanishing cycles*, *Invent. Math.* **213** (2018), no. 3, 1017–1097; doi:10.1007/s00222-018-0800-6.
- MS13 L. Migliorini and V. Shende, *A support theorem for Hilbert schemes of planar curves*, *J. Eur. Math. Soc. (JEMS)* **15** (2013), no. 6, 2353–2367; doi:10.4171/JEMS/423.
- Mor84 D. R. Morrison, *On K3 surfaces with large Picard number*, *Invent. Math.* **75** (1984), no. 1, 105–121; doi:10.1007/BF01403093.
- Nir08 F. Nironi, *Moduli Spaces of Semistable Sheaves on Projective Deligne–Mumford stacks*, 2008, arXiv:0811.1949.
- Obe18 G. Oberdieck, *Gromov–Witten invariants of the Hilbert schemes of points of a K3 surface*, *Geom. Topol.* **22** (2018), no. 1, 323–437; doi:10.2140/gt.2018.22.323.
- PT09 R. Pandharipande and R. P. Thomas, *Curve counting via stable pairs in the derived category*, *Invent. Math.* **178** (2009), no. 2, 407–447; doi:10.1007/s00222-009-0203-9.
- PT10 ———, *Stable pairs and BPS invariants*, *J. Amer. Math. Soc.* **23** (2010), no. 1, 267–297; doi:10.1090/S0894-0347-09-00646-8.
- PT16 ———, *The Katz–Klemm–Vafa conjecture for K3 surfaces*, *Forum Math. Pi* **4** (2016), e4; doi:10.1017/fmp.2016.2.
- Pie22 S. Pietromonaco, *G-invariant Hilbert schemes on Abelian surfaces and enumerative geometry of the orbifold Kummer surface*, *Res. Math. Sci.* **9** (2022), 1; doi:10.1007/s40687-021-00298-9.
- Pop17 M. Popa, *Derived equivalences of smooth stacks and orbifold Hodge numbers*, in *Higher dimensional algebraic geometry—in honour of Professor Yujiro Kawamata’s sixtieth birthday*, *Adv. Stud. Pure Math.*, vol. 74 (Math. Soc. Japan, Tokyo, 2017), 357–380; doi:10.2969/aspm/07410357.
- Ros14 S. C. F. Rose, *Counting hyperelliptic curves on an Abelian surface with quasi-modular forms*, *Commun. Number Theory Phys.* **8** (2014), no. 2, 243–293; doi:10.4310/CNTP.2014.v8.n2.a2.
- SY22 J. Shen and Q. Yin, *Topology of Lagrangian fibrations and Hodge theory of hyper-Kähler manifolds*, *Duke Math. J.* **171** (2022), no. 1, 209–24; doi:10.1215/00127094-2021-0010.

- Tod10 Y. Toda, *Curve counting theories via stable objects I. DT/PT correspondence*, J. Amer. Math. Soc. **23** (2010), no. 4, 1119–1157; doi:[10.1090/S0894-0347-10-00670-3](https://doi.org/10.1090/S0894-0347-10-00670-3).
- Toe99 B. Toen, *Théorèmes de Riemann–Roch pour les champs de Deligne–Mumford*, K-Theory **18** (1999), no. 1, 33–76; doi:[10.1023/A:1007791200714](https://doi.org/10.1023/A:1007791200714).

Jim Bryan jbryan@math.ubc.ca

Department of Mathematics, University of British Columbia, Room 121, 1984 Mathematics Road, Vancouver, B.C., V6T 1Z2, Canada

Stephen Pietromonaco spietro@umich.edu

Department of Mathematics, University of British Columbia, Room 121, 1984 Mathematics Road, Vancouver, B.C., V6T 1Z2, Canada

Current address: Department of Mathematics, University of Michigan, 530 Church Street, Ann Arbor, MI 48109, USA