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Cancellation theorems for Kähler differentials

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Abstract

We give an elementary calculation of the internal hom from Kähler differentials to Kähler differentials, and from Milnor K-theory to Kähler differentials in the category of presheaves with transfers. This answers a question of Merici and Saito.

1. Introduction

In their article "Cancellation theorems for reciprocity sheaves", [MS23], Merici and Saito generalise Voevodsky's cancellation theorem from homotopy invariant sheaves with transfers to reciprocity sheaves. As an application of their theorem, they prove the following.

Corollary 1 ([MS23, Corollary 1.3, Proposition 7.2]). *Assume* $\mathrm{ch}(k) = 0$. *For integers* $m, n \geqslant 0$, *there are natural isomorphisms of Nisnevich sheaves with transfers*

$$\underline{\mathrm{hom}}_{\mathrm{PST}}(\Omega^n, \Omega^m) \cong \Omega^{m-n} \oplus \Omega^{m-n-1}\,, \tag{1}$$

$$\underline{\mathrm{hom}}_{\mathrm{PST}}(\mathcal{K}^M_n, \Omega^m) \cong \Omega^{m-n}\,, \tag{2}$$

where $\Omega^i = 0$ *for* $i < 0$ *by convention.*

They write:

> It would be an interesting question if there is a direct proof of these formulas which does not use the machinery of modulus sheaves with transfers explained below.

We give such a direct proof.

2. Notation

Write Sm_k for the category of smooth k-schemes, with k a field, $\mathrm{PSh}(\mathrm{Sm}_k)$ for the category of presheaves of abelian groups on Sm_k, and $\mathrm{Shv}_{\mathrm{Zar}}(\mathrm{Sm}_k)$ for the category of Zariski sheaves. The free abelian presheaf associated with a k-scheme X is written $\mathbb{Z}(X)$. The tensor product on $\mathrm{PSh}(\mathrm{Sm}_k)$ is written $\otimes$. (This is simultaneously the Day convolution induced by the cartesian product on Sm_k, and the objectwise tensor product. So $\mathbb{Z}(X) \otimes \mathbb{Z}(Y) = \mathbb{Z}(X \times Y)$ and $(F \otimes G)(T) = F(T) \otimes G(T)$.) Voevodsky's category of transfers is denoted by SmCor_k. The category of additive presheaves of abelian groups on SmCor_k is written PST_k. The full subcategory of presheaves with transfers whose restrictions to Sm_k are Nisnevich sheaves is written NST_k. The presheaf with transfers represented by $X \in \mathrm{Sm}_k$ is written $\mathbb{Z}_{\mathrm{tr}}(X)$. The tensor product on NST_k is

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written $\otimes^{\mathrm{tr}}$. This is determined by $\mathbb{Z}_{\mathrm{tr}}(X)\otimes^{\mathrm{tr}}\mathbb{Z}_{\mathrm{tr}}(Y) = \mathbb{Z}_{\mathrm{tr}}(X\times Y)$ and the fact that it commutes with colimits in each variable.

By $\Omega^n \in \mathrm{Shv}_{\mathrm{Zar}}(\mathrm{Sm}_k)$ we mean absolute Kähler differentials. A popular[1] definition of absolute Kähler differentials of an affine schemes is the quotient

$$\Omega^n(\mathrm{Spec}(R)) = \frac{\bigoplus_{(b_1,\dots,b_n)\in R^n} R\, db_1 \dots db_n}{\left\langle \begin{array}{c} \text{additive in } b_i \text{ for } i=1,\dots,n, \\ \text{Leibniz in } b_i \text{ for } i=1,\dots,n, \\ \text{diagonals} \end{array} \right\rangle}$$

of the free R-module generated by formal symbols $db_1 \dots db_n$, where "diagonals" means $db_1 \dots db_i db_i \dots db_n = 0$.[2] We modify this definition slightly, replacing the free R-module with the free abelian group generated by symbols $adb_1 \dots db_n$ for $(a, b_1, \dots, b_n) \in R^{n+1}$ and also asking for additivity in a. (One checks directly that these two definitions agree: There is a canonical surjection of abelian groups from the latter to the former, which as a morphism of sets admits the canonical section $adb_1 \dots db_n \mapsto adb_1 \dots db_n$. This section is surjective by the linearity in a.)

