Algebraic Geometry 11 (2) (2024) 178–255
doi:10.14231/AG-2024-007

Motivic integration over wild Deligne–Mumford stacks

Takehiko Yasuda

Abstract

We develop the motivic integration theory over formal Deligne–Mumford stacks over a power series ring of arbitrary characteristic. This is a generalization of the corresponding theory for tame and smooth Deligne–Mumford stacks constructed in earlier papers of the author. As an application, we obtain the wild motivic McKay correspondence for linear actions of arbitrary finite groups, which has been known only for cyclic groups of prime order. In particular, this implies the motivic version of Bhargava's mass formula as a special case. In fact, we prove a more general result, the invariance of stringy motives of (stacky) log pairs under crepant morphisms.

1. Introduction

1.1 Background

The aim of this paper is to develop motivic integration over *wild* Deligne–Mumford (for short, DM) stacks, that is, DM stacks whose stabilizers may have order divisible by the characteristic of the base field. Motivic integration was introduced by Kontsevich during a lecture at Orsay in 1995 and developed by Denef and Loeser [DL99] for varieties in characteristic zero. In earlier papers [Yas04, Yas06], the author treated tame and smooth DM stacks, carrying forward the approach of Batyrev [Bat99b] and Denef–Loeser [DL02] to the McKay correspondence via motivic integration. An attempt of generalization to the wild case was started in [Yas14], and a grand design of the general theory was outlined in [Yas17a]. More work in this direction has been done in [WY15, Yas16, Yas17b, WY17, TY20, TY23], and the present work is based on that. Note that our theory is new even in characteristic zero since we allow DM stacks to be singular.

Motivic integration for formal schemes over a complete discrete valuation ring by Sebag [Seb04] is a generalization in another direction. In fact, the present paper treats formal DM stacks over a power series ring $k[\![t]\!]$. Thus we also generalize his theory except that we consider only the equal characteristic case and we use a different version of the complete Grothendieck ring of varieties.

Received 3 August 2020, accepted in final form 29 March 2023.
2020 Mathematics Subject Classification 14E18, 14D23, 14E16, 14B05, 11G25, 11S15.
Keywords: motivic integration, Deligne–Mumford stacks, McKay correspondence, singularities, stringy motives, wild ramification, Bhargava's mass formula.

This work was supported by JSPS KAKENHI Grant Numbers 16H06337, JP18H01112 and JP18K18710.

1.2 Twisted arcs

In a classical setting of motivic integration, given a k-variety X, one considers its *arcs*, that is, k-morphisms $\operatorname{Spec} L[\![t]\!] \to X$ from a formal disk to X, where L is an extension of k. The space of arcs, denoted by $\mathrm{J}_\infty X$, carries the so-called *motivic measure*. One studies integrals with respect to it, which leads to useful invariants of X. When X is a formal scheme over $k[\![t]\!]$, one considers $k[\![t]\!]$-morphisms $\operatorname{Spf} L[\![t]\!] \to X$ instead.

In the generalization to DM stacks, we need to consider *twisted arcs*. Roughly, it means that we replace a formal disk $\operatorname{Spec} L[\![t]\!]$ (or $\operatorname{Spf} L[\![t]\!]$) with twisted formal disks; a *twisted formal disk* is the quotient stack $[E/G]$ associated with a normal Galois cover of a formal disk with Galois group G. In the tame case, we only need to consider the μ_l-covers $\operatorname{Spec} L[\![t^{1/l}]\!] \to \operatorname{Spec} L[\![t]\!]$ because they are the only possible Galois covers if L is algebraically closed. In the wild case, there are infinitely many Galois covers even if we fix an algebraically closed coefficient field L and a Galois group. Moreover, they cannot be parameterized by any finite-dimensional space. We need to take all Galois extensions into account and to construct the moduli space of twisted arcs on a formal DM stack $\mathcal{X}$, which we denote by $\mathcal{J}_\infty\mathcal{X}$.

1.3 The change of variables formula

After defining the motivic measure on the space $\mathcal{J}_\infty\mathcal{X}$ of twisted arcs, we then need to prove the *change of variables formula*. In the case of schemes, the formula is written as

$$\int_{\mathrm{J}_\infty X} \mathbb{L}^{h}\, d\mu_X = \int_{\mathrm{J}_\infty Y} \mathbb{L}^{h\circ f_\infty - \mathrm{j}_f}\, d\mu_Y\,,$$

say for a proper birational morphism $f\colon Y \to X$. Here j_f is the Jacobian order function of f. We generalize the change of variables formula to DM stacks. To do so, we first clarify the meaning of a birational morphism of DM stacks; a morphism $\mathcal{Y} \to \mathcal{X}$ of DM stacks is said to be *birational* if it restricts to an isomorphism $\mathcal{Y}' \to \mathcal{X}'$ of open dense substacks $\mathcal{Y}' \subset \mathcal{X}$ and $\mathcal{X}' \subset \mathcal{X}$. We note that a birational morphism is not necessarily representable. The following theorem is one of our main results (see Theorems 11.13 and 16.1 for more general versions).

Theorem 1.1. *Let $\mathcal{Y}$ and $\mathcal{X}$ be DM stacks of finite type over k, and suppose that they have the same pure dimension and are generically smooth over k. Let $f\colon \mathcal{Y} \to \mathcal{X}$ be a proper birational morphism. Then, for a measurable function h on $\mathcal{J}_\infty\mathcal{X}$,*

$$\int_{\mathcal{J}_\infty\mathcal{X}} \mathbb{L}^{h+\mathfrak{s}_\mathcal{X}}\, d\mu_\mathcal{X} = \int_{\mathcal{J}_\infty\mathcal{Y}} \mathbb{L}^{h\circ f_\infty - \mathrm{j}_f + \mathfrak{s}_\mathcal{Y}}\, d\mu_\mathcal{Y}\,.$$

Note that this theorem as well as some theorems below generalize to formal DM stacks over $k[\![t]\!]$. Actually, we first prove them for formal DM stacks over $k[\![t]\!]$ and then specialize them to the case of DM stacks over k.

We call the new terms $\mathfrak{s}_\mathcal{X}$ and $\mathfrak{s}_\mathcal{Y}$ in the above formula *shift functions*. When $\mathcal{X}$ is tame and smooth, the function $\mathfrak{s}_\mathcal{X}$ is basically the same as the *age* invariant in the McKay correspondence or the *fermion shift* in physics (see [Rei02, Zas93]). In the wild and linear case, the function $\mathfrak{s}_\mathcal{X}$ has been determined in [Yas17a] and has turned out to be related to *Artin and Swan conductors* [WY15]. Furthermore, it later turned out to be essentially the same as Fröhlich's *module resolvent* [Frö76]. Determining $\mathfrak{s}_\mathcal{X}$ in the non-linear and singular cases (Definition 11.6) is also one of the new results in this paper.

1.4 Stringy motives

As an application of Theorem 1.1, we obtain the main property of stringy motives of stacky log pairs, that is, the invariance under crepant birational transforms. Consider a log pair $(\mathcal{X}, A)$ as is standard in birational geometry, allowing $\mathcal{X}$ to be a normal DM stack over k. We define its *stringy motive* to be the motivic integral

$$\mathrm{M}_{\mathrm{st}}(\mathcal{X}, A) := \int_{\mathcal{J}_\infty \mathcal{X}} \mathbb{L}^{\mathfrak{f}_{(\mathcal{X},A)}+\mathfrak{s}_{\mathcal{X}}}\, d\mu_{\mathcal{X}}\,,$$

where $\mathfrak{f}_{(\mathcal{X},A)}$ is a function measuring the difference of $\bigwedge^d \Omega_{\mathcal{X}/k}$ and the (log) canonical sheaf of $(\mathcal{X}, A)$ (see Definition 13.1). This generalizes the corresponding invariants previously considered in more restrictive situations (see for instance [Bat99b, DL02, Yas06]). When $\mathcal{X}$ is a smooth algebraic space and $A = 0$, this invariant specializes to the class $\{\mathcal{X}\}$ of $\mathcal{X}$ in our version of the complete Grothendieck ring of varieties. Using Theorem 1.1, we will prove the main property of this invariant, as follows.

Theorem 1.2 (Theorem 16.2). *For a proper birational* (*not necessarily representable*) *morphism* $\mathcal{Y} \to \mathcal{X}$ *giving a crepant morphism* $(\mathcal{Y}, B) \to (\mathcal{X}, A)$ *of log pairs, we have* $\mathrm{M}_{\mathrm{st}}(\mathcal{Y}, B) = \mathrm{M}_{\mathrm{st}}(\mathcal{X}, A)$.

This is a broad generalization of the theorem of Batyrev [Bat99a] and Kontsevich (stated at a lecture at Orsay in 1995) that K-equivalent complex smooth proper varieties have equal Hodge numbers as well as its counterpart for complex orbifolds [LP04, Yas04, Yas06].

1.5 The wild McKay correspondence

A typical example of a proper birational morphism of DM stacks is the morphism

$$f\colon \left[\mathbb{A}_k^d/G\right] \to \mathbb{A}_k^d/G$$

from the quotient stack to the quotient scheme associated with a faithful linear action of a finite group G on the affine space $\mathbb{A}_k^d$. Indeed, if $U \subset \mathbb{A}_k^d$ is the open locus where the G-action is free, then the quotient scheme U/G is an open subscheme of both $\left[\mathbb{A}_k^d/G\right]$ and $\mathbb{A}_k^d/G$, and hence the morphism f is birational. Note that f is not representable unless G is trivial. Treating such non-representable birational morphisms is the reason why we introduce the notion of twisted arcs. If the G-action has no pseudo-reflection, then the morphism is an isomorphism in codimension 1 and hence a crepant morphism in the sense that the relative canonical divisor $K_{[\mathbb{A}_k^d/G]/(\mathbb{A}_k^d/G)}$ vanishes. Theorem 1.2 implies the following theorem, which we call the *wild McKay correspondence*.

Theorem 1.3 (Corollaries 14.4 and 16.3). *Suppose that a finite group G acts linearly and faithfully on an affine space $\mathbb{A}_k^d$ without pseudo-reflection. Then*

$$\mathrm{M}_{\mathrm{st}}\left(\mathbb{A}_k^d/G\right) = \int_{\Delta_G} \mathbb{L}^{d-v}\,.$$

Here Δ_G denotes the moduli space of G-torsors over the punctured formal disk $\operatorname{Spec} k(\!(t)\!)$ and v is a function on Δ_G closely related to the shift function $\mathfrak{s}_{\mathcal{X}}$. The well-definedness of this integral was proved in [TY23]. The integral on the right side is interpreted as the stringy motive of the quotient stack $\mathcal{X} = \left[\mathbb{A}_k^d/G\right]$. When working over an algebraically closed field of characteristic zero, the integral on the right side reduces to the finite sum

$$\sum_{[g]\in \mathrm{Conj}(G)} \mathbb{L}^{d-\mathrm{age}(g)}\,.$$

Thus Theorem 1.3 is a generalization of the McKay correspondence by Batyrev [Bat99b] and Denef–Loeser [DL02] in characteristic zero. Note that Theorem 1.3 for the cyclic group of prime order equal to the characteristic was essentially obtained in [Yas14] and that the point-counting version of the theorem was proved in [Yas17b].

We give two applications of the wild McKay correspondence. The first application is an estimate of discrepancy of quotient singularities; the discrepancy is a basic invariant of singularities in the minimal model program. The following corollary generalizes results in [Yas14, Yas19] for the cyclic group of prime order to an arbitrary finite group.

COROLLARY 1.4 (Corollary 16.4). *Suppose that a finite group G acts linearly and faithfully on an affine space $\mathbb{A}_k^d$ and that G has no pseudo-reflections. Let $X := \mathbb{A}_k^d/G$.*

(1) *We have*

$$\operatorname{discrep}(\text{centers} \subset X_{\text{sing}}; X) = d - 1 - \max\left\{\dim X_{\text{sing}}, \dim \int_{\Delta_G \setminus \{o\}} \mathbb{L}^{d-v}\right\}.$$

Here $o \in \Delta_G$ is the point corresponding to the trivial G-torsor.

(2) *If the integral $\int_{\Delta_G} \mathbb{L}^{d-v}$ converges, then X is log terminal. If X has a log resolution, then the converse is also true.*

This corollary allows us to compute in theory the discrepancy by computing the integral $\int_{\Delta_G} \mathbb{L}^{d-v}$, which is more of arithmetic nature. This computation was carried out in [Yas19, Yas14] for the group $\mathbb{Z}/p\mathbb{Z}$ with p the characteristic of k. The case of $\mathbb{Z}/p^2\mathbb{Z}$ was studied in a paper by Tanno and the author [TY21].

The second application of the wild McKay correspondence is the following motivic version of Bhargava's mass formula [Bha07].

COROLLARY 1.5 (Corollary 16.5). *Let S_n be the nth symmetric group, and let $\mathbf{a}\colon \Delta_{S_n} \to \mathbb{Z}$ be the Artin conductor. Then*

$$\int_{\Delta_{S_n}} \mathbb{L}^{-\mathbf{a}} = \sum_{j=0}^{n-1} P(n, n-j)\mathbb{L}^{-j},$$

where $P(n, m)$ denotes the number of partitions of n into exactly m parts.

The proof of this is basically a translation of the proof given in [Yas17b] of the original formula into the motivic context.

1.6 Untwisting by Hom stacks

The key ingredient in the construction of our theory is the technique of *untwisting*, which reduces the study of twisted arcs to that of ordinary/untwisted arcs. The origin of this technique is found in [DL02]. In [Yas17a, Yas16], it was further generalized to wild linear actions and non-linear actions on affine schemes, respectively. In this paper, we develop it further and make it more intrinsic by using Hom stacks.

For untwisting, it is essential to work with formal DM stacks over $k[\![t]\!]$. Based on a work of Tonini and the author [TY20], we construct a moduli stack of twisted formal disks, denoted by Θ, and a universal family $\mathcal{E}_\Theta \to \operatorname{Spf} k[\![t]\!] \times \Theta$. For a formal DM stack $\mathcal{X}$ of finite type over $\operatorname{Spf} k[\![t]\!]$, we define its *untwisting stack* as

$$\operatorname{Utg}_\Theta(\mathcal{X}) := \underline{\operatorname{Hom}}^{\text{rep}}_{\operatorname{Spf} k[\![t]\!] \times \Theta}(\mathcal{E}_\Theta, \mathcal{X} \times \Theta),$$

the Hom stack of representable morphisms. For a technical reason, we modify it to what is denoted by $\mathrm{Utg}_\Gamma(\mathcal{X})^{\mathrm{pur}}$ by stratifying Θ to another stack Γ and killing t-torsions. Then the desired stack $\mathcal{J}_\infty\mathcal{X}$ of twisted arcs is obtained as the stack of untwisted arcs $\mathrm{J}_\infty \mathrm{Utg}_\Gamma(\mathcal{X})^{\mathrm{pur}}$ of $\mathrm{Utg}_\Gamma(\mathcal{X})^{\mathrm{pur}}$. We can give it the motivic measure in the same way as in the case of schemes.

The proof of Theorem 1.1 is also based on untwisting. We reduce the theorem to the case where $\mathcal{X}$ is a formal algebraic space X. The given morphism $f\colon \mathcal{Y} \to X$ induces a morphism

$$f^{\mathrm{utg}}\colon \mathrm{Utg}_\Gamma(\mathcal{Y})^{\mathrm{pur}} \to X\,,$$

which in turn induces a map

$$\mathcal{J}_\infty\mathcal{Y} = \mathrm{J}_\infty \mathrm{Utg}_\Gamma(\mathcal{Y})^{\mathrm{pur}} \to \mathrm{J}_\infty X\,.$$

Since this is a map of untwisted arc spaces, it is relatively easy to generalize the change of variables formula to this map. We will complete the proof by rewriting the formula in terms of f instead of f^{utg}. At this point, the shift function $\mathfrak{s}_\mathcal{Y}$ naturally appears as the difference of the Jacobian orders of f and f^{utg}.

1.7 Related topics

To conclude the introduction, we mention a few related topics. Understanding wild ramification in higher dimension is one of the central themes in arithmetic geometry (see for instance [Sai10, Ill15]). The theory developed in this paper would be regarded as a new approach to it. Given a wildly ramified finite morphism $f\colon Y \to X$ of some spaces, one natural method is to look at ramification of f along a general curve $C \to X$. In our approach, we look at all "curves" rather than a general one and integrate all information on ramification along them.

Recently Groechenig, Wyss and Ziegler [GWZ20] found an interesting application of a version of p-adic integration on DM stacks to the study of Hitchin fibration. The motivic version of their result was then obtained by Loeser and Wyss [LW21]. Although they considered only the tame case, it would be interesting to incorporate the wild action into the subject.

1.8 Terminology, notation and conventions

Here we fix our basic setup such as terminology, notation and conventions. For the reader's convenience, we explain some notions whose definitions will be given later.

We denote the set of non-negative integers by $\mathbb{N}$. For a category $\mathcal{C}$, we mean by $c \in \mathcal{C}$ that c is an object of $\mathcal{C}$. We use the symbol $\coprod$ to mean the coproduct (abstract disjoint union) and reserve the symbol $\bigsqcup$ to mean the disjoint union of subsets.

We fix a base field k of characteristic $p \geqslant 0$. In Section 4 and Section 5.3, we assume $p > 0$. At the beginning of these (sub)sections, we will repeat this assumption.

We denote by $\mathbf{Aff}$ or $\mathbf{Aff}_k$ the category of affine k-schemes. For an affine k-scheme $S = \operatorname{Spec} R$, we denote by $\mathbf{Aff}_R$ or $\mathbf{Aff}_S$ the category of affine schemes over S. Unless otherwise noted, all schemes, algebraic spaces and stacks as well as their morphisms are defined over k. Thus they are regarded as categories fibered in groupoids over $\mathbf{Aff}$. Similarly, unless otherwise noted, all rings and fields contain k, and their maps are supposed to be k-homomorphisms. We consider only *separated* schemes, algebraic spaces and (formal) DM stacks. Namely, their diagonal morphisms are proper and hence finite.

A DM stack $\mathcal{X}$ is said to be *almost of finite type* if $\mathcal{X}$ is isomorphic to the coproduct $\coprod_i \mathcal{X}_i$ of countably many DM stacks $\mathcal{X}_i$ of finite type.

For a k-algebra R, we denote by $R[\![t]\!]$ (respectively, $R(\!(t)\!)$) the ring of power series (respectively, Laurent power series) over R. Thus $R(\!(t)\!)$ is the localization $R[\![t]\!]_t$ of $R[\![t]\!]$ by t (not the total quotient ring of $R[\![t]\!]$). For an affine scheme $S = \operatorname{Spec} R$, we denote by D_S (respectively, D_S^*, Df_S) the affine scheme $\operatorname{Spec} R[\![t]\!]$ (respectively, the affine scheme $\operatorname{Spec} R(\!(t)\!)$, the formal affine scheme $\operatorname{Spf} R[\![t]\!]$). We often replace the subscript S with R. When $R = k$, we usually omit the subscript.

For a stack $\mathcal{X}$ over $\mathbf{Aff}$, by the 2-Yoneda lemma, we often identify an object of $\mathcal{X}$ over $S \in \mathbf{Aff}$ with a morphism $S \to \mathcal{X}$. We call it an *S-point* of $\mathcal{X}$. We denote the point set of $\mathcal{X}$ by $|\mathcal{X}|$; this is the set of equivalence classes of geometric points $\operatorname{Spec} K \to \mathcal{X}$. We give it the Zariski topology. For a field K, we denote the set of isomorphism classes of K-points, $\mathcal{X}(K)/\cong$, by $\mathcal{X}[K]$. More generally, for a subset $C \subset |\mathcal{X}|$, we denote by $C(K)$ the full subcategory of $\mathcal{X}(K)$ consisting of K-points $\operatorname{Spec} K \to \mathcal{X}$ mapping into C. Then $C[K]$ denotes its set of isomorphism classes, $C[K] := C(K)/\cong$.

For a formal DM stack $\mathcal{X}$ over Df and $n \in \mathbb{N}$, unless otherwise noted, we denote the fiber product $\mathcal{X} \times_{\mathrm{Df}} \operatorname{Spec} k[\![t]\!]/(t^{n+1})$ by $\mathcal{X}_n$.

We say that a morphism $\mathcal{Y} \to \mathcal{X}$ of DM stacks is a *thickening* if it is representable (by algebraic spaces) and a closed immersion such that the map $|\mathcal{Y}| \to |\mathcal{X}|$ is bijective.

We say that a morphism $\mathcal{Y} \to \mathcal{X}$ of DM stacks is *birational* if it restricts to an isomorphism $\mathcal{Y}' \to \mathcal{X}'$ of open dense substacks $\mathcal{Y}' \subset \mathcal{Y}$ and $\mathcal{X}' \subset \mathcal{X}$.

We say that a finite group G is *tame* (*with respect to p*) if p does not divide $\sharp G$. We say that a formal DM stack is *tame* if the automorphism group of every geometric point is tame.

2. Ind-DM stacks and formal DM stacks

In this section, we introduce formal DM stacks, which are the main object of our study, as a special case of ind-DM stacks. We then study their basic properties such as ones concerning coarse moduli spaces.

Definition 2.1. Let $f\colon \mathcal{Y} \to \mathcal{X}$ be a morphism of stacks. We say that f is *representable* if for every morphism $U \to \mathcal{X}$ from an algebraic space, the fiber product $U \times_{\mathcal{X}} \mathcal{Y}$ is isomorphic to an algebraic space. Let $\mathbf{P}$ be a property of morphisms of algebraic spaces which is stable under base change. We say that a representable morphism $f\colon \mathcal{Y} \to \mathcal{X}$ of stacks *has property* $\mathbf{P}$ if for every morphism $U \to \mathcal{X}$ from an algebraic space, the projection $\mathcal{Y} \times_{\mathcal{X}} U \to U$ has property $\mathbf{P}$.

Definition 2.2. An *ind-DM stack* (respectively, *ind-algebraic space, ind-scheme*, *ind-affine scheme*) is a fibered category over $\mathbf{Aff}$ isomorphic to the limit $\varinjlim \mathcal{X}_n$ of an inductive system $\mathcal{X}_0 \to \mathcal{X}_1 \to \cdots$ indexed by $\mathbb{N}$ of DM stacks (respectively, algebraic spaces, schemes, affine schemes) in the sense of [TY20, Appendix A]. An ind-DM stack (respectively, ind-algebraic space, ind-scheme, ind-affine scheme) is called a *formal DM stack* (respectively, *formal algebraic space, formal scheme*, *formal affine scheme*) if we can further assume that the transition maps $\mathcal{X}_n \to \mathcal{X}_{n+1}$ are (necessarily representable) thickenings.

By [TY20, Proposition A.5], an ind-DM stack is a stack. In [Sta20, tag 0AIH], a formal affine scheme is called a countably indexed affine formal algebraic space. There, it is showed that a formal affine scheme can be written as

$$\operatorname{Spf} A = \varinjlim \operatorname{Spec} A/I_n\,,$$

where A is a weakly admissible topological ring which has a fundamental system

$$I_0 \supset I_1 \supset \cdots$$

of neighborhoods of $0 \in A$ consisting of weak ideals of definition. In a linearly topologized ring A, a weak ideal of definition is an ideal consisting of topologically nilpotent elements (elements f such that $f^n \to 0$ as $n \to \infty$), and a weakly admissible ring is a separated and complete linearly topologized ring having a weak ideal of definition.

For a formal DM stack $\mathcal{X} = \varinjlim \mathcal{X}_n$, its point set $|\mathcal{X}|$ is defined in the same way as for a DM stack; it is the set of equivalence classes of geometric points $\operatorname{Spec} K \to \mathcal{X}$ (that is, such morphisms for algebraically closed fields K). It is identified with $|\mathcal{X}_n|$ for any n. Thus $|\mathcal{X}|$ has the topology induced from the Zariski topology on $|\mathcal{X}_n|$.

Lemma 2.3. *Let $\mathcal{X}$ be a formal DM stack.*

(1) *The diagonal morphism $\Delta \colon \mathcal{X} \to \mathcal{X} \times \mathcal{X}$ is representable, unramified and finite (and hence affine).*

(2) *For morphisms $U \to \mathcal{X}$ and $V \to \mathcal{X}$ from affine schemes (respectively, formal affine schemes), the fiber product $U \times_{\mathcal{X}} V$ is an affine scheme (respectively, a formal affine scheme).*

Proof. (1) Suppose $\mathcal{X} \cong \varinjlim \mathcal{X}_n$ for an inductive system $\mathcal{X}_n$, $n \in \mathbb{N}$, as in the definition. Let S be an affine scheme, and let $f \colon S \to \mathcal{X} \times \mathcal{X}$ be a morphism. By the definition of limits, f factors through $\mathcal{X}_n \times \mathcal{X}_n$ for $n \gg 0$. By [TY20, Proposition A.2], we have

$$S \times_{f, \mathcal{X} \times \mathcal{X}, \Delta} \mathcal{X} \cong \varinjlim (S \times_{\mathcal{X}_n \times \mathcal{X}_n} \mathcal{X}_n) .$$

However, the transition morphisms $S \times_{\mathcal{X}_n \times \mathcal{X}_n} \mathcal{X}_n \to S \times_{\mathcal{X}_{n+1} \times \mathcal{X}_{n+1}} \mathcal{X}_{n+1}$ are isomorphisms. Therefore, for $n \gg 0$, we have

$$S \times_{f, \mathcal{X} \times \mathcal{X}, \Delta} \mathcal{X} \cong S \times_{\mathcal{X}_n \times \mathcal{X}_n} \mathcal{X}_n .$$

Since the diagonal of $\mathcal{X}_n$ is representable, unramified and finite, the right side of this isomorphism is an affine scheme which is unramified and finite over S. This shows statement (1).

(2) By statement (1), if U and V are affine schemes, then

$$U \times_{\mathcal{X}} V \cong (U \times V) \times_{\mathcal{X} \times \mathcal{X}, \Delta} \mathcal{X}$$

is also an affine scheme. Next suppose that U and V are formal affine schemes and are written, respectively, as $\varinjlim U_n$ and $\varinjlim V_n$ as limits of affine schemes. Moreover, we may suppose that morphisms $U \to \mathcal{X}$ and $V \to \mathcal{X}$ are given by morphisms $U_n \to \mathcal{X}_n$ and $V_n \to \mathcal{X}_n$. Then

$$U \times_{\mathcal{X}} V \cong \varinjlim (U_n \times_{\mathcal{X}_n} V_n) .$$

Since the $U_n \times_{\mathcal{X}_n} V_n$ are affine schemes and their transition morphisms are thickenings, the limit is a formal affine scheme. ☐

Lemma 2.4. *Let $\mathcal{X}$ be a formal DM stack. There exists a family of representable étale morphisms $V_i \to \mathcal{X}$, $i \in I$, such that the V_i are formal affine schemes and $\coprod_i V_i \to \mathcal{X}$ is surjective.*

Proof. Suppose $\mathcal{X} = \varinjlim \mathcal{X}_n$, where the $\mathcal{X}_n$, $n \in \mathbb{N}$, are DM stacks. There exists a family of étale morphisms $V_{i,0} \to \mathcal{X}_0$ from affine schemes such that $\coprod_i V_{i,0} \to \mathcal{X}_0$ is surjective. By Lemma A.8, they uniquely extend to $V_{i,n} \to \mathcal{X}_n$ with the same property, and the diagrams

$$\begin{tikzcd} V_{i,n} \arrow[r] \arrow[d] & V_{i,n+1} \arrow[d] \\ \mathcal{X}_n \arrow[r] & \mathcal{X}_{n+1} \end{tikzcd}$$

are 2-Cartesian. Let $V_i := \varinjlim V_{i,n}$, which is a formal affine scheme. By [TY20, Corollary A.3], the diagrams

$$\begin{tikzcd} V_{i,n} \arrow[r] \arrow[d] & V_i \arrow[d] \\ \mathcal{X}_n \arrow[r] & \mathcal{X} \end{tikzcd}$$

are also 2-Cartesian. Therefore, the morphisms $V_i \to \mathcal{X}$ satisfy the desired property. □

Definition 2.5. We call a morphism $\coprod_i V_i \to \mathcal{X}$ or the formal scheme $\coprod_i V_i$ as above an *atlas* of $\mathcal{X}$.

Definition 2.6. We define $\mathrm{D}_n := \operatorname{Spec} k[\![t]\!]/(t^{n+1})$ and $\mathrm{Df} := \operatorname{Spf} k[\![t]\!] = \varinjlim \mathrm{D}_n$. For a DM stack Φ, we define $\mathrm{D}_{\Phi,n} := \mathrm{D}_n \times \Phi$ and $\mathrm{Df}_\Phi := \mathrm{Df} \times \Phi = \varinjlim \mathrm{D}_{\Phi,n}$. When $\Phi = \operatorname{Spec} R$, we also write them as $\mathrm{D}_{R,n}$ and Df_R; they are isomorphic to $\operatorname{Spec} R[\![t]\!]/(t^{n+1})$ and $\operatorname{Spf} R[\![t]\!]$. (We denote the affine scheme $\operatorname{Spec} R[\![t]\!]$ by D_R.)

Definition 2.7. We say that a morphism $f\colon \mathcal{Y} \to \mathcal{X}$ of formal DM stacks is *of finite type* if for every morphism $\mathcal{U} \to \mathcal{X}$ from a DM stack, the fiber product $\mathcal{U} \times_{\mathcal{X}} \mathcal{Y}$ is a DM stack of finite type over $\mathcal{U}$.

Notation 2.8. For a formal DM stack $\mathcal{X}$ of finite type over Df_Φ and $n \in \mathbb{N}$, we denote the fiber product $\mathcal{X} \times_{\mathrm{Df}} \mathrm{D}_n = \mathcal{X} \times_{\mathrm{Df}_\Phi} \mathrm{D}_{\Phi,n}$ by $\mathcal{X}_n$.

With this notation, we have

$$\mathcal{X} \cong \mathcal{X} \times_{\mathrm{Df}} \varinjlim \mathrm{D}_n \cong \varinjlim \mathcal{X}_n \,.$$

Recall that the restricted power series ring $k[\![t]\!]\{x_1, \dots, x_n\}$ is the t-adic completion of the ring $k[\![t]\!][x_1, \dots, x_n]$. A $k[\![t]\!]$-algebra is topologically finitely generated over $k[\![t]\!]$ if and only if it is isomorphic to a quotient ring of $k[\![t]\!]\{x_1, \dots, x_n\}$ for some $n \in \mathbb{N}$.

Lemma 2.9. *Let $\mathcal{X}$ be a formal DM stack of finite type over* Df. *Then there exists an atlas* $\operatorname{Spf} A \to \mathcal{X}$ *from a formal affine scheme such that A is a topologically finitely generated $k[\![t]\!]$-algebra.*

Proof. Let us take an atlas $\operatorname{Spec} B_0 \to \mathcal{X}_0$ from an affine scheme. The ring B_0 is finitely generated as a k-algebra. Using Lemma A.8, we can inductively construct the following 2-Cartesian diagrams for $n \in \mathbb{N}$:

$$\begin{tikzcd} \operatorname{Spec} B_n \arrow[r] \arrow[d] & \operatorname{Spec} B_{n+1} \arrow[d] \\ \mathcal{X}_n \arrow[r] & \mathcal{X}_{n+1} \,. \end{tikzcd}$$

Let $A := \varprojlim B_n$. By [GD60, Chapter 0, Proposition 7.5.5], the ring A is topologically finitely generated as a $k[\![t]\!]$-algebra. The morphism $\operatorname{Spf} A \to \mathcal{X}$ is an atlas. □

Lemma 2.10. *Let $\mathcal{X}_n$, $n \in \mathbb{N}$, be DM stacks of finite type over D_n, and let $\mathcal{X}_n \to \mathcal{X}_{n+1}$, $n \in \mathbb{N}$, be morphisms compatible with $\mathrm{D}_n \to \mathrm{D}_{n+1}$. Suppose that $\mathcal{X}_n \to \mathcal{X}_{n+1} \times_{\mathrm{D}_{n+1}} \mathrm{D}_n$ is an isomorphism. Then $\mathcal{X} := \varinjlim \mathcal{X}_n$ is a formal DM stack of finite type over* Df.

Proof. By the assumptions, $\mathcal{X}_n \to \mathcal{X}_{n+1}$ is a thickening. Therefore, $\mathcal{X}$ is a formal DM stack. To check that it is of finite type over Df, it suffices to consider a morphism $\mathcal{U} \to \mathrm{Df}$ from a quasi-compact DM stack. The morphism factors through D_n for $n \gg 0$. We have

$$\mathcal{U} \times_{\mathrm{Df}} \mathcal{X} \cong \mathcal{U} \times_{\mathrm{D}_n} \mathrm{D}_n \times_{\mathrm{Df}} \mathcal{X} \cong \mathcal{U} \times_{\mathrm{D}_n} \mathcal{X}_n \,,$$

which is of finite type over $\mathcal{U}$. □

Definition 2.11. A *coarse moduli space* of an ind-DM stack $\mathcal{X}$ is an ind-algebraic space X together with a morphism $\pi\colon \mathcal{X} \to X$ such that

(1) for any morphism $f\colon \mathcal{X} \to Y$ with Y an ind-algebraic space, there exists a unique morphism $g\colon X \to Y$ such that $g \circ \pi = f$;
(2) for any algebraically closed field K/k, the map

$$\mathcal{X}[K](:= \mathcal{X}(K)/\cong) \to Y(K)$$

is bijective.

Obviously, if it exists, a coarse moduli space of an ind-DM stack $\mathcal{X}$ is unique up to unique isomorphism. Therefore, we call it *the* coarse moduli space of $\mathcal{X}$.

Lemma 2.12. *For a formal DM stack $\mathcal{X} = \varinjlim \mathcal{X}_n$ with $\mathcal{X}_n$ quasi-compact, let X_n be the coarse moduli space of $\mathcal{X}_n$ (which exists by the Keel–Mori theorem [KM97]). The limit $X = \varinjlim X_n$ is the coarse moduli space of $\mathcal{X}$.*

Proof. Condition (2) in Definition 2.11 clearly holds. We will show the universality. Let $f\colon \mathcal{X} \to Y$ be a morphism with $Y = \varinjlim Y_i$ an ind-algebraic space. Since $\mathcal{X}_n$ is a quasi-compact DM stack, the composition $\mathcal{X}_n \to \mathcal{X} \to Y$ factors through $Y_{i(n)}$ for some $i(n)$. The morphism $\mathcal{X}_n \to Y_{i(n)}$ uniquely factors as $\mathcal{X}_n \to X_n \to Y_{i(n)}$. We can take a strictly increasing function as $i(n)$. The morphisms $X_n \to Y_{i(n)}$ define a morphism $g\colon X \to Y$ such that the composition $\mathcal{X} \to X \to Y$ is equal to f. It remains to show the uniqueness of the morphism $X \to Y$. Let $h\colon X \to Y$ be another morphism such that $\mathcal{X} \to X \xrightarrow{h} Y$ is equal to f. Suppose that h is represented by morphisms $h_n\colon X_n \to Y_{j(n)}$. Replacing both $i(n)$ and $j(n)$ with $\max\{i(n), j(n)\}$, we may suppose that $i(n) = j(n)$. By the universality of the coarse moduli space $\mathcal{X}_n \to X_n$, two morphisms $g_n, h_n\colon X_n \rightrightarrows Y_{i(n)}$ coincide. Therefore, $g = h$. □

Lemma 2.13. *Let $\mathcal{X}$ be a formal DM stack with the coarse moduli space $\mathcal{X} \to X$. Let $Y \to X$ be a morphism of ind-algebraic spaces which is representable by algebraic spaces and flat. Then Y is the coarse moduli space of $\mathcal{X} \times_X Y$.*

Proof. Let us write $\mathcal{X}$ as the limit $\varinjlim \mathcal{X}_n$ of DM stacks $\mathcal{X}_n$, and let X_n be the coarse moduli space of $\mathcal{X}_n$. Let $Y_n := Y \times_X X_n$ and $\mathcal{Y}_n := \mathcal{X}_n \times_{X_n} Y_n$. Since $Y_n \to X_n$ is a flat morphism of algebraic spaces, Y_n is the coarse moduli space of $\mathcal{Y}_n$. By [TY20, Proposition A.2] and Lemma 2.12, we have

$$Y \cong Y \times_X \varinjlim X_n \cong \varinjlim (Y \times_X X_n) \cong \varinjlim Y_n \,.$$

Similarly,

$$\mathcal{X} \times_X Y \cong \mathcal{X} \times_X \varinjlim Y_n \cong \varinjlim \mathcal{Y}_n \,.$$

By Lemma 2.12, the limit $Y \cong \varinjlim Y_n$ is the coarse moduli space of $\mathcal{X} \times_X Y \cong \varinjlim \mathcal{Y}_n$. □

For a further analysis of coarse moduli spaces, we recall the following result. See Section 4 for the relative Frobenius morphism.

Lemma 2.14 ([Fog80, Proposition 1 in § 2 (p. 1160) and Theorem in § 5 (p. 1172)]). *Let R be a Noetherian k-algebra such that the relative Frobenius $F_{R/k}\colon \operatorname{Spec} R \to (\operatorname{Spec} R)^{(1)}$ is a finite morphism (this holds for instance if R is a quotient of $k[\![t_1, \dots, t_m]\!]\{x_1, \dots, x_n\}$). Suppose that a finite group G acts on R as a k-algebra. Then the invariant subalgebra R^G is Noetherian and R is a finitely generated R^G-module.*

Lemma 2.15. *With the same assumption as in Lemma 2.14, for an ideal $I \subset R^G$ and for every n, we have*

$$R^G/I^n = \mathrm{Im}\big((R/I^{n'}R)^G \to (R/I^nR)^G\big) \quad (n' \gg n)\,.$$

In particular, the projective system $(R/I^nR)^G$, $n \in \mathbb{N}$, satisfies the Mittag-Leffler condition.

Proof. Since R is a finitely generated R^G-module, we can identify R with the quotient module

$$\big(R^G\big)^{\oplus l} / \sum_{q=1}^{c} R^G \cdot {}^t\big(f_1^{(q)}, \dots, f_l^{(q)}\big) \quad \big(f_j^{(q)} \in R^G\big)\,.$$

Here we regard elements of $\big(R^G\big)^{\oplus l}$ as column vectors. For each $g \in G$, we fix a lift $\tilde{g}\colon \big(R^G\big)^{\oplus l} \to \big(R^G\big)^{\oplus l}$ of $g\colon R \to R$ and suppose that it is given by the left multiplication with an l-by-l matrix $(\tilde{g}_{ij})_{1 \leqslant i,j \leqslant l}$, $\tilde{g}_{ij} \in R^G$. Then $(R/I^nR)^G$ consists of the residue classes $\overline{{}^t(x_1, \dots, x_l)}$ of those vectors ${}^t(x_1, \dots, x_l) \in \big(R^G\big)^{\oplus l}$ such that for some $y_j^{(q)} \in R^G$ ($1 \leqslant q \leqslant c$, $1 \leqslant j \leqslant l$), we have

$$x_i - \sum_{j=1}^{l} \tilde{g_{ij}} x_j + \sum_{q=1}^{c} \sum_{j=1}^{l} y_j^{(q)} f_j^{(q)} \equiv 0 \mod I^n \quad (g \in G\,,\ 1 \leqslant i \leqslant l)\,.$$

We regard these equations as R^G-linear equations with unknowns x_i, $y_j^{(q)}$ and apply a version of the approximation theorem for linear equations, [Ron06, Theorem 3.1]. There exists an $i_0 \in \mathbb{N}$ such that for a solution $\big(x_i, y_j^{(q)}\big)_{i,j,q} \in \big(R^G\big)^{\oplus(c+1)l}$ of the above equation for $n = i + i_0$, there exist $\big(\check{x}_i, \check{y}_j^{(q)}\big)_{i,j,q} \in \big(R^G\big)^{\oplus(c+1)l}$ satisfying

$$\check{x}_i - \sum_{j=1}^{l} \tilde{g_{ij}} \check{x}_j + \sum_{q=1}^{c} \sum_{j=1}^{l} \check{y}_j^{(q)} f_j^{(q)} = 0 \quad (g \in G\,,\ 1 \leqslant i \leqslant l)$$

and

$$\big(x_i, y_j^{(q)}\big)_{i,j,q} \equiv \big(\check{x}_i, \check{y}_j^{(q)}\big)_{i,j,q} \mod I^i\,.$$

This implies that if an element $\boldsymbol{x}$ of $\big(R/I^iR\big)^G$ lifts to an element of $\big(R/I^{i+i_0}R\big)^G$, then $\boldsymbol{x}$ also lifts to an element of R^G. The first assertion of the lemma follows. The second assertion is then obvious. □

Corollary 2.16. *Under the assumption of Lemma 2.14, for an ideal $I \subset R$, we have a natural isomorphism of ind-schemes*

$$\varinjlim \operatorname{Spec} R^G/I^n \cong \varinjlim \operatorname{Spec} (R/I^nR)^G\,.$$

Proof. Maps $R^G/I^n \to (R/I^nR)^G$ define a morphism

$$a\colon \varinjlim \operatorname{Spec} (R/I^nR)^G \to \varinjlim \operatorname{Spec} R^G/I^n\,.$$

Let $\beta\colon \mathbb{N} \to \mathbb{N}$ be a strictly increasing function such that $\mathrm{Im}\big(\big(R/I^{\beta(n)}R\big)^G \to (R/I^nR)^G\big) = R^G/I^n$. The maps $\big(R/I^{\beta(n)}R\big)^G \to R^G/I^n$ define a morphism

$$b\colon \varinjlim \operatorname{Spec} R^G/I^n \to \varinjlim \operatorname{Spec}\big(R/I^{\beta(n)}R\big)^G \cong \varinjlim \operatorname{Spec} (R/I^nR)^G\,.$$

We see that a and b are inverse to each other. □

Let $\operatorname{Spf} R$ be a formal affine scheme of finite type over Df with an action of a finite group G. The quotient stack $[(\operatorname{Spf} R)/G]$ is defined in the same way as in the case of non-formal schemes and isomorphic to $\varinjlim[(\operatorname{Spec} R/(t^n))/G]$.

Lemma 2.17. *With the above notation, suppose that R satisfies the assumption of Lemma 2.14. Then the coarse moduli space of $[(\operatorname{Spf} R)/G]$ is $\operatorname{Spf} R^G$. Moreover, if $\operatorname{Spf} R$ is of finite type over* Df, *then so is $\operatorname{Spf} R^G$.*

Proof. The coarse moduli space of $[(\operatorname{Spec} R/(t^n))/G]$ is $\operatorname{Spec}(R/(t^n))^G$. By Lemma 2.12, the limit $\varinjlim \operatorname{Spec}(R/(t^n))^G$ is the coarse moduli space of $[(\operatorname{Spf} R)/G]$. By Corollary 2.16, this is isomorphic to $\operatorname{Spf} R^G$. The first assertion has been proved. If $\operatorname{Spf} R$ is of finite type over Df, then since $(R/(t^n))^G \to R^G/(t^m)$ is surjective for $n \gg m$, we see that $\left(\operatorname{Spf} R^G\right)_m = \operatorname{Spec} R^G/(t^{m+1})$ is of finite type over k. By Lemma 2.10, the second assertion follows. □

The following proposition says that every formal DM stack of finite type over Df is locally a quotient stack.

Proposition 2.18. *Let $\mathcal{X}$ be a formal DM stack of finite type over* Df. *Then the coarse moduli space X of $\mathcal{X}$ is a formal algebraic space of finite type over* Df. *Moreover, there exists an atlas $\coprod_{\gamma=1}^{l} V_\gamma \to X$ such that each V_γ is a formal affine scheme of finite type over* Df *and $\mathcal{X} \times_X V_\gamma \cong [W_\gamma/H_\gamma]$ for a formal affine scheme W_γ with an action of a finite group H_γ. In particular, we have a stabilizer-preserving étale surjective morphism $\coprod_{\gamma=1}^{l}[W_\gamma/H_\gamma] \to \mathcal{X}$.*

Proof. For each n, let $\mathcal{X}_n := \mathcal{X} \times_D D_n$ and X_n be its coarse moduli space. By Lemma 2.12, the ind-algebraic space $X = \varinjlim X_n$ is the coarse moduli space of $\mathcal{X}$. As is well known (see [AV02, Lemma 2.2.3]), there exists an étale cover $(V_{0,\gamma} \to X_0)_{1\leqslant\gamma\leqslant l}$ such that for each γ, the scheme $V_{0,\gamma}$ is affine of finite type over k and $\mathcal{W}_{0,\gamma} := V_{0,\gamma} \times_{X_0} \mathcal{X}_0$ is isomorphic to $[W_{0,\gamma}/H_\gamma]$ for some affine scheme $W_{0,\gamma}$ and a finite constant group H_γ acting on it. The morphism $\mathcal{W}_{0,\gamma} = [W_{0,\gamma}/H_\gamma] \to \mathcal{X}_0$ is representable, étale, affine and stabilizer preserving, because $V_{0,\gamma} \to X_0$ is so. (Recall that we consider only separated algebraic spaces. Therefore, $V_{0,\gamma} \to X_0$ is automatically affine.) We get the diagram

$$W_{0,\gamma} \times H_\gamma \rightrightarrows W_{0,\gamma} \to \mathcal{W}_{0,\gamma} \to \mathcal{X}_0$$

of representable étale affine morphisms such that $W_{0,\gamma} \times H_\gamma \rightrightarrows W_{0,\gamma}$ defines a groupoid in schemes and $\mathcal{W}_{0,\gamma}$ is the associated stack. Let $U \to \mathcal{X}$ be an étale morphism with U a formal affine scheme. By pulling back by $U_0 \to \mathcal{X}_0$, we get a similar diagram of affine schemes

$$W_{0,\gamma,U_0} \times H_\gamma \rightrightarrows W_{0,\gamma,U_0} \to \mathcal{W}_{0,\gamma,U_0} \to U_0 \,.$$

Note that $\mathcal{W}_{0,\gamma,U_0}$ is a scheme in spite of its calligraphic letter. From the invariance of small étale sites [GD67, Theorem 18.1.2], we get the corresponding diagram over each U_n,

$$W_{n,\gamma,U_n} \times H_\gamma \rightrightarrows W_{n,\gamma,U_n} \to \mathcal{W}_{n,\gamma,U_n} \to U_n \,.$$

By [Sta20, tag 05YU], these are all affine schemes and give quasi-coherent sheaves on the small étale site on $\mathcal{X}_n$. Taking their relative spectra, we get the diagram of étale affine morphisms

$$W_{n,\gamma} \times H_\gamma \rightrightarrows W_{n,\gamma} \to \mathcal{W}_{n,\gamma} \to \mathcal{X}_n$$

such that $\mathcal{W}_{n,\gamma} = [W_{n,\gamma}/H_\gamma]$. Since $\mathcal{W}_{n,\gamma} \to \mathcal{X}_n$ is étale and stabilizer preserving, if $V_{n,\gamma}$ denotes the coarse moduli space of $\mathcal{W}_{n,\gamma}$, then $V_{n,\gamma} \to X_n$ is étale and $\mathcal{W}_{n,\gamma} \cong \mathcal{X}_n \times_{X_n} V_{n,\gamma}$. Moreover, we have natural morphisms $V_{n,\gamma} \to V_{n+1,\gamma}$. Let $W_\gamma := \varinjlim W_{n,\gamma}$, $\mathcal{W}_\gamma := \varinjlim \mathcal{W}_{n,\gamma}$ and $V_\gamma := \varinjlim V_{n,\gamma}$. We have $\mathcal{W}_\gamma \cong [W_\gamma/H_\gamma] \cong \mathcal{X} \times_X V_\gamma$. The W_γ are formal affine schemes of finite type over Df.

The morphisms $V_{n,\gamma} \to X_n$ are étale. The diagram

$$\begin{tikzcd} V_{n,\gamma} \arrow[r] \arrow[d] & V_{n+1,\gamma} \arrow[d] \\ X_n \arrow[r] & X_{n+1} \end{tikzcd}$$

is Cartesian. Indeed, since the horizontal arrows are universal homeomorphisms and the vertical arrows are étale, the morphism $V_n \to V_{n+1,\gamma} \times_{X_{n+1}} X_n$ is étale and a universal homeomorphism. Thus it is an isomorphism. By [TY20, Corollary A.3], the morphism $V_\gamma \to X$ is representable and étale. By Lemma 2.13, the ind-algebraic space V_γ is the coarse moduli space of $\mathcal{W}_\gamma$. By Lemma 2.17, the ind-algebraic spaces V_γ are formal affine schemes of finite type over Df.

It remains to show that X is a formal algebraic space. For $s \geqslant n$, let $Y_{n,s} \subset X_s$ be the scheme-theoretic image of X_n. By Lemma 2.15, there exists a strictly increasing function $s\colon \mathbb{N} \to \mathbb{N}$ such that for any n and any $s' \geqslant s(n)$, the morphism $Y_{n,s(n)} \to Y_{n,s'}$ is an isomorphism. The natural morphism $Y_{n,s(n)} \to Y_{n+1,s(n+1)}$ is a thickening, and we have $Y_{n+1,s(n+1)} \times_{\mathrm{D}_{n+1}} \mathrm{D}_n \cong Y_{n,s(n)}$. The morphisms $X_n \to Y_{n,s(n)}$ give a morphism $X \to \varinjlim Y_{n,s(n)}$, and the morphisms $Y_{n,s(n)} \to X_{s(n)}$ give a morphism $\varinjlim Y_{n,s(n)} \to X$. These are inverse to each other. Thus $X \cong \varinjlim Y_{n,s(n)}$, which is a formal algebraic space. By Lemma 2.10, it is of finite type over Df. $\square$

3. Galoisian group schemes

In this section, we study moduli stacks of finite group schemes of some class. In what follows, we often identify a finite group G with the constant group scheme $\coprod_{g\in G} \operatorname{Spec} F$ over the spectrum of a field F.

Definition 3.1. A finite group G is said to be *Galoisian* if there exist an algebraically closed field K and a Galois extension $L/K(\!(t)\!)$ such that $G \cong \mathrm{Gal}(L/K(\!(t)\!))$. We denote by GalGps a set of representatives of isomorphism classes of Galoisian groups.

A finite étale group scheme G over a scheme S is said to be *Galoisian* if for every geometric point $s\colon \operatorname{Spec} K \to S$, the fiber $G_s = G \times_S \operatorname{Spec} K$ is a Galoisian finite group. For a Galoisian group G, a finite étale group scheme $G' \to S$ is said to be *G-Galoisian* if every geometric fiber G'_s is isomorphic to G or, equivalently, if it is a twisted form of the constant group scheme $G \times S \to S$.

When $p = 0$, Galoisian groups are nothing but cyclic groups. When $p > 0$, a Galoisian group is isomorphic to the semidirect product $H \rtimes C$ of a p-group H and a tame cyclic group C. For these facts, see [Ser79, § IV.2, pp. 67–68]. In particular, a Galoisian group has a unique p-Sylow subgroup. Any subgroup of a Galoisian group, as well as any quotient group, is again Galoisian.

Definition 3.2. We define a category $\mathcal{A} \to \mathbf{Aff}$ fibered in groupoids as follows. An object over $S \in \mathbf{Aff}$ is a Galoisian group scheme $H \to S$. A morphism $(H \to S) \to (H' \to S')$ is the pair of a morphism $S \to S'$ and an isomorphism of group schemes $H \to H' \times_{S'} S$. We call $\mathcal{A}$ the *moduli stack of Galoisian group schemes.* For an isomorphism class $[G]$ of a Galoisian group G, we define $\mathcal{A}_{[G]} \subset \mathcal{A}$ to be the full subcategory of G-Galoisian group schemes.

We have $\mathcal{A} = \coprod_{G\in \mathrm{GalGps}} \mathcal{A}_{[G]}$. Let $\mathcal{G}$ be the universal Galoisian group scheme over $\mathcal{A}$. This is a category fibered in sets over $\mathcal{A}$ such that the fiber set over a Galoisian group scheme $G \to S$ is the set of its sections $G(S)$. The morphism $\mathcal{G} \to \mathcal{A}$ is representable and is finite and étale. In what follows, for a finite group H or a finite group scheme H over k, we denote by $\mathrm{B}(H)$ the quotient stack $[\operatorname{Spec} k/H]$ associated with the trivial H-action on $\operatorname{Spec} k$. The fiber $(\mathrm{B}(H))(S)$ over $S \in \mathbf{Aff}$ is the groupoid of H-torsors $T \to S$.

