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Néron blowups and low-degree cohomological applications

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Abstract

We define dilatations of general schemes and study their basic properties. Dilatations of group schemes are—in favorable cases—again group schemes, called Néron blowups. We give two applications to their cohomology in degree 0 (integral points) and degree 1 (torsors): we prove a canonical Moy–Prasad isomorphism that identifies the graded pieces in the congruent filtration of G with the graded pieces in its Lie algebra $\mathfrak{g}$, and we show that many level structures on moduli stacks of G-bundles are encoded in torsors under Néron blowups of G.

1. Introduction

1.1 Motivation and goals

Néron blowups (or dilatations) provide a tool to modify group schemes over the fibers of a given Cartier divisor on the base. After their initial introduction by Néron [Nér64], they where cast in the language of modern algebraic geometry by Raynaud [Ray66]. Classically, their integral points over discrete valuation rings appear as congruence subgroups of reductive groups over fields; see [Ana73, § 2.1.2], [WW80, Remark following Proposition 1.2, p. 551], [BLR90, § 3.2], [PY06, §§ 7.2–7.4] and [Yu15, § 2.8]. More recently Néron blowups have been used in conjunction with Tannakian methods to study differential Galois groups in [DHdS18] and [HdS21]. Over 2-dimensional base schemes they also appear in [PZ13, § 4.2].

This note extends the theory of dilatations—called Néron blowups in the case of group schemes—to general schemes. We give two applications involving their cohomology in degree 0 (integral points) and in degree 1 (torsors). Our results concerning their integral points lead to a general form of an isomorphism of Moy and Prasad, frequently used in representation theory. Our results concerning their torsors show that these naturally encode many level structures on moduli stacks of bundles. This is used to construct integral models of moduli stacks of shtukas

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with level structures as in [Dri87] and [Var04] which for parahoric level structures might be seen as function field analogues of the integral models of Shimura varieties in [KP18].

1.2 Results

Let S be a scheme. Let S_0 be an effective Cartier divisor on S, that is, a closed subscheme which is locally defined by a single non-zero divisor. We denote by $\mathrm{Sch}_S^{S_0\text{-reg}}$ the full subcategory of schemes $T \to S$ such that $T|_{S_0} := T \times_S S_0$ is an effective Cartier divisor on T. This category contains all flat schemes over S. For a group scheme $G \to S$ together with a closed subgroup $H \subset G|_{S_0}$ over S_0, we define the contravariant functor $\mathcal{G}\colon \mathrm{Sch}_S^{S_0\text{-reg}} \to \mathrm{Groups}$ given by all morphisms of S-schemes $T \to G$ such that the restriction $T|_{S_0} \to G|_{S_0}$ factors through H.

Theorem A. (1) *The functor $\mathcal{G}$ is representable by an open subscheme of the full scheme-theoretic blowup of G in H. The structure morphism $\mathcal{G} \to S$ is an object in $\mathrm{Sch}_S^{S_0\text{-reg}}$ (see Lemmas 2.3 and 2.4, Proposition 2.6)*

(2) *The canonical map $\mathcal{G} \to G$ is affine. Its restriction over $S\backslash S_0$ induces an isomorphism $\mathcal{G}|_{S\backslash S_0} \cong G|_{S\backslash S_0}$. Its restriction over S_0 factors as $\mathcal{G}|_{S_0} \to H \subset G|_{S_0}$ (see Lemmas 2.4 and 3.1).*

(3) *If $H \to S_0$ has connected fibers and $H \subset G|_{S_0}$ is regularly immersed, then $\mathcal{G}|_{S_0} \to S_0$ has connected fibers (see Proposition 2.16 and Theorem 3.2).*

(4) *If $G \to S$, $H \to S_0$ are flat and (locally) of finite presentation and $H \subset G|_{S_0}$ is regularly immersed, then $\mathcal{G} \to S$ is flat and (locally) of finite presentation. If both $G \to S$, $H \to S_0$ are smooth, then $\mathcal{G} \to S$ is smooth (see Proposition 2.16 and Theorem 3.2)*

(5) *Assume that $\mathcal{G} \to S$ is flat. Then its formation commutes with base change $S' \to S$ in $\mathrm{Sch}_S^{S_0\text{-reg}}$, and it carries the structure of a group scheme such that the canonical map $\mathcal{G} \to G$ is a morphism of S-group schemes (see Lemma 2.7 and Theorem 3.2).*

(6) *Assume that $G \to S$ is flat, finitely presented and $H \to S_0$ is flat, regularly immersed in $G|_{S_0}$. Locally over S_0, there is an exact sequence of S_0-group schemes $1 \to V \to \mathcal{G}|_{S_0} \to H \to 1$, where V is the vector bundle given by restriction to the unit section of an explicit twist of the normal bundle of H in $G|_{S_0}$. Assume moreover given a lifting of H to a flat S-subgroup scheme of G; then there is a canonical such sequence, which exists globally and is canonically split (see Theorem 3.5).*

We call $\mathcal{G} \to S$ the *Néron blowup* (or *dilatation*) *of G in H along S_0*. Note that $\mathcal{G} \to S$ is a group object in $\mathrm{Sch}_S^{S_0\text{-reg}}$ by item (1) but that $\mathcal{G} \to S$ is a group scheme only if the self-products $\mathcal{G} \times_S \mathcal{G}$ and $\mathcal{G} \times_S \mathcal{G} \times_S \mathcal{G}$ are objects in $\mathrm{Sch}_S^{S_0\text{-reg}}$, which holds for example in item (5); cf. § 3.1 for details. If S is the spectrum of a discrete valuation ring and if S_0 is defined by the vanishing of a uniformizer, then $\mathcal{G} \to S$ is the group scheme constructed in [Ana73, § 2.1.2], [WW80, § 1, p. 551], [BLR90, § 3.2], [PY06, §§ 7.2–7.4] and [Yu15, § 2.8]. For an example of Néron blowups over 2-dimensional base schemes, we refer to [PZ13, § 4.2]; cf. also Example 3.3.

We point out that most of the foundations of the study of dilatations can be settled in an absolute setting for schemes. That is, we initially develop the theory of affine blowups (or dilatations) for closed subschemes Z in a scheme X along a divisor D. It is only later that we specialize to relative schemes (over some base S, with the divisor coming from the base) and then further to group schemes.

The applications we give originate from a sheaf-theoretic viewpoint on Néron blowups. Write $j\colon S_0 \hookrightarrow S$ for the closed immersion of the Cartier divisor, and assume that $G \to S$ and $H \to S_0$ are flat, locally finitely presented groups. In this context, the dilatation $\mathcal{G} \to G$ sits in an exact sequence of sheaves of pointed sets on the small syntomic site of S (see Lemma 3.8):

$$1 \longrightarrow \mathcal{G} \longrightarrow G \longrightarrow j_*(G_0/H) \longrightarrow 1\,,$$

where $G_0 := G|_{S_0}$. If $G \to S$ and $H \to S_0$ are smooth, then the sequence is exact as a sequence of sheaves on the small étale site of S. Considering the associated sequence on global sections, we obtain the following statement, which generalizes and unifies several results sometimes found in the literature under the name of *Moy–Prasad isomorphisms* (Remark 4.4).

Corollary 1 (Theorem 4.3). *Let r, s be integers such that $0 \leqslant r/2 \leqslant s \leqslant r$. Let $(\mathcal{O}, \pi)$ be a henselian pair, where $\pi \subset \mathcal{O}$ is an invertible ideal. Let G be a smooth, separated $\mathcal{O}$-group scheme. Let G_r be the rth iterated dilatation of the unit section and $\mathfrak{g}_r$ its Lie algebra. If $\mathcal{O}$ is local or G is affine, there is a canonical and functorial isomorphism $G_s(\mathcal{O})/G_r(\mathcal{O}) \xrightarrow{\sim} \mathfrak{g}_s(\mathcal{O})/\mathfrak{g}_r(\mathcal{O})$.*

As another application, we are interested in comparing $\mathcal{G}$-torsors with G-torsors. In light of the above short exact sequence of sheaves, there is an equivalence between the category of $\mathcal{G}$-torsors and the category of G-torsors equipped with a section of their pushforward along $G \to j_*(G_0/H)$; see [Gir71, § III.3.2, Proposition 3.2.1].

This has consequences for moduli of torsors over curves. We thus specialize to the following setting; cf. § 4.2.1. Assume that X is a smooth, projective, geometrically irreducible curve over a field k with a Cartier divisor $N \subset X$, that $G \to X$ is a smooth, affine group scheme and that $H \to N$ is a smooth closed subgroup scheme of $G|_N$. In this case, the Néron blowup $\mathcal{G} \to X$ is a smooth, affine group scheme. Let Bun_G (respectively, $\mathrm{Bun}_{\mathcal{G}}$) denote the moduli stack of G-torsors (respectively, $\mathcal{G}$-torsors) on X. This is a quasi-separated, smooth algebraic stack locally of finite type over k (cf. for example [Hei10, Proposition 1] or [AH21, Theorem 2.5]). Pushforward of torsors along $\mathcal{G} \to G$ induces a morphism $\mathrm{Bun}_{\mathcal{G}} \to \mathrm{Bun}_G$, $\mathcal{E} \mapsto \mathcal{E} \times^{\mathcal{G}} G$. We also consider the stack $\mathrm{Bun}_{(G,H,N)}$ of G-torsors on X with level-(H,N)-structures; cf. Definition 4.5. Its k-points parametrize pairs $(\mathcal{E}, \beta)$ consisting of a G-torsor $\mathcal{E} \to X$ and a section β of the fppf quotient $(\mathcal{E}|_N/H) \to N$; that is, β is a reduction of $\mathcal{E}|_N$ to an H-torsor.

Corollary 2 (Corollary 4.7 and Theorem 4.8). *There is an equivalence of k-stacks*

$$\mathrm{Bun}_{\mathcal{G}} \xrightarrow{\cong} \mathrm{Bun}_{(G,H,N)}\,, \quad \mathcal{E} \longmapsto \big(\mathcal{E} \times^{\mathcal{G}} G,\, \beta_{\mathrm{can}}\big)\,,$$

where β_{can} denotes the canonical reduction induced from the factorization $\mathcal{G}|_N \to H \subset G|_N$ given in Theorem A(2).

If $H = \{1\}$ is trivial, then $\mathrm{Bun}_{(G,H,N)}$ is the moduli stack of G-torsors equipped with level-N-structures. If $G \to X$ is reductive, if N is reduced and if H is a parabolic subgroup in $G|_N$, then $\mathrm{Bun}_{(G,H,N)}$ is the moduli stack of G-torsors with quasi-parabolic structures as in [LS97]. In this case the restrictions of $\mathcal{G}$ to the completed local rings of X are parahoric group schemes in the sense of [BT84], and the previous corollary was pointed out in [PR10, § 2.a]. Thus, many level structures are encoded in torsors under Néron blowups. This construction is also compatible with the adelic viewpoint; cf. Corollary 4.11.

Now assume that k is a finite field. As a consequence of the corollary, one naturally obtains integral models for moduli stacks of G-shtukas on X with level structures over N as in [Dri87] for $G = \mathrm{GL}_n$ and in [Var04] for general split reductive G; cf. also [NN08, § 2.4]. General properties of moduli stacks of shtukas for smooth, affine group schemes are studied in [AH21, AH19, Bre19]. In § 4.2.2 below, we make the connection between G-shtukas with level structures as in [Dri87, Var04, Laf18] and $\mathcal{G}$-shtukas as in [AH21, AH19, Bre19]. We expect the point of view of $\mathcal{G}$-shtukas, as opposed to G-shtukas with level structures, to also be fruitful for investigations outside the case of parahoric level structures.

2. Dilatations

In this section, we define dilatations and give some properties. Dilatations (or affine blowups) are spectra of affine blowup algebras. We first introduce affine blowup algebras.

2.1 Definition

Fix a scheme X. Let $Z \subset D$ be closed subschemes in X, and assume that D is locally principal. Denote by $\mathcal{J} \subset \mathcal{I}$ the associated quasi-coherent sheaves of ideals in $\mathcal{O}_X$ so that $Z = V(\mathcal{I}) \subset V(\mathcal{J}) = D$. Let $\mathrm{Bl}_{\mathcal{I}}\mathcal{O}_X = \mathcal{O}_X \oplus \mathcal{I} \oplus \mathcal{I}^2 \oplus \cdots$ denote the Rees algebra; it is a quasi-coherent $\mathbb{Z}_{\geqslant 0}$-graded $\mathcal{O}_X$-algebra. If $\mathcal{J} = (b)$ is principal with $b \in \Gamma(X, \mathcal{J})$, then $\big(\mathrm{Bl}_{\mathcal{I}}\mathcal{O}_X\big)\big[\mathcal{J}^{-1}\big] := \big(\mathrm{Bl}_{\mathcal{I}}\mathcal{O}_X\big)\big[b^{-1}\big]$ is well defined independently of the choice of generators of $\mathcal{J}$. If $\mathcal{J}$ is only locally principal, then we define the localization $\big(\mathrm{Bl}_{\mathcal{I}}\mathcal{O}_X\big)\big[\mathcal{J}^{-1}\big]$ by glueing. This $\mathcal{O}_X$-algebra inherits a grading by giving local generators of $\mathcal{J}$ degree 1. In other words, the grading of $\big(\mathrm{Bl}_{\mathcal{I}}\mathcal{O}_X\big)\big[\mathcal{J}^{-1}\big]$ is given locally by $\deg(i/b^k) = n - k$ for $i \in \mathcal{I}^n$ and b a local generator of $\mathcal{J}$.