Explicitly, in the language of sheaves, Ω^n is defined as the image of a surjection $\mathbb{Z}(\mathbb{A}^{n+1}) \twoheadrightarrow \Omega^n$ in $\mathrm{Shv}_{\mathrm{Zar}}(\mathrm{Sm}_k)$ whose kernel is generated by the images of the following morphisms, cf. (3):

- (*Additivity*) if $n = 0$, the map $\mathrm{add}\colon \mathbb{Z}(\mathbb{A}^2) \to \mathbb{Z}(\mathbb{A}^1)$
$$(a, b) \mapsto (a + b) - (a) - (b)\,,$$
and for $n > 0$, the maps $\mathrm{add}_{n,i} = \mathrm{id}_{\mathbb{A}^i} \otimes \mathrm{add} \otimes \mathrm{id}_{\mathbb{A}^{n-i}}$, where $i = 0, \dots, n$;
- (*Diagonals*) the maps $\mathrm{diag}_{n,i} = \mathrm{id}_{\mathbb{A}^i} \otimes \mathrm{diag} \otimes \mathrm{id}_{\mathbb{A}^{n-i-1}}$ for $i = 1, \dots, n-1$, where $\mathrm{diag}\colon \mathbb{Z}(\mathbb{A}^1) \to \mathbb{Z}(\mathbb{A}^2)$ is
$$(a) \mapsto (a, a)\,;$$
- (*Leibniz*) if $n = 1$, the map $\mathrm{leib}\colon \mathbb{Z}(\mathbb{A}^3) \to \mathbb{Z}(\mathbb{A}^2)$
$$(a, b, c) \mapsto (a, bc) - (ab, c) - (ca, b)\,,$$
and for $n > 1$, the maps $\mathrm{leib}_n = \mathrm{leib} \otimes \mathrm{id}_{\mathbb{A}^{n-1}}$.[3]

Lecomte and Wach showed in [LW09] that Kähler differentials admit a structure of transfers[4] in the sense that there exist an $\Omega^n_{\mathrm{tr}} \in \mathrm{PST}_k$ and an isomorphism $\Omega^n_{\mathrm{tr}}|_{\mathrm{Sm}_k} \cong \Omega^n$.

By $\mathcal{K}^M_n \in \mathrm{Shv}_{\mathrm{Zar}}(\mathrm{Sm}_k)$ we mean the Milnor K-theory Zariski sheaf, which is defined by a surjection $\mathbb{Z}(\mathbb{G}^n_m) \twoheadrightarrow \mathcal{K}^M_n$ in $\mathrm{Shv}_{\mathrm{Zar}}(\mathrm{Sm}_k)$ whose kernel is generated by the images of the following morphisms:

- (*Linear*) if $n = 0$, the map $\mathrm{add}\colon \mathbb{Z}(\mathbb{G}^2_m) \to \mathbb{Z}(\mathbb{G}_m)$
$$(a, b) \mapsto (ab) - (a) - (b)\,,$$
and for $n > 0$, the maps $\mathrm{add}_{n,i} = \mathrm{id}_{\mathbb{G}^{i-1}_m} \otimes \mathrm{add} \otimes \mathrm{id}_{\mathbb{G}^{n-i}_m}$, where $i = 1, \dots, n$;
- (*Steinberg*) if $n = 2$, the map $\mathrm{stein}\colon \mathbb{Z}(\mathbb{G}_m \setminus \{1\}) \to \mathbb{Z}(\mathbb{G}^2_m)$
$$(x) \mapsto (x, 1 - x)\,,$$
and for $n > 2$, the maps $\mathrm{stein}_{n,i} = \mathrm{id}_{\mathbb{G}^{i-1}_m} \otimes \mathrm{stein} \otimes \mathrm{id}_{\mathbb{G}^{n-1-i}_m}$, where $i = 1, \dots, n-1$.

[1] Cf. [Har77, Section II.8 and Exercise II.5.16] or [Sta23, 00RM, 00DM].
[2] It will quickly become cumbersome to write $\wedge$, so we contract $dt_1 \wedge \dots \wedge dt_n$ to $dt_1 \dots dt_n$.
[3] Note we only need one leib thanks to the maps diag and add.
[4] Lecomte and Wach's result is exclusively characteristic zero. The existence of transfers is also known in positive characteristic and is due to Kay Rülling.

Déglise showed in [Dég06] that, as in the case of Ω^n, the Milnor K-theory sheaves admit a structure of transfers.

3. Proofs

First we prove the non-transfers version of Corollary 1.