Lemma 3.3. *For a Galoisian group G, we have $\mathcal{A}_{[G]} \cong \mathrm{B}(\operatorname{Aut} G)$.*

Proof. We define a functor $\Phi\colon \mathrm{B}(\operatorname{Aut} G) \to \mathcal{A}_{[G]}$. With an $\operatorname{Aut} G$-torsor $T \to S$, we associate the quotient $G_T/\operatorname{Aut} G$ of $G_T := G \times T$ by the diagonal action, which is a group scheme over S and a twisted form of the constant group scheme $G_S \to S$. Next we define a functor $\Psi\colon \mathcal{A}_{[G]} \to \mathrm{B}(\operatorname{Aut} G)$. With a G-Galoisian group scheme $H \to S$, we associate an $\operatorname{Aut} G$-torsor $\underline{\mathrm{Iso}}_S(G_S, H)$. Here $\alpha \in \operatorname{Aut}(G)$ acts by $\alpha \cdot \phi := \phi \circ \alpha^{-1}$.

Let us show that these functors are quasi-inverse to each other. For an $\operatorname{Aut} G$-torsor $T \to S$, we have a natural isomorphism

$$u_{T\to S}\colon T \to \underline{\mathrm{Iso}}_S(G_S, G_T/\operatorname{Aut} G)\,, \quad t \mapsto \big(g \mapsto \overline{(g,t)}\big)\,.$$

This is $\operatorname{Aut} G$-equivariant. Indeed, for $\alpha \in \operatorname{Aut} G$, the above functor sends $\alpha \cdot t$ to

$$\big(g \mapsto \overline{(g, \alpha\cdot t)}\big) = \big(g \mapsto \overline{(\alpha^{-1}g, t)}\big) = \alpha \cdot \big(g \mapsto \overline{(g,t)}\big)\,.$$

Since any equivariant morphism of torsors is an isomorphism, $u_{T\to S}$ is an isomorphism of $\operatorname{Aut} G$-torsors and defines a natural isomorphism $u\colon \mathrm{id}_{\mathrm{B}(\operatorname{Aut} G)} \to \Psi \circ \Phi$.

For a G-Galoisian group scheme $H \to S$, we have a morphism

$$G_S \times \underline{\mathrm{Iso}}_S(G_S, H) \to H\,, \quad (g, \phi) \mapsto \phi(g)\,.$$

This is $\operatorname{Aut} G$-invariant. Indeed, for $\alpha \in \operatorname{Aut} G$, we have

$$\alpha(g,\phi) = \big(\alpha(g), \phi \circ \alpha^{-1}\big) \mapsto \phi \circ \alpha^{-1}(\alpha(g)) = \phi(g)\,.$$

Thus we obtain a morphism

$$v_{H\to S}\colon (G_S \times \underline{\mathrm{Iso}}_S(G_S, H))/\operatorname{Aut} G \to H\,.$$

Étale locally on S, this is identified with the isomorphism

$$G_S \times \underline{\mathrm{Iso}}_S(G_S, G_S)/\operatorname{Aut} G \to G_S\,.$$

Therefore, $v_{H\to S}$ is an isomorphism. We obtain a natural isomorphism $v\colon \Phi\circ\Psi \to \mathrm{id}_{\mathcal{A}_{[G]}}$. □

Corollary 3.4. *For a Galoisian group G, the categories $\mathcal{A}_{[G]}$ and $\mathcal{G}_{[G]} := \mathcal{G} \times_{\mathcal{A}} \mathcal{A}_{[G]}$ are DM stacks étale, proper and quasi-finite over k.*

Proof. By Lemma 3.3, the category $\mathcal{A}_{[G]}$ is a DM stack étale, proper and quasi-finite over k. The morphism $\mathcal{G} \to \mathcal{A}$ is representable, étale and finite. Therefore, $\mathcal{G}_{[G]}$ is also a DM stack étale, proper and quasi-finite over k. □

4. Frobenius morphisms and ind-perfection

In this section, we assume $p > 0$ and discuss Frobenius morphisms and ind-perfection of DM stacks.

For an $\mathbb{F}_p$-algebra A, the pth power map $A \to A$, $f \mapsto f^p$ gives a scheme endomorphism $F_S\colon S \to S$ of the affine scheme $S = \operatorname{Spec} A$. This is called the *absolute Frobenius morphism* of S. Let $\mathcal{X}$ be a stack over $\mathbb{F}_p$. For each $S \in \mathbf{Aff}_{\mathbb{F}_p}$, we have the pullback functor $F_S^*\colon \mathcal{X}(S) \to \mathcal{X}(S)$ of groupoids. These functors give an endomorphism of $\mathcal{X}$ over $\mathbb{F}_p$, which we call the *absolute Frobenius morphism of* $\mathcal{X}$ and denote by $F_{\mathcal{X}}$. Let S be an affine scheme over $\mathbb{F}_p$ and $\mathcal{X}$ a stack over S with structure morphism $\pi\colon \mathcal{X} \to S$. We define the *relative Frobenius morphism* $F_{\mathcal{X}/S}$ to be the morphism

$$(F_{\mathcal{X}}, \pi)\colon \mathcal{X} \to \mathcal{X}^{(1)} := \mathcal{X} \times_{\pi,S,F_S} S\,.$$

We say that $\mathcal{X}$ is *perfect over S* if $F_{\mathcal{X}/S}$ is an isomorphism.

For each $n \in \mathbb{N}$, we also define the *n-iterated relative Frobenius morphism* $F^n_{\mathcal{X}/S}$ of $\mathcal{X}$ by

$$\left(F^n_{\mathcal{X}}, \pi\right)\colon \mathcal{X} \to \mathcal{X}^{(n)} := \mathcal{X} \times_{\pi,S,F^n_S} S\,.$$

We define the *ind-perfection* $\mathcal{X}^{\mathrm{iper}}$ of $\mathcal{X}$ to be the limit $\varinjlim \mathcal{X}^{(n)}$ as a category. This has the natural structure of a stack over S, see [TY20, Appendix A], which is clearly perfect over S. We have the expected universality, as follows.

Lemma 4.1. *For a morphism $f\colon \mathcal{Y} \to \mathcal{X}$ of stacks over S with $\mathcal{X}$ perfect over S, there exists an S-morphism $\mathcal{Y}^{\mathrm{iper}} \to \mathcal{X}$, unique up to 2-isomorphisms, such that the composition $\mathcal{Y} \to \mathcal{Y}^{\mathrm{iper}} \to \mathcal{X}$ is isomorphic to f.*

Proof. For $n \in \mathbb{N}$, we fix a quasi-inverse $\mathcal{X}^{(n+1)} \to \mathcal{X}^{(n)}$ of $\mathcal{X}^{(n)} \to \mathcal{X}^{(n+1)}$ so that we get the composition

$$\mathcal{X}^{(n)} \to \mathcal{X}^{(n-1)} \to \cdots \to \mathcal{X}^{(0)} = \mathcal{X}\,.$$

To $T \in \mathbf{Aff}_S$ and an object $y \in \mathcal{Y}^{\mathrm{iper}}(T)$ represented by $y' \in \mathcal{Y}^{(n)}(T)$, we assign its image by $\mathcal{Y}^{(n)}(T) \xrightarrow{f} \mathcal{X}^{(n)}(T) \xrightarrow{\sim} \mathcal{X}(T)$. The induced morphism $\mathcal{Y}^{\mathrm{iper}} \to \mathcal{X}$ is the desired morphism. □

Lemma 4.2. *Let $\mathcal{X}_0 \to \mathcal{X}_1 \to \cdots$ be an inductive system of DM stacks of finite type such that every transition map is representable and is a finite universal homeomorphism. Suppose that the limit $\mathcal{Y} := \varinjlim \mathcal{X}_i$ is perfect over k. Then the natural morphism $\mathcal{X}_0^{\mathrm{iper}} \to \mathcal{Y}$ is an isomorphism.*

Proof. Consider the inductive system $\mathcal{X}_i^{(n)}$, $(i,n) \in \mathbb{N} \times \mathbb{N}$. We have

$$\varinjlim_i \mathcal{X}_i^{\mathrm{iper}} \cong \varinjlim_i \varinjlim_n \mathcal{X}_i^{(n)} \cong \varinjlim_n \varinjlim_i \mathcal{X}_i^{(n)} \cong \varinjlim_n \mathcal{Y}^{(n)}\,.$$

Since $\mathcal{Y}$ is perfect, the morphism $\mathcal{Y}^{(n)} \to \mathcal{Y}^{(n+1)}$ are all isomorphisms. We have $\varinjlim_n \mathcal{Y}^{(n)} \cong \mathcal{Y}$. We claim that the morphisms $\mathcal{X}_i^{\mathrm{iper}} \to \mathcal{X}_{i+1}^{\mathrm{iper}}$ are isomorphisms. Indeed, for each i, there exists an $m \in \mathbb{N}$ such that the morphism $\mathcal{X}_i \to \mathcal{X}_i^{(m)}$ factors through $\mathcal{X}_{i+1}$. We obtain morphisms

$$\mathcal{X}_i \to \mathcal{X}_{i+1} \to \mathcal{X}_i^{(m)} \to \mathcal{X}_{i+1}^{(m)}\,.$$

Taking the ind-perfections, we obtain the following commutative diagram:

$$\begin{tikzcd} \mathcal{X}_i^{\mathrm{iper}} \arrow[r] \arrow[rr, bend left, "\mathrm{id}"] & \mathcal{X}_{i+1}^{\mathrm{iper}} \arrow[r] \arrow[rr, bend right, "\mathrm{id}"'] & \mathcal{X}_i^{\mathrm{iper}} \arrow[r] & \mathcal{X}_{i+1}^{\mathrm{iper}} \end{tikzcd}\,.$$

This shows the claim. From the claim, we deduce $\varinjlim_i \mathcal{X}_i^{\mathrm{iper}} \cong \mathcal{X}_0^{\mathrm{iper}}$, and the lemma follows. □

5. Formal torsors

Based upon results in [TY20], in this section, we construct moduli stacks of torsors over the punctured formal disk and a "universal" family of twisted formal disks.

5.1 The moduli stack $\boldsymbol{\Delta}$

For an affine scheme $S = \operatorname{Spec} R$, we denote by D_S^* or D_R^* the affine scheme $\operatorname{Spec} R(\!(t)\!)$. Here $R(\!(t)\!)$ denotes the localization $R[\![t]\!]_t$ of $R[\![t]\!]$ by t (rather than the total quotient ring of $R[\![t]\!]$). Namely, this is the ring of Laurent power series with coefficients in R.

Definition 5.1. For a finite étale group scheme $G \to S$ over an affine scheme S, we define a category fibered in groupoids $\Delta_G \to \mathbf{Aff}_S$ as follows: For an affine S-scheme T, the fiber category $\Delta_G(T)$ is the category of G-torsors over D_T^*. For a finite group G, we define $\Delta_G \to \mathbf{Aff}_k$ by regarding G as a constant group scheme over k.

Definition 5.2. We define a fibered category $\Delta \to \mathcal{A}$ as follows: For a Galoisian group scheme $G \to S$, the fiber category $\Delta(G \to S)$ is $\Delta_G(S)$. (Thus the whole category Δ is the category of pairs $(G \to S, P \to \mathrm{D}_S^*)$ of a Galoisian group scheme $G \to S$ and a G-torsor $P \to \mathrm{D}_S^*$.) For an isomorphism class $[G]$ of Galoisian groups, we define $\Delta_{[G]} := \Delta \times_{\mathcal{A}} \mathcal{A}_{[G]}$.

For a morphism $S \to \mathcal{A}$ corresponding to a Galoisian group scheme $G \to S$, we have $\Delta \times_{\mathcal{A}} S \cong \Delta_G$. Note that as a fibered category over $\mathbf{Aff}$, the scheme S is $\mathbf{Aff}_S$ and the projection $\Delta \times_{\mathcal{A}} S \to S = \mathbf{Aff}_S$ corresponds to the underlying functor $\Delta_G \to \mathbf{Aff}_S$.

5.2 Uniformization

Definition 5.3. For a finite étale group scheme $G \to S$, a G-torsor $P \to \mathrm{D}_S^*$ and a geometric point $x\colon \operatorname{Spec} F \to S$, we call the induced G_F-torsor

$$P_x := P \times_{\mathrm{D}_S^*} \mathrm{D}_F^* \to \mathrm{D}_F^*$$

the *fiber over* x. We say that $P \to \mathrm{D}_S^*$ or P is *fiberwise connected* or *has connected fibers* if for every geometric point x of S, the fiber P_x is connected.

Remark 5.4. The fiber defined above is not the fiber of $P \to S$ over x in the usual sense; P_x is not the fiber product $P \times_S \operatorname{Spec} F$ but the "complete fiber product."

The following notions are taken from [TY23], though we restrict ourselves to fiberwise connected torsors.

Definition 5.5. Let $G \to \operatorname{Spec} R$ be a Galoisian group scheme, and let $P = \operatorname{Spec} A \to \mathrm{D}_R^*$ be a fiberwise connected G-torsor. We say that this is *uniformizable* if there exists an isomorphism $A \cong R(\!(s)\!)$ such that the map $R(\!(t)\!) \to A \cong R(\!(s)\!)$ sends t to a series of the form $s^{\sharp G}u$ for $u \in R[\![s]\!]^*$. We call such an isomorphism $A \cong R(\!(s)\!)$ a *uniformization* of P or A.

When this is the case, the image of s in A gives a uniformizer of the discrete valuation field $A \otimes_{R(\!(t)\!)} K(\!(t)\!)$ for any point $\operatorname{Spec} K \to \operatorname{Spec} R$.

Lemma 5.6. *With notation as above, if R is reduced and A is uniformizable, then the image of $R[\![s]\!] \to A$ is independent of the choice of the uniformization $A \cong R(\!(s)\!)$.*

Proof. Let $A \cong R(\!(r)\!)$ be another isomorphism as in Definition 5.5. We regard $R[\![s]\!]$ and $R[\![r]\!]$ as subrings of A and write $r = su$ with $u \in A = R(\!(s)\!)$. For any point $\operatorname{Spec} K \to \operatorname{Spec} R$, the image of u in $K(\!(s)\!) = A \otimes_{R(\!(t)\!)} K(\!(t)\!)$ is a unit of $K[\![s]\!]$. This shows that $u \in R[\![s]\!]^*$ since R is reduced. Here we used the fact that for a reduced ring R, an element $u \in R$ is zero (respectively, a unit) if and only if for every point $\operatorname{Spec} K \to \operatorname{Spec} R$, the image of u in K is zero (respectively, a unit). It follows that $r \in R[\![s]\!]$. Similarly $s \in R[\![r]\!]$. Hence $R[\![s]\!] = R[\![r]\!]$. $\square$

Definition 5.7. With notation as above, when R is reduced and A is uniformizable, we define a subring $O_A \subset A$ to be the image of $R[\![s]\!]$ by a uniformization $A \cong R(\!(s)\!)$. We call it the *integral ring* of A or of the corresponding torsor $\operatorname{Spec} A \to \mathrm{D}_R^*$.

Lemma 5.8. *Under the assumption of Definition 5.7, the integral ring O_A is stable under the G-action on A.*

Proof. Let $A \cong R(\!(s)\!)$ be an isomorphism as above. For $g \in G$ and for every point $\operatorname{Spec} K \to \operatorname{Spec} R$, the image of $g(s)$ in $K(\!(s)\!)$ is a uniformizer. By the same argument as in the proof of Lemma 5.6, the element $g(s)$ is of the form su with $u \in R[\![s]\!]^*$. In particular, $g(s) \in O_A$, which implies the lemma. □

5.3 Ramification

In this subsection, we assume $p > 0$.

Let G be a Galoisian group, and let $N \subset H \subset G$ be a subgroups such that N is a normal subgroup of H. For a fiberwise connected G-torsor $P \to \mathrm{D}_S^*$ and a geometric point $x\colon \operatorname{Spec} F \to S$, we have an H/N-torsor $(P/N)_x \to (P/H)_x$ with connected source and target. This corresponds to a Galois extension L/K of complete discrete valuation fields with the common residue field F; its Galois group $\operatorname{Gal}(L/K)$ is identified with H/N. The Galois group is equipped with the (lower-numbering) ramification filtration

$$\operatorname{Gal}(L/K) = \operatorname{Gal}(L/K)_0 \supset \operatorname{Gal}(L/K)_1 \supset \cdots$$

such that $\operatorname{Gal}(L/K)_i = 1$ for $i \gg 0$ (see for instance [Ser79, Chapter IV]). When $\operatorname{Gal}(L/K) \cong H/N$ is a cyclic group of order p, there exists a unique i such that $\operatorname{Gal}(L/K)_i \neq \operatorname{Gal}(L/K)_{i+1}$. This i is called the *ramification jump* of the H/N-torsor $(P/N)_x \to (P/H)_x$. If we write $(P/H)_x = \operatorname{Spec} F(\!(s)\!)$, then by Artin–Schreier theory, we can write

$$(P/N)_x = \operatorname{Spec} F(\!(s)\!)[x]/(x^p - x - f)$$

for $f \in F(\!(s)\!)$ of the form

$$f = \sum_{j>0,\, p\nmid j} f_j s^{-j} \quad \bigl(f_j \in F\bigr).$$

The action of $G = \langle g \rangle$ is given by $g\colon x \mapsto x+1$. This f is uniquely determined, and the ramification jump is equal to $\max\{j \mid f_j \neq 0\}$. In particular, the ramification jump is prime to p. If two geometric points x and x' have the same image in S, then the ramification jumps of $(P/N)_x \to (P/H)_x$ and $(P/N)_{x'} \to (P/H)_{x'}$ are equal. Thus we obtain a function $|S| \to \mathbb{N}$.

Definition 5.9. Let G be a Galoisian group. A *ramification datum* for G is a function

$$\mathbf{r}\colon (H,N) \mapsto \mathbf{r}(H,N) \in \mathbb{N}$$

on pairs (H,N) of a subgroup $H \subset G$ and a normal subgroup $N \lhd H$ with $\sharp(H/N) = p$. We denote the set of ramification data for G by $\operatorname{Ram}(G)$.

Let $\mathbf{r}$ be a ramification datum for G. For a subgroup $G' \subset G$, restricting to pairs (H,N) with $H \subset G'$, we obtain a ramification datum on G'. When G' is a normal subgroup, we also have the induced ramification datum $\mathbf{s}$ for G/G'. For $N \lhd H \subset G/G'$, if $\tilde{N}, \tilde{H} \subset G$ are their preimages, then $\mathbf{s}(H,N) := \mathbf{r}\bigl(\tilde{H}, \tilde{N}\bigr)$.

Definition 5.10. When G is a Galoisian group and $P \to \mathrm{D}_F^*$ is a connected G-torsor with F an algebraically closed field, we define the ramification datum $\mathbf{r}_P$ so that $\mathbf{r}_P(H,N)$ is the ramification jump of $P/N \to P/H$. We say that a fiberwise connected G-torsor $P \to \mathrm{D}_S^*$ has *constant ramification* if the map

$$|S| \to \mathrm{Ram}(G)\,, \quad x \mapsto \mathbf{r}_{P_x}$$

is constant. When $G \to S$ is a Galoisian group scheme, we say that a G-torsor $P \to \mathrm{D}_S^*$ has *locally constant ramification* if there exists an étale cover $(S_i \to S)_{i\in I}$ such that for each i, the fiber product $G \times_S S_i$ is constant and $P_{S_i} \to \mathrm{D}_{S_i}^*$ has constant ramification.

For $n \in \mathbb{N}$ and $S \in \mathbf{Aff}$, let $S^{(-n)}$ be the scheme S regarded as a k-scheme by the morphism

$$S \to \operatorname{Spec} k \xrightarrow{(F_{\operatorname{Spec} k})^n} \operatorname{Spec} k\,.$$

The morphism

$$S^{(-n)} = S \xrightarrow{(F_S)^n} S$$

is a k-morphism with respect to the k-scheme structure $S^{(-n)}$. When k is perfect, we have canonical k-isomorphisms $\left(S^{(-n)}\right)^{(n)} \cong \left(S^{(n)}\right)^{(-n)} \cong S$.

Proposition 5.11. *Let G be a p-group.*

(1) *Let $\mathbf{r} \in \mathrm{Ram}(G)$, and let $P \to \mathrm{D}_S^*$ be a G-torsor. Let*

$$C := \{[x] \in |S| \mid P_x \text{ is connected and } \mathbf{r}_{P_x} = \mathbf{r}\}\,.$$

Then C is a locally closed subset of $|S|$.

(2) *Let $P \to \mathrm{D}_S^*$ be a fiberwise connected G-torsor having constant ramification. Suppose that S is reduced. For $n \gg 0$, the induced G-torsor $P_{S^{(-n)}} \to \mathrm{D}_{S^{(-n)}}^*$ is uniformizable.*

Proof. We first prove both assertions in the case $G = \mathbb{Z}/p$. Write $S = \operatorname{Spec} R$. There exists a Laurent polynomial of the form

$$f = f_j t^{-jp^m} + f_{j-1} t^{-(j-1)p^m} + \cdots + f_1 t^{-p^m} + f_0 \quad \left(f_i \in R\right),$$

such that for $i > 0$ with $p \mid i$, we have $f_i = 0$ and $P = \operatorname{Spec} R(\!(t)\!)[x]/(x^p - x + f)$ (see [TY20, Theorem 4.13] and the part of its proof concerning the essential surjectivity). Consider the iterated Frobenius $S^{(-m)} = \operatorname{Spec} R^{(-m)} \to S = \operatorname{Spec} R$ with corresponding map $\phi\colon R \to R^{(-m)}$. For each i, there exists a $g_i \in R^{(-m)}$ such that $(g_i)^{p^m} = \phi(f_i)$. By [TY20, Lemma 4.9], the torsor $P_{S^{(-m)}}$ is then isomorphic to the torsor associated with

$$g = g_j t^{-j} + g_{j-1} t^{-(j-1)} + \cdots + g_0\,.$$

For $j_0 > 0$, the locus in $S^{(-m)}$ of connected fibers and ramification jump j_0 is

$$\left(\bigcap_{j>j_0} V(g_j)\right) \setminus V(g_{j_0})\,.$$

This shows assertion (1) in the case $G = \mathbb{Z}/p$. Note that the locus of non-connected fibers is $\bigcap_{j>0} V(g_j)$. To show assertion (2) in this case, suppose that S is reduced and that P has constant ramification jump j_0. Then for every $j > j_0$, we have $V(g_j) = S^{(-m)}$, thus $g_j = 0$. As $V(g_{j_0}) = \emptyset$, the element g_{j_0} is invertible. Since j_0 is prime to p, there exist positive integers a, b such that $-aj + bp = 1$. Let us write $S^{(-m)} = \operatorname{Spec} R'$ and $P_{S^{(-m)}} = \operatorname{Spec} A = \operatorname{Spec} R'(\!(t)\!)[y]/(y^p - y + g)$, and let $s := y^a t^b \in A$. For each point $r\colon \operatorname{Spec} K \to S^{(-m)}$, the elements $s_r, y_r \in A_r$ deduced from s, y, respectively, have valuation 1 and $-j_0$ with respect to the normalized valuation of A_r.

We have

$$R'(\!(t)\!)[y]/(y^p - y + g) = \bigoplus_{l=0}^{p-1} R'(\!(t)\!) y^l \,,$$

and each s^i is uniquely written as $s^i = \sum_{l=0}^{p-1} t_{i,l} y^l$, $t_{i,l} \in R'(\!(t)\!)$. For each point r, the element $(t_{i,l})_r$ has valuation at least $i - lj_0$. Since R' is reduced, we have $(t_{i,l}) \in t^{i-lj_0} \cdot R'[\![t]\!]$. It follows that any power series $\sum_{i \geqslant 0} a_i s^i$, $a_i \in R'$, in the variable s converges. It follows that there exists a unique map $\phi\colon R'[\![u]\!] \to A$ sending u to s. Thus we may think of the image of ϕ as the power series ring $R[\![s]\!]$ with variable s. Clearly, $R'(\!(s)\!) \cong R'[\![s]\!] \otimes_{R'[\![t]\!]} R'(\!(t)\!) \cong A$. The image of t in A is written as us^p, $u \in R(\!(s)\!)$, such that at each point $r\colon \operatorname{Spec} K \to S$, the image $u_r \in K(\!(s)\!)$ of u is a unit of $K[\![s]\!]$. This means that $u \in R[\![s]\!]^*$. Thus the isomorphism $R'(\!(s)\!) \cong A$ is a uniformization. Assertion (2) holds when $G = \mathbb{Z}/p$.

Next we prove the general case by induction. Let $n \in \mathbb{N}$. We assume that both assertions hold if $\sharp G < n$ and will prove the case $\sharp G = n$. Let $P \to \mathrm{D}_S^*$ be a G-torsor. For every normal subgroup $N \lhd G$ of order p, the locus where the G/N-torsor $P/N \to \mathrm{D}_S^*$ has connected fibers, and the ramification datum induced from $\mathbf{r}$ is a locally closed subset, say $C \subset S$. We give it the reduced structure. For any affine open $U \subset C$ and for $m \gg 0$, the torsor $(P/N)_{U(-m)}$ is uniformizable. From the case $G = \mathbb{Z}/p$, it follows that the locus where the N-torsor $P_{U(-m)} \to (P/N)_{U(-m)} = \mathrm{D}_{S(-m)}^*$ has connected fibers and ramification jump induced by $\mathbf{r}$ is locally closed. This shows assertion (1). If P has connected fibers and constant ramification and if S is reduced, then by the induction hypothesis, $(P/N)_{S(-m)}$ is uniformizable for $m \gg 0$. The N-torsor $P_{S(-m)} \to (P/N)_{S(-m)} = \mathrm{D}_{S(-m)}^*$ also has constant ramification. Thus, by assertion (2) for the case of $\mathbb{Z}/p$, for $n \gg m$, the N-torsor $P_{S(-n)} \to (P/N)_{S(-n)} = \mathrm{D}_{S(-n)}^*$ is uniformizable, which implies assertion (2) for a general p-group. □

Definition 5.12. We define Λ (respectively, $\Lambda_{[G]}$, Λ_G) to be the full substack of Δ (respectively, $\Delta_{[G]}$, Δ_G) of fiberwise connected torsors with locally constant ramification. When G is constant and $\mathbf{r}$ is a ramification datum for G, we define $\Lambda_G^{\mathbf{r}}$ to be the full subcategory of Λ_G of torsors with ramification datum $\mathbf{r}$.

For a Galoisian group G, we have $\Lambda_{[G]} = \Lambda \times_{\mathcal{A}} \mathcal{A}_{[G]}$ and $\Lambda_G = \coprod_{\mathbf{r} \in \mathrm{Ram}(G)} \Lambda_G^{\mathbf{r}}$. For a Galoisian group scheme $G \to S$ corresponding to $S \to \mathcal{A}$, we have $\Lambda_G = \Lambda \times_{\mathcal{A}} S$.

Definition 5.13. For a ramification datum $\mathbf{r}$ for a Galoisian group G, let $[\mathbf{r}]$ be its $\mathrm{Aut}(G)$-orbit. We define $\Lambda_{[G]}^{[\mathbf{r}]}$ to be the essential image of the functor $\coprod_{\mathbf{s} \in [\mathbf{r}]} \Lambda_G^{\mathbf{s}} \to \Lambda_{[G]}$.

For $\operatorname{Spec} k \to \mathcal{A}_{[G]}$ corresponding to the constant group G, we have

$$\Lambda_{[G]}^{[\mathbf{r}]} \times_{\mathcal{A}_{[G]}} \operatorname{Spec} k \cong \coprod_{\mathbf{s} \in [\mathbf{r}]} \Lambda_G^{\mathbf{s}} \,.$$

Lemma 5.14. *For a Galoisian group G and a ramification datum $\mathbf{r}$, the stacks Δ_G and $\Lambda_G^{\mathbf{r}}$ are perfect over k.*

Proof. If we prove the lemma when $k = \mathbb{F}_p$, then the general case follows by base change along $\operatorname{Spec} k \to \operatorname{Spec} \mathbb{F}_p$. Thus we may suppose that k is perfect. For $S \in \mathbf{Aff}$, the Frobenius morphism

$$\Delta_G(S) \to \Delta_G\big(S^{(-1)}\big) = \Delta_G^{(1)}(S)$$

is an equivalence [TY20, Proposition 4.6]. By this equivalence, the property of having connected fibers is preserved. The ramification data are also preserved. This proves the lemma. □

Lemma 5.15. *For a Galoisian group G and a ramification datum $\mathbf{r}$ for G, the stack $\Lambda_G^{\mathbf{r}}$ is isomorphic to the ind-perfection $\mathcal{X}^{\mathrm{iper}}$ of a reduced DM stack $\mathcal{X}$ of finite type. The same is true for $\Lambda_{[G]}^{[\mathbf{r}]}$.*

Proof. Again we may suppose that k is perfect. By [TY20], we can write $\Delta_G = \varinjlim \mathcal{Y}_i$ as the limit of DM stacks of finite type such that the transition morphisms $\mathcal{Y}_i \to \mathcal{Y}_{i+1}$ are representable, finite and universally injective. In particular, we have $|\Delta_G| = \bigcup_i |\mathcal{Y}_i|$. Since torsors with the same ramification datum $\mathbf{r}$ have the same discriminant, we have $|\Lambda_G^{\mathbf{r}}| \subset |\mathcal{Y}_{i_0}|$ for some i_0; see [TY23]. (For each n, we can construct a family $P \to \mathrm{D}_S^*$ of totally ramified extensions of $k(\!(t)\!)$ with S of finite type which contains every totally ramified extension of discriminant exponent at most n. This shows that $\Lambda_G^{\mathbf{r}}$ is of finite type.) By Proposition 5.11, the set $|\Lambda_G^{\mathbf{r}}|$ is a locally closed subset of $|\mathcal{Y}_{i_0}|$. For $i \geqslant i_0$, let $\mathcal{X}_i \subset \mathcal{Y}_i$ be the reduced substack such that $|\mathcal{X}_i| = |\Lambda_G^{\mathbf{r}}|$.

Since Δ_G is perfect, we also have $\Delta_G \cong \varinjlim_{i,n} \mathcal{Y}_i^{(n)}$. We claim that

$$\Lambda_G^{\mathbf{r}} \cong \varinjlim_{i,n} \mathcal{X}_i^{(n)} .$$

To show this, we first observe that $\Lambda_G^{\mathbf{r}}$ is a full subcategory of Δ_G and $\varinjlim_{i,n} \mathcal{X}_i^{(n)}$ is a full subcategory of $\varinjlim_{i,n} \mathcal{Y}_i^{(n)}$. We need to show that their objects correspond through the isomorphism $\Delta_G \cong \varinjlim_{i,n} \mathcal{Y}_i^{(n)}$. It is clear that every object of $\varinjlim_{i,n} \mathcal{X}_i^{(n)}$ maps to an object of $\Lambda_G^{\mathbf{r}}$. It remains to show that every object of $\Lambda_G^{\mathbf{r}}$ comes from an object of $\mathcal{X}_i^{(n)}$ for some i, n. Let $P \to \mathrm{D}_S^*$ be an object of $\Lambda_G^{\mathbf{r}}$ over an affine scheme S. For some $i \geqslant i_0$, the morphism $S \to \Lambda_G^{\mathbf{r}}$ factors through $\mathcal{Y}_i$. The image of $|S| \to |\mathcal{Y}_i|$ is contained in $|\mathcal{X}_i|$. Let $\mathcal{U}_i \subset \mathcal{Y}_i$ be an open substack such that $\mathcal{X}_i$ is a closed substack of it. Let $\mathcal{I} \subset \mathcal{O}_{\mathcal{U}_i}$ be the defining ideal of $\mathcal{X}_i$. The morphism $S_{\mathrm{red}} \to S \to \mathcal{U}_i$ factors through $\mathcal{X}_i$, which means that $\mathcal{I}$ is pulled back to the zero ideal sheaf by this morphism. Since $\mathcal{U}_i$ is of finite type and $\mathcal{I}$ is locally finitely generated, for $n \gg 0$, the ideal $\mathcal{I}$ is also pulled back to the zero ideal sheaf by the morphism $S^{(-n)} \to S \to \mathcal{U}_i$. This means that the morphism $S^{(-n)} \to \mathcal{U}_i$ factors through $\mathcal{X}_i$; equivalently, $S \to \mathcal{U}_i^{(n)}$ factors through $\mathcal{X}_i^{(n)}$. We have proved the claim.

By Lemmas 5.14 and 4.2, if we put $\mathcal{X} := \mathcal{X}_{i_0}$, we have $\Lambda_G^{\mathbf{r}} \cong \mathcal{X}^{\mathrm{iper}}$. We have proved the first assertion. Since $\Lambda_{[G]}^{[\mathbf{r}]} \times_{\mathcal{A}_{[G]}} \operatorname{Spec} k \cong \coprod_{\mathbf{s} \in [\mathbf{r}]} \Lambda_G^{\mathbf{s}}$, by [TY20, Lemma 3.5], the stack $\Lambda_{[G]}^{[\mathbf{r}]}$ is also the inductive limit of DM stacks of finite type such that the transition morphisms $\mathcal{Y}_i \to \mathcal{Y}_{i+1}$ are representable, finite and universally injective. The Frobenius morphism of $\Lambda_{[G]}^{[\mathbf{r}]}$ becomes that of $\coprod_{\mathbf{s} \in [\mathbf{r}]} \Lambda_G^{\mathbf{s}}$ by the base change with $\operatorname{Spec} k \to \mathcal{A}_{[G]}$. Therefore, $\Lambda_{[G]}^{[\mathbf{r}]}$ is also perfect over k. The second assertion follows similarly. $\square$

5.4 The stack Λ in characteristic zero or in the tame case

We return to the situation of arbitrary characteristic.

Let C_l be the cyclic group of order l such that if $p > 0$, then $p \nmid l$. Every tame Galoisian group is of this form. In characteristic zero, we define Λ, $\Lambda_{[G]}$ and Λ_G as in Definition 5.12, removing the condition of locally constant ramification; they are simply the substacks of Δ, $\Delta_{[G]}$ and Δ_G, respectively, of fiberwise connected torsors. For the morphism $\mu_l \colon \operatorname{Spec} k \to \mathcal{A}_{[C_l]}$, we have

$$\Lambda_{[C_l]} \times_{\mathcal{A}_{[C_l]}, \mu_l} \operatorname{Spec} k \cong \Lambda_{\mu_l} \cong \coprod_{p \in (\mathbb{Z}/l\mathbb{Z})^*} \mathrm{B}\, \mu_l .$$

Here the second isomorphism follows from [TY20, Theorem B]. In particular, $\Lambda_{[C_l]}$ is a DM stack étale over k. The morphism

$$\Lambda_{\mu_l} = \Lambda_{[C_l]} \times_{\mathcal{A}_{[C_l]}} \operatorname{Spec} k \to \Lambda_{[C_l]}$$

is a torsor under the group $\operatorname{Aut}(C_l) = (\mathbb{Z}/l\mathbb{Z})^*$. The $(\mathbb{Z}/l\mathbb{Z})^*$-action permutes the components of Λ_{μ_l} freely and transitively. Therefore,

$$\Lambda_{[C_l]} \cong [\Lambda_{\mu_l}/(\mathbb{Z}/l\mathbb{Z})^*] \cong \mathrm{B}\,\mu_l$$

and, if $\Lambda_{\mu_l,i}$, $i \in (\mathbb{Z}/l\mathbb{Z})^*$, denotes the ith component, then the morphism

$$\Lambda_{\mu_l,i} \to \Lambda_{[C_l]} \tag{5.1}$$

is an isomorphism.

In characteristic zero, we have $\Lambda = \coprod_{l>0} \Lambda_{[C_l]}$, which is a reduced DM stack almost of finite type. In arbitrary characteristic, we put $\Lambda_{\mathrm{tame}} := \coprod_{l>0;\, p\nmid l} \Lambda_{[C_l]}$. Note that the component $\Lambda_{[1]} = \Lambda_{[C_1]}$ corresponding to the trivial cover $\mathrm{Df} \to \mathrm{Df}$ is isomorphic to $\operatorname{Spec} k$.

5.5 Integral models

Lemma 5.16. *Suppose $p > 0$ (respectively, $p = 0$). Let $(G \to S, P \to \mathrm{D}_S^*) \in \Lambda$, where $G \to S$ is a Galoisian group scheme and $P \to \mathrm{D}_S^*$ is a G-torsor. Suppose that S is reduced. Then there exists a finite étale cover $\coprod_{i=1}^n S_i \to S$ such that for each i, the group scheme G_{S_i} is constant, P_{S_i} has constant ramification and $P_{S_i^{(-m)}}$ is uniformizable for $m \gg 0$ (respectively, P_{S_i} is uniformizable).*

Proof. First consider the case $p > 0$. Since there exists a finite étale cover $U \to S$ such that G_U is Zariski-locally constant, we may suppose that G is constant. Let H be the unique p-Sylow subgroup of G. The quotient group G/H is a tame cyclic group, say of order l. Adding the lth roots of unity to k, we may also suppose that the group scheme μ_l is isomorphic to the constant group G/H. By [TY20, Theorem B], we have

$$\Delta_{G/H} \cong \coprod_{i \in \mathbb{Z}/l\mathbb{Z}} \mathrm{B}(G/H)\,,$$

and components for $i \in (\mathbb{Z}/l\mathbb{Z})^*$ correspond to connected torsors. For $i \in (\mathbb{Z}/l\mathbb{Z})^*$, if i' denotes the integer representing i with $0 < i' < l$, then the standard morphism onto the ith component

$$\operatorname{Spec} k \to \Delta_{G/H}$$

corresponds to the uniformizable torsor $\operatorname{Spec} k(\!(t)\!)[x]/\big(x^l - t^{i'}\big) \to \operatorname{Spec} k(\!(t)\!)$. Since this morphism is finite and étale, there exists a finite étale cover $V \to S$ such that the morphism $V \to S \to \Delta_{G/H}$ lifts to $\coprod_{i \in (\mathbb{Z}/l\mathbb{Z})^*} \operatorname{Spec} k$. Then $(P/H)_V$ is uniformizable. Thus we may also assume that P/H has uniformization $P/H \cong \mathrm{D}_S^*$. Since the H-torsor $P \to P/H \cong \mathrm{D}_S^*$ has locally constant ramification and S is quasi-compact (recall that S is affine), there exists a stratification $V = \bigsqcup_{i=1}^l S_i$ into open and closed subsets such that each P_{S_i} has constant ramification. By Proposition 5.11, for $m \gg 0$, the H-torsor $P_{S_i^{(-m)}} \to P_{S_i^{(-m)}}/H \cong \mathrm{D}^*_{S_i^{(-m)}}$ is uniformizable, and so is the G-torsor $P_{S_i^{(-m)}} \to \mathrm{D}^*_{S_i^{(-m)}}$. We have proved the lemma when $p > 0$.

When $p = 0$, the above argument for G/H shows the lemma. $\square$

Lemma 5.17. *Let $(G \to S = \operatorname{Spec} R, P = \operatorname{Spec} A \to \mathrm{D}_S^*) \in \Lambda$. Suppose that S is reduced and that there exists a finite étale cover $\coprod_{i \in I} S_i \to S$ such that every P_{S_i} is uniformizable. Then there exists a unique $R[\![t]\!]$-subalgebra $O \subset A$ such that*

(1) *O is a finite and flat $R[\![t]\!]$-module;*

(2) *$O_t = A$, where the left side is the localization of O by t;*

(3) *for any $U = \operatorname{Spec} B \to S$ such that P_U is uniformizable, the subring $O \otimes_{R[\![t]\!]} B[\![t]\!] \subset A \otimes_{R[\![t]\!]} B[\![t]\!]$ is the integral ring of P_U (see Definition 5.7).*

Proof. Let $S_{ij} = \operatorname{Spec} R_{ij} := S_i \times_S S_j$, and write $P_{S_i} = \operatorname{Spec} A_i$ and $P_{S_{ij}} = \operatorname{Spec} A_{ij}$. Let $O_i \subset A_i$ and $O_{ij} \subset A_{ij}$ be the integral rings. Since R_{ij} is a finitely presented R_i-module (recall that an étale morphism is by definition locally of finite presentation), by [TY20, Lemma 2.4] we have $R_i[\![t]\!] \otimes_{R_i} R_{ij} \cong R_{ij}[\![t]\!]$. Therefore,

$$O_{ij} = O_i \otimes_{R_i[\![t]\!]} R_{ij}[\![t]\!] = O_i \otimes_{R_i} R_{ij} \,,$$
$$A_{ij} = A_i \otimes_{R_i(\!(t)\!)} R_{ij}(\!(t)\!) = A_i \otimes_{R_i(\!(t)\!)} R_{ij}(\!(t)\!) = A_i \otimes_{R_i} R_{ij} \,.$$

By descent, we get a submodule $O \subset A$. Properties (1) and (2) hold since they hold after pullback to S_i.

Let $U \to S$ be as in property (3). Consider the finite étale cover $\coprod_i U \times_S S_i \to U$. The integral ring of P_U and $O \otimes_{R[\![t]\!]} B[\![t]\!]$ are the same because they become the same when pulled back to $U \times_S S_i$. □

Definition 5.18. For a torsor $P \to \mathrm{D}^*_S$ as in Lemma 5.17, we call $\operatorname{Spf} O$ the *integral model* of P and denote it by $\overline{P}$. We also call the morphism $\overline{P} \to \mathrm{Df}^*_S$ a *G-cover*.

Proposition 5.19. (1) *Suppose $p > 0$. Let G be a Galoisian group and $\mathbf{r}$ a ramification datum for G. Then there exist a reduced DM stack $\mathcal{X}$ of finite type and an isomorphism $\mathcal{X}^{\mathrm{iper}} \xrightarrow{\sim} \Lambda^{[\mathbf{r}]}_{[G]}$ such that for any morphism $S \to \mathcal{X}$ from an affine scheme, there exists an étale cover $T \to S$ by an affine scheme such that the composite morphism*

$$T \to \mathcal{X} \to \mathcal{X}^{\mathrm{iper}} \xrightarrow{\sim} \Lambda^{[\mathbf{r}]}_{[G]}$$

corresponds to a uniformizable torsor.

(2) *Let G be a tame cyclic group. Then $\Lambda_{[G]}$ is a reduced DM stack of finite type, and for any morphism $S \to \Lambda_{[G]}$ from an affine scheme, there exists a finite étale cover $T \to S$ (necessarily by an affine scheme) such that the induced morphism $T \to \Lambda_{[G]}$ corresponds to a uniformizable torsor.*

Proof. (1) First consider the case where $k = \mathbb{F}_p$. By Lemma 5.15, there exist a reduced DM stack $\mathcal{Y}$ of finite type and an isomorphism $\mathcal{Y}^{\mathrm{iper}} \cong \Lambda^{[\mathbf{r}]}_{[G]}$. Take an atlas $U \to \mathcal{Y}$ with U an affine scheme. By Lemma 5.16, there exists a finite étale cover $V \to U$ such that for $m \gg 0$, the morphism $V^{(-m)} \to \mathcal{Y} \to \Lambda^{[\mathbf{r}]}_{[G]}$ corresponds to a uniformizable torsor. The stack $\mathcal{X} = \mathcal{Y}^{(-m)}$ satisfies the desired property. Indeed, since k is perfect, $\mathcal{X}^{(-m)}$ is of finite type. Obviously, we have $\mathcal{X}^{\mathrm{iper}} \cong \Lambda^{[\mathbf{r}]}_{[G]}$. For any morphism $S \to \mathcal{X}$, the morphism $S \times_{\mathcal{X}} V^{(-m)} \to S$ is an étale cover and the morphism $S \times_{\mathcal{X}} V^{(-m)} \to \Lambda^{[\mathbf{r}]}_{[G]}$ is uniformizable since it factors through $V^{(-m)}$. For a general field k, we just take the base change $\mathcal{X} \otimes_{\mathbb{F}_p} k$.

(2) This is obvious. □

Definition 5.20. When $p > 0$, for each $[G]$ and $[\mathbf{r}]$, we choose a reduced DM stack of finite type $\Theta^{[\mathbf{r}]}_{[G]}$ as in Proposition 5.19(1). When G is tame, there is only one class $[\mathbf{r}]$, and we choose

$\Theta_{[G]}^{[\mathbf{r}]}$ to be $\Lambda_{[G]}$. We then define

$$\Theta := \coprod_{G \in \mathrm{GalGps}} \coprod_{[\mathbf{r}] \in \mathrm{Ram}(G)/\operatorname{Aut}(G)} \Theta_{[G]}^{[\mathbf{r}]},$$
$$\Theta_{[G]} := \Theta \times_{\mathcal{A}} \mathcal{A}_{[G]} = \coprod_{[\mathbf{r}] \in \mathrm{Ram}(G)/\operatorname{Aut}(G)} \Theta_{[G]}^{[\mathbf{r}]}.$$

When $p = 0$, we define $\Theta := \Lambda$ and $\Theta_{[G]} = \Lambda_{[G]}$. For a Galoisian group scheme $G \to S$, we define $\Theta_G := \Theta \times_{\mathcal{A}} S$.

All these stacks are almost of finite type; that is, each of them has only countably many connected components, each of which is of finite type. By Lemma 5.17, for any morphism $S \to \Theta$ from an affine scheme S, the induced torsor $P = \operatorname{Spec} A \to \mathrm{D}_S^*$ has the subalgebra $O \subset A$. Let $O_n := O/(t^{n+1})$.

Definition 5.21. We define a finite affine scheme $E_{\Theta,n}$ over Θ by the property $E_{\Theta,n} \times_\Theta S = \operatorname{Spec} O_n$ for each $S \to \Theta$. (This is also constructed as the relative spectrum $\underline{\mathrm{Spec}}_\Theta O_{\Theta,n}$ of the coherent sheaf $O_{\Theta,n}$ on Θ given by modules $O_{U,n}$ in Lemma 5.17.) There exists a natural morphism $E_{\Theta,n} \to \mathrm{D}_{\Theta,n} := \mathrm{D}_n \times \Theta$, which is representable, finite and flat. We define the formal DM stack $E_\Theta := \varinjlim E_{\Theta,n}$ and call this the *universal integral model* over Θ. There exists a natural morphism $E_\Theta \to \mathrm{Df}_\Theta$. For a morphism of stacks $\Sigma \to \Theta$, we call the induced morphism

$$E_\Sigma := E_\Theta \times_{\mathrm{Df}_\Theta} \mathrm{Df}_\Sigma = E_\Theta \times_\Theta \Sigma \to \mathrm{Df}_\Sigma$$

a *G-cover*.

For each point $x\colon \operatorname{Spec} K \to \Theta$, the fiber product $E_\Theta \times_\Theta \operatorname{Spec} K$ is the integral model of the G-torsor $P \to \mathrm{D}_K^*$ corresponding to x. Then $E_{\Theta,n} \times_\Theta \operatorname{Spec} K$ is its closed subscheme defined by t^{n+1}.

By Lemma 5.8, the group scheme $\mathcal{G}_\Theta := \mathcal{G} \times_{\mathcal{A}} \Theta$ over Θ acts on $E_{\Theta,n}$.

Definition 5.22. We define $\mathcal{E}_{\Theta,n}$ to be the quotient stack $[E_{\Theta,n}/\mathcal{G}_\Theta]$. More precisely, we define a stack $\mathcal{E}_{\Theta,n}$ as follows. For $U \in \mathbf{Aff}$, an object of $\mathcal{E}_{\Theta,n}(U)$ is a tuple $(U \to \Theta, P \to U, P \to E_{U,n})$, where $P \to U$ is a $\mathcal{G}_U$-torsor and $P \to E_{U,n}$ is a $\mathcal{G}_U$-equivariant morphism. We then define $\mathcal{E}_\Theta := \varinjlim \mathcal{E}_{\Theta,n}$ and call it the *universal twisted formal disk* over Θ.

For each morphism $U \to \Theta$ from an affine scheme, we have $\mathcal{E}_{\Theta,n} \times_\Theta U \cong [E_{U,n}/\mathcal{G}_U]$. We have natural morphisms of DM stacks $E_{\Theta,n} \to \mathcal{E}_{\Theta,n} \to \mathrm{D}_{\Theta,n}$ and ones of formal DM stacks $E_\Theta \to \mathcal{E}_\Theta \to \mathrm{Df}_\Theta$.

Definition 5.23. A *twisted formal disk over a field* K is a formal DM stack $\mathcal{E}$ over Df_K induced as the base change of $\mathcal{E}_\Theta \to \mathrm{Df}_\Theta$ by a K-point $\operatorname{Spec} K \to \Theta$. Two twisted formal disks $\mathcal{E}$ and $\mathcal{E}'$ over K are said to be *isomorphic* if there exists an isomorphism $\mathcal{E} \to \mathcal{E}'$ which makes the diagram

$$\begin{tikzcd} \mathcal{E} \arrow[r] \arrow[dr] & \mathcal{E}' \arrow[d] \\ & \mathrm{Df}_K \end{tikzcd}$$

2-commutative.

Lemma 5.24. *For an algebraically closed field K, we have one-to-one correspondences*

$$\Lambda[K] \leftrightarrow \Theta[K] \leftrightarrow \{\textit{twisted formal disk over } K\}/\cong.$$

Proof. The first correspondence is clear. An isomorphism in $\Lambda(K)$ gives an isomorphism $\alpha\colon G \cong G'$ of Galoisian groups and a D_K^*-isomorphism $P \cong P'$ compatible with α for a G-torsor $P \to \mathrm{D}_K^*$ and a G'-torsor $P' \to \mathrm{D}_K^*$. For their models $\overline{P}$ and $\overline{P'}$, the quotient stacks $[\overline{P}/G]$ and $[\overline{P'}/G']$ are clearly isomorphic twisted formal disks. Conversely, given an isomorphism $[E/G] \to [E'/G']$ of twisted formal disks over K, consider the fiber product $U := E \times_{[E'/G']} E'$. This is finite and étale over both E and E'. Therefore, U is isomorphic to the coproduct of copies of E and to the coproduct of copies of E'. Thus, if U_0 is a connected component of U, then we have isomorphisms $E \cong U_0 \cong E'$ over Df_K. It follows that the corresponding Galois extensions of $K(\!(t)\!)$ are isomorphic and determine isomorphic K-points of Λ. □

6. Untwisting stacks

In this section, we introduce our main technical ingredient, the untwisting stack. We define it as a certain Hom stack.

6.1 Some results on Hom stacks

Definition 6.1. Let $\mathcal{Y}, \mathcal{X}, \mathcal{S}$ be stacks over **Aff** with morphisms $\mathcal{Y} \to \mathcal{S}$ and $\mathcal{X} \to \mathcal{S}$. We define $\underline{\mathrm{Hom}}_{\mathcal{S}}(\mathcal{Y}, \mathcal{X})$ (respectively, $\underline{\mathrm{Hom}}_{\mathcal{S}}^{\mathrm{rep}}(\mathcal{Y}, \mathcal{X})$) to be the fibered category over $\mathcal{S}$ whose fiber category over an S-point $S \to \mathcal{S}$ is $\mathrm{Hom}_S(\mathcal{Y} \times_{\mathcal{S}} S, \mathcal{X} \times_{\mathcal{S}} S)$ (respectively, $\mathrm{Hom}_S^{\mathrm{rep}}(\mathcal{Y} \times_{\mathcal{S}} S, \mathcal{X} \times_{\mathcal{S}} S)$), the category of S-morphisms (respectively, representable S-morphisms).

Note that the canonical functor

$$\mathrm{Hom}_S^{\mathrm{rep}}(\mathcal{Y} \times_{\mathcal{S}} S, \mathcal{X} \times_{\mathcal{S}} S) \to \mathrm{Hom}_S^{\mathrm{rep}}(\mathcal{Y} \times_{\mathcal{S}} S, \mathcal{X})$$

is an equivalence. We often identify the two categories through this equivalence. A basic result on Hom stacks is the following one by Olsson.