Definition 2.1. We use the following terminology:

(1) The *affine blowup algebra* of $\mathcal{O}_X$ in $\mathcal{I}$ along $\mathcal{J}$ is the quasi-coherent sheaf of $\mathcal{O}_X$-algebras

$$\mathcal{O}_X\big[\tfrac{\mathcal{I}}{\mathcal{J}}\big] \overset{\text{def}}{=\!=} \Big[\big(\mathrm{Bl}_{\mathcal{I}}\mathcal{O}_X\big)\big[\mathcal{J}^{-1}\big]\Big]_{\deg=0}$$

obtained as the subsheaf of degree 0 elements in $\big(\mathrm{Bl}_{\mathcal{I}}\mathcal{O}_X\big)\big[\mathcal{J}^{-1}\big]$.

(2) The *dilatation* (or *affine blowup*) of X in Z along D is the X-affine scheme

$$\mathrm{Bl}_Z^D X \overset{\text{def}}{=\!=} \mathrm{Spec}\big(\mathcal{O}_X\big[\tfrac{\mathcal{I}}{\mathcal{J}}\big]\big)\,.$$

The subscheme Z, or the pair (Z, D), is called the *center* of the dilatation.

Remark 2.2. If X is affine, affine blowup algebras are defined in [Sta22, 052P]. In this case we write $B := \Gamma(X, \mathcal{O}_X)$, $I := \Gamma(X, \mathcal{I})$, $J := \Gamma(X, \mathcal{J})$ and $\mathrm{Bl}_I B := \Gamma(X, \mathrm{Bl}_{\mathcal{I}}\mathcal{O}_X) = \oplus_{n \geqslant 0} I^n$. Moreover, if $J = (b)$ is principal, then $B\big[\frac{I}{b}\big] := \Gamma\big(X, \mathcal{O}_X\big[\frac{\mathcal{I}}{\mathcal{J}}\big]\big)$ is the algebra whose elements are equivalence classes of fractions x/b^n with $x \in I^n$, where two representatives x/b^n, y/b^m with $x \in I^n$, $y \in I^m$ define the same element in $B\big[\frac{I}{b}\big]$ if and only if there exists an integer $l \geqslant 0$ such that

$$b^l(b^m x - b^n y) = 0 \quad \text{inside } B\,. \tag{2.1}$$

By [Sta22, 07Z3],

$$\text{the image of } b \text{ in } B\big[\tfrac{I}{b}\big] \text{ is a non-zero divisor}\,, \tag{2.2}$$

$$bB\big[\tfrac{I}{b}\big] = IB\big[\tfrac{I}{b}\big]\,, \quad \text{and} \tag{2.3}$$

$$B\big[\tfrac{I}{b}\big]\big[b^{-1}\big] = B\big[b^{-1}\big]\,. \tag{2.4}$$

In particular, the ring $B\big[\frac{I}{b}\big]$ is the B-subalgebra of $B\big[b^{-1}\big]$ generated by fractions x/b with $x \in I$.

2.2 Basic properties

We proceed with the notation from § 2.1. The following results generalize [BLR90, § 3.2, Proposition 1].

LEMMA 2.3. *The affine blowup* $\mathrm{Bl}_Z^D X$ *is the open subscheme of the blowup* $\mathrm{Bl}_Z X = \mathrm{Proj}(\mathrm{Bl}_{\mathcal{I}}\mathcal{O}_X)$ *defined by the complement of* $V_+(\overline{\mathcal{J}})$*, where* $\overline{\mathcal{J}}$ *is the sheaf of ideals generated by* $\mathcal{J} \subset \mathcal{I}$*, where* $\mathcal{I}$ *is the degree* 1 *part of* $\mathrm{Bl}_{\mathcal{I}}\mathcal{O}_X$.

Proof. By (2.2) the ideal sheaf $\mathcal{J}\mathcal{O}_X\big[\frac{\mathcal{I}}{\mathcal{J}}\big]$ is invertible, and by (2.3) the inclusion $\mathcal{J}\mathcal{O}_X\big[\frac{\mathcal{I}}{\mathcal{J}}\big] \subset \mathcal{I}\mathcal{O}_X\big[\frac{\mathcal{I}}{\mathcal{J}}\big]$ is an equality. In particular, $\mathcal{I}\mathcal{O}_X\big[\frac{\mathcal{I}}{\mathcal{J}}\big]$ is invertible, so by the universal property of the usual blowup, there is a map $\mathrm{Bl}_Z^D X \to \mathrm{Bl}_Z X$. Then the claim of the lemma is Zariski local on X. We reduce to the case where $X = \mathrm{Spec}(B)$ is affine and $J = (b)$ is principal. Then $B\big[\frac{I}{b}\big]$ is the homogenous localization of $B \oplus I \oplus I^2 \oplus \cdots$ at $b \in I$ viewed as an element in degree 1; cf. [Sta22, 052Q]. This shows that $\mathrm{Spec}(B[\frac{I}{b}])$ is the complement of $V_+(b)$ in $\mathrm{Proj}(\mathrm{Bl}_I B)$. □

LEMMA 2.4. *As closed subschemes of* $\mathrm{Bl}_Z^D X$*, one has*

$$\mathrm{Bl}_Z^D X \times_X Z = \mathrm{Bl}_Z^D X \times_X D\,,$$

which is an effective Cartier divisor on $\mathrm{Bl}_Z^D X$.

Proof. Our claim is Zariski local on X. We reduce to the case where $X = \mathrm{Spec}(B)$ is affine and $J = (b)$ is principal. We have to show that $bB\big[\frac{I}{b}\big] = IB\big[\frac{I}{b}\big]$ and that b is a non-zero divisor in $B\big[\frac{I}{b}\big]$. This is (2.2) and (2.3) above. □

When D is a Cartier divisor, we can also realize $\mathrm{Bl}_Z^D X$ as a closed subscheme of the affine projecting cone. Recall that, classically, this cone is defined as the relative spectrum of the Rees algebra $\mathrm{Bl}_{\mathcal{I}}\mathcal{O}_X$, so the blowup and its affine cone are complementary to each other in the completed projective cone; see [EGA, II, § 8.3]. However, twisting the blowup algebras with the invertible ideal sheaf $\mathcal{J}$ gives rise to different embeddings. Indeed, by [EGA, II, § 8.1.1] we have a canonical isomorphic presentation of the usual blowup as $\mathrm{Bl}_Z X = \mathrm{Proj}\big(\oplus_{n\geqslant 0}\mathcal{I}^n \otimes \mathcal{J}^{-n}\big)$. Here we define the *affine projecting cone* of $\mathrm{Bl}_Z X$ (with respect to the chosen presentation) as $\mathrm{C}_Z X := \mathrm{Spec}\big(\oplus_{n\geqslant 0}\mathcal{I}^n \otimes \mathcal{J}^{-n}\big)$.

LEMMA 2.5. *If* D *is a Cartier divisor, the affine blowup* $\mathrm{Bl}_Z^D X$ *is the closed subscheme of the affine projecting cone* $\mathrm{C}_Z X$ *defined by the equation* $\varrho - 1$*, where* $\varrho \in \mathcal{I} \otimes \mathcal{J}^{-1}$ *is the image of* 1 *under the inclusion* $\mathcal{O}_X = \mathcal{J} \otimes \mathcal{J}^{-1} \subset \mathcal{I} \otimes \mathcal{J}^{-1}$.

Proof. Let $\mathscr{A} = \oplus_{n\geqslant 0}\mathcal{I}^n \otimes \mathcal{J}^{-n}$. There is a surjective morphism of sheaves of algebras $\mathscr{A} \to \mathcal{O}_X\big[\frac{\mathcal{I}}{\mathcal{J}}\big]$ defined by mapping a local section $i \otimes j^{-1}$ in degree 1 to i/j. To check that $\varrho - 1$ goes to zero and generates the kernel, we may work locally on some affine open subscheme $U \subset X$ where the sheaf $\mathcal{J}$ is generated by a section b. Let $t = b^\vee$ be the generator for $\mathcal{J}^{-1}$ dual to b. Let $B = \Gamma(X, \mathcal{O}_X)$ and $I = \Gamma(X, \mathcal{I})$. Then the map $\mathscr{A}(U) \to \mathcal{O}_X\big[\frac{\mathcal{I}}{\mathcal{J}}\big](U)$ is given by

$$\big(\oplus_{n\geqslant 0} I^n t^n\big) \longrightarrow B\big[\tfrac{I}{b}\big]\,, \quad \sum_{n\geqslant 0} i_n t^n \longmapsto \sum_{n\geqslant 0} i_n/b^n\,.$$

This induces an isomorphism $\big(\oplus_{n\geqslant 0} I^n t^n\big)/(bt-1) \xrightarrow{\sim} B\big[\frac{I}{b}\big]$. □

2.3 Universal property

In this text we will use regular immersions in a possibly non-noetherian setting where the reference [EGA, IV$_4$, §§ 16.9 and 19] is inadequate. In this case we refer to the Stacks Project [Sta22]. There, four notions of regularity are studied: by decreasing order of strength, *regular*, *Koszul-regular*, H_1*-regular*, *quasi-regular* (see [Sta22, §§ 067M and 0638]. The useful ones for us are the first (which is regularity in its classical meaning) and the third: an H_1-regular sequence is a sequence whose Koszul complex has no homology in degree 1. In [Sta22], several results are stated under the weakest H_1-regular assumption. For simplicity we will state our results for regular immersions, although all of them also hold for H_1-regular immersions. Note that regularity and H_1-regularity coincide for sequences composed of one element x because for them, the Koszul complex has length 1 and the homology group in degree 1 is just the x-torsion. In particular, for locally principal subschemes, the three notions regular, Koszul-regular and H_1-regular are equivalent.

Let us denote by $\mathrm{Sch}_X^{D\text{-reg}}$ the full subcategory of schemes $T \to X$ such that $T \times_X D \subset T$ is regularly immersed, or equivalently is an effective Cartier divisor (possibly the empty set) on T. If $T' \to T$ is flat and $T \to X$ is an object in this category, so is the composition $T' \to T \to X$. In particular, the category $\mathrm{Sch}_X^{D\text{-reg}}$ can be equipped with the fpqc/fppf/étale/Zariski Grothendieck topology, so that the notion of sheaves is well defined.

As $\mathrm{Bl}_Z^D X \to X$ defines an object in $\mathrm{Sch}_X^{D\text{-reg}}$ by Lemma 2.4, the contravariant functor

$$\mathrm{Sch}_X^{D\text{-reg}} \longrightarrow \mathrm{Sets}\,, \quad (T \to X) \longmapsto \mathrm{Hom}_{X\text{-Sch}}\big(T, \mathrm{Bl}_Z^D X\big) \tag{2.5}$$

together with $\mathrm{id}_{\mathrm{Bl}_Z^D X}$ determines $\mathrm{Bl}_Z^D X \to X$ uniquely up to unique isomorphism. The next proposition gives the universal property of dilatations.

Proposition 2.6. *The affine blowup $\mathrm{Bl}_Z^D X \to X$ represents the contravariant functor $\mathrm{Sch}_X^{D\text{-reg}} \to$ Sets given by*

$$(f\colon T \to X) \longmapsto \begin{cases} \{*\} & \text{if } f|_{T\times_X D} \text{ factors through } Z \subset X\,, \\ \varnothing & \text{else}\,. \end{cases} \tag{2.6}$$

Proof. Let F be the functor defined by (2.6). If $T \to \mathrm{Bl}_Z^D X$ is a map of X-schemes, then the structure map $T \to X$ restricted to $T \times_X D$ factors through $Z \subset X$ by Lemma 2.4. This defines a map

$$\mathrm{Hom}_{X\text{-Sch}}\big(\text{-}\,, \mathrm{Bl}_Z^D X\big) \longrightarrow F \tag{2.7}$$

of contravariant functors $\mathrm{Sch}_X^{D\text{-reg}} \to \mathrm{Sets}$. We want to show that (2.7) is bijective when evaluated at an object $T \to X$ in $\mathrm{Sch}_X^{D\text{-reg}}$. As (2.7) is a morphism of Zariski sheaves, we reduce to the case where both $X = \mathrm{Spec}(B)$ and $T = \mathrm{Spec}(R)$ are affine and $J = (b)$ is principal. For the injectivity, let $g, g'\colon B\big[\frac{I}{b}\big] \to R$ be two B-algebra maps. We need to show $g = g'$. Indeed, since $B\big[b^{-1}\big] = B\big[\frac{I}{b}\big]\big[b^{-1}\big]$ by (2.4), we get $g\big[b^{-1}\big] = g'\big[b^{-1}\big]$. As b is a non-zero divisor in R by assumption, this implies $g = g'$. For the surjectivity, consider an element in $F(\mathrm{Spec}(R))$ which corresponds to a ring morphism $g\colon B \to R$ such that I is contained in the kernel of $B \to R \to R/bR$. We need to show that g extends (necessarily uniquely) to a B-algebra morphism $\tilde{g}\colon B\big[\frac{I}{b}\big] \to R$. Let $[x/b^n]$, $x \in I^n$ be a class in $B\big[\frac{I}{b}\big]$. Since $g(I^n) \subset (b^n)$ in R, the b-torsion-freeness of R implies that there is a unique element $r = r(x,n) \in R$ such that $g(x) = b^n \cdot r$. We define $\tilde{g}([x/b^n]) := r(x,n)$. This is well defined: if y/b^m, $y \in I^m$, is another representative of $[x/b^n]$, then applying g to equation (2.1) yields $b^m g(x) = b^n g(y)$ in R. It follows that $r(x,n) = r(y,m)$. Thus, $\tilde{g}$ is well defined. Similarly, one checks that $\tilde{g}$ defines a morphism of B-algebras. □

2.4 Functoriality

Let $Z' \subset D' \subset X'$ be another triple as in § 2.1. A morphism $X' \to X$ such that the preimage of D is equal to D' and the restriction to Z' factors through Z induces a unique morphism

$\mathrm{Bl}_{Z'}^{D'}X' \to \mathrm{Bl}_Z^D X$ such that the diagram of schemes

$$\begin{array}{ccc} \mathrm{Bl}_{Z'}^{D'}X' & \longrightarrow & \mathrm{Bl}_Z^D X \\ \downarrow & & \downarrow \\ X' & \longrightarrow & X \end{array}$$

commutes. Indeed, the existence of $\mathrm{Bl}_{Z'}^{D'}X' \to \mathrm{Bl}_Z^D X$ follows directly from Definition 2.1. The uniqueness can be tested Zariski locally on X and X', where it follows from (2.2) and (2.4).

