



Direct images of pluricanonical bundles and Frobenius stable canonical rings of fibers

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ABSTRACT

In this paper, we study an algebraic fiber space in positive characteristic whose generic fiber F has finitely generated canonical ring and sufficiently large Frobenius stable canonical ring. An example of such a case is when F is F -pure and its dualizing sheaf is invertible and ample. We treat a Fujita-type conjecture due to Popa and Schnell concerning direct images of pluricanonical bundles, and prove it under some additional hypotheses. As an application, we show the subadditivity of Kodaira dimensions in some new cases. We also prove an analog of Fujino’s result regarding his Fujita-type conjecture.

1. Introduction

We first recall a conjecture of Popa and Schnell [PS14] regarding direct images of pluricanonical bundles.

CONJECTURE 1.1 ([PS14, Conjecture 1.3]). *Let $f: X \rightarrow Y$ be a morphism of smooth projective varieties over an algebraically closed field of characteristic zero, with Y of dimension n , and let $\mathcal{L}$ be an ample line bundle on Y . Then, for every $m \geq 1$, the sheaf*

$$f_*\omega_X^m \otimes \mathcal{L}^l$$

is generated by its global sections for $l \geq m(n+1)$.

This is an extension of a famous conjecture due to Fujita [Fuj87, Conjecture] to the relative setting. Under the additional assumption that $\mathcal{L}$ is globally generated, Conjecture 1.1 follows from a result of Kollár [Kol87] when $m = 1$ and has been verified by Popa and Schnell [PS14, Theorem 1.4] when $m \geq 2$. If f is smooth over the complement of a normal crossing divisor on Y , then we can remove the above additional assumption by using a theorem of Kawamata [Kaw02, Theorem 1.7] when $m = 1$ and $\dim Y \leq 4$. Deng [Den21], Dutta [Dut20], Dutta–Murayama [DM19] and Iwai [Iwa20] have studied Conjecture 1.1, and they have given sufficient conditions, in terms of lower bounds on l , for the sheaf $f_*\omega_X^m \otimes \mathcal{L}^l$ to be (generically) globally generated.

Recently, Fujino [Fuj22] proposed a new generalization of Fujita’s freeness conjecture, as follows.

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CONJECTURE 1.2 ([Fuj22, Conjecture 1.3]). *Let $f: X \rightarrow Y$ be a surjective morphism of smooth projective varieties over an algebraically closed field of characteristic zero with $\dim Y = n$, and let $\mathcal{L}$ be an ample line bundle on Y . Then, for each $m \geq 1$, the sheaf*

$$f_*\omega_{X/Y}^m \otimes \omega_Y \otimes \mathcal{L}^l$$

is generated by its global sections on U for $l \geq n + 1$, where U is the largest Zariski open subset of Y such that f is smooth over U .

Fujino [Fuj22] has shown that the above sheaf is generically globally generated, assuming additionally that either $\mathcal{L}$ is globally generated or $l \geq n^2 + \min\{2, m\}$.

In positive characteristic, however, there exists a counterexample to Conjecture 1.1, even when we add the hypothesis that $\mathcal{L}$ is globally generated. Shentu and Zhang [SZ20] have observed that for a semi-stable fibration $g: S \rightarrow \mathbb{P}^1$ constructed by Moret-Bailly [Mor81], where S is a smooth projective surface, we have that $g_*\omega_S \otimes \mathcal{O}(2)$ is isomorphic to $\mathcal{O}(-1) \oplus \mathcal{O}(p)$. This kind of pathology has been treated more deeply in [SZ20, Proposition 3.16].

The purpose of this paper is to prove several generation results in positive characteristic. We treat morphisms whose generic fibers have sufficiently large Frobenius stable canonical rings. Recall that the Frobenius stable canonical ring, which was introduced in [HP16] (and named in [PST17]), is a homogeneous subring of the canonical ring of a regular variety V , whose degree m subgroup is $S^0(V, \omega_V^m)$. Also recall that $S^0(V, \omega_V^m)$ is the subspace of $H^0(V, \omega_V^m)$ defined by Schwede [Sch14] as the stable image under the trace maps of the iterations of the Frobenius morphism. This notion was generalized to the relative setting by Hacon and Xu [HX15].

We fix from now on the following notation. Let k be an algebraically closed field of positive characteristic. Let $f: X \rightarrow Y$ be a surjective morphism of projective varieties over k , with X smooth and Y of dimension n . Let X_η and $X_{\bar{\eta}}$ denote the generic fiber and the geometric generic fiber of f , respectively. We note that f is not necessarily separable.

Our first main theorem is then stated as follows.

THEOREM 1.3 (Theorem 6.3(2)). *Suppose that*

- (i) $\bigoplus_{m \geq 0} H^0(X_\eta, \omega_{X_\eta}^m)$ *is a finitely generated $k(\eta)$ -algebra and*
- (ii) *there exists an m_0 such that $S^0(X_\eta, \omega_{X_\eta}^m) = H^0(X_\eta, \omega_{X_\eta}^m)$ for $m \geq m_0$.*

Let $\mathcal{L}$ be a big and globally generated line bundle on Y . Then the sheaf

$$f_*\omega_X^m \otimes \mathcal{L}^l$$

is generically globally generated for $m \geq m_0$ and $l \geq m(n + 1)$.

One of the features of this theorem is that it does not impose any assumptions on the geometric generic fiber $X_{\bar{\eta}}$ of f . To the best of the author's knowledge, all results concerning the positivity of direct images of (relative) pluricanonical bundles impose at least that $X_{\bar{\eta}}$ is reduced (see, for example, [Szp79, Kol90, Mau14, Pat14, Ej17]).

The hypotheses of Theorem 1.3 hold if ω_{X_η} is ample, so we obtain the following.

THEOREM 1.4. *Suppose that ω_{X_η} is ample. Let $\mathcal{L}$ be an ample and globally generated line bundle on Y . Then there exists an m_0 such that the sheaf*

$$f_*\omega_X^m \otimes \mathcal{L}^l$$

is generically generated by its global sections for $m \geq m_0$ and $l \geq m(n+1)$.

Moreover, in the case where ω_X is f -ample, we prove a positive-characteristic analog of [PS14, Theorem 1.4] by the same method as that used to prove Theorem 1.3.

THEOREM 1.5 (Theorem 6.11(2)). *Suppose that ω_X is f -ample. Let $\mathcal{L}$ be an ample and globally generated line bundle on Y . Then there exists an m_1 such that the sheaf*

$$f_*\omega_X^m \otimes \mathcal{L}^l$$

is generated by its global sections for $m \geq m_1$ and $l \geq m(n+1)$.

The above theorems are proved by using two morphisms: the morphism $Y \rightarrow \mathbb{P}^n$ defined by a free linear system in $|\mathcal{L}|$ and a *separable* endomorphism π of $\mathbb{P}^n$. We use this π to introduce an invariant measuring the positivity of coherent sheaves (Definition 5.12). This is similar to the one introduced in [Eji17, § 4] but differs since π is separable while the Frobenius morphism is employed in [Eji17] instead of π . This difference allows us to investigate f without any assumptions on $X_{\bar{\eta}}$.

As long as Y has a generically finite morphism to a variety admitting a special endomorphism (Definition 5.1), the same method works. We apply it to the Albanese morphism of Y and use its consequence to study Iitaka's conjecture.

THEOREM 1.6. *Suppose either that the hypotheses of Theorem 1.3 hold or that ω_{X_η} is ample. In the latter case, let m_0 be a sufficiently large integer. Further suppose that Y is a smooth projective variety of maximal Albanese dimension.*

- (1) (Theorem 6.3(1)) *For each $m \geq m_0$, the sheaf $f_*\omega_X^m$ is weakly positive in the sense of [Vie83b].*
- (2) (Theorem 7.3) *Assume that f is separable. If Y either is a curve or is of general type, then*

$$\kappa(X) \geq \kappa(Y) + \kappa(X_\eta).$$

Here, a coherent sheaf $\mathcal{F}$ on Y is said to be *weakly positive* in the sense of [Vie83b] if for any ample line bundle $\mathcal{H}$ on Y and any $\alpha \geq 1$, there exists some $\beta \geq 1$ such that $(S^{\alpha\beta}(\mathcal{F}))^{**} \otimes \mathcal{H}^\beta$ is generically globally generated. Note that this definition is weaker than Viehweg's original one [Vie83a, Definition 1.2].

In the case where Y is a curve, Theorem 1.6(2) generalizes a result of the author [Eji17, Theorem 1.4] which indeed needs a stronger assumption than that of Theorem 1.6 to prove the weak positivity of sheaves of the form $f_*\omega_{X/Y}^m$ (see [Eji17, Theorem 1.1]). The positivity of these sheaves cannot be obtained from Theorem 1.3, unlike in the case of characteristic zero [PS14, Corollary 4.3]. In fact, there exists a fibration that satisfies the hypotheses in Theorem 1.3 but violates the weak positivity theorem (see [Ray78, Xie10, Muk13] or [Zha19, Example 1.14]).

We move on to results concerning Conjecture 1.2. For the same reason as above, we impose the same condition as that in [Eji17, Theorem 1.1], which is stronger than that of Theorem 1.3. The next theorem can be viewed as a positive-characteristic analog of a part of the above result due to Fujino [Fuj22, Theorem 1.5]. Recall that $X_{\bar{\eta}}$ denotes the geometric generic fiber of f .

THEOREM 1.7 (Theorem 6.7(3)). *Suppose that Y is smooth. Assume that*

- (i) $\bigoplus_{m \geq 0} H^0(X_{\bar{\eta}}, \omega_{X_{\bar{\eta}}}^m)$ *is a finitely generated $k(\bar{\eta})$ -algebra and*
- (ii') *there exists an m_0 such that $S^0(X_{\bar{\eta}}, \omega_{X_{\bar{\eta}}}^m) = H^0(X_{\bar{\eta}}, \omega_{X_{\bar{\eta}}}^m)$ for $m \geq m_0$.*

Let $\mathcal{L}$ be a big and globally generated line bundle. Then the sheaf

$$f_* \omega_{X/Y}^m \otimes \omega_Y \otimes \mathcal{L}^l$$

is generically globally generated for $m \geq m_0$ and $l \geq n + 1$.

The hypotheses of this theorem hold if $X_{\bar{\eta}}$ is F -pure (Definition 3.1) and has ample dualizing sheaf $\omega_{X_{\bar{\eta}}}$ (see [Eji17, § 3] for more examples), so we obtain the following.

THEOREM 1.8. *Suppose that Y is smooth. Assume that $X_{\bar{\eta}}$ is F -pure and $\omega_{X_{\bar{\eta}}}$ is ample. Then there exists an m_0 such that the sheaf*

$$f_* \omega_{X/Y}^m \otimes \omega_Y \otimes \mathcal{L}^l$$

is generically globally generated for $m \geq m_0$ and $l \geq n + 1$.

In the situation of this theorem, furthermore, we can find an open subset of Y , depending only the morphism f , on which the above sheaves are generated by their global sections.

THEOREM 1.9 (Theorem 6.9(3)). *Suppose that Y is smooth. Assume that there exists a dense open subset Y_0 of Y such that the following conditions hold:*

- *The morphism f is flat over Y_0 .*
- *The sheaf $\omega_X|_{f^{-1}(Y_0)}$ is ample over Y_0 .*
- *Every closed fiber of f over Y_0 is F -pure.*

Let $\mathcal{L}$ be an ample and globally generated line bundle on Y . Then there exists an m_0 such that the sheaf

$$f_* \omega_{X/Y}^m \otimes \omega_Y \otimes \mathcal{L}^l$$

is generated by its global sections on Y_0 for $m \geq m_0$ and $l \geq n + 1$.

This paper is organized as follows. In Section 2, we set up notation and terminology. Section 3 deals with the trace maps of iterations of (relative) Frobenius morphisms. In Section 4, we generalize and study different notions of base loci of coherent sheaves. In Section 5, using a special endomorphism, we introduce an invariant measuring the positivity of coherent sheaves. In Section 6, our main results are stated in a more general setting and proved. Section 7 is devoted to the study of Iitaka's conjecture in positive characteristic.

2. Notation and conventions

Let k be a field. By k -scheme, we mean a separated scheme of finite type over k . A *variety* is an integral k -scheme.

Let X be an equi-dimensional k -scheme of finite type satisfying S_2 and G_1 . Here, S_2 (respectively, G_1) stands for Serre's second condition (respectively, the condition that the scheme is Gorenstein in codimension 1). Let $\mathcal{K}$ be the sheaf of total quotient rings on X . An *AC divisor* (or *almost Cartier divisor*) on X is a reflexive coherent subsheaf of $\mathcal{K}$ that is invertible on an open subset U of X with $\text{codim}(X \setminus U) \geq 2$ (see [Har94, first definition following Proposition 2.4],

[MS12, Definition 2.1]). Let D be an AC divisor on X . We let $\mathcal{O}_X(D)$ denote the coherent sheaf defining D . We say that D is *effective* if $\mathcal{O}_X \subseteq \mathcal{O}_X(D)$. The set $\text{WSh}(X)$ of AC divisors on X forms naturally an additive group [Har94, Corollary 2.6]. In this paper, a *prime* AC divisor is an effective AC divisor that cannot be written as the sum of two non-zero effective AC divisors. A $\mathbb{Q}$ -AC divisor is an element of $\text{WSh}(X) \otimes_{\mathbb{Z}} \mathbb{Q}$. Let Δ be a $\mathbb{Q}$ -AC divisor. Then there are prime AC divisors Δ_i on X such that $\Delta = \sum_i \delta_i \Delta_i$ for $\delta_i \in \mathbb{Q}$. We define

$$\lfloor \Delta \rfloor := \sum_i \lfloor \delta_i \rfloor \Delta_i \quad \text{and} \quad \lceil \Delta \rceil := \sum_i \lceil \delta_i \rceil \Delta_i.$$

Note that $\lfloor \Delta \rfloor$ and $\lceil \Delta \rceil$ are not necessarily uniquely determined by Δ because the choice of the decomposition $\Delta = \sum_i \delta_i \Delta_i$ is not necessarily unique. We recall an example by Corti [Kol92, (16.1.2)]. Set $X := \text{Spec } k[x, y, z, z^{-1}]/(x^2 - y^2 z)$, $D := (x)$ and $E := (y)$. Then $D \neq E$ and $2D = 2E$, so $D = \frac{1}{2}D + \frac{1}{2}E$ as $\mathbb{Q}$ -Cartier divisors (and so also $\mathbb{Q}$ -AC divisors). *In this paper, given a $\mathbb{Q}$ -AC divisor Δ , we fix a decomposition $\Delta = \sum_i \delta_i \Delta_i$. If Δ is $\mathbb{Q}$ -Cartier, we also fix a decomposition into Cartier divisors.* We say that Δ is *effective* if $\delta_i \geq 0$ for each i . By $\Delta \geq \Delta'$, we mean that $\Delta - \Delta'$ is effective. If $\Delta = \alpha \Delta' + \beta \Delta''$ for $\alpha, \beta \in \mathbb{Q}$ and $\mathbb{Q}$ -AC divisors Δ' and Δ'' whose decompositions $\Delta' = \sum_j \delta'_j \Delta'_j$ and $\Delta'' = \sum_k \delta''_k \Delta''_k$ have already been given, then we choose the natural decomposition $\Delta = \sum_j \alpha \delta'_j \Delta'_j + \sum_k \beta \delta''_k \Delta''_k$. We say that a $\mathbb{Q}$ -AC divisor $\Delta = \sum_i \delta_i \Delta_i$ is *integral* if $\delta_i \in \mathbb{Z}$ for each i .

Let $\varphi: S \rightarrow T$ be a morphism of schemes, and let T' be a T -scheme. We denote by $S_{T'}$ and $\varphi_{T'}: S_{T'} \rightarrow T'$ the fiber product $S \times_T T'$ and its second projection, respectively. For an $\mathcal{O}_S$ -module $\mathcal{G}$, its pullback to $S_{T'}$ is denoted by $\mathcal{G}_{T'}$. We use the same notation for an AC or $\mathbb{Q}$ -AC divisor if its pullback is well defined.

3. Trace maps of Frobenius morphisms

In this section, we discuss several notions defined by using the trace maps of Frobenius morphisms. We work over an F -finite field k of characteristic $p > 0$, that is, a field k of characteristic $p > 0$ such that the extension k/k^p is finite.

DEFINITION 3.1. Let X be an equi-dimensional k -scheme satisfying S_2 and G_1 . Let Δ be an effective $\mathbb{Q}$ -AC divisor on X . We say that the pair (X, Δ) is *F-pure* if for each positive integer e , the composite

$$\mathcal{O}_X \xrightarrow{F_X^e \#} F_{X*}^e \mathcal{O}_X \hookrightarrow F_{X*}^e \mathcal{O}_X(\lfloor (p^e - 1)\Delta \rfloor)$$

locally splits as an $\mathcal{O}_X$ -module homomorphism.

Remark 3.2. (1) When X is a normal variety, Definition 3.1 is equivalent to that in [HW02].

(2) Let X be a variety satisfying S_2 and G_1 such that iK_X is Cartier for $i \in \mathbb{Z}_{>0}$ not divisible by p . Let X^N denote the normalization of X , and let B be the effective divisor corresponding to the conductor. Miller and Schwede [MS12] have proved that if X has hereditary surjective trace (see [MS12, Definition 3.5]), then the F -purity of X is equivalent to that of the pair (X^N, B) .

Let X and Δ be as in Definition 3.1. Suppose that $i\Delta$ is integral for some $i \in \mathbb{Z}_{>0}$ not divisible by p . Let e be the smallest positive integer such that $(p^e - 1)\Delta$ is integral. Let $g > 0$ be an integer such that $(p^g - 1)\Delta$ is integral, which is equivalent to $e|g$. Let L be an AC divisor on X . Applying the functor $\mathcal{H}om(\cdot, \mathcal{O}_X(L))$ to the morphism $\mathcal{O}_X \rightarrow F_{X*}^g \mathcal{O}_X((p^g - 1)\Delta)$ defined

in the same way as in Definition 3.1, we get the morphism

$$F_{X*}^g \mathcal{O}_X(p^g L + (1 - p^g)(K_X + \Delta)) \xrightarrow{\phi_{(X, \Delta)}^{(g)}(L)} \mathcal{O}_X(L)$$

by Grothendieck duality. Throughout this paper, we denote this morphism by $\phi_{(X, \Delta)}^{(g)}(L)$. The pair (X, Δ) is F -pure if and only if $\phi_{(X, \Delta)}^{(g)} := \phi_{(X, \Delta)}^{(g)}(0)$ is surjective for each $g \in \mathbb{Z}_{>0}$ with $e|g$.

DEFINITION 3.3 ([Sch14, §§3 and 4]). Let X be an equi-dimensional projective k -scheme satisfying S_2 and G_1 . Let Δ be an effective $\mathbb{Q}$ -AC divisor on X such that $i\Delta$ is integral for some $i \in \mathbb{Z}_{>0}$ not divisible by p . Let e be the smallest positive integer such that $(p^e - 1)\Delta$ is an AC divisor. Let L be an AC divisor on X . The k -vector space $S^0(X, \Delta; \mathcal{O}_X(L))$ of $H^0(X, \mathcal{O}_X(L))$ is defined as

$$\bigcap_{g>0, e|g} \text{Image}(H^0(X, F_{X*}^g \mathcal{O}_X(p^g L - (1 - p^g)(K_X + \Delta))) \rightarrow H^0(X, \mathcal{O}_X(L))),$$

where the morphism is induced from $\phi_{(X, \Delta)}^{(g)}(L)$. When $K_X + \Delta$ is $\mathbb{Q}$ -Cartier with index not divisible by p and L is Cartier, the above subspace is also denoted by $S^0(X, \sigma(X, \Delta) \otimes \mathcal{O}_X(L))$.

DEFINITION 3.4 ([HX15, Definition 2.14]). Let X be an equi-dimensional k -scheme satisfying S_2 and G_1 . Let Δ be an effective $\mathbb{Q}$ -AC divisor on X such that $i\Delta$ is integral for some $i \in \mathbb{Z}_{>0}$ not divisible by p . Let e be the smallest positive integer such that $(p^e - 1)\Delta$ is an AC divisor. Let $f: X \rightarrow Y$ be a proper morphism to a variety Y . Let L be an AC divisor on X . The subsheaf $S^0 f_*(\sigma(X, \Delta) \otimes \mathcal{O}_X(L))$ of $f_* \mathcal{O}_X(L)$ is defined as

$$\bigcap_{g>0, e|g} \text{Image}(F_{Y*}^g f_* \mathcal{O}_X(p^g L - (1 - p^g)(K_X + \Delta)) \rightarrow f_* \mathcal{O}_X(L)),$$

where the morphism is induced from $\phi_{(X, \Delta)}^{(g)}(L)$.