Theorem 6.2 ([Ols06, Ols07]). *Let S be a scheme, and let $\mathcal{X}$ and $\mathcal{Y}$ be DM stacks of finite presentation over S. Suppose that $\mathcal{Y}$ is proper and flat over S and that there exists a finite, finitely presented, flat and surjective morphism $Z \to \mathcal{Y}$ from an algebraic space. Then $\underline{\mathrm{Hom}}_S(\mathcal{Y}, \mathcal{X})$ is a DM stack locally of finite presentation over S, and $\underline{\mathrm{Hom}}_S^{\mathrm{rep}}(\mathcal{Y}, \mathcal{X})$ is an open substack of it.*

We prove a slight generalization of this theorem as well as auxiliary results on Hom stacks.

Lemma 6.3. *Let S be an algebraic space, and let Y, X, W be algebraic spaces of finite presentation over S. Suppose that Y is flat and proper over S. Let $X \hookrightarrow W$ be a closed immersion. Then the induced morphism*

$$\underline{\mathrm{Hom}}_S(Y, X) \to \underline{\mathrm{Hom}}_S(Y, W)$$

is also a closed immersion.

Proof. A closed immersion is characterized by three properties: universally closed, unramified and universally injective [Sta20, tag 04XV]. We check these properties.

Universally injective: This is obvious.

Unramified: Let $T \hookrightarrow T'$ be a thickening of S-schemes. We need to show that given the diagram of solid arrows

$$\begin{tikzcd} T \arrow[r] \arrow[d] & \underline{\mathrm{Hom}}_S(Y, X) \arrow[d] \\ T' \arrow[r] \arrow[ur, dashed] & \underline{\mathrm{Hom}}_S(Y, W) \end{tikzcd},$$

there exists at most one dashed arrow making the whole diagram commutative. This holds because for the corresponding diagram of solid arrows

$$\begin{tikzcd} Y \times_S T \arrow[r] \arrow[d] & X \arrow[d] \\ Y \times_S T' \arrow[r] \arrow[ur, dashed] & W \end{tikzcd},$$

there exists at most one dashed arrow since $X \to W$ is unramified.

Universally closed: Let R be a valuation ring, and let K be its fraction field. Suppose that we have the commutative diagram of solid arrows

$$\begin{tikzcd} \operatorname{Spec} K \arrow[r] \arrow[d] & \underline{\mathrm{Hom}}_S(Y,X) \arrow[d] \\ \operatorname{Spec} R \arrow[r] \arrow[ur, dashed] & \underline{\mathrm{Hom}}_S(Y,W) \end{tikzcd}.$$

We need to show the existence of the dashed arrow making the whole diagram commutative. Let $f\colon Y \times_S \operatorname{Spec} R \to W$ be the morphism corresponding to the bottom arrow. The induced morphism $g\colon Y \times_S \operatorname{Spec} K \to W$ factors through X. This means that $g^{-1}\mathcal{I}_X = 0$, where $\mathcal{I}_X \subset \mathcal{O}_W$ is the defining ideal sheaf of $X \subset W$. Since $Y \times_S \operatorname{Spec} R$ is flat over $\operatorname{Spec} R$, the structure sheaf of $Y \times_S \operatorname{Spec} R$ has no R-torsion. This implies that $f^{-1}\mathcal{I}_X = 0$ and that f factors through X. This implies the existence of the dashed arrow. □

Lemma 6.4. *Let S be an algebraic space of finite type, and let Y and X be algebraic spaces of finite type over S. Suppose that Y is flat and finite over S. Then $\underline{\mathrm{Hom}}_S(Y,X)$ is an algebraic space of finite type.*

Proof. First consider the case where S, Y, X are all affine schemes, say $S = \operatorname{Spec} R$, $X = \operatorname{Spec} A$ and $Y = \operatorname{Spec} B$ and B is a free R-module of rank m. Suppose that the scheme X is embedded in $\mathbb{A}^n_k$. By Lemma 6.3, the subspace $\underline{\mathrm{Hom}}_S(Y,X)$ of $\underline{\mathrm{Hom}}_S(Y,\mathbb{A}^n)$ is closed. An element $f \in \underline{\mathrm{Hom}}_S(Y,\mathbb{A}^n)(\operatorname{Spec} B)$ is given by $f^*\colon B[x_1,\dots,x_n] \to B^{\oplus m}$. Therefore, we have a universally injective morphism

$$\underline{\mathrm{Hom}}_S(Y,\mathbb{A}^n) \to \mathbb{A}^{nm}_S .$$

As a consequence, $\underline{\mathrm{Hom}}_S(Y,X)$ is quasi-compact and of finite type in this case.

We reduce the general case to the above special case step by step. Firstly, taking the base change by a surjective morphism $S' \to S$, we may suppose that S is an affine scheme, which implies that Y is also affine. By a further base change, we may suppose that if $S = \operatorname{Spec} R$ and $Y = \operatorname{Spec} B$, then B is free over R. There exists a surjective finite-type morphism $T \to S$ such that each connected component of $(T \times_S Y)_{\mathrm{red}}$ maps isomorphically onto T. This allows us to reduce to the case where $Y = \coprod_{i=1}^n Y_i$ and $Y_{i,\mathrm{red}} \to S$ is an isomorphism. Then $\underline{\mathrm{Hom}}_S(Y,X) = \prod_{i=1}^n \underline{\mathrm{Hom}}_S(Y_i,X)$. Thus we may further suppose that $Y_{\mathrm{red}} \to S$ is an isomorphism. The morphism $S \cong Y_{\mathrm{red}} \hookrightarrow Y$ induces a morphism $\underline{\mathrm{Hom}}_S(Y,X) \to \underline{\mathrm{Hom}}_S(S,X) = X$. For an étale morphism $U \to X$, we have

$$\underline{\mathrm{Hom}}_S(Y,X) \times_X U = \underline{\mathrm{Hom}}_S(Y,U) .$$

We can see this by looking at the diagram

$$\begin{tikzcd} T \arrow[r] \arrow[d, hook] & U \arrow[d] \\ Y_T \arrow[r] \arrow[ur, dashed] & X \end{tikzcd}$$

deduced from a morphism $T \to S$. Using this, we reduce the problem to the case where X is an affine scheme. This completes the proof. $\square$

Lemma 6.5. *Let $\mathcal{S}$ be a DM stack of finite type, and let $\mathcal{Y}$ and $\mathcal{X}$ be DM stacks of finite type over $\mathcal{S}$. Suppose that $\mathcal{Y}$ is flat and finite over $\mathcal{S}$ and that there exists a finite, étale and surjective morphism $\mathcal{U} \to \mathcal{Y}$ such that $\mathcal{U} \to \mathcal{S}$ is representable. Then $\underline{\mathrm{Hom}}_{\mathcal{S}}(\mathcal{Y},\mathcal{X})$ and $\underline{\mathrm{Hom}}_{\mathcal{S}}^{\mathrm{rep}}(\mathcal{Y},\mathcal{X})$ are DM stacks of finite type.*

Proof. Let $T \to \mathcal{S}$ be an atlas. Then

$$\underline{\mathrm{Hom}}_{\mathcal{S}}(\mathcal{Y},\mathcal{X}) \times_{\mathcal{S}} T \cong \underline{\mathrm{Hom}}_{T}(\mathcal{Y} \times_{\mathcal{S}} T, \mathcal{X} \times_{\mathcal{X}} T)\,,$$
$$\underline{\mathrm{Hom}}_{\mathcal{S}}^{\mathrm{rep}}(\mathcal{Y},\mathcal{X}) \times_{\mathcal{S}} T \cong \underline{\mathrm{Hom}}_{T}^{\mathrm{rep}}(\mathcal{Y} \times_{\mathcal{S}} T, \mathcal{X} \times_{\mathcal{X}} T)$$

are DM stacks locally of finite type. The stacks $\underline{\mathrm{Hom}}_{\mathcal{S}}(\mathcal{Y},\mathcal{X})$ and $\underline{\mathrm{Hom}}_{\mathcal{S}}^{\mathrm{rep}}(\mathcal{Y},\mathcal{X})$ are also DM stacks locally of finite type, by Lemma A.3. To show that they are of finite type, it suffices to show that $\underline{\mathrm{Hom}}_{\mathcal{S}}(\mathcal{Y},\mathcal{X})$ is of finite type in the case where $\mathcal{S} = S$ is a scheme. Let X be the coarse moduli space of $\mathcal{X}$. Then the morphism $\underline{\mathrm{Hom}}_{S}(\mathcal{Y},\mathcal{X}) \to \underline{\mathrm{Hom}}_{S}(\mathcal{Y},X)$ is of finite type by [Ols07]. By the assumption, there exists a finite, étale and surjective morphism $U \to \mathcal{Y}$ from an algebraic space. Let $V := U \times_{\mathcal{Y}} U$. We have a short exact sequence

$$\underline{\mathrm{Hom}}_{S}(\mathcal{Y},X) \to \underline{\mathrm{Hom}}_{S}(U,X) \rightrightarrows \underline{\mathrm{Hom}}_{S}(V,X)\,.$$

By Lemma 6.4, the right two Hom spaces are of finite type. It follows that $\underline{\mathrm{Hom}}_{S}(\mathcal{Y},X)$ is of finite type. Therefore, $\underline{\mathrm{Hom}}_{S}(\mathcal{Y},\mathcal{X})$ is also of finite type. $\square$

Lemma 6.6. *Let $\mathcal{C}$ be a DM stack, and let $\mathcal{E}$, $\mathcal{X}$, $\mathcal{Y}$, $\mathcal{Z}$ be DM stacks over $\mathcal{C}$. Let $f\colon \mathcal{Y} \to \mathcal{X}$ and $g\colon \mathcal{Z} \to \mathcal{X}$ be $\mathcal{C}$-morphisms. Then there exists an isomorphism*

$$\underline{\mathrm{Hom}}_{\mathcal{C}}(\mathcal{E}, \mathcal{Y} \times_{\mathcal{X}} \mathcal{Z}) \to \underline{\mathrm{Hom}}_{\mathcal{C}}(\mathcal{E},\mathcal{Y}) \times_{\underline{\mathrm{Hom}}_{\mathcal{C}}(\mathcal{E},\mathcal{X})} \underline{\mathrm{Hom}}_{\mathcal{C}}(\mathcal{E},\mathcal{Z})\,.$$

If f and g are representable morphisms of formal DM stacks, then there exists an isomorphism

$$\underline{\mathrm{Hom}}_{\mathcal{C}}^{\mathrm{rep}}(\mathcal{E}, \mathcal{Y} \times_{\mathcal{X}} \mathcal{Z}) \to \underline{\mathrm{Hom}}_{\mathcal{C}}^{\mathrm{rep}}(\mathcal{E},\mathcal{Y}) \times_{\underline{\mathrm{Hom}}_{\mathcal{C}}^{\mathrm{rep}}(\mathcal{E},\mathcal{X})} \underline{\mathrm{Hom}}_{\mathcal{C}}^{\mathrm{rep}}(\mathcal{E},\mathcal{Z})\,.$$

Proof. Let $S \to \mathcal{C}$ be a morphism from an affine scheme. We define $\mathcal{E}_S$ to be the base change of $\mathcal{E}$ by $S \to \mathcal{C}$, and similarly for $\mathcal{X}_S$, $\mathcal{Y}_S$ and $\mathcal{Z}_S$. Let $\gamma\colon \mathcal{E}_S \to \mathcal{Y} \times_{\mathcal{X}} \mathcal{Z}$ be a $\mathcal{C}$-morphism. We get the induced morphisms $p_{\mathcal{Y}} \circ \gamma\colon \mathcal{E}_S \to \mathcal{Y}$, $p_{\mathcal{Z}} \circ \gamma\colon \mathcal{E}_S \to \mathcal{Z}$ and the isomorphism $f \circ p_{\mathcal{Y}} \circ \gamma \to g \circ p_{\mathcal{Z}} \circ \gamma$. This defines a morphism

$$\underline{\mathrm{Hom}}_{\mathcal{C}}(\mathcal{E}, \mathcal{Y} \times_{\mathcal{X}} \mathcal{Z}) \to \underline{\mathrm{Hom}}_{\mathcal{C}}(\mathcal{E},\mathcal{Y}) \times_{\underline{\mathrm{Hom}}_{\mathcal{C}}(\mathcal{E},\mathcal{X})} \underline{\mathrm{Hom}}_{\mathcal{C}}(\mathcal{E},\mathcal{Z})\,.$$

We need to show that

$$\mathrm{Hom}_S(\mathcal{E}_S, (\mathcal{Y} \times_{\mathcal{X}} \mathcal{Z})_S) \to \mathrm{Hom}_S(\mathcal{E}_S, \mathcal{Y}_S) \times_{\mathrm{Hom}_S(\mathcal{E}_S,\mathcal{X}_S)} \mathrm{Hom}_S(\mathcal{E}_S, \mathcal{Z}_S)$$

is an isomorphism. This follows from

$$\mathcal{Y}_S \times_{\mathcal{X}_S} \mathcal{Z}_S \cong (\mathcal{Y} \times_{\mathcal{X}} \mathcal{X}_S) \times_{\mathcal{X}_S} (\mathcal{Z} \times_{\mathcal{X}} \mathcal{X}_S) \cong \mathcal{Y} \times_{\mathcal{X}} \mathcal{Z} \times_{\mathcal{X}} \mathcal{X}_S \cong \mathcal{Y} \times_{\mathcal{X}} \mathcal{Z} \times_{\mathcal{C}} S$$

and the fact that

$$\mathrm{Hom}_S(\mathcal{E}_S, \mathcal{Y}_S \times_{\mathcal{X}_S} \mathcal{Z}_S) \to \mathrm{Hom}_S(\mathcal{E}_S, \mathcal{Y}_S) \times_{\mathrm{Hom}_S(\mathcal{E}_S,\mathcal{X}_S)} \mathrm{Hom}_S(\mathcal{E}_S, \mathcal{Z}_S)$$

is an isomorphism by [Ols16, § 3.4, pp. 82–83].

For the second assertion, since f and g are representable, $\underline{\mathrm{Hom}}_{\mathcal{C}}^{\mathrm{rep}}(\mathcal{E},\mathcal{Y})$ and $\underline{\mathrm{Hom}}_{\mathcal{C}}^{\mathrm{rep}}(\mathcal{E},\mathcal{Z})$ map into $\underline{\mathrm{Hom}}_{\mathcal{C}}^{\mathrm{rep}}(\mathcal{E},\mathcal{X})$. Let $\beta\colon \mathcal{E}_S \to \mathcal{Y}_S \times_{\mathcal{X}_S} \mathcal{Z}_S$ be a morphism over S. It suffices to show that β is representable if and only if the induced morphisms $\mathcal{E}_S \to \mathcal{Y}_S$ and $\mathcal{E}_S \to \mathcal{Z}_S$ are representable.

This reduces to proving the following fact: Suppose that $B \to A$ and $C \to A$ are injective homomorphisms of groups and that D is another group. Then a homomorphism $D \to B \times_A C$ is injective if and only if $D \to B$ and $D \to C$ are injective. But this is obvious. $\square$

6.2 Untwisting stacks

Definition 6.7. Let $\mathcal{X}$ be a formal DM stack of finite type over Df. We fix a stack Θ as in Definition 5.20. We define the *total untwisting stack* $\mathrm{Utg}_\Theta(\mathcal{X})$ to be the Hom stack $\underline{\mathrm{Hom}}^{\mathrm{rep}}_{\mathrm{Df}_\Theta}(\mathcal{E}_\Theta, \mathcal{X}_\Theta)$, where $\mathcal{X}_\Theta := \mathcal{X} \times_k \Theta$. This is a fibered category over Df_Θ such that for a morphism $U \to \mathrm{Df}_\Theta$ from an affine scheme, the fiber category $\underline{\mathrm{Hom}}^{\mathrm{rep}}_{\mathrm{Df}_\Theta}(\mathcal{E}_\Theta, \mathcal{X}_\Theta)(U)$ is the category $\mathrm{Hom}^{\mathrm{rep}}_U(\mathcal{E}_\Theta \times_{\mathrm{Df}_\Theta} U, \mathcal{X}_\Theta \times_{\mathrm{Df}_\Theta} U)$ of representable morphisms $\mathcal{E}_\Theta \times_{\mathrm{Df}_\Theta} U \to \mathcal{X}_\Theta \times_{\mathrm{Df}_\Theta} U$ over U. For any morphism $\sigma\colon \Sigma \to \Theta$ of stacks, we define

$$\mathrm{Utg}_\Sigma(\mathcal{X}) = \mathrm{Utg}_\sigma(\mathcal{X}) := \underline{\mathrm{Hom}}^{\mathrm{rep}}_{\mathrm{Df}_\Theta}(\mathcal{E}_\Theta, \mathcal{X}_\Theta) \times_\Theta \Sigma \big(\cong \underline{\mathrm{Hom}}^{\mathrm{rep}}_{\mathrm{Df}_\Sigma}(\mathcal{E}_\Sigma, \mathcal{X}_\Sigma)\big)\,,$$

where $\mathcal{E}_\Sigma := \mathcal{E}_\Theta \times_\Theta \Sigma$.

Lemma 6.8. *For a Galoisian group G and a ramification datum $\mathbf{r}$ for G, the stack $\mathrm{Utg}_{\Theta^{[\mathbf{r}]}_{[G]}}(\mathcal{X})$ is a formal DM stack of finite type over $\mathrm{Df}_{\Theta^{[\mathbf{r}]}_{[G]}}$.*

Proof. First note that $\Theta^{[\mathbf{r}]}_{[G]}$ is a DM stack of finite type. Let $V \to \Theta^{[\mathbf{r}]}_{[G]}$ be an atlas. For each $n \in \mathbb{N}$,

$$\mathrm{Utg}_V(\mathcal{X})_n = \underline{\mathrm{Hom}}^{\mathrm{rep}}_{\mathrm{D}_{V,n}}(\mathcal{E}_{V,n}, \mathcal{X}_{V,n}) = \mathrm{Utg}_{\Theta^{[\mathbf{r}]}_{[G]}}(\mathcal{X})_n \times_{\Theta^{[\mathbf{r}]}_{[G]}} V$$

is a DM stack of finite type by Lemma 6.5. By Lemma A.3, the stack $\mathrm{Utg}_{\Theta^{[\mathbf{r}]}_{[G]}}(\mathcal{X})_n$ is a DM stack of finite type. The lemma follows from Lemma 2.10. $\square$

By Lemma 5.24, for an algebraically closed field K, the set of K-points

$$\mathrm{Utg}_\Theta(\mathcal{X})[K] = \mathrm{Utg}_\Theta(\mathcal{X})_0[K]$$

is identified with the isomorphism classes of pairs $(\mathcal{E}, \mathcal{E}_0 \to \mathcal{X})$ of a twisted formal disk $\mathcal{E}$ over K and a representable morphism $\mathcal{E}_0 \to \mathcal{X}$. Here two pairs $(\mathcal{E}, a\colon \mathcal{E}_0 \to \mathcal{X})$ and $(\mathcal{E}', b\colon \mathcal{E}'_0 \to \mathcal{X})$ are *isomorphic* if there exists an isomorphism $h\colon \mathcal{E} \to \mathcal{E}'$ such that a is isomorphic to

$$\mathcal{E}_0 \xrightarrow{h \times_{\mathrm{Df}} \mathrm{D}_0} \mathcal{E}'_0 \xrightarrow{b} \mathcal{X}\,.$$

Lemma 6.9. *Let $S = \operatorname{Spec} R$ be an affine scheme, and let $\mathcal{E}$ and $\mathcal{X}$ be DM stacks over $\mathrm{D}_S = \operatorname{Spec} R[\![t]\!]$. Let $\hat{\mathcal{E}}$ and $\hat{\mathcal{X}}$ be their t-adic completions, which are formal DM stacks over $\mathrm{Df}_S = \operatorname{Spf} R[\![t]\!]$. Then we have an isomorphism of stacks over Df_S*

$$\underline{\mathrm{Hom}}^{\mathrm{rep}}_{\mathrm{Df}_S}(\hat{\mathcal{E}}, \hat{\mathcal{X}}) \cong \widehat{\underline{\mathrm{Hom}}^{\mathrm{rep}}_{\mathrm{D}_S}}(\mathcal{E}, \mathcal{X})\,.$$

Proof. For $n \in \mathbb{N}$, we put $\mathcal{E}_n := \mathcal{E} \times_{\mathrm{D}_S} \mathrm{D}_{S,n} = \hat{\mathcal{E}} \times_{\mathrm{Df}_S} \mathrm{D}_{S,n}$ and similarly for $\mathcal{X}_n$. Any morphism $U \to \mathrm{Df}_S$ from an affine scheme factors through $\mathrm{D}_{S,n}$ for some $n \in \mathbb{N}$. Therefore, we have natural isomorphisms

$$\begin{aligned}
\underline{\mathrm{Hom}}^{\mathrm{rep}}_{\mathrm{Df}_S}(\hat{\mathcal{E}}, \hat{\mathcal{X}})(U) &\cong \big(\underline{\mathrm{Hom}}^{\mathrm{rep}}_{\mathrm{Df}_S}(\hat{\mathcal{E}}, \hat{\mathcal{X}}) \times_{\mathrm{Df}_S} \mathrm{D}_{S,n}\big)(U) \\
&\cong \underline{\mathrm{Hom}}^{\mathrm{rep}}_{\mathrm{D}_{S,n}}(\mathcal{E}_n, \mathcal{X}_n)(U) \\
&\cong \big(\underline{\mathrm{Hom}}^{\mathrm{rep}}_{\mathrm{D}_S}(\mathcal{E}, \mathcal{X}) \times_{\mathrm{D}_S} \mathrm{D}_{S,n}\big)(U) \\
&\cong \widehat{\underline{\mathrm{Hom}}^{\mathrm{rep}}_{\mathrm{D}_S}}(\mathcal{E}, \mathcal{X})(U)\,.
\end{aligned}$$

$\square$

Lemma 6.10. *If $\mathcal{Y} \to \mathcal{X}$ is representable and étale, then $\mathrm{Utg}_\Theta(\mathcal{Y}) \to \mathrm{Utg}_\Theta(\mathcal{X})$ is also representable and étale. If $\mathcal{Y} \to \mathcal{X}$ is stabilizer preserving, étale and surjective, then $\mathrm{Utg}_\Theta(\mathcal{Y}) \to \mathrm{Utg}_\Theta(\mathcal{X})$ is surjective.*

Proof. Let $\operatorname{Spec} K \to \mathrm{Df}_\Theta$ be a geometric point, and let $r\colon \operatorname{Spec} K \to \mathrm{Utg}_\Theta(\mathcal{Y})$ be a lift of it. The latter corresponds to a representable morphism $\mathcal{E}_{\operatorname{Spec} K} \to \mathcal{Y}$. The automorphism group of this morphism is equal to $\mathrm{Aut}_{\mathrm{Utg}_\Theta(\mathcal{Y})/\mathrm{Df}_\Theta}(r)$, the group of those automorphisms of the object r that map to the identity morphism in Df_Θ. By Lemma A.4, the map $\mathrm{Aut}_{\mathrm{Utg}_\Theta(\mathcal{Y})/\mathrm{Df}_\Theta}(r) \to \mathrm{Aut}_{\mathrm{Utg}_\Theta(\mathcal{X})/\mathrm{Df}_\Theta}(r')$ is injective, where $r'\colon \operatorname{Spec} K \to \mathrm{Utg}_\Theta(\mathcal{X})$ is the image of r and corresponds to the composition $\mathcal{E}_{\operatorname{Spec} K} \to \mathcal{Y} \to \mathcal{X}$. By Lemma A.1, the map $\mathrm{Aut}_{\mathrm{Utg}_\Theta(\mathcal{Y})}(r) \to \mathrm{Aut}_{\mathrm{Utg}_\Theta(\mathcal{X})}(r')$ is also injective. Therefore, for each n, the morphism $\mathrm{Utg}_\Theta(\mathcal{Y})_n \to \mathrm{Utg}_\Theta(\mathcal{X})_n$ of DM stacks is representable. We conclude that the morphism $\mathrm{Utg}_\Theta(\mathcal{Y}) \to \mathrm{Utg}_\Theta(\mathcal{X})$ is also representable.

Consider a 2-commutative diagram

$$\begin{tikzcd}
S \arrow[rr] \arrow[d, hook, "\iota"'] & & \mathrm{Utg}_\Theta(\mathcal{Y}) \arrow[d] \\
S' \arrow[r] \arrow[urr, Rightarrow] & U \arrow[r] & \mathrm{Utg}_\Theta(\mathcal{X})
\end{tikzcd}$$

such that S, S', U are algebraic spaces and ι is a thickening. Ignoring U, we obtain the corresponding diagram

$$\begin{tikzcd}
\mathcal{E}_S \arrow[r] \arrow[d, hook] & \mathcal{Y} \arrow[d] \\
\mathcal{E}_{S'} \arrow[r] \arrow[ur, Rightarrow] & \mathcal{X}\,.
\end{tikzcd} \tag{6.1}$$

Since $\mathcal{Y} \to \mathcal{X}$ is representable and étale, by Lemma A.7 there exist a morphism $\mathcal{E}_{S'} \to \mathcal{Y}$ and two 2-morphisms forming the 2-commutative diagram

$$\begin{tikzcd}
\mathcal{E}_S \arrow[r] \arrow[d, hook] \arrow[dr, phantom, "\Nearrow", shift right=2ex] & \mathcal{Y} \arrow[d] \\
\mathcal{E}_{S'} \arrow[r] \arrow[ur] \arrow[ur, phantom, "\Nearrow", shift left=2ex] & \mathcal{X}
\end{tikzcd}$$

which induces (6.1). Moreover, such a triple of a morphism $\mathcal{E}_{S'} \to \mathcal{Y}$ and two 2-morphisms is unique up to unique isomorphism (see the uniqueness assertion in Lemma A.7). We have the corresponding diagram

$$\begin{tikzcd}
S \arrow[rr] \arrow[d, hook, "\iota"'] & & \mathrm{Utg}_\Theta(\mathcal{Y}) \arrow[d] \\
S' \arrow[r] \arrow[urr] & U \arrow[r] & \mathrm{Utg}_\Theta(\mathcal{X})\,.
\end{tikzcd}$$

This shows that there exists a unique dashed arrow fitting into the commutative diagram of algebraic spaces

$$\begin{tikzcd}
S \arrow[r] \arrow[d, hook] & U \times_{\mathrm{Utg}_\Theta(\mathcal{X})} \mathrm{Utg}_\Theta(\mathcal{Y}) \arrow[d] \\
S' \arrow[r] \arrow[ur, dashed] & U\,.
\end{tikzcd}$$

Therefore, $U \times_{\mathrm{Utg}_\Theta(\mathcal{X})} \mathrm{Utg}_\Theta(\mathcal{Y}) \to U$ is étale for any morphism $U \to \mathrm{Utg}_\Theta(\mathcal{X})$ from an algebraic space. It follows that $\mathrm{Utg}_\Theta(\mathcal{Y}) \to \mathrm{Utg}_\Theta(\mathcal{X})$ is étale.

Now suppose that $\mathcal{Y} \to \mathcal{X}$ is stabilizer preserving, étale and surjective. Let $\operatorname{Spec} K \to$

$\mathrm{Utg}_\Theta(\mathcal{X})$ be any geometric point. This induces a geometric point

$$x\colon \operatorname{Spec} K \to \mathcal{E}_{\operatorname{Spec} K} \to \mathcal{X}\,.$$

Let $y\colon \operatorname{Spec} K \to \mathcal{Y}$ be a lift of x. For some finite group G and a scheme E with $E_{\mathrm{red}} = \operatorname{Spec} K$, we have $\mathcal{E}_{\operatorname{Spec} K} \cong [E/G]$. The representable morphism $\mathcal{E}_{\operatorname{Spec} K} \to \mathcal{X}$ induces an injection $G \to \operatorname{Aut}_{\mathcal{X}}(x)$. Since $\mathcal{Y} \to \mathcal{X}$ is stabilizer preserving, the composition

$$\mathrm{B}\,G = [E_{\mathrm{red}}/G] \hookrightarrow \mathcal{E}_{\operatorname{Spec} K} \to \mathcal{X}$$

lifts to $\mathrm{B}\,G \to \mathcal{Y}$. We obtain a 2-commutative diagram

$$\begin{tikzcd} \operatorname{Spec} K \arrow[r] & \mathrm{B}\,G \arrow[r] \arrow[d] & \mathcal{Y} \arrow[d] \\ & \mathcal{E}_{\operatorname{Spec} K} \arrow[r] & \mathcal{X}\,. \end{tikzcd} \tag{6.2}$$

The morphism $\mathrm{B}\,G \to \mathcal{E}_{\operatorname{Spec} K}$ is a thickening. By Lemma A.7, there exists a morphism $\mathcal{E}_{\operatorname{Spec} K} \to \mathcal{Y}$ fitting into (6.2). It gives a K-point of $\mathrm{Utg}_\Theta(\mathcal{Y})$ lying over the chosen K-point of $\mathrm{Utg}_\Theta(\mathcal{X})$. This shows the desired surjectivity. □

6.3 The case of quotient stacks

For a formal scheme X of finite type over Df with an automorphism α, the α-fixed locus X^α is defined to be

$$X \times_{(\mathrm{id}_X,\alpha), X\times_{\mathrm{Df}} X, \Delta} X\,.$$

This becomes a closed subscheme of X by the first projection. When a finite group H acts on X, we define the *H-fixed locus* X^H to be $\bigcup_{h\in H} X^h$.

Definition 6.11. Let G be a Galoisian group, let V be a formal scheme over Df with an action of a finite group H, and let $\iota\colon G \to H$ be an embedding of finite groups. We define $\mathrm{Utg}_{\Theta_G,\iota}(V)$ to be the G-fixed locus of $\underline{\mathrm{Hom}}_{\mathrm{D}_{\Theta_G}}(E_{\Theta_G}, V_{\Theta_G})$ by the action $g(f) := \iota(g) f g^{-1}$.

Note that the Hom stack $\underline{\mathrm{Hom}}_{\mathrm{D}_{\Theta_G}}(E_{\Theta_G}, V_{\Theta_G})$ is identical to the Weil restriction

$$\mathrm{R}_{E_{\Theta_G}/\mathrm{Df}_{\Theta_G}}(V \times_{\mathrm{Df}_{\Theta_G}} E_{\Theta_G})\,.$$

An object of $\mathrm{Utg}_{\Theta_G,\iota}(V)$ over an S-point $S \to \mathrm{Df}_{\Theta_G}$ is an ι-equivariant morphism $E_{\Theta_G} \times_{\mathrm{Df}_{\Theta_G}} S \to V$. This induces a representable morphism

$$[(E_{\Theta_G} \times_{\mathrm{Df}_{\Theta_G}} S)/G] \cong \mathcal{E}_{\Theta_G} \times_{\mathrm{Df}_{\Theta_G}} S \to [V/\iota(G)] \to [V/H]\,.$$

In turn, this leads to a morphism

$$\coprod_{\iota\in \mathrm{Emb}(G,H)} \mathrm{Utg}_{\Theta_G,\iota}(V) \to \mathrm{Utg}_{\Theta_G}(\mathcal{X})\,,$$

where $\mathrm{Emb}(G,H)$ is the set of embeddings $G \hookrightarrow H$.

We define actions of H and $\operatorname{Aut}(G)$ on $\coprod_\iota \mathrm{Utg}_{\Theta_G,\iota}(V)$. For $h \in H$, let $c_h \in \operatorname{Aut}(H)$ be the automorphism $f \mapsto hfh^{-1}$. Given an ι-equivariant morphism $\psi\colon F \to V$, the composition $F \xrightarrow{\psi} V \xrightarrow{h} V$ is $c_h\iota$-equivariant. Sending $F \to V$ to $F \to V \xrightarrow{h} V$ gives a morphism $\mathrm{Utg}_{\Theta_G,\iota}(V) \to \mathrm{Utg}_{\Theta_G,c_h\iota}(V)$ and defines an H-action on $\coprod_\iota \mathrm{Utg}_{\Theta_G,\iota}(V)$.

For $c \in \operatorname{Aut}(G)$, let $F^{(c)}$ be the scheme F with the new G-action such that the automorphism $c^{-1}(g)\colon F \to F$ for the old action is the new g-action. Then the same scheme morphism $\psi\colon F^{(c)} \to V$ is now ιc^{-1}-equivariant. Sending $F \to V$ to $F^{(c)} \to V$ gives a morphism $\mathrm{Utg}_{\Theta_G,\iota}(V) \to \mathrm{Utg}_{\Theta_G,\iota c^{-1}}(V)$ and defines an $\mathrm{Aut}(G)$-action on $\coprod_\iota \mathrm{Utg}_{\Theta_G,\iota}(V)$.

The two actions commute, and we obtain the $\mathrm{Aut}(G)\times H$-action. The morphism $\coprod_\iota \mathrm{Utg}_{\Theta_G,\iota}(V) \to \mathrm{Utg}_{\Theta_G}(\mathcal{X})$ is invariant for the H-action and equivariant for the $\mathrm{Aut}(G)$- and $\mathrm{Aut}(G)\times H$-actions. From the universality of quotient stacks [Rom05] (similar to that of quotient schemes), we obtain morphisms

$$\Big[\Big(\coprod_{\iota\in\mathrm{Emb}(G,H)} \mathrm{Utg}_{\Theta_G,\iota}(V)\Big)/H\Big] \to \mathrm{Utg}_{\Theta_G}(\mathcal{X})\,, \tag{6.3}$$

$$\Big[\Big(\coprod_{\iota\in\mathrm{Emb}(G,H)} \mathrm{Utg}_{\Theta_G,\iota}(V)\Big)/\,\mathrm{Aut}(G)\times H\Big] \to \mathrm{Utg}_{\Theta_{[G]}}(\mathcal{X})\,. \tag{6.4}$$

Note that $\Theta_G \to \Theta_{[G]}$, as well as $\mathrm{Utg}_{\Theta_G}(\mathcal{X}) \to \mathrm{Utg}_{\Theta_{[G]}}(\mathcal{X})$, is an $\mathrm{Aut}(G)$-torsor since it is a base change of $\operatorname{Spec} k \to \mathcal{A}_{[G]} \cong \mathrm{B}\,\mathrm{Aut}(G)$. Since $\mathrm{Aut}(G)$ acts freely on $\mathrm{Emb}(G,H)$, if $\mathrm{Emb}(G,H)/\,\mathrm{Aut}(G)$ denotes a set of representatives of $\mathrm{Aut}(G)$-orbits, we have

$$\Big[\Big(\coprod_{\iota\in\mathrm{Emb}(G,H)} \mathrm{Utg}_{\Theta_G,\iota}(V)\Big)/\,\mathrm{Aut}(G)\Big] \cong \coprod_{\iota\in\mathrm{Emb}(G,H)/\,\mathrm{Aut}(G)} \mathrm{Utg}_{\Theta_G,\iota}(V)\,.$$

The $\mathrm{Aut}(G)\times H$-action on $\mathrm{Emb}(G,H)$ is not generally free. The stabilizer of ι is

$$\{(\iota^* c_h, h) \mid h \in \mathrm{N}_H(\iota(G))\}\ (\cong \mathrm{N}_H(\iota(G)))\,.$$

Here $\mathrm{N}_H(\iota(G))$ is the normalizer of $\iota(G)$ in H, and $\iota^* c_h$ is the automorphism of G corresponding to $c_h|_{\iota(G)}$ via ι. We have the induced $\mathrm{N}_H(\iota(G))$-action on $\mathrm{Utg}_{\Theta_G,\iota}(V)$; for $h \in \mathrm{N}_H(\iota(G))$,

$$h\colon\ \mathrm{Utg}_{\Theta_G,\iota}(V) \to \mathrm{Utg}_{\Theta_G,c_h\iota}(V) \to \mathrm{Utg}_{\Theta_G,c_h\iota(\iota^*c_h)^{-1}}(V) = \mathrm{Utg}_{\Theta_G,\iota}(V)$$

is the composition of morphisms given above. Therefore,

$$\begin{aligned}&\Big[\Big(\coprod_{\iota\in\mathrm{Emb}(G,H)} \mathrm{Utg}_{\Theta_G,\iota}(V)\Big)/(\mathrm{Aut}(G)\times H)\Big]\\ &\qquad\qquad \cong \coprod_{\iota\in\mathrm{Emb}(G,H)/(\mathrm{Aut}(G)\times H)} [\mathrm{Utg}_{\Theta_G,\iota}(V)/\,\mathrm{N}_H(\iota(G))]\,.\end{aligned} \tag{6.5}$$

Note that $\mathrm{Emb}(G,H)/(\mathrm{Aut}(G)\times H)$ is in one-to-one correspondence with the set of H-conjugacy classes of subgroups $G' \subset H$ such that $G' \cong G$. In particular, if G is a cyclic group of order l, then it is in one-to-one correspondence with the set of conjugacy classes of H that have order l. We also have

$$\Big[\Big(\coprod_{\iota\in\mathrm{Emb}(G,H)} \mathrm{Utg}_{\Theta_G,\iota}(V)\Big)/H\Big] \cong \coprod_{\iota\in\mathrm{Emb}(G,H)/H} [\mathrm{Utg}_{\Theta_G,\iota}(V)/\,\mathrm{C}_H(\iota(G))]\,.$$

Here $\mathrm{C}_H(\iota(G))$ denotes the centralizer of $\iota(G)$ in H.

Lemma 6.12. *The morphisms* (6.3) *and* (6.4) *are isomorphisms.*

Proof. It suffices to show that $\coprod_{\iota\in\mathrm{Emb}(G,H)} \mathrm{Utg}_{\Theta_G,\iota}(V) \to \mathrm{Utg}_{\Theta_G}(\mathcal{X})$ is an H-torsor. This is true because the base change by $E_{\Theta_G} \to \mathrm{Df}_{\Theta_G}$ is an H-torsor, as proved in Lemma 6.13. □

Lemma 6.13. *Let $\mathcal{X} := [V/H]$. The diagram*

$$\begin{tikzcd}
\coprod_{\iota\in\mathrm{Emb}(G,H)} \mathrm{Utg}_{\Theta_G,\iota}(V)\times_{\mathrm{Df}_{\Theta_G}} E_{\Theta_G} \arrow[r] \arrow[d] & V \arrow[d] \\
\mathrm{Utg}_{\Theta_G}(\mathcal{X})\times_{\mathrm{Df}_{\Theta_G}} E_{\Theta_G} \arrow[r] & \mathcal{X}
\end{tikzcd}$$

is 2-Cartesian. In particular, the left vertical arrow is an H-torsor.

Proof. Since $V\to\mathcal{X}=[V/H]$ is an H-torsor, the second assertion follows from the first. Let $s\colon S\to \mathrm{Df}_{\Theta_G}$ be a morphism from an affine scheme, let $E_S := E_{\Theta_G}\times_{\Theta_G,s} S$ and $\mathcal{E}_S := \mathcal{E}_{\Theta_G}\times_{\Theta_G,s} S$, and let $V_S := V\times_{\mathrm{Df}} S$ and $\mathcal{X}_S := \mathcal{X}\times_{\mathrm{Df}} S$. The fiber over s of the fibered category $(\mathrm{Utg}_{\Theta_G}(\mathcal{X})\times_{\mathrm{Df}_{\Theta_G}} E_{\Theta_G})\times_{\mathcal{X}} V\to\mathrm{Df}_{\Theta_G}$ is regarded as the category of 2-commutative diagrams of the form

$$\begin{tikzcd}
S \arrow[rr] \arrow[d] & & V_S \arrow[d, dotted] \\
E_S \arrow[r, dotted] \arrow[urr, Rightarrow] & \mathcal{E}_S \arrow[r] & \mathcal{X}_S\,,
\end{tikzcd} \tag{6.6}$$

where the dotted arrows are the canonical ones, the normal arrows are S-morphisms and the thick arrow is a witnessing 2-isomorphism. On the other hand, the fiber over s of the fibered category $\mathrm{Utg}_{\Theta_G,\iota}(V)\times_{\mathrm{Df}_{\Theta_G}} E_{\Theta_G}\to\mathrm{Df}_{\Theta_G}$ is identified with the category of diagrams of S-morphisms

$$\begin{tikzcd}
S \arrow[d] & V_S \\
E_S \arrow[ur, "\iota\text{-eq.}"'] &
\end{tikzcd} \tag{6.7}$$

where $E_S\to V_S$ is ι-equivariant.

Given a diagram of the form (6.7), we obtain dashed arrows and thick arrows forming the 2-commutative diagram

$$\begin{tikzcd}
S \arrow[rrr, dashed] \arrow[d] & & & V_S \arrow[d, dotted] \\
E_S \arrow[r, dotted] \arrow[urrr] \arrow[ur, Rightarrow] & \mathcal{E}_S \arrow[r, dashed] \arrow[urr, Rightarrow] & {[V_S/\iota(H)]} \arrow[r, dotted] & \mathcal{X}_S\,.
\end{tikzcd}$$

Again, the dotted arrows are the canonical ones. Thus we obtain a diagram of the form (6.6) and a morphism

$$\mathrm{Utg}_{\Theta_G,\iota}(V)\times_{\mathrm{Df}_{\Theta_G}} E_{\Theta_G}\to(\mathrm{Utg}_{\Theta_G}(\mathcal{X})\times_{\mathrm{Df}_{\Theta_G}} E_{\Theta_G})\times_{\mathcal{X}} V\,.$$

Conversely, given a diagram of the form (6.6), by Lemma A.7, we obtain a dashed arrow and a 2-isomorphism for each triangle in

$$\begin{tikzcd}
S \arrow[rr] \arrow[d] & & V_S \arrow[d, dotted] \\
E_S \arrow[r, dotted] \arrow[urr, dashed] \arrow[ur, Rightarrow] & \mathcal{E}_S \arrow[r] \arrow[ur, Rightarrow] & \mathcal{X}_S\,,
\end{tikzcd}$$

which induces the given 2-isomorphism for the rectangle. We claim that there exists a unique decomposition $S=\coprod_{\iota\in\mathrm{Emb}(G,H)} S_\iota$ into open and closed subschemes such that the morphism $E_S\to V_S$ restricts to ι-equivariant morphisms $E_{S_\iota}\to V_{S_\iota}$. The subdiagram

$$\begin{tikzcd}
E_S \arrow[r] \arrow[d] & V_S \arrow[d] \\
\mathcal{E}_S=[E_S/H] \arrow[r] & \mathcal{X}_S=[V_S/G]
\end{tikzcd}$$

gives a morphism of groupoids $(E_S\times H \rightrightarrows E_S)\to(V_S\times G\rightrightarrows V_S)$. Therefore, there exists a unique decomposition $S=\coprod_{\iota\in\mathrm{Emb}(G,H)}S_\iota$ into open and closed subschemes such that for each ι, the morphism $E_S\times H\to V_S\times G$ restricts to a morphism $E_{S_\iota}\times H\to V_{S_\iota}\times G$ compatible with ι. This shows the claim. Thus we have obtained a morphism

$$(\mathrm{Utg}_{\Theta_G}(\mathcal{X})\times_{\mathrm{D}_{\Theta_G}}\mathcal{E}_{\Theta_G})\times_{\mathcal{X}}V\to\coprod_{\iota\in\mathrm{Emb}(G,H)}\mathrm{Utg}_{\Theta_G,\iota}(V)\times_{\mathrm{D}_{\Theta_G}}E_{\Theta_G}\,.$$

We can check that the resulting morphisms between

$$(\mathrm{Utg}_{\Theta_G}(\mathcal{X})\times_{\mathrm{D}_{\Theta_G}}\mathcal{E}_{\Theta_G})\times_{\mathcal{X}}V\quad\text{and}\quad\coprod_{\iota\in\mathrm{Emb}(G,H)}\mathrm{Utg}_{\Theta_G,\iota}(V)\times_{\mathrm{D}_{\Theta_G}}E_{\Theta_G}$$

are quasi-inverse to each other by using the uniqueness in Lemma A.7. □

6.4 Variants of inertia stacks

Recall from Section 3 that $\mathcal{A}$ denotes the moduli stack of Galoisian group schemes and $\mathcal{G}\to\mathcal{A}$ is the universal group scheme.

Definition 6.14. We define a fibered category $\mathrm{B}\,\mathcal{G}\to\mathcal{A}$ as follows: for an object $G\to S$ of $\mathcal{A}$, which is by definition a Galoisian group scheme over an affine scheme, the fiber category $(\mathrm{B}\,\mathcal{G})(G\to S)$ is $(\mathrm{B}\,G)(S)$, that is, the groupoid of G-torsors over S. For a Galoisian group G, we define $\mathrm{B}\,\mathcal{G}_{[G]}:=\mathrm{B}\,\mathcal{G}\times_{\mathcal{A}}\mathcal{A}_{[G]}$.

Definition 6.15. Let $\mathcal{Z}$ be a DM stack of finite type over k. For a Galoisian group scheme $G\to\operatorname{Spec}k$, we define $\mathrm{I}^{(G)}\,\mathcal{Z}:=\underline{\mathrm{Hom}}^{\mathrm{rep}}_{\operatorname{Spec}k}(\mathrm{B}\,G,\mathcal{Z})$. We define $\mathrm{I}^{(\bullet)}\,\mathcal{Z}:=\coprod_{G\in\mathrm{GalGps}}\mathrm{I}^{(G)}\,\mathcal{Z}$.

For a Galoisian group G, we define $\mathrm{I}^{[G]}\,\mathcal{Z}:=\underline{\mathrm{Hom}}^{\mathrm{rep}}_{\mathcal{A}_{[G]}}(\mathrm{B}\,\mathcal{G}_{[G]},\mathcal{Z})$. We also define $\mathrm{I}^{[\bullet]}\,\mathcal{Z}:=\coprod_{G\in\mathrm{GalGps}}\mathrm{I}^{[G]}\,\mathcal{Z}$.

These are variants of inertia stacks, and their relation is closer in the tame case, as we will see in Section 15. For the morphism $\operatorname{Spec}k\to\mathcal{A}_{[G]}$ given by $G\to\operatorname{Spec}k$, we have $\mathrm{I}^{(G)}\,\mathcal{Z}\cong\mathrm{I}^{[G]}\,\mathcal{Z}\times_{\mathcal{A}_{[G]}}\operatorname{Spec}k$.

Lemma 6.16. *Let $\mathcal{Z}$ be a quotient stack $[V/H]$ associated with a finite group action $H\curvearrowright V$ on an algebraic space. For a Galoisian group G, we have*

$$\mathrm{I}^{(G)}\,\mathcal{Z}\cong\left[\left(\coprod_{\iota\in\mathrm{Emb}(G,H)}V^{\iota(G)}\right)/H\right]\cong\coprod_{\iota\in\mathrm{Emb}(G,H)/H}\left[V^{\iota(G)}/\,\mathrm{C}_H(\iota(G))\right]$$

and

$$\mathrm{I}^{[G]}\,\mathcal{Z}\cong\left[\left(\coprod_{\iota\in\mathrm{Emb}(G,H)}V^{\iota(G)}\right)/(\mathrm{Aut}(G)\times H)\right]\cong\coprod_{\iota\in\mathrm{Emb}(G,H)/(\mathrm{Aut}(G)\times H)}\left[V^{\iota(G)}/\,\mathrm{N}_H(\iota(G))\right].$$

Proof. We apply the same arguments as in the proof of Lemma 6.12 to $\mathrm{B}\,G=[\operatorname{Spec}k/G]$, $V^{\iota(G)}=\underline{\mathrm{Hom}}^{\iota}_{\operatorname{Spec}k}(\operatorname{Spec}k,V)$ (the right side is the scheme of ι-equivariant morphisms) and $\mathrm{I}^{[G]}\,\mathcal{Z}$. □

For a formal DM stack $\mathcal{X}$ of finite type over Df, we have a morphism

$$\mathrm{Utg}_{\Theta_{[G]}}(\mathcal{X})_0\to\mathrm{I}^{[G]}(\mathcal{X}_0)$$

as follows. Let $S\to\mathrm{D}_{0,\Theta_{[G]}}$ be a morphism with $S\in\mathbf{Aff}$ and $G_S\to S$ the associated group scheme and

$$\mathcal{E}_S:=\mathcal{E}_{\Theta_{[G]}}\times_{\mathrm{Df}_{\Theta_{[G]}}}S\to\mathcal{X}$$

a representable morphism, which give an S-point of $\mathrm{Utg}_{\Theta_{[G]}}(\mathcal{X})_0$. The given section $S \to \mathcal{E}_S$ has automorphism group scheme G_S. Therefore, we have a closed immersion $\mathrm{B}(G_S/S) \hookrightarrow \mathcal{E}_S$. We send the above S-point of $\mathrm{Utg}_{\Theta}(\mathcal{X})_0$ to the S-point of $\mathrm{I}^{[G]}(\mathcal{X})$ defined by the induced $\mathrm{B}(G_S/S) \to \mathcal{X}_0$ and $G_S \colon S \to \mathcal{A}_{[G]}$.

By base change along the morphism $\operatorname{Spec} k \to \mathcal{A}_{[G]}$, given by a Galoisian group G, we get a morphism

$$\mathrm{Utg}_{\Theta_G}(\mathcal{X})_0 \to \mathrm{I}^{(G)}(\mathcal{X}_0)\,.$$

6.5 More on untwisting stacks

In this and the next subsection, we use a few notions from rigid geometry, for which we refer the reader to [Abb10]. With a formal scheme X of finite type over Df, we can associate a coherent rigid space X^{rig}, which may be regarded as the "generic fiber" of $X \to \mathrm{Df}$.

Definition 6.17 ([Abb10, Proposition 6.4.12]). Let U and V be formal affine schemes of finite type over Df, and let $f \colon V \to U$ be a Df-morphism. We say that f is *rig-étale* if the induced morphism $f^{\mathrm{rig}} \colon V^{\mathrm{rig}} \to U^{\mathrm{rig}}$ of coherent rigid spaces is étale.

Lemma 6.18. *Let $g \colon Z \to Y$ and $f \colon Y \to X$ be morphisms of coherent rigid spaces.*

(1) *If $f \circ g$ is étale and f is unramified, then f is étale.*
(2) *If $f \circ g$ is étale, g is étale and the map $\langle Z \rangle \to \langle Y \rangle$ of sets of rigid points is surjective, then f is étale.*

Proof. (1) This is [Abb10, Proposition 6.4.3(vi)].

(2) Let $y \in \langle Y \rangle$ be any rigid point, let $x \in \langle X \rangle$ be its image, and let $z \in \langle Z \rangle$ be a lift of y. Let Z_y denote the fiber of $Z \to Y$ over y, and similarly for Z_x and Y_x. By [Abb10, Proposition 6.4.24(v)], the extensions $\mathcal{O}_{Z_y,z}/\kappa(y)$ and $\mathcal{O}_{Z_x,z}/\kappa(x)$ are finite separable field extensions. If $\mathfrak{m}$ denotes the maximal ideal of $\mathcal{O}_{Y_x,y}$, then since $\mathcal{O}_{Z_x,z}$ is a field, we have $\mathfrak{m}\mathcal{O}_{Z_x,z} = 0$. Since $\mathcal{O}_{Y_x,y} \to \mathcal{O}_{Z_x,z}$ is faithfully flat, it is injective. Therefore, $\mathfrak{m} = 0$ and $\mathcal{O}_{Y_x,y}$ is a field. This is a finite separable extension of $\kappa(x)$ since $\mathcal{O}_{Z_x,z}$ is so. Therefore, f is étale. □

Lemma 6.19. *Let*

$$\begin{tikzcd} \tilde{V} \arrow[r, "\tilde{f}"] \arrow[d] & \tilde{U} \arrow[d] \\ V \arrow[r, "f"'] & U \end{tikzcd}$$

be a commutative diagram of formal schemes of finite type over Df. *Suppose that the vertical arrows are étale and surjective. Then f is rig-étale if and only if $\tilde{f}$ is rig-étale.*

Proof. If either f or $\tilde{f}$ is rig-étale, then $\tilde{V} \to U$ is rig-étale. Since $\tilde{V} \to V$ is étale and surjective, the map $\langle \tilde{V} \rangle \to \langle V \rangle$ is surjective. Now the "if" and "only if" parts follow from assertions (2) and (1) of Lemma 6.18, respectively. □

Definition 6.20. Let $\mathcal{Y}$ and $\mathcal{X}$ be formal DM stacks of finite type over Df. We say that a Df-morphism $\mathcal{Y} \to \mathcal{X}$ is *rig-étale* if it fits into some 2-commutative diagram

$$\begin{tikzcd} V \arrow[r] \arrow[d] & U \arrow[d] \\ \mathcal{Y} \arrow[r] & \mathcal{X} \end{tikzcd}$$

such that vertical arrows are atlases from formal affine schemes and the upper arrow is rig-étale.