Proof that $\underline{\mathrm{hom}}_{\mathrm{Shv}_{\mathrm{Zar}}}(\Omega^n, \Omega^m) \cong \Omega^{m-n} \oplus \Omega^{m-n-1}$ *in* $\mathrm{Shv}_{\mathrm{Zar}}(\mathrm{Sm}_k)$. Recall that Ω^n is defined by a presentation in $\mathrm{Shv}_{\mathrm{Zar}}(\mathrm{Sm}_k)$,

$$\oplus_\Lambda \mathbb{Z}(\mathbb{A}^{n_\lambda}) \xrightarrow{\mathrm{rel}} \mathbb{Z}(\mathbb{A}^{n+1}) \to \Omega^n \to 0\,, \tag{3}$$

where rel is the sum of the $\mathrm{add}_{n,i}$, $\mathrm{diag}_{n,i}$, leib_n for appropriate i. Since $\underline{\mathrm{hom}}(-, \Omega^m)$ is right exact, it suffices to construct isomorphisms

$$\Omega^{m-n}(X) \oplus \Omega^{m-n-1}(X) \xrightarrow{\sim} \ker\left(\Omega^m(X \times \mathbb{A}^{n+1}) \xrightarrow{\mathrm{rel}^*} \oplus_\Lambda \Omega^m(X \times \mathbb{A}^{n_\lambda})\right)$$

for smooth affine X which are functorial in X. We claim that

$$(\alpha, \beta) \mapsto \alpha t_0 dt_1 \dots dt_n + \beta dt_0 \dots dt_n \tag{4}$$

defines such isomorphisms. One sees readily that (4) lies in the kernel of rel^* and that it is functorial in X. So it suffices to see that every element of the kernel is in the image. It is well known that we can write a general $\omega \in \Omega^m(X \times \mathbb{A}^{n+1})$ uniquely as

$$\omega = \sum \omega_{i,j} t_0^{i_0} \dots t_n^{i_n} dt_{j_0} \dots dt_{j_q}$$

with $\omega_{i,j} \in \Omega_X^{m-q-1}$, where the sum is over all functions $i\colon \{0, \dots, n\} \to \mathbb{N}$ and all monomorphisms of totally ordered sets $j\colon \{0, \dots, q\} \to \{0, \dots, n\}$ for $q = -1, \dots, n$.

The condition $\mathrm{add}_{n,k}^* \omega = 0$ implies that $\omega_{i,j} = 0$ unless either $i_k = 1$ and $k \notin \mathrm{im}(j)$, or $i_k = 0$ and $k \in \mathrm{im}(j)$ (here one uses that the Ω_X^μ are $\mathbb{Q}$-linear). Combining these conditions for all k, we see that our element must have the form

$$\omega = \alpha dt_0 \dots dt_n + \sum_{i \in \{0,\dots,n\}} \beta_i t_i dt_0 \dots \widehat{dt_i} \dots dt_n\,.$$

The condition $\mathrm{diag}_{n,k}^* \omega = 0$ implies $\beta_k = -\beta_{k+1}$. Combining these conditions over $k = 1, \dots, n-1$ gives $\beta_k = (-1)^{k-1}\beta_1$ for $k > 1$. Finally, the condition $\mathrm{leib}_n^* \omega = 0$ implies that $\beta_1 = 0$. □

Proof that $\underline{\mathrm{hom}}_{\mathrm{Shv}_{\mathrm{Zar}}}(\mathcal{K}_n^M, \Omega^m) \cong \Omega^{m-n}$ *in* $\mathrm{Shv}_{\mathrm{Zar}}(\mathrm{Sm}_k)$. The same argument as above reduces our task to constructing isomorphisms

$$\Omega^{m-n}(X) \xrightarrow{\sim} \ker\left(\Omega^m(X \times \mathbb{G}_m^n) \xrightarrow{\mathrm{rel}^*} \oplus_\Lambda \Omega^m(X \times G_\lambda)\right)$$

for smooth affine X which are functorial in X, where rel^* is now the sum of the morphisms induced by $\mathrm{add}_{n,i}$ and $\mathrm{stein}_{n,i}$ for appropriate i, and $G_\lambda = \mathbb{G}_m^{n+1}$ or $\mathbb{G}_m^{i-1} \times (\mathbb{G}_m \setminus \{1\}) \times \mathbb{G}_m^{n-1-i}$ as appropriate. We claim that

$$\omega \mapsto \omega \,\mathrm{dlog}\, t_1 \dots \mathrm{dlog}\, t_n \tag{5}$$

defines such isomorphisms. One sees readily that (5) lies in the kernel of rel^* and that it is functorial in X. So it suffices to see that every element of the kernel is in the image. It is well known that we can write a general $\omega \in \Omega^m(X \times \mathbb{G}_m^n)$ uniquely as

$$\omega = \sum \omega_{i,j} t_1^{i_1} \dots t_n^{i_n} \,\mathrm{dlog}\, t_{j_1} \dots \mathrm{dlog}\, t_{j_q}$$