Considering the cohomology and base change theorem, one can easily check that the stalk of $S^0 f_*(\sigma(X, \Delta) \otimes \mathcal{O}_X(L))$ at the generic point η of Y is isomorphic to $S^0(X_\eta, \Delta|_{X_\eta}; \mathcal{O}_{X_\eta}(L|_{X_\eta}))$.

Next, we consider the trace maps of relative Frobenius morphisms. Let X be an equi-dimensional k -scheme satisfying S_2 and G_1 . Let Δ be an effective $\mathbb{Q}$ -AC divisor on X such that $i\Delta$ is integral for some $i \in \mathbb{Z}_{>0}$ not divisible by p . Let L be an AC divisor on X . Let $f: X \rightarrow Y$ be a morphism to a *regular* affine variety Y . Let U be the maximal open subset of X such that U is Gorenstein and $L|_U$ is Cartier. Replacing X with U , we assume that X is Gorenstein and L is Cartier. Let e be the smallest positive integer such that $(p^e - 1)\Delta$ is an AC divisor. Fix $g \in \mathbb{Z}_{>0}$ with $e|g$. We now have the following commutative diagram:

$$\begin{array}{ccccc} & & X^g & & \\ & \searrow F_{X/Y}^{(g)} & \downarrow & \searrow F_X^g & \\ f^{(g)} \swarrow & & X_{Y^g} & \xrightarrow{w^{(g)}} & X \\ & \downarrow f_{Y^g} & & & \downarrow f \\ & Y^g & \xrightarrow{F_Y^g} & Y & \end{array}$$

We consider the relative dualizing sheaf $\omega_{F_{X/Y}^{(g)}}$ of $F_{X/Y}^{(g)}$. Since Y is regular, F_Y is flat, so it is

a Gorenstein morphism [Har66, § V.9]. Then $\omega_{w^{(g)}} \cong f_{Y^g}^* \omega_{F_Y^g} \cong f_{Y^g}^* \omega_Y^{1-p^g}$ by [Con00, Theorem 3.6.1], and so

$$\begin{aligned}
 \omega_{F_{X/Y}^{(g)}} &= (F_{X/Y}^{(g)})^! \mathcal{O}_{X_{Y^g}} \cong (F_{X/Y}^{(g)})^! (\omega_{w^{(g)}} \otimes \omega_{w^{(g)}}^*) \\
 &\cong (F_{X/Y}^{(g)})^! (\omega_{w^{(g)}} \otimes f_{Y^g}^* \omega_{Y^g}^{p^g-1}) && \text{since } \omega_{w^{(g)}} \cong f_{Y^g}^* \omega_Y^{1-p^g} \\
 &\cong ((F_{X/Y}^{(g)})^! \omega_{w^{(g)}}) \otimes f^{(g)*} \omega_{Y^g}^{p^g-1} && \text{by [Har66, Proposition III.6.9(a)]} \\
 &\cong (F_X^g)^! \mathcal{O}_X \otimes f^{(g)*} \omega_{Y^g}^{p^g-1} && \text{since } F_{X/Y}^{(g) \dagger} w^{(g) \dagger} \cong F_X^g \\
 &\cong \omega_{X^g}^{1-p^g} \otimes f^{(g)*} \omega_{Y^g}^{p^g-1} \cong \omega_{X^g/Y^g}^{1-p^g}.
 \end{aligned}$$

Therefore, for a Cartier divisor M on X_{Y^g} , applying $\mathcal{H}om(\cdot, \mathcal{O}_{X_{Y^g}}(M))$ to the composite

$$\mathcal{O}_{X_{Y^g}} \rightarrow F_{X/Y}^{(g)} \mathcal{O}_{X^g} \hookrightarrow F_{X/Y}^{(g)} \mathcal{O}_{X^g}((p^g - 1)\Delta),$$

we obtain from Grothendieck duality the following morphism:

$$\phi_{(X/Y, \Delta)}^{(g)}(M): F_{X/Y}^{(g)} \mathcal{O}_{X^g}(F_{X/Y}^{(g)*} M + (1 - p^g)(K_{X/Y} + \Delta)) \rightarrow \mathcal{O}_{X_{Y^g}}(M).$$

Here $K_{X/Y} := K_X - f^* K_Y$. Using this morphism, we discuss the surjectivity of $f_* \phi_{(X, \Delta)}^{(g)}(L)$. By the above diagram, we obtain the commutative diagram

$$\begin{array}{ccc}
 F_{X^g}^g \mathcal{O}_X((p^g - 1)\Delta) & & \\
 \uparrow & \swarrow & \\
 w^{(g)*} \mathcal{O}_{X_{Y^g}} & \xleftarrow{\alpha} & \mathcal{O}_X.
 \end{array}$$

Applying $\mathcal{H}om(\cdot, \mathcal{O}_X(L))$ to the above diagram, we get

$$\begin{array}{ccc}
 F_{X^g}^g \mathcal{O}_X(p^g L + (1 - p^g)(K_X + \Delta)) & & \\
 \gamma \downarrow & \searrow \phi_{(X, \Delta)}^{(g)}(L) & \\
 w^{(g)*} \mathcal{O}_{X_{Y^g}}(L_{Y^g} + (1 - p^g)f_{Y^g}^* K_Y) & \xrightarrow{\beta} & \mathcal{O}_X(L).
 \end{array}$$

Note that here we used the isomorphisms

$$\mathcal{H}om(w^{(g)*} \mathcal{O}_{X_{Y^g}}, \mathcal{O}_X) \cong w^{(g)*} \mathcal{H}om(\mathcal{O}_{X_{Y^g}}, w^{(g) \dagger} \mathcal{O}_X) \cong w^{(g)*} w^{(g) \dagger} \mathcal{O}_X \cong w^{(g)*} f_{Y^g}^* \omega_Y^{1-p^g},$$

where the first isomorphism follows from Grothendieck duality. Put $M := L_{Y^g} + (1 - p^g)f_{Y^g}^* K_Y$. Then $\gamma \cong w^{(g)*} \phi_{(X/Y, \Delta)}^{(g)}(M)$. Hence,

$$f_* \phi_{(X, \Delta)}^{(g)}(L) \cong (f_* \beta) \circ f_* (w^{(g)*} \phi_{(X/Y, \Delta)}^{(g)}(M)) = (f_* \beta) \circ F_{Y^g}^g (f_{Y^g*} \phi_{(X/Y, \Delta)}^{(g)}(M)).$$

Since Y is affine, α splits, and hence so does β , which means that $f_* \beta$ is surjective. Thus, we see that if the morphism $f_{Y^g*} \phi_{(X/Y, \Delta)}^{(g)}(M)$ is surjective (respectively, the zero map), then so is $f_* \phi_{(X, \Delta)}^{(g)}(L)$.

Let l be an F -finite field that is an extension of k , let W be a regular l -scheme, and let $a: W \rightarrow Y$ be a k -morphism. Set $V := X \times_Y W$, and let $h: V \rightarrow W$ be the second projection.

We next consider the following commutative diagram:

$$\begin{array}{ccc}
 V^g & \longrightarrow & X^g \\
 F_{V/W}^{(g)} \downarrow & & \downarrow F_{X/Y}^{(g)} \\
 V_{W^g} & \xrightarrow{b} & X_{Y^g} \\
 h_{W^g} \downarrow & & \downarrow f_{Y^g} \\
 W^g & \xrightarrow{a} & Y^g .
 \end{array}$$

Note that the squares in the diagram are cartesian. By [PSZ18, Lemma 2.16], we see that

$$\phi_{(V/W, \Delta_V)}^{(g)}(M_{W^g}) \cong b^* \phi_{(X/Y, \Delta)}^{(g)}(M) ,$$

where M is as above. We now assume that a is flat. Then by the above isomorphism, we get

$$h_{W^g *} \phi_{(V/W, \Delta_V)}^{(g)}(M_{W^g}) \cong a^* f_{Y^g *} \phi_{(X/Y, \Delta)}^{(g)}(M) .$$

This means that if $f_{Y^g *} \phi_{(X/Y, \Delta)}^{(g)}(M)$ is surjective, then so is $h_{W^g *} \phi_{(V/W, \Delta_V)}^{(g)}(M_{W^g})$. The converse holds when a is surjective. In particular, we get the following lemma.

LEMMA 3.5. *Let X be an equi-dimensional projective k -scheme satisfying S_2 and G_1 . Let Δ be an effective $\mathbb{Q}$ -AC divisor on X such that $i\Delta$ is integral for some $i \in \mathbb{Z}_{>0}$ not divisible by p . Let L be an AC divisor on X . Let $\bar{k}$ be the algebraic closure of k . If*

$$S^0(X_{\bar{k}}, \Delta_{\bar{k}}; \mathcal{O}_{X_{\bar{k}}}(L_{\bar{k}})) = H^0(X_{\bar{k}}, \mathcal{O}_{X_{\bar{k}}}(L_{\bar{k}})) ,$$

then

$$S^0(X, \Delta; \mathcal{O}_X(L)) = H^0(X, \mathcal{O}_X(L)) .$$

LEMMA 3.6. *Let X be an equi-dimensional projective k -scheme satisfying S_2 and G_1 . Let Δ be an effective $\mathbb{Q}$ -AC divisor on X such that*

- $K_X + \Delta$ is $\mathbb{Q}$ -Cartier,
- $i\Delta$ is integral for some $i \in \mathbb{Z}_{>0}$ not divisible by p ,
- (X, Δ) is F -pure.

Let $f: X \rightarrow Y$ be a projective morphism to a variety Y . Let L be an f -ample Cartier divisor on X . Then there exists an integer m_0 such that

$$S^0 f_*(\sigma(X, \Delta) \otimes \mathcal{O}_X(mL + N)) = f_* \mathcal{O}_X(mL + N)$$

for each $m \geq m_0$ and every f -nef Cartier divisor N on X .

Proof. The proof is basically the same as that of [Pat14, Corollary 2.23]. Let e be the smallest positive integer such that $(p^e - 1)\Delta$ is integral. Let us prove that there is an $m_0 \in \mathbb{Z}_{>0}$ such that

$$f_*(\phi_{(X, \Delta)}^{(g)}(mL + N)) : f_* F_{X^g}^g \mathcal{O}_X(p^g(mL + N) + (1 - p^g)(K_X + \Delta)) \rightarrow f_* \mathcal{O}_X(mL + N)$$

is surjective for each $g \in \mathbb{Z}_{>0}$ with $e|g$, for each $m \geq m_0$ and for every f -nef Cartier divisor N

on X . By definition, $\phi_{(X,\Delta)}^{(g)}(mL + N)$ is decomposed into

$$\begin{aligned}
 & \phi_{(X,\Delta)}^{(g)} : F_{X*}^g \mathcal{O}_X(p^g(mL + N) + (1 - p^g)(K_X + \Delta)) \\
 & \xrightarrow{F_X^{g-e} * \phi_{(X,\Delta)}^{(e)}(p^{g-e}(mL + N) + (1 - p^{g-e})(K_X + \Delta))} F_X^{g-e} * \mathcal{O}_X(p^{g-e}(mL + N) + (1 - p^{g-e})(K_X + \Delta)) \\
 & \xrightarrow{F_X^{g-2e} * \phi_{(X,\Delta)}^{(e)}(p^{g-2e}(mL + N) + (1 - p^{g-2e})(K_X + \Delta))} F_X^{g-2e} * \mathcal{O}_X(p^{g-2e}(mL + N) + (1 - p^{g-2e})(K_X + \Delta)) \\
 & \quad \vdots \\
 & \xrightarrow{F_X^e * \phi_{(X,\Delta)}^{(e)}(p^e(mL + N) + (1 - p^e)(K_X + \Delta))} F_X^e * \mathcal{O}_X(p^e(mL + N) + (1 - p^e)(K_X + \Delta)) \\
 & \xrightarrow{\phi_{(X,\Delta)}^{(e)}(mL + N)} \mathcal{O}_X(mL + N).
 \end{aligned}$$

Note that the above morphisms are surjective since (X, Δ) is F -pure. Hence, it is enough to show that there is an $m_0 \in \mathbb{Z}_{>0}$ such that for each $m \geq m_0$, for each $g \in \mathbb{Z}_{>0}$ with $e|g$ and for every f -nef Cartier divisor N on X ,

$$R^1 f_* (\mathcal{B}^e(p^g(mL + N) + (1 - p^g)(K_X + \Delta))) = 0, \quad (*)$$

where $\mathcal{B}^e(M)$ denotes the kernel of $\phi_{(X,\Delta)}^{(e)}(M)$ for every AC divisor M on X . Let d be the smallest positive integer such that $d(K_X + \Delta)$ is Cartier. Let q_g and r_g be, respectively, the quotient and remainder of the division of $p^g - 1$ by d . Then

$$\begin{aligned}
 & p^g(mL + N) + (1 - p^g)(K_X + \Delta) \\
 & = mL + N + (p^g - 1)(mL + N - K_X - \Delta) \\
 & = mL + N + dq_g(mL + N - K_X - \Delta) + r_g(mL + N - K_X - \Delta).
 \end{aligned}$$

Note that $r_g(mL + N - K_X - \Delta)$ is integral. Let m_1 be an integer such that $m_1 L - K_X - \Delta$ is f -nef. Then $N_{m,g} := N + dq_g(mL + N - K_X - \Delta)$ is an f -nef Cartier divisor for each $m \geq m_1$ and $g \in \mathbb{Z}_{>0}$. Since we have

$$\begin{aligned}
 & \mathcal{B}^e(p^g(mL + N) + (1 - p^g)(K_X + \Delta)) \\
 & = \mathcal{B}^e(mL + N + dq_g(mL + N - K_X - \Delta) + r_g(mL + N - K_X - \Delta)) \\
 & = \mathcal{B}^e(mL + N_{m,g} + r_g(mL + N - K_X - \Delta)) \\
 & \cong \mathcal{B}^e(r_g(mL + N - K_X - \Delta)) \otimes \mathcal{O}_X(mL + N_{m,g})
 \end{aligned}$$

and $0 \leq r_g < d$, it follows from Keeler's relative Fujita vanishing theorem [Kee03, Theorem 1.5] that there is an $m_0 \geq m_1$ such that $(*)$ holds for each $m \geq m_0$, for each $g \in \mathbb{Z}_{>0}$ with $e|g$ and for every f -nef Cartier divisor on X . $\square$

LEMMA 3.7. *Let X be an equi-dimensional k -scheme satisfying S_2 and G_1 . Let Δ be an effective $\mathbb{Q}$ -AC divisor such that $i(K_X + \Delta)$ is Cartier for some $i \in \mathbb{Z}_{>0}$ not divisible by p . Let $f: X \rightarrow Y$ be a flat projective morphism to a regular variety Y . Suppose that the following conditions hold:*

- *There exists a Gorenstein open subset $U \subseteq X$ such that $\text{codim}_{X_y}(X_y \setminus U_y) \geq 2$ for every $y \in Y$.*
- *The support $\text{Supp}(\Delta)$ does not contain any irreducible component of any fiber of f .*
- *The pair $(X_{\bar{y}}, \overline{\Delta}|_{U_{\bar{y}}})$ is F -pure for every $y \in Y$, where $X_{\bar{y}}$ is the geometric fiber over y and $\overline{\Delta}|_{U_{\bar{y}}}$ is the $\mathbb{Q}$ -AC divisor on $X_{\bar{y}}$ that is the extension of $\Delta|_{U_{\bar{y}}}$.*

Let A be an f -ample Cartier divisor. Then there exists an $m_0 \in \mathbb{Z}_{>0}$ such that

$$S^0 f_{Y^e*}(\sigma(X_{Y^e}, \Delta_{Y^e}) \otimes \mathcal{O}_{X_{Y^e}}(mA_{Y^e} + N_{Y^e})) = f_{Y^e*} \mathcal{O}_{X_{Y^e}}(mA_{Y^e} + N_{Y^e})$$

for each $e \geq 0$, each $m \geq m_0$ and every f -nef Cartier divisor N .

Proof. Fix $e \geq 0$. By the argument above, it is enough to show that the morphism

$$f_{Y^{d+e}*}(\phi_{(X_{Y^e}/Y^e, \Delta_{Y^e})}^{(d)}(mA_{Y^e} + N_{Y^e}))$$

is surjective for each $d \geq 0$. By the argument above again, we have

$$f_{Y^{d+e}*}(\phi_{(X_{Y^e}/Y^e, \Delta_{Y^e})}^{(d)}(mA_{Y^e} + N_{Y^e})) \cong F_Y^e f_{Y^d*}(\phi_{(X/Y, \Delta)}^{(d)}(mA + N))$$

for each $d \geq 0$. The surjectivity of $f_{Y^e*}(\phi_{(X/Y, \Delta)}^{(d)}(mA + N))$ follows from [PSZ18, Theorem C] or [Eji19, Lemma. 3.7]. $\square$

4. Positivity of coherent sheaves

4.1 Base loci of coherent sheaves

The restricted base locus and the augmented base locus of an $\mathbb{R}$ -Cartier divisor were introduced and studied in [ELM⁺06]. These notions have been generalized to vector bundles in [BKK⁺15]. In this subsection, we further generalize these notions to coherent sheaves in the same way as in [BKK⁺15, § 2] and [FM21, § 6].

Let k be a field, let X be a quasi-projective variety over k , and let $\mathrm{sp}(X)$ denote the underlying topological space of X . Let $\mathcal{F}$ be a coherent sheaf on X . The *base locus* $\mathrm{Bs}(\mathcal{F})$ of $\mathcal{F}$ is defined as the subset of $\mathrm{sp}(X)$ consisting of points at which $\mathcal{F}$ is *not* generated by its global sections. If

$$\varphi: H^0(X, \mathcal{F}) \otimes \mathcal{O}_X \rightarrow \mathcal{F}$$

denotes the evaluation map, then $\mathrm{Bs}(\mathcal{F}) = \mathrm{Supp}(\mathrm{Coker} \varphi)$. One can check that

- $\mathrm{Bs}(\mathcal{F} \otimes \mathcal{G}) \subseteq \mathrm{Bs}(\mathcal{F}) \cup \mathrm{Bs}(\mathcal{G})$ for a coherent sheaf $\mathcal{G}$,
- $\mathrm{Bs}(\mathcal{G}) \subseteq \mathrm{Bs}(\mathcal{F}) \cup \mathrm{Supp}(\mathrm{Coker} \psi)$ for a morphism $\psi: \mathcal{F} \rightarrow \mathcal{G}$ and
- $\mathrm{Bs}(\pi^* \mathcal{F}) \subseteq \pi^{-1}(\mathrm{Bs}(\mathcal{F}))$ for a morphism $\pi: Y \rightarrow X$.

Take $S \subseteq \mathrm{sp}(X)$. We say that $\mathcal{F}$ is *globally generated over S* (respectively, *generically globally generated*) if $\mathrm{Bs}(\mathcal{F}) \cap S = \emptyset$ (respectively, $\mathrm{Bs}(\mathcal{F}) \neq X$).

Let D be a $\mathbb{Q}$ -Cartier divisor on X . Fix a decomposition $D = \sum_{j=1}^n d_j D_j$ into Cartier divisors, where $d_1, \dots, d_n \in \mathbb{Q}$. Let i be the smallest positive integer such that $id_1, \dots, id_n \in \mathbb{Z}$. Note that $iD = \sum_{j=1}^n id_j D_j$ is a Cartier divisor. Then we define

$$\mathbb{B}(\mathcal{F} + D) := \bigcap_{l \geq 1} (\mathrm{Bs}(S^{il}(\mathcal{F}) \otimes \mathcal{O}_X(ilD))) \subseteq \mathrm{sp}(X).$$

Set $\mathbb{B}(\mathcal{F}) := \mathbb{B}(\mathcal{F} + 0)$. When $\mathcal{F}$ is locally free, $\mathbb{B}(\mathcal{F})$ is called the *stable base locus* of $\mathcal{F}$. We put $\mathbb{B}(\mathcal{F} - D) := \mathbb{B}(\mathcal{F} + (-D))$ and $\mathbb{B}(D) := \mathbb{B}(\mathcal{O}_X + D)$. Take $l, m \in \mathbb{Z}_{>0}$. The natural morphisms

$$S^l(S^{im}(\mathcal{F}) \otimes \mathcal{O}_Y(imD)) \cong S^l(S^{im}(\mathcal{F})) \otimes \mathcal{O}_Y(ilmd) \twoheadrightarrow S^{ilm}(\mathcal{F}) \otimes \mathcal{O}_Y(ilmd)$$

imply that

$$\begin{aligned} \mathrm{Bs}(S^{im}(\mathcal{F}) \otimes \mathcal{O}_Y(imD)) &\supseteq \mathrm{Bs}(S^l(S^{im}(\mathcal{F})) \otimes \mathcal{O}_Y(ilmd)) \\ &\supseteq \mathrm{Bs}(S^{ilm}(\mathcal{F}) \otimes \mathcal{O}_Y(ilmd)), \end{aligned}$$

so we get an integer $n > 0$ such that

$$\mathbb{B}(\mathcal{F} + D) = \text{Bs}(S^{\text{in}}(\mathcal{F}) \otimes \mathcal{O}_Y(\text{in}D)).$$

One can check that

- $\mathbb{B}(\mathcal{F} + (D + E)) \subseteq \mathbb{B}(\mathcal{F} + D) \cup \mathbb{B}(E)$ for a $\mathbb{Q}$ -Cartier divisor E and
- $\mathbb{B}(\pi^*\mathcal{F} + \pi^*D) \subseteq \pi^{-1}(\mathbb{B}(\mathcal{F} + D))$ for a morphism $\pi: Y \rightarrow X$ such that π^*D can be defined.