By Lemma 6.19, "some" in the last condition can be replaced by "any."

Lemma 6.21. *Let G be a finite group, and let V be a formal scheme of finite type over* Df *with an action of G. Then the natural morphism*

$$\mathrm{Utg}_{\Theta_G,\iota}(V)\times_{\mathrm{Df}_{\Theta_G}} E_{\Theta_G}\to V\times\Theta_G$$

is rig-étale.

Proof. By the construction of Θ_G (see Definition 5.20), there exists an atlas $T=\operatorname{Spec}R\to\Theta_G$ from a formal affine scheme which corresponds to a uniformizable torsor

$$P_T=\operatorname{Spec}A\to\mathrm{D}^*_T=\operatorname{Spec}R(\!(t)\!)\,.$$

Let $E_T=\operatorname{Spf}O_A\to\mathrm{Df}_T$ be its integral model, and let $E^{\mathrm{rig}}\to\mathrm{Df}_T^{\mathrm{rig}}$ be the induced morphism of rigid spaces, which is an étale G-torsor.

Let $V_T:=V\times T$. It suffices to show that the morphism of formal schemes $\mathrm{Utg}_{T,\iota}(V)\times_{\mathrm{Df}_T}E_T\to V_T$ is rig-étale. Here $\mathrm{Utg}_{T,\iota}(V):=\mathrm{Utg}_{\Theta_G,\iota}(V)\times_{\Theta_G}T$. By definition, $\mathrm{Utg}_{T,\iota}(V)$ is the G-fixed subscheme of the Weil restriction $\mathrm{R}_{E_T/\mathrm{Df}_T}(V_T\times_{\mathrm{Df}_T}E_T)$. By [Ber00, Proposition 1.22], the rigid space $\mathrm{Utg}_{T,\iota}(V)^{\mathrm{rig}}$ is an open subspace of $\mathrm{R}_{E^{\mathrm{rig}}/\mathrm{Df}_T^{\mathrm{rig}}}\big(V_T^{\mathrm{rig}}\times_{\mathrm{Df}_T^{\mathrm{rig}}}E^{\mathrm{rig}}\big)^G$. It suffices to show that

$$\theta\colon\ \mathrm{R}_{E^{\mathrm{rig}}/\mathrm{Df}_T^{\mathrm{rig}}}\big(V_T^{\mathrm{rig}}\times_{\mathrm{Df}_T^{\mathrm{rig}}}E^{\mathrm{rig}}\big)^G\times_{\mathrm{Df}_T^{\mathrm{rig}}}E^{\mathrm{rig}}\to V_T^{\mathrm{rig}}$$

is formally étale. Let us consider the commutative diagram of solid arrows

$$\begin{tikzcd} S \arrow[r, "\phi"] \arrow[d, hook] & \mathrm{R}_{E^{\mathrm{rig}}/\mathrm{Df}_T^{\mathrm{rig}}}\big(V_T^{\mathrm{rig}}\times_{\mathrm{Df}_T^{\mathrm{rig}}}E^{\mathrm{rig}}\big)^G\times_{\mathrm{Df}_T^{\mathrm{rig}}}E^{\mathrm{rig}} \arrow[d, "\theta"] \\ S' \arrow[r, "\phi'"'] \arrow[ur, dashed] & V_T^{\mathrm{rig}}\,, \end{tikzcd} \tag{6.8}$$

where $S\hookrightarrow S'$ is a thickening of coherent rigid spaces over $\mathrm{Df}_T^{\mathrm{rig}}$. The S-point ϕ is given by the pair (f,s) of a G-equivariant morphism $f\colon S\times_{\mathrm{Df}_T^{\mathrm{rig}}}E^{\mathrm{rig}}\to V_T^{\mathrm{rig}}$ and a $\mathrm{Df}_T^{\mathrm{rig}}$-morphism $s\colon S\to E^{\mathrm{rig}}$. Since the G-torsor $S\times_{\mathrm{Df}_T^{\mathrm{rig}}}E^{\mathrm{rig}}\to S$ has the section $\sigma:=(\mathrm{id}_S,s)$, we have the decomposition

$$S\times_{\mathrm{Df}_T^{\mathrm{rig}}}E^{\mathrm{rig}}=\coprod_{g\in G}g\sigma(S)\,. \tag{6.9}$$

The morphism θ sends the S-point ϕ to the S-point

$$\psi\colon S\xrightarrow{\sigma}S\times_{\mathrm{Df}_T^{\mathrm{rig}}}E^{\mathrm{rig}}\xrightarrow{f}V_T^{\mathrm{rig}}\,.$$

This ψ extends an S'-point $\phi'\colon S'\to V_T^{\mathrm{rig}}$. Thanks to the above decomposition (6.9), the morphism ϕ' uniquely extends to the G-equivariant morphism $f'\colon S'\times_{\mathrm{Df}_T^{\mathrm{rig}}}E^{\mathrm{rig}}\to V_T^{\mathrm{rig}}$, which restricts to f on $S\times_{\mathrm{Df}_T^{\mathrm{rig}}}E^{\mathrm{rig}}$. On the other hand, since $E^{\mathrm{rig}}\to\mathrm{Df}_T^{\mathrm{rig}}$ is étale, the morphism s also uniquely extends to $s'\colon S'\to E^{\mathrm{rig}}$. The pair (f',s') gives the unique dashed arrow fitting in diagram (6.8). Thus θ is formally étale. ∎

Lemma 6.22. *The morphism* $\mathrm{Utg}_\Theta(\mathcal{X})\times_{\mathrm{Df}_\Theta}\mathcal{E}_\Theta\to\mathcal{X}\times\Theta$ *is rig-étale.*

Proof. It suffices to show that for each Galoisian group G, the morphism

$$\mathrm{Utg}_{\Theta_G}(\mathcal{X})\times_{\mathrm{Df}_{\Theta_G}}\mathcal{E}_{\Theta_G}\to\mathcal{X}\times\Theta_G$$

is rig-étale. Moreover, by Lemmas 2.18 and 6.10, we can reduce to the case where $\mathcal{X}$ is a quotient stack $[V/H]$ associated with a formal scheme V with an action of a finite group H. Consider the following 2-commutative diagram:

$$\begin{tikzcd} \coprod_{\iota\in\mathrm{Emb}(G,H)} \mathrm{Utg}_{\Theta_G,\iota}(V) \times_{\mathrm{Df}_{\Theta_G}} E_{\Theta_G} \arrow[r] \arrow[d] & V \times \Theta_G \arrow[d] \\ \mathrm{Utg}_{\Theta_G}(\mathcal{X}) \times_{\mathrm{Df}_{\Theta_G}} \mathcal{E}_{\Theta_G} \arrow[r] & \mathcal{X} \times \Theta_G\,. \end{tikzcd}$$

Since the vertical arrows are étale and the upper arrow is rig-étale, we conclude that the bottom arrow is also rig-étale. □

Remark 6.23. In the non-formal setting, the "untwisting stack" for a uniformizable G-torsor does not change the generic fiber. Lemma 6.22 is an analog of this fact. More precisely, let $\mathcal{X}$ be a (non-formal) DM stack of finite type over the genuine scheme $\mathrm{D} = \operatorname{Spec} k[\![t]\!]$, let T be an affine scheme, and let $\operatorname{Spec} A \to \mathrm{D}^*_T$ is a uniformizable G-torsor. Consider the non-formal counterpart

$$\mathcal{U}_T := \underline{\mathrm{Hom}}^{\mathrm{rep}}_{\mathrm{D}_T}([\operatorname{Spec} O_A/G], \mathcal{X} \times_{\mathrm{D}} \mathrm{D}_T)$$

of the untwisting stack $\mathrm{Utg}_T(\mathcal{X})$. We have

$$\begin{aligned} \mathcal{U}_T \times_{\mathrm{D}} \mathrm{D}^* &\cong \underline{\mathrm{Hom}}^{\mathrm{rep}}_{\mathrm{D}_T}([\operatorname{Spec} O_A/G], \mathcal{X} \times_{\mathrm{D}} \mathrm{D}_T) \times_{\mathrm{D}_T} \mathrm{D}^*_T \\ &\cong \underline{\mathrm{Hom}}^{\mathrm{rep}}_{\mathrm{D}^*_T}([\operatorname{Spec} O_A/G] \times_{\mathrm{D}_T} \mathrm{D}^*_T, \mathcal{X} \times_{\mathrm{D}} \mathrm{D}^*_T) \\ &\cong \underline{\mathrm{Hom}}^{\mathrm{rep}}_{\mathrm{D}^*_T}([\operatorname{Spec} A/G], \mathcal{X} \times_{\mathrm{D}} \mathrm{D}^*_T) \\ &\cong \underline{\mathrm{Hom}}^{\mathrm{rep}}_{\mathrm{D}^*_T}(\mathrm{D}^*_T, \mathcal{X} \times_{\mathrm{D}} \mathrm{D}^*_T) \\ &\cong \mathcal{X} \times_{\mathrm{D}} \mathrm{D}^*_T\,. \end{aligned}$$

In particular, if $T = \operatorname{Spec} k$, the two D-stacks $\mathcal{U}_{\operatorname{Spec} k}$ and $\mathcal{X}$ share the generic fiber.

Lemma 6.24. *Let $\mathcal{Y} \hookrightarrow \mathcal{X}$ be a closed immersion of formal DM stacks of finite type over* Df. *The induced morphism* $\mathrm{Utg}_\Theta(\mathcal{Y}) \to \mathrm{Utg}_\Theta(\mathcal{X})$ *is a closed immersion.*

Proof. For each $G \in \mathrm{GalGps}$, we prove that $\mathrm{Utg}_{\Theta_{[G]}}(\mathcal{Y}) \to \mathrm{Utg}_{\Theta_{[G]}}(\mathcal{X})$ is a closed immersion. By base change, it suffices to show that $\mathrm{Utg}_{\Theta_G}(\mathcal{Y}) \to \mathrm{Utg}_{\Theta_G}(\mathcal{X})$ is a closed immersion. Let $\mathcal{X}' := \coprod_\gamma [W_\gamma/H_\gamma] \to \mathcal{X}$ be as in Lemma 2.18, which is a stabilizer-preserving, étale and surjective morphism, and let $\mathcal{Y}' := \mathcal{Y} \times_{\mathcal{X}} \mathcal{X}'$. By Lemma 6.6, the diagram

$$\begin{tikzcd} \mathrm{Utg}_{\Theta_G}(\mathcal{Y}') \arrow[r] \arrow[d] & \mathrm{Utg}_{\Theta_G}(\mathcal{X}') \arrow[d] \\ \mathrm{Utg}_{\Theta_G}(\mathcal{Y}) \arrow[r] & \mathrm{Utg}_{\Theta_G}(\mathcal{X}) \end{tikzcd}$$

is 2-Cartesian. The upper arrow is a closed immersion by Lemmas 6.13 and 6.3. By Lemma 6.10, the vertical arrows are étale and surjective. Therefore, the bottom arrow is also a closed immersion. □

Let $\mathcal{X}$ be a formal DM stack of finite type and $\mathcal{Y}$ a closed substack of $\mathcal{X}$ defined by an ideal sheaf $\mathcal{I} \subset \mathcal{O}_{\mathcal{X}}$. Let $\mathcal{Y}_{(n)}$ denote the closed substack defined by $\mathcal{I}^{n+1}$. We get an inductive system of closed substacks

$$\mathcal{Y} = \mathcal{Y}_{(0)} \hookrightarrow \mathcal{Y}_{(1)} \hookrightarrow \cdots .$$

We define the ind-DM stack $\widehat{\mathcal{X}}$ to be the limit $\varinjlim \mathcal{Y}_{(n)}$ and call it the *completion* of $\mathcal{X}$ along $\mathcal{Y}$.

Lemma 6.25. *Let* Γ *be a DM stack of finite type, and let* $\Gamma \to \Theta_{[G]}$ *be any morphism. Suppose* $\mathcal{Y} \subset \mathcal{X}_m := \mathcal{X} \times_{\mathrm{Df}} \mathrm{Df}_m$ *for some* $m \in \mathbb{N}$. *Then* $\mathrm{Utg}_\Gamma(\widehat{\mathcal{X}})$ *is the completion of* $\mathrm{Utg}_\Gamma(\mathcal{X})$ *along* $\mathrm{Utg}_\Gamma(\mathcal{Y}_{(i)})$ *for any* $i \geqslant \sharp G - 1$.

Proof. The stack $\mathrm{Utg}_\Gamma(\widehat{\mathcal{X}})$ is the limit of $\mathrm{Utg}_\Gamma(\mathcal{Y}_{(n)})$, $n \in \mathbb{N}$, while the completion of $\mathrm{Utg}_\Gamma(\mathcal{X})$ along $\mathrm{Utg}_\Gamma(\mathcal{Y}_{(i)})$ is the limit of $\mathrm{Utg}_\Gamma(\mathcal{Y}_{(i)})_{(n)}$, $n \in \mathbb{N}$. It suffices to show that for every n, there exists an n' such that

$$\mathrm{Utg}_\Gamma(\mathcal{Y}_{(n)}) \subset \mathrm{Utg}_\Gamma(\mathcal{Y}_{(i)})_{(n')}\,, \tag{6.10}$$

$$\mathrm{Utg}_\Gamma(\mathcal{Y}_{(i)})_{(n)} \subset \mathrm{Utg}_\Gamma(\mathcal{Y}_{(n')})\,. \tag{6.11}$$

To show these, we may suppose $\Gamma = \Theta_{[G]}$; in particular, we may suppose that Γ is reduced. Since $\mathcal{Y} \subset \mathcal{X}_m$, the morphism $\mathrm{Utg}_\Gamma(\mathcal{Y}_{(n)})_{\mathrm{red}} \to \mathrm{Df}_\Gamma$ factors through $\mathrm{D}_{\Gamma,0}$ and

$$\mathrm{Utg}_\Gamma(\mathcal{Y}_{(n)})_{\mathrm{red}} \times_{\mathrm{Df}_\Gamma} \mathcal{E}_\Gamma \cong \mathrm{Utg}_\Gamma(\mathcal{Y}_{(n)})_{\mathrm{red}} \times_{\mathrm{D}_{\Gamma,0}} \mathcal{E}_{\Gamma,0}\,.$$

Let $\mathcal{N}$ be the nilradical of the structure sheaf of $\mathrm{Utg}_\Gamma(\mathcal{Y}_{(n)})_{\mathrm{red}} \times_{\mathrm{D}_{\Gamma,0}} \mathcal{E}_{\Gamma,0}$, which is the pullback of the nilradical $\mathcal{N}'$ of the structure sheaf of $\mathcal{E}_{\Gamma,0}$. Since $(\mathcal{N}')^{\sharp G} = 0$, we have $\mathcal{N}^{\sharp G} = 0$. Consider the universal morphisms

$$u\colon\ \mathrm{Utg}_\Gamma(\mathcal{Y}_{(n)})_{\mathrm{red}} \times_{\mathrm{Df}_\Gamma} \mathcal{E}_\Gamma \to \mathcal{Y}_{(n)}\,.$$

If $\mathcal{I}$ denotes the defining ideal of $\mathcal{Y} \subset \mathcal{Y}_{(n)}$, which is nilpotent, then $u^{-1}\mathcal{I} \subset \mathcal{N}$. It follows that $u^{-1}\mathcal{I}^{\sharp G} = 0$. This means that u factors through $\mathcal{Y}_{(i)}$ whenever $i \geqslant \sharp G - 1$. In turn, this shows that

$$\mathrm{Utg}_\Gamma(\mathcal{Y}_{(n)})_{\mathrm{red}} \subset \mathrm{Utg}_\Gamma(\mathcal{Y}_{(i)})\,.$$

Inclusion (6.10) follows.

Inclusion (6.11) holds in a more general situation. For any closed substack $\mathcal{Z} \subset \mathcal{X}$, consider the 2-commutative diagram

$$\begin{tikzcd} \mathrm{Utg}_\Gamma(\mathcal{Z}) \times_{\mathrm{Df}_\Gamma} \mathcal{E}_\Gamma \arrow[r] \arrow[d, hook] \arrow[dr, "b"] & \mathcal{Z} \arrow[d, hook] \\ \mathrm{Utg}_\Gamma(\mathcal{Z})_{(n)} \times_{\mathrm{Df}_\Gamma} \mathcal{E}_\Gamma \arrow[r, "c"'] & \mathcal{X}\,. \end{tikzcd}$$

Let $\mathcal{J}$ and $\mathcal{I}_\mathcal{Z}$ be the defining ideals of the left and right vertical arrows, respectively. We have $b^{-1}\mathcal{I}_\mathcal{Z} = 0$, $c^{-1}\mathcal{I}_\mathcal{Z} \subset \mathcal{J}$ and $\mathcal{J}^{n+1} = 0$. Therefore, $c^{-1}\mathcal{I}_\mathcal{Z}^{n+1} = 0$. This shows that c factors through $\mathcal{Z}_{(n)}$. Therefore, $\mathrm{Utg}_\Gamma(\mathcal{Z})_{(n)} \subset \mathrm{Utg}_\Gamma(\mathcal{Z}_{(n)})$. In particular,

$$\mathrm{Utg}_\Gamma(\mathcal{Y}_{(i)})_{(n)} \subset \mathrm{Utg}_\Gamma\left((\mathcal{Y}_{(i)})_{(n)}\right) = \mathrm{Utg}_\Gamma(\mathcal{Y}_{((i+1)(n+1)-1)})\,. \qquad \square$$

6.6 Rig-pure formal stacks and flattening stratification

Definition 6.26 ([Abb10, § 1.8.30, p. 43]). Let A be a $k[\![t]\!]$-algebra. We say that A is *rig-pure* if one of the following equivalent conditions holds:

(1) The element $t \in A$ is not a zerodivisor.

(2) The localization map $A \to A_t$ is injective.

(3) The scheme-theoretic closure of the open subscheme $\operatorname{Spec} A_t \subset \operatorname{Spec} A$ in $\operatorname{Spec} A$ is $\operatorname{Spec} A$.

For a $k[\![t]\!]$-algebra A, the *associated rig-pure algebra* of A, denoted by A^{pur}, is defined to be the image of the map $A \to A_t$.

This construction is characterized by the universality: the map $A \to A^{\mathrm{pur}}$ is the universal map from A to a rig-pure algebra.

Lemma 6.27. *Let A be an $R[\![t]\!]$-algebra and $A \to B$ a flat homomorphism of rings. Then there exists a natural isomorphism $B^{\mathrm{pur}} \cong A^{\mathrm{pur}} \otimes_A B$. In particular, B is rig-pure if A is rig-pure, and the converse is also true if $A \to B$ is faithfully flat.*

Proof. This is [Abb10, Corollary 5.1.13 and Lemma 5.11.9]. For the sake of completeness, we give a proof. Let C be an A-algebra which is rig-pure. Then the map $C \to C_t$ is injective. Applying $-\otimes_A B$, we get a map $C \otimes_A B \to C_t \otimes_A B = (C \otimes_A B)_t$. Since $A \to B$ is flat, this is injective, which means that $C \otimes_A B$ is rig-pure. We apply this to $C = A^{\mathrm{pur}}$ and get that $A^{\mathrm{pur}} \otimes_A B$ is rig-pure. From the commutative diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[d] \\ A^{\mathrm{pur}} \arrow[r] & B^{\mathrm{pur}}, \end{tikzcd}$$

we get a map $A^{\mathrm{pur}} \otimes_A B \to B^{\mathrm{pur}}$. By the universality of B^{pur}, this map has an inverse, and the first assertion follows.

If A is rig-pure, then we have $A \cong A^{\mathrm{pur}}$, which implies $B \cong B^{\mathrm{pur}}$, that is, B is rig-pure. Conversely, if B is rig-pure, then $B = A \otimes_A B \to A^{\mathrm{pur}} \otimes_A B \cong B^{\mathrm{pur}}$ is an isomorphism. Now, if $A \to B$ is faithfully flat, this implies that $A \to A^{\mathrm{pur}}$ is also an isomorphism and A is rig-pure. $\square$

Lemma 6.28. *Let A be a $R[\![t]\!]$-algebra. Suppose that A^{pur} is flat over $R[\![t]\!]$. Then, for any ring map $R \to S$, we have $A^{\mathrm{pur}} \otimes_{R[\![t]\!]} S[\![t]\!] \cong (A \otimes_{R[\![t]\!]} S[\![t]\!])^{\mathrm{pur}}$.*

Proof. As a base change of $A \to A^{\mathrm{pur}} \to A_t$, we get $A_S \to (A^{\mathrm{pur}})_S \to (A_S)_t$, where the subscript S means the base change by $R[\![t]\!] \to S[\![t]\!]$. Therefore, we get surjective maps $A_S \to (A^{\mathrm{pur}})_S \to (A_S)^{\mathrm{pur}}$. If $(A_S)_{t\text{-tor}}$ denotes the submodule of t-torsions of A_S, we have

$$(A_S)_{t\text{-tor}} = \mathrm{Ker}(A_S \to (A_S)^{\mathrm{pur}}) \supset \mathrm{Ker}(A_S \to (A^{\mathrm{pur}})_S) \supset (A_S)_{t\text{-tor}}\,.$$

The last inclusion holds since $(A^{\mathrm{pur}})_S$ is flat over $S[\![t]\!]$ as well as over $k[\![t]\!]$ and has no t-torsion. Thus the two kernels are equal. It follows that $(A^{\mathrm{pur}})_S \to (A_S)^{\mathrm{pur}}$ is an isomorphism. We have proved the lemma. $\square$

Corollary 6.29. *Let $\mathcal{X}$ be a formal DM stack of finite type over* Df. *Then there exists a unique closed substack $\mathcal{X}^{\mathrm{pur}} \subset \mathcal{X}$ such that for any flat finite-type morphism* $\mathrm{Spf}\, A \to \mathcal{X}$, *the scheme-theoretic preimage of $\mathcal{X}^{\mathrm{pur}}$ in* $\mathrm{Spf}\, A$ *is* $\mathrm{Spf}\, A^{\mathrm{pur}}$.

Proof. Let $\mathrm{Spf}\, B \to \mathcal{X}$ be an atlas. By Lemma 2.10,

$$\mathrm{Spf}\, B \times_{\mathcal{X}} \mathrm{Spf}\, B \left(\cong \varinjlim (\mathrm{Spf}\, B)_n \times_{\mathcal{X}_n} (\mathrm{Spf}\, B)_n\right)$$

is a formal affine scheme of finite type; we write this as $\mathrm{Spf}\, C$. The groupoid $\mathrm{Spf}\, C \rightrightarrows \mathrm{Spf}\, B$ induces the groupoid $\mathrm{Spf}\, C^{\mathrm{pur}} \rightrightarrows \mathrm{Spf}\, B^{\mathrm{pur}}$. Let $\mathcal{X}^{\mathrm{pur}}$ be the stack associated with the last groupoid, which is regarded as a closed substack of $\mathcal{X}$. We show that this satisfies the desired property. Let $\mathrm{Spf}\, A \to \mathcal{X}$ be a flat finite-type morphism. Let $\mathrm{Spf}\, R := \mathrm{Spf}\, A \times_{\mathcal{X}} \mathrm{Spf}\, B$. The preimage of $\mathcal{X}^{\mathrm{pur}}$ in $\mathrm{Spf}\, B$ is $\mathrm{Spf}\, B^{\mathrm{pur}}$ by construction. Since $B \to R$ is flat, by Lemma 6.27, the preimage of $\mathcal{X}^{\mathrm{pur}}$ in $\mathrm{Spf}\, R$ is $\mathrm{Spf}\, R^{\mathrm{pur}}$. Since $\mathrm{Spf}\, R \to \mathrm{Spf}\, A$ is étale and surjective, a closed subscheme of $\mathrm{Spf}\, A$ is uniquely determined by its preimage in $\mathrm{Spf}\, R$. The preimage of $\mathcal{X}^{\mathrm{pur}}$ in $\mathrm{Spf}\, A$ and the closed subscheme $\mathrm{Spf}\, A^{\mathrm{pur}} \subset \mathrm{Spf}\, A$ are both pulled back to $\mathrm{Spf}\, R^{\mathrm{pur}}$. Therefore, the two closed subschemes are the same. The uniqueness of $\mathcal{X}^{\mathrm{pur}}$ is clear. $\square$

Definition 6.30. We call $\mathcal{X}^{\mathrm{pur}}$ in the last corollary the *associated rig-pure stack* of $\mathcal{X}$.

Lemma 6.31. *Let Σ be a DM stack of finite type, and let $\mathcal{Z} \to \mathrm{Df}_\Sigma$ be a morphism of formal DM stacks of finite type over Df_Σ. Suppose that $\mathcal{Z}^{\mathrm{pur}}$ is flat over Df_Σ. Then, for any morphism of DM stacks $\Sigma' \to \Sigma$, we have $(\mathcal{Z}_{\Sigma'})^{\mathrm{pur}} = (\mathcal{Z}^{\mathrm{pur}})_{\Sigma'}$.*

Proof. This is a direct consequence of Lemma 6.28. □

Lemma 6.32. *Let Σ be a reduced DM stack of finite type, and let $\mathcal{X}$ be a rig-pure formal stack of finite type over Df_Σ. Then there exists an open dense substack $\Sigma^\circ \subset \Sigma$ such that the induced morphism $\mathcal{X} \times_\Sigma \Sigma^\circ \to \mathrm{Df}_{\Sigma^\circ}$ is flat.*

Proof. By the base change along an atlas $S \to \Sigma$, we can reduce the lemma to the case where Σ is a scheme. Restricting to an irreducible component of Σ, we may further assume that Σ is integral. Let $\eta \to \Sigma$ be its generic point. Since $\mathcal{X}$ is rig-pure over Df_Σ, the morphism $\mathcal{X} \times_\Sigma \eta \to \mathrm{Df}_\eta$ is flat. Let $U \subset |\mathcal{X}|$ be the flat locus of the morphism $\mathcal{X} \to \mathrm{Df}_\Sigma$, which is an open subset by [Abb10, Corollary 5.3.10]. The image of $|\mathcal{X}| \setminus U$ in $|\Sigma| = |\mathrm{Df}_\Sigma|$ is a constructible subset which does not contain the generic point. Removing the closure of this image, we obtain an open dense subscheme $\Sigma^\circ \subset \Sigma$ having the desired property. □

Corollary 6.33. *Let Σ be a DM stack of finite type, and let $\mathcal{Z}$ be a formal DM stack of finite type over Df_Σ. Then there exist finitely many locally closed reduced substacks $\Sigma_i \hookrightarrow \Sigma$, $i \in I$, such that $|\Sigma| = \bigsqcup_{i\in I} |\Sigma_i|$ and for every $i \in I$, the associated rig-pure stack $(\mathcal{Z}_{\Sigma_i})^{\mathrm{pur}}$ of $\mathcal{Z}_{\Sigma_i} := \mathcal{Z} \times_\Sigma \Sigma_i$ is flat over Df_{Σ_i}.*

Proof. This follows from Lemma 6.32 and the fact that $|\Sigma|$ is a Noetherian topological space. □

Definition 6.34. For a formal DM stack $\mathcal{X}$ of finite type over Df and for each $\Theta_{[G]}^{[\mathbf{r}]}$, we take locally closed reduced substacks $\Theta_{[G],i}^{[\mathbf{r}]} \subset \Theta_{[G]}^{[\mathbf{r}]}$ such that $\left|\Theta_{[G]}^{[\mathbf{r}]}\right| = \bigsqcup_i \left|\Theta_{[G],i}^{[\mathbf{r}]}\right|$ and $\mathrm{Utg}_{\Theta_{[G],i}^{[\mathbf{r}]}}(\mathcal{X})^{\mathrm{pur}}$ is flat over $\mathrm{Df}_{\Theta_{[G],i}^{[\mathbf{r}]}}$. We write $\Gamma_{\mathcal{X}} = \Gamma := \coprod_{[G],[\mathbf{r}],i} \Theta_{[G],i}^{[\mathbf{r}]}$. For each G, we write $\Gamma_{[G]} := \coprod_{[\mathbf{r}],i} \Theta_{[G],i}^{[\mathbf{r}]}$ and $\Gamma_G := \Gamma_{[G]} \times_{\mathcal{A}_{[G]}} \operatorname{Spec} k$.

Thus $\Gamma = \Gamma_{\mathcal{X}}$ is a DM stack of finite type given with a morphism $\Gamma \to \Theta$ such that the map $|\Gamma| \to |\Theta|$, as well as the maps $\Gamma[K] \to \Theta[K]$ for algebraically closed fields K, is bijective and $\mathrm{Utg}_\Gamma(\mathcal{X})^{\mathrm{pur}}$ is flat over Df_Γ.

For a tame cyclic group C_l, we have

$$\Gamma_{[C_l]} = \Theta_{[C_l]} = \Lambda_{[C_l]} \quad \text{and} \quad \Gamma_{C_l} = \Theta_{C_l} = \Lambda_{C_l}\,.$$

In particular, Γ contains Λ_{tame} as an open and closed substack.

Lemma 6.35. *Suppose that $\mathcal{X}$ is flat over Df and has pure relative dimension. Let $\Gamma = \Gamma_{\mathcal{X}}$ be as above. Then $\mathrm{Utg}_\Gamma(\mathcal{X})^{\mathrm{pur}}$ has pure relative dimension over Df_Γ equal to the relative dimension of $\mathcal{X}$ over Df.*

Proof. This follows from Lemma 6.22. □

Proposition 6.36. *Let $\mathcal{Y}$ be a formal DM stack of finite type over Df and X a formal algebraic space over Df. Let $f\colon \mathcal{Y} \to X$ be a Df-morphism, and let $\Gamma = \Gamma_{\mathcal{Y}}$ be as in Definition 6.34. For each locally finite-type morphism of DM stacks $\Sigma \to \Gamma$, there exists a Df-morphism*

$$f_\Sigma^{\mathrm{utg}}\colon \mathrm{Utg}_\Sigma(\mathcal{Y})^{\mathrm{pur}} \to X$$

satisfying the following properties:

(1) *If* $\operatorname{Spf} A \to \operatorname{Utg}_\Sigma(\mathcal{Y})^{\mathrm{pur}}$ *is a flat morphism corresponding to a representable morphism* $\mathcal{E}_\Sigma \times_{\mathrm{Df}_\Sigma} \operatorname{Spf} A \to \mathcal{Y}$*, then the morphism* $\operatorname{Spf} A \to \overline{\mathcal{Y}} \to X$ *deduced from this by taking coarse moduli spaces is the same as the composition* $\operatorname{Spf} A \to \operatorname{Utg}_\Sigma(\mathcal{Y})^{\mathrm{pur}} \to X$*. Here* $\overline{\mathcal{Y}}$ *is the coarse moduli space of* $\mathcal{Y}$*. (Note that since* $\operatorname{Spf} A \to \mathrm{Df}_\Sigma$ *is flat, by Lemma 2.13, the coarse moduli space of* $\mathcal{E}_\Sigma \times_{\mathrm{Df}_\Sigma} \operatorname{Spf} A$ *is* $\operatorname{Spf} A$*.)*

(2) *For a locally finite-type morphism* $\Sigma' \to \Sigma$ *of DM stacks, the composition*

$$\operatorname{Utg}_{\Sigma'}(\mathcal{Y})^{\mathrm{pur}} \to \operatorname{Utg}_\Sigma(\mathcal{Y})^{\mathrm{pur}} \xrightarrow{f_\Sigma^{\mathrm{utg}}} X$$

is the same as $f_{\Sigma'}^{\mathrm{utg}}$*.*

Proof. Let $W \to \Gamma$ be an atlas, let $\mathcal{U} := \operatorname{Utg}_\Gamma(\mathcal{Y})^{\mathrm{pur}}$ and $\mathcal{U}_W := \operatorname{Utg}_W(\mathcal{Y})^{\mathrm{pur}}$, and let $V_0 \to \mathcal{U}_W$ be an atlas from a formal scheme and $V_1 := V_0 \times_{\mathcal{U}} V_0$ so that we have the groupoid in formal schemes $V_1 \rightrightarrows V_0$. The morphisms $V_i \to \mathcal{U}$, $i = 0, 1$, correspond to representable Df-morphisms $\gamma_i\colon \mathcal{E}_\Gamma \times_{\mathrm{Df}_\Gamma} V_i = \mathcal{E}_W \times_{\mathrm{Df}_W} V_i \to \mathcal{Y}$. The coarse moduli space of $\mathcal{E}_W$ is Df_W. Since $V_i \to \mathrm{Df}_W$ is flat, by Lemma 2.13, the moduli space of $\mathcal{E}_\Gamma \times_{\mathrm{Df}_\Gamma} V_i$ is V_i. Therefore, the morphisms $f \circ \gamma_i\colon \mathcal{E}_\Gamma \times_{\mathrm{Df}_\Gamma} V_i \to X$ induce morphisms $\beta_i\colon V_i \to X$. The morphisms β_i define a morphism $(V_1 \rightrightarrows V_0) \to (X \rightrightarrows X)$ of groupoids. Finally, this induces a morphism of the associated stacks,

$$\mathcal{U} = [V_1/V_0] \to X = [X/X]\,.$$

For any morphism $\Sigma \to \Gamma$, by base change, the groupoid $V_1 \rightrightarrows V_0$ induces a groupoid $V_{1,\Sigma} \rightrightarrows V_{0,\Sigma}$ whose associated stack is $\operatorname{Utg}_\Sigma(\mathcal{Y})^{\mathrm{pur}}$. We define $\operatorname{Utg}_\Sigma(f)$ to be the stack associated with the morphism of groupoids $(V_{1,\Sigma} \rightrightarrows V_{0,\Sigma}) \to (X \rightrightarrows X)$. We easily see that these morphisms satisfy the desired properties. □

Remark 6.37. The reason why we need a flattening stratification as in Corollary 6.33 and Definition 6.34 is that the coarse moduli space does not commute with non-flat base change in the case of wild DM stacks. This prevents us from having a natural morphism $\operatorname{Utg}_\Theta(\mathcal{Y}) \to \operatorname{Utg}_\Theta(X)$. For instance, a $\mathrm{D}_{R,n}$-point of $\operatorname{Utg}_\Theta(\mathcal{Y})$ over $\mathrm{D}_{R,n} \hookrightarrow \mathrm{Df}_R$, a typical non-flat morphism, corresponds to a representable morphism $\mathcal{E}_n = [E_n/G] \to \mathcal{Y}$ for some G-covering $E \to \mathrm{Df}_R$. But the G-action on E_n may be trivial, and the coarse moduli space of $\mathcal{E}_n$ may be E_n rather than $\mathrm{D}_{R,n}$. Thus we would get a morphism $E_n \to X$ rather than a desired morphism $\mathrm{D}_{R,n} \to X$.

Remark 6.38. For a morphism $\mathcal{Y} \to \mathcal{X}$ of formal DM stacks of finite type over Df and for a suitable choice of $\Gamma_{\mathcal{Y}}$ and $\Gamma_{\mathcal{X}}$, we would like to have a morphism

$$\operatorname{Utg}_{\Gamma_{\mathcal{Y}}}(\mathcal{Y}) \to \operatorname{Utg}_{\Gamma_{\mathcal{X}}}(\mathcal{X})\,.$$

However, obtaining this was technically difficult, and we are content with restricting ourselves to the case where $\mathcal{X}$ is a formal algebraic space.

7. Stacks of jets and arcs

In this section, we first define stacks of untwisted jets and arcs on formal DM stacks. Next we define stacks of twisted jets and arcs, using untwisting stacks and stacks of untwisted jets and arcs.

Definition 7.1. Let Φ be a DM stack almost of finite type and $\mathcal{X}$ a formal DM stack of finite type over Df_Φ. For $n \in \mathbb{N}$, we define the *stack of (untwisted) n-jets on* $\mathcal{X}$, denoted by $\mathrm{J}_{\Phi,n}\,\mathcal{X}$ or $\mathrm{J}_n\,\mathcal{X}$, to be the Weil restriction $\mathrm{R}_{\mathrm{D}_{\Phi,n}/\Phi}(\mathcal{X}_n)$. Namely, $\mathrm{J}_n\,\mathcal{X}$ is a fibered category over Φ such that the fiber category over an S-point $s\colon S \to \Phi$ is

$$(\mathrm{J}_n\,\mathcal{X})(s) = \mathrm{Hom}_{\mathrm{D}_{S,n}}(\mathrm{D}_{S,n}, \mathcal{X}_n \times_\Phi S) = \mathrm{Hom}_{\mathrm{Df}_\Phi}(\mathrm{D}_{S,n}, \mathcal{X})\,.$$

LEMMA 7.2. *The stack $\mathrm{J}_n\,\mathcal{X}$ is a DM stack almost of finite type, and the morphism $\mathrm{J}_n\,\mathcal{X} \to \Phi$ is of finite type.*

Proof. We claim

$$\mathrm{J}_n\,\mathcal{X} \cong \underline{\mathrm{Hom}}_\Phi(\mathrm{D}_{\Phi,n}, \mathcal{X}_n) \times_{\underline{\mathrm{Hom}}_\Phi(\mathrm{D}_{\Phi,n},\mathrm{D}_{\Phi,n})} \Phi\,,$$

where $\Phi \to \underline{\mathrm{Hom}}_\Phi(\mathrm{D}_{\Phi,n}, \mathrm{D}_{\Phi,n})$ is the morphism induced by the identity morphism $\mathrm{Df}_{\Phi,n} \to \mathrm{Df}_{\Phi,n}$. For an object $s \in \mathrm{D}_{\Phi,n}(S)$, if $\{s\}$ denotes the category consisting of s and its identity morphism, then

$$\begin{aligned}(\underline{\mathrm{Hom}}_\Phi(\mathrm{D}_{\Phi,n}, \mathcal{X}_n) \times_{\underline{\mathrm{Hom}}_\Phi(\mathrm{D}_{\Phi,n},\mathrm{D}_{\Phi,n})} \Phi)(s) &\cong \mathrm{Hom}_S(\mathrm{D}_{S,n}, \mathcal{X}_n \times_\Phi S) \times_{\mathrm{Hom}_S(\mathrm{D}_{S,n},\mathrm{D}_{S,n})} \{s\} \\ &\cong \mathrm{Hom}_{\mathrm{D}_{S,n}}(\mathrm{D}_{S,n}, \mathcal{X}_n \times_\Phi S) \\ &\cong (\mathrm{J}_n\,\mathcal{X})(s)\,.\end{aligned}$$

Thus the claim holds. Applying Lemma 6.5 to each connected component of Φ, we get the lemma. ☐

For $n = 0$, we have $\mathrm{J}_{\Phi,0}\,\mathcal{X} = \mathcal{X}_0$. For $n \geqslant m$, the closed immersion $\mathrm{D}_m \hookrightarrow \mathrm{D}_n$ induces a morphism $\pi^n_m\colon\ \mathrm{J}_n\,\mathcal{X} \to \mathrm{J}_m\,\mathcal{X}$.

LEMMA 7.3. *The morphism π^n_m is representable and affine.*

Proof. For an atlas $U \to \Phi$, we have $(\mathrm{J}_{\Phi,n}\,\mathcal{X}) \times_\Phi U \cong \mathrm{J}_{U,n}(\mathcal{X} \times_\Phi U)$. Therefore, we may suppose that $\Phi = U$ is a scheme. For an atlas $V \to \mathcal{X}$, we have $(\mathrm{J}_{U,n}\,\mathcal{X}) \times_{\mathcal{X}} V \cong \mathrm{J}_{U,n}\,V$ as in the case of étale morphisms of schemes. Indeed, for the 2-commutative diagram of solid arrows

$$\begin{tikzcd} \operatorname{Spec} R \arrow[r] \arrow[d, hook] & V \arrow[d] \\ \operatorname{Spec} R[\![t]\!]/(t^{n+1}) \arrow[r] \arrow[ur, dashed] & \mathcal{X}\,, \end{tikzcd}$$

there exists a unique dashed arrow fitting into the diagram. This shows the last isomorphism. Thus we may also suppose that $\mathcal{X} = V$ is a formal scheme. The problem is now reduced to the case of formal schemes. The lemma is well known in this case (see for instance [CNS18, § 3.3.4, p. 249]). ☐

From this lemma, we get a projective system $(\mathrm{J}_n\,\mathcal{X}, \pi^n_m)$ of DM stacks almost of finite type having affine morphisms as transition morphisms. Therefore, the projective limit

$$\mathrm{J}_{\Phi,\infty}\,\mathcal{X} = \mathrm{J}_\infty\,\mathcal{X} := \varprojlim \mathrm{J}_n\,\mathcal{X}$$

exists and is a DM stack.

DEFINITION 7.4. We call $\mathrm{J}_\infty\,\mathcal{X}$ the *stack of* (*untwisted*) *arcs* on $\mathcal{X}$.

A point of $|\mathrm{J}_\infty\mathcal{X}|$ corresponds to an equivalence class of pairs of a geometric point $\operatorname{Spec} K \to \Phi$ and a lift $\mathrm{Df}_K \to \mathcal{X}$ of the morphism $\mathrm{Df}_K \to \mathrm{Df}_\Phi$.

Stacks of untwisted jets are functorial in $\mathcal{X}$. Let Ψ and Φ be DM stacks almost of finite type, and let $\mathcal{Y}$ and $\mathcal{X}$ be formal DM stacks of finite type over Df_Ψ and Df_Φ, respectively. Let $\Psi \to \Phi$

be a morphism, and let $f\colon \mathcal{Y} \to \mathcal{X}$ be a morphism compatible with the induced morphism $\mathrm{Df}_\Psi \to \mathrm{Df}_\Phi$. Then we obtain a morphism

$$f_n\colon\ \mathrm{J}_{\Psi,n}\,\mathcal{Y} \to \mathrm{J}_{\Phi,n}\,\mathcal{X}$$

compatible with $\Psi \to \Phi$. This sends an S-point of $\mathrm{J}_{\Psi,n}\,\mathcal{Y}$ given by a pair $(S \to \Psi, \mathrm{D}_{S,n} \to \mathcal{Y})$ to the induced pair $(S \to \Psi \to \Phi, \mathrm{D}_{S,n} \to \mathcal{Y} \to \mathcal{X})$.

DEFINITION 7.5. A *twisted arc of* $\mathcal{X}$ is a representable morphism $\mathcal{E} \to \mathcal{X}$, where $\mathcal{E}$ is a twisted formal disk over some algebraically closed field. We say that two twisted arcs $\mathcal{E} \to \mathcal{X}$ and $\mathcal{E}' \to \mathcal{X}$, say over K and K', are *equivalent* if there exist morphisms $\operatorname{Spec} K'' \to \operatorname{Spec} K$ and $\operatorname{Spec} K'' \to \operatorname{Spec} K'$ with K'' an algebraically closed field and an isomorphism $\mathcal{E}_{K''} \to \mathcal{E}'_{K''}$ such that

$$\begin{tikzcd} \mathcal{E}_{K''} \arrow[r] \arrow[dr] & \mathcal{E}'_{K''} \arrow[d] \\ & \mathcal{X} \end{tikzcd}$$

is 2-commutative.

DEFINITION 7.6. Let $\mathcal{X}$ be a formal DM stack of finite type over Df, and let $\Gamma = \Gamma_{\mathcal{X}}$ be as in Definition 6.34. For $n \in \mathbb{N}$, we define the *stack of twisted* n*-jets* on $\mathcal{X}$, denoted by $\mathcal{J}_{\Gamma,n}\mathcal{X}$ or $\mathcal{J}_n\mathcal{X}$, to be $\mathrm{J}_{\Gamma,n}\operatorname{Utg}_\Gamma(\mathcal{X})^{\mathrm{pur}}$. We define the *stack of twisted arcs* on $\mathcal{X}$, denoted by $\mathcal{J}_{\Gamma,\infty}\mathcal{X}$ or $\mathcal{J}_\infty\mathcal{X}$, to be $\mathrm{J}_{\Gamma,\infty}\operatorname{Utg}_\Gamma(\mathcal{X})^{\mathrm{pur}}$.

Let K be an algebraically closed field, let $E \to \mathrm{Df}_K$ a G-cover for some Galoisian group G corresponding to a K-point $e\colon \operatorname{Spec} K \to \Gamma$, and let $\mathcal{E} = [E/G]$ be the induced twisted formal disk. We have

$$\begin{aligned}
(\mathcal{J}_\infty\mathcal{X})(e) &= \varprojlim \operatorname{Hom}_{\mathrm{Df}}(\mathrm{D}_{K,n}, \operatorname{Utg}_{\operatorname{Spec} K}(\mathcal{X})^{\mathrm{pur}}) \\
&= \operatorname{Hom}_{\mathrm{Df}}(\mathrm{Df}_K, \operatorname{Utg}_{\operatorname{Spec} K}(\mathcal{X})^{\mathrm{pur}}) \\
&= \operatorname{Hom}_{\mathrm{Df}}(\mathrm{Df}_K, \operatorname{Utg}_{\operatorname{Spec} K}(\mathcal{X})) \\
&= \varprojlim \operatorname{Hom}_{\mathrm{Df}}(\mathrm{D}_{K,n}, \operatorname{Utg}_{\operatorname{Spec} K}(\mathcal{X})) \\
&= \varprojlim \operatorname{Hom}^{\mathrm{rep}}_{\mathrm{Df}}(\mathcal{E}_n, \mathcal{X}) \\
&= \operatorname{Hom}^{\mathrm{rep}}_{\mathrm{Df}}(\mathcal{E}, \mathcal{X})\,.
\end{aligned}$$

It follows that the point set $|\mathcal{J}_\infty\mathcal{X}|$ is in a one-to-one correspondence with the set of equivalence classes of twisted arcs of $\mathcal{X}$. This also shows that $|\mathcal{J}_\infty\mathcal{X}|$ is independent of the choice of Γ.

Over the component $\operatorname{Spec} k \cong \Lambda_{[1]} \subset \Gamma$ corresponding to the trivial Galoisian group $G = 1$, we have

$$\operatorname{Utg}_{\Lambda_{[1]}}(\mathcal{X}) = \underline{\operatorname{Hom}}^{\mathrm{rep}}_{\mathrm{Df}}(\mathrm{Df}, \mathcal{X}) = \mathcal{X}\,.$$

Therefore, the stack of untwisted arcs $\mathrm{J}_\infty\,\mathcal{X}$ is regarded as an open and closed substack of $\mathcal{J}_\infty\mathcal{X}$.

If $f\colon \mathcal{Y} \to X$ is a morphism from a formal DM stack $\mathcal{Y}$ to a formal algebraic space X which are both of finite type over Df, then we have the map

$$f_\infty\colon |\mathcal{J}_\infty\mathcal{Y}| \to |\mathrm{J}_\infty X|$$

which sends the class of a twisted arc $\mathcal{E} \to \mathcal{Y}$ to the class of the arc $\mathrm{Df}_K \to X$ obtained from the composite $\mathcal{E} \to \mathcal{Y} \to X$ and the universality of the coarse moduli space $\mathcal{E} \to \mathrm{Df}_K$. This map is nothing but the map

$$(f^{\mathrm{utg}})_\infty\colon |\mathrm{J}_\infty \operatorname{Utg}_\Gamma(\mathcal{Y})| \to |\mathrm{J}_\infty X|\,,$$

which is associated with the morphism $f^{\mathrm{utg}}\colon \operatorname{Utg}_\Gamma(\mathcal{Y}) \to X$ obtained in Proposition 6.36.

8. Jacobian order functions

In this section, we define Jacobian order functions, denoted by $\mathfrak{j}$, of formal DM stacks and of morphisms between them.

Let Φ be a DM stack of finite type, and let $\mathcal{X}$ be a formal DM stack flat and of finite type over Df_Φ.

Definition 8.1. For a field K, let

$$\mathrm{ord}\colon K(\!(t)\!) \to \mathbb{Z} \cup \{\infty\}$$

be the order function (the normalized additive valuation) with convention $\mathrm{ord}\, 0 = \infty$. For a finite extension $L/K(\!(t)\!)$ of degree r, we continue to denote by ord the unique extension of ord to L, which takes values in $\frac{1}{r}\mathbb{Z} \cup \{\infty\}$. Let O_L be the integral closure of $K[\![t]\!]$ in L. For an ideal I of O_L, say generated by a, we define $\mathrm{ord}\, I := \mathrm{ord}\, a$.

Definition 8.2. Let $\mathcal{I}$ be a coherent ideal sheaf on (the small étale site of) $\mathcal{X}$. For a finite extension $L/K(\!(t)\!)$ and a Df-morphism $\beta\colon \operatorname{Spf} O_L \to \mathcal{X}$, the pullback $\beta^{-1}\mathcal{I}$ is an ideal of O_L, and we define

$$\mathrm{ord}_{\mathcal{I}}(\beta) := \mathrm{ord}\, \beta^{-1}\mathcal{I} \in \frac{1}{r}\mathbb{Z} \cup \{\infty\}\,.$$

For a twisted arc $\gamma\colon \mathcal{E} \to \mathcal{X}$ with $\mathcal{E}$ defined over K, let us take a finite Df_K-morphism $\operatorname{Spf} O_L \to \mathcal{E}$ for some finite extension $L/K(\!(t)\!)$, and let $\beta\colon \operatorname{Spf} O_L \to \mathcal{E} \to \mathcal{X}$ be the composition. We define

$$\mathrm{ord}_{\mathcal{I}}(\gamma) := \mathrm{ord}_{\mathcal{I}}(\beta)\,.$$

The last definition is independent of the choice of $\operatorname{Spf} O_L \to \mathcal{E}$. This follows from the fact that if $\beta\colon \operatorname{Spf} O_L \to \mathcal{X}$ is a morphism as above and $\beta'\colon \operatorname{Spf} O_{L'} \to \operatorname{Spf} O_L \to \mathcal{X}$ is the one deduced from β and a finite extension L'/L, then $\mathrm{ord}_{\mathcal{I}}(\beta) = \mathrm{ord}_{\mathcal{I}}(\beta')$.

We use sheaves of differentials. This well-known notion for schemes is generalized to formal schemes (see for instance [TLR07, CNS18, Abb10]). We generalize it further to formal DM stacks.

Definition 8.3. We define the *sheaf of differentials* of $\mathcal{X}$ over Df_Φ, denoted by $\Omega_{\mathcal{X}/\mathrm{Df}_\Phi}$, as follows. For each atlas $u\colon U \to \mathcal{X}$ from a formal scheme which fits into a 2-commutative diagram

$$\begin{tikzcd} U \arrow[r] \arrow[d] & \mathcal{X} \arrow[d] \\ \mathrm{Df}_T \arrow[r] & \mathrm{Df}_\Phi \end{tikzcd}$$

for some atlas $T \to \Phi$, the sheaf Ω_{U/Df_T} is independent of the choice of $T \to \Phi$; we denote it by $\Omega_{U/\mathrm{Df}_\Phi}$. For such an atlas $U \to \mathcal{X}$, the morphism $U \times_{\mathcal{X}} U \to \mathcal{X}$ is also an atlas satisfying the same condition. For projections $p_1, p_2\colon U \times_{\mathcal{X}} U \rightrightarrows U$, we have canonical isomorphisms

$$p_1^*\Omega_{U/\mathrm{Df}_\Phi} \cong \Omega_{U\times_{\mathcal{X}} U/\mathrm{Df}_\Phi} \cong p_2^*\Omega_{U/\mathrm{Df}_\Phi}\,.$$

We define $\Omega_{\mathcal{X}/\mathrm{Df}_\Phi}$ to be the coherent $\mathcal{O}_{\mathcal{X}}$-module associated with these data.

Definition 8.4. Suppose that $\mathcal{X}$ has pure relative dimension d over Df_Φ. We define the *Jacobian ideal sheaf* $\mathrm{Jac}_{\mathcal{X}/\mathrm{Df}_\Phi} \subset \mathcal{O}_{\mathcal{X}}$ to be the dth Fitting ideal sheaf of $\Omega_{\mathcal{X}/\mathrm{Df}_\Phi}$. We denote the associated order function $\mathrm{ord}_{\mathrm{Jac}_{\mathcal{X}/\mathrm{Df}_\Phi}}$ on $|\mathcal{J}_\infty\mathcal{X}|$ by $\mathfrak{j}_{\mathcal{X}} = \mathfrak{j}_{\mathcal{X}/\Phi}$. The *singular locus* of $\mathcal{X}$ over Φ is the closed substack defined by $\mathrm{Jac}_{\mathcal{X}/\mathrm{Df}_\Phi}$; it is denoted by $\mathcal{X}_{\mathrm{sing}}$.