2.5 Base change

Now let $X' \to X$ be a map of schemes, and denote by $Z' \subset D' \subset X'$ the preimage of $Z \subset D \subset X$. Then $D' \subset X'$ is locally principal, so that the affine blow $\mathrm{Bl}_{Z'}^{D'}X' \to X'$ is well defined. By § 2.4 there is a canonical morphism of X'-schemes

$$\mathrm{Bl}_{Z'}^{D'}X' \longrightarrow \mathrm{Bl}_Z^D X \times_X X'. \tag{2.8}$$

LEMMA 2.7. *If $\mathrm{Bl}_Z^D X \times_X X' \to X'$ is an object of $\mathrm{Sch}_{X'}^{D'\text{-reg}}$, then (2.8) is an isomorphism.*

Proof. Our claim is Zariski local on X and X'. We reduce to the case where both $X = \mathrm{Spec}(B)$ and $X' = \mathrm{Spec}(B')$ are affine and $J = (b)$ is principal. We set $Z' = \mathrm{Spec}(B'/I')$ and $D' = \mathrm{Spec}(B'/J')$. Then $J' = (b')$ is principal as well, where b' is the image of b under $B \to B'$. We need to show that the map of B'-algebras $B' \otimes_B B\left[\frac{I}{b}\right] \longrightarrow B'\left[\frac{I'}{b'}\right]$ is an isomorphism. However, this map is surjective with kernel the b'-torsion elements [Sta22, 0BIP]. As b' is a non-zero divisor in $B' \otimes_B B\left[\frac{I}{b}\right]$ by assumption, the lemma follows. □

COROLLARY 2.8. *If the morphism $X' \to X$ is flat and has some property $\mathcal{P}$ which is stable under base change, then $\mathrm{Bl}_{Z'}^{D'}X' \to \mathrm{Bl}_Z^D X$ is flat and has $\mathcal{P}$.*

Proof. Since flatness is stable under base change, the projection $p\colon \mathrm{Bl}_Z^D X \times_X X' \to \mathrm{Bl}_Z^D X$ is flat and has property $\mathcal{P}$. By Lemma 2.7, it is enough to check that the closed subscheme $\mathrm{Bl}_Z^D X \times_X D'$ defines an effective Cartier divisor on $\mathrm{Bl}_Z^D X \times_X X'$. But this closed subscheme is the preimage of the effective Cartier divisor $\mathrm{Bl}_Z^D X \times_X D$ under the flat map p, and hence is an effective Cartier divisor as well. □

2.6 Exceptional divisor

For closed subschemes $Z \subset D$ in X with D locally principal, we saw in Lemma 2.4 that the preimage of the center $\mathrm{Bl}_Z^D X \times_X Z = \mathrm{Bl}_Z^D X \times_X D$ is an effective Cartier divisor in $\mathrm{Bl}_Z^D X$. It is called the *exceptional divisor* of the affine blowup. In order to describe it, as before we denote by $\mathcal{I}$ and $\mathcal{J}$ the sheaves of ideals of Z and D in $\mathcal{O}_X$. Also, we let $\mathcal{C}_{Z/D} = \mathcal{I}/(\mathcal{I}^2 + \mathcal{J})$ and $\mathcal{N}_{Z/D} = \mathcal{C}_{Z/D}^\vee$ be the conormal and normal sheaves of Z in D.

PROPOSITION 2.9. *Assume that $D \subset X$ is an effective Cartier divisor and $Z \subset D$ is a regular immersion. Write $\mathcal{J}_Z := \mathcal{J}|_Z$.*

(1) *The exceptional divisor $\mathrm{Bl}_Z^D X \times_X Z \to Z$ is an affine bundle (that is, a torsor under a vector bundle), Zariski locally over Z isomorphic to $\mathbb{V}(\mathcal{C}_{Z/D} \otimes \mathcal{J}_Z^{-1}) \to Z$.*

(2) *If $H^1(Z, \mathcal{N}_{Z/D} \otimes \mathcal{J}_Z) = 0$ (for example if Z is affine), then $\mathrm{Bl}_Z^D X \times_X Z \to Z$ is globally isomorphic to $\mathbb{V}(\mathcal{C}_{Z/D} \otimes \mathcal{J}_Z^{-1}) \to Z$.*

(3) *If Z is a transversal intersection in the sense that there is a cartesian square of closed subschemes whose vertical maps are regular immersions*

$$\begin{array}{ccc} W & \hookrightarrow & X \\ \uparrow & \square & \uparrow \\ Z & \hookrightarrow & D\,, \end{array}$$

then $\mathrm{Bl}_Z^D X \times_X Z \to Z$ *is* globally and canonically *isomorphic to* $\mathbb{V}(\mathcal{C}_{Z/D} \otimes \mathcal{J}_Z^{-1}) \to Z$.

Proof. Using Lemma 2.5, we can view the affine blowup $\mathrm{Bl}_Z^D X$ as the closed subscheme with equation $\varrho - 1 = 0$ of the affine projecting cone $\mathrm{C}_Z X = \mathrm{Spec}(\mathscr{A})$ with $\mathscr{A} = \oplus_{n \geqslant 0} \mathcal{I}^n \otimes \mathcal{J}^{-n}$. First we compute the preimage $\mathrm{C}_Z X \times_X Z$. We have $\mathscr{A} \otimes \mathcal{O}_X/\mathcal{I} = \oplus_{k \geqslant 0} (\mathcal{I}^k/\mathcal{I}^{k+1}) \otimes \mathcal{J}^{-k}$. Since the immersions $Z \subset D \subset X$ are regular, the conormal sheaf $\mathcal{C}_{Z/X} = \mathcal{I}/\mathcal{I}^2$ is finite locally free, and the canonical surjective morphism of $\mathcal{O}_X$-algebras $\mathrm{Sym}^\bullet(\mathcal{C}_{Z/X}) \to \mathrm{Gr}^\bullet_{\mathcal{I}}(\mathcal{O}_X)$ is an isomorphism, that is, $\mathrm{Sym}^k(\mathcal{C}_{Z/X}) \to \mathcal{I}^k/\mathcal{I}^{k+1}$ is an isomorphism for each $k \geqslant 0$. Taking into account that $\mathcal{J}$ is an invertible sheaf, we obtain $\mathrm{Sym}^k(\mathcal{C}_{Z/X} \otimes \mathcal{J}_Z^{-1}) \xrightarrow{\sim} (\mathcal{I}^k/\mathcal{I}^{k+1}) \otimes \mathcal{J}^{-k}$. It follows that $\mathrm{Spec}(\mathscr{A} \otimes \mathcal{O}_X/\mathcal{I}) = \mathbb{V}(\mathcal{C}_{Z/X} \otimes \mathcal{J}_Z^{-1})$ and that $\mathrm{Bl}_Z^D X \times_X Z$ is the closed subscheme cut out by $\varrho - 1$ inside the latter. Now consider the sequence of conormal sheaves

$$0 \longrightarrow \mathcal{C}_{D/X}|_Z \longrightarrow \mathcal{C}_{Z/X} \longrightarrow \mathcal{C}_{Z/D} \longrightarrow 0\,.$$

By [EGA, IV$_4$, Proposition 16.9.13] or [Sta22, 063N] in the non-noetherian case, this sequence is exact and locally split (beware that $\mathcal{C}_{Z/X}$ is denoted by $\mathscr{N}_{Z/X}$ in [EGA, IV$_4$]). Using the fact that $\mathcal{C}_{D/X}|_Z \otimes \mathcal{J}_Z^{-1} \simeq \mathcal{O}_Z$ is freely generated by ϱ as a subsheaf of $\mathcal{C}_{Z/X} \otimes \mathcal{J}_Z^{-1}$, we obtain an extension

$$0 \longrightarrow \varrho\mathcal{O}_Z \longrightarrow \mathcal{C}_{Z/X} \otimes \mathcal{J}_Z^{-1} \longrightarrow \mathcal{C}_{Z/D} \otimes \mathcal{J}_Z^{-1} \longrightarrow 0\,. \tag{2.9}$$

Now we consider the three cases listed in the proposition.

(1) Since $\mathcal{C}_{Z/D} \otimes \mathcal{J}_Z^{-1}$ is locally free, locally over Z we can choose a splitting of the exact sequence (2.9) of conormal sheaves:

$$\mathcal{C}_{Z/X} \otimes \mathcal{J}_Z^{-1} = \varrho\mathcal{O}_Z \oplus \mathcal{C}_{Z/D} \otimes \mathcal{J}_Z^{-1}\,.$$

Mapping $\varrho \mapsto 1$ yields a morphism of $\mathcal{O}_Z$-modules

$$\mathcal{C}_{Z/X} \otimes \mathcal{J}_Z^{-1} \longrightarrow \mathcal{O}_Z \oplus (\mathcal{C}_{Z/D} \otimes \mathcal{J}_Z^{-1}) \subset \mathrm{Sym}(\mathcal{C}_{Z/D} \otimes \mathcal{J}_Z^{-1})\,,$$

which extends to a surjection of algebras with kernel $(\varrho - 1)$:

$$\mathrm{Sym}(\mathcal{C}_{Z/X} \otimes \mathcal{J}_Z^{-1}) \longrightarrow \mathrm{Sym}(\mathcal{C}_{Z/D} \otimes \mathcal{J}_Z^{-1})\,.$$

This identifies $\mathrm{Bl}_Z^D X \times_X Z$ with the affine bundle $\mathbb{V}(\mathcal{C}_{Z/D} \otimes \mathcal{J}_Z^{-1})$, locally over Z.

(2) The exact sequence defines a class in $\mathrm{Ext}^1_{\mathcal{O}_X}(\mathcal{C}_{Z/D} \otimes \mathcal{J}_Z^{-1}, \mathcal{O}_Z)$. Because the conormal sheaf is locally free, we have

$$\mathrm{Ext}^1_{\mathcal{O}_Z}(\mathcal{C}_{Z/D} \otimes \mathcal{J}_Z^{-1}, \mathcal{O}_Z) \simeq \mathrm{Ext}^1_{\mathcal{O}_Z}(\mathcal{O}_Z, \mathcal{C}_{Z/D}^\vee \otimes \mathcal{J}_Z) \simeq H^1(Z, \mathcal{N}_{Z/D} \otimes \mathcal{J}_Z)\,.$$

By assumption this vanishes, and we obtain a global splitting. From this one concludes as before.

(3) If Z is the transversal intersection of W and D, then we have two exact, locally split sequences:

$$
\begin{array}{ccccccccc}
0 & & & & & & & & 0 \\
 & \searrow & & & & & & \nearrow & \\
 & & \mathcal{C}_{D/X}|_Z & & & & \mathcal{C}_{Z/W} & & \\
 & & & \searrow & & \nearrow & & & \\
 & & & & \mathcal{C}_{Z/X} & & & & \\
 & & & \nearrow & & \searrow & & & \\
 & & \mathcal{C}_{W/X}|_Z & & \dashrightarrow & & \mathcal{C}_{Z/D} & & \\
 & \nearrow & & & & & & \searrow & \\
0 & & & & & & & & 0\,.
\end{array}
$$

We claim that the dashed arrow is an isomorphism. To see this, write $\mathcal{I}$, $\mathcal{J}$, $\mathcal{K}$ for the defining ideals of Z, D, W. The composition $\mathcal{C}_{W/X}|_Z \to \mathcal{C}_{Z/D}$ is the map $\mathcal{K}/(\mathcal{K}^2+\mathcal{I}\mathcal{K}) \to \mathcal{I}/(\mathcal{I}^2+\mathcal{J})$. From the fact that $\mathcal{I}=\mathcal{J}+\mathcal{K}$, we deduce

- $\mathcal{K}^2+\mathcal{I}\mathcal{K}=\mathcal{K}^2+\mathcal{J}\mathcal{K}$; hence $\mathcal{K}/(\mathcal{K}^2+\mathcal{I}\mathcal{K})=\mathcal{K}/(\mathcal{K}^2+\mathcal{J}\mathcal{K})$;
- $\mathcal{I}^2=\mathcal{J}^2+\mathcal{J}\mathcal{K}+\mathcal{K}^2$; hence $\mathcal{I}^2+\mathcal{J}=\mathcal{K}^2+\mathcal{J}$ and $\mathcal{I}/(\mathcal{I}^2+\mathcal{J})=(\mathcal{J}+\mathcal{K})/(\mathcal{K}^2+\mathcal{J})=\mathcal{K}/(\mathcal{K}^2+\mathcal{J}\cap\mathcal{K})$.