Let A be a semi-ample $\mathbb{Q}$ -Cartier divisor on X . Then for all rational numbers $r_1 > r_2$, we have

$$\begin{aligned} \mathbb{B}(\mathcal{F} + (D + r_1A)) &\subseteq \mathbb{B}(\mathcal{F} + (D + r_2A)) \cup \mathbb{B}((r_1 - r_2)A) \\ &= \mathbb{B}(\mathcal{F} + (D + r_2A)). \end{aligned}$$

In this paper, we use the following notation:

$$\mathbb{B}_-^A(\mathcal{F} + D) := \bigcup_{r \in \mathbb{Q}_{>0}} \mathbb{B}(\mathcal{F} + (D + rA)), \quad \mathbb{B}_+^A(\mathcal{F} + D) := \bigcap_{r \in \mathbb{Q}_{>0}} \mathbb{B}(\mathcal{F} + (D - rA)).$$

One can check that the following conditions hold:

- We have $\mathbb{B}_-^A(\mathcal{F} + D) \subseteq \mathbb{B}(\mathcal{F} + D) \subseteq \mathbb{B}_+^A(\mathcal{F} + D)$.
- For every $r \in \mathbb{Q}_{>0}$, we have

$$\mathbb{B}_-^{rA}(\mathcal{F} + D) = \mathbb{B}_-^A(\mathcal{F} + D) \quad \text{and} \quad \mathbb{B}_+^{rA}(\mathcal{F} + D) = \mathbb{B}_+^A(\mathcal{F} + D).$$

- For a sequence $r_1, r_2, \dots \in \mathbb{Q}_{>0}$ converging to 0, we have

$$\mathbb{B}_-^A(\mathcal{F} + D) = \bigcup_{j \geq 1} \mathbb{B}(\mathcal{F} + (D + r_jA)).$$

- There is an $r \in \mathbb{Q}_{>0}$ such that for every $r' \in (0, r] \cap \mathbb{Q}$,

$$\mathbb{B}_+^A(\mathcal{F} + D) = \mathbb{B}(\mathcal{F} + (D - r'A)).$$

LEMMA 4.1. *Let $X, \mathcal{F}$ and D be as above. Let A and B be semi-ample $\mathbb{Q}$ -Cartier divisors on X . Then*

$$\mathbb{B}_-^A(\mathcal{F} + D) \subseteq \mathbb{B}_-^B(\mathcal{F} + D) \cup \mathbb{B}_+^B(A) \quad \text{and} \quad \mathbb{B}_+^B(\mathcal{F} + D) \subseteq \mathbb{B}_+^A(\mathcal{F} + D) \cup \mathbb{B}_+^B(A).$$

Proof. Replacing B with εB for some small $\varepsilon \in \mathbb{Q}_{>0}$, we may assume $\mathbb{B}_+^B(A) = \mathbb{B}(A - B)$. Then for every $r \in \mathbb{Q}_{>0}$,

$$\mathbb{B}(\mathcal{F} + D + rA) \subseteq \mathbb{B}(\mathcal{F} + D + rB) \cup \mathbb{B}(rA - rB) = \mathbb{B}(\mathcal{F} + D + rB) \cup \mathbb{B}_+^B(A),$$

which proves the first assertion. By an argument similar to the above, one can prove the second assertion. $\square$

We define $\mathbb{B}_-(\mathcal{F} + D) := \mathbb{B}_-^A(\mathcal{F} + D)$ and $\mathbb{B}_+(\mathcal{F} + D) = \mathbb{B}_+^A(\mathcal{F} + D)$ for an ample $\mathbb{Q}$ -Cartier divisor A on X . This definition is independent of the choice of A . Indeed, for another ample $\mathbb{Q}$ -Cartier divisor B on X , we have $\mathbb{B}_+^A(B) = \mathbb{B}_+^B(A) = \emptyset$, so Lemma 4.1 shows that

$$\mathbb{B}_-^A(\mathcal{F} + D) = \mathbb{B}_-^B(\mathcal{F} + D) \quad \text{and} \quad \mathbb{B}_+^A(\mathcal{F} + D) = \mathbb{B}_+^B(\mathcal{F} + D).$$

We set $\mathbb{B}_-(\mathcal{F}) := \mathbb{B}_-(\mathcal{F} + 0)$ and $\mathbb{B}_+(\mathcal{F}) := \mathbb{B}_+(\mathcal{F} + 0)$. When $\mathcal{F}$ is locally free, we call $\mathbb{B}_-(\mathcal{F})$ the *restricted base locus* (or *diminished base locus*) of $\mathcal{F}$. We also call $\mathbb{B}_+(\mathcal{F})$ the *augmented base locus* of $\mathcal{F}$. We also set $\mathbb{B}_-(D) := \mathbb{B}_-(\mathcal{O}_X + D)$ and $\mathbb{B}_+(D) := \mathbb{B}_+(\mathcal{O}_X + D)$, which coincide with the usual ones.

As a corollary of Lemma 4.1, we get the following.

COROLLARY 4.2. *Let X , $\mathcal{F}$ and D be as above. Let L be a semi-ample $\mathbb{Q}$ -Cartier divisor on X . Then the following hold:*

- (1) $\mathbb{B}_-(\mathcal{F} + D) \subseteq \mathbb{B}_-^L(\mathcal{F} + D) \subseteq \mathbb{B}(\mathcal{F} + D) \subseteq \mathbb{B}_+^L(\mathcal{F} + D) \subseteq \mathbb{B}_+(\mathcal{F} + D)$.
- (2) $\mathbb{B}_-^L(\mathcal{F} + D) \subseteq \mathbb{B}_-(\mathcal{F} + D) \cup \mathbb{B}_+(L)$.
- (3) $\mathbb{B}_+^L(\mathcal{F} + D) \subseteq \mathbb{B}_+(\mathcal{F} + D) \cup \mathbb{B}_+(L)$.

Example 4.3. For each inclusion in Corollary 4.2(1), we give an example such that it is a proper inclusion. Let X be a projective variety of positive dimension over a field k . Then $\mathbb{B}(-A) = X$ for any ample Cartier divisor A on X , so we get from the definition of $\mathbb{B}_+^L$ that

$$\begin{cases} \mathbb{B}(0) = \emptyset \subsetneq X = \mathbb{B}_+^L(0) & \text{if } L \text{ is an ample Cartier divisor on } X, \\ \mathbb{B}_+^L(0) = \emptyset \subsetneq X = \mathbb{B}_+(0) & \text{if } L = 0. \end{cases}$$

Next, suppose that k is uncountable and algebraically closed, and let E be an elliptic curve over k . Since k is uncountable, there is a line bundle $\mathcal{N}$ on E such that $\deg \mathcal{N} = 0$ and $H^0(E, \mathcal{N}^m) = 0$ for each $m > 0$. Then $\mathcal{N}$ is nef and $\mathbb{B}(\mathcal{N}) = E$, so

$$\begin{cases} \mathbb{B}_-(\mathcal{N}) = \emptyset \subsetneq E = \mathbb{B}_-^L(\mathcal{N}) & \text{if } L = 0, \\ \mathbb{B}_-^L(\mathcal{N}) = \emptyset \subsetneq E = \mathbb{B}(\mathcal{N}) & \text{if } L \text{ is an ample Cartier divisor on } E. \end{cases}$$

Example 4.4. We give an example satisfying $\mathbb{B}_-^L(\mathcal{F}) \neq \mathbb{B}_-^M(\mathcal{F})$ when L and M are neither ample nor linearly trivial. Let k , E and $\mathcal{N}$ be as in the second part of Example 4.3. Let C be a smooth projective curve over k . Let A_C (respectively, A_E) be an ample divisor on C (respectively, E). Put $X := C \times_k E$. Let $c: X \rightarrow C$ and $e: X \rightarrow E$ denote the natural projections. Set

$$L := c^*A_C, \quad M := e^*A_E \quad \text{and} \quad \mathcal{F} := e^*\mathcal{N}.$$

In this setting, we prove that

$$\mathbb{B}_-^L(\mathcal{F}) = X \quad \text{and} \quad \mathbb{B}_-^M(\mathcal{F}) = \emptyset.$$

It is enough to show that $\mathbb{B}(\mathcal{F} + n^{-1}L) \stackrel{(1)}{=} X$ and $\mathbb{B}(\mathcal{F} + n^{-1}M) \stackrel{(2)}{=} \emptyset$ for each $n > 0$. Equality (1) follows from the fact that we have

$$H^0(X, S^{ln}(\mathcal{F}) \otimes \mathcal{O}_X(lL)) \cong H^0(X, (e^*\mathcal{N}^{ln}) \otimes c^*\mathcal{O}_C(lA_C)) \cong H^0(E, \mathcal{N}^{ln}) \otimes H^0(C, lA_C) = 0$$

for each $l > 0$ by the choice of $\mathcal{N}$. Equality (2) is obtained from that

$$(S^{ln}\mathcal{F}) \otimes (lM) \cong (e^*\mathcal{N}^{ln}) \otimes e^*\mathcal{O}_E(lA_E) \cong e^*(\mathcal{N}^n(A_E))^l$$

is globally generated for each $l \gg 0$. Note that $\mathcal{N}^n(A_E)$ is an ample line bundle.

Furthermore, putting $\mathcal{G} := \mathcal{O}_X(M)$, we get from an argument similar to the above that

$$\mathbb{B}_+^L(\mathcal{G}) = X \quad \text{and} \quad \mathbb{B}_+^M(\mathcal{G}) = \emptyset.$$

In this paper, we use the following terminology, which is a natural extension of the positivity conditions of a $\mathbb{Q}$ -Cartier divisor on a projective variety.

DEFINITION 4.5. Let X be a quasi-projective variety, and let D be a $\mathbb{Q}$ -Cartier divisor on X . We say that D is *pseudo-effective* (respectively, *big*) if $\mathbb{B}_-(D) \neq X$ (respectively, $\mathbb{B}_+(D) \neq X$).

In the rest of this subsection, we prove several lemmas on base loci that are used in Sections 5 and 6.

LEMMA 4.6. *Let X , D and $\mathcal{F}$ be as above. Let $g: X \rightarrow Z$ be a projective morphism to a quasi-projective variety Z , let H and D be $\mathbb{Q}$ -Cartier divisors on Z , and assume that H is ample. Then*

- (1) $\mathbb{B}_+(g^*H) = \{x \in \text{sp}(X) \mid \dim g^{-1}(g(x)) \geq 1\}$,
- (2) $\text{Bs}(\mathcal{F}) \subseteq g^{-1}(\text{Bs}(g_*\mathcal{F})) \cup \mathbb{B}_+(g^*H)$, and
- (3) for a semi-ample $\mathbb{Q}$ -Cartier divisor L on Z ,

$$\begin{aligned} \mathbb{B}_-^{g^*L}(\mathcal{F} + g^*D) &\subseteq g^{-1}(\mathbb{B}_-^L(g_*\mathcal{F} + D) \cup \mathbb{B}(L)) \cup \mathbb{B}_+(g^*H) \quad \text{and} \\ \mathbb{B}_+^{g^*L}(\mathcal{F} + g^*D) &\subseteq g^{-1}(\mathbb{B}_+^L(g_*\mathcal{F} + D) \cup \mathbb{B}(L)) \cup \mathbb{B}_+(g^*H). \end{aligned}$$

Proof. Let $A \geq 0$ be an ample Cartier divisor on X . Fix $l \in \mathbb{Z}_{>0}$ such that lH is Cartier, $\mathbb{B}_+(g^*H) = \mathbb{B}(g^*lH - A)$, and $(g_*\mathcal{O}_X(-A))(lH)$ is generated by its global sections. Take a point $x \in \text{sp}(X)$. If $g_x^\#$ is finite, then we see from Lemma 4.7 below that the natural morphism $g^*((g_*\mathcal{O}_X(-A))(lH)) \rightarrow \mathcal{O}_X(g^*lH - A)$ is surjective over x , so $x \notin \text{Bs}(g^*lH - A) \supseteq \mathbb{B}(g^*lH - A)$.

LEMMA 4.7. *Let $h: V \rightarrow W$ be a projective morphism between quasi-projective varieties, and let $\mathcal{G}$ be a coherent sheaf on V . Let v be a point in $\text{sp}(V)$ such that $\dim h^{-1}(h(v)) = 0$. Then the natural morphism $\varphi: h^*h_*\mathcal{G} \rightarrow \mathcal{G}$ is surjective over v .*

Proof. Let $V \xrightarrow{a} W' \xrightarrow{b} W$ be the Stein factorization of h . Then b is an affine morphism, so the natural morphism $\varphi': b^*b_*(a_*\mathcal{G}) \rightarrow a_*\mathcal{G}$ is surjective. Since φ can be decomposed as $h^*h_*\mathcal{G} \xrightarrow{a^*\varphi'} a^*a_*\mathcal{G} \xrightarrow{\varphi''} \mathcal{G}$, it is enough to show that φ'' is surjective, so we may assume that every fiber of h is connected. Then $\{v\} = h^{-1}(h(v))$, so h is finite over $h(v)$, and hence φ is surjective in a neighborhood of v , which completes the proof. $\square$

We return to the proof of Lemma 4.6. Suppose that $x \notin \mathbb{B}(g^*lH - A)$. Then there is an effective $\mathbb{Q}$ -Cartier divisor E with $E \sim_{\mathbb{Q}} g^*lH - A$ and $x \notin \text{Supp}(E)$. Let F be an irreducible component of the fiber of g over $g(x)$. We may assume that F is not contained in A . Then $E|_F \sim_{\mathbb{Q}} (g^*lH - A)|_F \sim -A|_F$, so $A|_F \sim 0$ since F is projective over $g(x)$. This means that $\dim F = 0$, so $g_x^\#$ is finite, and the proof of assertion (1) is complete.

Next, take an $l \in \mathbb{Z}_{>0}$ such that lD is Cartier. We then have the morphisms

$$g^*(S^l(g_*\mathcal{F})(lD)) \cong S^l(g^*g_*\mathcal{F})(lg^*D) \rightarrow S^l(\mathcal{F})(lg^*D),$$

and assertion (1) tells us that the cokernel of the composite is supported on $\mathbb{B}_+(g^*H)$. From this, we can prove assertions (2) and (3). $\square$

LEMMA 4.8. *Let X be a quasi-projective variety, let D be a Cartier divisor on X , and let $\mathcal{F}$ be a coherent sheaf on X . Then there exists an integer $n_0 = n_0(\mathcal{F}, D)$ such that*

$$\text{Bs}(\mathcal{F} \otimes \mathcal{O}_X(nD)) \subseteq \mathbb{B}_+(D)$$

for each $n \geq n_0$.

Proof. Let A be an ample Cartier divisor on X . Fix $l, m \in \mathbb{Z}_{>0}$ such that

$$\mathbb{B}_+(D) = \mathbb{B}(lD - A) = \text{Bs}(m(lD - A)).$$

Set $\mathcal{F}' := \bigoplus_{0 \leq r < lm} \mathcal{F}(rD)$. Take $n \in \mathbb{Z}_{>0}$. Let q and r denote the quotient and the remainder of the division of n by lm , respectively. Then

$$\begin{aligned} \text{Bs}(\mathcal{F}(nD)) &= \text{Bs}(\mathcal{F}(rD + qmA + qm(lD - A))) \\ &\subseteq \text{Bs}(\mathcal{F}(rD + qmA)) \cup \text{Bs}(qm(lD - A)) \\ &\subseteq \text{Bs}(\mathcal{F}'(qmA)) \cup \mathbb{B}_+(D). \end{aligned}$$

When $n \gg 0$, we have $\text{Bs}(\mathcal{F}'(qmA)) = \emptyset$, so $\text{Bs}(\mathcal{F}(nD)) \subseteq \mathbb{B}_+(D)$. $\square$

LEMMA 4.9. *Let X be a quasi-projective variety, let A and D be $\mathbb{Q}$ -Cartier divisors on X , and let $\mathcal{F}$ be a coherent sheaf on X . Let $\pi: Y \rightarrow X$ be a finite surjective morphism, and let U denote the maximal open subset of X such that $\pi|_{\pi^{-1}(U)}: \pi^{-1}(U) \rightarrow U$ is flat. Set $C := X \setminus U$. Then*

$$\mathbb{B}(\mathcal{F} + (D + A)) \subseteq \pi(\mathbb{B}(\pi^*\mathcal{F} + \pi^*D)) \cup \mathbb{B}_+(A) \cup C.$$

Proof. Set $\mathcal{G} := (\pi_*\mathcal{O}_Y)^* \otimes \pi_*\mathcal{O}_Y$. Then there is a natural morphism $\alpha: \mathcal{G} \rightarrow \mathcal{O}_Y$ whose cokernel is supported on C . Take $i \in \mathbb{Z}_{>0}$ so that iA is Cartier. Thanks to Lemma 4.8, we get an $n_0 \geq 0$ such that $\text{Bs}(\mathcal{G}(inA)) \subseteq \mathbb{B}_+(A)$ for each $n \geq n_0$. Fix $m \in \mathbb{Z}$ with $m \geq n_0$ that is divisible enough. Then we have a morphism

$$\bigoplus^j \mathcal{O}_Y \xrightarrow{\beta} (S^m(\pi^*\mathcal{F}))(m\pi^*D)$$

whose cokernel is supported on $\mathbb{B}(\pi^*\mathcal{F} + \pi^*D)$. Applying $(\pi_*\mathcal{O}_Y)^*(mA) \otimes \pi_*(\cdot)$ to β , we get the following sequence of morphisms whose cokernels are supported on $\pi(\mathbb{B}(\pi^*\mathcal{F} + \pi^*D)) \cup C$:

$$\begin{aligned} \bigoplus^j \mathcal{G}(mA) &\rightarrow (\pi_*\mathcal{O}_Y)^*(mA) \otimes \pi_*(\pi^*(S^m(\mathcal{F})(mD))) && \text{induced by } \beta \\ &\cong (\pi_*\mathcal{O}_Y)^*(mA) \otimes (\pi_*\mathcal{O}_Y) \otimes S^m(\mathcal{F})(mD) && \text{since } \pi \text{ is affine} \\ &\cong \mathcal{G} \otimes S^m(\mathcal{F})(m(D + A)) \\ &\rightarrow S^m(\mathcal{F})(m(D + A)). && \text{induced by } \alpha \end{aligned}$$

Note that since π is affine, $\text{Coker } \pi_*\beta = \pi_*(\text{Coker } \beta)$. We then get that

$$\text{Bs}(S^m(\mathcal{F})(m(D + A))) \subseteq \pi(\mathbb{B}(\pi^*\mathcal{F} + \pi^*D)) \cup \mathbb{B}_+(A) \cup C.$$

The left-hand side is equal to $\mathbb{B}(\mathcal{F} + (D + A))$ since m is divisible enough. $\square$

PROPOSITION 4.10. *Let k be an F -finite field. Let W be a projective variety over k of dimension n , and let H be a big Cartier divisor on W with $|H|$ free. Let Y be a dense open subset of W , and let $\mathcal{F}_1, \dots, \mathcal{F}_N$ and $\mathcal{G}$ be coherent sheaves on Y . Fix $\varepsilon, \varepsilon_1, \dots, \varepsilon_N \in \mathbb{Q}_{>0}$. Put $B := \bigcup_{1 \leq i \leq N} \mathbb{B}(\mathcal{F}_i - \varepsilon_i H|_Y)$. Then there exists a positive integer e_0 such that*

$$\text{Bs}(F_{Y*}^e(S^{l_1}(\mathcal{F}_1) \otimes \dots \otimes S^{l_n}(\mathcal{F}_n) \otimes \mathcal{G})) \subseteq \mathbb{B}_+(H) \cup B$$

for each $e \geq e_0$ and each $l_1, \dots, l_n \in \mathbb{Z}_{>0}$ with $\sum_{1 \leq i \leq N} \varepsilon_i l_i \geq (n + \varepsilon)p^e$.