Definition 8.5. We say that for a formal DM stack $\mathcal{X}$ of finite type and flat over Df_K with K a field, a closed substack $\mathcal{Y} \subset \mathcal{X}$ is *narrow* if for every geometric point $y\colon \operatorname{Spec} L \to \mathcal{Y}$, we have

$\dim \mathcal{O}_{\mathcal{Y},y} < \dim \mathcal{O}_{\mathcal{X},y}$. When Φ is a DM stack almost of finite type and $\mathcal{X}$ is a formal DM stack of finite type and flat over Df_Φ, we say that a closed substack $\mathcal{Y} \subset \mathcal{X}$ is *narrow* if for every geometric point $\operatorname{Spec} K \to \Phi$, the fiber product $\mathcal{Y} \times_\Phi \operatorname{Spec} K$ is a narrow closed substack of $\mathcal{X} \times_\Phi \operatorname{Spec} K$.

Definition 8.6. We say that $\mathcal{X}$ is *fiberwise generically smooth over* Df_Φ if $\mathcal{X}_{\mathrm{sing}} \subset \mathcal{X}$ is narrow.

Remark 8.7. When $\Phi = \operatorname{Spec} k$ and $\mathcal{X}$ is a formal affine scheme $\operatorname{Spf} A$, the formal scheme $\mathcal{X}$ is said to be *rig-smooth* [Abb10, Definition 1.14.7] if we have the inclusion $V(\mathrm{Jac}_{\mathcal{X}/\mathrm{Df}}) \subset V(t)$ of closed subsets of $\operatorname{Spec} A$, where we regard $\mathrm{Jac}_{\mathcal{X}/\mathrm{Df}}$ as an ideal of A. When $\mathcal{X}$ is rig-pure, "rig-smooth" implies "generically smooth," but the converse is not true.

If $\mathcal{X}$ is fiberwise generically smooth and has pure relative dimension d over Df_Φ, then $\Omega_{\mathcal{X}/\mathrm{Df}_\Phi}$ has generic rank d and $\Omega^d_{\mathcal{X}/\mathrm{Df}_\Phi} := \bigwedge^d \Omega_{\mathcal{X}/\mathrm{Df}_\Phi}$ has generic rank 1. (By this, we mean that for every étale morphism $u\colon \operatorname{Spf} A \to \mathcal{X}$, the associated A-module has the indicated generic rank.)

Definition 8.8. Let $\Psi \to \Phi$ be a morphism of DM stacks almost of finite type. Let $\mathcal{Y}$ and $\mathcal{X}$ be formal DM stacks of finite type, flat and fiberwise generically smooth over Df_Ψ and Df_Φ, respectively, both of which have pure relative dimension d. Let $f\colon \mathcal{Y} \to \mathcal{X}$ be a morphism compatible with $\mathrm{Df}_\Psi \to \mathrm{Df}_\Phi$. Let $\gamma\colon \mathcal{E} = [E/G] \to \mathcal{Y}$ be a twisted arc, let $\beta\colon E \to \mathcal{Y}$ be the induced morphism, and let O_E be the coordinate ring of E. Suppose that we have $\mathfrak{j}_{\mathcal{Y}/\Psi}(\beta) < \infty$ and $\mathfrak{j}_{\mathcal{X}/\Phi}(f \circ \beta) < \infty$. Let

$$\beta^\flat \Omega^d_{\mathcal{Y}/\mathrm{Df}_\Psi} := \big(\beta^* \Omega^d_{\mathcal{Y}/\mathrm{Df}_\Psi}\big)/\mathrm{tors}\,,$$
$$(f \circ \beta)^\flat \Omega^d_{\mathcal{X}/\mathrm{Df}_\Phi} := \big((f \circ \beta)^* \Omega^d_{\mathcal{X}/\mathrm{Df}_\Phi}\big)/\mathrm{tors}$$

be flat pullbacks, which are free O_E-modules of rank 1. We define

$$\mathfrak{j}_f(\gamma) = \mathfrak{j}_f(\beta) := \frac{1}{\sharp G} \operatorname{length} \frac{\beta^\flat \Omega^d_{\mathcal{Y}/\mathrm{Df}_\Psi}}{\operatorname{Im}\big((f \circ \beta)^\flat \Omega^d_{\mathcal{X}/\mathrm{Df}_\Phi} \to \beta^\flat \Omega^d_{\mathcal{Y}/\mathrm{Df}_\Psi}\big)} \in \mathbb{N} \cup \{\infty\}\,.$$

When either $\mathfrak{j}_{\mathcal{Y}/\Psi}(\beta) = \infty$ or $\mathfrak{j}_{\mathcal{X}/\Phi}(f \circ \beta) = \infty$, we define $\mathfrak{j}_f(\gamma) = \mathfrak{j}_f(\beta) := \infty$.

Similarly, we define

$$\beta^\flat \Omega_{\mathcal{Y}/\mathrm{Df}_\Psi} := \big(\beta^* \Omega_{\mathcal{Y}/\mathrm{Df}_\Psi}\big)/\mathrm{tors}\,,$$
$$(f \circ \beta)^\flat \Omega_{\mathcal{X}/\mathrm{Df}_\Phi} := \big((f \circ \beta)^* \Omega_{\mathcal{X}/\mathrm{Df}_\Phi}\big)/\mathrm{tors}\,,$$

which are free O_E-modules of rank d. For suitable bases, the map $(f \circ \beta)^\flat \Omega_{\mathcal{X}/\mathrm{Df}_\Phi} \to \beta^\flat \Omega_{\mathcal{Y}/\mathrm{Df}_\Psi}$ is represented by a diagonal matrix $\operatorname{diag}(\pi^{a_1}, \dots, \pi^{a_d})$, where π is a uniformizer of O_E and we have $\sharp G \cdot \mathfrak{j}_f(\gamma) = \sum_{i=1}^d a_i$.

Definition 8.9. We define the *Jacobian ideal sheaf* $\mathrm{Jac}_f \subset \mathcal{O}_\mathcal{Y}$ of f to be the 0th Fitting ideal of $\Omega_{\mathcal{Y}/(\mathcal{X} \times_\Phi \Psi)}$.

Lemma 8.10. *We have* $\mathfrak{j}_f \leqslant \mathrm{ord}_{\mathrm{Jac}_f}$. *Equality holds if* $\mathcal{Y}$ *is smooth over* Df_Ψ.

Proof. We first recall two properties of Fitting ideals. Firstly, Fitting ideals commute with base change [Eis95, Corollary 20.5]. Secondly, for a discrete valuation ring R with maximal ideal (π) and for an R-module $M = R^{\oplus r} \oplus \bigoplus_{i=1}^l R/(\pi^{a_i})$ of rank r, we see

$$\mathrm{Fitt}_r(M) = \big(\pi^{\sum_{i=1}^l a_i}\big)$$

by a direct computation. From these properties, we have

$$\sharp G \cdot \mathrm{ord}_{\mathrm{Jac}_f}(\beta) = \mathrm{length}\, \beta^* \Omega_{\mathcal{Y}/(\mathcal{X}\times_\Phi \Psi)}\,. \tag{8.1}$$

Consider the exact sequence

$$(f\circ\beta)^*\Omega_{\mathcal{X}/\mathrm{Df}_\Phi} \xrightarrow{\alpha} \beta^*\Omega_{\mathcal{Y}/\mathrm{Df}_\Psi} \to \beta^*\Omega_{\mathcal{Y}/(\mathcal{X}\times_\Phi\Psi)} \to 0\,.$$

The map α induces $(f\circ\beta)^\flat\Omega_{\mathcal{X}/\mathrm{Df}_\Phi} \to \beta^\flat\Omega_{\mathcal{Y}/\mathrm{Df}_\Psi}$ by taking quotients by the torsion parts. If this is represented by a diagonal matrix $\mathrm{diag}(\pi^{a_1},\dots,\pi^{a_d})$, then since its cokernel is isomorphic to $\bigoplus_{i=1}^d O_E/(\pi^{a_i})$, we have

$$\sharp G\cdot \mathfrak{j}_f(\beta) = \mathrm{length}\, \frac{\beta^\flat\Omega_{\mathcal{Y}/\mathrm{Df}_\Psi}}{\mathrm{Im}((f\circ\beta)^\flat\Omega_{\mathcal{X}/\mathrm{Df}_\Phi}\to\beta^\flat\Omega_{\mathcal{Y}/\mathrm{Df}_\Psi})}\,. \tag{8.2}$$

The last quotient module is a quotient of $\mathrm{Coker}(\alpha)\cong\gamma^*\Omega_{\mathcal{Y}/\mathcal{X}\times_\Phi\Psi}$. The first assertion follows from (8.1) and (8.2). If $\mathcal{Y}\to\mathrm{Df}_\Psi$ is smooth, then $\gamma^\flat\Omega_{\mathcal{Y}/\mathrm{Df}_\Psi} = \gamma^*\Omega_{\mathcal{Y}/\mathrm{Df}_\Psi}$ and

$$\frac{\gamma^\flat\Omega_{\mathcal{Y}/\mathrm{Df}_\Psi}}{\mathrm{Im}((f\circ\beta)^\flat\Omega_{\mathcal{X}/\mathrm{Df}_\Phi}\to\beta^\flat\Omega_{\mathcal{Y}/\mathrm{Df}_\Psi})} = \gamma^*\Omega_{\mathcal{Y}/\mathcal{X}\times_\Phi\Psi}\,,$$

which shows the second assertion. □

Lemma 8.11. *Let $f\colon\mathcal{Y}\to\mathcal{X}$ be as above, and let $g\colon\mathcal{Z}\to\mathcal{Y}$ be another morphism satisfying the same condition. Let $\beta\colon\mathrm{Spf}\,O_L\to\mathcal{Z}$ be a* Df*-morphism for $L/K(\!(t)\!)$ as before. Let $\beta_{\mathcal{Y}} = g\circ\beta\colon\mathrm{Spf}\,O_L\to\mathcal{Y}$ and $\beta_{\mathcal{X}} = f\circ g\circ\gamma\colon\mathrm{Spf}\,O_L\to\mathcal{X}$ be the induced morphisms. Suppose that $\mathfrak{j}_{\mathcal{Z}}(\beta)$, $\mathfrak{j}_{\mathcal{Y}}(\beta_{\mathcal{Y}})$ and $\mathfrak{j}_{\mathcal{X}}(\beta_{\mathcal{X}})$ have finite values. Then*

$$\mathfrak{j}_{f\circ g}(\beta) = \mathfrak{j}_g(\beta)+\mathfrak{j}_f(\beta_{\mathcal{Y}})\,.$$

Proof. Let $r:=[L:K(\!(t)\!)]$, and let $\mathfrak{m}\subset O_L$ be the maximal ideal. Suppose that $\mathcal{Z}$ is defined over Df_Υ. We have

$$\begin{aligned}(\beta_{\mathcal{Y}})^\flat\Omega^d_{\mathcal{Y}/\mathrm{Df}_\Psi} &= \mathfrak{m}^{\mathfrak{j}_g(\beta)r}\beta^\flat\Omega^d_{\mathcal{Z}/\mathrm{Df}_\Upsilon}\,,\\ (\beta_{\mathcal{X}})^\flat\Omega^d_{\mathcal{X}/\mathrm{Df}_\Phi} &= \mathfrak{m}^{\mathfrak{j}_{f\circ g}(\beta)r}\beta^\flat\Omega^d_{\mathcal{Z}/\mathrm{Df}_\Upsilon}\,,\\ (\beta_{\mathcal{X}})^\flat\Omega^d_{\mathcal{X}/\mathrm{Df}_\Phi} &= \mathfrak{m}^{\mathfrak{j}_f(\beta_{\mathcal{Y}})r}(\beta_{\mathcal{Y}})^\flat\Omega^d_{\mathcal{Y}/\mathrm{Df}_\Psi}\,.\end{aligned}$$

The first and third equalities imply $(\beta_{\mathcal{X}})^\flat\Omega^d_{\mathcal{X}/\mathrm{Df}_\Phi} = \mathfrak{m}^{\mathfrak{j}_f(\beta_{\mathcal{Y}})r+\mathfrak{j}_g(\beta)r}\beta^\flat\Omega^d_{\mathcal{Z}/\mathrm{Df}_\Upsilon}$. Comparing this with the second, we get the lemma. □

9. Grothendieck rings and realizations

In this section, we define several rings constructed from the Grothendieck ring of varieties. We will define motivic measures to take values in these rings in later sections.

Definition 9.1. For an algebraic space S of finite type over k, we denote by $\mathbf{Var}_S$ the category of algebraic spaces of finite type over S. We define $K_0(\mathbf{Var}_S)$ to be the quotient of the free $\mathbb{Z}$-module generated by isomorphism classes $\{X\}$ of $X\in\mathbf{Var}_S$ modulo the scissor relation: if $Y\subset X$ is a closed subspace, then $\{X\}=\{X\setminus Y\}+\{X\}$.

Remark 9.2. We do not use the usual notation $[X]$ to avoid the confusion that the symbol $[X/G]$ would have two meanings, a quotient stack and the class of a quotient space X/G in the Grothendieck ring.

Remark 9.3. The usual definition of the Grothendieck ring of varieties uses schemes rather than algebraic spaces. But every algebraic space X of finite type admits a stratification $X = \bigsqcup_i X_i$ into finitely many locally closed subspaces X_i which are schemes. Therefore, the use of algebraic spaces causes no essential difference. But algebraic spaces are more natural for our purpose because they appear as coarse moduli spaces of DM stacks.

Remark 9.4. One may consider the Grothendieck *semiring* instead of the Grothendieck ring so that one can obtain slightly finer invariants (see for instance [Yas06, § 3.1, p. 725], [CNS18, Chapter 2, Definition 1.2.1]).

The scissor relation in particular implies that $\{X\} = \{X_{\mathrm{red}}\}$. For a constructible subset C of $X \in \mathbf{Var}_S$, if $C = \bigsqcup_{i=1}^n C_i$ is a decomposition into finitely many locally closed subsets, then we define $\{C\} := \sum_{i=1}^n \{C_i\} \in K_0(\mathbf{Var}_S)$, which is independent of the decomposition.

The additive group $K_0(\mathbf{Var}_S)$ becomes a commutative ring by the multiplication $\{X\}\{Y\} := \{X \times_S Y\}$. We denote $\{\mathbb{A}^1_S\} \in K_0(\mathbf{Var}_S)$ by $\mathbb{L}$. For a positive integer r, we define a ring $K_0(\mathbf{Var}_S)_r$ by adjoining an rth root $\mathbb{L}^{1/r}$ of $\mathbb{L}$ formally. Namely, $K_0(\mathbf{Var}_S)_r = K_0(\mathbf{Var}_S)[x]/(x^r - \mathbb{L})$, and we denote the class of x by $\mathbb{L}^{1/r}$. We define $\mathcal{M}_S$ and $\mathcal{M}_{S,r}$ to be the localization of $K_0(\mathbf{Var}_S)$ and $K_0(\mathbf{Var}_S)_r$ by $\mathbb{L}$, respectively.

Definition 9.5. We say that a morphism $f\colon Y \to X$ between algebraic spaces of finite type is a *pseudo-$\mathbb{A}^r$-bundle* if for every geometric point $x\colon \operatorname{Spec} K \to X$, its fiber is universally homeomorphic over K to $\mathbb{A}^r_K/H$ for some finite group action $H \curvearrowright \mathbb{A}^r_K$. In particular, a universal bijection $Y \to X$ between algebraic spaces of finite type is a pseudo-$\mathbb{A}^0$-bundle. More generally, let $f\colon \mathcal{Y} \to \mathcal{X}$ be a morphism between DM stacks of finite type, and let $D \subset |\mathcal{Y}|$ and $C \subset |\mathcal{X}|$ be constructible subsets with $f(D) \subset C$. We say that the map $f|_D\colon D \to C$ is a *pseudo-$\mathbb{A}^r$-bundle* if there exist stratification $D = \bigsqcup_{i=1}^n D_i$ and $C = \bigsqcup_{i=1}^n C_i$ into locally closed subsets such that for every i, we have $f(D_i) \subset C_i$ and if $\mathcal{D}_i \subset \mathcal{Y}$ and $\mathcal{C}_i \subset \mathcal{X}$ are the associated reduced locally closed substacks, then the induced morphisms of coarse moduli spaces, $\overline{\mathcal{D}_i} \to \overline{\mathcal{C}_i}$, are pseudo-$\mathbb{A}^r$-bundles.

Definition 9.6. We define the quotient group $K_0(\mathbf{Var}_S)'_r$ by the following relation: for a pseudo-$\mathbb{A}^r$-bundle $Y \to X$ of algebraic spaces of finite type over S, we have $\{Y\} = \{X\}\mathbb{L}^r$.

In particular, for a universally bijective morphism $Y \to X$ in $\mathbf{Var}_S$, we have $\{Y\} = \{X\}$ in $K_0(\mathbf{Var}_S)'_r$. If $\mathcal{X}$ is a DM stack of finite type over S and $\overline{\mathcal{X}}$ is its coarse moduli space, then we define $\{\mathcal{X}\} := \{\overline{\mathcal{X}}\} \in K_0(\mathbf{Var}_S)'_r$. If $C \subset |\mathcal{X}|$ is a constructible subset, then we define $\{C\} \in K_0(\mathbf{Var}_S)'_r$ either as the class $\{\overline{C}\}$ of the image $\overline{C} \subset |\overline{\mathcal{X}}|$ of C or as $\sum_{i=1}^n \{\mathcal{C}_i\}$ for locally closed substacks $\mathcal{C}_i \subset \mathcal{X}$ such that $C = \bigsqcup_{i=1}^n |\mathcal{C}_i|$. These coincide thanks to the relation above. Indeed, if $C_i \subset \overline{\mathcal{X}}$ is the image of $\mathcal{C}_i$, say with the structure of a reduced algebraic space, then the morphism $\overline{\mathcal{C}_i} \to C_i$ is a universal homeomorphism and $\{\mathcal{C}_i\} = \{C_i\}$. We define $\mathcal{M}'_S$ and $\mathcal{M}'_{S,r}$ to be the localizations of $K_0(\mathbf{Var}_S)'$ and $K_0(\mathbf{Var}_S)'_r$ by $\mathbb{L}$, respectively.

Lemma 9.7. *Let S be an algebraic space of finite type, let $\mathcal{Y}$ and $\mathcal{X}$ be DM stacks of finite type over S, and let $D \subset |\mathcal{Y}|$ and $C \subset |\mathcal{X}|$ be constructible subsets. Let $f\colon \mathcal{Y} \to \mathcal{X}$ be an S-morphism such that $f(D) \subset C$ and $f|_D\colon D \to C$ is a pseudo-$\mathbb{A}^d$-bundle. Then, for every positive integer r, we have*

$$\{D\} = \{C\}\mathbb{L}^d \in K_0(\mathbf{Var}_S)'_r\,.$$

Proof. We take stratifications $D = \bigsqcup_i D_i$ and $C = \bigsqcup_i C_i$ as in Definition 9.5. We then have

$$\{D\} = \sum_i \{D_i\} = \sum_i \{\overline{\mathcal{D}_i}\} = \sum_i \{\overline{\mathcal{C}_i}\}\mathbb{L}^d = \sum_i \{C_i\}\mathbb{L}^d = \{C\}\mathbb{L}^d\,. \qquad \square$$

For a morphism $f\colon S \to T$ between algebraic spaces of finite type, we have a map $K_0(\mathbf{Var}_S) \to K_0(\mathbf{Var}_T)$ sending the class $\{\phi\colon X \to S\}$ of an S-space to the class $\{f\circ\phi\colon X \to T\}$ of the induced T-space. This induces maps $K_0(\mathbf{Var}_S)' \to K_0(\mathbf{Var}_T)'$ and $\mathcal{M}'_S \to \mathcal{M}'_T$. We denote these maps by $\int_{S\to T}$. When $T = \operatorname{Spec} k$, we also denote them by $\int_S$. We call them *integration maps*.

Next consider the case where S is only *locally* of finite type over k. We define

$$\mathcal{M}'_S := \varinjlim_U \mathcal{M}'_U\,,$$

where U runs over quasi-compact open subspaces of S. Let $f\colon S \to T$ be a morphism between algebraic spaces, locally of finite type over k. Let $U \subset U' \subset S$ and $V \subset V' \subset T$ be finite-type open subspaces such that $f(U) \subset V$ and $f(U') \subset V'$. Then we have the following commutative diagram of integration maps:

$$\begin{tikzcd} \mathcal{M}'_U \arrow[r] \arrow[d] \arrow[dr] & \mathcal{M}'_{U'} \arrow[d] \\ \mathcal{M}'_V \arrow[r] & \mathcal{M}'_{V'}\,. \end{tikzcd}$$

Therefore, taking limits, we get the integration map $\int_{S\to T}\colon \mathcal{M}'_S \to \mathcal{M}'_T$.

Given an S-scheme X which is of finite type over k, we define $\{X\} \in \mathcal{M}'_{S,r}$ in the obvious way: If a finite-type open subset $U \subset S$ contains the image of X, then $\{X\} \in \mathcal{M}'_{S,r}$ is defined to be the image of $\{X\} \in \mathcal{M}'_{U,r}$. For $n \in \frac{1}{r}\mathbb{Z}$, we also define $\{X\}\mathbb{L}^n \in \mathcal{M}'_{S,r}$ to be the image of $\{X\}\mathbb{L}^n \in \mathcal{M}'_{U,r}$. Then $\mathcal{M}'_{S,r}$ is generated by elements of this form.

For $m \in \frac{1}{r}\mathbb{Z}$, we define $F_m \subset \mathcal{M}'_{S,r}$ to be the subgroup generated by elements of the form $\{X\}\mathbb{L}^n$, $n \in \frac{1}{r}\mathbb{Z}$, such that $\dim X + n \leqslant -m$. Here $\dim X$ means the dimension of the scheme X, not the relative dimension over S. These groups form a descending filtration of $\mathcal{M}'_{S,r}$. We define

$$\widehat{\mathcal{M}'_{S,r}} := \varprojlim \mathcal{M}'_{S,r}/F_m\,.$$

The integration map $\int_{S\to T}$ extends to the completions:

$$\int_{S\to T}\colon\ \widehat{\mathcal{M}'_{S,r}} \to \widehat{\mathcal{M}'_{T,r}}\,.$$

To extract cohomological or numerical data from elements of the Grothendieck ring of varieties or rings derived from it as above, one often considers realization maps. We first consider the l-adic realization. Suppose that k is finitely generated. Let l be a prime number different from p. Let $G_k = \operatorname{Gal}(k^{\mathrm{sep}}/k)$ be the absolute Galois group of k, and let $\mathbf{WRep}_{G_k}(\mathbb{Q}_l)$ be the abelian category of mixed l-adic representations of G_k (see [NS11, Example 2.9, p. 152]). With this abelian category, we associate the Grothendieck group $K_0(\mathbf{WRep}_{G_k}(\mathbb{Q}_l))$ which has the ring structure induced by tensor products. We then have the realization map

$$K_0(\mathbf{Var}_k) \to K_0(\mathbf{WRep}_{G_k}(\mathbb{Q}_l))\,, \quad \{X\} \mapsto \chi_l(X) := \sum_i (-1)^i \big[\operatorname{H}^i_c(X_{\bar{k}}, \mathbb{Q}_l)\big]\,. \tag{9.1}$$

This extends to a map of complete rings

$$\widehat{\mathcal{M}_k} \to K_0\big(\widehat{\mathbf{WRep}_{G_k}}(\mathbb{Q}_l)\big)\,. \tag{9.2}$$

Here the completion $K_0\big(\widehat{\mathbf{WRep}_{G_k}}(\mathbb{Q}_l)\big)$ is with respect to a filtration defined in terms of weights.

Lemma 9.8. *Let $\mathcal{X}$ be a DM stack of finite type and X its coarse moduli space. Then, for* *every $i \in \mathbb{Z}$, we have isomorphisms of cohomology groups (with compact support)*

$$\mathrm{H}^i(X_{\bar{k}}, \mathbb{Q}_l) \cong \mathrm{H}^i(\mathcal{X}_{\bar{k}}, \mathbb{Q}_l) \quad \textit{and} \quad \mathrm{H}^i_c(X_{\bar{k}}, \mathbb{Q}_l) \cong \mathrm{H}^i_c(\mathcal{X}_{\bar{k}}, \mathbb{Q}_l)$$

which are compatible with $\mathrm{Gal}(\bar{k}/k)$-action.

Proof. We literally follow the proof of [Sun12, Proposition 7.3.2], a similar result for $\overline{\mathbb{Q}_l}$-coefficient cohomology of stacks over a finite field. Let $\pi\colon \mathcal{X} \to X$ be the given morphism. Since π is proper and quasi-finite, $\mathrm{R}^i\pi_!\mathbb{Q}_l = \mathrm{R}^i\pi_*\mathbb{Q}_l = 0$ for $i > 0$. Moreover, the natural map $\mathbb{Q}_l \to \pi_*\mathbb{Q}_l = \pi_!\mathbb{Q}_l$ is an isomorphism. The proposition follows from the degeneration of the spectral sequences $\mathrm{H}^i\big(X_{\bar{k}}, \mathrm{R}^j\pi_*\mathbb{Q}_l\big) \Rightarrow \mathrm{H}^{i+j}(\mathcal{X}_{\bar{k}}, \mathbb{Q}_l)$ and $\mathrm{H}^i_c\big(X_{\bar{k}}, \mathrm{R}^j\pi_!\mathbb{Q}_l\big) \Rightarrow \mathrm{H}^{i+j}_c(\mathcal{X}_{\bar{k}}, \mathbb{Q}_l)$. □

LEMMA 9.9. *Let $f\colon \mathcal{Y} \to \mathcal{X}$ be a representable morphism between DM stacks of finite type, and let $d \in \mathbb{N}$. Suppose that for every geometric point $\operatorname{Spec} K \to \mathcal{X}$, the fiber $\mathcal{Y} \times_{\mathcal{X}} \operatorname{Spec} K$ is isomorphic to $\mathbb{A}^d_K$ over K. Then*

$$\mathrm{H}^{i+2d}_c(\mathcal{Y}_{\bar{k}}, \mathbb{Q}_l) \cong \mathrm{H}^i_c(\mathcal{X}_{\bar{k}}, \mathbb{Q}_l)(-d)\,.$$

Proof. Since $\mathrm{H}^i_c\big(\mathbb{A}^d_K, \mathbb{Q}_l\big) = 0$ for $i \neq 2d$, we have $\mathrm{R}^i f_!\mathbb{Q}_l = 0$ for $i \neq 2d$, and the trace map $\mathrm{R}^{2d} f_!\mathbb{Q}_l(d) \to \mathbb{Q}_l$, see [Ols15, Theorem 4.1], is an isomorphism. Thus

$$\mathrm{H}^{i+2d}_c(\mathcal{Y}_{\bar{k}}, \mathbb{Q}_l) \cong \mathrm{H}^i_c\big(\mathcal{X}_{\bar{k}}, \mathrm{R}^{2d} f_!\mathbb{Q}_l\big) \cong \mathrm{H}^i_c(\mathcal{X}_{\bar{k}}, \mathbb{Q}_l)(-d)\,. \qquad \square$$

LEMMA 9.10. *For a finite group action $G \curvearrowright \mathbb{A}^d_k$, we have*

$$\mathrm{H}^i_c\big(\big(\mathbb{A}^d_k/G\big)_{\bar{k}}, \mathbb{Q}_l\big) \cong \mathrm{H}^i_c\big(\mathbb{A}^d_{\bar{k}}, \mathbb{Q}_l\big) \cong \begin{cases} \mathbb{Q}_l(-d) & (i = 2d)\,, \\ 0 & (i \neq 2d)\,. \end{cases}$$

Proof. We have

$$\mathrm{H}^i_c\big(\big(\mathbb{A}^d_k/G\big)_{\bar{k}}, \mathbb{Q}_l\big) \cong \mathrm{H}^i_c\big(\big[\mathbb{A}^d_k/G\big]_{\bar{k}}, \mathbb{Q}_l\big) \cong \mathrm{H}^{i-2d}_c((\mathrm{B}\,G)_{\bar{k}}, \mathbb{Q}_l)(-d) \cong \mathrm{H}^{i-2d}_c(\operatorname{Spec}\bar{k}, \mathbb{Q}_l)(-d)\,,$$

where the first and last isomorphisms follow from Proposition 9.8 and the middle one from Lemma 9.9. □

LEMMA 9.11. *Let $f\colon \mathcal{Y} \to \mathcal{X}$ be a representable morphism between DM stacks of finite type, and let $d \in \mathbb{N}$. Suppose that for every geometric point $\operatorname{Spec} K \to \mathcal{X}$, the fiber $\mathcal{Y} \times_{\mathcal{X}} \operatorname{Spec} K$ is universally homeomorphic to $\mathbb{A}^d_K/G$ over K for some finite group action $G \curvearrowright \mathbb{A}^d_K$ over K. Then*

$$\mathrm{H}^{i+2d}_c(\mathcal{Y}_{\bar{k}}, \mathbb{Q}_l) \cong \mathrm{H}^i_c(\mathcal{X}_{\bar{k}}, \mathbb{Q}_l)(-d)\,.$$

Proof. Thanks to Lemma 9.10, the same proof as that of Lemma 9.9 is valid. □

In particular, this shows that for a pseudo-$\mathbb{A}^r$-bundle $Y \to X$, we have

$$\chi_l(Y) = \chi_l(X)\chi_l(\mathbb{A}^r_k) = \chi_l(X)[\mathbb{Q}_l(-r)]\,.$$

Therefore, maps (9.1) and (9.2) induce maps

$$K_0(\mathbf{Var}_k)' \to K_0(\mathbf{WRep}_{G_k}(\mathbb{Q}_l))\,, \quad \widehat{\mathcal{M}'_k} \to K_0(\widehat{\mathbf{WRep}_{G_k}}(\mathbb{Q}_l))\,.$$

For each $r \in \mathbb{N}$, we also have a map

$$\widehat{\mathcal{M}'_{k,r}} \to K_0\big(\widehat{\mathbf{WRep}_{G_k}}(\mathbb{Q}_l)\big)_r\,.$$

Here we define $K_0(\mathbf{WRep}_{G_k}(\mathbb{Q}_l))_r$ by formally adjoining an rth root of $[\mathbb{Q}_l(-1)] = \chi_l\big(\mathbb{A}^1_k\big)$. If there exists a 1-dimensional representation $V \in \mathbf{WRep}_{G_k}(\mathbb{Q}_l)$ such that $V^{\otimes r} \cong \mathbb{Q}_l(-1)$, then we also have a map

$$\widehat{\mathcal{M}'_{k,r}} \to K_0\big(\widehat{\mathbf{WRep}_{G_k}}(\mathbb{Q}_l)\big)$$

sending $\mathbb{L}^{1/r}$ to $[V]$. By [Ito04, § 5.3, p. 1510], such a V exists if we replace the base field k with a finite extension of it.

Similarly, when k is a subfield of $\mathbb{C}$, we can define

$$\widehat{\mathcal{M}'_{k,r}} \to K_0\big(\widehat{\mathbf{MHS}^{1/r}}\big)\,,$$

where $\mathbf{MHS}^{1/r}$ is the abelian category of $\frac{1}{r}\mathbb{Z}$-indexed mixed Hodge structures [Yas06, § 3.8, p. 741].

Over any field k, we have the Poincaré polynomial realization

$$\widehat{\mathcal{M}'_{k,r}} \to \mathbb{Z}\big[T^{1/r}\big]\llbracket T^{-1/r}\rrbracket$$

(see [Nic11, Appendix]). When k is a subfield of $\mathbb{C}$, we also have the E-polynomial (Hodge–Deligne polynomial) realization,

$$\widehat{\mathcal{M}'_{k,r}} \to \mathbb{Z}\big[u^{1/r}, v^{1/r}\big]\llbracket u^{-1/r}, v^{-1/r}\rrbracket\,.$$

For other realization maps, we refer the reader to [NS11, CNS18].

10. Motivic integration for untwisted arcs

In this section, we develop the motivic integration over formal DM stacks by restricting ourselves to untwisted arcs. We can do this in a way quite similar to that used in the less general cases of schemes and formal schemes. Throughout the section, we make the following assumption.

Assumption-Notation 10.1. Let Ψ and Φ be reduced DM stacks almost of finite type, and let $\mathcal{Y}$ and $\mathcal{X}$ be formal DM stacks of finite type and flat over Df_Ψ and Df_Φ, respectively, which have pure relative dimension d and are fiberwise generically smooth. Let $\Psi \to \Phi$ be a morphism, and let $f\colon \mathcal{Y} \to \mathcal{X}$ be a morphism compatible with $\mathrm{Df}_\Psi \to \mathrm{Df}_\Phi$. We suppose that the Jacobian ideal sheaf Jac_f defines a narrow closed substack of $\mathcal{Y}$.

We also fix a positive integer r so that functions on arc spaces and motivic measures will take values in $\frac{1}{r}\mathbb{Z} \cup \{\infty\}$ and $\widehat{\mathcal{M}'_{Z,r}}$, respectively.

We first note that for an étale morphism $V \to \mathcal{X}$ from a formal scheme and a geometric point $\operatorname{Spec} K \to \Phi$, we have

$$(\mathrm{J}_n\,\mathcal{X}) \times_{\mathcal{X}_0} V_0 \cong \mathrm{J}_n\,V \quad \text{and} \quad (\mathrm{J}_{\Phi,n}\,V) \times_\Phi \operatorname{Spec} K \cong \mathrm{J}_{\operatorname{Spec} K,n}(V \times_\Phi \operatorname{Spec} K)\,.$$

Therefore, we can often reduce problems on untwisted arcs and jets to the case of formal schemes over Df_K, which was treated in [Seb04, CNS18]. We denote the morphism $\mathrm{J}_\infty\,\mathcal{X} \to \mathrm{J}_n\,\mathcal{X}$ by π_n.

Definition 10.2. Let $C \subset |\mathrm{J}_\infty \mathcal{X}|$ be a subset. We say that C is *cylindrical of level n* ($n \in \mathbb{N}$) if there exists a quasi-compact constructible subset $C_n \subset |\mathrm{J}_n \mathcal{X}|$ such that $\pi_n^{-1}(C_n) = C$. We say that C is *cylindrical* if it is cylindrical of level n for some $n \in \mathbb{N}$.

Note that by [CNS18, Appendix, Proposition 1.3.3], a subset C is cylindrical if and only if it is a quasi-compact constructible subset.

Definition 10.3. We say that a subset $C \subset |\mathrm{J}_\infty \mathcal{X}|$ is *stable of level n* if

(1) C is cylindrical of level n and
(2) for every $n' \geqslant n$, we have that $\pi_{n'+1}(C) \to \pi_{n'}(C)$ is a pseudo-$\mathbb{A}^d$-bundle.

We say that C is *stable* if it is stable of level n for some $n \in \mathbb{N}$.

Definition 10.4. Let Z be an algebraic space almost of finite type, and let $\Phi \to Z$ be a morphism. For a stable subset C of level n, we define its measure by

$$\mu_{\mathcal{X},Z}(C) = \{\pi_n(C)\}\mathbb{L}^{-nd} \in \widehat{\mathcal{M}'_{Z,r}}\,.$$

Clearly, we have $\int_Z \mu_{\mathcal{X},Z}(C) = \mu_{\mathcal{X},k}(C)$.

For a cylindrical subset C and $e \in \mathbb{N}$, let

$$C^{(e)} := \{c \in C \mid \mathfrak{j}_{\mathcal{X}}(c) \leqslant e\}\,.$$

By Lemma 10.15 below, this is a stable subset.

Definition 10.5. For a cylindrical subset C, we define

$$\mu_{\mathcal{X},Z}(C) := \lim_{e\to\infty} \mu_{\mathcal{X},Z}\big(C^{(e)}\big)\,.$$

The limit exists by the same argument as in [DL02, Appendix A.2].

Recall that $\widehat{\mathcal{M}'_{Z,r}}$ is defined as the completion $\varprojlim \mathcal{M}'_{Z,r}/F_m$. This has the filtration $F_\bullet\widehat{\mathcal{M}'_{Z,r}}$ given by $F_m\widehat{\mathcal{M}'_{Z,r}} := \varprojlim_{n\geqslant m} F_m/F_n$.

Definition 10.6. We define the seminorm

$$\widehat{\mathcal{M}'_{Z,r}} \mapsto \mathbb{R}_{\geqslant 0}\,,\quad a \mapsto \|a\| := 2^{-m}\,,$$

where m is the largest $m \in \frac{1}{r}\mathbb{Z}$ such that $a \in F_m\widehat{\mathcal{M}'_{Z,r}}$.

For an element $\{V\}\mathbb{L}^r$, $r \in \frac{1}{m}\mathbb{Z}$, we have $\|\{V\}\mathbb{L}^r\| = 2^{\dim V + r}$, which is independent of Z.

Definition 10.7. A subset $C \subset |\mathrm{J}_\infty\mathcal{X}|$ is said to be *measurable* if for every $\epsilon > 0$, there exist cylindrical subsets $C(\epsilon)$ and $A_i(\epsilon)$, $i \in \mathbb{N}$, such that $(C \triangle C(\epsilon)) \subset \bigcup_i A_i(\epsilon)$ and $\|\mu_{\mathcal{X},Z}(A_i(\epsilon))\| \leqslant \epsilon$ for every i. For a measurable subset C, we define

$$\mu_{\mathcal{X},Z}(C) := \lim_{\epsilon\to 0} \mu_{\mathcal{X},Z}(C(\epsilon)) \in \widehat{\mathcal{M}'_{Z,r}}\,.$$

We define a *negligible subset* to be a measurable subset of measure zero.

Note that $\|\mu_{\mathcal{X},Z}(-)\|$ is independent of Z. Therefore, the measurability and negligibility are also independent of Z. Clearly, cylindrical subsets are measurable. We have a map

$$\mu_{\mathcal{X},Z}\colon \{\text{measurable subset of } |\mathrm{J}_\infty\mathcal{X}|\} \to \widehat{\mathcal{M}'_{Z,r}}\,.$$

Let C_i, $i \in \mathbb{N}$, be a countable family of mutually disjoint measurable subsets of $|\mathrm{J}_\infty\mathcal{X}|$. Then $\bigsqcup_i C_i$ is measurable if and only if $\lim_{i\to\infty}\|\mu_{\mathcal{X},Z}(C_i)\| = 0$ (see [CNS18, Chapter 6, Proposition 3.4.3]). If this is the case, we have

$$\mu_{\mathcal{X},Z}\Big(\bigsqcup_i C_i\Big) = \sum_i \mu_{\mathcal{X},Z}(C_i)\,.$$

Lemma 10.8. *Let $\mathcal{Z} \subset \mathcal{X}$ be a narrow closed substack. Then $|\mathrm{J}_\infty\mathcal{Z}| \subset |\mathrm{J}_\infty\mathcal{X}|$ is negligible.*

Proof. This is known for the case of formal schemes over Df_K and essentially follows from the fact that fibers of the map

$$\pi_{n+1}(|\mathcal{J}_\infty\mathcal{Z}|) \to \pi_n(|\mathcal{J}_\infty\mathcal{Z}|)$$

have dimension less than d. We can show that this fact also holds in our situation; we can reduce it to the classical case by the base change along $\operatorname{Spec} K \to \Phi$ and an atlas $V \to \mathcal{X} \times_\Phi \operatorname{Spec} K$. The fact similarly also implies the lemma in our situation. □

Definition 10.9. Let $A \subset |\mathrm{J}_\infty \mathcal{X}|$ be a subset and $h\colon A \to \frac{1}{r}\mathbb{Z} \cup \{\infty\}$ be a function. We say that h is *measurable* if there exists a decomposition $A = \bigsqcup_{i\in\mathbb{N}} A_i$ into countably many measurable subsets such that for each i, the restriction $h|_{A_i}$ is constant and if $h(A_i) = \infty$, then A_i is negligible. If h is measurable, then for A_i as above, we define

$$\int_A \mathbb{L}^h \, d\mu_{\mathcal{X},Z} := \sum_{i\in\mathbb{N}} \mu_{\mathcal{X},Z}(A_i)\mathbb{L}^{h(A_i)} \in \widehat{\mathcal{M}'_{Z,r}} \cup \{\infty\}\,.$$

This is independent of the decomposition $A = \bigsqcup_i A_i$. For a morphism $Z \to Z'$ of algebraic spaces almost of finite type, we have

$$\int_{Z\to Z'} \int_A \mathbb{L}^h \, d\mu_{\mathcal{X},Z} = \int_A \mathbb{L}^h \, d\mu_{\mathcal{X},Z'}\,.$$

Notation 10.10. In the rest of this section, for a ring R, we denote the ideal $\left(t^{n+1}\right) \subset R[\![t]\!]$ by $\mathfrak{t}^n_R$ and the quotient $\left(t^{m+1}\right)/\left(t^{n+1}\right)$ for $n \geqslant m$ by $\mathfrak{t}^m_{n,R}$. For an arc $\gamma\colon \mathrm{Df}_K \to \mathcal{X}$, we denote by $\gamma^\flat \Omega_{\mathcal{X}/\mathrm{Df}_\Phi}$ the flat pullback $\left(\gamma^* \Omega_{\mathcal{X}/\mathrm{Df}_\Phi}\right)/\mathrm{tors}$. By γ_n, we denote the n-jet $\mathrm{D}_{K,n} \to \mathcal{X}$ deduced from γ.

Let $v\colon V = \operatorname{Spf} R \to \mathcal{X}$ be an étale morphism. For $n', n \in \mathbb{N} \cup \{\infty\}$ with $n' \geqslant n$, we have the following 2-Cartesian diagram:

$$\begin{tikzcd} \mathrm{J}_{n'} V \arrow[r] \arrow[d] & \mathrm{J}_{n'} \mathcal{X} \arrow[d] \\ \mathrm{J}_n V \arrow[r] & \mathrm{J}_n \mathcal{X}\,. \end{tikzcd}$$

Let K be an algebraically closed field, and let $\beta \in (\mathrm{J}_\infty V)(K)$ and $\gamma := v(\beta) \in (\mathrm{J}_\infty \mathcal{X})(K)$. In particular, a K-point $\beta' \in (\mathrm{J}_\infty V)(K)$ over β_n is identified with a pair (γ', α) of $\gamma' \in (\mathrm{J}_\infty \mathcal{X})(K)$ and an isomorphism $\alpha\colon \gamma_n \xrightarrow{\sim} \gamma'_n$. For such β', the map

$$(\beta')^* - \beta^*\colon R \to K[\![t]\!]$$

has image in $\mathfrak{t}^{n+1}$, and the induced map $R \to \mathfrak{t}^{n+1}_{2(n+1)}$ is a $k[\![t]\!]$-derivation. This corresponds to a $K[\![t]\!]$-module homomorphism

$$\beta^* \Omega_{V/\mathrm{Df}_\Phi} = \gamma^* \Omega_{\mathcal{X}/\mathrm{Df}_\Phi} \to \mathfrak{t}^{n+1}_{2(n+1)}\,.$$

Suppose $a := \mathfrak{j}_{\mathcal{X}}(\gamma) < \infty$. The torsion part of $\gamma^* \Omega_{\mathcal{X}/\mathrm{Df}_\Phi}$ has length a and is of the form $\bigoplus_i K[\![t]\!]/\mathfrak{t}^{a_i}$, $\sum_i a_i = a$. If $2(n+1) - (n+c+1) \geqslant a$, equivalently if $c \leqslant n - a + 1$, then the composite

$$\gamma^* \Omega_{\mathcal{X}/\mathrm{Df}_\Phi} \to \mathfrak{t}^{n+1}_{2(n+1)} \to \mathfrak{t}^{n+1}_{n+c+1}$$

kills the torsion part of $\gamma^* \Omega_{\mathcal{X}/\mathrm{Df}_\Phi}$ and factors through the flat pullback $\gamma^\flat \Omega_{\mathcal{X}/\mathrm{Df}_\Phi}$.

Notation 10.11. We denote the corresponding element of

$$\operatorname{Hom}_{K[\![t]\!]}\left(\gamma^\flat \Omega_{\mathcal{X}/\mathrm{Df}_\Phi}, \mathfrak{t}^{n+1}_{n+c+1}\right)$$

by $\delta^{n+c}_n(\beta; \beta')$ or $\delta^{n+c}_n(\gamma; \gamma', \alpha)$.

For another such $\beta'' \in (\mathrm{J}_\infty V)(K)$, we have

$$\beta'_{n+c} = \beta''_{n+c} \Leftrightarrow \delta^{n+c}_n(\beta; \beta') = \delta^{n+c}_n(\beta; \beta'')\,. \tag{10.1}$$

Let F denote the fiber of $\pi^{n+c}_n\colon \mathrm{J}_{n+c} V \to \mathrm{J}_n V$ over β_n, and let $F' := F \cap \pi_n((\mathrm{J}_\infty V) \times_\Phi \operatorname{Spec} K)$, the locus of n-jets in F that lift to arcs. Since $2n+1 \geqslant n+c$, by [CNS18, Proposition 2.2.6],

$F(K)$ is identified with $\mathrm{Hom}_{K[\![t]\!]}(\gamma^*\Omega_{\mathcal{X}/\mathrm{Df}_\Phi}, \mathfrak{t}_{n+c+1}^{n+1}) \cong K^r$, with r some non-negative integer. Moreover, F is isomorphic to $\mathbb{A}_K^r$.

Lemma 10.12. *If $c \leqslant n-a+1$, the map $\beta' \mapsto \delta_n^{n+c}(\beta;\beta')$ gives a bijection*

$$F'(K) \to \mathrm{Hom}_{K[\![t]\!]}(\gamma^\flat\Omega_{\mathcal{X}/\mathrm{Df}_\Phi}, \mathfrak{t}_{n+c+1}^{n+1}) .$$

Proof. The injectivity follows from (10.1). When $c=1$, we have $F' \cong \mathbb{A}_K^d$ (see [Loo02, Lemma 9.1]). We denote F' by F'_c to specify the dependence on c. For $0 < c' \leqslant c$, by the case $c=1$, the map $F'_{c'} \to F'_{c'-1}$ is a pseudo-$\mathbb{A}^d$-bundle. By Lemma 9.7, we have $\{F'_{c'}\} = \{F'_{c'-1}\}\mathbb{L}^d$ and $\{F'_c\} = \mathbb{L}^{dc}$. Since

$$F_n^{n+c} \to \mathrm{Hom}_{K[\![t]\!]}(\gamma^\flat\Omega_{\mathcal{X}/\mathrm{Df}_\Phi}, \mathfrak{t}_{n+c+1}^{n+1}) = K^{dc}$$

is injective, its image is a constructible subset with class $\mathbb{L}^{dc}$ in $K_0(\mathbf{Var}_k)'$. Therefore, the complement of the image has class $\mathbb{L}^{dc} - \mathbb{L}^{dc} = 0$. This shows that it is empty; thus the map is surjective. □

From this lemma, we obtain the following.

Lemma 10.13. *The locus F' is a d-dimensional linear subspace of $F \cong \mathbb{A}_K^r$. In particular, F' is a closed subscheme of F and is isomorphic to $\mathbb{A}_K^d$.*

Lemma 10.14. *Let $h\colon \mathcal{W} \to \mathcal{V}$ be a map between DM stacks of finite type, and let $D \subset |\mathcal{W}|$ and $C \subset |\mathcal{V}|$ be constructible subsets such that $h(D) \subset C$. Suppose that for every $c \in |C|$, the intersection $h^{-1}(c) \cap D$ is a closed subset of $h^{-1}(c)$. Then there exists a stratification $C = \bigsqcup_{i=1}^n C_i$ such that $h^{-1}(C_i) \cap D$ is a locally closed subset of $|\mathcal{W}|$.*

Proof. By taking an arbitrary stratification $C = \bigsqcup_{i=1}^n C_i$ and restricting to each C_i, we can reduce the lemma to the case $|\mathcal{V}| = C$. Moreover, we may suppose that $\mathcal{V}$ is irreducible. Under these assumptions, it suffices to show that there exists an open dense subset $U \subset |\mathcal{V}|$ such that $h^{-1}(U) \cap D$ is a locally closed subset of $|\mathcal{W}|$. Let $\eta \in |\mathcal{V}|$ be the generic point, and let $\overline{h^{-1}(\eta) \cap D}$ be the closure of $h^{-1}(\eta) \cap D$ in $|\mathcal{W}|$. Since $\overline{h^{-1}(\eta) \cap D}$ and D coincide over η, the symmetric difference

$$\overline{h^{-1}(\eta) \cap D} \triangle D := \left(\overline{f^{-1}(\eta) \cap D} \setminus D\right) \cup \left(D \setminus \overline{f^{-1}(\eta) \cap D}\right)$$

has non-dense image in $|\mathcal{V}|$. Let $U \subset C$ be the complement of the Zariski closure of this image. Then

$$h^{-1}(U) \cap D = h^{-1}(U) \cap \overline{h^{-1}(\eta) \cap D}$$

is a closed subset of $h^{-1}(U)$ and a locally closed subset of $|\mathcal{W}|$. □

Lemma 10.15. *For $a \in \mathbb{N}$ and any quasi-compact substack $\Upsilon \subset \Phi$, the set*

$$\mathrm{j}_{\mathcal{X}}^{-1}(a) \cap ((\mathrm{J}_\infty \mathcal{X}) \times_\Phi \Upsilon)$$

is stable of level a.