Hence, the map above is an isomorphism if and only if $\mathcal{J}\mathcal{K}=\mathcal{J}\cap\mathcal{K}$, which holds because W cuts D transversally (this is another way of saying that a local equation for $\mathcal{J}$ remains a non-zero divisor in $\mathcal{O}_W$). This provides a canonical splitting $\mathcal{C}_{Z/X}=\mathcal{C}_{D/X}|_Z\oplus\mathcal{C}_{Z/D}$. One concludes as before. ☐

Remark 2.10. In the course of the proof, we saw that the exceptional divisor has the following explicit description: as an affine bundle over Z, its local sections over an open $U\subset Z$ are the $\mathcal{O}_U$-linear maps $\varphi\colon \mathcal{C}_{U/X}\otimes\mathcal{J}_U^{-1}\to\mathcal{O}_U$ such that $\varphi(\varrho)=1$.

2.7 Iterated dilatations

Here we study the behavior of dilatations under iteration. Namely, we will prove that when the center Z of the affine blowup is a transversal intersection $W\cap D$, it can be dilated any finite number of times, and the result of r dilatations can be seen as the dilatation of the single "thickened" center rZ (to be defined below) inside the multiple Cartier divisor rD. To make this precise, we place ourselves in the following situation.

Assumption 2.11. *We are given a cartesian diagram of closed subschemes*

$$
(\mathscr{D})\colon\quad
\begin{array}{ccc}
W & \xhookrightarrow{\;i\;} & X \\
\uparrow & \square & \uparrow \\
Z & \hookrightarrow & D
\end{array}
$$

such that the vertical maps are Cartier divisor inclusions.

In this situation, the subscheme W lifts along the dilatation, and we have control over the ideal sheaf of the lift.

Lemma 2.12. *Let $X':=\mathrm{Bl}_Z^D X$ and $i'\colon W\to X'$ be the lift of i given by the universal property of the dilatation. Let $\mathcal{J}$ (respectively, $\mathcal{K}$) be the ideal sheaf of D (respectively, W) in X. Then i' is a closed immersion with sheaf of ideals $\mathcal{K}\mathcal{O}_{X'}\otimes(\mathcal{J}\mathcal{O}_{X'})^{-1}$.*

Proof. First of all, i' is automatically a monomorphism of schemes, and a proper map because i is proper and $X'\to X$ is separated. Therefore, i' is a closed immersion by [EGA, IV$_4$, Corollaire 18.12.6]. The computation of the ideal sheaf is a local matter, so we can suppose that

$X = \mathrm{Spec}(A)$ is affine and the ideal sheaf $\mathcal{J}$ is generated by a section b. We write $I, J, K \subset A$ for the ideals defining Z, D, W and $t := b^\vee$ for the generator of $\mathcal{J}^{-1}$ dual to b. The assumptions of the lemma mean that $I + K = J + K$ and b is a non-zero divisor in A/K. From Lemma 2.5, we know that X' is the spectrum of the ring $A' = (\oplus_{e\geqslant 0} I^e t^e)/(bt-1)$. In the present local situation, the map $i' \colon W \to X'$ is given by a lifting of $i^\sharp \colon A \to A/K$ to a map $(i')^\sharp \colon A' \to A/K$. Since A' is generated by It as an A-algebra, this map is determined by the formula $(i')^\sharp(it) = j^\sharp(a)$ for all $i \in I$ written $i = ab + k \in I \subset bA + K$. In particular, we see that $(i')^\sharp(Kt) = 0$. Now working modulo $(bt-1) + Kt$ in the ring $C = \oplus_{e\geqslant 0} I^e t^e$, we have $It \subset btA + Kt \equiv A + Kt \equiv A$, which sits in the degree 0 part of C, whence a surjection $A \hookrightarrow C \to A'/KtA'$. Moreover, $bKt \equiv K$ implies that K in degree 0 belongs to the ideal generated by Kt; hence finally $A'/KtA' \xrightarrow{\sim} A/K$, as desired. □

Continuing with the same situation, we can construct a sequence of dilatations

$$\cdots \longrightarrow X_r \longrightarrow X_{r-1} \longrightarrow \cdots \longrightarrow X_1 \longrightarrow X_0 = X$$

and closed immersions $i_r \colon W \hookrightarrow X_r$, as follows. We let $\mathscr{D}_0 = \mathscr{D}$ (the cartesian square of Assumption 2.11), $X_0 = X$, $D_0 = D$ and $i_0 = i \colon W \hookrightarrow X_0$. Let $u_1 \colon X_1 \to X_0$ be the dilatation of Z in (X_0, D_0) and D_1 the preimage of D_0 in X_1. By Lemma 2.12, there is a closed immersion $i_1 \colon W \to X_1$ lifting i_0 and $(i_1)^{-1}(D_1) = (i_1)^{-1}(u_1^{-1}(D)) = i^{-1}(D) = Z$. That is, we again have a cartesian diagram

$$(\mathscr{D}_1)\colon \quad \begin{array}{ccc} W & \xhookrightarrow{\ i_1\ } & X_1 \\ \uparrow & \square & \uparrow \\ Z & \hookrightarrow & D_1 , \end{array}$$

where the vertical maps are Cartier divisor inclusions. Our sequence is obtained by iterating this construction.

Lemma 2.13. *Under Assumption 2.11, denote by $\mathcal{I}$, $\mathcal{J}$, $\mathcal{K}$ the ideal sheaves of Z, D, W in $\mathcal{O}_X$. Let rD be the rth multiple of D as a Cartier divisor and $rZ := W \cap rD$. Then the dilatation $v_r \colon X'_r \to X$ of (rZ, rD) in X is characterized as being universal among all morphisms $V \to X$ with the following two properties:*

1. *The sheaf $\mathcal{J}\mathcal{O}_V$ is invertible.*
2. *The sheaf $\mathcal{K}\mathcal{O}_V$ is divisible by $\mathcal{J}^r\mathcal{O}_V$; that is, we have $\mathcal{K}\mathcal{O}_V = \mathcal{J}^r\mathcal{O}_V \cdot \mathcal{K}_r$ for some sheaf of ideals $\mathcal{K}_r \subset \mathcal{O}_V$.*

Proof. The defining properties of the dilatation v_r say that it is universal among morphisms $V \to X$ such that $rZ \times_X V = rD \times_X V$ is a Cartier divisor. Since the ideal sheaves of rD and rZ are $\mathcal{J}^r$ and $\mathcal{J}^r + \mathcal{K}$, respectively, these properties mean that the ideal $\mathcal{J}^r\mathcal{O}_V$ is invertible and $\mathcal{J}^r\mathcal{O}_V = (\mathcal{J}^r + \mathcal{K})\mathcal{O}_V$. But the properties "$\mathcal{J}$ is invertible" and "$\mathcal{J}^r$ is invertible" are equivalent, as follows from the isomorphism between the blowup of $\mathcal{J}$ and the blowup of $\mathcal{J}^r$; see [EGA, II, Définition 8.1.3]. This takes care of property (1). Besides, $\mathcal{J}^r\mathcal{O}_V = (\mathcal{J}^r + \mathcal{K})\mathcal{O}_V$ means that $\mathcal{K}\mathcal{O}_V \subset \mathcal{J}^r\mathcal{O}_V$, and in the situation where $\mathcal{J}\mathcal{O}_V$ is invertible, this is the same as saying that $\mathcal{K}\mathcal{O}_V = \mathcal{J}^r\mathcal{O}_V \cdot \mathcal{K}_r$ with $\mathcal{K}_r = (\mathcal{K}\mathcal{O}_V : \mathcal{J}^r\mathcal{O}_V)$ as an ideal of $\mathcal{O}_V$. (Note that in this case, $(\mathcal{K}\mathcal{O}_V : \mathcal{J}^r\mathcal{O}_V) \simeq (\mathcal{K} \otimes \mathcal{J}^{-r})\mathcal{O}_V$ as an $\mathcal{O}_V$-module.) This takes care of property (2). □

We finish this subsection with a result which will be crucial for the proof of Theorem 4.3. Also note that this gives a positive answer to [DHdS18, Question 5.19].

Proposition 2.14. *In the situation of Assumption 2.11, let*

$$\cdots \longrightarrow X_r \longrightarrow X_{r-1} \longrightarrow \cdots \longrightarrow X_1 \longrightarrow X_0 = X$$

be the sequence of dilatations constructed above. Let rD be the rth multiple of D as a Cartier divisor and $rZ := W \cap rD$. Then the composition $X_r \to X$ is the dilatation of (rZ, rD) inside X.

Proof. According to Lemma 2.13, dilating (rZ, rD) means making $\mathcal{J}$ invertible and $\mathcal{K}$ divisible by $\mathcal{J}^r$, all of this in a universal way. This can be done by the steps

- make $\mathcal{J}$ invertible, and make $\mathcal{K}_0 = \mathcal{K}$ divisible by $\mathcal{J}$;
- keep $\mathcal{J}$ invertible, and make $\mathcal{K}_1 := (\mathcal{K}_0 : \mathcal{J}) \simeq \mathcal{K} \otimes \mathcal{J}^{-1}$ divisible by $\mathcal{J}$;
- keep $\mathcal{J}$ invertible, and make $\mathcal{K}_2 := (\mathcal{K}_1 : \mathcal{J}) \simeq \mathcal{K}_1 \otimes \mathcal{J}^{-1} \simeq \mathcal{K} \otimes \mathcal{J}^{-2}$ divisible by $\mathcal{J}$; etc., and finally
- keep $\mathcal{J}$ invertible, and make $\mathcal{K}_{r-1} := (\mathcal{K}_{r-2} : \mathcal{J}) \simeq K \otimes \mathcal{J}^{-(r-1)}$ divisible by $\mathcal{J}$.

In view of Lemma 2.12, these steps amount to

- dilate Z in (X, D);
- dilate Z in (X_1, D_1);
- dilate Z in (X_2, D_2); etc., until
- dilate Z in (X_{r-1}, D_{r-1}).

In this way, we see the equivalence between the dilatation of the thick pair (rZ, rD) and the sequence of dilatations of Z constructed after Assumption 2.11. □

2.8 Flatness and smoothness

Flatness and smoothness properties of blowups are discussed in [EGA, IV_4, § 19.4]. Here we need slightly different versions. We proceed with the notation from § 2.1. We assume further that there exists a scheme S under X together with a locally principal closed subscheme $S_0 \subset S$ fitting into a commutative diagram of schemes

$$\begin{array}{ccccc} Z & \longrightarrow & D & \longrightarrow & X \\ & \searrow & \downarrow & & \downarrow \\ & & S_0 & \longrightarrow & S\,, \end{array} \tag{2.10}$$

where the square is cartesian; that is, $D \to X_0 := X \times_S S_0$ is an isomorphism.

Lemma 2.15. *Assume that S_0 is an effective Cartier divisor in S. Let $f\colon Y \to S$ be a morphism of schemes such that $Y_0 := Y \times_S S_0$ is a Cartier divisor in Y. Assume that both restrictions of f above $S \backslash S_0$ and S_0 are flat. If one of the conditions*

(1) *S and Y are locally noetherian,*

(2) *$Y \to S$ is locally of finite presentation*

holds, then f is flat.

Proof. Since by assumption u is flat at all points above the open subscheme $S \backslash S_0$, it is enough to prove that u is flat at all points $y \in Y$ lying above a point $s \in S_0$.

In case (1), the local criterion for flatness [EGA, III_1, Chapter 0, § 10.2.2] (cf. also [Sta22, 00ML]) shows that $\mathcal{O}_{S,s} \to \mathcal{O}_{Y,y}$ is flat, and we are done.

In case (2), we may localize around y and s and hence assume that Y and S are affine and small enough so that the ideal sheaf of S_0 in S is generated by an element $f \in A = \Gamma(S, \mathcal{O}_S)$. We write $A = \mathrm{colim} A_i$ as the union of its subrings of finite type over $\mathbb{Z}$. In each A_i, the element f is a non-zero divisor. Write $S_i := \mathrm{Spec}(A_i)$ and $S_{i,0} := \mathrm{Spec}(A_i/f)$. Using the results of [EGA, IV$_3$, § 8] on limits of schemes, we can find an index i and a morphism of finite presentation $Y_i \to S_i$ such that $Y_{i,0} := Y_i \times_{S_i} S_{i,0}$ is a Cartier divisor in Y_i and $Y_{i,0} \to S_{i,0}$ is flat, and such that the situation (S, S_0, Y) is a pullback of $(S_i, S_{i,0}, Y_i)$ by $S \to S_i$. More in detail, using the following results, we find indices which we increase at each step in order to have all the conditions simultaneously met: use [EGA, IV$_3$, Théorème 8.8.2] to find morphisms $Y_i \to S_i$ and $Y_{i,0} \to S_{i,0}$, use [EGA, IV$_3$, Corollaire 8.8.2.5] to make $Y_i \times_{S_i} S_{i,0}$ and $Y_{i,0}$ isomorphic over $S_{i,0}$, use [EGA, IV$_3$, Théorème 11.2.6] to ensure $Y_{i,0} \to S_{i,0}$ flat, and use [EGA, IV$_3$, Proposition 8.5.6] to ensure that f is a non-zero divisor in $\mathcal{O}_{Y_i}$, that is, $Y_{i,0} \subset Y_i$ is a Cartier divisor. Since A_i is noetherian, for $(S_i, S_{i,0}, Y_i)$ we can apply case (1), and the result follows by base change. □

For our conventions on regular immersions, the reader is referred back to § 2.3.