Remark 4.11. We consider the following case: $Y = W$, H is ample and $|H|$ is free, $N = 1$, $\mathcal{F}_1 = \mathcal{O}_W(H)$, and $\varepsilon_1 = \varepsilon = 1$. Then, Proposition 4.10 is equivalent to the following well-known fact: $(F_{W*}^e \mathcal{G})((n + 1)H)$ is generated by its global sections for each $e \gg 0$. One can use this fact to verify Fujita's freeness conjecture in positive characteristic in the case where the ample line bundle is globally generated [Har05, Kee03]. This special case of Fujita's conjecture was first proved by Smith [Smi97].

Proof. Step 1. We first define e_0 . Fix an $m \in \mathbb{Z}_{>0}$ such that $\varepsilon_i m \in \mathbb{Z}$ and that

$$\mathbb{B}(\mathcal{F}_i - \varepsilon_i H|_Y) = \text{Bs}(S^m(\mathcal{F}_i)(-m\varepsilon_i H|_Y))$$

for each $i = 1, \dots, N$. Set $\mathbb{S} := \{(r_1, \dots, r_N) \mid 0 \leq r_i < m \text{ for each } i\}$ and

$$\mathcal{G}' := \mathcal{G} \otimes \bigoplus_{(r_1, \dots, r_N) \in \mathbb{S}} \left(\bigotimes_{1 \leq i \leq N} S^{r_i}(\mathcal{F}_i) \right).$$

Take a coherent sheaf $\mathcal{G}''$ on W such that $\mathcal{G}''|_Y \cong \mathcal{G}'$. Since $|H|$ is free, there exist a generically finite surjective morphism $g: W \rightarrow Z$ and an ample Cartier divisor L on Z such that $|L|$ is free and $H \sim g^*L$. Then, by Serre's vanishing theorem, we find an $s_0 \in \mathbb{Z}_{>0}$ such that $H^j(Z, (g_*\mathcal{G}'')(sL)) = 0$ for each $1 \leq j \leq n$ and $s \geq s_0$. We define $e_0 := \min\{e \in \mathbb{Z}_{>0} \mid \varepsilon p^e \geq s_0 + m \sum_i \varepsilon_i\}$.

Step 2. Take $e \geq e_0$ and $l_1, \dots, l_N \in \mathbb{Z}_{>0}$ with $\sum_i \varepsilon_i l_i \geq (n + \varepsilon)p^e$. For each i , let q_i and r_i be integers such that $l_i = mq_i + r_i$ and $0 \leq r_i < m$. Put $\mu := \sum_{1 \leq i \leq N} \varepsilon_i mq_i$ and $\mathcal{E} := F_{W*}^e(\mathcal{G}''(\mu H))$. In this step, we prove that $\text{Bs}(\mathcal{E}|_Y) \subseteq \mathbb{B}_+(H)$. Since $mq_i > -m + l_i$, we have

$$\mu = \sum_{1 \leq i \leq N} \varepsilon_i mq_i > - \left(\sum_{1 \leq i \leq N} \varepsilon_i m \right) + \sum_{1 \leq i \leq N} \varepsilon_i l_i \geq s_0 - \varepsilon p^{e_0} + (n + \varepsilon)p^e \geq s_0 + np^e,$$

so $\mu - jp^e \geq s_0$ for each $0 < j \leq n$. Hence, we see from the projection formula that

$$\begin{aligned} H^j(Z, \mathcal{O}_Z(-jL) \otimes g_*\mathcal{E}) &\cong H^j(Z, \mathcal{O}_Z(-jL) \otimes F_{Z*}^e((g_*\mathcal{G}'')(\mu L))) \\ &\cong H^j(Z, F_{Z*}^e((g_*\mathcal{G}'')((\mu - jp^e)L))) \\ &\cong H^j(Z, (g_*\mathcal{G}'')((\mu - jp^e)L)) = 0, \end{aligned}$$

which means that $g_*\mathcal{E}$ is 0-regular with respect to $\mathcal{O}_Z(L)$. Thus, $g_*\mathcal{E}$ is generated by its global sections as shown in [Laz04, Theorem 1.8.5]. Lemma 4.6(2) then tells us that $\text{Bs}(\mathcal{E}) \subseteq \mathbb{B}_+(g^*L)$, which proves the claim since $\text{Bs}(\mathcal{E}|_Y) \subseteq \text{Bs}(\mathcal{E})$.

Step 3. We show the assertion. Put

$$\mathcal{D} := \bigotimes_{1 \leq i \leq N} S^{mq_i}(\mathcal{F}_i)(-mq_i \varepsilon_i H|_Y) \cong \left(\bigotimes_{1 \leq i \leq N} S^{mq_i}(\mathcal{F}_i) \right)(-\mu H|_Y).$$

By the definition of $\mathcal{D}$ and $\mathcal{G}'$, we get

$$\begin{aligned} F_{Y*}^e(\mathcal{G}'(\mu H|_Y) \otimes \mathcal{D}) &\cong F_{Y*}^e \left(\mathcal{G}' \otimes \left(\bigotimes_{1 \leq i \leq N} S^{mq_i}(\mathcal{F}_i) \right) \right) \\ &\rightarrow F_{Y*}^e \left(\mathcal{G} \otimes \left(\bigotimes_{1 \leq i \leq N} S^{mq_i}(\mathcal{F}_i) \otimes S^{r_i}(\mathcal{F}_i) \right) \right) \\ &\rightarrow F_{Y*}^e \left(\mathcal{G} \otimes \left(\bigotimes_{1 \leq i \leq N} S^{l_i}(\mathcal{F}_i) \right) \right) =: \mathcal{C}. \end{aligned}$$

Furthermore, we see from the choice of m that there is a morphism $\bigoplus \mathcal{O}_Y \rightarrow \mathcal{D}$ whose cokernel is supported on B , which induces the morphism

$$\bigoplus F_{Y*}^e(\mathcal{G}'(\mu H|_Y)) \cong F_{Y*}^e(\mathcal{G}'(\mu H|_Y) \otimes (\bigoplus \mathcal{O}_Y)) \rightarrow F_{Y*}^e(\mathcal{G}'(\mu H|_Y) \otimes \mathcal{D})$$

whose cokernel is supported on B . It then follows from $\mathcal{E}|_Y \cong F_{Y*}^e(\mathcal{G}'(\mu H|_Y))$ that

$$\text{Bs}(\mathcal{C}) \subseteq \text{Bs}(\mathcal{E}|_Y) \cup B \stackrel{\text{Step 2}}{\subseteq} \mathbb{B}_+(H) \cup B,$$

which completes the proof. $\square$

4.2 Weak positivity

Let k be a field. A notion of weak positivity was introduced by Viehweg [Vie83a].

DEFINITION 4.12 ([Vie95, Variant 2.13]). Let Y be a normal quasi-projective variety, and let $\mathcal{G}$ be a coherent sheaf on Y . Let $\mathcal{G}'$ denote the quotient of $\mathcal{G}$ by the torsion submodule, and let Y_1 be the maximal open subset such that $\mathcal{G}'|_{Y_1}$ is locally free. Let Y_0 be a dense open subset of Y_1 . We say that $\mathcal{G}$ is *weakly positive over Y_0* if for every ample line bundle $\mathcal{H}$ on Y and every positive integer α , there exists a positive integer β such that

$$S^{\alpha\beta}(\mathcal{G}'|_{Y_1}) \otimes \mathcal{H}|_{Y_1}^{\beta} \quad (4.12.1)$$

is globally generated over Y_0 . We simply say that $\mathcal{G}$ is *weakly positive* if it is weakly positive over a dense open subset of Y_1 .

The sheaf $\mathcal{G}$ is often said to be weakly positive if (4.12.1) is generically globally generated (cf. [Vie83b, Kaw85]). In order to distinguish this terminology from Definition 4.12, we employ the following definition.

DEFINITION 4.13. Let Y , $\mathcal{G}$, $\mathcal{G}'$ and Y_1 be as in Definition 4.12. In this paper, we say that $\mathcal{G}$ is *pseudo-effective* if for every ample line bundle $\mathcal{H}$ on Y and every positive integer α , there exists a positive integer β such that sheaf (4.12.1) is generated by its global sections at the generic point η of Y .

The weak positivity and the pseudo-effectivity of coherent sheaves are rephrased in terms of restricted base loci.

LEMMA 4.14. Let Y , $\mathcal{G}$, $\mathcal{G}'$, Y_1 and Y_0 be as in Definition 4.12.

- (1) The sheaf $\mathcal{G}$ is weakly positive if and only if $\mathbb{B}_-(\mathcal{G}'|_{Y_1})$ and Y_0 do not intersect.
- (2) If $\mathbb{B}_-(\mathcal{G})$ and Y_0 do not intersect, then $\mathcal{G}$ is weakly positive over Y_0 .
- (3) The sheaf $\mathcal{G}$ is pseudo-effective if and only if $\mathbb{B}_-(\mathcal{G}'|_{Y_1})$ is not equal to Y_1 , or equivalently, $\mathbb{B}_-(\mathcal{G}'|_{Y_1})$ does not contain the generic point η of Y .
- (4) If $\mathbb{B}_-(\mathcal{G})$ does not contain η , then $\mathcal{G}$ is pseudo-effective.
- (5) The converses of statements (2) and (4) hold if $\mathcal{G}$ is locally free.

$$\begin{array}{ccccc}
 & & \mathcal{G} \text{ is locally free} & & \\
 & & \curvearrowright & & \\
 \mathcal{G} \text{ is weakly positive over } Y_0 & \iff & \mathbb{B}_-(\mathcal{G}'|_{Y_1}) \cap Y_0 = \emptyset & \iff & \mathbb{B}_-(\mathcal{G}) \cap Y_0 = \emptyset \\
 \Downarrow & & \Downarrow & & \Downarrow \\
 \mathcal{G} \text{ is pseudo-effective} & \iff & \eta \notin \mathbb{B}_-(\mathcal{G}'|_{Y_1}) & \iff & \eta \notin \mathbb{B}_-(\mathcal{G}) \\
 & & \curvearrowright & & \mathcal{G} \text{ is locally free}
 \end{array}$$

Proof. Statements (1), (3) and (5) are obvious. Statements (2) and (4) follow from the inclusion $\mathbb{B}_-(\mathcal{G}|_{Y_1}) \subseteq \mathbb{B}_-(\mathcal{G}) \cap Y_1$. $\square$

The following example shows that Lemma 4.14(5) does not hold if $\mathcal{G}$ is not locally free.

Example 4.15. Let Y be a regular projective surface, let $y \in \text{sp}(Y)$ be a closed point, and let $\pi: Y' \rightarrow Y$ be the blow-up of Y along y . Let $\mathcal{I}$ be the ideal sheaf of y , and let A be an ample Cartier divisor on Y . Take $l \gg 0$ so that $\pi^*A - lE$ is not pseudo-effective, that is,

$\mathbb{B}_-(\mathcal{O}_{Y'}(\pi^*A - lE)) = Y'$, and set $\mathcal{G} := \mathcal{I}^l \otimes \mathcal{O}_Y(A)$. Then the natural map $\pi^*\mathcal{G} \rightarrow \mathcal{O}_{Y'}(\pi^*A - lE)$ shows that

$$\mathbb{B}_-(\mathcal{G}) = \mathbb{B}_+(\mathcal{G}) = Y.$$

Put $Y_1 := Y \setminus \{y\}$. This is the maximal open subset such that $\mathcal{G}|_{Y_1}$ is locally free. Since $\mathcal{I}|_{Y_1} \cong \mathcal{O}_{Y_1}$, we have

$$\mathbb{B}_-(\mathcal{G}|_{Y_1}) = \mathbb{B}_+(\mathcal{G}|_{Y_1}) = \emptyset.$$

In particular, $\mathcal{G}$ is weakly positive but $\mathbb{B}_-(\mathcal{G}) = Y$.

Remark 4.16. Similarly to the argument in this subsection, one can discuss the bigness of coherent sheaves, using $\mathbb{B}_+$ instead of $\mathbb{B}_-$. We say that a coherent sheaf $\mathcal{F}$ on a projective variety X is *V-big* (or *Viehweg-big*) if $\mathbb{B}_+(\mathcal{F}) \neq X$ (see [BKK⁺15, Definition 6.1]). This condition is stronger than the one that the tautological bundle $\mathcal{O}_{\mathbb{P}(\mathcal{F})}(1)$ is big [Jab09, Examples 1.7 and 1.8.]. We do not treat the above notions further since they are not used in this paper.

5. An invariant of coherent sheaves

In this section, we introduce an invariant of coherent sheaves that we use to study the positivity of coherent sheaves. The invariant is defined by using a morphism to a variety admitting a special endomorphism. Throughout this section, we work over an F -finite field k of characteristic $p > 0$.

DEFINITION 5.1. Let Y be a quasi-projective variety, let H be a big Cartier divisor on Y , and let S be a non-empty subset of $\mathrm{sp}(Y)$. Fix a non-negative rational number a . We say that the pair (S, H) *satisfies condition $(*)_a$* if all the following conditions hold:

- I. There exist a smooth projective variety Z and a projective morphism $g: Y \rightarrow U$ to a dense open subset U of Z such that
 - I-1. $g^{-1}(s)$ is a finite set for every $s \in g(S)$ and
 - I-2. $S = \bigcup_{s \in g(S)} g^{-1}(s)$.

We do not distinguish between $g: Y \rightarrow U$ and the composite $Y \xrightarrow{g} U \hookrightarrow Z$.

- II. There exists a big Cartier divisor L on Z such that
 - II-1. $H \sim g^*L$ and
 - II-2. $\mathbb{B}_-(K_Z + aL) \cap g(S) = \mathbb{B}_+(L) \cap g(S) = \emptyset$.

- III. There exists a separable finite flat endomorphism $\pi: Z \rightarrow Z$ such that
 - III-1. $\pi^*L \sim qL$ for an integer $q \geq 2$ and
 - III-2. π^d is étale over a neighborhood of every point in $g(S)$ for each $d \geq 1$.

Remark 5.2. (1) We note that q in condition III does not stand for a power p^e of the characteristic p . We employ the same notation as that in some papers dealing with polarized endomorphisms (cf. [NZ10]).

- (2) Let η denote the generic point of Y . When $S = \{\eta\}$, condition I-2 follows from condition I-1.
- (3) When $S = \{\eta\}$ and g is dominant, condition III-2 is always satisfied.
- (4) Assume $S = \{\eta\}$ and that g is dominant. Then condition II-2 is equivalent to saying that $K_Z + aL$ is pseudo-effective. In particular, $(*)_a$ holds if a is at least Fujita's invariant (or a -constant) $a(Z, L) = \inf\{t > 0 \mid K_Z + tL \text{ is big}\}$.

Remark 5.3. Let $\mathcal{F}$ be a coherent sheaf on Y , and let D be a $\mathbb{Q}$ -Cartier divisor on Y . Suppose that (S, H) satisfies $(*)_a$. Take an ample Cartier divisor B on Z . By Corollary 4.2, we have

$$\mathbb{B}_-(\mathcal{F} + D) \subseteq \mathbb{B}_-^{g^*B}(\mathcal{F} + D) \subseteq \mathbb{B}_-(\mathcal{F} + D) \cup \mathbb{B}_+(g^*B).$$

By conditions I-1 and II-2, it follows from Lemma 4.6 that $\mathbb{B}_+(g^*B) \cap S = \emptyset$, so

$$\mathbb{B}_-^{g^*B}(\mathcal{F} + D) \cap S = \mathbb{B}_-(\mathcal{F} + D) \cap S.$$

Similarly, we can check that $\mathbb{B}_+^{g^*B}(\mathcal{F} + D) \cap S = \mathbb{B}_+(\mathcal{F} + D) \cap S$.

The next lemma follows immediately from Definition 5.1.

LEMMA 5.4. *Let Y be a quasi-projective variety, let H be a big Cartier divisor on Y , and let S be a subset of $\mathrm{sp}(Y)$ such that (S, H) satisfies condition $(*)_a$ for some $a \in \mathbb{Q}_{\geq 0}$. Let $f: Y' \rightarrow Y$ be a projective morphism such that $f|_{f^{-1}(V)}: f^{-1}(V) \rightarrow V$ is finite for some open subset $V \subseteq Y$ containing S . Then $(f^{-1}(S), f^*H)$ also satisfies condition $(*)_a$.*

Proof. Conditions I-1 and I-2 hold obviously. Conditions II and III follow from the inclusion $(g \circ f)(f^{-1}(S)) \subseteq g(S)$. $\square$

Example 5.5. Fix $N \in \mathbb{Z}_{>0}$. Let $Y \subseteq \mathbb{P}^N$ be a subvariety, and let $S \subseteq \mathrm{sp}(Y)$ be a subset. Let $L \subset \mathbb{P}^N$ be a hyperplane with $Y \not\subseteq L$, and set $H := L|_Y$. We show that there exist open subsets $U_1, \dots, U_l$ of Y such that $S \subseteq \bigcup_{i=1}^l U_i$ and $(S \cap U_i, H)$ satisfies $(*)_{N+1}$ for each i . Note that since $g: Y \rightarrow \mathbb{P}^N =: Z$ is injective and $K_Z + (N+1)L \sim 0$, conditions I and II hold. Fix $s \in S$. Choose a basis $x_0, x_1, \dots, x_N$ of $H^0(\mathbb{P}^N, \mathcal{O}(1))$ so that

$$s \notin B := \{z \in \mathrm{sp}(\mathbb{P}^N) \mid x_i \text{ vanishes at } z \text{ for some } i\}.$$

Take $q \in \mathbb{Z}_{\geq 2}$ with $p \nmid q$. Then $x_0^q, \dots, x_N^q$ define a separable flat endomorphism $\pi: \mathbb{P}^N \rightarrow \mathbb{P}^N$ of degree q^N such that $\pi^*L \sim qL$. One can check that π^d is étale over $\mathbb{P}^N \setminus B$ for each $d \geq 1$, so $(S \setminus B, H)$ satisfies $(*)_{N+1}$. Since Y is noetherian, our claim follows.

Example 5.6. Suppose that k is an F -finite infinite field. Let Y be a projective variety of dimension n , and let H be a big Cartier divisor with $|H|$ free. Since k is infinite, one can find a free linear system $\mathfrak{d} \subseteq |H|$ of dimension n . Let $g: Y \rightarrow \mathbb{P}^n =: Z$ be the morphism defined by $\mathfrak{d}$, and let L be a hypersurface of Z . Then $g^*L \sim H$, so g is finite over $\mathrm{sp}(Z) \setminus g(\mathbb{B}_+(H))$ by Lemma 4.6. Fix $S' \subseteq \mathrm{sp}(Z) \setminus g(\mathbb{B}_+(H))$, and put $S := g^{-1}(S')$. Combining the argument in Example 5.5 with Lemma 5.4, one can prove that there exist subsets $S_1, \dots, S_l$ of S such that $S = \bigcup_{i=1}^l S_i$ and (S_i, H) satisfies $(*)_{n+1}$ for each i . Note that if $\mathcal{L}$ is ample (that is, if $\mathbb{B}_+(\mathcal{L}) = \emptyset$), we can take $S = Y$.

Example 5.7. Let A be an abelian variety, and let L be a symmetric ample divisor on A (that is, an ample divisor L with $(-1_A)^*L \sim L$). Then (S, L) satisfies $(*)_0$ for every subset $S \subseteq \mathrm{sp}(A)$. Indeed, we have $K_A + 0L \sim 0$, and for some $m \in \mathbb{Z}_{\geq 2}$ with $p \nmid m$, the morphism $\pi := m_A: A \rightarrow A$ is an étale endomorphism with the property that $\pi^*L \sim m^2L$.

Example 5.8. Let Y be a projective variety, and let $g: Y \rightarrow A$ be a morphism to an abelian variety A such that $\dim Y = \dim g(Y)$. (For example, suppose that k is algebraically closed, and let Y be a normal projective variety of maximal Albanese dimension.) Let $V \subseteq A$ be the maximal open subset such that g is finite over V . Take $S' \subseteq \mathrm{sp}(V)$, and put $S := g^{-1}(S')$. Combining Example 5.7 with Lemma 5.4, we see that (S, g^*L) satisfies $(*)_0$ for every symmetric ample divisor L on A .

Example 5.9. Let Y be a smooth projective toric variety, and set $L := -K_Y$. Fix a subset $S \subseteq \text{sp}(T) \setminus \mathbb{B}_+(-K_Y)$, where T is the dense open subset of Y isomorphic to the n -dimensional algebraic torus $(k^\times)^n$. Then (S, L) satisfies $(*)_1$. Indeed, we have $K_Y + 1L = 0$, and for some $q \in \mathbb{Z}_{\geq 2}$ with $p \nmid q$, the q th toric Frobenius morphism $\pi: Y \rightarrow Y$ is étale over S and satisfies $\pi^*L \sim qL$.

We need the following notion in order to define our invariant (Definition 5.12).