Proof. Let us denote this set by C. Then $\pi_a(C)$ is quasi-compact and constructible. It suffices to show that for every $b \geqslant a$, the map $\pi_{b+1}(C) \to \pi_b(C)$ is a pseudo-$\mathbb{A}^d$-bundle. By Lemmas 10.14 and 10.13, there exists a stratification $\pi_b(C) = \bigsqcup_i \pi_b(C)_i$ into locally closed subsets such that the preimage $\pi_{b+1}(C)_i \subset \pi_{b+1}(C)$ of $\pi_b(C)_i$ is also locally closed. We regard these locally closed subsets as substacks of $\mathrm{J}_b \mathcal{X}$ or $\mathrm{J}_{b+1} \mathcal{X}$. For a geometric point $\xi\colon \operatorname{Spec} K \to \pi_b(C)_i$, if G is its automorphism group, we have the following 2-commutative diagram of stacks over K:

$$\begin{tikzcd} \mathbb{A}^d_K \arrow[r] \arrow[d] & {[\mathbb{A}^d_K/G]} \arrow[r, hook] \arrow[d] & \pi_{b+1}(C)_i \otimes_k K \arrow[d] \\ \operatorname{Spec} K \arrow[r] & \mathrm{B}\, G \arrow[r, hook] & \pi_b(C)_i \otimes_k K\,. \end{tikzcd}$$

Here the two squares are 2-Cartesian, and the arrows of the form $\hookrightarrow$ are closed immersions. This induces a universally injective morphism of coarse moduli spaces

$$\mathbb{A}^d_K/G \to \overline{\pi_{b+1}(C)_i \otimes_k K}\,.$$

The image of this map is the fiber of $\overline{\pi_{b+1}(C)_i \otimes_k K} \to \overline{\pi_b(C)_i \otimes_k K}$ over the point induced by ξ. We conclude that $\pi_{b+1}(C) \to \pi_b(C)$ is a pseudo-$\mathbb{A}^d$-bundle. $\square$

Lemma 10.16. *Let $\gamma \in (\mathrm{J}_\infty \mathcal{Y})(K)$ and $\eta \in (\mathrm{J}_\infty \mathcal{X})(K)$ with K an algebraically closed field. Let $e := \mathfrak{j}_f(\gamma)$ and $a := \mathfrak{j}_{\mathcal{X}}(f_\infty(\gamma))$, and let $n \in \mathbb{N}$ be such that $n \geqslant 2e + a$. Suppose that there exists an isomorphism $\sigma\colon f_n(\gamma_n) \cong \eta_n$. Then there exists a $\xi \in (\mathrm{J}_\infty \mathcal{Y})(K)$ such that $\gamma_{n-e} \cong \xi_{n-e}$ and $f_\infty(\xi) \cong \eta$.*

Proof. We have a $K[\![t]\!]$-module homomorphism

$$\delta_n^{n+1}(\gamma \circ f; \eta, \sigma)\colon (\gamma \circ f)^\flat \Omega_{\mathcal{X}/\mathrm{Df}_\Phi} \to \mathfrak{t}_{n+2}^{n+1}\,.$$

For suitable bases, the map of free modules $(\gamma \circ f)^\flat \Omega_{\mathcal{X}/\mathrm{Df}_\Phi} \to \gamma^\flat \Omega_{\mathcal{Y}/\mathrm{Df}_\Psi}$ is represented by a diagonal matrix $\mathrm{diag}(t^{e_1}, \dots, t^{e_d})$ with $\sum_{i=1}^d e_i = e$. Therefore, $\delta_n^{n+1}(\gamma \circ f; \eta)$ is induced by some map $\gamma^\flat \Omega_{\mathcal{Y}/\mathrm{Df}_\Psi} \to \mathfrak{t}_{n+2}^{n-e+1}$. Since $n \geqslant 2e + a$, we have

$$(n+2) - (n-e+1) = e + 1 \leqslant (n-e) - a + 1\,.$$

By Lemma 10.12, this map is $\delta_{n-e}^{n+1}\big(\gamma; \xi^{(1)}, \alpha^{(1)}\big)$ for some $\xi^{(1)} \in (\mathrm{J}_\infty \mathcal{Y})(K)$ and an isomorphism $\alpha^{(1)}\colon \gamma_{n-e} \cong \xi^{(1)}_{n-e}$. We have an isomorphism $f_{n+1}\big(\xi^{(1)}_{n+1}\big) \cong \eta_{n+1}$ compatible with σ and $\alpha^{(1)}$. Applying the same argument to $\xi^{(1)}$ and η instead of γ and η, we obtain $\xi^{(2)} \in (\mathrm{J}_\infty \mathcal{Y})(K)$ and isomorphisms $\xi^{(1)}_{n-e+1} \cong \xi^{(2)}_{n-e+1}$ and $f_{n+2}\big(\xi^{(2)}_{n+2}\big) \cong \eta_{n+2}$. Repeating this procedure, we obtain a sequence $\xi^{(i)} \in (\mathrm{J}_\infty \mathcal{Y})(K)$, $i > 0$, with isomorphisms $\xi^{(i)}_{n-e+i} \cong \xi^{(i+1)}_{n-e+i}$ and $f_{n+i}\big(\xi^{(i)}_{n+i}\big) \cong \eta_{n+i}$. The sequence $\xi^{(i)}_{n-e+i} \in (\mathrm{J}_{n-e+i} \mathcal{Y})(K)$ gives the limit $\xi \in (\mathrm{J}_\infty \mathcal{Y})(K)$ such that $\gamma_{n-e} \cong \xi_{n-e}$ and $f_\infty(\xi) \cong \eta$. $\square$

Remark 10.17. If γ and γ' correspond to twisted arcs $\mathcal{E}_K \to \mathcal{Y}$ and $\mathcal{E}'_K \to \mathcal{Y}$, respectively, then the equality $\pi_{n-e}(\gamma) \cong \pi_{n-e}(\gamma')$ especially implies that $\mathcal{E}_K \cong \mathcal{E}'_K$.

Definition 10.18. For a subset $C \subset |\mathrm{J}_\infty \mathcal{Y}|$, we say that $f_\infty|_C$ is *geometrically injective* if for every algebraically closed field K, the map $f_\infty|_{C[K]}\colon C[K] \to (\mathrm{J}_\infty \mathcal{X})[K]$ is injective. We say that $f|_C$ is *almost geometrically injective* if there exists a negligible subset $C' \subset C$ such that $f|_{C \setminus C'}$ is geometrically injective.

Let $B \subset |\mathrm{J}_\infty \mathcal{X}|$ be a subset such that $f_\infty(C) \subset B$. We say that $f_\infty|_C\colon C \to B$ is *almost geometrically bijective* if there exist negligible subsets $B' \subset B$ and $C' \subset C$ such that for each algebraically closed field K, the map f_∞ induces a bijection $(C \setminus C')[K] \to (B \setminus B')[K]$.

Lemma 10.19. *Let e, $n \in \mathbb{N}$ with $n \geqslant e$, and let $C \subset |\mathrm{J}_\infty \mathcal{Y}|$ be a stable subset of level $n - e$. Suppose that $\mathfrak{j}_f|_C$ takes constant value e. Suppose $n \geqslant 2e + \mathfrak{j}_{\mathcal{X}}(\gamma)$ for every $\gamma \in C$. Suppose that $f_\infty|_C$ is geometrically injective. Then $\pi_n(C) \to f_n \pi_n(C)$ is a pseudo-$\mathbb{A}^e$-bundle.*

Proof. Let $\gamma, \gamma' \in C(K)$ be such that $f_n(\gamma_n) \cong f_n(\gamma'_n)$. Applying Lemma 10.16 for $\eta = f_\infty(\gamma')$, we see that there exists a $\xi \in (\mathrm{J}_\infty \mathcal{Y})(K)$ such that $\gamma_{n-e} \cong \xi_{n-e}$ and $f_\infty(\xi) \cong f_\infty(\gamma')$. By the injectivity assumption, we have $\xi \cong \gamma'$; we conclude that $\gamma_{n-e} \cong \gamma'_{n-e}$. Namely, a fiber of $\pi_n(C) \to f_n\pi_n(C)$ is contained in a fiber of $\pi_n(C) \to \pi_{n-e}(C)$. We may suppose that $\pi_n(C)$ and $\pi_{n-e}(C)$ are locally closed subsets. We regard them as substacks. From $\gamma_{n-e}\colon \operatorname{Spec} K \to \pi_{n-e}(C)$, we obtain the following 2-Cartesian diagram as in the proof of Lemma 10.15:

$$\begin{tikzcd}
\mathbb{A}_K^{de} \arrow[r] \arrow[d] & {[\mathbb{A}_K^{de}/G]} \arrow[r, hook] \arrow[d] & \pi_n(C) \otimes_k K \arrow[d] \\
\operatorname{Spec} K \arrow[r] & \mathrm{B}\, G \arrow[r, hook] & \pi_{n-e}(C) \otimes_k K\,.
\end{tikzcd}$$

By Lemma 10.12, the K-point set of the above $\mathbb{A}_K^{de}$ is identified with

$$\operatorname{Hom}_{K[\![t]\!]}\big(\gamma^\flat \Omega_{\mathcal{Y}/\mathrm{Df}_\Psi}, \mathfrak{t}_{n+1}^{n-e+1}\big)\,.$$

The kernel of the map

$$\operatorname{Hom}_{K[\![t]\!]}\big(\gamma^\flat \Omega_{\mathcal{Y}/\mathrm{Df}_\Psi}, \mathfrak{t}_{n+1}^{n-e+1}\big) \to \operatorname{Hom}_{k[\![t]\!]}\big((\gamma\circ f)^\flat \Omega_{\mathcal{X}/\mathrm{Df}_\Phi}, \mathfrak{t}_{n+1}^{n-e+1}\big)$$

parametrizes pairs (γ'_n, α_n) of $\gamma'_n \in \pi_n((\mathrm{J}_\infty \mathcal{X})(K))$ and an isomorphism $\gamma'_{n-e} \cong \gamma_{n-e}$ such that the induced isomorphism $f_{n-e}(\gamma_{n-e}) \cong f_{n-e}(\gamma'_{n-e})$ lifts to $f_n(\gamma_n) \cong f_n(\gamma'_n)$. This map is represented by a diagonal matrix $\operatorname{diag}(t^{e_1}, \dots, t^{e_d})$ with $\sum_{i=1}^d e_i = e$. The kernel is isomorphic to $\bigoplus_{i=1}^d \mathfrak{t}_{n+1}^{n-e_i+1} \cong K^{\oplus e}$. The image of the corresponding subspace $\mathbb{A}_K^e \cong W \subset \mathbb{A}_K^{de}$ in $\pi_n(C)\otimes_k K$ is the fiber of

$$\pi_n(C) \otimes_k K \to f_n\pi_n(C) \otimes_k K$$

over the point $f_n(\gamma_n)$. We see that W is stable under the G-action. We obtain a closed immersion $[W/G] \to \pi_n(C) \otimes_k K$. The induced morphism of coarse moduli spaces $W/G \to \overline{\pi_n(C) \otimes_k K}$ is a universally injective morphism onto the fiber of $\overline{\pi_n(C)} \to \overline{f_n\pi_n(C)}$ over the image of $f_n(\gamma_n)$. We have proved the lemma. $\square$

Lemma 10.20. *Let $C \subset |\mathrm{J}_\infty \mathcal{Y}|$ be a measurable subset such that $\mathfrak{j}_f$ has constant value $e < \infty$ on C. Suppose that $f_\infty|_C$ is almost geometrically injective. Then*

$$\mu_{\mathcal{Y},Z}(C) = \mu_{\mathcal{X},Z}(f_\infty(C))\mathbb{L}^e\,.$$

Proof. By a standard limit argument, we can reduce to the case where $C \subset |\mathrm{J}_\infty \mathcal{Y}|$ is a stable subset such that $\mathfrak{j}_f$ and $\mathfrak{j}_\mathcal{X} \circ f_\infty$ are nowhere infinity on C and the $f|_{C[K]}$ are injective. For $n \gg 0$, by Lemma 10.19, we have

$$\mu_{\mathcal{Y},Z}(C) = \{\pi_n(C)\}\mathbb{L}^{-dn} = \{\pi_n f_\infty(C)\}\mathbb{L}^{-dn+e} = \mu_{\mathcal{X},Z}(f_\infty(C))\mathbb{L}^e\,. \qquad \square$$

Corollary 10.21. *Let $C \subset |\mathrm{J}_\infty \mathcal{Y}|$ be a stable subset. Suppose that $\operatorname{ord}_{\mathrm{Jac}_f}$ and $\mathfrak{j}_\mathcal{X} \circ f_\infty$ are nowhere infinity on C. Then $f_\infty(C)$ is also stable.*

Proof. For $e \in \mathbb{N}$, let C_e be the locus in C with $\operatorname{ord}_{\mathrm{Jac}_f} = e$. This is a stable subset since whether we have $\operatorname{ord}_{\mathrm{Jac}_f}(\gamma) = e$ is determined by the n-jet γ_n. We have $C = \bigsqcup_e C_e$. By [Loo02, Lemma 2.3] (although this paper assumes that the base field has characteristic zero, this lemma is characteristic-free), C is covered by finitely many C_e. Thus $\operatorname{ord}_{\mathrm{Jac}_f}|_C$ is bounded. By Lemma 8.10, the function $\mathfrak{j}_f$ is also bounded. By the same argument, $(\mathfrak{j}_\mathcal{X} \circ f_\infty)|_C$ is also bounded. Let us take $e, a, n \in \mathbb{N}$ such that $n \geqslant 2e+a$, $\mathfrak{j}_f|_C \leqslant e$, $(\mathfrak{j}_\mathcal{X} \circ f_\infty)|_C \leqslant a$ and C is stable of level $n-e$. Let $\gamma \in C$ and $\eta \in |\mathrm{J}_\infty \mathcal{X}|$. If $f_n(\gamma_n) = \eta_n$, then by Lemma 10.16, if $f_n(\gamma_n) = \eta_n$, there exists a $\xi \in |\mathrm{J}_\infty \mathcal{X}|$ such that $f_\infty(\xi) = \eta$ and $\gamma_{n-e} = \xi_{n-e}$. Since C is stable of level $n-e$, we have $\xi \in C$. Therefore,

$\eta \in f_\infty(C)$. We have showed that $f_\infty(C)$ is cylindrical of level n. Since $\mathrm{j}_{\mathcal{X}}|_{f_\infty(C)}$ is bounded, we can deduce from Lemma 10.15 that $f_\infty(C)$ is also stable. $\square$

Lemma 10.22. *Let $C \subset |\mathrm{J}_\infty \mathcal{Y}|$ be a measurable subset. Then $f_\infty(C)$ is also measurable and $\|\mu_{\mathcal{X}}(f_\infty(C))\| \leqslant \|\mu_{\mathcal{Y}}(C)\|$. In particular, if C is negligible, then so is $f_\infty(C)$.*

Proof. Let $C_\infty \subset C$ be the locus where either j_f or $\mathrm{j}_{\mathcal{X}} \circ f_\infty$ has value ∞. Under Assumption-Notation 10.1, by Lemma 8.10, there exists a narrow closed substack $\mathcal{Z} \subset \mathcal{Y}$ such that $C_\infty \subset |\mathrm{J}_\infty \mathcal{Z}|$. Let us show that $f_\infty(|\mathrm{J}_\infty \mathcal{Z}|)$ is negligible, hence so is $f_\infty(C_\infty)$. Since a countable union of negligible subsets is negligible, we may suppose that $\mathcal{Z}$ is contained in $\mathcal{Y} \times_\Psi \Psi_0$ for some connected component Ψ_0 of Ψ. For each geometric point $\psi\colon \operatorname{Spec} K \to \Psi_0$, consider the induced morphism $f_K\colon \mathcal{Y}_K \to \mathcal{X}_K$ of formal DM stacks of finite type over Df_K and the induced narrow substack $\mathcal{Z}_K \subset \mathcal{Y}_K$. The scheme-theoretic image $\mathcal{W}_K$ of $\mathcal{Z}_K$ in $\mathcal{X}_K$ is again narrow. Therefore,

$$\dim \pi_n \circ f_\infty(|\mathrm{J}_\infty \mathcal{Z}_K|) \leqslant \dim \pi_n(|\mathrm{J}_\infty \mathcal{W}_K|) \leqslant (d-1)(n+1)\,.$$

This shows $\dim \pi_n \circ f_\infty(|\mathrm{J}_\infty \mathcal{Z}|) \leqslant (d-1)(n+1) + \dim \Psi_0$. Therefore, the measure of the cylindrical subset $\pi_n^{-1}(\pi_n \circ f_\infty(|\mathrm{J}_\infty \mathcal{Z}|))$ tends to zero as n tends to infinity. Since

$$f_\infty(|\mathrm{J}_\infty \mathcal{Z}|) \subset \bigcap_n \pi_n^{-1}(\pi_n \circ f_\infty(|\mathrm{J}_\infty \mathcal{Z}|))\,,$$

we see that $f_\infty(|\mathrm{J}_\infty \mathcal{Z}|)$ is negligible.

We can write $C \setminus C_\infty = \bigsqcup_{i \in \mathbb{N}} C_i$ such that for each i, the set C_i is measurable and j_f and $\mathrm{j}_{\mathcal{X}} \circ f_\infty$ are bounded on C_i. For each C_i, we can take stable subsets $C_i(\epsilon)$ and $A_{ij}(\epsilon)$, $j \in \mathbb{N}$, of $|\mathrm{J}_\infty \mathcal{Y}|$ as in Definition 10.6 such that j_f and $\mathrm{j}_{\mathcal{X}} \circ f_\infty$ are bounded on each of these stable subsets. We have

$$f_\infty(C_i) \triangle f_\infty(C_i(\epsilon)) \subset f_\infty(C_i \triangle C_i(\epsilon)) \subset \bigcup_j f_\infty(A_{ij}(\epsilon))\,.$$

By Corollary 10.21, the subsets $f_\infty(C_i(\epsilon))$ and $f_\infty(A_{ij}(\epsilon))$ are stable, and we have

$$\|\mu_{\mathcal{X}}(f_\infty(C_i(\epsilon)))\| \leqslant \|\mu_{\mathcal{Y}}(C_i(\epsilon))\| \text{ and } \|\mu_{\mathcal{X}}(f_\infty(A_{ij}(\epsilon))\| \leqslant \|\mu_{\mathcal{Y}}(A_{ij}(\epsilon))\|\,.$$

This shows that $f_\infty(C_i)$ is measurable with $\|\mu_{\mathcal{X}}(f_\infty(C_i))\| \leqslant \|\mu_{\mathcal{X}}(C_i)\|$. Since $\lim_{i\to\infty} \|\mu_{\mathcal{Y}}(C_i)\| = 0$, we have $\lim_{i\to\infty} \|\mu_{\mathcal{X}}(f_\infty(C_i))\| = 0$. Therefore, $f_\infty(C) = \bigsqcup_{i \in \mathbb{N}\cup\{\infty\}} f_\infty(C_i)$ is also measurable. The inequality of the lemma is clear. $\square$

Lemma 10.23. (1) *Let $A \subset |\mathrm{J}_\infty \mathcal{X}|$ be a subset with $A \subset f_\infty(|\mathrm{J}_\infty \mathcal{Y}|)$, and let $h\colon A \to \frac{1}{r}\mathbb{Z} \cup \{\infty\}$ be a function. If $h \circ f_\infty\colon f_\infty^{-1}(A) \to \frac{1}{r}\mathbb{Z} \cup \{\infty\}$ is measurable, then so is h.*

(2) *Let $B \subset |\mathrm{J}_\infty \mathcal{Y}|$ be a measurable subset such that $f_\infty|_B$ is almost geometrically injective. If a function $h\colon f_\infty(B) \to \frac{1}{r}\mathbb{Z} \cup \{\infty\}$ is measurable, then $h \circ f_\infty\colon B \to \frac{1}{r}\mathbb{Z} \cup \{\infty\}$ is also measurable.*

Proof. (1) Let $f_\infty^{-1}(A) = \bigsqcup_{i\in\mathbb{N}} B_i$ be a decomposition as in the definition of measurable functions. Then h is constant on each $f_\infty(B_i)$. By Lemma 10.22, the subsets $f_\infty(B_i)$ are measurable, and $\mu_{\mathcal{X}}(f_\infty(B_i)) = 0$ if $\mu_{\mathcal{Y}}(B_i) = 0$. If we put $A_i := f_\infty(B_i) \setminus \bigcup_{j<i} f_\infty(B_j)$, we have

$$A = \bigcup_{i\in\mathbb{N}} f_\infty(B_i) = \bigsqcup_{i\in\mathbb{N}} A_i\,.$$

This decomposition of A shows that h is measurable.

(2) Removing the negligible subset $(\mathrm{ord}_{\mathrm{Jac}_f})^{-1}(\infty) \cup (\mathrm{j}_{\mathcal{X}} \circ f_\infty)^{-1}(\infty)$, we may suppose that these two functions are nowhere infinity. For $e, a \in \mathbb{N}$, let

$$B_{e,a} := (\mathrm{ord}_{\mathrm{Jac}_f})^{-1}(e) \cap (\mathrm{j}_{\mathcal{X}} \circ f_\infty)^{-1}(a)\,,$$

which is a measurable subset. Then $h|_{f_\infty(B_{e,a})}\colon f_\infty(B) \to \frac{1}{r}\mathbb{Z} \cup \{\infty\}$ is a measurable function. Let $f_\infty(B_{e,a}) = \bigsqcup_i C_i$ be a decomposition into countably many measurable subsets such that for every i, the restriction $h|_{C_i}$ is constant and if $h|_{C_i} \equiv 0$, then C_i is negligible. From Lemma 10.19, we deduce that $f_\infty^{-1}(C_i)$ is also measurable and if C_i is negligible, so is $f_\infty^{-1}(C_i)$. This shows the assertion. □

Lemma 10.24. *Let $\mathcal{I} \subset \mathcal{O}_\mathcal{X}$ be an ideal sheaf defining a narrow closed substack. Then*

$$\mathrm{ord}_\mathcal{I}\colon |\mathrm{J}_\infty\mathcal{X}| \to \mathbb{Z} \cup \{\infty\}$$

is a measurable function.

Proof. It suffices to show that for each connected component Υ of Φ, the restriction of $\mathrm{ord}_\mathcal{I}$ to $(\mathrm{J}_\infty\,\mathcal{X}) \times_\Phi \Upsilon$ is measurable. Therefore, we may suppose that Φ is of finite type. Let $C_{a,b} := \{\mathrm{ord}_\mathcal{I} = a,\ \mathfrak{j}_\mathcal{X} = b\}$. We have $|\mathrm{J}_\infty\mathcal{X}| = \bigsqcup_{0\leqslant a,b \leqslant \infty} C_{a,b}$. If $a = \infty$ or $b = \infty$, then $C_{a,b}$ is negligible. Otherwise, $C_{a,b}$ is stable of level $\max\{a, b\}$ by Lemma 10.15. The lemma follows. □

Lemma 10.25. *With the above notation, the function $\mathfrak{j}_f\colon |\mathrm{J}_\infty\mathcal{Y}| \to \mathbb{Z} \cup \{\infty\}$ is measurable .*

Proof. Let $h\colon \mathcal{Z} \to \mathcal{Y}$ be the blowup along $\Omega^d_{\mathcal{Y}/\mathrm{Df}_\Psi} \oplus f^*\Omega^d_{\mathcal{X}/\mathrm{Df}_\Phi}$ (in the sense of [Vil06]), so that

$$h^\flat\Omega^d_{\mathcal{Y}/\mathrm{Df}_\Psi} := h^*\Omega^d_{\mathcal{Y}/\mathrm{Df}_\Psi}/\mathrm{tors}\,,$$
$$(f\circ h)^\flat\Omega^d_{\mathcal{X}/\mathrm{Df}_\Phi} := (f\circ h)^*\Omega^d_{\mathcal{X}/\mathrm{Df}_\Phi}/\mathrm{tors}$$

are invertible sheaves. By Assumption-Notation 10.1, the morphism $h_\infty\colon |\mathrm{J}_\infty\mathcal{Z}| \to |\mathrm{J}_\infty\mathcal{Y}|$ is almost geometrically bijective. The image of $(f\circ h)^\flat\Omega^d_{\mathcal{X}/\mathrm{Df}_\Phi} \to h^\flat\Omega^d_{\mathcal{Y}/\mathrm{Df}_\Psi}$ is written as $\mathcal{I}\cdot h^\flat\Omega^d_{\mathcal{Y}/\mathrm{Df}_\Psi}$ for some locally principal ideal sheaf $\mathcal{I}$, which defines a narrow substack. Then $\mathfrak{j}_f \circ h_\infty$ is equal to $\mathrm{ord}_\mathcal{I}$ outside the negligible subset $\{\mathfrak{j}_\mathcal{X} \circ f_\infty \circ h_\infty = \infty\} \cup \{\mathfrak{j}_\mathcal{Y} \circ h_\infty = \infty\}$. The function $\mathrm{ord}_\mathcal{I}$ is measurable. By Lemma 10.23, the function $\mathfrak{j}_f$ is measurable. □

The following is the *change of variables formula* (also called the *transformation rule*) for untwisted arcs in the present setting.

Theorem 10.26. *We keep Assumption-Notation 10.1. Let $A \subset |\mathrm{J}_\infty\mathcal{Y}|$ be a measurable subset. Suppose that $f_\infty|_A$ is almost geometrically injective. Let $h\colon f_\infty(A) \to \frac{1}{r}\mathbb{Z} \cup \{\infty\}$ be a function such that $h \circ f_\infty$ is measurable. Then*

$$\int_{f_\infty(A)} \mathbb{L}^h \, d\mu_{\mathcal{X},Z} = \int_A \mathbb{L}^{h\circ f_\infty - \mathfrak{j}_f}\, d\mu_{\mathcal{Y},Z}\,.$$

Proof. Removing a negligible subset from A, we may suppose that all relevant measurable functions are nowhere infinity. There exists a decomposition $A = \bigsqcup_i A_i$ into countably many measurable subsets A_i such that $h \circ f_\infty$ and $\mathfrak{j}_f$ are constant on each A_i. By Lemma 10.19,

$$\begin{aligned}\int_{f_\infty(A)} \mathbb{L}^h \, d\mu_{\mathcal{X},Z} &= \sum_i \mu_{\mathcal{X},Z}(f_\infty(A_i))\mathbb{L}^{h(f_\infty(A_i))}\\ &= \sum_i \mu_{\mathcal{Y},Z}(A_i)\mathbb{L}^{h(f_\infty(A_i))-\mathfrak{j}_f(A_i)} = \int_A \mathbb{L}^{h\circ f_\infty - \mathfrak{j}_f}\, d\mu_{\mathcal{Y},Z}\,.\end{aligned}$$ □

11. Motivic integration for twisted arcs

In this section, we develop the motivic integration over formal DM stacks for twisted arcs. Using untwisting stacks, this is reduced to the motivic integration for untwisted arcs.

Definition 11.1. Let $\mathcal{X}$ be a formal DM stack of finite type over Df, and let $\mathrm{I}\,\mathcal{X}$ be its inertia stack with trivial section $\epsilon\colon \mathcal{X} \to \mathrm{I}\,\mathcal{X}$. We define the *stacky locus* $\mathcal{X}_{\mathrm{st}}$ of $\mathcal{X}$ to be the scheme-theoretic image of the finite morphism $\mathrm{I}\,\mathcal{X} \setminus \epsilon(\mathcal{X}) \to \mathcal{X}$. We say that $\mathcal{X}$ is *spacious* if $\mathcal{X}_{\mathrm{st}}$ is a narrow closed substack of $\mathcal{X}$.

In the rest of this section, we make the following assumption.

Assumption-Notation 11.2. Let $\mathcal{X}$ and $\mathcal{Y}$ be formal DM stacks of finite type, flat and of pure relative dimension d over Df. Suppose that they are spacious and generically smooth over Df. Let $f\colon \mathcal{Y} \to \mathcal{X}$ be a morphism over Df such that the Jacobian ideal sheaf Jac_f defines a narrow closed substack of $\mathcal{Y}$.

Let X be the coarse moduli space of $\mathcal{X}$, and let $\overline{\mathcal{X}_{\mathrm{st}}} \subset X$ be the scheme-theoretic image of $\mathcal{X}_{\mathrm{st}}$.

Lemma 11.3. *The subspace $\overline{\mathcal{X}_{\mathrm{st}}} \subset X$ is narrow.*

Proof. By Lemma 2.18, we can reduce the lemma to the case where $\mathcal{X}$ is a quotient stack $[\mathrm{Spf}\,R/G]$ with G a finite group. Let $I \subset R$ be the ideal defining the preimage of $\mathcal{X}_{\mathrm{st}}$ in $\mathrm{Spf}\,R$. The assumption that $\mathcal{X}$ is spacious means that $V(I) \subset \mathrm{Spec}\,R$ is nowhere dense. Then the image of $V(I)$ in $\mathrm{Spec}\,R^G$ is also nowhere dense. This shows the lemma. □

From the morphism $\pi_\Gamma^{\mathrm{utg}}\colon \mathrm{Utg}_\Gamma(\mathcal{X})^{\mathrm{pur}} \to X$ as in Proposition 6.36, we obtain maps

$$|\mathcal{J}_\infty \mathcal{X}| \to |\mathrm{J}_\infty X| \tag{11.1}$$

and

$$(\mathcal{J}_\infty \mathcal{X})[K] \to (\mathrm{J}_\infty X)(K) \tag{11.2}$$

for each algebraically closed field K. These maps send a twisted arc $\mathcal{E} \to \mathcal{X}$ to the arc of X induced by taking coarse moduli spaces. The notion of twisted arcs is necessary for the following proposition to holds.

Proposition 11.4. *Let $\overline{\mathcal{X}_{\mathrm{st}}} \subset X$ be the scheme-theoretic image of $\mathcal{X}_{\mathrm{st}}$. For every algebraically closed field K, the map*

$$(\mathcal{J}_\infty \mathcal{X})[K] \setminus (\mathcal{J}_\infty \mathcal{X}_{\mathrm{st}})[K] \to (\mathrm{J}_\infty X)(K) \setminus \big(\mathrm{J}_\infty \overline{\mathcal{X}_{\mathrm{st}}}\big)(K)$$

is bijective. In particular, the map $|\mathcal{J}_\infty \mathcal{X}| \to |\mathrm{J}_\infty X|$ is almost geometrically bijective.

Proof. First note that if a twisted arc $\mathcal{E} \to \mathcal{X}$ with $\mathcal{E}$ a twisted formal disk over K does not factor through $\mathcal{X}_{\mathrm{st}}$, then the induced arc $\mathrm{Df}_K \to X$ does not factor through $\overline{\mathcal{X}_{\mathrm{st}}}$ either. Therefore, we have the map of the lemma. Conversely, given an arc $\mathrm{Df}_K \to X$ which does not factor through $\overline{\mathcal{X}_{\mathrm{st}}}$, we can construct a twisted arc of $\mathcal{X}$ as follows. Let $\mathcal{E}$ be the normalization of $(\mathcal{X} \times_X \mathrm{Df}_K)^{\mathrm{pur}}$. Since $\mathcal{X}$ is spacious, so is $\mathcal{E}$. We conclude that $\mathcal{E}$ is a twisted formal disk over K. The morphism $\mathcal{E} \to \mathcal{X}$ is a twisted arc over K, which maps to the given arc $\mathrm{Df}_K \to X$. Therefore, the map of the lemma is surjective.

To show the injectivity, let $\mathcal{E}' \to \mathcal{X}$ be a twisted arc and $\mathrm{Df}_K \to X$ the induced arc. Let $\mathcal{E} \to \mathcal{X}$ be the twisted arc induced by this arc on X by the above construction. Let us show that the induced morphism $\mathcal{E}' \to \mathcal{E}$ is an isomorphism. Let E and E' be Galois covers of Df_K corresponding to $\mathcal{E}$ and $\mathcal{E}'$, respectively. Since the morphism $\mathcal{E}' \to \mathcal{E}$ is representable, the Galois group of $E' \to \mathrm{Df}_K$ can be embedded into the one of $E \to \mathrm{Df}_K$. In particular, the degree of $E \to \mathrm{Df}_K$ is greater than or equal to that of $E' \to \mathrm{Df}_K$. The morphism $E' \times_{\mathcal{E}} E \to E'$ is étale and finite. Therefore, each component of $E' \times_{\mathcal{E}} E$ is isomorphic to E'. For a component F, the morphism

$E' \cong F \to E$ is a finite dominant Df-morphism. But, the degree of $E \to \mathrm{Df}_K$ is greater than or equal to that of $E' \to \mathrm{Df}_K$. Therefore, the morphism $E' \to E$ is an isomorphism. This shows that the morphism $\mathcal{E}' \to \mathcal{E}$ is an isomorphism. As a consequence, the isomorphism class of the twisted arc $\mathcal{E} \to \mathcal{X}$ is determined by the induced arc $\mathrm{Df}_K \to X$. This implies the desired injectivity. We have proved the first assertion of the proposition. The second assertion immediately follows. □

Definition 11.5. Let $\Gamma = \Gamma_{\mathcal{X}}$ be as in Definition 6.34. Let $\Gamma \to Z$ be a morphism with Z an algebraic space almost of finite type. We define the *motivic measure* $\mu_{\mathcal{X},Z}$ on $|\mathcal{J}_\infty \mathcal{X}| = |\mathrm{J}_\infty \mathrm{Utg}_\Gamma(\mathcal{X})^{\mathrm{pur}}|$ to be $\mu_{\mathrm{Utg}_\Gamma(\mathcal{X})^{\mathrm{pur}}}$. For a measurable function $h\colon A \to \frac{1}{r}\mathbb{Z} \cup \{\infty\}$ on a subset $A \subset |\mathcal{J}_\infty \mathcal{X}|$, the integral $\int_A \mathbb{L}^h \,\mathrm{d}\mu_{\mathcal{X},Z} \in \widehat{\mathcal{M}'_{Z,r}} \cup \{\infty\}$ is defined as in Definition 10.9.

Note that this motivic measure is independent of the choice of Γ.

Consider the following commutative diagram of natural morphisms among formal DM stacks over Df_Γ:

$$\begin{tikzcd} \mathrm{Utg}_\Gamma(\mathcal{X})^{\mathrm{pur}} \times_{\mathrm{Df}_\Gamma} \mathcal{E}_\Gamma \arrow[r, "s"] \arrow[d, "r"'] & \mathrm{Utg}_\Gamma(\mathcal{X})^{\mathrm{pur}} \arrow[d, "\pi^{\mathrm{utg}}"] \\ \mathcal{X} \arrow[r, "\pi"'] & X\,. \end{tikzcd} \tag{11.3}$$

Here r is the universal morphism, and s is the projection. A twisted arc $\gamma\colon \mathcal{E} \to \mathcal{X}$, say over an algebraically closed field K, corresponds to an arc $\gamma'\colon \mathrm{Df}_K \to \mathrm{Utg}_K(\mathcal{X})^{\mathrm{pur}}$. These lift to the same $\mathcal{E}$-morphism $\tilde{\gamma}\colon \mathcal{E} \to \mathrm{Utg}_K(\mathcal{X})^{\mathrm{pur}} \times_{\mathrm{Df}_K} \mathcal{E}$. We define the Jacobian ideal sheaves $\mathfrak{j}_s$ and $\mathfrak{j}_r$, regarding $\mathrm{Utg}_\Gamma(\mathcal{X})^{\mathrm{pur}} \times_{\mathrm{Df}_\Gamma} \mathcal{E}_\Gamma$ and $\mathrm{Utg}_\Gamma(\mathcal{X})^{\mathrm{pur}}$ as formal DM stacks over Df_Γ and regarding $\mathcal{X}$ as a formal DM stack over Df.

Definition 11.6. We define the *shift number* $\mathfrak{s}_{\mathcal{X}}(\gamma)$ of γ to be

$$\mathfrak{s}_{\mathcal{X}}(\gamma) := \mathfrak{j}_s(\tilde{\gamma}) - \mathfrak{j}_r(\tilde{\gamma}) \in \mathbb{Z}\,.$$

Note that if $\mathcal{X}$ is a formal algebraic space, then $\mathfrak{s}_{\mathcal{X}} \equiv 0$. This follows from

$$\mathrm{Utg}_\Gamma(\mathcal{X})^{\mathrm{pur}} = \mathrm{Utg}_{\Gamma_{[1]}}(\mathcal{X})^{\mathrm{pur}} = \mathcal{X}\,.$$

Since r and s are rig-étale (see Lemma 6.22), we always have $\mathfrak{j}_s(\tilde{\gamma}) \neq \infty$ and $\mathfrak{j}_r(\tilde{\gamma}) \neq \infty$. If $\mathfrak{j}_\pi(\gamma) = \mathfrak{j}_{\tilde{\pi}}(\gamma) < \infty$, then $\mathfrak{j}_{\pi'}(\gamma) < \infty$. By Lemma 8.11, outside the negligible subset $\mathfrak{j}_\pi^{-1}(\infty)$, we have

$$\mathfrak{s}_{\mathcal{X}}(\gamma) = \mathfrak{j}_\pi(\gamma) - \mathfrak{j}_{\pi^{\mathrm{utg}}}(\gamma')\,. \tag{11.4}$$

Lemma 11.7. *Let $\mathcal{I} \subset \mathcal{O}_{\mathcal{X}}$ be an ideal sheaf defining a narrow closed substack. Then the function $\mathrm{ord}_{\mathcal{I}}$ on $|\mathcal{J}_\infty \mathcal{X}|$ is measurable.*

Proof. Note that for the case of formal schemes, the lemma is well known and easy to show. It is straightforward to generalize it to formal DM stacks as far as *untwisted* arcs are concerned. We will reduce the lemma to the case of untwisted arcs.

It suffices to show that for each connected component $\Gamma_0 \subset \Gamma$ and $a \in \mathbb{Q}_{\geqslant 0}$, the set

$$C := (\mathrm{ord}_{\mathcal{I}})^{-1}(a) \cap |\mathcal{J}_{\Gamma_0,\infty}\mathcal{X}|$$

is cylindrical. Whether a twisted arc $\gamma\colon \mathcal{E} \to \mathcal{X}$ has the value $\mathrm{ord}_{\mathcal{I}} = a$ is determined by the induced twisted n-jet for $n \geqslant \lfloor a \rfloor$. Namely, we have $C = \pi_n^{-1}\pi_n(C)$. We need to show that for $n \gg 0$, the subset $C_n := \pi_n(C)$ is constructible. To do this, let us take an atlas

$\operatorname{Spec} R \to \Gamma_0$ such that the corresponding torsor over D_R^* is uniformizable. Then we can write $\mathcal{E}_R = [\operatorname{Spf} R[\![s]\!]/G]$. For each n, we have maps

$$\begin{tikzcd} \mathcal{J}_{\operatorname{Spec} R,n}\mathcal{X} \arrow[r, "\alpha"] \arrow[d, "\beta"'] & \mathcal{J}_{\Gamma_0,n}\mathcal{X} \\ \mathrm{J}_{\sharp G\cdot(n+1)-1}(\mathcal{X} \times_{\mathrm{Df}} \operatorname{Spf} k[\![s]\!]) . & \end{tikzcd}$$

The map α is surjective. The map β sends a twisted n-jet $\left[\left(\operatorname{Spec} K[\![s]\!]/\left(t^{n+1}\right)\right)/G\right] \to \mathcal{X}$ of $\mathcal{X}$ to the induced untwisted $(\sharp G(n+1)-1)$-jet of $\mathcal{X} \times_{\mathrm{Df}} \operatorname{Spf} k[\![s]\!]$

$$\operatorname{Spec} K[\![s]\!]/\left(t^{n+1}\right) = \operatorname{Spec} K[\![s]\!]/\left(s^{\sharp G(n+1)}\right) \to \left[\left(\operatorname{Spec} K[\![s]\!]/\left(t^{n+1}\right)\right)/G\right] \to \mathcal{X} \times_{\mathrm{Df}} \operatorname{Spf} k[\![s]\!] .$$

Let $\mathcal{I}'$ be the pullback of $\mathcal{I}$ to $\mathcal{X} \times_{\mathrm{Df}} \operatorname{Spf} k[\![s]\!]$. For $n \gg 0$, let $B_a \subset |\mathrm{J}_{\sharp G\cdot(n+1)-1}(\mathcal{X} \times_{\mathrm{Df}} \operatorname{Spf} k[\![s]\!])|$ be the locus where $\operatorname{ord} \mathcal{I}'$ has value $\sharp G \cdot a$, which is constructible. We see that $C_a = \alpha\left(\beta^{-1}(B_a)\right)$ is also constructible, which completes the proof. □

Lemma 11.8. *The Jacobian order function $\mathfrak{j}_\pi$ on $|\mathcal{J}_\infty \mathcal{X}|$ associated with the coarse moduli space $\pi\colon \mathcal{X} \to X$ is measurable.*

Proof. As in the proof of Lemma 10.25, we take a blowup $h\colon \mathcal{Z} \to \mathcal{X}$ such that $h^\flat \Omega^d_{\mathcal{X}/\mathrm{Df}}$ and $(\pi \circ h)^\flat \Omega^d_{X/\mathrm{Df}}$ are invertible sheaves. The image of $(\pi \circ h)^\flat \Omega^d_{X/\mathrm{Df}} \to h^\flat \Omega^d_{\mathcal{X}/\mathrm{Df}}$ is written as $\mathcal{I} \cdot h^\flat \Omega^d_{\mathcal{X}/\mathrm{Df}}$ for some locally principal ideal sheaf $\mathcal{I} \subset \mathcal{O}_{\mathcal{Z}}$ that defines a narrow substack. Then $\mathfrak{j}_\pi \circ h_\infty$ is equal to $\operatorname{ord}_{\mathcal{I}}$ outside a negligible subset. Now the lemma follows from Lemmas 11.7 and 10.23. □

Corollary 11.9. *The function $\mathfrak{s}_{\mathcal{X}}$ on $|\mathcal{J}_\infty \mathcal{X}|$ is measurable.*

Proof. The functions $\mathfrak{j}_\pi$ and $\mathfrak{j}_{\pi'}$ are measurable by Lemmas 11.8 and 10.25, respectively. From (11.4), we deduce that $\mathfrak{s}_{\mathcal{X}}$ is also measurable. □

Lemma 11.10. *Outside a negligible subset of $|\mathcal{J}_\infty \mathcal{Y}|$, we have*

$$\mathfrak{s}_{\mathcal{Y}} - \mathfrak{s}_{\mathcal{X}} \circ f_\infty = \mathfrak{j}_f - \mathfrak{j}_{(\pi \circ f)^{\mathrm{utg}}} + \mathfrak{j}_{\pi^{\mathrm{utg}}} \circ f_\infty .$$

In particular, if $\mathcal{X}$ is a formal algebraic space, then $\mathfrak{s}_{\mathcal{Y}} = \mathfrak{j}_f - \mathfrak{j}_{f^{\mathrm{utg}}}$.

Proof. In this proof, we ignore negligible subsets, and equalities below hold outside some negligible subset. First consider the case where $\mathcal{X}$ is a formal algebraic space. Given a twisted arc γ on $\mathcal{Y}$, we define $\tilde{\gamma}$ and γ' as above. Consider a diagram similar to (11.3) with f in place of π. By Lemma 8.11, we have

$$\mathfrak{j}_f(\gamma) + \mathfrak{j}_r(\tilde{\gamma}) = \mathfrak{j}_{f\circ r}(\tilde{\gamma}) = \mathfrak{j}_{f^{\mathrm{utg}} \circ s}(\tilde{\gamma}) = \mathfrak{j}_s(\tilde{\gamma}) + \mathfrak{j}_{f^{\mathrm{utg}}}(\gamma') .$$

Therefore,

$$\mathfrak{s}_{\mathcal{Y}}(\gamma) = \mathfrak{j}_s(\tilde{\gamma}) - \mathfrak{j}_r(\tilde{\gamma}) = \mathfrak{j}_f(\gamma) - \mathfrak{j}_{f^{\mathrm{utg}}}(\gamma') .$$

Thus the second assertion of the lemma holds. We then apply this to π and $\pi \circ f$ to get $\mathfrak{s}_{\mathcal{Y}} = \mathfrak{j}_{\pi\circ f} - \mathfrak{j}_{(\pi\circ f)^{\mathrm{utg}}}$ and $\mathfrak{s}_{\mathcal{X}} = \mathfrak{j}_\pi - \mathfrak{j}_{\pi^{\mathrm{utg}}}$. Therefore,

$$\begin{aligned} \mathfrak{s}_{\mathcal{Y}} - \mathfrak{s}_{\mathcal{X}} \circ f_\infty &= \mathfrak{j}_{\pi\circ f} - \mathfrak{j}_\pi \circ f_\infty - \mathfrak{j}_{(\pi\circ f)^{\mathrm{utg}}} + \mathfrak{j}_{\pi^{\mathrm{utg}}} \circ f_\infty \\ &= \mathfrak{j}_f - \mathfrak{j}_{(\pi\circ f)^{\mathrm{utg}}} + \mathfrak{j}_{\pi^{\mathrm{utg}}} \circ f_\infty , \end{aligned}$$

where the last equality follows from Lemma 8.11. We have obtained the first equality of the lemma. □

Remark 11.11. If we constructed $f^{\mathrm{utg}}\colon \mathrm{Utg}_{\Gamma_{\mathcal{Y}}}(\mathcal{Y})^{\mathrm{pur}} \to \mathrm{Utg}_{\Gamma_{\mathcal{X}}}(\mathcal{X})^{\mathrm{pur}}$ (see Remark 6.38), then we would get $\mathfrak{s}_{\mathcal{Y}} - \mathfrak{s}_{\mathcal{X}} \circ f_\infty = \mathfrak{j}_f - \mathfrak{j}_{f^{\mathrm{utg}}}$ in place of the equality of Lemma 11.10.

Corollary 11.12. *The function $\mathfrak{j}_f$ on $|\mathcal{J}_\infty \mathcal{Y}|$ is measurable.*

Proof. By Lemma 11.10, outside a negligible subset, we have

$$\mathfrak{j}_f = \mathfrak{s}_{\mathcal{Y}} - \mathfrak{s}_{\mathcal{X}} \circ f_\infty + \mathfrak{j}_{(\pi\circ f)^{\mathrm{utg}}} - \mathfrak{j}_{\pi^{\mathrm{utg}}} \circ f_\infty\,.$$

By Lemmas 10.23 and 10.25 and Corollary 11.9, the four functions on the right side are measurable. Therefore, $\mathfrak{j}_f$ is also measurable. □

Generalizing maps (11.1) and (11.2) associated with $\pi\colon \mathcal{X} \to X$, we now define maps $|\mathcal{J}_\infty \mathcal{Y}| \to |\mathcal{J}_\infty \mathcal{X}|$ and $(\mathcal{J}_\infty \mathcal{Y})[K] \to (\mathcal{J}_\infty \mathcal{X})[K]$ associated with the given morphism $f\colon \mathcal{Y} \to \mathcal{X}$. For a twisted arc $\gamma\colon \mathcal{E} \to \mathcal{Y}$ over an algebraically closed field K, let

$$\mathcal{E} \to \mathcal{E}' \xrightarrow{\gamma'} \mathcal{X}$$

be the canonical factorization of the composition $\mathcal{E} \to \mathcal{Y} \to \mathcal{X}$ as in [Yas06, Lemma 25] (cf. [AOV11, Theorem 3.1]) so that γ' is a twisted arc. Note that the paper [Yas06] treats the non-formal situation, but it is straightforward to generalize it to the formal situation. If $\mathcal{E} = [E/G]$ for a G-cover $E \to \mathrm{Df}_K$ and if y and x denote the induced K-points of $\mathcal{Y}$ and $\mathcal{X}$, respectively, then we have maps

$$G \to \mathrm{Aut}_{\mathcal{Y}}(y) \to \mathrm{Aut}_{\mathcal{X}}(x)\,,$$

the left one being injective. Let $N \lhd G$ be the kernel of this composite map. Then

$$\mathcal{E}' \cong [(E/N)/(G/N)]\,.$$

We have the map

$$(\mathcal{J}_\infty \mathcal{Y})[K] \to (\mathcal{J}_\infty \mathcal{X})[K], \qquad [\gamma] \mapsto [\gamma']$$

as well as

$$|\mathcal{J}_\infty \mathcal{Y}| \to |\mathcal{J}_\infty \mathcal{X}|\,.$$

We denote these maps by f_∞. We say that $f_\infty|_A$ is *geometrically injective* if for every algebraically closed field K, the map $f_\infty|_{A[K]}$ is injective. We say that $f_\infty|_A$ is *almost geometrically injective* if there exists a negligible subset A' such that $f_\infty|_{A\setminus A'}$ is geometrically injective.

The following is the *change of variables formula* for twisted arcs.

Theorem 11.13. *We keep Assumption-Notation 11.2. Let $A \subset |\mathcal{J}_\infty \mathcal{Y}|$ be a subset such that $f_\infty|_A$ is almost geometrically injective. Let $h\colon f_\infty(A) \to \frac{1}{r}\mathbb{Z} \cup \{\infty\}$ be a measurable function. Then*

$$\int_{f_\infty(A)} \mathbb{L}^{h+\mathfrak{s}_{\mathcal{X}}}\, d\mu_{\mathcal{X}} = \int_A \mathbb{L}^{h\circ f_\infty - \mathfrak{j}_f + \mathfrak{s}_{\mathcal{Y}}}\, d\mu_{\mathcal{Y}} \in \widehat{\mathcal{M}'_{k,r}} \cup \{\infty\}\,.$$

Proof. First consider the case where $\mathcal{X}$ is a formal algebraic space. By Lemma 6.22, the morphism $f^{\mathrm{utg}}\colon \mathrm{Utg}_{\Gamma_{\mathcal{Y}}}(\mathcal{Y})^{\mathrm{pur}} \to \mathcal{X}$ satisfies the assumption of Assumption-Notation 10.1 with $\Psi = \Gamma_{\mathcal{Y}}$ and $\Phi = \operatorname{Spec} k$.

By definition, $|\mathcal{J}_\infty \mathcal{Y}|$ is identical to $|\mathrm{J}_\infty \mathrm{Utg}_{\Gamma_{\mathcal{Y}}}(\mathcal{Y})^{\mathrm{pur}}|$. Moreover, the two measures $\mu_{\mathrm{Utg}_{\Gamma_{\mathcal{Y}}}(\mathcal{Y})^{\mathrm{pur}}}$ and $\mu_{\mathcal{Y}}$ on this space are equal. Applying Theorem 10.26 to f^{utg}, we obtain

$$\int_{f_\infty(A)} \mathbb{L}^{h}\, d\mu_{\mathcal{X}} = \int_A \mathbb{L}^{h\circ f_\infty - \mathfrak{j}_{f^{\mathrm{utg}}}}\, d\mu_{\mathrm{Utg}_{\Gamma_{\mathcal{Y}}}(\mathcal{Y})^{\mathrm{pur}}} = \int_A \mathbb{L}^{h\circ f_\infty - \mathfrak{j}_{f^{\mathrm{utg}}}}\, d\mu_{\mathcal{Y}}\,.$$

By Lemma 11.10, this is equal to $\int_A \mathbb{L}^{h\circ f_\infty - \mathfrak{j}_f + \mathfrak{s}_\mathcal{Y}}\, d\mu_\mathcal{Y}$. Thus the theorem holds in this case.

For the general case, let $\pi\colon \mathcal{X} \to X$ be the coarse moduli space. Let $\overline{h}$ and $\overline{\mathfrak{j}_\pi}$ be the functions corresponding to h and $\mathfrak{j}_\pi$, respectively, through the almost geometric bijection $\pi_\infty\colon |\mathcal{J}_\infty\mathcal{X}| \to |\mathrm{J}_\infty X|$ (see Lemma 11.4). By Lemma 8.11, outside a negligible subset, we have

$$\overline{\mathfrak{j}_\pi} \circ (\pi\circ f)_\infty - \mathfrak{j}_{\pi\circ f} = \mathfrak{j}_\pi \circ f_\infty - \mathfrak{j}_{\pi\circ f} = -\mathfrak{j}_f\,.$$

Therefore, by the special case discussed above, we have

$$\begin{aligned}\int_A \mathbb{L}^{h\circ f_\infty - \mathfrak{j}_f + \mathfrak{s}_\mathcal{Y}}\, d\mu_\mathcal{Y} &= \int_A \mathbb{L}^{\overline{h}\circ(\pi\circ f)_\infty + \overline{\mathfrak{j}_\pi}\circ(\pi\circ f)_\infty - \mathfrak{j}_{\pi\circ f} + \mathfrak{s}_\mathcal{Y}}\, d\mu_\mathcal{Y} = \int_{(\pi\circ f)_\infty(A)} \mathbb{L}^{\overline{h}+\overline{\mathfrak{j}_\pi}}\, d\mu_X \\ &= \int_{f_\infty(A)} \mathbb{L}^{\overline{h}\circ\pi_\infty + \overline{\mathfrak{j}_\pi}\circ\pi_\infty - \mathfrak{j}_\pi + \mathfrak{s}_\mathcal{X}}\, d\mu_\mathcal{X} = \int_{f_\infty(A)} \mathbb{L}^{h+\mathfrak{s}_\mathcal{X}}\, d\mu_\mathcal{X}\,. \qquad \square\end{aligned}$$

12. Log pairs

In birational geometry, one often consider log pairs; a log pair means the pair of a normal variety X and a $\mathbb{Q}$-divisor D on X such that $K_X + D$ is $\mathbb{Q}$-Cartier. In this section, we generalize this notion by allowing X to be a formal DM stack. In the next section, we will associate an invariant with a generalized log pair and apply the change of variables formula to study properties of this invariant.

Let $\mathcal{X}$ be a formal stack over Df_Φ as in Assumption-Notation 10.1. Moreover, we suppose that $\mathcal{X}$ is normal; the local rings $\mathcal{O}_{\mathcal{X},x}$ of all geometric points are normal.

Definition 12.1. A closed substack $\mathcal{H} \subset \mathcal{X}$ is said to be *irreducible* if for any closed substacks $\mathcal{H}_i \subset \mathcal{X}$, $i = 1, 2$, such that $\mathcal{H} = \mathcal{H}_1 \cup \mathcal{H}_2$, we have either $\mathcal{H} = \mathcal{H}_1$ or $\mathcal{H} = \mathcal{H}_2$. A *prime divisor* on $\mathcal{X}$ means an irreducible and reduced closed substack $B \subset \mathcal{X}$ of codimension 1. A *divisor* (respectively, $\mathbb{Q}$-*divisor*) A on $\mathcal{X}$ is a formal combination $\sum_{i=1}^{l} c_i A_i$, $c_i \in \mathbb{Z}$ (respectively, $c_i \in \mathbb{Q}$), of prime divisors A_i on $\mathcal{X}$. We say that a ($\mathbb{Q}$-)divisor $A = \sum_i c_i A_i$ is *narrow* if for every i with $c_i \neq 0$, the closed substack $A_i \subset \mathcal{X}$ is narrow. The support $\operatorname{Supp} A$ of A is defined to be the reduced closed substack $\bigcup_{c_i\neq 0} A_i$.