Proposition 2.16. *Assume that S_0 is an effective Cartier divisor on S.*

(1) *If $Z \subset D$ is regular, then $\mathrm{Bl}_Z^D X \to X$ is of finite presentation.*

(2) *If $Z \subset D$ is regular, the fibers of $\mathrm{Bl}_Z^D X \times_S S_0 \to S_0$ are connected (respectively, irreducible, geometrically connected, geometrically irreducible) if and only if the fibers of $Z \to S_0$ are.*

(3) *If $X \to S$ is flat and if, moreover, one of the conditions*

  (i) *$Z \subset D$ is regular, $Z \to S_0$ is flat and S, X are locally noetherian;*

  (ii) *$Z \subset D$ is regular, $Z \to S_0$ is flat and $X \to S$ is locally of finite presentation;*

  (iii) *the local rings of S are valuation rings*

  holds, then $\mathrm{Bl}_Z^D X \to S$ is flat.

(4) *If both $X \to S$ and $Z \to S_0$ are smooth, then $\mathrm{Bl}_Z^D X \to S$ is smooth.*

Proof. For item (1), recall that the blowup of a regularly immersed subscheme has an explicit structure, where generating relations between local generators of the blown up ideal are the obvious ones, in finite number; see [Sta22, 0BIQ]. This shows that $\mathrm{Bl}_Z^D X \to X$ is locally of finite presentation. Being also affine, it is of finite presentation.

For item (2) about connectedness and irreducibility, recall from Proposition 2.9(1) that the exceptional divisor is an affine bundle over Z. In particular, it is a submersion, so that the elementary topological lemma [EGA, IV$_2$, Lemme 4.4.2] (cf. also [Sta22, 0377]) gives the assertion.

For item (3)(i)–(ii), we apply Lemma 2.15 to $Y := \mathrm{Bl}_Z^D X$. The preimage of S_0 under the affine blowup $f \colon Y \to S$ is equal to $\mathrm{Bl}_Z^D X \times_X D = \mathrm{Bl}_Z^D X \times_X Z$ by Lemma 2.4. This implies that the restriction $f|_{f^{-1}(S \backslash S_0)}$ is equal to $X \backslash D \to S \backslash S_0$, which is flat by assumption. It remains to show flatness in points of $\mathrm{Bl}_Z^D X$ lying over S_0. For this, note that the restriction $f|_{f^{-1}(S_0)}$ factors as

$$\mathrm{Bl}_Z^D X \times_X D = \mathrm{Bl}_Z^D X \times_X Z \longrightarrow Z \longrightarrow S_0\,, \tag{2.11}$$

where the first map is smooth by Proposition 2.9 and the second map is flat by assumption. Then Lemma 2.15 applies and gives the flatness of $Y \to S$.

For item (3)(iii), we can work locally at a point of S and hence assume that S is the spectrum of a valuation ring R. We use the fact that flat R-modules are the same as torsion-free R-modules. Locally over an open subscheme $\mathrm{Spec}(B) \subset X$, the Rees algebra $\mathrm{Bl}_I B = B[It]$ is a subalgebra of the polynomial algebra $B[t]$, and the affine blowup algebra is a localization of the latter. It follows that if B is R-torsion-free, then the affine blowup algebra also is; hence it is flat.

For item (4), assume that $X \to S$ and $Z \to S_0$ are smooth. Then item (4) follows from [EGA, IV_4, Théorème 17.5.1] (cf. also [Sta22, 01V8]) once we know that $\text{Bl}_Z^D X \to S$ is locally of finite presentation, flat and has smooth fibers. Applying [EGA, IV_4, Proposition 19.2.4] to the commutative triangle in (2.10), we see that $Z \subset D$ is regularly immersed. Therefore, $\text{Bl}_Z^D X \to S$ is flat and locally of finite presentation by items (1) and (3). The smoothness of the fiber over points in $S \backslash S_0$ is clear and follows from (2.11) over points in S_0. This proves item (4). □

3. Néron blowups

We extend the theory of Néron blowups of affine group schemes over discrete valuation rings as in [Ana73, § 2.1.2], [WW80, § 1, p. 551], [BLR90, § 3.2], [Yu15, § 2.8] and [PY06, §§ 7.2–7.4] to group schemes over arbitrary bases.

3.1 Definition

Let S be a scheme, and let $G \to S$ be a group scheme. Let $S_0 \subset S$ be a locally principal, closed subscheme, and consider the base change $G_0 := G \times_S S_0$. Let $H \subset G_0$ be a closed subgroup scheme over S_0. Let $\mathcal{G} := \text{Bl}_H^{G_0} G \to G$ be the dilatation of G in H along the locally principal, closed subscheme $G_0 \subset G$ in the sense of Definition 2.1. In this case, we also call $\mathcal{G} \to S$ the *Néron blowup of G in H (along S_0)*. We denote by $\mathcal{G}_0 := \mathcal{G} \times_S S_0 \to S_0$ its exceptional divisor.

Let $\text{Sch}_S^{S_0\text{-reg}}$ be the full subcategory of schemes $T \to S$ such that $T_0 := T \times_S S_0$ defines an effective Cartier divisor on T. By Lemma 2.4, the structure morphism $\mathcal{G} \to S$ defines an object in $\text{Sch}_S^{S_0\text{-reg}}$.

Lemma 3.1. *Let $\mathcal{G} \to S$ be the Néron blowup of G in H along S_0.*

(1) *The scheme $\mathcal{G} \to S$ represents the contravariant functor $\text{Sch}_S^{S_0\text{-reg}} \to \text{Sets}$ given for $T \to S$ by the set of all S-morphisms $T \to G$ such that the induced morphism $T_0 \to G_0$ factors through $H \subset G_0$.*

(2) *The map $\mathcal{G} \to G$ is affine. Its restriction over $S \backslash S_0$ induces an isomorphism $\mathcal{G}|_{S \backslash S_0} \cong G|_{S \backslash S_0}$. Its restriction over S_0 factors as $\mathcal{G}_0 \to H \subset G_0$.*

Proof. Item (1) is a reformulation of Proposition 2.6, and item (2) is immediate from Lemmas 2.3 and 2.4. □

By virtue of Lemma 3.1(1), the (forgetful) map $\mathcal{G} \to G$ defines a subgroup functor when restricted to the category $\text{Sch}_S^{S_0\text{-reg}}$. As $\mathcal{G} \to S$ is an object in $\text{Sch}_S^{S_0\text{-reg}}$, it is a group object in this category. Here we note that products in the category $\text{Sch}_S^{S_0\text{-reg}}$ exist and are computed as $\text{Bl}_{S_0}(X_1 \times_S X_2)$ by the universal property of the blowup [Sta22, 085U]. This is the closed subscheme of $X_1 \times_S X_2$ which is locally defined by the ideal of a-torsion elements for a local equation a of S_0 in S. In particular, if $\mathcal{G} \to S$ is flat, then it is equipped with the structure of a group scheme such that $\mathcal{G} \to G$ is a morphism of S-group schemes.

3.2 Properties

We continue with the notation of § 3.1. Additionally, assume that S_0 is an effective Cartier divisor in S. Again recall our conventions on regular immersions from § 2.3. The following summarizes the main properties of Néron blowups.

Theorem 3.2. *Let $\mathcal{G} \to G$ be the Néron blowup of G in H along S_0.*

1. *If $G \to S$ is (quasi-)affine, then $\mathcal{G} \to S$ is (quasi-)affine.*
2. *If $G \to S$ is (locally) of finite presentation and $H \subset G_0$ is regular, then $\mathcal{G} \to S$ is (locally) of finite presentation.*
3. *If $H \to S_0$ has connected fibers and $H \subset G_0$ is regular, then $\mathcal{G} \times_S S_0 \to S_0$ has connected fibers.*
4. *Assume that $G \to S$ is flat and one of the following holds:*
   (i) *The scheme $H \subset G_0$ is regular, $H \to S_0$ is flat, and S, G are locally noetherian.*
   (ii) *The scheme $H \subset G_0$ is regular, $H \to S_0$ is flat, and $G \to S$ is locally of finite presentation.*
   (iii) *The local rings of S are valuation rings.*

   Then $\mathcal{G} \to S$ is flat.
5. *If both $G \to S$ and $H \to S_0$ are smooth, then $\mathcal{G} \to S$ is smooth.*
6. *Assume that $\mathcal{G} \to S$ is flat. If $S' \to S$ is a scheme such that $S'_0 := S' \times_S S_0$ is an effective Cartier divisor on S', then the base change $\mathcal{G} \times_S S' \to S'$ is the Néron blowup of $G \times_S S'$ in $H \times_{S_0} S'_0$ along S'_0.*

In items (4) and (5), the map $\mathcal{G} \to S$ is a group scheme.

Proof. The map $\mathcal{G} \to G$ is affine by Lemma 3.1(2) which implies item (1). Items (2) through (5) are a direct transcription of Proposition 2.16, noting for item (3) that for schemes equipped with a section, the properties "with connected fibers" and "with geometrically connected fibers" are equivalent [EGA, IV_2, Corollaire 4.5.14] (cf. also [Sta22, 04KV]). Item (6) follows from Lemma 2.7, noting that the preimage of S'_0 under the flat map $\mathcal{G} \times_S S' \to S'$ defines an effective Cartier divisor. □

Example 3.3. Let $G_0 \to \mathrm{Spec}(\mathbb{Z})$ be a split reductive group scheme with connected fibers. Consider the base change $G := G_0 \times_{\mathbb{Z}} \mathbb{A}^1_{\mathbb{Z}}$ to the affine line. Let $S_0 = \mathrm{Spec}(\mathbb{Z})$, considered as the effective Cartier divisor defined by the zero section of $S = \mathbb{A}^1_{\mathbb{Z}}$. Let $P_0 \subset G_0$ be a parabolic subgroup. By Theorem 3.2(3) and (5), the Néron blowup $\mathcal{G} \to \mathbb{A}^1_{\mathbb{Z}}$ of G in P_0 is a smooth, affine group scheme with connected fibers. In fact, it is an easy special case of the group schemes constructed in [PZ13, § 4] and [Lou23, § 2]. Let ϖ denote a global coordinate on $\mathbb{A}^1_{\mathbb{Z}}$. By Theorem 3.2(6) the base changes have the following properties:

1. If k is any field, then $\mathcal{G}(k[\![\varpi]\!])$ is the subgroup of those elements in $G(k[\![\varpi]\!]) = G_0(k[\![\varpi]\!])$ whose reduction modulo ϖ lies in $P_0(k)$.
2. If p is any prime number, then $\mathcal{G}_{\varpi\mapsto p}(\mathbb{Z}_p)$ is subgroup of those elements in $G_{\varpi\mapsto p}(\mathbb{Z}_p) = G_0(\mathbb{Z}_p)$ whose reduction modulo p lies in $P_0(\mathbb{F}_p)$.

In other words, the respective base changes $\mathcal{G} \times_{\mathbb{A}^1_{\mathbb{Z}}} \mathrm{Spec}(k[\![\varpi]\!])$ and $\mathcal{G} \times_{\mathbb{A}^1_{\mathbb{Z}}, \varpi\mapsto p} \mathrm{Spec}(\mathbb{Z}_p)$ are parahoric group schemes in the sense of [BT84]; cf. also [PZ13, Corollary 4.2] and [Lou23, § 2.6]. Thus, the Néron blowup $\mathcal{G} \to \mathbb{A}^1_{\mathbb{Z}}$ can be viewed as a family of parahoric group schemes.

3.3 Group structure on the exceptional divisor

We continue with the notation of § 3.1. In this subsection, we take up the description of the exceptional divisor from Proposition 2.9, in the context of group schemes: we explain the interplay between the ambient group structure and the vector bundle structure on the exceptional divisor.

From here on, for any scheme X, we write $\Gamma(X) = H^0(X, \mathcal{O}_X)$ for its ring of global functions.

Lemma 3.4. *Assume that S is affine. Let $\mathcal{G}$ be an S-group scheme and $\mathcal{G}_0 := \mathcal{G} \times_S S_0$. Denote by $i\colon \mathcal{G}_0 \hookrightarrow \mathcal{G}$ the closed immersion and $\mathcal{K}$ the corresponding ideal sheaf. Let $m, \mathrm{pr}_1, \mathrm{pr}_2 \colon \mathcal{G} \times_S \mathcal{G} \to \mathcal{G}$ be the multiplication and the projections, with corresponding morphisms*

$$m^\sharp, \mathrm{pr}_1^\sharp, \mathrm{pr}_2^\sharp \colon \Gamma(\mathcal{G}) \longrightarrow \Gamma(\mathcal{G} \times_S \mathcal{G}),$$
$$(i \times i)^\sharp \colon \Gamma(\mathcal{G} \times_S \mathcal{G}) \longrightarrow \Gamma(\mathcal{G}_0 \times_{S_0} \mathcal{G}_0).$$

If $\delta := m^\sharp - \mathrm{pr}_1^\sharp - \mathrm{pr}_2^\sharp$, we have $\delta\big(H^0(\mathcal{G}, \mathcal{K})\big) \subset \ker\big((i \times i)^\sharp\big)$.