with $\omega_{i,j} \in \Omega_X^{m-q}$, where the sum is over all functions $i\colon \{1,\dots,n\} \to \mathbb{Z}$ and all monomorphisms of totally ordered sets $j\colon \{1,\dots,q\} \to \{1,\dots,n\}$ for $q = 0,\dots,n$. The condition $\mathrm{add}_{n,k}^* \omega = 0$ implies that $\omega_{i,j} = 0$ unless $i_k = 0$ or $q \neq 0$. Combining these conditions for all k, we see that our element must be of the form $\omega = \sum \omega_j \operatorname{dlog} t_{j_1} \dots \operatorname{dlog} t_{j_q}$ with $\omega_j \in \Omega_X^{m-q}$, where the sum is over all totally ordered monomorphisms $\{1,\dots,q\} \to \{1,\dots,n\}$ with $q > 0$. The condition $\mathrm{stein}_{n,i}^* \omega = 0$ implies that $\omega_j = 0$ if exactly one of k or $k+1$ is in the image of j. Combining these conditions for all k shows that only the $\mathrm{id}\colon \{1,\dots,n\} \to \{1,\dots,n\}$ summand remains. □

To upgrade the above $\mathrm{Shv}_{\mathrm{Zar}}(\mathrm{Sm}_k)$ isomorphisms to NST_k isomorphisms, one applies [MS23, Lemma 3.1], attributed to Kay Rülling. We reproduce it for the convenience of the reader since it is particularly elementary in characteristic zero.

Lemma 1. *For any $F \in \mathrm{NST}_k$, $T \in \mathrm{Sm}_k$, and $m \in \mathbb{Z}$, there are canonical isomorphisms*

$$\hom_{\mathrm{NST}}(F, \underline{\hom}(\mathbb{Z}_{\mathrm{tr}}(T), \Omega^m_{\mathrm{tr}})) \cong \hom_{\mathrm{PSh}}(F|_{\mathrm{Sm}_k}, \underline{\hom}(\mathbb{Z}(T), \Omega^m))\,, \tag{6}$$

$$\underline{\hom}(F, \Omega^m_{\mathrm{tr}})|_{\mathrm{Sm}} \cong \underline{\hom}(F|_{\mathrm{Sm}}, \Omega^m)\,. \tag{7}$$

Proof. The surjectivity of (6) follows from the following fact: Since we are in characteristic zero, for any correspondence $\alpha \in \hom_{\mathrm{SmCor}_k}(Y, X)$, there exists some dominant étale morphism $f\colon V \to Y$ such that $\alpha \circ f\colon V \to X$ is a sum of morphisms in Sm_k. More explicitly, since $\Omega^m(Y \times T) \to \Omega^m(V \times T)$ is injective, if we have morphisms $\eta\colon F(W) \to \Omega^m(W \times T)$ functorial in $W \in \mathrm{Sm}_k$, they are also functorial in SmCor_k:

$$\begin{tikzcd}
F(X) \arrow[r, "\alpha^*"] \arrow[d, "\eta"'] & F(Y) \arrow[r, "f^*"] \arrow[d, "\eta"'] & F(V) \arrow[d, "\eta"'] \\
\Omega^m(X \times T) \arrow[r, "\alpha^*"'] & \Omega^m(Y \times T) \arrow[r, "\subseteq", "f^*"'] & \Omega^m(V \times T)\,.
\end{tikzcd}$$

The injectivity of (6) is because $\mathrm{Sm}_k \to \mathrm{SmCor}_k$ is essentially surjective. The bijectivity of (7) now follows by evaluating at some $X \in \mathrm{Sm}_k$:

$$\begin{array}{ccccc}
\underline{\hom}(F, \Omega^m_{\mathrm{tr}})(X) & = & \hom_{\mathrm{NST}}(F \otimes^{\mathrm{tr}} \mathbb{Z}_{\mathrm{tr}}(X), \Omega^m_{\mathrm{tr}}) & = & \hom_{\mathrm{NST}}(F, \underline{\hom}(\mathbb{Z}_{\mathrm{tr}}(X), \Omega^m_{\mathrm{tr}}))\,, \\
 & & & & {\scriptstyle (6)}\,\| \\
\underline{\hom}(F|_{\mathrm{Sm}}, \Omega^m)(X) & = & \hom_{\mathrm{PSh}}(F|_{\mathrm{Sm}} \otimes \mathbb{Z}(X), \Omega^m) & = & \hom_{\mathrm{PSh}}(F|_{\mathrm{Sm}}, \underline{\hom}(\mathbb{Z}(X), \Omega^m))\,.
\end{array}$$

□

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