We need to consider fractional ideals of rings which are not necessarily integral domains; for this notion which is not so common in this generality, we refer the reader to [LM71].

Definition 12.2. Let $\mathcal{S} \subset \mathcal{O}_\mathcal{X}$ be the subsheaf of sections which are not zerodivisors at the stalk $\mathcal{O}_{\mathcal{X},x}$ of every geometric point $x\colon \operatorname{Spec} K \to \mathcal{X}$. We define the *sheaf of total quotient rings*, denoted by $\mathcal{K}_\mathcal{X}$, to be the sheaf associated with the presheaf $U \mapsto \mathcal{S}(U)^{-1}\mathcal{O}_\mathcal{X}(U)$. A *fractional ideal sheaf* on $\mathcal{X}$ is a coherent $\mathcal{O}_\mathcal{X}$-submodule $\mathcal{L} \subset \mathcal{K}_\mathcal{X}$ such that for every geometric point $\operatorname{Spec} K \to \mathcal{X}$, there exist an étale neighborhood $U \to \mathcal{X}$ and an $f \in \mathcal{S}(U)$ such that $f\cdot \mathcal{L}|_U \subset \mathcal{O}_\mathcal{X}|_U$.

Let $\mathcal{F}$ be a torsion-free coherent sheaf on $\mathcal{X}$ of generic rank 1. A *fractional submodule* of $\mathcal{F}$ is a coherent $\mathcal{O}_\mathcal{X}$-submodule $\mathcal{L}$ of $\mathcal{S}^{-1}\mathcal{F} = \mathcal{F}\otimes_{\mathcal{O}_\mathcal{X}} \mathcal{K}_\mathcal{X}$ such that for every geometric point $\operatorname{Spec} K \to \mathcal{X}$, there exist an étale neighborhood $U \to \mathcal{X}$ and an $f \in \mathcal{S}(U)$ such that $f\cdot\mathcal{L}|_U \subset \mathcal{F}|_U$.

Definition 12.3. For a divisor $A = \sum_i c_i A_i$, we define a fractional ideal sheaf $\mathcal{O}_\mathcal{X}(A)$ in the usual way: a local section s of $\mathcal{K}_\mathcal{X}$ belongs to $\mathcal{O}_\mathcal{X}(A)$ if and only if for each i, the order of s along A_i is at least $-c_i$.

Definition 12.4. We define the *canonical sheaf* $\omega_{\mathcal{X}/\mathrm{Df}_\Phi}$ to be $\big(\Omega^d_{\mathcal{X}/\mathrm{Df}_\Phi}\big)^{\vee\vee}$, the double dual (that is, the reflexive hull) of $\Omega^d_{\mathcal{X}/\mathrm{Df}_\Phi}$. For $r\in\mathbb{N}$, we define $\omega^{[r]}_{\mathcal{X}/\mathrm{Df}_\Phi}$ to be the reflexive rth power $\big((\omega_{\mathcal{X}/\mathrm{Df}_\Phi})^{\otimes r}\big)^{\vee\vee}$. For a divisor A, we define $\omega^{[r]}_{\mathcal{X}/\mathrm{Df}_\Phi}(A)$ to be the double dual $\big(\omega^{[r]}_{\mathcal{X}/\mathrm{Df}_\Phi}\otimes\mathcal{O}_{\mathcal{X}}(A)\big)^{\vee\vee}$. A *log pair* means the pair $(\mathcal{X},A)$ of a formal stack $\mathcal{X}$ as above and a narrow $\mathbb{Q}$-divisor A on it such that for some $r>0$, the divisor rA has integral coefficients and $\omega^{[r]}_{\mathcal{X}/\mathrm{Df}_\Phi}(rA)$ is an invertible sheaf. (Namely, $K_{\mathcal{X}/\mathrm{Df}_\Phi}+A$ is $\mathbb{Q}$-Cartier, according to the terminology which is little inaccurate but common in birational geometry.) We call A a *boundary divisor* or simply *boundary*. If $\omega^{[r]}_{\mathcal{X}/\mathrm{Df}_\Phi}$ is invertible for some $r>0$, then we identify $\mathcal{X}$ with the log pair $(\mathcal{X},0)$.

A *morphism* $(\mathcal{Y},B)\to(\mathcal{X},A)$ of log pairs is just a morphism $\mathcal{Y}\to\mathcal{X}$ of ambient stacks as in Assumption-Notation 10.1. We say that a morphism $f\colon(\mathcal{Y},B)\to(\mathcal{X},A)$ is *crepant* if the morphism $f^*\Omega^d_{\mathcal{X}/\mathrm{Df}_\Phi}\to\Omega^d_{\mathcal{Y}/\mathrm{Df}_\Psi}$ induces an isomorphism $f^*\omega^{[r]}_{\mathcal{X}/\mathrm{Df}_\Phi}(rA)\to\omega^{[r]}_{\mathcal{Y}/\mathrm{Df}_\Psi}(rB)$ of fractional modules for sufficiently factorial r (that is, $f^*(K_{\mathcal{X}/\mathrm{Df}_\Phi}+A)=K_{\mathcal{Y}/\mathrm{Df}_\Psi}+B$).

Lemma 12.5. *Let $(\mathcal{X},A)$ be a log pair and X the coarse moduli space of $\mathcal{X}$. There exists a unique boundary $\overline{A}$ on X such that the morphism $(\mathcal{X},A)\to\big(X,\overline{A}\big)$ is crepant.*

Proof. Consider the case where $\mathcal{X}$ is a quotient stack $[V/G]$ for a finite group G. Then $X=V/G$. We have the relative canonical divisor $K_{V/X}$, a divisor on V. Let $\tilde{A}$ be the $\mathbb{Q}$-divisor on V corresponding to A, and let

$$\overline{A}:=\frac{1}{\sharp G}\pi_*\big(K_{V/X}+\tilde{A}\big)$$

with π denoting the morphism $V\to X$. Then

$$\pi^*\big(K_X+\overline{A}\big)=\pi^*K_X+K_{V/X}+\tilde{A}=K_V+\tilde{A}\,.$$

This means that $(\mathcal{X},A)\to\big(X,\overline{A}\big)$ is crepant. It is clear that $\overline{A}$ is the unique boundary with this property. The uniqueness allows us to glue boundaries $\overline{A}$ defined étale locally to get the desired boundary $\overline{A}$ in the general case. □

Definition 12.6. Let $(\mathcal{X},A)$ be a log pair, and let r be a positive integer such that $\omega^{[r]}_{\mathcal{X}/\mathrm{Df}_\Phi}(rA)$ is invertible. Let $L/K(\!(t)\!)$ be a finite extension, and let $\beta\colon\operatorname{Spf}O_L\to\mathcal{X}$ be a Df-morphism which does not factor through $\operatorname{Supp}A$. Then $\beta^*\omega^{[r]}_{\mathcal{X}/\mathrm{Df}_\Phi}(rA)$ is a non-zero fractional submodule of $\beta^\flat\big(\Omega^d_{\mathcal{X}/\mathrm{Df}_\Phi}\big)^{\otimes r}$. We define

$$\mathfrak{f}_{(\mathcal{X},A)}(\beta):=\frac{1}{r}\operatorname{ord}\big[\beta^\flat\big(\Omega^d_{\mathcal{X}/\mathrm{Df}_\Phi}\big)^{\otimes r}:\beta^*\omega^{[r]}_{\mathcal{X}/\mathrm{Df}_\Phi}(rA)\big]\,.$$

Here $[M:N]:=\{s\in O_L\mid sN\subset M\}$, a fractional ideal of O_L. When $\mathcal{X}$ is defined over Df, for a twisted arc $\gamma\colon\mathcal{E}\to\mathcal{X}$ which does not factor through $\operatorname{Supp}A$, we define $\mathfrak{f}_{(\mathcal{X},A)}(\gamma)$ to be $\mathfrak{f}_{(\mathcal{X},A)}(\beta)$ for $\beta\colon\operatorname{Spf}O_L\to\mathcal{E}\to\mathcal{X}$ induced by any finite dominant Df-morphism $\operatorname{Spf}O_L\to\mathcal{E}$.

Lemma 12.7. *The function $\mathfrak{f}_{(\mathcal{X},A)}$ on $|\mathrm{J}_{\Phi,\infty}\mathcal{X}|$ is measurable. When $\Phi=\operatorname{Spec}k$, the function $\mathfrak{f}_{(\mathcal{X},A)}$ on $|\mathcal{J}_\infty\mathcal{X}|$ is also measurable.*

Proof. There exists an ideal sheaf $\mathcal{J}$ such that

$$\mathcal{J}\big(\Omega^d_{\mathcal{X}/\mathrm{Df}_\Phi}\big)^{\otimes r}\subset\omega^{[r]}_{\mathcal{X}/\mathrm{Df}_\Phi}(rA)\,.$$

As in the proof of Lemma 10.25, there exists a blowup $h\colon\mathcal{Z}\to\mathcal{X}$ such that $h^{-1}\mathcal{J}$ and $h^\flat\big(\Omega^d_{\mathcal{X}/\mathrm{Df}_\Phi}\big)^{\otimes r}$ are invertible. Then, for some locally principal ideal sheaf $\mathcal{I}$ on $\mathcal{Z}$,

$$h^{-1}\mathcal{J}\cdot h^\flat\big(\Omega^d_{\mathcal{X}/\mathrm{Df}_\Phi}\big)^{\otimes r}=\mathcal{I}\cdot h^*\omega^{[r]}_{\mathcal{X}/\mathrm{Df}_\Phi}(rA)\,.$$

Outside a negligible subset, we have $\mathfrak{f}_{(\mathcal{X},A)} \circ h_\infty = \frac{1}{r}(\mathrm{ord}_{\mathcal{I}} - \mathrm{ord}_{h^{-1}\mathcal{J}})$. By Lemmas 10.24 and 11.7, the right side is measurable, and so is the left side. By Lemma 10.23, the function $\mathfrak{f}_{(\mathcal{X},A)}$ is measurable. □

Lemma 12.8. *Let $f\colon (\mathcal{Y},B) \to (\mathcal{X},A)$ be a crepant morphism of log pairs. We have the following equality between functions on $|\mathcal{J}_\infty \mathcal{Y}|$ outside a negligible subset:*

$$\mathfrak{f}_{(\mathcal{X},A)} \circ f_\infty = \mathfrak{f}_{(\mathcal{Y},B)} + \mathfrak{j}_f\,.$$

Proof. Let $\beta\colon \operatorname{Spf} O_L \to \mathcal{Y}$ be a morphism as before. For a sufficiently factorial r, we consider three O_L-modules

$$M_1 := \beta^* f^* \omega^{[r]}_{\mathcal{X}/\mathrm{Df}_\Phi}(rA) = \beta^* \omega^{[r]}_{\mathcal{Y}/\mathrm{Df}_\Psi}(rB)\,, \quad M_2 := \beta^\flat\big(\Omega^d_{\mathcal{Y}/\mathrm{Df}_\Psi}\big)^{\otimes r}\,, \quad M_3 := (f\circ\beta)^\flat\big(\Omega^d_{\mathcal{X}/\mathrm{Df}_\Psi}\big)^{\otimes r}\,.$$

Ignoring a negligible subset, we may suppose that they are free of rank 1. We have

$$[M_2 : M_1] = \mathfrak{m}_E^{\sharp G\cdot r\cdot \mathfrak{f}_{(\mathcal{Y},B)}(\beta)}\,, \quad [M_3 : M_1] = \mathfrak{m}_E^{\sharp G\cdot r\cdot \mathfrak{f}_{(\mathcal{X},A)}(f\circ\beta)}\,, \quad [M_3 : M_2] = \mathfrak{m}_E^{\sharp G\cdot r\cdot \mathfrak{j}_f(\beta)}\,.$$

The lemma follows from the equality $[M_3 : M_1] = [M_3 : M_2][M_2 : M_1]$. □

For a log pair $(\mathcal{X},A)$, let X be the coarse moduli space of $\mathcal{X}$, and let $\overline{A}$ be the $\mathbb{Q}$-divisor such that $(\mathcal{X},A) \to \big(X,\overline{A}\big)$ is crepant. Let $\mathrm{Utg}_\Gamma(\mathcal{X})^{\mathrm{nor}}$ be the normalization of $\mathrm{Utg}_\Gamma(\mathcal{X})^{\mathrm{pur}}$. Let A^{utg} be the unique $\mathbb{Q}$-divisor such that $\big(\mathrm{Utg}_\Gamma(\mathcal{X})^{\mathrm{nor}}, A^{\mathrm{utg}}\big) \to \big(X,\overline{A}\big)$ is crepant; with the conventional notation, if $\pi\colon \mathcal{X} \to X$ is the given morphism and if $\phi\colon \mathrm{Utg}_\Gamma(\mathcal{X})^{\mathrm{nor}} \to X$ is the induced morphism, then

$$A^{\mathrm{utg}} = \phi^*\big(K_{X/\mathrm{Df}} + \overline{A}\big) - K_{\mathrm{Utg}_\Gamma(\mathcal{X})^{\mathrm{nor}}/\mathrm{Df}_\Gamma}\,.$$

Lemma 12.9. *If ν denotes the normalization morphism $\mathrm{Utg}_\Gamma(\mathcal{X})^{\mathrm{nor}} \to \mathrm{Utg}_\Gamma(\mathcal{X})$, then we have the following equality of functions on $|\mathrm{J}_\infty \mathrm{Utg}_\Gamma(\mathcal{X})^{\mathrm{nor}}|$ outside a negligible subset:*

$$\mathfrak{s}_{\mathcal{X}} \circ \nu_\infty = \mathfrak{f}_{(\mathrm{Utg}_\Gamma(\mathcal{X})^{\mathrm{nor}},A^{\mathrm{utg}})} - \mathfrak{f}_{(\mathcal{X},A)} \circ \nu_\infty + \mathfrak{j}_\nu\,.$$

(*Note that $\mathfrak{f}_{(\mathcal{X},A)}$ is a function on $|\mathcal{J}_\infty\mathcal{X}| = |\mathrm{J}_\infty \mathrm{Utg}_\Gamma(\mathcal{X})^{\mathrm{pur}}|$.*)

Proof. By Lemma 11.10, we have $\mathfrak{s}_{\mathcal{X}} = \mathfrak{j}_\pi - \mathfrak{j}_{\pi^{\mathrm{utg}}}$. By Lemma 12.8, we have

$$\mathfrak{j}_\pi = \mathfrak{f}_{(X,\overline{A})} \circ \pi_\infty - \mathfrak{f}_{(\mathcal{X},A)}\,, \quad \mathfrak{j}_\phi = \mathfrak{f}_{(X,\overline{A})} \circ \phi_\infty - \mathfrak{f}_{(\mathrm{Utg}_\Gamma(\mathcal{X})^{\mathrm{nor}},A^{\mathrm{utg}})}\,.$$

By Lemma 8.11, we have $\mathfrak{j}_\phi = \mathfrak{j}_\nu + \mathfrak{j}_{\pi^{\mathrm{utg}}} \circ \nu_\infty$. Thus

$$\begin{aligned}
\mathfrak{s}_{\mathcal{X}} \circ \nu_\infty &= \big(\mathfrak{f}_{(X,\overline{A})} \circ \pi_\infty - \mathfrak{f}_{(\mathcal{X},A)}\big) \circ \nu_\infty - \mathfrak{j}_{\pi^{\mathrm{utg}}} \circ \nu_\infty \\
&= \mathfrak{f}_{(X,\overline{A})} \circ \phi_\infty - \mathfrak{f}_{(\mathcal{X},A)} \circ \nu_\infty + \mathfrak{j}_\nu - \mathfrak{j}_\phi \\
&= \mathfrak{f}_{(\mathrm{Utg}_\Gamma(\mathcal{X})^{\mathrm{nor}},A^{\mathrm{utg}})} - \mathfrak{f}_{(\mathcal{X},A)} \circ \nu_\infty + \mathfrak{j}_\nu\,.
\end{aligned}$$
□

13. Stringy motives and the wild McKay correspondence

We associate stringy motives with log pairs and reformulate the change of variables formula as a property of stringy motives. This invariant is a common generalization of stringy E-function and orbifold E-function [Bat99b]. Following Denef–Loeser's approach [DL02], we define this invariant as an integral on the space of twisted arcs, $|\mathcal{J}_\infty\mathcal{X}|$, even if the ambient formal stack $\mathcal{X}$ is singular.

We fix a sufficiently factorial positive integer r so that all relevant functions on spaces of (twisted or untwisted) arcs have values in $\frac{1}{r}\mathbb{Z} \cup \{\infty\}$.

Definition 13.1. Let $(\mathcal{X},A)$ be a log pair over Df, and let $C \subset |\mathcal{X}|$ be a constructible subset. Let $|\mathcal{J}_\infty\mathcal{X}|_C \subset |\mathcal{J}_\infty\mathcal{X}|$ be the preimage of C by π_0. Let $\Gamma = \Gamma_{\mathcal{X}} \to Z$ be a morphism to an algebraic space almost of finite type. We define the *stringy motive of* $(\mathcal{X},A)$ *along* C as

$$\mathrm{M}_{\mathrm{st},Z}(\mathcal{X},A)_C := \int_{|\mathcal{J}_\infty\mathcal{X}|_C} \mathbb{L}^{\mathfrak{f}_{(\mathcal{X},A)}+\mathfrak{s}_{\mathcal{X}}}\,\mathrm{d}\mu_{\mathcal{X},Z} \in \widehat{\mathcal{M}'_{Z,r}} \cup \{\infty\}\,.$$

Definition 13.2. Let $(\mathcal{X},A)$ be a log pair over Df_Φ with Φ a reduced DM stack almost of finite type, and let $C \subset |\mathcal{X}|$ be a constructible subset. Let $|\mathrm{J}_\infty\mathcal{X}|_C \subset |\mathrm{J}_\infty\mathcal{X}|$ be the preimage of C by π_0. Let $\Phi \to Z$ be a morphism to an algebraic space almost of finite type. We define the *untwisted stringy motive of* $(\mathcal{X},A)$ *along* C as

$$\mathrm{M}^{\mathrm{utd}}_{\mathrm{st},Z}(\mathcal{X},A)_C := \int_{|\mathrm{J}_\infty\mathcal{X}|_C} \mathbb{L}^{\mathfrak{f}_{(\mathcal{X},A)}}\,\mathrm{d}\mu_{\mathcal{X},Z} \in \widehat{\mathcal{M}'_{Z,r}} \cup \{\infty\}\,.$$

For these invariants, we usually omit the subscripts Z when $Z = \operatorname{Spec} k$ and C when $C = |\mathcal{X}|$.

We have

$$\int_{Z\to Z'} \mathrm{M}_{\mathrm{st},Z}(\mathcal{X},A)_C = \mathrm{M}_{\mathrm{st},Z'}(\mathcal{X},A)_C\,;$$

in particular,

$$\int_Z \mathrm{M}_{\mathrm{st},Z}(\mathcal{X},A)_C = \mathrm{M}_{\mathrm{st}}(\mathcal{X},A)\,,$$

and similarly for the untwisted stringy motive. When $\mathcal{X}$ is a formal algebraic space over Df, obviously $\mathrm{M}_{\mathrm{st},Z}(\mathcal{X},A)_C = \mathrm{M}^{\mathrm{utd}}_{\mathrm{st},Z}(\mathcal{X},A)_C$.

Let $(\mathcal{X},A)$ be a log pair over Df, and let $\big(X,\overline{A}\big)$ and $\big(\mathrm{Utg}_\Gamma(\mathcal{X})^{\mathrm{nor}},A^{\mathrm{utg}}\big)$ be the induced log pairs (see Section 12). Let $C \subset |\mathcal{X}|$ be a constructible subset, $\overline{C} \subset |X|$ its image of C and $C^{\mathrm{utg}} \subset |\mathrm{Utg}_\Gamma(\mathcal{X})^{\mathrm{nor}}|$ the preimage of $\overline{C}$.

Theorem 13.3. *We keep the notation above. Let $\overline{\Gamma}$ be the coarse moduli space of Γ. We have*

$$\begin{aligned}\mathrm{M}_{\mathrm{st}}(\mathcal{X},A)_C &= \mathrm{M}_{\mathrm{st}}\big(X,\overline{A}\big)_{\overline{C}}\\ &= \mathrm{M}^{\mathrm{utd}}_{\mathrm{st}}\big(\mathrm{Utg}_\Gamma(\mathcal{X})^{\mathrm{nor}},A^{\mathrm{utg}}\big)_{C^{\mathrm{utg}}}\\ &= \int_{\overline{\Gamma}} \mathrm{M}^{\mathrm{utd}}_{\mathrm{st},\overline{\Gamma}}\big(\mathrm{Utg}_\Gamma(\mathcal{X})^{\mathrm{nor}},A^{\mathrm{utg}}\big)_{C^{\mathrm{utg}}}\,.\end{aligned}$$

In particular, if $A=0$ and the stacky locus $\mathcal{X}_{\mathrm{st}} \subset \mathcal{X}$ has codimension at least 2, then

$$\mathrm{M}_{\mathrm{st}}(\mathcal{X})_C = \mathrm{M}_{\mathrm{st}}(X)_{\overline{C}}\,.$$

Proof. By the change of variables (Lemma 11.10) and by Lemma 12.8, we have

$$\begin{aligned}\int_{|\mathcal{J}_\infty\mathcal{X}|_C} \mathbb{L}^{\mathfrak{f}_{(\mathcal{X},A)}+\mathfrak{s}_{\mathcal{X}}}\,d\mu_{\mathcal{X}} &= \int_{|\mathrm{J}_\infty X|_{\overline{C}}} \mathbb{L}^{\mathfrak{f}_{(X,\overline{A})}}\,d\mu_X = \int_{|\mathrm{J}_\infty \mathrm{Utg}_\Gamma(\mathcal{X})^{\mathrm{nor}}|_{C^{\mathrm{utg}}}} \mathbb{L}^{\mathfrak{f}_{(\mathrm{Utg}_\Gamma(\mathcal{X})^{\mathrm{nor}},A^{\mathrm{utd}})}}\,d\mu_{\mathrm{Utg}_\Gamma(\mathcal{X})}\\ &= \int_{\overline{\Gamma}}\int_{|\mathrm{J}_\infty \mathrm{Utg}_\Gamma(\mathcal{X})^{\mathrm{nor}}|_{C^{\mathrm{utg}}}} \mathbb{L}^{\mathfrak{f}_{(\mathrm{Utg}_\Gamma(\mathcal{X})^{\mathrm{nor}},A^{\mathrm{utd}})}}\,d\mu_{\mathrm{Utg}_\Gamma(\mathcal{X}),\overline{\Gamma}}\,.\end{aligned}$$

These equalities show the first assertion. If $A=0$ and the stacky locus $\mathcal{X}_{\mathrm{st}} \subset \mathcal{X}$ has codimension at least 2, then $\overline{A}=0$ and the second assertion follows. □

We apply this theorem to the case of a quotient stack. Let V be a formal scheme of finite type, flat and generically smooth over Df. Suppose that a finite group H acts on V and that the quotient stack $\mathcal{X} := [V/H]$ is spacious, equivalently that if V^h, $h \in H$, are the h-fixed subschemes, then $\bigcup_{h\in H\setminus\{1\}} V^h$ is a narrow subscheme of V. Let (V, A) be a log pair such that A is stable under the H-action. Let $(V/H, \overline{A})$ be the induced log pair such that $(V, A) \to (V/H, \overline{A})$ is crepant. Let $\Gamma := \Gamma_{\mathcal{X}}$ for $\mathcal{X}$, and let $\Gamma_G := \Gamma \times_\Theta \Theta_G$. Then $\Gamma_G \to \Theta_G$ is representable and universally bijective. Moreover, $\mathrm{Utg}_{\Gamma_{G,\iota}}(V)^{\mathrm{pur}}$ is flat over Df_{Γ_G} for every $\iota \in \mathrm{Emb}(G, H)$. We denote by $\mathrm{Utg}_{\Gamma_{G,\iota}}(V)^{\mathrm{nor}}$ its normalization. Let $C \subset |V|$ be an H-stable constructible subset and $\overline{C} \subset |V/H|$ its image. For an embedding $\iota\colon G \hookrightarrow H$ of a Galoisian group, let A_ι be the restriction of A^{utg} to $[\mathrm{Utg}_{\Gamma_{G,\iota}}(V)^{\mathrm{nor}}/\,\mathrm{N}_H(\iota(G))]$, and let $C_\iota \subset \big|[\mathrm{Utg}_{\Gamma_{G,\iota}}(V)^{\mathrm{nor}}/\,\mathrm{N}_H(\iota(G))]\big|$ be the preimage of $\overline{C}$.

Corollary 13.4. *We have*

$$\mathrm{M}_{\mathrm{st}}\big(V/H, \overline{A}\big)_{\overline{C}} = \sum_{G\in\mathrm{GalGps}} \sum_{\iota\in\frac{\mathrm{Emb}(G,H)}{\mathrm{Aut}(G)\times H}} \mathrm{M}_{\mathrm{st}}^{\mathrm{utd}}\left(\left[\frac{\mathrm{Utg}_{\Gamma_{G,\iota}}(V)^{\mathrm{nor}}}{\mathrm{N}_H(\iota(G))}\right], A_\iota\right)_{C_\iota}.$$

Proof. Let $\mathcal{X} = [V/H]$, and define $\big(\mathrm{Utg}_\Gamma(\mathcal{X})^{\mathrm{nor}}, A^{\mathrm{utg}}\big)$ as before. By Theorem 13.3, we have

$$\mathrm{M}_{\mathrm{st}}\big(V/H, \overline{A}\big)_{\overline{C}} = \mathrm{M}_{\mathrm{st}}^{\mathrm{utd}}\big(\mathrm{Utg}_\Gamma(\mathcal{X})^{\mathrm{nor}}, A^{\mathrm{utg}}\big)_{C^{\mathrm{utg}}}.$$

By Lemma 6.12 and (6.5), we have

$$\begin{aligned}\mathrm{Utg}_\Gamma(\mathcal{X}) &= \coprod_{G\in\mathrm{GalGps}} \mathrm{Utg}_{\Gamma_{[G]}}(\mathcal{X})\\ &\cong \coprod_{G\in\mathrm{GalGps}} \left[\Big(\coprod_{\iota\in\mathrm{Emb}(G,H)} \mathrm{Utg}_{\Gamma_{G,\iota}}(V)\Big)\Big/\,\mathrm{Aut}(G)\times H\right]\\ &\cong \coprod_{G\in\mathrm{GalGps}} \coprod_{\iota\in\mathrm{Emb}(G,H)/(\mathrm{Aut}(G)\times H)} [\mathrm{Utg}_{\Theta_{G,\iota}}(V)/\,\mathrm{N}_H(\iota(G))].\end{aligned}$$

Therefore, $\mathrm{M}_{\mathrm{st}}^{\mathrm{utd}}\big(\mathrm{Utg}_\Gamma(\mathcal{X})^{\mathrm{nor}}, A^{\mathrm{utg}}\big)_{C^{\mathrm{utg}}}$ is equal to the right side of the equality of the lemma. ☐

Remark 13.5. Corollary 13.4 is the motivic version of the main result of [Yas17b], formulated by using stacks. Note that the untwisting stack considered in this paper is essentially the same as the one considered in [Yas16, Yas17b] (see Remark 14.5).

Definition 13.6. Let $f\colon \mathcal{Y} \to \mathcal{X}$ be a morphism of formal DM stacks as in Assumption-Notation 11.2. We say that $f\colon \mathcal{Y} \to \mathcal{X}$ is a *pseudo-modification* if $f_\infty\colon |\mathcal{J}_\infty\mathcal{X}| \to |\mathcal{J}_\infty\mathcal{X}|$ is almost geometrically bijective.

Example 13.7. The following are examples of pseudo-modifications:

(1) the t-adic completion of a (not necessarily representable) proper birational morphism $\mathcal{W} \to \mathcal{V}$ of DM stacks of finite type over $\mathrm{D} = \operatorname{Spec} k[\![t]\!]$,
(2) the blowup along a closed substack,
(3) the relative coarse moduli space $\mathcal{Y} \to \mathcal{X}$ of some morphism $\mathcal{Y} \to \mathcal{Z}$ of formal DM stacks of finite type over Df.

A composition of finitely many such morphisms is also a pseudo-modification.

Theorem 13.8. *Let $f\colon \mathcal{Y} \to \mathcal{X}$ be a pseudo-modification which induces a crepant morphism $(\mathcal{Y}, B) \to (\mathcal{X}, A)$ of log pairs over* Df. *Let $C \subset |\mathcal{X}|$ be a constructible subset and $\tilde{C} \subset |\mathcal{Y}|$ its preimage. Then*

$$\mathrm{M}_{\mathrm{st}}(\mathcal{X}, A)_C = \mathrm{M}_{\mathrm{st}}(\mathcal{Y}, B)_{\tilde{C}}.$$

Proof. With notation as before, we have $\mathrm{M}_{\mathrm{st}}(\mathcal{X},A)_C = \mathrm{M}_{\mathrm{st}}\big(X,\overline{A}\big)_{\overline{C}}$. Applying Theorem 11.13 to the crepant morphism $(\mathcal{Y},B)\to\big(X,\overline{A}\big)$, we see that $\mathrm{M}_{\mathrm{st}}\big(X,\overline{A}\big)_{\overline{C}} = \mathrm{M}_{\mathrm{st}}(\mathcal{Y},B)_{\tilde{C}}$. □

14. Linear actions

In this section, we focus on a linear action of a finite group on an affine space over Df and the associated quotient stack/scheme. We specialize results obtained in earlier sections to this situation and get a more explicit formula, the wild McKay correspondence.

Let K be an algebraically closed field, let M be a free $K[\![t]\!]$-module of rank d, and let $S^\bullet M$ be its symmetric algebra over $K[\![t]\!]$, which is isomorphic to the polynomial ring $K[\![t]\!][x_1,\dots,x_d]$. Suppose that M is given a $K[\![t]\!]$-linear action of a finite group H, which induces an H-action on the scheme $W=\operatorname{Spec} S^\bullet M\cong\mathbb{A}^d_{\mathrm{D}_K}$ and one on the formal scheme $V:=\operatorname{Spf}\widehat{S^\bullet M}\cong\mathbb{A}^d_{\mathrm{Df}_K}$, where $\widehat{S^\bullet M}$ is the t-adic completion of $S^\bullet M$. Let $\iota\colon G\hookrightarrow H$ be an embedding of a Galoisian group. For a G-cover $E\to\mathrm{Df}_K$, we define the *tuning module*

$$\Xi_{E,\iota}:=\operatorname{Hom}^{\iota}_{K[\![t]\!]}(M,O_E)\,,$$

the module of ι-equivariant $K[\![t]\!]$-linear maps [Yas16, § 4, p. 127] (cf. [WY15, Definition 3.1]). This is again a free $K[\![t]\!]$-module of rank d. Moreover, this is also a saturated $K[\![t]\!]$-submodule of $\operatorname{Hom}_{K[\![t]\!]}(M,O_E)$; that is, the quotient module $\operatorname{Hom}_{K[\![t]\!]}(M,O_E)/\Xi_E$ is a torsion-free (hence free) $K[\![t]\!]$-module. It follows that the map of dual $K[\![t]\!]$-modules

$$\operatorname{Hom}_{K[\![t]\!]}(M,O_E)^\vee\to\Xi_{E,\iota}^\vee$$

is surjective. Let

$$M_\iota^{|E|}:=\Xi^\vee_{E,\iota}\,,\quad W_\iota^{|E|}:=\operatorname{Spec} S^\bullet\big(M_\iota^{|E|}\big)\quad\text{and}\quad V_\iota^{|E|}:=\operatorname{Spf}\widehat{S^\bullet\big(M_\iota^{|E|}\big)}=\widehat{W_\iota^{|E|}}\,.$$

We have the following commutative diagram of natural morphisms [Yas16, diagram (4.1), p. 130]:

$$\begin{tikzcd} W_\iota^{|E|}\otimes_{K[\![t]\!]}O_E \arrow[r]\arrow[d] & W_\iota^{|E|}\arrow[d]\\ W\arrow[r] & W/H\,. \end{tikzcd}$$

Taking t-adic completion, we have the corresponding diagram for V in place of W.

Let $E\colon\operatorname{Spec}K\to\Theta_G$ be a morphism corresponding to our G-cover $E\to\mathrm{Df}_K$, and let $\mathrm{Utg}_{E,\iota}(V):=\mathrm{Utg}_{\Theta_G,\iota}(V)\times_{\Theta_G,E}\operatorname{Spec}K$.

Lemma 14.1. *We have a* Df*-isomorphism* $\mathrm{Utg}_{E,\iota}(V)^{\mathrm{pur}}\cong V_\iota^{|E|}$ *compatible with the morphisms to* V/H. *In particular,* $\mathrm{Utg}_{E,\iota}(V)^{\mathrm{pur}}$ *is smooth over* Df.

Proof. Let $E^{\mathrm{a}}:=\operatorname{Spec}O_E$ be the scheme corresponding to the formal scheme E. The formal scheme $\mathrm{Utg}_{E,\iota}(V)^{\mathrm{pur}}$ is the completion of the closure of $\underline{\mathrm{Hom}}^{\iota}_{\mathrm{D}}(E^{\mathrm{a}},W)\times_{\mathrm{D}}\mathrm{D}^*$ in $\underline{\mathrm{Hom}}_{\mathrm{D}}(E^{\mathrm{a}},W)$ (see Remark 6.23). We note that

$$\begin{aligned} M\otimes O_E^\vee &= \operatorname{Hom}_{K[\![t]\!]}\big(M^\vee,K[\![t]\!]\big)\otimes_{K[\![t]\!]}O_E^\vee\\ &= \operatorname{Hom}_{K[\![t]\!]}\big(M^\vee,O_E^\vee\big)\\ &= \operatorname{Hom}_{K[\![t]\!]}\big(M^\vee\otimes_{K[\![t]\!]}O_E,K[\![t]\!]\big)\\ &= \operatorname{Hom}_{K[\![t]\!]}(M,O_E)^\vee\,. \end{aligned}$$

Here duals are taken as $K[\![t]\!]$-modules. The D-scheme $\underline{\mathrm{Hom}}_{\mathrm{D}}(E^{\mathrm{a}},W)$ is identified with

$$\operatorname{Spec} S^\bullet\bigl(M\otimes_{K[\![t]\!]} O_E^\vee\bigr)=\operatorname{Spec} S^\bullet \operatorname{Hom}_{K[\![t]\!]}(M,O_E)^\vee .$$

Indeed, for a morphism $\operatorname{Spec} R\to D$,

$$\begin{aligned}\underline{\mathrm{Hom}}_{\mathrm{D}}(E^{\mathrm{a}},W)(\operatorname{Spec} R)&=\operatorname{Hom}_{K[\![t]\!]\text{-alg}}(S^\bullet M,O_E\otimes_{K[\![t]\!]}R)\\&=\operatorname{Hom}_{K[\![t]\!]\text{-mod}}(M,O_E\otimes_{K[\![t]\!]}R)\\&=\operatorname{Hom}_{K[\![t]\!]\text{-mod}}\bigl(M,\operatorname{Hom}_{K[\![t]\!]}(O_E^\vee,R)\bigr)\\&=\operatorname{Hom}_{K[\![t]\!]\text{-mod}}\bigl(M\otimes_{K[\![t]\!]}O_E^\vee,R\bigr)\\&=\operatorname{Hom}_{K[\![t]\!]\text{-alg}}\bigl(S^\bullet\bigl(M\otimes_{K[\![t]\!]}O_E^\vee\bigr),R\bigr)\\&=\bigl(\operatorname{Spec} S^\bullet\bigl(M\otimes_{K[\![t]\!]}O_E^\vee\bigr)\bigr)(\operatorname{Spec} R).\end{aligned}$$

Therefore,

$$\underline{\mathrm{Hom}}_{\mathrm{D}}(E^{\mathrm{a}},W)\times_{\mathrm{D}}\mathrm{D}^*=\operatorname{Spec} S^\bullet \operatorname{Hom}_{K(\!(t)\!)}(M\otimes_{K[\![t]\!]}K(\!(t)\!),O_E\otimes_{K[\![t]\!]}K(\!(t)\!))^\vee ,$$

and for $\iota\colon G\hookrightarrow H$,

$$\underline{\mathrm{Hom}}^\iota_{\mathrm{D}}(E^{\mathrm{a}},W)\times_{\mathrm{D}}\mathrm{D}^*=\operatorname{Spec} S^\bullet \operatorname{Hom}^\iota_{K(\!(t)\!)}(M\otimes_{K[\![t]\!]}K(\!(t)\!),O_E\otimes_{K[\![t]\!]}K(\!(t)\!))^\vee .$$

Since $\operatorname{Hom}_{K[\![t]\!]}(M,O_E)^\vee\to\Xi_{E,\iota}^\vee$ is surjective, we may regard $W_\iota^{|E|}$ as a closed subscheme of $\operatorname{Spec} S^\bullet(M\otimes_{K[\![t]\!]}O_E^\vee)$. This closed subscheme is integral and over D^* coincides with the fiber product $\underline{\mathrm{Hom}}^\iota_{\mathrm{D}}(E^{\mathrm{a}},W)\times_{\mathrm{D}}\mathrm{D}^*$. Therefore, $W_\iota^{|E|}$ is the closure of $\underline{\mathrm{Hom}}^\iota_{\mathrm{D}}(E^{\mathrm{a}},W)\times_{\mathrm{D}}\mathrm{D}^*$. Since the completion of $W_\iota^{|E|}$ is $V_\iota^{|E|}$, the lemma follows. □

Following [WY15, Definitions 3.3] and [Yas16, Definition 5.4] (cf. [Yas17a, Definition 6.5]), we define

$$v(E):=\frac{1}{\sharp G}\operatorname{length}\frac{\operatorname{Hom}_{K[\![t]\!]}(M,O_E)}{O_E\cdot\Xi_{E,\iota}} .$$

Lemma 14.2. *Let $\beta\colon E\to V$ be an ι-equivariant Df_K-morphism, let $\mathcal{X}:=[V/H]$, and let $\gamma\colon\mathcal{E}\to\mathcal{X}$ be the induced twisted arc. Then $\mathfrak{s}_{\mathcal{X}}(\gamma)=-v(E)$.*

Proof. Let B be the $\mathbb{Q}$-divisor on V/H such that $V\to(V/H,B)$ is crepant. Then the induced boundary on $\mathrm{Utg}_{E,\iota}(V)^{\mathrm{pur}}\cong\mathbb{A}^d_{\mathrm{Df}_K}$ is the relative anti-canonical divisor

$$-K_{\mathrm{Utg}_{E,\iota}(V)^{\mathrm{pur}}/(V/H,B)}:=-K_{\mathrm{Utg}_{E,\iota}(V)^{\mathrm{pur}}/\mathrm{Df}_K}+u^*(K_{(V/H)/\mathrm{Df}_K}+B),$$

where u is the morphism $\mathrm{Utg}_{E,\iota}(V)^{\mathrm{pur}}\to V/H$. (Note that since $\mathrm{Utg}_{E,\iota}(V)^{\mathrm{pur}}$ is already normal, we do not need to normalize it.) This is the vertical divisor $\mathrm{Utg}_{E,\iota}(V)^{\mathrm{pur}}_0\cong\mathbb{A}^d_K$ with multiplicity $-v(E)$ (see [Yas16, Lemma 6.5]). Since V is smooth over Df_K, we have $\Omega^d_{V/\mathrm{Df}}=\omega_{V/\mathrm{Df}}$ and $\mathfrak{f}_{([V/H],0)}\equiv 0$. Therefore, by Lemma 12.9, we have

$$\begin{aligned}\mathfrak{s}_{\mathcal{X}}(\gamma)&=\mathfrak{f}_{(\mathrm{Utg}_{\mathcal{E}}(\mathcal{X})^{\mathrm{pur}},A^{\mathrm{utg}}|_{\mathrm{Utg}_{\mathcal{E}}(\mathcal{X})^{\mathrm{pur}}})}(\gamma')-\mathfrak{f}_{([V/H],0)}\\&=\mathfrak{f}_{(\mathrm{Utg}_{E,\iota}(V)^{\mathrm{pur}},-v(E)\cdot\mathrm{Utg}_{E,\iota}(V)^{\mathrm{pur}}_0)}(\beta)\\&=-v(E).\end{aligned}$$

Note that the last equality holds because $\mathrm{Utg}_{E,\iota}(V)^{\mathrm{pur}}_0$, as well as its pullback by β, is a closed subscheme defined by $t\in K[\![t]\!]$. □

The following lemma was proved in [TY23] by a different and more direct method.

Lemma 14.3. *Let us fix an embedding* $\iota\colon G \hookrightarrow H$. *The map*

$$|\Lambda_G| \mapsto \mathbb{Q},\ [E] \mapsto v(E)$$

is locally constructible; for every substack $\mathcal{W} \subset \Lambda_G$ *of finite type, the restriction of this function to* $|\mathcal{W}|$ *is constructible* (*that is, every fiber is a constructible subset*).

Proof. Let Γ_G be as in the paragraph before Corollary 13.4. We further assume that each connected component of it is irreducible. We claim that for a suitable choice of Γ_G, the boundary on $\mathrm{Utg}_{\Gamma_G}(V)$ is vertical. Then the function v on $|\Lambda_G| = |\Gamma_G|$ is constant on each connected component of Γ_G, and the lemma follows.

To show the claim, let $\Gamma_{G,0}$ be a connected component of Γ_G. Let us write the boundary on $\mathrm{Utg}_{\Gamma_{G,0}}(V)$ as $A = \sum_{i=0}^{l} c_i A_i$ with A_i prime divisors and $c_i \in \mathbb{Q}$, where A_0 is the vertical divisor $\mathrm{Utg}_{\Gamma_{G,0}}(V)$. For a geometric generic point $\bar\eta\colon \operatorname{Spec} K \to \Gamma_{G,0}$, the pullback of A to $\mathrm{Utg}_{\bar\eta}(V)$ is the vertical divisor. Therefore, the image of $A_0 \cap \bigcup_{i\geqslant 1} A_i$ on $\Gamma_{G,0}$ is a constructible subset which does not containing the generic point. Removing its closure, we get an open dense substack $\Upsilon \subset \Gamma_{G,0}$ such that the boundary on $\mathrm{Utg}_{\Upsilon}(V)$ is vertical. This shows the above claim. □

By Lemma 14.3, for each $s \in \frac{1}{\sharp H}\mathbb{Z}$, there exist a DM stack $\mathcal{C}_s$ almost of finite type, and a stabilizer-preserving and geometrically injective map $\mathcal{C}_s \to \Delta_H$ which maps onto $v^{-1}(s)$. We define $\{v^{-1}(s)\}$ to be $\{\mathcal{C}_s\}$ if $\mathcal{C}_s$ is of finite type. If not, we put $\{v^{-1}(s)\} := \infty$. We can define

$$\int_{\Delta_H} \mathbb{L}^{d-v} := \sum_{s\in\frac{1}{\sharp H}\mathbb{Z}} \{v^{-1}(s)\}\mathbb{L}^{d-s}.$$

In [TY23], we study this kind of integrals more systematically and especially prove the well-definedness of the above integral.

Corollary 14.4. *Suppose that a finite group* H *acts on* $\mathbb{A}^d_{\mathrm{Df}}$ *linearly. Let* $X := \mathbb{A}^d_{\mathrm{Df}}/H$, *and let* A *be the boundary on* X *such that* $\mathbb{A}^d_{\mathrm{Df}} \to (X, A)$ *is crepant.* (*Note that if the morphism* $\mathbb{A}^d_{\mathrm{Df}} \to X$ *is étale in codimension* 1, *then* $A = 0$.) *Then we have*

$$\mathrm{M}_{\mathrm{st}}(X,A) = \int_{\Delta_H} \mathbb{L}^{d-v}\left(:= \sum_{s\in\frac{1}{\sharp H}\mathbb{Z}} \{v^{-1}(s)\}\mathbb{L}^{d-s}\right).$$

Proof. As in the proof of Lemma 14.3, we can choose $\Gamma_{[G]}$ so that the function v on $|\Gamma_{[G]}|$ is locally constant. Let $v_{G,i}$ be the value of v on a connected component $|\Gamma_{[G],i}|$. By Corollary 13.4, we have

$$\begin{aligned}\mathrm{M}_{\mathrm{st}}(X,A) = \mathrm{M}_{\mathrm{st}}(\mathcal{X}) &= \int_{|\mathcal{J}_\infty \mathcal{X}|} \mathbb{L}^{-v}\, d\mu_{\mathcal{X}} \\ &= \sum_{G\in\mathrm{GalGps}} \sum_i \mu_{\mathcal{X}}(\mathcal{J}_{\Gamma_{[G],i},\infty}\mathcal{X})\mathbb{L}^{-v_{G,i}} \\ &= \sum_{G\in\mathrm{GalGps}} \sum_i \{\mathrm{Unt}_{\Gamma_{[G],i}}(\mathcal{X})^{\mathrm{pur}}_0\}\mathbb{L}^{-v_{G,i}}.\end{aligned}$$

By Lemma 6.12 and (6.5), we have

$$\left\{\mathrm{Unt}_{\Gamma_{[G],i}}(\mathcal{X})^{\mathrm{pur}}_0\right\} = \coprod_{\iota\in\frac{\mathrm{Emb}(G,H)}{\mathrm{Aut}(G)\times H}} \{\mathrm{Utg}_{\Gamma_{G,i,\iota}}(V)^{\mathrm{pur}}_0 / \mathrm{N}_H(\iota(G))\}.$$

Here $\Gamma_{G,i} := \Gamma_{[G],i} \times_{\mathcal{A}_G} \operatorname{Spec} k$. Since $\mathrm{Utg}_{\Gamma_{G,i},\iota}(V)_0^{\mathrm{pur}} \to \Gamma_{G,i}$ is $\mathrm{N}_H(\iota(G))$-equivariant $\mathbb{A}^d$-bundle, we have

$$\{\mathrm{Utg}_{\Gamma_{G,i},\iota}(V)_0^{\mathrm{pur}}/N_H(\iota(G))\} = \{\Gamma_{G,i}/\mathrm{N}_H(\iota(G))\}\mathbb{L}^d .$$

We conclude that

$$\mathrm{M}_{\mathrm{st}}(X,\mathcal{P}) = \sum_{G\in\mathrm{GalGps}} \sum_{i\in\mathbb{N}} \coprod_{\iota\in\frac{\mathrm{Emb}(G,H)}{\mathrm{Aut}(G)\times H}} \{\Gamma_{G,i}/N_H(\iota(G))\}\mathbb{L}^{d-v_{G,i}} .$$

The map

$$\coprod_{G\in\mathrm{GalGps}} \coprod_{i\in\mathbb{N}} \coprod_{\iota\in\frac{\mathrm{Emb}(G,H)}{\mathrm{Aut}(G)\times H}} \Gamma_{G,i}/\mathrm{N}_H(\iota(G)) \to \Delta_H$$

is geometrically bijective. Therefore, $\mathrm{M}_{\mathrm{st}}(X,\mathcal{P}) = \int_{\Delta_H} \mathbb{L}^{d-v}$. ☐

Remark 14.5. If X is a formal affine scheme of finite type over Df with an action of a finite group H, then we can embed it by an equivariant closed immersion into an affine space $\mathbb{A}^n_{\mathrm{Df}}$ with a linear action of H (see [Yas16, Remark 7.1]). By Lemma 6.24, the stack $\mathrm{Utg}_\Gamma([X/H])^{\mathrm{pur}}$ is a closed substack of $\mathrm{Utg}_\Gamma([\mathbb{A}^n_{\mathrm{Df}}/H])^{\mathrm{pur}}$. Therefore, for a twisted formal disk $\mathcal{E}$ over K, the fiber $\mathrm{Utg}_{\mathcal{E}}([X/H])^{\mathrm{pur}} = \mathrm{Utg}_\Gamma([X/H])^{\mathrm{pur}} \times_{\Gamma,\mathcal{E}} \operatorname{Spec} K$ is essentially the same as the untwisting variety denoted by $\mathbf{v}^{|F|}$ in [Yas16].

15. The tame case

In this section, we suppose that $\mathcal{X}$ is a *tame* formal DM stack of finite type over Df; that is, the automorphism group $\mathrm{Aut}(x)$ of every geometric point $x\colon \operatorname{Spec} K \to \mathcal{X}$ has order prime to p (we follow the convention that 0 is prime to every positive integer so that the above assumption holds whenever $p = 0$). We also suppose that $\mathcal{X}$ is spacious and generically smooth over Df, as before. Specializing to this situation, we see how to recover results in [Yas06] from the more general ones in the present paper.

For a wild Galoisian group G, the stack $\mathrm{Utg}_{\Gamma_{[G]}}(\mathcal{X})$ is empty. Hence

$$\mathrm{Utg}_\Gamma(\mathcal{X}) = \mathrm{Utg}_{\Lambda_{\mathrm{tame}}}(\mathcal{X}) = \coprod_{l>0;\, p\nmid l} \mathrm{Utg}_{\Lambda_{[C_l]}}(\mathcal{X}) .$$

Theorem 13.3 is now rephrased as follows.