Proof. Each of the maps $f \in \{m, \mathrm{pr}_1, \mathrm{pr}_2\}$ fits in a commutative diagram

$$\begin{array}{ccc} \mathcal{G}_0 \times_{S_0} \mathcal{G}_0 & \xrightarrow{i \times i} & \mathcal{G} \times_S \mathcal{G} \\ {\scriptstyle f}\big\downarrow & & \big\downarrow{\scriptstyle f} \\ \mathcal{G}_0 & \xrightarrow{\quad i \quad} & \mathcal{G}\,. \end{array}$$

Since $H^0(\mathcal{G}, \mathcal{K})$ is the kernel of the map $i^\sharp \colon \Gamma(\mathcal{G}) \to \Gamma(\mathcal{G}_0)$, by taking global sections, we obtain $\big((i \times i)^\sharp f^\sharp\big)\big(H^0(\mathcal{G}, \mathcal{K})\big) = 0$. $\square$

We recall that for a group scheme $G \to S$ with unit section $e\colon S \to G$, the Lie algebra $\mathrm{Lie}(G/S)$ is the S-group scheme $\mathbb{V}\big(e^*\Omega^1_{G/S}\big)$.

Theorem 3.5. *With the notation of §3.1, assume that $G \to S$ is flat, locally finitely presented and $H \to S_0$ is flat, regularly immersed in G_0. Let $\mathcal{G} \to G$ be the dilatation of G in H with exceptional divisor $\mathcal{G}_0 := \mathcal{G} \times_S S_0$. Let $\mathcal{J}$ be the ideal sheaf of G_0 in G and $\mathcal{J}_H := \mathcal{J}|_H$. Let V be the restriction of the normal bundle $\mathbb{V}\big(\mathcal{C}_{H/G_0} \otimes \mathcal{J}_H^{-1}\big) \to H$ along the unit section $e_0 \colon S_0 \to H$.*

1. *Locally over S_0, there is an exact sequence of S_0-group schemes $1 \to V \to \mathcal{G}_0 \to H \to 1$.*
2. *Assume given a lifting of H to a flat S-subgroup scheme of G. Then there is globally an exact, canonically split sequence $1 \to V \to \mathcal{G}_0 \to H \to 1$.*
3. *If $G \to S$ is smooth, separated and $\mathcal{G} \to G$ is the dilatation of the unit section of G, there is a canonical isomorphism of smooth S_0-group schemes $\mathcal{G}_0 \xrightarrow{\sim} \mathrm{Lie}(G_0/S_0) \otimes \mathrm{N}_{S_0/S}^{-1}$, where $\mathrm{N}_{S_0/S}$ is the normal bundle of S_0 in S.*

Proof. (1) Let $\mathcal{F} = \mathcal{C}_{H/G_0} \otimes \mathcal{J}_H^{-1}$. According to Proposition 2.9(1), locally over S_0 we have an isomorphism of S_0-schemes $\psi\colon \mathcal{G} \times_G H \xrightarrow{\sim} \mathbb{V}(\mathcal{F})$. Let $K = \ker(\mathcal{G}_0 \to H)$. To obtain the exact sequence of the statement, it is enough to prove that the restriction of ψ along the unit section $e_0 \colon S_0 \to H$ is an isomorphism of S_0-group schemes $e_0^*\psi \colon K \xrightarrow{\sim} V$. For this we may localize further around a point of S_0, hence assume that S and S_0 are affine and small enough so that $\mathcal{J}$ is trivial. Proving that $e_0^*\psi$ is a morphism of groups is equivalent to checking an equality between two morphisms $K \times_{S_0} K \to V$. Since $V = \mathrm{Spec}(\mathrm{Sym}(\mathcal{F}_0))$, where $\mathcal{F}_0 := e_0^*\mathcal{F}$, this is the same as checking an equality between two maps of $\Gamma(S_0)$-modules $H^0(S_0, \mathcal{F}_0) \to \Gamma(K \times_{S_0} K)$. More precisely, since K is affine, we have $\Gamma(K \times_{S_0} K) = \Gamma(K) \otimes_{\Gamma(S_0)} \Gamma(K)$, and what we have to check is that $m^\sharp(x) = x \otimes 1 + 1 \otimes x$ for all $x \in H^0(S_0, \mathcal{F}_0)$, with m the multiplication of K. That is, we want to prove that $\delta\big(H^0(S_0, \mathcal{F}_0)\big) = 0$, where $\delta\colon \Gamma(K) \to \Gamma(K \times_{S_0} K)$ is defined by $\delta = m^\sharp - \mathrm{pr}_1^\sharp - \mathrm{pr}_2^\sharp$. In order to prove this, let $\mathcal{I}$ be the ideal sheaf of the closed immersion $H \hookrightarrow G$, and let $f^*\mathcal{I}$ be its preimage as a module under the dilatation morphism $f\colon \mathcal{G} \to G$.

Consider the closed immersions $K \hookrightarrow \mathcal{G}_0$ and $i\colon \mathcal{G}_0 \hookrightarrow \mathcal{G}$ and the diagram

$$\begin{array}{ccccc} H^0(\mathcal{G}, f^*\mathcal{I}) & \longrightarrow & \Gamma(\mathcal{G}) & \xrightarrow{\ \delta\ } & \Gamma(\mathcal{G} \times_S \mathcal{G}) \\ & & & & \big\downarrow{\scriptstyle (i\times i)^\sharp} \\ \Big\downarrow & & \Big\downarrow & & \Gamma(\mathcal{G}_0 \times_{S_0} \mathcal{G}_0) \\ & & & & \big\downarrow \\ H^0(S_0, \mathcal{F}_0) & \longrightarrow & \Gamma(K) & \xrightarrow{\ \delta\ } & \Gamma(K \times_{S_0} K)\,. \end{array}$$

We claim that the vertical map $H^0(\mathcal{G}, f^*\mathcal{I}) \to H^0(S_0, \mathcal{F}_0)$ is surjective. To prove this, let $e_{\mathcal{G}}\colon S \to \mathcal{G}$ be the unit section of $\mathcal{G}$ and $j\colon S_0 \hookrightarrow S$ be the closed immersion, and decompose the said map as follows:

$$H^0(\mathcal{G}, f^*\mathcal{I}) \xrightarrow{e_{\mathcal{G}}^*} H^0(S, e^*\mathcal{I}) \xrightarrow{j^*} H^0(S_0, j^*e^*\mathcal{I}) \longrightarrow H^0(S_0, \mathcal{F}_0)\,.$$

The first map is surjective because it has the section σ^*, where $\sigma\colon \mathcal{G} \to S$ is the structure map. The second map is surjective because it is obtained by taking global sections on the affine scheme S of the surjective map of sheaves $e^*\mathcal{I} \to j_*j^*e^*\mathcal{I}$. To show that the third map is surjective, start from the surjection of sheaves $\mathcal{I}|_H \to \mathcal{F}$. Since pullback is right exact, this gives rise to a surjection $j^*e^*\mathcal{I} = e_0^*\mathcal{I}|_H \to e_0^*\mathcal{F} = \mathcal{F}_0$. Taking global sections on the affine scheme S_0, we obtain the desired surjection. Now let $\mathcal{I}\mathcal{O}_{\mathcal{G}} = f^{-1}\mathcal{I} \cdot \mathcal{O}_{\mathcal{G}}$ be the preimage of $\mathcal{I}$ as an ideal. Note that by the universal property of the dilatation, we have $\mathcal{I}\mathcal{O}_{\mathcal{G}} = \mathcal{J}\mathcal{O}_{\mathcal{G}} =: \mathcal{K}$. Therefore, according to Lemma 3.4, we have $\delta\big(H^0(\mathcal{G}, \mathcal{I}\mathcal{O}_{\mathcal{G}})\big) = \delta\big(H^0(\mathcal{G}, \mathcal{K})\big) \subset \ker\big((i \times i)^\sharp\big)$. Precomposing with the surjection $f^*\mathcal{I} \to \mathcal{I}\mathcal{O}_{\mathcal{G}}$, we find that $H^0(\mathcal{G}, f^*\mathcal{I})$ is mapped into $\ker(i \times i)^\sharp$ by δ. As a result, $H^0(\mathcal{G}, f^*\mathcal{I})$ goes to zero in $\Gamma(K \times_{S_0} K)$. Since $H^0(\mathcal{G}, f^*\mathcal{I}) \to H^0(S_0, \mathcal{F}_0)$ is surjective, the commutativity of the diagram implies that $\delta\big(H^0(S_0, \mathcal{F}_0)\big) = 0$ in $\Gamma(K \times_{S_0} K)$, as desired. Hence $e_0^*\psi\colon K \to V$ is an isomorphism of groups. This proves item (1).

(2) If $\tilde{H} \subset G$ is a flat S-subgroup scheme lifting H, we have a transversal intersection $H = \tilde{H} \cap G_0$. By Proposition 2.9(3), the preceding construction of the short exact sequence can be performed globally over S_0. Moreover, by the universal property of the dilatation, the map $\tilde{H} \to G$ lifts to a map $\tilde{H} \to \mathcal{G}$. In restriction to S_0, this splits the short exact sequence previously obtained.

(3) Finally, if $G \to S$ is smooth, the unit section is a regular immersion with conormal sheaf $\omega_{G/S} = e^*\Omega^1_{G/S}$. In restriction to S_0 the group V is the Lie algebra, whence the canonical isomorphism $\mathcal{G}_0 \xrightarrow{\sim} \mathrm{Lie}(G_0/S_0) \otimes \mathrm{N}^{-1}_{S_0/S}$. □

Remark 3.6. In the case where the base scheme is the spectrum of a discrete valuation ring, Theorem 3.5 recovers [WW80, Theorems 1.5 and 1.7] (in a precised form).

Remark 3.7. In the situation of Theorem 3.5(2), the group H acts by conjugation on $V = \mathbb{V}\big(e_0^*\mathcal{C}_{H/G_0} \otimes \mathcal{J}_{S_0}^{-1}\big)$. We checked on examples that this additive action is linear and is in fact none other than the "adjoint" representation of H on its normal bundle as in [SGA3-1, Exposé I, Proposition 6.8.6]. In fact, when the base scheme is the spectrum of a discrete valuation ring, this is proved in [DHdS18, Proposition 2.7].

3.4 Néron blowups as syntomic sheaves

We continue with the notation of § 3.1. Additionally, assume that $j\colon S_0 \hookrightarrow S$ is an effective Cartier divisor, that $G \to S$ is a flat, locally finitely presented group scheme and that $H \subset G_0 := G \times_S S_0$ is a flat, locally finitely presented, closed S_0-subgroup scheme. In this context, there is another viewpoint on the dilatation $\mathcal{G}$ of G in H, namely as the kernel of a certain map of syntomic sheaves.

To explain this, let $f\colon G_0 \to G_0/H$ be the morphism to the fppf quotient sheaf, which by Artin's theorem (see [Art74, Corollary 6.3] and [Sta22, 04S6]) is representable by an algebraic space. By the structure theorem for algebraic group schemes (see [SGA3-1, Exposé VII$_{\mathrm{B}}$, Corollaire 5.5.1]), the morphisms $G \to S$ and $H \to S_0$ are syntomic. Since $f\colon G_0 \to G_0/H$ makes G_0 an H-torsor, it follows that f is syntomic too.

Lemma 3.8. *Let S_{syn} be the small syntomic site of S. Let $\eta\colon G \to j_*j^*G$ be the adjunction map in the category of sheaves on S_{syn}, and consider the composition $v = (j_*f) \circ \eta$:*

$$G \xrightarrow{\ \eta\ } j_*j^*G = j_*G_0 \xrightarrow{\ j_*f\ } j_*(G_0/H)\,.$$

Then the dilatation $\mathcal{G} \to G$ is the kernel of v. More precisely, we have an exact sequence of sheaves of pointed sets in S_{syn}:

$$1 \longrightarrow \mathcal{G} \longrightarrow G \xrightarrow{\ v\ } j_*(G_0/H) \longrightarrow 1\,.$$

If $G \to S$ and $H \to S_0$ are smooth, then the sequence is exact as a sequence of sheaves on the small étale site of S.

Proof. That $\mathcal{G} \to G$ is the kernel of v follows directly from the universal property of the dilatation, restricted to syntomic S-schemes. It remains to prove that the map of sheaves v is surjective. It is enough to prove that both maps η and j_*f are surjective. For η, this is because if $T \to S$ is a syntomic morphism, any point $t\colon T \to j_*G_0$ lifts tautologically to G after the syntomic refinement $T' = G \times_S T \to T$. For j_*f, we start from a syntomic morphism $T \to S$ and a point $t\colon T \to j_*(G_0/H)$, that is, a morphism $T_0 \to G_0/H$. Using that $G_0 \to G_0/H$ is syntomic, we can find as before a syntomic refinement $T_0' \to T_0$ and a lift $T_0' \to G_0$. Using that syntomic coverings lift across closed immersions (see [Sta22, 04E4]), there is a syntomic covering $T'' \to T$ such that T_0'' refines T_0'. This provides a lift of t to j_*G_0.

Finally, if $G \to S$ and $H \to S_0$ are smooth, the existence of étale sections for smooth morphisms [EGA, IV$_4$Corollaire 17.16.3] and the possibility to lift étale coverings across closed immersions (see [Sta22, 04E4] again) show that the sequence is also exact in the étale topology. □

4. Applications

Here we give two applications in cohomological degree 0 and 1 of the theory developed so far: integral points and torsors. In § 4.1, we consider integral points of Néron blowups and discuss the isomorphism relating the graded pieces of the congruent filtration of G to the graded pieces of its Lie algebra $\mathfrak{g}$. In § 4.2, we discuss torsors under Néron blowups and apply this in § 4.2.1 to the construction of level structures on moduli stacks of G-bundles on curves, and in § 4.2.2 to the construction of integral models of moduli stacks of shtukas.