DEFINITION 5.10. Let Z be a quasi-projective variety, and let $\pi: Z \rightarrow Z$ be a surjective endomorphism of Z . Let $\mathcal{G}$ be a coherent sheaf on a dense open subset U of Z . Set $U_d := (\pi^d)^{-1}(U)$ and $\pi_U^d := (\pi^d)|_{U_d}: U_d \rightarrow U$ for each $d \geq 1$. Let D be a $\mathbb{Q}$ -Cartier divisor on U such that $(\pi_U^{d_0})^*D$ is Cartier for some d_0 . Set

$$B_d := \pi_U^d(\text{Bs}((\pi_U^d)^*\mathcal{G} \otimes_{\mathcal{O}_{U_d}} (\pi_U^d)^*D)) \subseteq \text{sp}(U).$$

We then define

$$\mathbb{B}^\pi(\mathcal{G} + D) := \bigcap_{d \geq d_0} B_d \subseteq \text{sp}(U).$$

Note that for each $d \geq d_0$, we have $B_d \supseteq B_{d+1}$, so $\mathbb{B}^\pi(\mathcal{G} + D) = B_d$ for some $d \gg d_0$.

PROPOSITION 5.11. *Let the notation be as in Definition 5.10. Let C denote the set of points $z \in \text{sp}(U)$ such that π^d is not flat over z for some d . Let A be a $\mathbb{Q}$ -Cartier divisor on U . Then*

$$\mathbb{B}(\mathcal{G} + (D + A)) \subseteq \mathbb{B}^\pi(\mathcal{G} + D) \cup \mathbb{B}_+(A) \cup C.$$

Proof. This follows immediately from Lemma 4.9. $\square$

The next invariant plays an important role in Section 6.

DEFINITION 5.12. Let Y be a quasi-projective variety, let H be a big Cartier divisor on Y , and let S be a non-empty subset of $\text{sp}(Y)$. Fix a non-negative rational number a . With notation as in Definition 5.1, suppose that (S, H) satisfies condition $(*)_a$. Let $\mathcal{F}$ be a coherent sheaf on Y . We define

$$\begin{aligned} T(\mathcal{F}) &:= T_S(g, \pi, \mathcal{F}) := \{r \in \mathbb{Z}[q^{-1}] \mid \mathbb{B}^\pi(g_*\mathcal{F} - rL|_U) \cap g(S) = \emptyset\} \quad \text{and} \\ t(\mathcal{F}) &:= t_S(g, \pi, \mathcal{F}) := \sup T_S(g, \pi, \mathcal{F}). \end{aligned}$$

We note that $g: Y \rightarrow U$ is a projective morphism to a dense open subset U of Z .

Remark 5.13. (1) If $r \ll 0$, then $\mathbb{B}^\pi(g_*\mathcal{F} - rL|_U) \subseteq \text{Bs}(g_*\mathcal{F}(-rL|_U)) \subseteq Z \setminus g(S)$ by Lemma 4.8, so $T(\mathcal{F}) \neq \emptyset$. Hence, we have $t(\mathcal{F}) \in \mathbb{R} \cup \{+\infty\}$.

(2) One can easily check that $T(\mathcal{F}) \setminus \{t(\mathcal{F})\} = \mathbb{Z}[q^{-1}] \cap (-\infty, t(\mathcal{F}))$.

PROPOSITION 5.14. *Let the notation be as in Definition 5.12. Let r be a rational number.*

- (1) *If $r \leq t(\mathcal{F})$, then $\mathbb{B}_-(\mathcal{F} - rH) \cap S = \emptyset$.*
- (2) *If $r < t(\mathcal{F})$, then $\mathbb{B}_+(\mathcal{F} - rH) \cap S = \emptyset$.*

Note that we have $\mathbb{B}_-^{g^*B}(\cdot + \cdot') \cap S = \mathbb{B}_-(\cdot + \cdot') \cap S$ and $\mathbb{B}_+^{g^*B}(\cdot + \cdot') \cap S = \mathbb{B}_+(\cdot + \cdot') \cap S$ for some ample Cartier divisor B on Z as explained in Remark 5.3.

Proof. We use the same notation as in Definition 5.1. We first prove assertion (1). Since $g|_{g^{-1}(V)} : g^{-1}(V) \rightarrow V$ is finite for some open subset $V \subseteq Z$ by Definition 5.1, we see from Lemma 4.6(1),(3) that it is enough to show that $\mathbb{B}_-(g_*\mathcal{F} - rL|_U) \cap g(S) = \emptyset$. Fix $\alpha \in \mathbb{Q}_{>0}$ and an ample Cartier divisor B on U . We show that

$$\mathbb{B}(g_*\mathcal{F} + (-rL|_U + \alpha B)) \cap g(S) = \emptyset.$$

Take $\beta \in \mathbb{Q}_{>0}$ so that $\alpha B - \beta L|_U$ is ample and $r - \beta \in \mathbb{Z}[q^{-1}]$. Note that $r - \beta \in T(\mathcal{F})$. Applying Proposition 5.11 with $A := \alpha B - \beta L|_U$, we obtain

$$\begin{aligned} \mathbb{B}(g_*\mathcal{F} + (-rL|_U + \alpha B)) &= \mathbb{B}(g_*\mathcal{F} + ((-r + \beta)L|_U + \underbrace{\alpha B - \beta L|_U}_{=A})) \\ &\subseteq \mathbb{B}^\pi(g_*\mathcal{F} + (-r + \beta)L|_U) \subseteq Z \setminus g(S). \end{aligned}$$

Note that C in Proposition 5.11 is empty since π is flat by definition.

Next, we show assertion (2). By an argument similar to the above, we only need to show that $\mathbb{B}_+(g_*\mathcal{F} - rL|_U) \cap g(S) = \emptyset$. Fix an $\alpha \in \mathbb{Q}_{>0}$ such that $r + \alpha \in T(\mathcal{F})$. Take $\beta \in \mathbb{Q}_{>0}$ so that $\mathbb{B}_+(\alpha L|_U - \beta B) = \mathbb{B}_+(L|_U)$. Using Proposition 5.11 with $A = \alpha L|_U - \beta B$, we get

$$\begin{aligned} \mathbb{B}_+(g_*\mathcal{F} - rL|_U) &\subseteq \mathbb{B}(g_*\mathcal{F} - (rL|_U + \beta B)) \\ &= \mathbb{B}(g_*\mathcal{F} - ((r + \alpha)L|_U - \underbrace{\alpha L|_U + \beta B}_{=-A})) \\ &\subseteq \mathbb{B}^\pi(g_*\mathcal{F} - (r + \alpha)L|_U) \cup \mathbb{B}_+(L|_U). \end{aligned}$$

Since $\mathbb{B}_+(L|_U) \subseteq \mathbb{B}_+(L)$, our claim follows from Definitions 5.1 and 5.12. $\square$

PROPOSITION 5.15. *Let Y be a dense open subset of an n -dimensional projective variety W , and take $S \subseteq \text{sp}(Y)$. Let $\mathcal{E}, \mathcal{F}$ and $\mathcal{G}$ be coherent sheaves on Y . Let Λ be an infinite set of positive integers. Fix $\delta, M \in \mathbb{R}_{\geq 0}$. Suppose that for every $e \in \Lambda$, there exist a positive integer h_e with $|\delta p^e - h_e| \leq M$ and a morphism*

$$\psi^{(e)} : F_{Y*}^e(\mathcal{E} \otimes S^{h_e}(\mathcal{F})) \rightarrow \mathcal{G}$$

that is surjective over S . Let H be a big Cartier divisor on Y , let A' be a big Cartier divisor on W with $|A'|$ free, and set $A := A'|_Y$. Take two rational numbers r, r' so that $\delta r + r' > 0$. Then

$$\text{Bs}(\mathcal{G}(H)) \cap S \subseteq \mathbb{B}_-(\mathcal{F} - rA) \cup \mathbb{B}_-(H - (n + r')A) \cup \mathbb{B}_+(A).$$

In particular, $\text{Bs}(\mathcal{G}(H)) \cap S \subseteq \mathbb{B}(\mathcal{F} - rA) \cup \mathbb{B}(H - (n + r')A) \cup \mathbb{B}_+(A)$.

Proof. We first show the second assertion. The morphism $\psi^{(e)}$ induces

$$\mathcal{D}_e := F_{Y*}^e(\mathcal{E} \otimes S^{h_e}(\mathcal{F}) \otimes \mathcal{O}_Y(p^e H)) \xrightarrow{\psi^{(e)} \otimes \mathcal{O}_Y(H)} \mathcal{G}(H),$$

which is surjective over S , so $\text{Bs}(\mathcal{G}(H)) \cap S \subseteq \text{Bs}(\mathcal{D}_e)$. Take $s \in \mathbb{Q}$ so that $-r' < s < \delta r$. Then for each $0 \ll e \in \Lambda$, we have

$$\begin{aligned} h_e r + (n + r')p^e &\geq \delta p^e r - |rM| + (n + r')p^e \\ &= (n + r' + s)p^e + (\delta r - s)p^e - |rM| \\ &> (n + r' + s)p^e \quad \text{since } e \gg 0. \end{aligned}$$

Applying Proposition 4.10 for

$$(\mathcal{F}_1, \varepsilon_1, l_1; \mathcal{F}_2, \varepsilon_2, l_2; \varepsilon) := (\mathcal{F}, r, h_e; \mathcal{O}_Y(H), n + r', p^e; r' + s),$$

we obtain

$$\mathrm{Bs}(\mathcal{G}(H)) \cap S \subseteq \mathrm{Bs}(\mathcal{D}_e) \stackrel{\text{Prop. 4.10}}{\subseteq} \mathbb{B}(\mathcal{F} - rA) \cup \mathbb{B}(H - (n + r')A) \cup \mathbb{B}_+(A).$$

Let us prove the first claim. Since the condition $\delta r + r' > 0$ is open with respect to r and r' , it follows from the above argument that

$$\mathrm{Bs}(\mathcal{G}(H)) \cap S \subseteq \mathbb{B}_-^A(\mathcal{F} - rA) \cup \mathbb{B}_-^A(H - (n + r')A) \cup \mathbb{B}_+(A),$$

so the first claim follows from Corollary 4.2(2). $\square$

The following two lemmas are used in the proof of Proposition 5.18.

LEMMA 5.16. *Let $f: X \rightarrow Y$ be a separable finite flat morphism between smooth varieties. Let $V \subseteq Y$ be the largest open subset over which f is étale. Let R denote the ramification divisor of f . Fix $e \in \mathbb{Z}_{>0}$. Then there exists a morphism*

$$F_{X/Y}^{(e)} \mathcal{O}_{X^e}((1 - p^e)R) \rightarrow \mathcal{O}_{X_{Y^e}}$$

of coherent sheaves on X_{Y^e} that is an isomorphism over $f_{Y^e}^{-1}(V^e)$.

Proof. Set $U := f^{-1}(V)$. We have the following commutative diagram:

$$\begin{array}{ccccc} U^e & \hookrightarrow & X^e & & \\ F_{U/V}^{(e)} \downarrow & & F_{X/Y}^{(e)} \downarrow & \searrow F_X^e & \\ U_{V^e} & \hookrightarrow & X_{Y^e} & \xrightarrow{w^{(e)}} & X \\ f_{V^e} \downarrow & & f_{Y^e} \downarrow & & \downarrow f \\ V^e & \hookrightarrow & Y^e & \xrightarrow{F_Y^e} & Y. \end{array}$$

Here, each square in the diagram is cartesian. Since $R \sim K_{X/Y}$, applying the argument after Definition 3.4 to the morphism f , we obtain the morphism

$$\phi_{(X/Y,0)}^{(e)}(0): F_{X/Y}^{(e)} \mathcal{O}_{X^e}((1 - p^e)R) \rightarrow \mathcal{O}_{X_{Y^e}}.$$

We see from the choice of V that $F_{U/V}^{(e)}$ is an isomorphism (see, for example, [Hou77, §1.2, Proposition 2(c)(2)]), so it follows from the definition that $\phi_{(X/Y,0)}^{(e)}(0)$ is an isomorphism over $U_{V^e} = f_{V^e}^{-1}(V^e)$. $\square$

LEMMA 5.17. *Let $\pi: X \rightarrow Y$ be a separable finite flat morphism between smooth varieties. Let R denote the ramification divisor of π , and let $V \subseteq Y$ be the largest open subset over which π is étale. Let $\mathcal{G}$ be a coherent sheaf on Y , and let $\mathcal{M}$ be a line bundle on X . Then for each $e \in \mathbb{Z}_{>0}$, there exists a morphism*

$$F_{X*}^e(\mathcal{M}^{p^e} \otimes \mathcal{O}_{X^e}((1 - p^e)R) \otimes \pi^{(e)*} \mathcal{G}) \rightarrow \mathcal{M} \otimes \pi^* F_{Y*}^e \mathcal{G}$$

that is an isomorphism over $\pi^{-1}(V)$.

Proof. Fix $e \in \mathbb{Z}_{>0}$. We have the following commutative diagram:

$$\begin{array}{ccccc}
 & & X^e & & \\
 & \searrow^{F_{X/Y}^{(e)}} & \downarrow & \searrow^{F_X^e} & \\
 \pi^{(e)} & & X_{Y^e} & \xrightarrow{w^{(e)}} & X \\
 & \searrow^{\pi_{Y^e}} & \downarrow & & \downarrow \pi \\
 & & Y^e & \xrightarrow{F_Y^e} & Y.
 \end{array}$$

Here, the square in the diagram is cartesian. We get

$$\begin{aligned}
 & F_{X*}^e(\mathcal{M}^{p^e} \otimes \mathcal{O}_{X^e}((1-p^e)R) \otimes \pi^{(e)*}\mathcal{G}) \\
 & \cong \mathcal{M} \otimes w^{(e)*} F_{X/Y*}^{(e)}(\mathcal{O}_{X^e}((1-p^e)R) \otimes F_{X/Y}^{(e)*} \pi_{Y^e*}\mathcal{G}) && \text{projection formula} \\
 & \cong \mathcal{M} \otimes w^{(e)*}((F_{X/Y}^{(e)} \mathcal{O}_{X^e}((1-p^e)R)) \otimes \pi_{Y^e*}\mathcal{G}) && \text{projection formula} \\
 & \rightarrow \mathcal{M} \otimes w^{(e)*} \pi_{Y^e*}\mathcal{G} && \text{Lemma 5.16} \\
 & \cong \mathcal{M} \otimes \pi^* F_{Y*}^e \mathcal{G}. && \pi \text{ is flat}
 \end{aligned}$$

Note that the projection formula holds for coherent sheaves if the morphism is finite. Set $U := \pi^{-1}(V)$. Then $U = w^{(e)}(\pi_{V^e}^{-1}(V^e))$. Hence, the assertion follows from Lemma 5.16. $\square$

The next proposition plays a key role in the proofs of our main theorems. The situation is similar to that of Proposition 5.15.

PROPOSITION 5.18. *Let Y be a dense open subset of an n -dimensional projective variety W , let H be a big Cartier divisor on Y , and take $S \subseteq \text{sp}(Y)$ so that (S, H) satisfies condition $(*)_a$ for some rational number $a \geq 0$. Let $\mathcal{E}, \mathcal{F}$ and $\mathcal{G}$ be coherent sheaves on Y . Let Λ be an infinite set of positive integers. Fix $\delta, M \in \mathbb{R}_{\geq 0}$. Suppose that for every $e \in \Lambda$, there exist a positive integer h_e with $|\delta p^e - h_e| \leq M$ and a morphism*

$$\psi^{(e)}: F_{Y*}^e(\mathcal{E} \otimes S^{h_e}(\mathcal{F})) \rightarrow \mathcal{G}$$

that is surjective over S . Then

- (1) $\delta \cdot t(\mathcal{F}) \leq t(\mathcal{G}) + a$, and
- (2) if $H = H'|_Y$ for some big Cartier divisor H' on W with $|H'|$ free, then $\text{Bs}(\mathcal{G}(lH)) \cap S = \emptyset$ for each $l \in \mathbb{Z}$ with $\delta \cdot t(\mathcal{F}) + l > n$.

Proof. We first prove assertion (2). Set $r' := l - n$. Take $r \in \mathbb{Q}$ so that $r < t(\mathcal{F})$ and $\delta r + r' > 0$. Then

$$\text{Bs}(\mathcal{G}(lH)) \cap S \stackrel{\text{Prop. 5.15}}{\subseteq} \mathbb{B}_-(\mathcal{F} - rH) \cup \underbrace{\mathbb{B}_-(lH - (n + r')H)}_{=\mathbb{B}_-(0)=\emptyset} \cup \mathbb{B}_+(H).$$

Since $\mathbb{B}_-(\mathcal{F} - rH) \cap S = \emptyset$ by Proposition 5.14 and $\mathbb{B}_+(H) \cap S = \emptyset$ by Definition 5.1, the assertion follows. Next, we prove assertion (1). Let the notation be as in Definition 5.1. Replacing $\mathcal{E}, \mathcal{F}$ and $\mathcal{G}$ with $g_*\mathcal{E}, g_*\mathcal{F}$ and $g_*\mathcal{G}$, respectively, we may assume $Y = U$ and $g = \text{id}_U$. Note that $\psi^{(e)}$ can be replaced by the composite

$$F_{Z*}^e(S^{h_e}(g_*\mathcal{F}) \otimes g_*\mathcal{E}) \rightarrow F_{Z*}^e g_*(S^{h_e}(\mathcal{F}) \otimes \mathcal{E}) \xrightarrow{g_*\psi^{(e)}} g_*\mathcal{G},$$

where each morphism is surjective over $g(S)$ because $g|_{g^{-1}(V)}: g^{-1}(V) \rightarrow V$ is finite for some open subset V of Z containing $g(S)$. By Definition 5.1, we have

$$\mathbb{B}_-(K_Z + aL) \cap S = \emptyset. \quad (5.18.1)$$

Let R_d denote the ramification divisor of π^d for each $d > 0$. Fix $a' \in \mathbb{Z}[q^{-1}]$ with $a' > a$. Take $a'' \in \mathbb{Z}[q^{-1}] \cap (a, a')$ and $d_0 \in \mathbb{Z}_{>0}$ so that $a'q^{d_0}, a''q^{d_0} \in \mathbb{Z}$. Set $S_d := (\pi^d)^{-1}(S)$. We show the following claim.

CLAIM 1. *There is a $d_1 \geq d_0$ such that for each $d \geq d_1$, we have $\mathbb{B}(-R_d + a'q^dL) \cap S_d = \emptyset$.*

For each $d \geq d_0$, we see that

$$\begin{aligned} -R_d + a'q^dL &\sim -(K_Z - (\pi^d)^*K_Z) + (a' - a'')q^dL + a''(\pi^d)^*L \\ &\sim -K_Z + q^dL' + (\pi^d)^*(K_Z + a''L), \quad \text{where } L' := (a' - a'')L. \end{aligned}$$

Hence,

$$\begin{aligned} \mathbb{B}(-R_d + a'q^dL) &\subseteq B_1 \cup B_2, \quad \text{where} \\ B_1 &:= \mathbb{B}(-K_Z + q^dL') \quad \text{and} \quad B_2 := \mathbb{B}((\pi^d)^*(K_Z + a''L)). \end{aligned}$$

We show that $(B_1 \cup B_2) \cap S_d = \emptyset$ for every $d \gg 0$. By Lemma 4.8, there is a $d_1 \geq d_0$ such that for each $d \geq d_1$, we have

$$B_1 \subseteq \text{Bs}(-K_Z + q^dL') \stackrel{\text{Lem. 4.8}}{\subseteq} \mathbb{B}_+(L') = \mathbb{B}_+(\pi^{d*}L) \stackrel{\text{Lem. 4.6}}{\subseteq} (\pi^d)^{-1}(\mathbb{B}_+(L)),$$

so

$$(B_1 \cup B_2) \cap S_d \subseteq (\pi^d)^{-1}((\mathbb{B}_+(L) \cup \mathbb{B}(K_Z + a''L)) \cap S),$$

and

$$\mathbb{B}(K_Z + a''L) \stackrel{a'' > a}{\subseteq} \mathbb{B}_-(K_Z + aL) \stackrel{\text{Lem. 4.1}}{\subseteq} \mathbb{B}_-(K_Z + aL) \cup \mathbb{B}_+(L) \stackrel{(5.18.1)}{\subseteq} Z \setminus S,$$

which proves Claim 1.