Proposition 15.1. *Let $(\mathcal{X}, A)$ be a log pair over* Df *such that $\mathcal{X}$ is tame. Let A_l^{utg} be the induced boundary divisor on $\mathrm{Utg}_{\Lambda_{[C_l]}}(\mathcal{X})^{\mathrm{pur}}$. Let $C \subset |\mathcal{X}|$ be a constructible subset and $C_l^{\mathrm{utg}} \subset |\mathrm{Utg}_{\Lambda_{[C_l]}}(\mathcal{X})^{\mathrm{pur}}|$ the induced constructible subsets. Then we have*

$$\mathrm{M}_{\mathrm{st}}(\mathcal{X}, A)_C = \sum_{l>0;\, l\nmid p} \mathrm{M}_{\mathrm{st}}^{\mathrm{utd}}\big(\mathrm{Utg}_{\Lambda_{[C_l]}}(\mathcal{X})^{\mathrm{pur}}, A_l^{\mathrm{utg}}\big)_{C_l^{\mathrm{utg}}} .$$

Lemma 15.2. *Suppose that a tame finite group G acts on $k[\![t]\!][\![x_1,\dots,x_n]\!]$ over $k[\![t]\!]$. Then there exist coordinates $y_1,\dots,y_n \in k[\![t]\!][\![x_1,\dots,x_n]\!]$ (that is, $k[\![t]\!][\![y_1,\dots,y_n]\!] = k[\![t]\!][\![x_1,\dots,x_n]\!]$) such that the action preserves the subring $k[\![y_1,\dots,y_n]\!]$ and the induced action on $k[\![y_1,\dots,y_n]\!]$ is k-linear with respect to $y_1,\dots,y_n$.*

Proof. As is well known, we can linearize the action over k; there exists a regular system of parameters $z_0,\dots,z_n$ of $k[\![t]\!][\![x_1,\dots,x_n]\!]$ for which the G-action is k-linear. Since G is tame, by

Maschke's theorem, linear G-representations over k are semisimple. If $\mathfrak{m}$ denotes the maximal ideal of $k[\![t]\!][\![x_1,\dots,x_n]\!]$, then the image of the map of linear representations

$$(t)/(t^2) \to \mathfrak{m}/\mathfrak{m}^2 \cong \bigoplus_i k \cdot z_i$$

is a direct summand. Let $V \subset \bigoplus_i k \cdot z_i$ be a complementary subrepresentation of it, and let $y_1,\dots,y_n$ be a basis of V. Since $t, y_1,\dots,y_n$ generates $\mathfrak{m}/\mathfrak{m}^2$, we have

$$k[\![t]\!][\![x_1,\dots,x_n]\!] = k[\![t]\!][\![y_1,\dots,y_n]\!] .$$

We easily see that $y_1,\dots,y_n$ satisfies the desired property. □

Lemma 15.3. *If $\mathcal{X}$ is smooth of pure relative dimension d over* Df, *then for every l, the stack* $\mathrm{Utg}_{\Lambda_{[C_l]}}(\mathcal{X})^{\mathrm{pur}}$ *is smooth of pure relative dimension d over* $\mathrm{Df}_{\Lambda_{[C_l]}}$, *unless it is empty.*

Proof. We may assume that k is algebraically closed. A k-point of $\mathrm{Utg}_{\Lambda_{[C_l]}}(\mathcal{X})$ corresponds to a morphism $\mathcal{E}_{\Lambda_{[C_l]},0} \to \mathcal{X}$. Let $x\colon \operatorname{Spec} k \to \mathcal{X}$ be the induced k-point. We have the corresponding closed embedding $\mathcal{Y} := \mathrm{B}\operatorname{Aut}(x) \hookrightarrow \mathcal{X}$. The completion $\widehat{\mathcal{X}}$ of $\mathcal{X}$ along this closed substack is isomorphic to $[\operatorname{Spf} k[\![t]\!][\![x_1,\dots,x_d]\!]/\operatorname{Aut}(x)]$. By Lemma 15.2, we may suppose that the $\operatorname{Aut}(x)$-action on $k[\![t]\!][\![x_1,\dots,x_d]\!]$ is the scalar extension of a k-linear action on $k[\![x_1,\dots,x_d]\!]$. For $i \gg 0$, the given k-point of $\mathrm{Utg}_{\Lambda_{[C_l]}}(\mathcal{X})$ sits in $\mathrm{Utg}_{\Lambda_{[C_l]}}(\mathcal{Y}_{(i)})$, where $\mathcal{Y}_{(i)}$ is the ith infinitesimal neighborhood of $\mathcal{Y}$. By Lemma 6.25, the completion of $\mathrm{Utg}_{\Lambda_{[C_l]}}(\mathcal{X})$ along $\mathrm{Utg}_{\Lambda_{[C_l]}}(\mathcal{Y}_{(i)})$ is isomorphic to $\mathrm{Utg}_{\Lambda_{[C_l]}}(\widehat{\mathcal{X}})$. On the other hand, $\mathrm{Utg}_{\Lambda_{[C_l]}}(\widehat{\mathcal{X}})$ is also isomorphic to the completion of $\mathrm{Utg}_{\Lambda_{[C_l]}}\big([\mathbb{A}^d_{\mathrm{Df}}/\operatorname{Aut}(x)]\big)$ along some closed substack. The lemma follows from Lemma 14.1. □

Recall $\Lambda_{\mu_l} \cong \coprod_{(\mathbb{Z}/l\mathbb{Z})^*} \mathrm{B}\,\mu_l$ (see Section 5.4). If $\Lambda_{\mu_l,1}$ denotes its first component, then the morphism $\Lambda_{\mu_l,1} \to \Lambda_{[C_l]}$ is an isomorphism (see (5.1)). Hence the morphism

$$\mathrm{Utg}_{\Lambda_{\mu_l,1}}(\mathcal{X})_0 \to \mathrm{Utg}_{\Lambda_{[C_l]}}(\mathcal{X})_0$$

is also an isomorphism. We define a morphism

$$\mathrm{Utg}_{\Lambda_{\mu_l,1}}(\mathcal{X})_0 \to \mathrm{I}^{(\mu_l)}\,\mathcal{X}_0$$

as follows. For an S-point of $\mathrm{Utg}_{\Lambda_{\mu_l,1}}(\mathcal{X})$ corresponding to $\mathcal{E}_S = [E_S/\mu_l] \to \mathcal{X}$, we have the induced closed immersion $\mathrm{B}\,\mu_l \times S \hookrightarrow \mathcal{E}_S$. With the above S-point of $\mathrm{Utg}_{\Lambda_{\mu_l,1}}(\mathcal{X})$, we associate the S-point of $\mathrm{I}^{(\mu_l)}\,\mathcal{X}$ given by the composition $\mathrm{B}\,\mu_l \times S \to \mathcal{E}_S \to \mathcal{X}$.

Now suppose that $\mathcal{X}$ is smooth and of pure relative dimension d over Df. Let $\alpha\colon \operatorname{Spec} K \to \mathrm{I}^{(\mu_l)}\,\mathcal{X}_0$ be a geometric point. The tangent space $T_{\mathcal{X}_{0,K},x}$ at the induced point $x\colon \operatorname{Spec} K \to \mathcal{X}_{0,K}$, which is a d-dimensional K-vector space, has an $\operatorname{Aut}(x)$-action. The point α induces an injection $C_l \cong \mu_{l,K} \hookrightarrow \operatorname{Aut}(x)$ and a $\mu_{l,K}$-action on $T_{\mathcal{X}_{0,K},x}$.

Lemma 15.4. *With notation as above, the morphism* $\mathrm{Utg}_{\Lambda_{\mu_l,1}}(\mathcal{X})_0 \to \mathrm{I}^{(\mu_l)}\,\mathcal{X}_0$ *is representable, and its fiber over* $\alpha \in \big(\mathrm{I}^{(\mu_l)}\,\mathcal{X}_0\big)(K)$ *is isomorphic to* $\mathbb{A}^c_K$, *where c is the codimension of the fixed locus* $(T_{\mathcal{X}_{0,K},x})^{\mu_{l,K}}$ *in* $T_{\mathcal{X}_{0,K},x}$.

Proof. By Lemma A.5, the map $\operatorname{Aut}(\mathcal{E}_S \to \mathcal{X}_0) \to \operatorname{Aut}(\mathrm{B}\,\mu_l \times S \to \mathcal{X}_0)$ is injective. Therefore, the morphism of the lemma is representable.

Let $x\colon \operatorname{Spec} K \to \mathcal{X}_0$ be the point induced by α. Suppose that the completion of $\mathcal{X}_{0,K}$ at x is $[\operatorname{Spf} K[\![x_1,\dots,x_d]\!]/G]$, where G acts linearly on $K[\![x_1,\dots,x_d]\!]$. Then $\mu_{l,K}$ also acts on this power series ring through α. We can choose coordinates $x_1,\dots,x_d$ so that $\mu_{l,K}$ acts diagonally on $\bigoplus_i Kx_i$; for $\zeta\in\mu_{l,K}$, we have $\zeta\cdot x_i=\zeta^{a_i}x_i$ $(0\leqslant a_i<l)$. A K-point of the fiber over α corresponds to a $\mu_{l,K}$-equivariant morphism

$$\gamma\colon\ \operatorname{Spec} K\llbracket t^{1/l}\rrbracket/(t)\to \operatorname{Spf} K\llbracket x_1,\dots,x_d\rrbracket\,.$$

Such a morphism is determined by the images of x_i in $K\llbracket t^{1/l}\rrbracket/(t)$, which should be of the form

$$\gamma^*(x_i)\begin{cases}\in K\cdot t^{a_i/l} & (a_i\neq 0)\,,\\ =0 & (a_i=0)\,.\end{cases}$$

Note that γ^* is a morphism of local rings and x_i should be mapped into the maximal ideal $\left(t^{1/l}\right)$; thus $\gamma^*(x_i)$ cannot be a non-zero scalar in $K\cdot t^0$. This shows that the fiber over α is an affine space of dimension $\sharp\{i\mid \alpha_i\neq 0\}$, which is equal to the codimension of $(T_{\mathcal{X}_{0,K},x})^{\mu_{l,K}}$. □

Note that $\operatorname{Utg}_{\Lambda_{\mu_l,1}}(\mathcal{X})_0$ has pure dimension d, while $\mathrm{I}^{(\mu_l)}\,\mathcal{X}$ has the same dimension as $(T_{\mathcal{X}_{0,K},x})^{\mu_{l,K}}$ at α. Lemma 15.4 is compatible with this fact. The codimension of $(T_{\mathcal{X}_{0,K},x})^{\mu_{l,K}}$ depends only on the connected component of $\mathrm{I}^{(\mu_l)}\,\mathcal{X}_0$ on which α lies. The value of the function $\mathfrak{s}_{\mathcal{X}}$ as a point of $|\mathcal{J}_\infty\mathcal{X}|$ also depends only on the connected component of its image on $\mathrm{I}^{(\mu_l)}\,\mathcal{X}_0$ through the map

$$\mathcal{J}_\infty\mathcal{X}\to\mathcal{J}_0\mathcal{X}=\operatorname{Utg}_{\Lambda_{\mathrm{tame}}}(\mathcal{X})_0\to\mathrm{I}^{(\mu_l)}\,\mathcal{X}_0\,.$$

For a connected component $\mathcal{Z}\subset\mathrm{I}^{(\mu_l)}\,\mathcal{X}_0$, we write these values as $c(\mathcal{Z})$ and $\mathfrak{s}_{\mathcal{X}}(\mathcal{Z})$, respectively.

Corollary 15.5 (cf. [Yas06, Corollary 72]). *If $\mathcal{X}$ is smooth over* Df, *we have*

$$\mathrm{M}_{\mathrm{st}}(\mathcal{X})_C=\sum_{l>0;\,p\nmid l}\ \sum_{\mathcal{Z}\subset\mathrm{I}^{(\mu_l)}\,\mathcal{X}_0}\{C_{\mathcal{Z}}\}\mathbb{L}^{c(\mathcal{Z})+\mathfrak{s}_{\mathcal{X}}(\mathcal{Z})}\,,$$

where $\mathcal{Z}$ runs over connected components of $\mathrm{I}^{(\mu_l)}\,\mathcal{X}_0$ and $C_{\mathcal{Z}}$ denotes the preimage of C in $|\mathcal{Z}|$.

Proof. By Proposition 15.1, we have

$$\mathrm{M}_{\mathrm{st}}(\mathcal{X})_C=\sum_{l\nmid p}\mathrm{M}_{\mathrm{st}}^{\mathrm{utg}}\big(\operatorname{Utg}_{\Lambda_{[C_l]}}(\mathcal{X})^{\mathrm{pur}},A_l^{\mathrm{utg}}\big)_{C_l^{\mathrm{utg}}}\,.$$

Locally at the level of completion, $\mathcal{X}$ is the quotient stack associated with a linear action. The boundary A_l^{utg} is a vertical divisor; on each connected component $\mathcal{W}$ of $\operatorname{Utg}_{\Lambda_{[C_l]}}(\mathcal{X})$, the coefficient of this divisor is $-\mathfrak{s}_{\mathcal{X}}(\mathcal{W})$ (see the proof of Lemma 14.2). Therefore,

$$\mathrm{M}_{\mathrm{st}}(\mathcal{X})=\sum_{l\nmid p}\ \sum_{\mathcal{W}\subset\operatorname{Utg}_{\Lambda_{[C_l]}}(\mathcal{X})}\{\mathcal{W}\}\mathbb{L}^{\mathfrak{s}_{\mathcal{X}}(\mathcal{W})}\,.$$

By Lemma 15.4, this is equal to $\sum_{l\nmid p}\sum_{\mathcal{Z}\subset\mathrm{I}^{(\mu_l)}(\mathcal{X})}\{\mathcal{Z}\}\mathbb{L}^{c(\mathcal{Z})+\mathfrak{s}_{\mathcal{X}}(\mathcal{Z})}$. □

Remark 15.6. In the present paper, we adopt a definition of twisted jets slightly different from the one in [Yas06]. A twisted n-jet is a morphism from

$$\left[\left(\operatorname{Spec} K\llbracket t^{1/l}\rrbracket/\left(t^{n+1}\right)\right)/\mu_l\right]$$

in the present paper, while it is a morphism from

$$\left[\left(\operatorname{Spec} K\llbracket t^{1/l}\rrbracket/\left(t^{(nl+1)/l}\right)\right)/\mu_l\right]$$

in [Yas06]. Our stack $\mathrm{I}^{(\mu_l)}\mathcal{X}$ was the stack of twisted 0-jets of order l in [Yas06], which was denoted by $\mathcal{J}_0^l\mathcal{X}$. Lemma 15.4 shows that our motivic measure on $|\mathcal{J}_\infty\mathcal{X}|$ and the one given in [Yas06] differ by the factor of $\mathbb{L}^{c(\mathcal{Z})}$. The value of $\mathfrak{s}$ also differs in the present paper and the paper [Yas06] by $c(\mathcal{Z})$.

Remark 15.7. Corollary 15.5 readily implies a refinement of the Reid–Shepherd-Barron–Tai criterion for terminal/canonical tame quotient singularities, the homological McKay correspondence and Ruan's conjecture (and its l-adic cohomology version) on orbifold cohomology [Yas06, Corollaries 5–8]. (There is an error in [Yas06, Corollary 5]; we need to take the minimal discrepancy of divisors over X *having centers in the singular locus* rather than all divisors over X. See [Yas19] for the relation between this version of minimal discrepancy and stringy invariants.)

16. DM stacks over k

People are more interested in DM stacks of finite type over k than in formal DM stacks of finite type over Df. One reason for our use of formal stacks is that it is essential both in our construction of stacks of twisted arcs and in the proof of the change of variables formula. However, many results are also formulated for DM stacks of finite type over k and regarded as a special case of those results for formal stacks. For the reader's convenience, we state them below. We also give a few applications of them.

Let $\mathcal{X}$ be a generically smooth DM stack of finite type over k which has pure dimension d. This induces the formal DM stack $\mathcal{X}\times\mathrm{Df}$ of finite type over Df. We define the *stack of twisted arcs* on $\mathcal{X}$, denoted by $\mathcal{J}_\infty\mathcal{X}$, to be $\mathcal{J}_\infty(\mathcal{X}\times\mathrm{Df})$. We define the measure $\mu_\mathcal{X}$ and the shift function $\mathfrak{s}_\mathcal{X}$ on it to be $\mu_{\mathcal{X}\times\mathrm{Df}}$ and $\mathfrak{s}_{\mathcal{X}\times\mathrm{Df}}$, respectively. A point of $|\mathcal{J}_\infty\mathcal{X}|$ corresponds to a class of a representable morphism $\mathcal{E}\to\mathcal{X}$ from a twisted formal disk $\mathcal{E}$. With the obvious translation of notions to this setting, from Theorem 11.13, we obtain the following change of variables formula.

Theorem 16.1. *Let $\mathcal{Y}$ and $\mathcal{X}$ be generically smooth DM stacks of finite type over k which have pure dimension d. Let $f\colon\mathcal{Y}\to\mathcal{X}$ be a morphism such that Jac_f defines a closed substack of positive codimension. Let $A\subset|\mathcal{J}_\infty\mathcal{Y}|$ be such that $f_\infty|_A$ is almost geometrically injective. Let $h\colon f_\infty(A)\to\frac{1}{r}\mathbb{Z}\cup\{\infty\}$ be a measurable function. Then*

$$\int_{f_\infty(A)}\mathbb{L}^{h+\mathfrak{s}_\mathcal{X}}\,d\mu_\mathcal{X}=\int_A\mathbb{L}^{h\circ f_\infty-\mathrm{j}_f+\mathfrak{s}_\mathcal{Y}}\,d\mu_\mathcal{Y}\in\widehat{\mathcal{M}'_{k,r}}\cup\{\infty\}\,.$$

We can also define a log pair $(\mathcal{X},A)$ for a DM stack $\mathcal{X}$ of finite type over k which is generically smooth and normal. We define its stringy motive $\mathrm{M}_{\mathrm{st},Z}(\mathcal{X},A)\in\widehat{\mathcal{M}'_{Z,r}}\cup\{\infty\}$, where Z is an algebraic space almost of finite type, given with a morphism $\Gamma_\mathcal{X}\to Z$. In this setting, it is more natural to consider (not necessarily representable) proper birational morphisms instead of pseudo-modifications. Recall that a morphism $\mathcal{Y}\to\mathcal{X}$ of DM stacks is said to be *birational* if it restricts to an isomorphism $\mathcal{Y}'\to\mathcal{X}'$ of open dense substacks $\mathcal{Y}'\subset\mathcal{X}$ and $\mathcal{X}'\subset\mathcal{X}$. From Theorem 13.8, we obtain the following.

Theorem 16.2. *Let $f\colon\mathcal{Y}\to\mathcal{X}$ be a proper birational morphism of DM stacks of finite type over k which induces a crepant morphism $(\mathcal{Y},B)\to(\mathcal{X},A)$ of log pairs over k. Let $C\subset|\mathcal{X}|$ be a constructible subset and $\tilde{C}\subset|\mathcal{Y}|$ its preimage. Then*

$$\mathrm{M}_{\mathrm{st}}(\mathcal{X},A)_C=\mathrm{M}_{\mathrm{st}}(\mathcal{Y},B)_{\tilde{C}}\,.$$

From Corollary 14.4, we obtain the following.

Corollary 16.3 (Wild McKay correspondence). *Suppose that a finite group G acts linearly on $\mathbb{A}^d_k$. Let $X := \mathbb{A}^d_k/G$, and let A be the boundary on X such that $\mathbb{A}^d_k \to (X, A)$ is crepant.* (*Note that if the morphism $\mathbb{A}^d_k \to X$ is étale in codimension* 1, *then $A = 0$.*) *Then we have*

$$\mathrm{M}_{\mathrm{st}}(X, A) = \int_{\Delta_G} \mathbb{L}^{d-v}.$$

We can use the wild McKay correspondence to evaluate the discrepancy of quotient singularities, an invariant of singularities appearing in the minimal model program. Consider the invariant

$$\mathrm{discrep}(\mathrm{centers} \subset X_{\mathrm{sing}}; X)$$

defined to be the infimum of the discrepancies of divisors over X with center included in the singular locus X_{sing} (see [Kol13, Definition 4.7.4] or [Yas19, Introduction]). Let $o \in \Delta_G$ be the point corresponding to the trivial torsor, and let $\Delta_G \setminus \{o\} = \bigsqcup_i C_i$ be a stratification into countably many constructible subsets (of finite dimensions) such that v is constant on each stratum. We define

$$\dim \int_{\Delta_G \setminus \{o\}} \mathbb{L}^{d-v} = \sup_i \{\dim C_i + d - v(C_i)\} \in \mathbb{Q} \cup \{\infty\}.$$

This is independent of the choice of the stratification.

Corollary 16.4. *We keep the notation of Corollary 16.3. Suppose that the morphism $\mathbb{A}^d_k \to X$ is étale in codimension* 1.

(1) *We have*

$$\mathrm{discrep}(\mathrm{centers} \subset X_{\mathrm{sing}}; X) = d - 1 - \max\left\{\dim X_{\mathrm{sing}}, \dim \int_{\Delta_G \setminus \{o\}} \mathbb{L}^{d-v}\right\}.$$

(2) *If the integral $\int_{\Delta_G} \mathbb{L}^{d-v}$ converges, then X is log terminal. If X has a log resolution, then the converse is also true.*

Proof. Assertions (1) and (2) can be proved in the same way as they are proved in [Yas19] and [Yas14], respectively, for $G = \mathbb{Z}/p\mathbb{Z}$. We only give a sketch.

(1) We can similarly define the dimension of $\mathrm{M}_{\mathrm{st}}(X)_{X_{\mathrm{sing}}} \in \mathbb{Z} \cup \{\infty\}$, whether the integral diverges or not. Evaluating it by using infinitely many modifications $Y \to X$, we get

$$\mathrm{discrep}(\mathrm{centers} \subset X_{\mathrm{sing}}; X) = d - 1 - \dim \mathrm{M}_{\mathrm{st}}(X)_{X_{\mathrm{sing}}}$$

(see [Yas19, Proposition 2.1]). On the other hand, the version of the wild McKay correspondence for $\mathrm{M}_{\mathrm{st}}(X)_{X_{\mathrm{sing}}}$ shows

$$\mathrm{M}_{\mathrm{st}}(X)_{X_{\mathrm{sing}}} = \{X_{\mathrm{sing}}\} + \int_{\Delta_G \setminus \{o\}} \mathbb{L}^{d-v}.$$

Note that the term $\{X_{\mathrm{sing}}\}$ on the right side is the contribution of the trivial torsor to $\mathrm{M}_{\mathrm{st}}(X)_{X_{\mathrm{sing}}}$. Assertion (1) follows from these.

(2) The convergence of $\mathrm{M}_{\mathrm{st}}(X)$ implies that X is log terminal. The converse is true if X has a log resolution. By the wild McKay correspondence, the convergence of $\mathrm{M}_{\mathrm{st}}(X)$ is equivalent to that of $\int_{\Delta_G} \mathbb{L}^{d-v}$. □

Let us consider the case where $G = S_n$, the symmetric group, and let $\mathbf{a}\colon \Delta_{S_n} \to \mathbb{Z}$ be the Artin conductor. This function associates with a geometric point $\operatorname{Spec} K \to \Delta_{S_n}$ the Artin conductor (see [Ser79, Chapter VI]) of the corresponding map $G_{K(\!(t)\!)} \to S_n \subset \mathrm{GL}_n(\mathbb{C})$, where $G_{K(\!(t)\!)}$ denotes the absolute Galois group of $K(\!(t)\!)$ and S_n is embedded into $\mathrm{GL}_n(\mathbb{C})$ by the standard permutation representation. As an application of Corollary 16.3, we obtain the motivic version of Bhargava's mass formula [Bha07].

Corollary 16.5. *We have the following equality in* $\widehat{\mathcal{M}'_k}$*:*

$$\int_{\Delta_{S_n}} \mathbb{L}^{-\mathbf{a}} = \sum_{i=0}^{n-1} P(n, n-i)\mathbb{L}^{-i}\,.$$

Here $P(n,m)$ *denotes the number of partitions of* n *into exactly* m *parts.*

Proof. Consider the action of S_n on $\mathbb{A}_k^{2n} = \left(\mathbb{A}_k^2\right)^n$ by permutation. The quotient variety $X = \mathbb{A}_k^{2n}/S_n$ is the symmetric product $S^n\mathbb{A}_k^2$; it has a crepant resolution by the Hilbert scheme of n points on $\mathbb{A}_k^2$, denoted by $\mathrm{Hilb}^n\left(\mathbb{A}_k^2\right) \to X$; see [Bea83, BK05, KT01]. From a cell decomposition of $\mathrm{Hilb}^n\left(\mathbb{A}_k^2\right)$ as in [ES87] (see also [CV08] for the case of a general base field), we can deduce

$$\mathrm{M}_{\mathrm{st}}(X) = \left\{\mathrm{Hilb}^n\left(\mathbb{A}_k^2\right)\right\} = \sum P(n, n-i)\mathbb{L}^{2n-i}\,.$$

On the other hand, by [WY15, Theorem 4.8], the function $\mathbf{a}$ is equal to the v-function associated with the action $S_n \curvearrowright \mathbb{A}_k^{2n}$. The corollary now follows from Corollary 16.3. □

Appendix. General results on stacks

In this appendix, we collect general results on (mainly DM) stacks which are necessary in this paper. Some of them would be well known to specialists, but the author could not find them in the literature.

We first recall that a morphism $f\colon \mathcal{Y} \to \mathcal{X}$ of DM stacks is representable if and only if for every geometric point $y\colon \operatorname{Spec} K \to \mathcal{Y}$, the group homomorphism $\mathrm{Aut}_{\mathcal{Y}}(y) \to \mathrm{Aut}_{\mathcal{X}}(f(y))$ is injective [AV02, Lemma 4.4.3]. We say that f is *stabilizer preserving* if for every y, the map $\mathrm{Aut}_{\mathcal{Y}}(y) \to \mathrm{Aut}_{\mathcal{X}}(f(y))$ is an isomorphism. If $\mathcal{Y}$ lies over another stack $\mathcal{S}$, then we can consider the kernel $\mathrm{Aut}_{\mathcal{Y}/\mathcal{S}}(y) \subset \mathrm{Aut}_{\mathcal{Y}}(y)$ of $\mathrm{Aut}_{\mathcal{Y}}(y) \to \mathrm{Aut}_{\mathcal{S}}(\bar{y})$ with $\bar{y} \in \mathcal{S}$ the image of y.

Lemma A.1. *Suppose that* $f\colon \mathcal{Y} \to \mathcal{X}$ *is an* $\mathcal{S}$*-morphism, and let* y *be a geometric point of* $\mathcal{Y}$*. Then* $\mathrm{Aut}_{\mathcal{Y}}(y) \to \mathrm{Aut}_{\mathcal{X}}(f(y))$ *is injective if and only if* $\mathrm{Aut}_{\mathcal{Y}/\mathcal{S}}(y) \to \mathrm{Aut}_{\mathcal{X}/\mathcal{S}}(f(y))$ *is injective.*

Proof. The "only if" is clear. We prove the "if" part. For $\alpha \in \mathrm{Aut}_{\mathcal{S}}(\bar{y})$, if $\mathrm{Aut}_{\mathcal{Y}}(y)_\alpha$ and $\mathrm{Aut}_{\mathcal{X}}(f(y))_\alpha$ denote the preimages of α in $\mathrm{Aut}_{\mathcal{Y}}(y)$ and $\mathrm{Aut}_{\mathcal{X}}(f(y))$, respectively, and if these preimages are not empty, then each map $\mathrm{Aut}_{\mathcal{Y}}(y)_\alpha \to \mathrm{Aut}_{\mathcal{X}}(f(y))_\alpha$ is isomorphic to the map

$$\mathrm{Aut}_{\mathcal{Y}}(y)_0 = \mathrm{Aut}_{\mathcal{Y}/\mathcal{S}}(y) \to \mathrm{Aut}_{\mathcal{X}}(f(y))_0 = \mathrm{Aut}_{\mathcal{X}/\mathcal{S}}(f(y))$$

as a map of sets. This shows the "if" part. □

Lemma A.2. *Let* $g\colon \mathcal{Z} \to \mathcal{Y}$ *and* $f\colon \mathcal{Y} \to \mathcal{X}$ *be morphisms of stacks. Let* $\mathbf{P}$ *be a property of morphisms of stacks which is stable under base change and composition. If the diagonals* Δ_g *and* Δ_f *have property* $\mathbf{P}$*, then so does* $\Delta_{f\circ g}$*.*

Proof. Consider the 2-commutative diagram

$$\begin{tikzcd} & \mathcal{Z} \arrow[d, "\Delta_g"] \arrow[dd, bend right=60, "\Delta_{f\circ g}"'] & \\ & \mathcal{Z} \times_{\mathcal{Y}} \mathcal{Z} \arrow[r] \arrow[d, "\phi"] & \mathcal{Y} \arrow[d, "\Delta_f"] \\ & \mathcal{Z} \times_{\mathcal{X}} \mathcal{Z} \arrow[r] & \mathcal{Y} \times_{\mathcal{X}} \mathcal{Y}\,. \end{tikzcd}$$

The square in the diagram is 2-Cartesian. Therefore, ϕ has property $\mathbf{P}$. It follows that $\Delta_{f\circ g} \cong \phi \circ \Delta_g$ has property $\mathbf{P}$. □

Lemma A.3. *Let*

$$\begin{tikzcd} \mathcal{X}_T \arrow[r] \arrow[d] & \mathcal{X} \arrow[d] \\ T \arrow[r] & \mathcal{S} \end{tikzcd}$$

be a 2-Cartesian diagram of stacks. Suppose that $\mathcal{S}$ and $\mathcal{X}_T$ are DM stacks and $T \to \mathcal{S}$ is an atlas. Then $\mathcal{X}$ is a DM stack.

Proof. By the assumption, the diagonal $\Delta_{\mathcal{X}_T/T}$ is a representable, unramified and finite morphism. This is the base change of $\Delta_{\mathcal{X}/\mathcal{S}}$ along $T \to \mathcal{S}$. Therefore, $\Delta_{\mathcal{X}/\mathcal{S}}$ is also a representable, unramified and finite morphism. By Lemma A.2, the morphism $\Delta_{\mathcal{X}} = \Delta_{\mathcal{X}/\operatorname{Spec} k}$ is also representable, unramified and finite. If $U \to \mathcal{X}_T$ is an atlas, then so is $U \to \mathcal{X}_T \to \mathcal{X}$. We have proved the lemma. □

Lemma A.4. *Let $h\colon \mathcal{Z} \to \mathcal{Y}$ and $f\colon \mathcal{Y} \to \mathcal{X}$ be morphisms of DM stacks. Suppose that f is representable* (*respectively, stabilizer preserving*) *and unramified. Then $\operatorname{Aut}(h) \to \operatorname{Aut}(f \circ h)$ is injective* (*respectively, bijective*).

Proof. Consider the morphisms

$$\mathrm{I}\,\mathcal{Y} \xrightarrow{b} \mathrm{I}\,\mathcal{X} \times_{\mathcal{X}} \mathcal{Y} \xrightarrow{a} \mathcal{Y}\,.$$

Since $a \circ b$ is representable, so is b. Since $a \circ b$ is finite and a is separated, b is finite. Since $a \circ b$ and a are unramified, b is unramified. Thus b is representable, finite and unramified. When f is representable, b is also universally injective. By [Sta20, tag 04XV], the morphism b is a closed immersion. When f is stabilizer preserving, by [Sta20, tag 0DU9], the morphism b is an isomorphism. Therefore, for each object $u \in \mathcal{Z}$, say over $U \in \mathbf{Aff}$, we have an injection (respectively, an isomorphism) $\underline{\operatorname{Aut}}(h(u)) \to \underline{\operatorname{Aut}}(f \circ h(u))$ of group schemes over U and an injection (respectively, an isomorphism) $\operatorname{Aut}(h(u)) \to \operatorname{Aut}(f \circ h(u))$ of groups. An automorphism $\operatorname{Aut}(h)$ is by definition a collection of automorphisms $\alpha_u \in \operatorname{Aut}(h(u))$, $u \in \mathcal{Z}$, with a suitable compatibility. Therefore, the lemma follows. □

Lemma A.5. *Let $h\colon \mathcal{Z} \to \mathcal{Y}$ and $f\colon \mathcal{Y} \to \mathcal{X}$ be morphisms of DM stacks. Suppose that h is a thickening. Then the map $\operatorname{Aut}(f) \to \operatorname{Aut}(f \circ h)$ is injective.*

Proof. Let $s\colon S \to \mathcal{Y}$ be a morphism from an affine scheme, and let $t\colon T := S \times_{\mathcal{Y}} \mathcal{Z} \to \mathcal{Z}$ be the projection. Automorphisms $\alpha \in \operatorname{Aut}(f \circ s)$ and $\beta \in \operatorname{Aut}(f \circ h \circ t)$ with $\alpha \mapsto \beta$ induce the following commutative diagram:

$$\begin{tikzcd} \underline{\operatorname{Aut}}_T(f \circ h \circ t) \arrow[r] \arrow[d, shift right] & \underline{\operatorname{Aut}}_S(f \circ s) \arrow[d, shift right] \\ T \arrow[r, hook] \arrow[u, shift right, "\beta_s"'] & S\,. \arrow[u, shift right, "\alpha_s"'] \end{tikzcd}$$

Since $\mathcal{X}$ is a DM stack, the morphism $\underline{\mathrm{Aut}}_S(f \circ s) \to S$ is unramified. Since $T \to S$ is a thickening, α_s is determined by β_s. Since we have this for each object $s \in \mathcal{Y}(S)$, we have proved the lemma. □

Lemma A.6. *Let $f\colon \mathcal{Y} \to \mathcal{X}$ be a stabilizer-preserving étale morphism of DM stacks such that for every algebraically closed field K, the map $\mathcal{Y}[K] \to \mathcal{X}[K]$ is bijective. Then f is an isomorphism.*

Proof. Let $V \to \mathcal{X}$ be an atlas. It suffices to show that $V \times_{\mathcal{X}} \mathcal{Y} \to V$ is an isomorphism. This is an étale surjective morphism of algebraic spaces. It suffices to show that $(V \times_{\mathcal{X}} \mathcal{Y})(K) \to V(K)$ is injective. In turn, it suffices to show that for every geometric point $x\colon \operatorname{Spec} K \to \mathcal{X}$, we have $(\operatorname{Spec} K \times_{\mathcal{X}} \mathcal{Y})_{\mathrm{red}} \cong \operatorname{Spec} K$. Let G be the automorphism group of x. Then we have the induced closed immersion $\mathrm{B}\,G = [\operatorname{Spec} K/G] \hookrightarrow \mathcal{X}_K$. Since $\mathcal{Y} \to \mathcal{X}$ is stabilizer preserving, we have

$$(\mathrm{B}\,G \times_{\mathcal{X}_K} \mathcal{Y}_K)_{\mathrm{red}} \cong (\mathrm{B}\,G \times_{\mathcal{X}} \mathcal{Y})_{\mathrm{red}} \cong \mathrm{B}\,G\,.$$

Therefore, we have

$$(\operatorname{Spec} K \times_{\mathcal{X}} \mathcal{Y})_{\mathrm{red}} \cong (\operatorname{Spec} K \times_{\mathrm{B}\,G} \mathrm{B}\,G \times_{\mathcal{X}} \mathcal{Y})_{\mathrm{red}} \cong \operatorname{Spec} K\,.$$ □

Lemma A.7. *Let $f\colon \mathcal{Y} \to \mathcal{X}$ be a representable morphism of DM stacks which is formally étale (in the sense that for every morphism $V \to \mathcal{X}$ from algebraic space, $\mathcal{Y} \times_{\mathcal{X}} V \to V$ is formally étale). Let $\iota\colon \mathcal{D} \hookrightarrow \mathcal{E}$ be a thickening of DM stacks. Suppose that we have the following 2-commutative diagram:*

$$\begin{tikzcd} \mathcal{D} \arrow[r] \arrow[d, "\iota"'] & \mathcal{Y} \arrow[d, "f"] \\ \mathcal{E} \arrow[r] \arrow[ur, phantom, "\eta"] & \mathcal{X}\,. \end{tikzcd}$$

Then there exist a dashed arrow γ (1-morphism) and two thick arrows α, β (2-morphisms) forming a 2-commutative diagram

$$\begin{tikzcd} \mathcal{D} \arrow[rr] \arrow[d, "\iota"'] \arrow[dr, phantom, "\alpha"] & & \mathcal{Y} \arrow[d, "f"] \\ \mathcal{E} \arrow[rru, dashed, "\gamma"] \arrow[rr] & \arrow[ur, phantom, "\beta"] & \mathcal{X} \end{tikzcd}$$

which induces η. Moreover, for two such tuples (γ, α, β) and $(\gamma', \alpha', \beta')$, there exists a unique 2-isomorphism $\gamma \Rightarrow \gamma'$ through which α and α' as well as β and β' correspond to each other.

Proof. Consider the 2-commutative diagram

$$\begin{tikzcd} \mathcal{D} \arrow[dr] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\ & \mathcal{E} \times_{\mathcal{X}} \mathcal{Y} \arrow[r] \arrow[d] & \mathcal{Y} \arrow[d, "f"] \\ & \mathcal{E} \arrow[r] & \mathcal{X}\,. \end{tikzcd}$$

The two triangles and the square in this diagram are given witnessing 2-isomorphisms which induce η. The image of $|\mathcal{D}| \to |\mathcal{E} \times_{\mathcal{X}} \mathcal{Y}|$ is an open and closed subset. Let $\mathcal{E}'$ be the corresponding open and closed substack. By Lemma A.6, the morphism $\mathcal{E}' \to \mathcal{E}$ is an isomorphism, which we denote by a. We choose the quasi-inverse $b\colon \mathcal{E} \to \mathcal{E}'$ of the natural isomorphism $a\colon \mathcal{E}' \to \mathcal{E}$ and witnessing 2-isomorphisms α and β of the two right triangles in the following diagram which induce the identity of a:

$$\begin{tikzcd}
\mathcal{D} \arrow[dr] \arrow[ddr, bend right] \arrow[drrrr, bend left] \\
& \mathcal{E}' \arrow[rr, "\mathrm{id}"] \arrow[d, "a"'] & & \mathcal{E}' \arrow[r] \arrow[d, "a"] & \mathcal{Y} \arrow[d] \\
& \mathcal{E} \arrow[rr, "\mathrm{id}"'] \arrow[urr, dashed, "b"] \arrow[ur, phantom, "\Nearrow\alpha"] & \arrow[ur, phantom, "\Nearrow\beta"] & \mathcal{E} \arrow[r] & \mathcal{X}\,.
\end{tikzcd}$$

Indeed, such a choice exists. First we choose any isomorphism $\alpha\colon \mathrm{id}_{\mathcal{E}'} \to b \circ a$, which induces $a(\alpha)\colon a \to a \circ b \circ a$ and an isomorphism $\mathrm{Iso}(a \circ b \circ a, a) \to \mathrm{Aut}(a)$. We also have an isomorphism $\mathrm{Aut}(a \circ b, \mathrm{id}_{\mathcal{E}}) \to \mathrm{Iso}(a \circ b \circ a, a)$ induced by a. We choose $\beta \in \mathrm{Aut}(a \circ b, \mathrm{id}_{\mathcal{E}})$ to be the one corresponding to id_a through the two isomorphisms. The first assertion of the lemma follows. For another quasi-inverse $b'\colon \mathcal{E} \to \mathcal{E}'$ and an isomorphism $\alpha'\colon \mathrm{id}_{\mathcal{E}'} \to b' \circ a$, we obtain a unique isomorphism $\xi\colon b \to b'$ compatible with α and α'; for any $e' \in \mathcal{E}'$, the morphism $\xi\colon b(a(e')) \to b'(a(e'))$ is given by $\alpha' \circ \alpha^{-1}$. $\square$

Lemma A.8. *Let $\mathcal{Y} \to \mathcal{X}$ be a thickening of DM stacks, and let $U \to \mathcal{Y}$ be an étale morphism from an algebraic space (respectively, scheme, affine scheme). There exist a thickening $U \hookrightarrow \tilde{U}$ of algebraic spaces (respectively, schemes, affine schemes) and an étale morphism $\tilde{U} \to \mathcal{X}$ such that the diagram*

$$\begin{tikzcd}
U \arrow[r] \arrow[d] & \tilde{U} \arrow[d] \\
\mathcal{Y} \arrow[r] & \mathcal{X}
\end{tikzcd}$$

is 2-Cartesian. For another pair $\left(U \hookrightarrow \tilde{U}', \tilde{U}' \to \mathcal{X}\right)$ of such morphisms, there exists a unique isomorphism $\tilde{U} \xrightarrow{\sim} \tilde{U}'$ compatible with these morphisms.

Proof. We take an étale groupoid in affine schemes, $W \rightrightarrows V$ inducing $\mathcal{X}$. Let $W_{\mathcal{Y}} \rightrightarrows V_{\mathcal{Y}}$ and $W_U \rightrightarrows V_U$ be its base changes by $\mathcal{Y} \to \mathcal{X}$ and $U \to \mathcal{X}$, respectively. These are again étale groupoids in affine schemes. The morphisms $W_U \to W_{\mathcal{Y}}$ and $V_U \to V_{\mathcal{Y}}$ are étale, and the morphisms $W_{\mathcal{Y}} \to W$ and $V_{\mathcal{Y}} \to V$ are thickenings. By the invariance of the small étale site by a thickening [Sta20, tag 05ZH], they fit into the unique Cartesian diagrams

$$\begin{tikzcd}
V_U \arrow[r] \arrow[d] & V_{\tilde{U}} \arrow[d] & W_U \arrow[r] \arrow[d] & W_{\tilde{U}} \arrow[d] \\
V_{\mathcal{Y}} \arrow[r] & V\,, & W_{\mathcal{Y}} \arrow[r] & W\,.
\end{tikzcd}$$

Here horizontal arrows are thickenings and vertical arrows are étale. We obtain an étale groupoid $W_{\tilde{U}} \rightrightarrows V_{\tilde{U}}$, which induces a DM stack $\tilde{U}$. It fits into the Cartesian diagram

$$\begin{tikzcd}
U \arrow[r] \arrow[d] & \tilde{U} \arrow[d] \\
\mathcal{Y} \arrow[r] & \mathcal{X}
\end{tikzcd}$$

with horizontal arrows being thickenings and vertical arrows being étale. It follows that $\tilde{U}$ is an algebraic space. By [Sta20, tags 06AD and 05ZR], if U is a scheme (respectively, an affine scheme), then so is $\tilde{U}$. We have proved the first assertion.

For the second assertion, let $\tilde{U}'' \subset \tilde{U} \times_{\mathcal{X}} \tilde{U}'$ be the image of U, which is an open and closed subscheme. We get the third pair $(U \hookrightarrow \tilde{U}'', \tilde{U}'' \to \mathcal{X})$ with the same property. The projections $\tilde{U}'' \to \tilde{U}$ and $\tilde{U}'' \to \tilde{U}'$ are isomorphisms, giving the desired isomorphism $\tilde{U} \to \tilde{U}'$. □

Acknowledgements

This work relies on results obtained in my joint work with Fabio Tonini. I thank him for a fruitful cooperation. I started to write the present paper after the first draft of [TY23] was completed. I thank also Takashi Suzuki for helpful discussions.

References

Abb10 A. Abbes, *Éléments de géométrie rigide. Vol. I. Construction et étude géométrique des espaces rigides*, Progr. Math., vol. 286 (Birkhäuser, Basel, 2010); doi:10.1007/978-3-0348-0012-9.

AOV11 D. Abramovich, M. Olsson and A. Vistoli, *Twisted stable maps to tame Artin stacks*, J. Algebraic Geom. **20** (2011), no. 3, 399–477; doi:10.1090/S1056-3911-2010-00569-3.

AV02 D. Abramovich and A. Vistoli, *Compactifying the space of stable maps*, J. Amer. Math. Soc. **15** (2002), no. 1, 27–75; doi:10.1090/S0894-0347-01-00380-0.

Bat99a V. V. Batyrev, *Birational Calabi–Yau n-folds have equal Betti numbers*, in *New Trends in Algebraic Geometry*, London Math. Soc. Lecture Note Ser., vol. 264 (Cambridge Univ. Press, Cambridge, 1999), 1–11; doi:10.1017/CBO9780511721540.002.

Bat99b ———, *Non-Archimedean integrals and stringy Euler numbers of log-terminal pairs*, J. Eur. Math. Soc. (JEMS) **1** (1999), no. 1, 5–33; doi:10.1007/PL00011158.

Bea83 A. Beauville, *Variétés Kähleriennes dont la première classe de Chern est nulle*, J. Differential Geom. **18** (1983), no. 4, 755–782; doi:10.4310/jdg/1214438181.

Ber00 A. Bertapelle, *Formal Néron models and Weil restriction*, Math. Ann. **316** (2000), no. 3, 437–463; doi:10.1007/s002080050338.

Bha07 M. Bhargava, *Mass formulae for extensions of local fields, and conjectures on the density of number field discriminants*, Int. Math. Res. Not. IMRN **2007** (2007), no. 17, paper no. rnm052; doi:10.1093/imrn/rnm052.

BK05 M. Brion and S. Kumar, *Frobenius splitting methods in geometry and representation theory*, Progr. Math., vol. 231 (Birkhäuser, Boston, MA, 2005); doi:10.1007/b137486.

CNS18 A. Chambert-Loir, J. Nicaise and J. Sebag, *Motivic integration*, Progr. Math., vol. 325 (Birkhäuser, New York, 2018); doi:10.1007/978-1-4939-7887-8.

CV08 A. Conca and G. Valla, *Canonical Hilbert–Burch matrices for ideals of $k[x, y]$*, Michigan Math. J. **57** (2008), 157–172; doi:10.1307/mmj/1220879402.

DL99 J. Denef and F. Loeser, *Germs of arcs on singular algebraic varieties and motivic integration*, Invent. Math. **135** (1999), no. 1, 201–232; doi:10.1007/s002220050284.

DL02 ———, *Motivic integration, quotient singularities and the McKay correspondence*, Compos. Math. **131** (2002), no. 3, 267–290; doi:10.1023/A:1015565912485.

Eis95 D. Eisenbud, *Commutative algebra*, Grad. Texts in Math., vol. 150 (Springer, New York, 1995); doi:10.1007/978-1-4612-5350-1.

ES87 G. Ellingsrud and S. A. Strømme, *On the homology of the Hilbert scheme of points in the plane*, Invent. Math. **87** (1987), no. 2, 343–352; doi:10.1007/BF01389419.

Fog80 J. Fogarty, *Kähler differentials and Hilbert's fourteenth problem for finite groups*, Amer. J. Math. **102** (1980), no. 6, 1159–1175; doi:10.2307/2374183.

Frö76 A. Fröhlich, *Module conductors and module resolvents*, Proc. Lond. Math. Soc. (3) **32** (1976), no. 2, 279–321; doi:10.1112/plms/s3-32.2.279.

GWZ20 M. Groechenig, D. Wyss and P. Ziegler, *Mirror symmetry for moduli spaces of Higgs bundles via p-adic integration*, Invent. Math. **221** (2020), no. 2, 505–596; doi:10.1007/s00222-020-00957-8.

GD60 A. Grothendieck and J. Dieudonné, *Éléments de géométrie algébrique. I. Le langage des schémas*, Inst. Hautes Études Sci. Publ. Math. **4** (1960).

GD67 ______, *Éléments de géométrie algébrique. IV. Étude locale des schémas et des morphismes de schémas IV*, Inst. Hautes Études Sci. Publ. Math. **32** (1967).

Ill15 L. Illusie, *From Pierre Deligne's secret garden: looking back at some of his letters*, Jpn. J. Math. **10** (2015), no. 2, 237–248; doi:10.1007/s11537-015-1514-9.

Ito04 T. Ito, *Stringy Hodge numbers and p-adic Hodge theory*, Compos. Math. **140** (2004), no. 6, 1499–1517; doi:10.1112/S0010437X04001095.

KM97 S. Keel and S. Mori, *Quotients by groupoids*, Ann. of Math. (2) **145** (1997), no. 1, 193–213; doi:10.2307/2951828.

Kol13 J. Kollár, *Singularities of the minimal model program*, Cambridge Tracts in Math., vol. 200 (Cambridge Univ. Press, Cambridge, 2013); doi:10.1017/CBO9781139547895.

KT01 S. Kumar and J. F. Thomsen, *Frobenius splitting of Hilbert schemes of points on surfaces*, Math. Ann. **319** (2001), no. 4, 797–808; doi:10.1007/PL00004459.

LM71 M. D. Larsen and P. J. McCarthy, *Multiplicative theory of ideals*, Pure Appl. Math., vol. 43 (Academic Press, New York, 1971).

LW21 F. Loeser and D. Wyss, *Motivic integration on the Hitchin fibration*, Algebr. Geom. **8** (2021), no. 2, 196–230; doi:10.14231/ag-2021-004.

Loo02 E. Looijenga, *Motivic measures*, Astérisque **276** (2002), 267–297.

LP04 E. Lupercio and M. Poddar, *The global McKay–Ruan correspondence via motivic integration*, Bull. London Math. Soc. **36** (2004), no. 4, 509–515; doi:10.1112/S002460930300290X.

Nic11 J. Nicaise, *A trace formula for varieties over a discretely valued field*, J. reine angew. Math. **650** (2011), 193–238; doi:10.1515/CRELLE.2011.008.

NS11 J. Nicaise and J. Sebag, *The Grothendieck ring of varieties*, in *Motivic Integration and Its Interactions with Model Theory and Non-Archimedean Geometry. Vol. I*, London Math. Soc. Lecture Note Ser., vol. 383 (Cambridge Univ. Press, Cambridge, 2011), 145–188; doi:10.1017/CBO9780511667534.005.

Ols06 M. C. Olsson, $\underline{\mathrm{Hom}}$*-stacks and restriction of scalars*, Duke Math. J. **134** (2006), no. 1, 139–164; doi:10.1215/S0012-7094-06-13414-2.

Ols07 ______, *A boundedness theorem for Hom-stacks*, Math. Res. Lett. **14** (2007), no. 6, 1009–1021; doi:10.4310/MRL.2007.v14.n6.a9.

Ols15 ______, *Fujiwara's theorem for equivariant correspondences*, J. Algebraic Geom. **24** (2015), no. 3, 401–497; doi:10.1090/S1056-3911-2014-00628-7.

Ols16 ______, *Algebraic spaces and stacks*, Amer. Math. Soc. Colloq. Publ., vol. 62 (Amer. Math. Soc., Providence, RI, 2016); doi:10.1090/coll/062.

Rei02 M. Reid, *La correspondance de McKay*, Astérisque **276** (2002), 53–72.

Rom05 M. Romagny, *Group actions on stacks and applications*, Michigan Math. J. **53** (2005), no. 1, 209–236; doi:10.1307/mmj/1114021093.

Ron06 G. Rond, *Lemme d'Artin–Rees, théorème d'Izumi et fonction de Artin*, J. Algebra **299** (2006), no. 1, 245–275; doi:10.1016/j.jalgebra.2005.06.037.

Sai10 T. Saito, *Wild ramification of schemes and sheaves*, Proc. Internat. Congress of Math., Vol. 2 (Hindustan Book Agency, New Delhi, 2010), 335–356.

Seb04 J. Sebag, *Intégration motivique sur les schémas formels*, Bull. Soc. Math. France **132** (2004), no. 1, 1–54; doi:10.24033/bsmf.2458.

Ser79 J.-P. Serre, *Local fields*, Grad. Texts in Math., vol. 67 (Springer, New York, 1979); doi:10.1007/978-1-4757-5673-9.

Sta20 The Stacks Project Authors, *The Stacks Project*, 2020, https://stacks.math.columbia.edu.

Sun12 S. Sun, *L-series of Artin stacks over finite fields*, Algebra Number Theory **6** (2012), no. 1, 47–122; doi:10.2140/ant.2012.6.47.

TY21 M. Tanno and T. Yasuda, *The wild McKay correspondence for cyclic groups of prime power order*, Illinois J. Math. **65** (2021), no. 3, 619–654; doi:10.1215/00192082-9402078.

TLR07 L. A. Tarrío, A. J. López and M. P. Rodríguez, *Infinitesimal lifting and Jacobi criterion for smoothness on formal schemes*, Comm. Algebra **35** (2007), no. 4, 1341–1367; doi:10.1080/00927870601115823.

TY20 F. Tonini and T. Yasuda, *Moduli of formal torsors*, J. Algebraic Geom. **29** (2020), no. 4, 753–801; doi:10.1090/jag/771.

TY23 ———, *Moduli of formal torsors II*, Ann. Inst. Fourier **73** (2023), no. 2, 511–558; doi:10.5802/aif.3533.

Vil06 O. Villamayor U., *On flattening of coherent sheaves and of projective morphisms*, J. Algebra **295** (2006), no. 1, 119–140; doi:10.1016/j.jalgebra.2005.01.022.

WY15 M. M. Wood and T. Yasuda, *Mass formulas for local Galois representations and quotient singularities. I: a comparison of counting functions*, Int. Math. Res. Not. IMRN **2015** (2015), no. 23, 12590–12619; doi:10.1093/imrn/rnv074.

WY17 ———, *Mass formulas for local Galois representations and quotient singularities II: Dualities and resolution of singularities*, Algebra Number Theory **11** (2017), no. 4, 817–840; doi:10.2140/ant.2017.11.817.

Yas04 T. Yasuda, *Twisted jets, motivic measures and orbifold cohomology*, Compos. Math. **140** (2004), no. 2, 396–422; doi:10.1112/S0010437X03000368.

Yas06 ———, *Motivic integration over Deligne–Mumford stacks*, Adv. Math. **207** (2006), no. 2, 707–761; doi:10.1016/j.aim.2006.01.004.

Yas14 ———, *The p-cyclic McKay correspondence via motivic integration*, Compos. Math. **150** (2014), no. 7, 1125–1168; doi:10.1112/S0010437X13007781.

Yas16 ———, *Wilder McKay correspondences*, Nagoya Math. J. **221** (2016), no. 1, 111–164; doi:10.1017/nmj.2016.3.

Yas17a ———, *Toward motivic integration over wild Deligne–Mumford stacks*, in *Higher Dimensional Algebraic Geometry—in Honour of Professor Yujiro Kawamata's Sixtieth Birthday*, Adv. Stud. Pure Math., vol. 74 (Math. Soc. Japan, Tokyo, 2017), 407–437; doi:10.2969/aspm/07410407.

Yas17b ———, *The wild McKay correspondence and p-adic measures*, J. Eur. Math. Soc. (JEMS) **19** (2017), no. 12, 3709–3734; doi:10.4171/JEMS/751.

Yas19 ———, *Discrepancies of p-cyclic quotient varieties*, J. Math. Sci. Univ. Tokyo **26** (2019), no. 1, 1–14.

Zas93 E. Zaslow, *Topological orbifold models and quantum cohomology rings*, Comm. Math. Phys. **156** (1993), no. 2, 301–331; doi:10.1007/BF02098485.

Takehiko Yasuda yasuda.takehiko.sci@osaka-u.ac.jp
Department of Mathematics, Graduate School of Science, Osaka University, Toyonaka, Osaka 560-0043, Japan