4.1 Integral points and the congruent Moy–Prasad isomorphism

In this subsection, we prove an isomorphism describing the graded pieces of the filtration by congruence subgroups on the integral points of reductive group schemes. For the benefit of the interested reader, we provide comments on the literature on this topic in Remark 4.4 below.

We start with the following lemma.

Lemma 4.1. *Let $\mathcal{O}$ be a ring and $\pi \subset \mathcal{O}$ an invertible ideal such that $(\mathcal{O}, \pi)$ is a henselian pair. Let G be a smooth, separated $\mathcal{O}$-group scheme and $\mathcal{G} \to G$ the dilatation of the trivial subgroup over $\mathcal{O}/\pi$. If either $\mathcal{O}$ is local or G is affine, then the exact sequence of Lemma 3.8 induces an exact sequence of groups*

$$1 \longrightarrow \mathcal{G}(\mathcal{O}) \longrightarrow G(\mathcal{O}) \longrightarrow G(\mathcal{O}/\pi) \longrightarrow 1\,.$$

Proof. Write $S = \mathrm{Spec}(\mathcal{O})$ and $S_0 = \mathrm{Spec}(\mathcal{O}/\pi)$, and set $G_0 = G \times_S S_0$, $\mathcal{G}_0 = \mathcal{G} \times_S S_0$. Consider the short exact sequence of Lemma 3.8 on the étale site, and take the global sections over S. It is then enough to prove that the map $G(\mathcal{O}) \to G(\mathcal{O}/\pi)$ is surjective. For this start with an $(\mathcal{O}/\pi)$-point of G, that is, a section $u_0 \colon S_0 \to G_0$ to the map $G_0 \to S_0$. If either $\mathcal{O}$ is local or G is affine, u_0 factors through an open affine subscheme $U \subset G$. In this situation, the classical existence result for lifting of sections for smooth schemes over a henselian local ring (as in, for example, [BLR90, § 2.3, Proposition 5]) extends to henselian pairs; see [Gru72, Théorème 1.8]. In this way, we see that u_0 lifts to a section u of $G \to S$. □

In the proof of Theorem 4.3, we also use the following simple observation, which allows us to handle Lie algebras and dilatations in whichever order.

Lemma 4.2. *Let $G \to S$ be a smooth group scheme. Let G_r be the rth iterated dilatation of the unit section. Then the canonical map $\mathrm{Lie}(G_r) \to \mathrm{Lie}(G)_r$ is an isomorphism.*

Proof. Indeed, since G and G_r are smooth, their Lie algebras also are. It follows that both $\mathrm{Lie}(G_r)$ and $\mathrm{Lie}(G)_r$ represent the functor on smooth maps $\mathrm{Spec}(R) \to S$ which to such R assigns the set of elements of $G(R[\epsilon])$ (with $\epsilon^2 = 0$) that modulo ϵ and π^r are trivial. □

Theorem 4.3. *Let r, s be integers such that $0 \leqslant r/2 \leqslant s \leqslant r$. Let $(\mathcal{O}, \pi)$ be a henselian pair, where $\pi \subset \mathcal{O}$ is an invertible ideal. Let G be a smooth, separated $\mathcal{O}$-group scheme. Let G_r be the rth iterated dilatation of the unit section and $\mathfrak{g}_r$ its Lie algebra. If $\mathcal{O}$ is local or G is affine, there is a canonical and functorial isomorphism*

$$G_s(\mathcal{O})/G_r(\mathcal{O}) \xrightarrow{\ \sim\ } \mathfrak{g}_s(\mathcal{O})/\mathfrak{g}_r(\mathcal{O})\,.$$

Proof. The $(r-s)$th iterated dilatation of G_s is naturally G_r. But as we observed in Proposition 2.14, the group scheme G_r can also be seen as the dilatation of $\{1\}$ in $(\mathcal{O}, \pi^r)$. For an integer $n \geqslant 0$, we write $\mathcal{O}_n := \mathcal{O}/\pi^n$. Combining these remarks, we see that Lemma 4.2 applied to the dilatation of the group scheme $G = G_s$ with respect to $(\mathcal{O}, \pi^{r-s})$ yields an isomorphism

$$G_s(\mathcal{O})/G_r(\mathcal{O}) \xrightarrow{\ \sim\ } G_s(\mathcal{O}_{r-s})\,. \tag{4.1}$$

We now consider the statement of the theorem. If $s = 0$, we have $r = 0$, hence the left-hand side and the right-hand side are equal to $\{1\}$ and the result is clear. Therefore, we may assume $s > 0$. Theorem 3.5 applied to the dilatation of $\{1\}$ in $(\mathcal{O}, \pi^s)$ provides a canonical isomorphism

$$G_s|_{\mathcal{O}_s} \xrightarrow{\ \sim\ } \mathrm{Lie}(G|_{\mathcal{O}_s}) \otimes \mathrm{N}^{-1}_{\mathcal{O}_s/\mathcal{O}}\,.$$

The Lie algebra of a vector bundle $\mathbb{V}(\mathscr{E})$ is canonically isomorphic to $\mathbb{V}(\mathscr{E})$ itself [SGA3-1, Exposé II, Exemple 4.4.2]. Taking Lie algebras on both sides, we deduce a canonical isomorphism $G_s|_{\mathcal{O}_s} \xrightarrow{\sim} \mathfrak{g}_s|_{\mathcal{O}_s}$. Since $r - s \leqslant s$, the ring $\mathcal{O}_{r-s}$ is an $\mathcal{O}_s$-algebra, and we can take $\mathcal{O}_{r-s}$-valued points in the previous isomorphism to obtain $G_s(\mathcal{O}_{r-s}) \xrightarrow{\sim} \mathfrak{g}_s(\mathcal{O}_{r-s})$. Using (4.1) once for G_s and once for $\mathfrak{g}_s$, we end up with $G_s(\mathcal{O})/G_r(\mathcal{O}) \xrightarrow{\sim} \mathfrak{g}_s(\mathcal{O})/\mathfrak{g}_r(\mathcal{O})$, which is the desired canonical isomorphism. Finally, the functoriality follows from the functoriality of Néron blowups and Lie algebras. □

Remark 4.4. For reductive groups over non-archimedean local fields, similar results appeared in the literature. See for instance [Ser68, Proposition 6(b)] for the multiplicative group, [How77, proof of Lemma 6, p. 442, l. 1], [BK93, § 1.1, p. 22] for general linear groups, [Séc04, § 1, equation (8)] for general linear groups over division algebras and [PR84, § 2], [MP94, § 2], [MP96], [Adl98, § 1], [Yu01, § 1] for general reductive groups. In these examples, the isomorphisms are defined using ad hoc explicit formulas. In the case of an affine, smooth group scheme over a discrete valuation ring, the isomorphism of Theorem 4.3 appears without proof in [Yu15, proof of Lemma 2.8].

4.2 Torsors and level structures

In this subsection, we adopt notation more specific to the study of torsors over curves. Let X be a scheme, and let $N \subset X$ be an effective Cartier divisor. Let $G \to X$ be a smooth, finitely presented group scheme, and let $H \subset G|_N$ be an N-smooth closed subgroup. We denote by $\mathcal{G} \to G$ the Néron blowup of G in H (over N), which is a smooth, finitely presented X-group scheme by Theorem 3.2.

For a scheme $T \to X$, let $BG(T)$ (respectively, $B\mathcal{G}(T)$) denote the groupoid of right G-torsors (respectively, $\mathcal{G}$-torsors) on T in the fppf topology. Here we note that every such torsor is representable by a smooth algebraic space (of finite presentation) and hence admits sections étale locally. Whenever convenient, we may therefore work in the étale topology as opposed to the fppf topology.

Pushforward of torsors along $\mathcal{G} \to G$ induces a morphism of contravariant functors $\mathrm{Sch}_X \to$ Groupoids given by

$$B\mathcal{G} \longrightarrow BG\,, \quad \mathcal{E} \longmapsto \mathcal{E} \times^{\mathcal{G}} G\,. \tag{4.2}$$

Definition 4.5. For a scheme $T \to X$, let $B(G, H, N)(T)$ be the groupoid whose objects are pairs $(\mathcal{E}, \beta)$, where $\mathcal{E} \to T$ is a right fppf G-torsor and β is a section of the fppf quotient

$$\big(\mathcal{E}|_{T_N} / H|_{T_N}\big) \longrightarrow T_N\,,$$

where $T_N := T \times_X N$, that is, β is a reduction of $\mathcal{E}|_{T_N}$ to an H-torsor. Morphisms $(\mathcal{E}, \beta) \to (\mathcal{E}', \beta')$ are given by isomorphisms of torsors $\varphi\colon \mathcal{E} \cong \mathcal{E}'$ such that $\bar{\varphi} \circ \beta = \beta'$, where $\bar{\varphi}$ denotes the induced map on the quotients. Note that if $T_N = \varnothing$, then there is no condition on the compatibility of β and β'.

Each of the contravariant functors $\mathrm{Sch}_X \to$ Groupoids induced by $B\mathcal{G}$, BG and $B(G, H, N)$ defines a stack over X in the fppf topology. We call $B(G, H, N)$ the *stack of G-torsors equipped with level-(H, N)-structures*.

Lemma 4.6. *The map (4.2) factors as a map of X-stacks*

$$B\mathcal{G} \longrightarrow B(G, H, N) \longrightarrow BG\,, \tag{4.3}$$

where the second arrow denotes the forgetful map.

Proof. By Lemma 3.1(1), the map $\mathcal{G}|_N \to G|_N$ factors as $\mathcal{G}|_N \to H \subset G|_N$. Thus, given a $\mathcal{G}$-torsor $\mathcal{E} \to T$, we get the H-equivariant map

$$\mathcal{E} \times^{\mathcal{G}|_{T_N}} H|_{T_N} \subset \mathcal{E} \times^{\mathcal{G}|_{T_N}} G|_{T_N}\,.$$

Passing to the fppf quotient for the right H-action defines the section β_{can}. The association $\mathcal{E} \mapsto \big(\mathcal{E} \times^{\mathcal{G}} G, \beta_{\mathrm{can}}\big)$ induces the desired map $B\mathcal{G} \to B(G, H, N)$. □

Proposition 4.7. *The map* (4.3) *induces an equivalence of contravariant functors* $\mathrm{Sch}_X^{N\text{-reg}} \to \mathrm{Groupoids}$ *given by* $B\mathcal{G} \cong B(G,H,N)$, $\mathcal{E} \mapsto \big(\mathcal{E} \times^{\mathcal{G}} G, \beta_{\mathrm{can}}\big)$.

Proof. For $T \to X$ in $\mathrm{Sch}_X^{N\text{-reg}}$, we need to show that $B\mathcal{G}(T) \to B(G,H,N)(T)$ is an equivalence of groupoids. Since $\mathcal{G} \to X$ is smooth, in particular flat, its formation commutes with base change along $T \to X$ by Theorem 3.2. Hence we may reduce to the case where $T = X$. Now recall from Lemma 3.8 the exact sequence of sheaves of pointed sets on the étale site of X,

$$1 \longrightarrow \mathcal{G} \longrightarrow G \longrightarrow j_*(G|_N/H) \longrightarrow 1\,,$$

where $j\colon N \subset X$ denotes the inclusion. The desired equivalence is a consequence of [Gir71, § III.3.2, Proposition 3.2.1]. A quasi-inverse is given by pulling back a section $\beta\colon X \to j_*(G|_N/H)$ along the $\mathcal{G}$-torsor $G \to j_*(G|_N/H)$. Here we have used that by the smoothness of the group schemes $G \to X$, $\mathcal{G} \to X$, $H \to N$ and consequently of the quotient $G|_N/H$, we are allowed to work with the étale topology as opposed to the fppf topology. □

4.2.1 *Level structures on moduli stacks of bundles on curves.* We continue with the same notation and additionally assume that X is a smooth, projective, geometrically irreducible curve over a field k and that $G \to X$, and hence $\mathcal{G} \to X$ is affine.

Let $\mathrm{Bun}_G := \mathrm{Res}_{X/k} BG$ (respectively, $\mathrm{Bun}_{\mathcal{G}} := \mathrm{Res}_{X/k} B\mathcal{G}$) be the moduli stack of G-torsors (respectively, $\mathcal{G}$-torsors) on X; here $\mathrm{Res}_{X/k}$ stands for the Weil restriction along $X \to \mathrm{Spec}(k)$. This is a quasi-separated, smooth algebraic stack locally of finite type over k; cf. for example [Hei10, Proposition 1] or [AH21, Theorem 2.5]. Similarly, let $\mathrm{Bun}_{(G,H,N)} := \mathrm{Res}_{X/k} B(G,H,N)$ be the stack parametrizing G-torsors over X with level-(H,N)-structures as in Definition 4.5.

Theorem 4.8. *The map* (4.3) *induces equivalences of contravariant functors* $\mathrm{Sch}_k \to \mathrm{Groupoids}$ *given by* $\mathrm{Bun}_{\mathcal{G}} \cong \mathrm{Bun}_{(G,H,N)}$, $\mathcal{E} \mapsto \big(\mathcal{E} \times^{\mathcal{G}} G,\, \beta_{\mathrm{can}}\big)$.

Proof. For any k-scheme T, the projection $X \times_k T \to X$ is flat and hence defines an object in $\mathrm{Sch}_X^{N\text{-reg}}$. The theorem follows from Proposition 4.7. □

Example 4.9. If $H = \{1\}$ is trivial, then $\mathrm{Bun}_{(G,H,N)}$ is the moduli stack of G-torsors on X with level-N-structures. If $G \to X$ is split reductive, if N is reduced and if H a parabolic subgroup in $G|_N$, then $\mathrm{Bun}_{(G,H,N)}$ is the moduli stack of G-torsors with quasi-parabolic structures in the sense of Laszlo–Sorger [LS97]; cf. [PR10, § 2.a] and [Hei10, § 1, Example (2)].