To state a second claim, we fix the following data:

- (i) an ample Cartier divisor A on Z with $|A|$ free,
- (ii) $0 < \varepsilon_1 \in \mathbb{Q}$ with $\mathbb{B}(L - \varepsilon_1 A) = \mathbb{B}_+(L)$,
- (iii) $0 < \nu \in \mathbb{Z}$ such that $\dim Z + 1 \leq \nu \varepsilon_1 \in \mathbb{Z}$,
- (iv) $0 \neq r \in \mathbb{Z}[q^{-1}]$ with $-r \in T(\mathcal{F})$,
- (v) $\delta' \in \mathbb{Z}[q^{-1}]$ such that $\delta'r > \delta r$,
- (vi) $d \gg d_1$ such that $a'q^d, \delta'q^d, \delta'q^dr$ and q^dr are integers and

$$\text{Bs}((\rho_d^*\mathcal{F}) \otimes \mathcal{O}_V(q^drL|_V)) \cap S_d = \emptyset,$$

where $V := (\pi^d)^{-1}(U)$ and $\rho_d := (\pi^d)|_V: V \rightarrow U$,

- (vii) $\mu := (a' + \delta'r)q^d + \nu \in \mathbb{Z}$.

CLAIM 2. $\text{Bs}((\rho_d^*\mathcal{G}) \otimes \mathcal{O}_V(\mu L|_V)) \cap S_d = \emptyset$.

If this holds, then $\mathbb{B}^\pi(\mathcal{G} + q^{-d}\mu L|_U) \cap S = \emptyset$, so $-q^{-d}\mu \in T(\mathcal{G})$, and hence

$$t(\mathcal{G}) \geq -q^{-d}\mu = -a' - \delta'r - q^{-d}\nu \xrightarrow{d \rightarrow \infty, \delta' \rightarrow \delta, a' \rightarrow a} -a - \delta r,$$

which implies $t(\mathcal{G}) \geq -a + \delta t(\mathcal{F})$. To prove Claim 2, we use Proposition 4.10. Set

$$(\mathcal{F}_1, \varepsilon_1; \mathcal{F}_2, \varepsilon_2; \mathcal{F}_3, \varepsilon_3) := (\mathcal{O}_V(L|_V), \varepsilon_1; \mathcal{O}_V(-R + a'q^dL|_V), 0; (\rho_d^*\mathcal{F}) \otimes \mathcal{O}_V(q^drL|_V), 0),$$

where ε_1 is that in item (ii) and $R := R_d|_V$. Then B in Proposition 4.10 is equal to

$$\underbrace{\mathbb{B}(L|_V - \varepsilon_1 A|_V)}_{= \mathbb{B}_+(L) \text{ by (ii)}} \cup \underbrace{\mathbb{B}(\mathcal{O}_V(-R + a'q^dL|_V))}_{\subseteq Z \setminus S_d \text{ by Claim 1}} \cup \underbrace{\mathbb{B}((\rho_d^*\mathcal{F}) \otimes \mathcal{O}_V(q^drL|_V))}_{\subseteq Z \setminus S_d \text{ by (vi)}},$$

so $B \cap S_d = \mathbb{B}_+(L) \cap S_d = \emptyset$ by Definition 5.1. Take $0 \ll e \in \Lambda$. Set

$$(l_1; l_2; l_3) := (\mu p^e - a'(p^e - 1)q^d - h_e q^d r; p^e - 1; h_e).$$

Then

$$\begin{aligned} l_1 &\stackrel{\text{(vii)}}{=} ((a' + \delta' r)q^d + \nu)p^e - a'(p^e - 1)q^d - h_e q^d r \\ &= a'q^d + (\delta' p^e - h_e)q^d r + \nu p^e > (\delta' p^e - h_e)q^d r + \nu p^e. \end{aligned}$$

We have $(\delta' p^e - h_e)q^d r > 0$ since

$$\delta' p^e r = \delta p^e r + (\delta' r - \delta r)p^e \geq h_e r - |r|M + \underbrace{(\delta' r - \delta r)p^e}_{>0} \underset{e \gg 0}{>} h_e r.$$

Hence, we get $l_1 > \nu p^e$ and

$$\sum_{1 \leq i \leq 3} \varepsilon_i l_i = \varepsilon_1 l_1 > \varepsilon_1 \nu p^e \geq (\dim Z + 1)p^e.$$

Proposition 4.10 shows that

$$\text{Bs}\left(F_{V*}^e\left((\rho_d^*\mathcal{E}) \otimes \bigotimes_{1 \leq i \leq 3} S^{l_i}(\mathcal{F}_i)\right)\right) \subseteq B \subseteq Z \setminus S_d.$$

The sheaf $F_{V*}^e((\rho_d^*\mathcal{E}) \otimes \bigotimes_{1 \leq i \leq 3} S^{l_i}(\mathcal{F}_i))$ is isomorphic to

$$F_{V*}^e(\mathcal{O}_V(\mu p^e L|_V) \otimes (\rho_d^*\mathcal{E})((1 - p^e)R) \otimes S^{h_e}(\rho_d^*\mathcal{F})) =: \mathcal{D}.$$

Here, we used the equality $l_1 + l_2(a'q^d) + l_3(q^d r) = \mu p^e$, which follows from the choice of $(l_1; l_2; l_3)$. Therefore, we obtain $\text{Bs}(\mathcal{D}) \cap S_d = \emptyset$. Now, we have the following sequence of morphisms:

$$\begin{aligned} \mathcal{D} &\cong F_{V*}^e(\mathcal{O}_V(\mu p^e L|_V + (1 - p^e)R) \otimes \rho_d^*(\mathcal{E} \otimes S^{h_e}(\mathcal{F}))) && \text{definition of } \mathcal{D} \\ &\xrightarrow{u} \mathcal{O}_V(\mu L|_V) \otimes \rho_d^* F_{V*}^e(\mathcal{E} \otimes S^{h_e}(\mathcal{F})) && \text{Lemma 5.17} \\ &\xrightarrow{v} \mathcal{O}_V(\mu L|_V) \otimes \rho_d^* \mathcal{G}. && \text{induced by } \psi^{(e)} \end{aligned}$$

Then, u is surjective over S_d since ρ_d is étale over S . Also, v is surjective over S_d because of the assumption on $\psi^{(d)}$, so we get $\text{Bs}(\mathcal{O}_V(\mu L|_V) \otimes \rho_d^* \mathcal{G}) \cap S_d = \emptyset$, which is our claim. $\square$

6. Positivity of direct images

In this section, using the invariant studied in Section 5, we discuss the positivity of direct images of (relative) pluricanonical bundles.

6.1 Direct images of pluricanonical bundles

In this subsection, we prove the main theorems of this paper. We work over an F -finite field k of characteristic $p > 0$. To begin with, we define the following notation.

DEFINITION 6.1. Let Y be a variety. Fix $S \subseteq \text{sp}(Y)$. Let $\mathcal{R} = \bigoplus_{d \geq 0} \mathcal{R}_d$ be a graded $\mathcal{O}_Y$ -algebra such that each $\mathcal{R}_d$ is a coherent sheaf on Y . In this paper, we say that $\mathcal{R}$ is *finitely generated over S* if there exists a positive integer N such that for each $d \geq 0$, the natural morphism

$$\bigoplus_{\substack{i_1, \dots, i_N \geq 0 \\ \sum_{j=1}^N i_j = d}} \left(\bigotimes_{j=1}^N S^{i_j}(\mathcal{R}_j) \right) \rightarrow \mathcal{R}_d$$

is surjective over S . If we can take $N = 1$, then we say that $\mathcal{R}$ is *generated by $\mathcal{R}_1$ over S* . Let $\mathcal{M} = \bigoplus_{d \geq 0} \mathcal{M}_d$ be a graded $\mathcal{R}$ -module such that each $\mathcal{M}_d$ is a coherent sheaf on Y . In this paper, we say that $\mathcal{M}$ is *finitely generated over S as $\mathcal{R}$ -module* if there exists a positive integer N such that for each $d \geq N$, the natural morphism

$$\bigoplus_{i=0}^N \mathcal{R}_{d-i} \otimes \mathcal{M}_i \rightarrow \mathcal{M}_d$$

is surjective over S .

When $S = \{\eta\}$, where η is the generic point of Y , the $\mathcal{O}_Y$ -algebra $\mathcal{R}$ is finitely generated over S if and only if $\mathcal{R}_\eta = \bigoplus_{d \geq 0} (\mathcal{R}_d)_\eta$ is a finitely generated $k(\eta)$ -algebra.

We first prove our main result in a general setting.

THEOREM 6.2. Let X be a quasi-projective equi-dimensional k -scheme satisfying S_2 and G_1 , and let Δ be an effective $\mathbb{Q}$ -AC divisor on X such that $i\Delta$ is integral for some $i > 0$ not divisible by p . Let Y be a dense open subset of a projective variety W , let H be a big Cartier divisor on Y , and fix subsets $S \subseteq S' \subseteq \text{sp}(Y)$ such that (S, H) satisfies condition $(*)_a$ for some rational number $a \geq 0$. Let $f: X \rightarrow Y$ be a surjective projective morphism. Let M and N be AC divisors on X such that $\delta N \sim_{\mathbb{Q}} M - (K_X + \Delta) =: M'$ for a rational number $\delta \geq 0$. Let $\mathcal{R}(N)$ and $\mathcal{R}(M')$ denote the $\mathcal{O}_Y$ -algebras $\bigoplus_{l \geq 0} f_* \mathcal{O}_X(lN)$ and $\bigoplus_{l \geq 0} f_* \mathcal{O}_X(\lfloor lM' \rfloor)$, respectively. Suppose that the following conditions hold:

- (i) The $\mathcal{O}_Y$ -algebra $\mathcal{R}(N)$ is generated by $f_* \mathcal{O}_X(N)$ over S' .
- (ii) The $\mathcal{R}(M')$ -module $\bigoplus_{l \geq 0} f_* \mathcal{O}_X(\lfloor lM' + K_X + \Delta \rfloor)$ is finitely generated over S' .
- (iii) The inclusion $S^0 f_*(\sigma(X, \Delta) \otimes \mathcal{O}_X(M)) \hookrightarrow f_* \mathcal{O}_X(M)$ induces an isomorphism of stalks at every point in S' .

Then

- (1) $\delta \cdot t(f_* \mathcal{O}_X(N)) \leq t(f_* \mathcal{O}_X(M)) + a$, and
- (2) if $H = H'|_Y$ for a big Cartier divisor H' on W with $|H'|$ free, then for each $l \in \mathbb{Z}$ with $\delta \cdot t(f_* \mathcal{O}_X(N)) + l > \dim Y$, the sheaf

$$f_* \mathcal{O}_X(M) \otimes \mathcal{O}_Y(lH)$$

is generated by its global sections at every point in S .

Proof. Take $e \in \mathbb{Z}_{>0}$ so that $(p^e - 1)\Delta$ is integral. We use the morphism defined after Remark 3.2, that is, the morphism

$$\phi_{(X, \Delta)}^{(e)}(M): F_{X*}^e \mathcal{O}_X(p^e M + (1 - p^e)(K_X + \Delta)) \rightarrow \mathcal{O}_X(M).$$

By assumption (iii), the push-forward

$$F_{Y*}^e f_* \mathcal{O}_X(p^e M + (1 - p^e)(K_X + \Delta)) \xrightarrow{f_* \phi_{(X, \Delta)}^{(e)}(M)} f_* \mathcal{O}_X(M) \quad (6.2.1)$$

is surjective over S for each $e \in \mathbb{Z}_{>0}$. Take $m \in \mathbb{Z}_{>0}$ so that mM' is integral, $\delta m \in \mathbb{Z}$ and $\delta m N \sim_{\mathbb{Z}} mM'$. Put $\mathcal{F} := f_* \mathcal{O}_X(N)$. Then for each $l \in \mathbb{Z}_{>0}$, the natural morphism

$$S^{\delta lm}(\mathcal{F}) = S^{\delta lm}(f_* \mathcal{O}_X(N)) \rightarrow f_* \mathcal{O}_X(\delta lm N) \cong f_* \mathcal{O}_X(lm M') \quad (6.2.2)$$

is surjective over S by assumption (i). Here, we put $S^{\delta lm}(\mathcal{F}) := f_* \mathcal{O}_X$ when $\delta = 0$. Replacing m if necessary, we see from assumption (ii) that there is an $n_0 \in \mathbb{Z}_{>0}$ such that the natural morphism

$$f_* \mathcal{O}_X(lm M') \otimes f_* \mathcal{O}_X(\lfloor nM' + (K_X + \Delta) \rfloor) \rightarrow f_* \mathcal{O}_X(\lfloor (lm + n)M' + (K_X + \Delta) \rfloor) \quad (6.2.3)$$

is surjective over S for each $l \geq 0$ and $n \geq n_0$. Hence, for each $l \geq 0$ and $n \geq n_0$, we get the morphism

$$S^{\delta lm}(\mathcal{F}) \otimes f_* \mathcal{O}_X(\lfloor nM' + K_X + \Delta \rfloor) \rightarrow f_* \mathcal{O}_X(\lfloor (lm + n)M' + K_X + \Delta \rfloor) \quad (6.2.4)$$

that is surjective over S . Let q_e and r_e be integers such that $p^e = mq_e + r_e$ and $n_0 \leq r_e < m + n_0$. Then

$$p^e M + (1 - p^e)(K_X + \Delta) = p^e M' + K_X + \Delta = (mq_e + r_e)M' + K_X + \Delta, \quad (6.2.5)$$

and so $r_e M' + K_X + \Delta$ is integral. Put $\mathcal{G} := \bigoplus_{n_0 \leq r < m + n_0} f_* \mathcal{O}_X(\lfloor rM' + K_X + \Delta \rfloor)$. We now have the following sequence of morphisms:

$$\begin{aligned} F_{Y*}^e(S^{\delta mq_e}(\mathcal{F}) \otimes \mathcal{G}) &\xrightarrow{\text{def of } \mathcal{G}} F_{Y*}^e(S^{\delta mq_e}(\mathcal{F}) \otimes f_* \mathcal{O}_X(r_e M' + K_X + \Delta)) \\ &\xrightarrow{(6.2.4)} F_{Y*}^e f_* \mathcal{O}_X((mq_e + r_e)M' + K_X + \Delta) \\ &\stackrel{(6.2.5)}{=} F_{Y*}^e f_* \mathcal{O}_X(p^e M + (1 - p^e)(K_X + \Delta)) \xrightarrow{(6.2.1)} f_* \mathcal{O}_X(M). \end{aligned}$$

The composite $\psi^{(e)}$ is also surjective over S since each morphism is. Set $h_e := \delta mq_e$. Then $|h_e - \delta p^e| = \delta |mq_e - p^e| = \delta r_e \leq \delta(m + n_1)$ for each $e \in \mathbb{Z}_{>0}$, so we can apply Proposition 5.18, which completes the proof. $\square$

The next theorem can be viewed as an analog of [PS14, Theorem 1.4].

THEOREM 6.3. *Let $X, \Delta, W, Y, S, S', H$ and f be as in Theorem 6.2. Suppose that*

- (i) *the $\mathcal{O}_Y$ -algebra $\bigoplus_{l \geq 0} f_* \mathcal{O}_X(\lfloor l(K_X + \Delta) \rfloor)$ is finitely generated over S' and*
- (ii) *there exists an integer $m_0 \geq 0$ such that the inclusion*

$$S^0 f_*(\sigma(X, \Delta) \otimes \mathcal{O}_X(m(K_X + \Delta))) \hookrightarrow f_* \mathcal{O}_X(m(K_X + \Delta))$$

induces an isomorphism of stalks at every point in S' and each $m \geq m_0$ such that $m\Delta$ is integral.

Take $m \geq m_0$ so that $m\Delta$ is integral, and set $\mathcal{F}_m := f_* \mathcal{O}_X(m(K_X + \Delta))$.

- (1) *Then $\mathbb{B}_-(\mathcal{F}_m + amH) \cap S = \emptyset$. In particular, if Y is normal, then for each integer $l \geq am$, the sheaf $\mathcal{F}_m \otimes \mathcal{O}_Y(lH)$ is pseudo-effective in the sense of Definition 4.13.*
- (2) *If $H = H'|_Y$ for a big Cartier divisor H' on W with $|H'|$ free, then for each integer $l > a(m - 1) + \dim Y$, the sheaf $\mathcal{F}_m \otimes \mathcal{O}_Y(lH)$ is generated by its global sections at every point in S .*

Remark 6.4. When $a = \dim Y + 1$ (as in Example 5.6), we have

$$l \geq am \Leftrightarrow l > a(m-1) + \dim Y \Leftrightarrow l \geq m(\dim Y + 1).$$

This condition on l is the same as that in [PS14, Theorem 1.4].

Proof. Let $i > 0$ be the minimum integer such that $i\Delta$ is integral. For simplicity, put $t_m := t(\mathcal{F}_m)$ for each $m \in \mathbb{Z}_{>0}$ with $i|m$. Let $\mu \geq m_0$ be a divisible enough integer. We first show that $-a \leq \mu^{-1}t_\mu$. Set $M := N := \mu(K_X + \Delta)$ and $\delta := \mu^{-1}(\mu - 1)$. Then one can check that all the assumptions in Theorem 6.2 hold, so the theorem shows that $\delta t_\mu \leq t_\mu + a$, which means that $-a \leq \mu^{-1}t_\mu$. Next, we take $m \geq m_0$ with $i|m$. Put $M := m(K_X + \Delta)$, $N := \mu(K_X + \Delta)$ and $\delta' := \mu^{-1}(m-1)$. Theorem 6.2(1) then says that $\delta' t_\mu \leq t_m + a$. Combining this with $-a \leq \mu^{-1}t_\mu$, we obtain that $-am \leq t_m$, so we see from Proposition 5.14 that $\mathbb{B}_-(\mathcal{F}_m + lH) \cap S = \emptyset$ for each $l \geq am$. Furthermore, since $a(m-1) + \dim Y \geq -\delta' t_\mu + \dim Y$, the second assertion follows from Theorem 6.2(2). $\square$

COROLLARY 6.5. *Let X, Δ, W, Y, S, H and f be as in Theorem 6.2. Let $Y_0 \subseteq Y$ be a dense open subset containing S , and put $X_0 := f^{-1}(Y_0)$. Suppose that*

- $K_{X_0} + \Delta|_{X_0}$ is a $\mathbb{Q}$ -Cartier divisor that is relatively ample over Y_0 and
- $(X_0, \Delta|_{X_0})$ is F -pure.

Let i be the smallest positive integer such that $i\Delta$ is integral and $i(K_{X_0} + \Delta|_{X_0})$ is Cartier. Set $\mathcal{F}_m := f_\mathcal{O}_X(m(K_X + \Delta))$ for each $m \geq 0$ with $i|m$. Then there exists an integer $m_0 \geq 0$ such that the following hold:*

- (1) *The sets $\mathbb{B}_-(\mathcal{F}_m + amH)$ and S do not intersect for $m \geq m_0$ with $i|m$. Furthermore, if Y is normal and $S \subseteq Y$ is open, then there exists an integer $m_1 \geq m_0$ such that $\mathcal{F}_m \otimes \mathcal{O}_Y(lH)$ is weakly positive over $S \cap Y_1$ for $m \geq m_1$ and $l \geq am$, where Y_1 is the maximal open subset of Y such that f is flat over Y_1 .*
- (2) *If $H = H'|_Y$ for a big Cartier divisor H' on W with $|H'|$ free, then for each integer $l > a(m-1) + \dim Y$, the sheaf $\mathcal{F}_m \otimes \mathcal{O}_Y(lH)$ is generated by its global sections at every point in S .*

Proof. Put $S' := Y_0$. Take a sufficiently large integer m_0 . We only need to check that assumptions (i) and (ii) in Theorem 6.3 hold. Since $K_{X_0} + \Delta|_{X_0}$ is ample over S' , we see that assumption (i) holds, and assumption (ii) follows from Lemma 3.6. Hence, we can apply Theorem 6.3. Note that Definition 4.12 requires the local freeness of $\mathcal{F}_m|_{Y_1}$, which is ensured by the choice of Y_1 . $\square$

6.2 Direct images of relative pluricanonical bundles

In this subsection, we deal with the positivity of the direct images of relative pluricanonical bundles. We fix an infinite F -finite field k of characteristic $p > 0$.

The following lemma is used to prove the main theorem (Theorem 6.7) of this subsection. The author learned the proof from the referee.

