



Factorization centers in dimension 2 and the Grothendieck ring of varieties

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ABSTRACT

We initiate the study of factorization centers of birational maps and complete it for surfaces over a perfect field in this article. We prove that for every birational automorphism $\phi: X \dashrightarrow X$ of a smooth projective surface X over a perfect field k , the blowup centers are isomorphic to the blowdown centers in every weak factorization of ϕ . This implies that nontrivial L-equivalences of zero-dimensional varieties cannot be constructed based on birational automorphisms of a surface. It also implies that rationality centers are well defined for every rational surface X ; namely, there exists a zero-dimensional variety intrinsic to X , which is blown up in any rationality construction of X .

1. Introduction

One source of motivation in birational geometry comes from studying groups of birational automorphisms of algebraic varieties, in particular the Cremona groups $\mathrm{Cr}_n(k) = \mathrm{Bir}(\mathbb{P}_k^n)$. Each birational automorphism blows up some subschemes and contracts some exceptional divisors. The primary question we study in this paper is the following.

Question 1.1. Let $\phi: X \dashrightarrow X$ be a birational automorphism of a smooth projective variety. Do centers blown up by ϕ correspond, up to stable birational equivalence, to the exceptional divisors blown down by ϕ ?

We note that exceptional divisors are ruled over the corresponding blowup centers, so asking about stable birational equivalence classes is a natural way to compare exceptional divisors with the blowup centers. We give a complete answer to a stronger version of Question 1.1 for surfaces over an arbitrary perfect field; see Theorem 1.3 below. Our proof is an application of the 2-dimensional minimal model program [Isk79, Isk96] combined with the Grothendieck ring of varieties, étale cohomology groups and the so-called Gassmann equivalence of Galois sets.

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There are three main reasons to study Question 1.1. First of all, it is the structure of birational automorphisms and Cremona groups, in particular their generation by involutions or regularizable elements. In the recent paper [LS22], we explain that the answer to Question 1.1 is negative in various contexts in dimension $n \geq 3$ and give applications to the structure of the higher Cremona groups. Secondly, Question 1.1 has a tight relationship to the structure of the Grothendieck ring of varieties.

Question 1.2. (a) (Larsen–Lunts [LL03], slightly reformulated) If classes of smooth projective varieties $[X]$ and $[Y]$ coincide in the Grothendieck ring, how can we compare the geometry of X and Y ? For instance, are X and Y birational?

(b) [KS18] What is the geometric meaning of L-equivalence? For instance, if zero-dimensional schemes are L-equivalent, do they have to be isomorphic?

(c) [Dom15] Does the Grothendieck ring of varieties $K_0(\text{Var}/k)$ have torsion elements?

In the direction of item (a), the main result of [LL03], which also follows from [Bit04], is that if $\text{char}(k) = 0$, the equality of classes $[X], [Y]$ of smooth projective connected varieties implies that X and Y are *stably* birational. On the other hand, for nonprojective smooth connected varieties, the second part of Question 1.2(a) is known to have a negative answer (the first such example is [Bor18, proof of Theorem 2.13]). For item (b), see [KS18] for conjectural relations to derived equivalence. In Section 2.2, we explain that L-equivalence of smooth zero-dimensional schemes implies Gassmann equivalence of the corresponding Galois sets, but this does not rule out the possibility of nontrivial L-equivalence between such schemes. Nothing is known about item (c). As the Grothendieck ring is a colimit of the truncated groups $K_0(\text{Var}^{\leq n}/k)$ generated by varieties of dimension up to n , we can ask each of the questions in these truncated groups. As a direct consequence of our positive answer to Question 1.1 in dimension 2, we are able to answer Question 1.2 completely for $K_0(\text{Var}^{\leq 2}/k)$; see Corollary 3.11. Namely, (a) equality of classes of smooth projective varieties in $K_0(\text{Var}^{\leq 2}/k)$ implies birationality, (b) L-equivalence in $K_0(\text{Var}^{\leq 2}/k)$ is trivial, and (c) $K_0(\text{Var}^{\leq 2}/k)$ is a free abelian group. We expect that studying Question 1.1 in dimension up to n would lead to good control over Question 1.2 for $K_0(\text{Var}^{\leq n}/k)$.