We end § 4.2 by discussing Weil uniformizations. Let $|X| \subset X$ be the set of closed points, and let $\eta \in X$ be the generic point. We denote by $F = \kappa(\eta)$ the function field of X. For each $x \in |X|$, we let $\mathcal{O}_x$ be the completed local ring at x with fraction field F_x and residue field $\kappa(x) = \mathcal{O}_x/\mathfrak{m}_x$. Let $\mathbb{A} := \prod'_{x\in|X|} F_x$ be the ring of adeles with subring of integral elements $\mathbb{O} = \prod_{x\in|X|} \mathcal{O}_x$. As in [Ngô06, Lemme 1.1] or [Laf18, Remarque 8.21], one has the following result.

Proposition 4.10. *Assume either that k is a finite field and $G \to X$ has connected fibers, or that k is a separably closed field. Then there is an equivalence of groupoids*

$$\mathrm{Bun}_G(k) \;\simeq\; \coprod_{\gamma} G_\gamma(F)\big\backslash\big(G_\gamma(\mathbb{A})/G(\mathbb{O})\big)\,, \tag{4.4}$$

where γ ranges over $\ker^1(F,G) := \ker\big(H^1_{\text{ét}}(F,G) \to \prod_{x\in|X|} H^1_{\text{ét}}(F_x,G)\big)$, *and where G_γ denotes the associated pure inner form of $G|_F$. The identification* (4.4) *is functorial in G among maps of X-group schemes which are isomorphisms in the generic fiber.*

Proof. Under our assumptions, Hensel's lemma implies that $H^1_{\text{ét}}(\mathcal{O}_x, G)$ is trivial for all $x \in |X|$: use that $H^1_{\text{ét}}(\kappa(x), G)$ is trivial (by Lang's lemma if $\kappa(x)$ is finite), then Hensel's lemma using the smoothness of $G \to X$. In particular, for every G-torsor $\mathcal{E} \to X$, the class of its generic fiber $[\mathcal{E}|_F]$ lies in $\ker^1(F, G)$. For each $\gamma \in \ker^1(F, G)$, we fix a G-torsor $\mathcal{E}^0_\gamma \to \mathrm{Spec}(F)$ of class γ. We denote by G_γ its group of automorphisms, which is an inner form of G. We also fix an identification $G_\gamma(F_x) = G(F_x)$ for all $x \in |X|$, $\gamma \in \ker^1(F, G)$. In particular, $G_\gamma(\mathbb{A}) = G(\mathbb{A})$, so that the right-hand quotient in (4.4) is well defined. Now consider the groupoid

$$\Sigma_\gamma := \left\{(\mathcal{E}, \delta, (\epsilon_x)_{x\in|X|}) \mid \delta\colon \mathcal{E}|_F \simeq \mathcal{E}^0_\gamma,\ \epsilon_x\colon \mathcal{E}^0 \simeq \mathcal{E}|_{\mathcal{O}_x}\right\}.$$

For each $x \in |X|$, we have

$$g_x := \delta|_{F_x} \circ \epsilon_x|_{F_x} \in \mathrm{Aut}\big(\mathcal{E}^0_\gamma|_{F_x}\big) = G_\gamma(F_x) = G(F_x)\,,$$

and further $g_x \in G(\mathcal{O}_x)$ for almost all $x \in |X|$. Thus, the collection $(g_x)_{x\in|X|}$ defines a point in $G(\mathbb{A}) = G_\gamma(\mathbb{A})$. In this way, we obtain an $(G_\gamma(F) \times G(\mathbb{O}))$-equivariant map $\pi_\gamma\colon \Sigma_\gamma \to G_\gamma(\mathbb{A})$ and thus a commutative diagram of groupoids

$$\begin{array}{ccc} \bigsqcup_\gamma \Sigma_\gamma & \xrightarrow{\ \sqcup_\gamma \pi_\gamma\ } & \bigsqcup_\gamma G_\gamma(\mathbb{A}) \\ \downarrow & & \downarrow \\ \mathrm{Bun}_G & \dashrightarrow & \bigsqcup_\gamma G_\gamma(F)\backslash\big(G_\gamma(\mathbb{A})/G(\mathbb{O})\big)\,. \end{array}$$

As the vertical maps are disjoint unions of $(G_\gamma(F) \times G(\mathbb{O}))$-torsors, the dashed arrow is fully faithful. Hence it suffices to show that it is a bijection on isomorphism classes, that is, a bijection of sets. We construct an inverse of the dashed arrow as follows: Given a representative $(g_x)_{x\in|X|} \in G_\gamma(\mathbb{A}) = G(\mathbb{A})$ of some class, there is a non-empty open subset $U \subset X$ such that $g_x \in G(\mathcal{O}_x)$ for all $x \in |U|$ and such that $\mathcal{E}^0_\gamma$ is defined over U. Let $X\backslash U = \{x_1, \ldots, x_n\}$ for some $n \geqslant 0$. We define the associated G-torsor by gluing the torsor $\mathcal{E}^0_\gamma$ on U with the trivial G-torsor on

$$\mathrm{Spec}(\mathcal{O}_{x_1}) \sqcup \cdots \sqcup \mathrm{Spec}(\mathcal{O}_{x_n})$$

using the elements $g_{x_1}, \ldots, g_{x_n}$ and the identification $G_\gamma(F_x) = G(F_x)$. The gluing is justified by the Beauville–Laszlo lemma [BL95], or alternatively [Hei10, Lemma 5]. This shows (4.4). From the construction of the map $\sqcup_\gamma \pi_\gamma$, one sees that (4.4) is functorial in G among generic isomorphisms. □

Note that N defines an effective Cartier divisor on $\mathrm{Spec}(\mathbb{O})$ so that the map of groups $\mathcal{G}(\mathbb{O}) \to G(\mathbb{O})$ is injective. As subgroups of $G(\mathbb{O})$, we have

$$\mathcal{G}(\mathbb{O}) = \ker\big(G(\mathbb{O}) \longrightarrow G(\mathcal{O}_N) \longrightarrow G(\mathcal{O}_N)/H(\mathcal{O}_N)\big)\,, \tag{4.5}$$

where $\mathcal{O}_N$ denotes the ring of functions on N viewed as a quotient ring $\mathbb{O}_X \to \mathcal{O}_N$.

Corollary 4.11. *Under the assumptions of Proposition 4.10, the Néron blowup $\mathcal{G} \to X$ is smooth, affine with connected fibers by Theorem 3.2, and there is a commutative diagram of groupoids*

$$\begin{array}{ccc} \mathrm{Bun}_{\mathcal{G}}(k) & \xrightarrow{\ \simeq\ } & \bigsqcup_\gamma G_\gamma(F)\backslash\big(G_\gamma(\mathbb{A})/\mathcal{G}(\mathbb{O})\big) \\ \downarrow & & \downarrow \\ \mathrm{Bun}_G(k) & \xrightarrow{\ \simeq\ } & \bigsqcup_\gamma G_\gamma(F)\backslash\big(G_\gamma(\mathbb{A})/G(\mathbb{O})\big) \end{array}$$

identifying the vertical maps as the level maps.

Remark 4.12. Let $G|_F$ be reductive.

(1) If k is algebraically closed, then $H^1(F,G)$, and hence $\ker^1(F,G)$, is trivial by [BS68, § 8.6].

(2) If k is a finite field and if $G|_F$ is either simply connected or split reductive, then $\ker^1(F,G)$ is trivial by [Laf18, Remarques 8.21 and 12.2] and the references cited therein.

4.2.2 *Integral models of moduli stacks of shtukas.* Here we point out that Theorem 4.8 immediately applies to construct certain integral models of moduli stacks of shtukas. We proceed with the notation of § 4.2.1 and additionally assume that k is a finite field.

For any partition $I = I_1 \sqcup \cdots \sqcup I_r$, $r \in \mathbb{Z}_{\geqslant 1}$, of a finite index set, the *moduli stack of iterated G-shtukas* is the contravariant functor of groupoids $\mathrm{Sch}_k \to \mathrm{Groupoids}$ given by

$$\mathrm{Sht}_{G,I_\bullet} \overset{\mathrm{def}}{=\!=} \left\{ \mathcal{E}_r \underset{I_r}{\overset{\alpha_r}{\dashrightarrow}} \mathcal{E}_{r-1} \underset{I_{r-1}}{\overset{\alpha_r}{\dashrightarrow}} \cdots \underset{I_2}{\overset{\alpha_2}{\dashrightarrow}} \mathcal{E}_1 \underset{I_1}{\overset{\alpha_1}{\dashrightarrow}} \mathcal{E}_0 = {}^\tau\mathcal{E}_r \right\}, \tag{4.6}$$

where ${}^\tau\mathcal{E} := (\mathrm{id}_X \times \mathrm{Frob}_{T/k})^*\mathcal{E}$ denotes the pullback under the relative Frobenius $\mathrm{Frob}_{T/k}$. The dashed arrows in (4.6) indicate that the maps α_j between G-bundles are rationally defined. More precisely, $\mathrm{Sht}_{G,I_\bullet}(T)$ classifies data $\big((\mathcal{E}_j)_{j=1,\dots,r}, \{x_i\}_{i\in I}, (\alpha_j)_{j=1,\dots,r}\big)$, where the $\mathcal{E}_j \in \mathrm{Bun}_G(T)$ are torsors, the $\{x_i\}_{i\in I} \in X^I(T)$ are points and the

$$\alpha_j \colon \mathcal{E}_j|_{X_T\setminus(\cup_{i\in I_j}\Gamma_{x_i})} \longrightarrow \mathcal{E}_{j-1}|_{X_T\setminus(\cup_{i\in I_j}\Gamma_{x_i})}$$

are isomorphisms of torsors. Here $\Gamma_{x_i} \subset X_T$ denotes the graph of x_i viewed as a relative effective Cartier divisor on $X_T \to T$. We have a forgetful map $\mathrm{Sht}_{G,I_\bullet} \to X^I$. Similarly, we have the moduli stack $\mathrm{Sht}_{\mathcal{G},I_\bullet} \to X^I$ defined by replacing G with $\mathcal{G}$. By [Var04] for split reductive groups and by [AH21, Theorem 3.15] for general smooth, affine group schemes, both stacks are ind-(Deligne–Mumford) stacks which are ind-(separated and of locally finite type) over k. Furthermore, pushforward of torsors along $\mathcal{G} \to G$ induces a map of X^I-stacks

$$\mathrm{Sht}_{\mathcal{G},I_\bullet} \longrightarrow \mathrm{Sht}_{G,I_\bullet}\,; \tag{4.7}$$

cf. [Bre19]. We also consider the *moduli stack of iterated G-shtukas with level-(H,N)-structures*,

$$\mathrm{Sht}_{(G,H,N),I_\bullet} \longrightarrow X^I\,;$$

that is, the groupoid $\mathrm{Sht}_{(G,H,N),I_\bullet}(T)$ classifies data $\big((\mathcal{E}_j,\beta_j)_{j=1,\dots,r}, \{x_i\}_{i\in I}, (\alpha_j)_{j=1,\dots,r}\big)$, where the $(\mathcal{E}_j,\beta_j)$ in $\mathrm{Bun}_{(G,H,N)}(T)$ are G-torsors with a level-(H,N)-structure, $\{x_i\}_{i\in I} \in X^I(T)$ are points and the

$$\alpha_j \colon\ (\mathcal{E}_j,\beta_j)|_{X_T\setminus(\cup_{i\in I_j}\Gamma_{x_i})} \longrightarrow (\mathcal{E}_{j-1},\beta_{j-1})|_{X_T\setminus(\cup_{i\in I_j}\Gamma_{x_i})} \tag{4.8}$$

are maps of G-torsors with a level-(H,N)-structure, where $(\mathcal{E}_0,\beta_0) := ({}^\tau\mathcal{E}_r, {}^\tau\beta_r)$. We have a forgetful map of X^I-stacks

$$\mathrm{Sht}_{(G,H,N),I_\bullet} \longrightarrow \mathrm{Sht}_{G,I_\bullet}\ . \tag{4.9}$$

Corollary 4.13. *Let G, H, N, $\mathcal{G}$ and $I = I_1 \sqcup \cdots \sqcup I_r$ be as above. Then the equivalence in Theorem 4.8 induces an equivalence of X^I-stacks*

$$\mathrm{Sht}_{\mathcal{G},I_\bullet} \xrightarrow{\cong} \mathrm{Sht}_{(G,H,N),I_\bullet}\,,$$

which is compatible with the maps (4.7) and (4.9).

Loosely speaking, $\mathrm{Sht}_{\mathcal{G},I_\bullet} \cong \mathrm{Sht}_{(G,H,N),I_\bullet}$ over X^I is an integral model for

$$\mathrm{Sht}_{(G,H,N),I_\bullet}|_{(X\setminus N)^I}\,,$$

which however needs modification outside the case of parahoric group schemes $\mathcal{G} \to X$. Concretely, if the characteristic places x_i, $i \in I$, of the shtuka divide the level N, then there is simply no compatibility condition on the β_j in (4.8). The fibers of the map (4.7), respectively (4.9), over such places are certain quotients of positive loop groups by [Bre19, Theorem 3.20]. These fibers are (in general) of strictly positive dimension and are (in general) not proper if $\mathcal{G} \to X$ is not parahoric.

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