LEMMA 6.6. *Let X be an equi-dimensional quasi-projective k -scheme satisfying S_2 and G_1 , let Y be a regular quasi-projective variety, and let $f: X \rightarrow Y$ be a surjective projective morphism. Let e be a positive integer. Then X_{Y^e} is also an equi-dimensional quasi-projective k -scheme satisfying S_2 and G_1 .*

Proof. Since F_Y^e is a universal homeomorphism, the projection $w^{(e)}: X_{Y^e} \rightarrow X$ is homeomorphic, so X_{Y^e} is equi-dimensional. Also, since Y is regular, F_Y^e is a flat morphism with Gorenstein fibers, and hence so is the base change $w^{(e)}$ (see [WIT069, Corollary 2']), which means that X_{Y^e} satisfies S_2 and G_1 (see [RF72, Proposition 1(ii)]). $\square$

THEOREM 6.7. *Let X be an equi-dimensional quasi-projective k -scheme satisfying S_2 and G_1 , let Y be a dense normal open subset of a projective variety W of dimension n , and let η (respectively, $\bar{\eta}$) be the generic (respectively, geometric generic) point of Y . Let $f: X \rightarrow Y$ be a surjective projective morphism, and let Δ be an effective $\mathbb{Q}$ -AC divisor on X such that $i(K_X + \Delta)$ is Cartier for some $i > 0$ not divisible by p . Suppose that the following conditions hold:*

- (i) *The sum $\bigoplus_{m \geq 0} H^0(X_{\bar{\eta}}, \mathcal{O}_{X_{\bar{\eta}}}(\lfloor m(K_{X_{\bar{\eta}}} + \Delta|_{X_{\bar{\eta}}}) \rfloor))$ is a finitely generated $k(\bar{\eta})$ -algebra.*
- (ii) *There exists an integer $m_0 \geq 0$ such that*

$$S^0(X_{\bar{\eta}}, \Delta|_{X_{\bar{\eta}}}; \mathcal{O}_{X_{\bar{\eta}}}(m(K_{X_{\bar{\eta}}} + \Delta|_{X_{\bar{\eta}}})) = H^0(X_{\bar{\eta}}, \mathcal{O}_{X_{\bar{\eta}}}(m(K_{X_{\bar{\eta}}} + \Delta|_{X_{\bar{\eta}}}))$$

for each $m \geq m_0$ such that $m\Delta$ is integral.

Define $\mathcal{G}_m := f_* \mathcal{O}_X(m(K_X + \Delta)) \otimes \omega_Y^{[-m]}$ for an integer $m \geq m_0$ such that $m\Delta$ is integral.

- (1) *The sheaf $\mathcal{G}_m$ is pseudo-effective in the sense of Definition 4.13.*
- (2) *If Y is regular, then $\mathbb{B}_-(\mathcal{G}_m) \neq Y$.*
- (3) *Let A be a big Cartier divisor on W with $|A|$ free, and let H be a Cartier divisor on Y such that $H - nA|_Y$ is big. If Y is regular, then the sheaf*

$$\mathcal{G}_m \otimes \mathcal{O}_Y(K_Y + H)$$

is generated by its global sections at η .

Remark 6.8. Statement (1) in the theorem is a positive-characteristic analog of the weak positivity theorem due to Viehweg [Vie83a, Theorem III]. The first weak positivity theorem in positive characteristic is due to Patakfalvi [Pat14, Theorem 1.1]. In [Eji17], based on Patakfalvi's techniques, the author proved a higher-dimensional version of [Pat14, Theorem 1.1].

Proof of Theorem 6.7. We first consider statement (1). Let $\mathcal{G}'_m$ be the torsion-free part of $\mathcal{G}_m$, and let $Y_1 \subseteq Y$ be the maximal open subset on which $\mathcal{G}'_m$ is locally free. Let Y_{reg} be the regular locus of Y . Then clearly $\mathbb{B}_-(\mathcal{G}'_m|_{Y_1}) \cap Y_{\text{reg}} = \mathbb{B}_-(\mathcal{G}'_m|_{Y_1 \cap Y_{\text{reg}}})$, so statement (1) follows from statement (2). To prove statement (2), we use Theorem 6.3. Let H be a very ample Cartier divisor on Y , and put $S := S' := \{\eta\}$. As shown in Example 5.5, the pair (S, H) satisfies condition $(*)_a$ for some $a \in \mathbb{Z}_{>0}$. Take $e \in \mathbb{Z}_{>0}$. By Lemma 3.5, one can check that $f_{Y^e}: X_{Y^e} \rightarrow Y^e$ and Δ_{Y^e} satisfy conditions (i) and (ii) in Theorem 6.3. Note that X_{Y^e} is an equi-dimensional quasi-projective k -scheme satisfying S_2 and G_1 by Lemma 6.6. Set $L := (am - a + \dim Y + 1)H + mK_Y$. Since F_Y^e is flat, we have

$$\begin{aligned} (F_Y^e \mathcal{G}_m) \otimes \mathcal{O}_{Y^e}(L) &= (F_Y^e f_* \mathcal{O}_X(m(K_{X/Y} + \Delta))) \otimes \mathcal{O}_{Y^e}(L) \\ &\cong f_{Y^e*} \mathcal{O}_{X_{Y^e}}(m(K_{X_{Y^e}/Y^e} + \Delta_{Y^e})) \otimes \mathcal{O}_{Y^e}(L) \\ &\cong f_{Y^e*} \mathcal{O}_{X_{Y^e}}(m(K_{X_{Y^e}} + \Delta_{Y^e})) \otimes \mathcal{O}_{Y^e}((am - a + \dim Y + 1)H), \end{aligned}$$

so Theorem 6.3(2) tells us that $\eta \notin \text{Bs}((F_Y^e \mathcal{G}_m) \otimes \mathcal{O}_{Y^e}(L))$. This means that

$$\eta \notin \mathbb{B}^{F_Y}(\mathcal{G}_m + p^{-e}L).$$

Let M be an ample divisor on Y such that $L + M$ is ample. Thanks to Proposition 5.11, we get $\eta \notin \mathbb{B}(\mathcal{G}_m + p^{-e}(L + M))$, so

$$\mathbb{B}_-(\mathcal{G}_m) = \bigcup_{e>0} \mathbb{B}(\mathcal{G}_m + p^{-e}(L + M)) \not\ni \eta.$$

Let us show statement (3). Let $\mu \geq m_0$ be an integer divisible enough. By an argument similar to that in the proof of Theorem 6.2, we have an $n_0 \in \mathbb{Z}_{>0}$ such that for each $e \in \mathbb{Z}_{>0}$, there is the morphism

$$F_{Y*}^e(S^{q_e}(f_*\mathcal{O}_X(\mu(K_X + \Delta))) \otimes f_*\mathcal{O}_X(r_e(K_X + \Delta))) \rightarrow f_*\mathcal{O}_X(m(K_X + \Delta)) \quad (6.8.1)$$

that is surjective over S , where q_e and r_e are integers such that $(m-1)p^e + 1 = \mu q_e + r_e$ and $n_0 \leq r_e < n_0 + \mu$. (If $m = 1$, then we put $n_0 := 1$, $q_e = 0$ and $S^0(\cdot) := f_*\mathcal{O}_X(\cdot)$.) Note that $r_e(K_X + \Delta)$ is integral by definition. For each $l \in \mathbb{Z}_{>0}$, we denote by $\mathcal{G}_l$ the sheaf $f_*\mathcal{O}_X([l(K_X/Y + \Delta)])$. Taking the tensor product of (6.8.1) and ω_Y^{1-m} , we obtain

$$F_{Y*}^e(S^{q_e}(\mathcal{G}_\mu) \otimes \mathcal{G}_{r_e} \otimes \omega_Y) \rightarrow \mathcal{G}_m \otimes \omega_Y \quad (6.8.2)$$

by the projection formula. Putting $\mathcal{E} := \bigoplus_{n_0 \leq r < n_0 + \mu} \mathcal{G}_r(K_Y)$, we get the morphism

$$F_{Y*}^e(S^{q_e}(\mathcal{G}_\mu) \otimes \mathcal{E}) \rightarrow \mathcal{G}_m \otimes \omega_Y,$$

which is surjective over S . By the assumption, we find an $r' \in \mathbb{Q}_{>0}$ such that $H - (n + r')A|_Y$ is big. We use Proposition 5.15 with the following data:

$$(\mathcal{E}, \mathcal{F}, \mathcal{G}; h_e, \delta; r, r') := \left(\mathcal{E}, \mathcal{G}_\mu, \mathcal{G}_m \otimes \omega_Y; q_e, \frac{m-1}{\mu}; 0, r' \right)$$

Then $|\delta p^e - h_e| = \mu^{-1}|(m-1)p^e - \mu q_e| \leq \mu^{-1}(n_0 + \mu)$, so we get

$$\text{Bs}(\mathcal{G}_m(K_Y + H)) \cap S \subseteq \mathbb{B}_-(\mathcal{G}_\mu) \cup \underbrace{\mathbb{B}(H - (n + r')A|_Y)}_{\text{big}} \cup \underbrace{\mathbb{B}_+(A|_Y)}_{\text{big}}$$

by Proposition 5.15. Hence, assertion (2) implies that $\text{Bs}(\mathcal{G}_m(K_Y + H)) \cap S = \emptyset$. $\square$

Next, we prove the weak positivity of the direct images of relative pluricanonical bundles in the case where the geometric generic fiber is F -pure and has ample dualizing sheaf.

THEOREM 6.9. *Let X, Δ, W, Y and f be as in Theorem 6.7. Let $U \subseteq X$ be the largest Gorenstein open subset. Let $Y_0 \subseteq Y$ be the subset consisting of points $y \in Y$ with the following properties:*

- y is a regular point;
- f is flat at every point in $f^{-1}(y)$;
- X_y satisfies S_2 and G_1 ;
- $\text{Supp}(\Delta)$ does not contain any irreducible component of $f^{-1}(y)$;
- $(X_{\overline{y}}, \overline{\Delta|_{U_{\overline{y}}}})$ is F -pure, where $X_{\overline{y}}$ is the geometric fiber of f over y and $\overline{\Delta|_{U_{\overline{y}}}}$ is the $\mathbb{Q}$ -AC divisor on $X_{\overline{y}}$ that is the extension of $\Delta|_{U_{\overline{y}}}$ to $X_{\overline{y}}$;
- $K_X + \Delta$ is $\mathbb{Q}$ -Cartier in a neighborhood of every point in $f^{-1}(y)$;
- $K_{X_y} + \overline{\Delta|_{U_{\overline{y}}}}$ is ample.

Define $\mathcal{G}_m := f_*\mathcal{O}_X(m(K_X + \Delta)) \otimes \omega_Y^{[-m]}$ for each positive integer $m \geq m_0$ with $i|m$. Then there exists a positive integer m_0 such that the following properties hold:

- (0) The set Y_0 is an open subset of Y .

- (1) The sheaf $\mathcal{G}_m$ is weakly positive over Y_0 for each $m \geq m_0$ with $i|m$.
- (2) If Y is regular, then $\mathbb{B}_-(\mathcal{G}_m) \cap Y_0 = \emptyset$ for each $m \geq m_0$ with $i|m$.
- (3) Let A be an ample Cartier divisor on W with $|A|$ free, and let H be a Cartier divisor on Y such that $H - nA|_Y$ is ample. If Y is regular, then the sheaf

$$\mathcal{G}_m \otimes \mathcal{O}_Y(K_Y + H)$$

is globally generated over Y_0 for each $m \geq m_0$ with $i|m$.

Proof. For property (0), one can check that each condition on y is open on Y . Note that the openness of the F -purity of fibers follows from [PSZ18, Theorem 3.28]. We see that property (1) follows from property (2), applying the same argument as that in the proof of Theorem 6.7(1). Let us prove that properties (2) and (3) hold. By Lemma 3.7, there is an $m_0 \in \mathbb{Z}_{>0}$ such that the natural inclusion

$$S^0 f_{Y^e*}(\sigma(X_{Y^e}, \Delta_{Y^e}) \otimes \mathcal{O}_{X_{Y^e}}(m(K_X + \Delta)_{Y^e})) \hookrightarrow f_{Y^e*} \mathcal{O}_{X_{Y^e}}(m(K_X + \Delta)_{Y^e})$$

is an isomorphism over Y_0 for each $e \in \mathbb{Z}_{>0}$ and $m \geq m_0$ with $i|m$. Hence, replacing the generic point η with a point in Y_0 , we can apply the same argument as that in the proof of Theorem 6.7. $\square$

6.3 Conclusions

In this subsection, for the reader's convenience, we summarize the conclusions in Subsections 6.1 and 6.2 in the case where the log canonical divisor on the generic fiber is ample. We use the following notation.

NOTATION 6.10. Let k be an algebraically closed field of characteristic $p > 0$. Let X be a normal projective variety over k , and let Δ be an effective $\mathbb{Q}$ -Weil divisor on X such that $K_X + \Delta$ is $\mathbb{Q}$ -Cartier. Let i_W (respectively, i_C) be the smallest positive integer such that $i_W \Delta$ (respectively, $i_C(K_X + \Delta)$) is integral (respectively, Cartier). Note that $i_W|i_C$. Let $U \subseteq X$ be the largest Gorenstein open subset. Let $f: X \rightarrow Y$ be a surjective morphism to a projective n -dimensional variety Y over k .

Let $Y_1 \subseteq Y$ be the subset consisting of points y with the following properties:

- (X, Δ) is F -pure in a neighborhood of every point in $f^{-1}(y)$;
- $(K_X + \Delta)|_{X_y}$ is ample.

Let $Y_0 \subseteq Y$ be the subset consisting of points y with the following properties:

- Y is regular at y ;
- f is flat at every point in $f^{-1}(y)$;
- X_y satisfies S_2 and G_1 ;
- $\text{Supp}(\Delta)$ does not contain any irreducible component of $f^{-1}(y)$;
- $(X_{\overline{y}}, \overline{\Delta}|_{U_{\overline{y}}})$ is F -pure, where $\overline{\Delta}|_{U_{\overline{y}}}$ is the effective $\mathbb{Q}$ -AC divisor that is the extension of $\Delta|_{U_{\overline{y}}}$;
- $(K_X + \Delta)|_{X_y}$ is ample.

We note that

- Y_0 and Y_1 are open subsets of Y such that $Y_0 \subseteq Y_1 \subseteq Y$, and
- if f is not separable, then Y_0 is empty, but Y_1 may be not empty.

Let H be an ample Cartier divisor on Y with $|H|$ free.

THEOREM 6.11. We use Notation 6.10.

- (1) (Corollary 6.5) Suppose $p \nmid i_W$. Then there exists a positive integer m_1 such that for each $m \geq m_1$ with $i_C|m$, the sheaf

$$f_*\mathcal{O}_X(m(K_X + \Delta)) \otimes \mathcal{O}_Y(lH)$$

is globally generated over Y_1 for each $l \geq m(n+1)$.

- (2) (Theorem 6.7) Suppose $p \nmid i_C$, that Y is normal and that K_Y is $\mathbb{Q}$ -Cartier. Then there exists an integer $m_0 \geq m_1$ such that for each $m \geq m_0$ with $i_C|m$, the sheaf

$$f_*\mathcal{O}_X(m(K_{X/Y} + \Delta))$$

is weakly positive over Y_0 and

$$f_*\mathcal{O}_X(m(K_{X/Y} + \Delta)) \otimes \mathcal{O}_Y(K_Y + lH)$$

is globally generated over Y_0 for each $l \geq n+1$.

Proof. We only prove the first statement. The second is proved by the same argument. As shown in Example 5.6, we have an open covering $\{S_1, \dots, S_d\}$ of Y such that for each $i = 1, \dots, d$, the pair (S_i, H) satisfies condition $(*)_{n+1}$, and then so does the pair $(S_i \cap Y_1, H)$. Then by Corollary 6.5, we get an integer m_1 such that

$$\text{Bs}(f_*\mathcal{O}_X(m(K_X + \Delta)) \otimes \mathcal{O}_Y(lH)) \subseteq Y \setminus \bigcup_i (S_i \cap Y_1) = Y \setminus Y_1$$

for each $m \geq m_1$ with $i_C|m$, which completes the proof. $\square$

7. Iitaka's conjecture

Iitaka [Iit72] has proposed the following conjecture.

CONJECTURE 7.1. Let k be an algebraically closed field of characteristic zero, let $f: X \rightarrow Y$ be a surjective morphism between smooth projective varieties with connected fibers, and let F denote the geometric generic fiber of f . Then

$$\kappa(X) \geq \kappa(Y) + \kappa(F). \quad (\text{I})$$

This conjecture has proved in several cases including the following:

- X is a surface, by Ueno [Uen06];
- F is a curve, by Viehweg [Vie77];
- X is a threefold, by Viehweg [Vie80];
- Y is a curve, by Kawamata [Kaw82];
- Y is of general type, by Viehweg [Vie82];
- F has a good minimal model, by Kawamata [Kaw85];
- F is of general type, by Kollár [Kol87];
- Y is of maximal Albanese dimension, by Cao and Păun [CP17] and Hacon–Popa–Schnell [HPS18].

In [HPS18], Conjecture 7.1 is proved when Y is of maximal Albanese dimension, based on [CP17] that has shown the conjecture when Y is an abelian variety, but according to [HPS18], the work from [CP17] to [HPS18] is “very little extra work.”

In this section, we study inequality (I) in positive characteristic. To explain several known results, we assume that F is smooth. Then inequality (I) has proved in the following cases:

- X is a surface, by Chen and Zhang [CZ15];
- F is a curve, by Chen and Zhang [CZ15];
- Y is of general type and F has non-nilpotent Hasse–Witt matrix, by Patakfalvi [Pat18];
- F satisfies conditions (i) and (ii') in Theorem 1.7, and Y either is a curve or is of general type, by the author [Eji17];
- X is a threefold and $p \geq 7$, by the author and Zhang [EZ18] (the case where $k = \overline{\mathbb{F}}_p$ is due to [BCZ18]; see [Zha19] for the log version).

In this section, we deal with an algebraic fiber space whose general fibers may have “bad” singularities. More precisely, we study inequality (I) under some assumptions on the generic fiber, but we do not impose any condition on general fibers. Note that, in such a situation, counterexamples to inequality (I) have been found by the recent work of Cascini, Kollár, Zhang and the author [CEKZZ21]. The main theorem (Theorem 7.3) in this section gives sufficient conditions for inequality (I) to hold. To prove it, we need the following theorem.

THEOREM 7.2. *Let k be an F -finite field of characteristic $p > 0$. Let X be a quasi-projective normal variety, let Y be a regular quasi-projective variety, and let $f: X \rightarrow Y$ be a separable surjective projective morphism. Let Δ be an effective $\mathbb{Q}$ -Weil divisor on X , and let i_W be the smallest positive integer such that $i_W \Delta$ is integral. Assume that i_W is not divisible by p . Suppose that*

- (i) $\bigoplus_{l \geq 0} H^0(X_\eta, \mathcal{O}_{X_\eta}([l(K_{X_\eta} + \Delta|_{X_\eta})]))$ is a finitely generated $k(\eta)$ -algebra, where η is the generic point of Y and
- (ii) there exists a non-negative integer m_0 with $i_W | m_0$ such that

$$S^0(X_\eta, \Delta|_{X_\eta}; \mathcal{O}_{X_\eta}(m(K_{X_\eta} + \Delta|_{X_\eta}))) = H^0(X_\eta, \mathcal{O}_{X_\eta}(m(K_{X_\eta} + \Delta|_{X_\eta})))$$

for each $m \geq m_0$ with $i_W | m$.

Let H be a big and semi-ample Cartier divisor on Y such that $(\{\eta\}, H)$ satisfies condition $(*)_a$. Let X_0 be the largest open subset such that $K_{X_0} + \Delta|_{X_0}$ is $\mathbb{Q}$ -Cartier, and let $f_0: X_0 \rightarrow Y$ denote the induced morphism. Set

$$D := m_0(K_X + \Delta) - f^*K_Y + a(m_0 - 1)f^*H.$$

Then $\mathbb{B}_-^{f_0^*H}(D|_{X_0}) \neq X_0$ (see Section 4 for the definition of $\mathbb{B}_-^{f^*H}$). In particular, if $K_X + \Delta$ is $\mathbb{Q}$ -Cartier, then $\kappa(X, D + \varepsilon f^*H) \geq 0$ for every $0 < \varepsilon \in \mathbb{Q}$.

Note that we cannot remove the assumption that f is separable. (For example, the theorem does not hold for the Frobenius morphism of a smooth projective curve of genus at least 2.)

Proof. We first prove the assertion in the case where f is flat. For simplicity, we put $\mathcal{F}_m := f_*\mathcal{O}_X([m(K_X + \Delta)])$ for each $m \geq 0$. Fix integers μ and n_0 that are large and divisible enough. By condition (i), for $e \gg 0$, the natural morphisms

$$\mathcal{F}_{\mu q_e} \otimes \mathcal{E} \twoheadrightarrow \mathcal{F}_{\mu q_e} \otimes \mathcal{F}_{r_e} \rightarrow \mathcal{F}_{(m_0-1)p^e+1} \tag{7.2.1}$$

are generically surjective, where q_e and r_e are integers such that

$$(m_0 - 1)p^e + 1 = \mu q_e + r_e \quad \text{and} \quad n_0 \leq r_e < n_0 + \mu,$$

and $\mathcal{E} := \bigoplus_{n_0 \leq r < n_0 + \mu} \mathcal{F}_r$. Put $l_e := a(\mu(q_e + 1) + n_0 - 1) + 1$ for each $e \in \mathbb{Z}_{>0}$. Then

$$l_e = a(\mu q_e + r_e - 1 + \mu + n_0 - r_e) + 1 = a(m_0 - 1)p^e + \underbrace{a(\mu + n_0 - r_e)}_{\geq 0} + 1,$$

so $0 < \varepsilon_e := l_e p^{-e} - a(m_0 - 1) \xrightarrow{e \rightarrow \infty} 0$. Fix some $\nu \in \mathbb{Z}_{>0}$, and put $L := m_0(K_X + \Delta)$. Let us show that

$$\mathbb{B}(p^e(L|_{X_0} - f_0^* K_Y) + (l_e + \nu)f_0^* H) \neq X_0$$

for each $e \gg 0$. If this holds, then

$$\begin{aligned} \mathbb{B}_-^{f_0^* H}(D|_{X_0}) &= \mathbb{B}_-^{f_0^* H}(L|_{X_0} - f_0^* K_Y + a(m_0 - 1)f_0^* H) \\ &= \bigcup_{e \gg 0} \mathbb{B}\left(L|_{X_0} - f_0^* K_Y + \underbrace{\left(a(m_0 - 1) + \varepsilon_e + \frac{\nu}{p^e}\right)}_{= \frac{l_e}{p^e}} f_0^* H\right) \\ &= \bigcup_{e \gg 0} \mathbb{B}(p^e(L|_{X_0} - f_0^* K_Y) + (l_e + \nu)f_0^* H) \neq X_0, \end{aligned}$$

which prove the assertion.

Since f is flat, we may apply the argument from Section 3. Put $M := L_{Y^e} + (1 - p^e)f_{Y^e}^* K_Y$. By assumption (ii), we see that $f_* \phi_{(X, \Delta)}^{(e)}(L)$ is non-zero, so

$$\underbrace{f^{(e)}_* \mathcal{O}_{X^e}(p^e L + (1 - p^e)(K_X + \Delta))}_{= \mathcal{F}_{(m_0 - 1)p^e + 1}} \xrightarrow{f_{Y^e}^* \phi_{(X/Y, \Delta)}^{(e)}(M)} f_{Y^e}^* \mathcal{O}_{X_{Y^e}}(M) \quad (7.2.2)$$

is also non-zero, as explained in Section 3. Applying $f_{Y^e}^*(\cdot \otimes \mathcal{O}_{X_{Y^e}}(M))$ to the natural morphism $\mathcal{O}_{X_{Y^e}} \hookrightarrow F_{X/Y}^{(e)} \mathcal{O}_{X^e}$, we get

$$f_{Y^e}^* \mathcal{O}_{X_{Y^e}}(M) \hookrightarrow f^{(e)}_* \mathcal{O}_{X^e}(F_{X/Y}^{(e)*} M). \quad (7.2.3)$$

Combining morphisms (7.2.1), (7.2.2) and (7.2.3), we get the morphisms

$$\mathcal{F}_{\mu q_e} \otimes \mathcal{E} \rightarrow f_* \mathcal{O}_X(F_{X/Y}^{(e)*} M) \cong f_* \mathcal{O}_X(p^e L + (1 - p^e)f^* K_Y) \quad (7.2.4)$$

whose composite is non-zero. Take $\nu_1 \in \mathbb{Z}_{>0}$ so that $K_Y \leq \nu_1 H$. Then, we may replace the right-hand side of (7.2.4) with $f_* \mathcal{O}_X(p^e(L - f^* K_Y) + \nu_1 f^* H)$. Pick $\nu_2 \in \mathbb{Z}_{>0}$ so that $\mathcal{E}(\nu_2 H)$ is generically globally generated, and put $\nu := \nu_1 + \nu_2$. By (7.2.4), we have the non-zero morphism

$$\mathcal{F}_{\mu q_e} \otimes \mathcal{E}(\nu_2 H) \rightarrow f_* \mathcal{O}_X(p^e(L - f^* K_Y) + \nu f^* H) = f_* \mathcal{O}_X(D_e), \quad (7.2.5)$$

where $D_e := p^e(L - f^* K_Y) + \nu f^* H$. Note that ν is independent of e . By the choice of ν_2 , there exists a non-zero morphism $\mathcal{F}_{\mu q_e} \rightarrow f_* \mathcal{O}_X(D_e)$; we denote its adjoint by $\varphi: f^* \mathcal{F}_{\mu q_e} \rightarrow \mathcal{O}_X(D_e)$. Put $C := \text{Supp}(\text{Coker } \varphi)$. Then $C \neq X$ and

$$\mathbb{B}(\mathcal{O}_X(D_e) + l_e f^* H) \stackrel{\text{by } \varphi}{\subseteq} \mathbb{B}(f^* \mathcal{F}_{\mu q_e} + l_e f^* H) \cup C \subseteq f^{-1}(\mathbb{B}(\mathcal{F}_{\mu q_e} + l_e H)) \cup C.$$

Since $l_e > a\mu q_e$ by definition, we have

$$\begin{aligned} \mathbb{B}(\mathcal{F}_{\mu q_e} + l_e H) &\subseteq \mathbb{B}_-^H(\mathcal{F}_{\mu q_e} + a\mu q_e H) \\ &\stackrel{\text{Cor. 4.2}}{\subseteq} \mathbb{B}_-(\mathcal{F}_{\mu q_e} + a\mu q_e H) \cup \mathbb{B}_+(H) \stackrel{\text{Thm. 6.3}}{\neq} Y, \end{aligned}$$

so $\mathbb{B}(\mathcal{O}_X(D_e) + l_e f^* H) \neq X$, and hence $\mathbb{B}(D_e|_{X_0} + l_e f_0^* H) \neq X_0$, which is our claim.

Next, we show the assertion in the case where f is not necessarily flat. Let $X' \rightarrow Y'$ be the flattening of f . Let Y'' be the normalization of Y' , let X'' be the normalization of the main component of $X' \times_{Y'} Y''$, and let $f'' : X'' \rightarrow Y''$ be the induced morphism. Let V be the regular open subset of Y'' such that $g := f''|_U : U \rightarrow V$ is flat, where $U := f''^{-1}(V)$. Now we have the following commutative diagram:

$$\begin{array}{ccc} X & \xleftarrow{\rho} & U \\ f \downarrow & & \downarrow g \\ Y & \xleftarrow{\sigma} & V. \end{array}$$

Note that $(\{\eta_V\}, \sigma^*H)$ satisfies $(*)_a$, as shown in Lemma 5.4, where η_V is the generic point of V . Put $\Delta' := \rho_*^{-1}\Delta$. Since $\sigma^*K_Y \leq K_V$, by the above argument, we see that for each $t \in \mathbb{Q}_{>0}$, there is a $\mathbb{Q}$ -Weil divisor $E_t \geq 0$ on U such that

$$E_t \sim_{\mathbb{Q}} m_0(K_U + \Delta') - g^*\sigma^*K_Y + (a(m_0 - 1) + t)g^*\sigma^*H.$$

Applying ρ_* , we get

$$0 \leq \rho_*E_t \sim_{\mathbb{Q}} m_0(K_X + \Delta) - f^*K_Y + (a(m_0 - 1) + t)f^*H,$$

which completes the proof. $\square$

THEOREM 7.3. *Let k be an algebraically closed field of characteristic $p > 0$. Let X be a normal projective variety, and let Δ be an effective $\mathbb{Q}$ -Weil divisor on X such that $i_C(K_X + \Delta)$ is Cartier for an integer $i_C > 0$ not divisible by p . Let Y be a smooth projective variety of maximal Albanese dimension, and let $f : X \rightarrow Y$ be a separable surjective morphism. Suppose that the following conditions hold:*

- (i) *The sum $\bigoplus_{l \geq 0} H^0(X_\eta, \mathcal{O}_{X_\eta}(\lfloor l(K_{X_\eta} + \Delta|_{X_\eta}) \rfloor))$ is a finitely generated $k(\eta)$ -algebra, where η is the generic point of Y .*
- (ii) *There exists an integer $m_0 \geq 0$ with $i_C|m_0$ such that*

$$S^0(X_\eta, \Delta|_{X_\eta}; \mathcal{O}_{X_\eta}(m(K_{X_\eta} + \Delta|_{X_\eta}))) = H^0(X_\eta, \mathcal{O}_{X_\eta}(m(K_{X_\eta} + \Delta|_{X_\eta})))$$

for each $m \geq m_0$ with $i_C|m$;

- (iii) *The variety Y either is a curve or is of general type.*

Then

$$\kappa(X, K_X + \Delta) \geq \kappa(Y) + \kappa(X_\eta, K_{X_\eta} + \Delta|_{X_\eta}).$$

Proof. We may assume $\kappa(Y) \geq 0$ and $\kappa(X_\eta, K_{X_\eta} + \Delta|_{X_\eta}) \geq 0$. Since Y is of maximal Albanese dimension, $(\{\eta\}, H)$ satisfies condition $(*)_0$ for some ample Cartier divisor H on Y , as shown in Example 5.8.

First, we deal with the case where Y is of general type. The proof is similar to that of [Eji17, Theorem 7.2], which is based on the proof of [Pat18, Theorem 1.7]. Let H be an ample Cartier divisor on Y . Put $D := m_0(K_X + \Delta) - f^*K_Y$,

$$\begin{aligned} S &:= \{\varepsilon \in \mathbb{Q} \mid \kappa(X, D - \varepsilon f^*H) \geq 0\} \quad \text{and} \\ S' &:= \{\varepsilon \in \mathbb{Q} \mid \kappa(X, D - \varepsilon f^*H) \geq \kappa(X_\eta, D_\eta) + \dim Y\}. \end{aligned}$$

Then $S' \subseteq S$. Let us show that $S' \neq \emptyset$. By assumption (i), there is a $\mu \in \mathbb{Z}_{>0}$ such that the $k(\eta)$ -algebra $\bigoplus_{l \geq 0} H^0(X_\eta, \mathcal{O}_{X_\eta}(l\mu D_\eta))$ is generated by $H^0(X_\eta, \mu D_\eta)$. Using the projection formula,

we can find a $\nu \in \mathbb{Z}_{>0}$ such that $f_*\mathcal{O}_X(\mu D + \nu f^*H)$ is generated by its global sections. By the choice of μ , the natural morphism

$$\bigotimes^n f_*\mathcal{O}_X(\mu D + \nu f^*H) \rightarrow f_*\mathcal{O}_X(n(\mu D + \nu f^*H))$$

is generically surjective for each $n \in \mathbb{Z}_{>0}$, so $f_*\mathcal{O}_X(n(\mu D + \nu f^*H))$ is generically generated by its global sections. From this, we get the injective morphism

$$\bigoplus^{\text{rank } f_*\mathcal{O}_X(n\mu D)} \mathcal{O}_Y(nH) \rightarrow f_*\mathcal{O}_X(n(\mu D + (\nu + 1)f^*H)).$$

Since $\text{rank } f_*\mathcal{O}_X(n\mu D) = \dim H^0(X_\eta, n\mu D_\eta)$, we obtain that

$$\dim H^0(Y, nH) \times \dim H^0(X_\eta, n\mu D_\eta) \leq \dim H^0(X, n(\mu D + (\nu + 1)f^*H))$$

for each $n \in \mathbb{Z}_{>0}$, so

$$\dim Y + \kappa(X_\eta, D_\eta) = \kappa(Y, H) + \kappa(X_\eta, \mu D_\eta) \leq \kappa(X, \mu D + (\nu + 1)f^*H),$$

and hence $\varepsilon_0 := -\mu^{-1}(\nu + 1) \in S'$.

Let us prove that $\sup S' = \sup S$. The inequality $\leq$ is obvious. Let us show $\geq$. Take $\varepsilon \in S$. Then $D - \varepsilon f^*H$ is $\mathbb{Q}$ -linearly equivalent to an effective $\mathbb{Q}$ -Cartier divisor, so for every $\delta \in \mathbb{Q}_{>0}$, we have

$$\kappa(X, (1 + \delta)D - (\varepsilon + \delta\varepsilon_0)f^*H) \geq \kappa(X, \delta(D - \varepsilon_0 f^*H)) \geq \kappa(X_\eta, D_\eta) + \dim Y.$$

Hence, we get $S' \ni (1 + \delta)^{-1}(\varepsilon + \delta\varepsilon_0) \xrightarrow{\delta \rightarrow 0} \varepsilon$, which means that $\sup S' \geq \varepsilon$, and so $\sup S' \geq \sup S$.

Let us prove the assertion. By Theorem 7.2, we see that $\mathbb{Q}_{<0} \subseteq S$, so $0 \leq \sup S = \sup S'$. Therefore, we can find an $\varepsilon \in S'$ such that $K_Y + \varepsilon H$ is $\mathbb{Q}$ -linearly equivalent to an effective $\mathbb{Q}$ -divisor. Note that K_Y is big. Hence, we get

$$\begin{aligned} \kappa(X, K_X + \Delta) &= \kappa(X, D + f^*K_Y) \\ &\geq \kappa(X, D + f^*K_Y - f^*(K_Y + \varepsilon H)) \\ &= \kappa(X, D - \varepsilon f^*H) \\ &\geq \kappa(X_\eta, D_\eta) + \dim Y \\ &= \kappa(X_\eta, K_{X_\eta} + \Delta_\eta) + \kappa(Y). \end{aligned}$$

Next, we consider the case where Y is an elliptic curve. Let μ be an integer that is large and divisible enough. Set $\mathcal{G} := f_*\mathcal{O}_X(\mu(K_X + \Delta))$. Then $\mathcal{G}$ is a nef vector bundle by Theorem 6.3. For a nef vector bundle $\mathcal{V}$ on Y , let $\mathbb{L}(\mathcal{V})$ denote the subset of $\text{Pic}^0(Y)$ consisting of line bundles $\mathcal{L}$ on Y that can be obtained as a quotient bundle of $\mathcal{V}$. Let G be the subgroup of $\text{Pic}^0(Y)$ generated by $\mathbb{L}(\mathcal{G})$. Let us prove that G is a finite group. If this holds, then each $\mathcal{L} \in \mathbb{L}(\mathcal{G})$ is a torsion line bundle, which means that there is a finite morphism $\pi: Y' \rightarrow Y$ from an elliptic curve Y' such that $\pi^*\mathcal{G}$ is generated by its global sections, and hence we can prove the assertion by applying the same argument as that in the proof of [Eji17, Theorem 7.6].

Here we use the classification of vector bundles on an elliptic curve [Ati57, Oda71]. See [Eji17, Theorem 7.3] for a summary. Considering the decomposition of $\mathcal{G}$ into indecomposable vector bundles, we see that

- $\mathcal{G}$ is isomorphic to the direct sum of an ample vector bundle $\mathcal{G}^+$ and a nef vector bundle $\mathcal{E}$ of degree 0 and

- $\mathcal{E} \cong \bigoplus_{1 \leq j \leq \nu} \mathcal{E}_{r_j,0} \otimes \mathcal{L}_j$, where $\mathcal{L}_j \in \text{Pic}^0(Y)$ and $\mathcal{E}_{r_j,0}$ is an indecomposable vector bundle of rank r_j and degree 0 having a non-zero global section.

Note that, to show the first statement, we used the fact that indecomposable vector bundles of positive degree are ample, which is obtained by, for example, combining [Eji17, Theorem 7.3(3)] and [Oda71, Theorem 2.16]. Since $\mathcal{E}_{r,0}$ is an extension of $\mathcal{E}_{r-1,0}$ by $\mathcal{O}_Y$, we get a filtration

$$0 = \mathcal{G}_0 \subset \mathcal{G}_1 \subset \cdots \subset \mathcal{G}_{\rho+1} = \mathcal{G} \quad (\rho := \text{rank}(\mathcal{E}))$$

of $\mathcal{G}$ such that $\{\mathcal{G}_1/\mathcal{G}_0, \dots, \mathcal{G}_\rho/\mathcal{G}_{\rho-1}\} = \mathbb{L}(\mathcal{E}) (= \mathbb{L}(\mathcal{G}))$ and $\mathcal{G}_{\rho+1}/\mathcal{G}_\rho \cong \mathcal{G}^+$.

Put $\mathbb{L}(\mathcal{E}) = \{\mathcal{L}_1, \dots, \mathcal{L}_\lambda\}$. For each $m \geq 0$, set

$$G(m) := \{\mathcal{L}_1^{m_1} \otimes \cdots \otimes \mathcal{L}_\lambda^{m_\lambda} \mid |m_1| + \cdots + |m_\lambda| \leq m\}.$$

Then for each $\mathcal{N}_1 \in G(n_1)$ and $\mathcal{N}_2 \in G(n_2)$, we have $\mathcal{N}_1 \otimes \mathcal{N}_2 \in G(n_1 + n_2)$. Set

$$\mathcal{F} := \bigoplus_{n_0 \leq r < n_0 + \mu \text{ and } i_C | r} f_* \mathcal{O}_X(r(K_X + \Delta))$$

for some $n_0 \gg 0$. Then $\mathcal{F}$ is a nef vector bundle by Theorem 6.3. Since $\mathbb{L}(\mathcal{F})$ is a finite set, there is a $\nu \in \mathbb{Z}_{>0}$ such that $\mathbb{L}(\mathcal{F}) \cap G \subseteq G(\nu)$. Take $e \gg 0$ so that $p^e - q_e > \nu$, where q_e is an integer such that $(\mu - 1)p^e + 1 = \mu q_e + r_e$ for an integer r_e with $n_0 \leq r_e < n_0 + \mu$. We have the generically surjective morphisms

$$F_{Y*}^e(S^{q_e}(\mathcal{G}) \otimes \mathcal{F}) \rightarrow F_{Y*}^e(S^{q_e}(\mathcal{G}) \otimes f_* \mathcal{O}_X(r_e(K_X + \Delta))) \rightarrow \mathcal{G}$$

by the same argument as that in the proof of Theorem 6.2. For each $\mathcal{L}_j \in \mathbb{L}(\mathcal{E})$, we have $\mathcal{G} \twoheadrightarrow \mathcal{L}_j$, which induces the non-zero morphism

$$F_{Y*}^e(S^{q_e}(\mathcal{G}) \otimes \mathcal{F}) \rightarrow \mathcal{L}_j.$$

Since $\omega_Y \cong \mathcal{O}_Y$, by [Har66, Proposition III.6.9(a)], we get

$$F_Y^e! \mathcal{L}_j \cong (F_Y^e! \mathcal{O}_Y) \otimes F_Y^e \mathcal{L}_j \cong \omega_Y^{1-p^e} \otimes \mathcal{L}_j^{p^e} \cong \mathcal{L}_j^{p^e},$$

so it follows from Grothendieck duality that

$$0 \neq \text{Hom}(F_{Y*}^e(S^{q_e}(\mathcal{G}) \otimes \mathcal{F}), \mathcal{L}_j) \cong \text{Hom}(S^{q_e}(\mathcal{G}) \otimes \mathcal{F}, \mathcal{L}_j^{p^e}),$$

and hence we get the non-zero morphism

$$S^{q_e}(\mathcal{G}) \otimes \mathcal{F} \rightarrow \mathcal{L}_j^{p^e}.$$

This is surjective since the source is a nef vector bundle and $\deg \mathcal{L}_j^{p^e} = 0$. Considering the above filtration, we find $m_1, \dots, m_\lambda \in \mathbb{Z}_{\geq 0}$ with $\sum_{j=1}^\lambda m_j = q_e$ and $\mathcal{M} \in \mathbb{L}(\mathcal{F})$ such that

$$\mathcal{L}_1^{m_1} \otimes \cdots \otimes \mathcal{L}_\lambda^{m_\lambda} \otimes \mathcal{M} \cong \mathcal{L}_j^{p^e},$$

which means that $\mathcal{L}_j^{p^e} \in G(q_e + \nu)$.

Set $N := \lambda(p^e - 1)$. Suppose that there is an $\mathcal{L} \in G \setminus G(N)$. Let M be the minimal integer such that $\mathcal{L} \in G(M)$. Then we find $m_1, \dots, m_\lambda$ with $\sum_{j=1}^\lambda |m_j| = M$ such that $\mathcal{L} \cong \mathcal{L}_1^{m_1} \otimes \cdots \otimes \mathcal{L}_\lambda^{m_\lambda}$. We see from $M > N$ that $|m_j| \geq p^e$ for some j , so $\mathcal{L}_j^{|m_j|} \cong \mathcal{L}_j^{|m_j| - p^e} \otimes \mathcal{L}_j^{p^e}$, which means that $\mathcal{L}_j^{m_j} \in G(|m_j| - p^e + q_e + \nu)$, and hence

$$\mathcal{L} \in G(|m_j| - p^e + q_e + \nu + M - |m_j|) = G(M - (p^e - q_e - \nu)),$$

but this contradicts the choice of M since $p^e - q_e - \nu > 0$. Hence, we conclude that $G = G(N)$, which proves our claim. $\square$

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