Finally, answering Question 1.1 positively, or explaining all ways in which it can fail, allows one to control all rationality constructions for every rational variety; see the discussion of rationality centers below and Section 5.2 for more details.

We now explain our answer to Question 1.1 in dimension 2 over perfect fields. Our main result can be stated in the following way.

THEOREM 1.3 (see Theorem 3.5). *Let k be a perfect field. Let X/k be a smooth projective surface, and let $\phi \in \text{Bir}(X)$ be a birational automorphism. For any factorization of ϕ into a sequence of blowups and blowdowns at connected smooth zero-dimensional subschemes, let $Z_1, \dots, Z_r$ (respectively, $Z'_1, \dots, Z'_{r'}$) be the centers which get blown up (respectively, blown down). Then $r = r'$, and there is a reordering under which $Z_i \simeq Z'_i$ over k for all $i = 1, \dots, r$.*

Note that Theorem 1.3 is easily seen to hold when k is algebraically closed field, as the Galois actions are trivial and the number of points blown up is equal to the number of exceptional divisors contracted. Over an arbitrary perfect field, Theorem 1.3 is a nontrivial statement. In particular, by this we mean that it is not possible to recover centers of the blowups simply from the Galois action on the cohomology of the surface; see Remark 2.11 for the technical formulation of this statement in terms of Chow motives and Example 2.14 for an explicit construction. Neither

does there seem to exist a straightforward geometric argument; see Example 3.8 for an illustration of the birational geometry involved.

To put Theorem 1.3 into an appropriate context, we introduce a general invariant $c(\phi)$, keeping track of the *factorization centers* of the birational map ϕ , which is a homomorphism from the groupoid of birational types of surfaces to a free abelian group generated by reduced k -schemes of dimension zero; see Corollary 3.2 for an axiomatic definition of $c(\phi)$. We then show that c is constant on each $\text{Bir}(X, Y)$ and as a consequence $c(\phi) = 0$ for every self-map, which implies Theorem 1.3.

To prove that c is constant on $\text{Bir}(X, Y)$, we have to consider each birational type of surfaces that can occur, with geometrically rational (and, especially, rational) surfaces being the most interesting ones. For geometrically rational surfaces, by the 2-dimensional minimal model program, we have to consider birational maps between del Pezzo surfaces and conic bundles. Our proof for the uniqueness of factorization centers uses two ingredients: *Sarkisov links* and *Gassmann equivalence*. Sarkisov links are certain elementary birational transformations between del Pezzo surfaces and conic bundles which generate the groupoid of birational maps between geometrically rational surfaces. In dimension 2, the existence of decomposition into links has been proved, and all the links have been classified by Iskovskikh into a finite list [Isk96]. The largest variety of links occurs for what we call *models of large degree*: these include all minimal geometrically rational surfaces with $K_X^2 \geq 5$.

We derive Theorem 1.3 for geometrically rational surfaces from a uniform claim we make for all links between minimal geometrically rational surfaces (see Proposition 5.5), which we check for each link in Iskovskikh's classification. The latter uniform claim is formulated in terms of the (virtual) Néron–Severi Galois set, which is closely related to the Néron–Severi lattice of the surface as a Galois module. These Néron–Severi sets are defined in terms of the Galois action on linear systems of rational curves of low degree (typically pencils of conics and nets of twisted cubics) on del Pezzo surfaces.

On top of the classification of links, to prove the result we use étale cohomology, permutation modules and the so-called Gassmann triples from group theory: these are triples (G, H, H') with H and H' subgroups of a finite group G such that $\mathbb{C}[G/H]$ and $\mathbb{C}[G/H']$ are isomorphic G -representations (this holds if H, H' are conjugate, but the converse is false). Gassmann triples are used to produce arithmetically equivalent fields, see for example [Per77, BdS02], isospectral manifolds [Sun85], as well as curves with isomorphic Jacobians [Pra17]. In our dealing with Gassmann equivalence, we follow the approach of [Par13], which generalizes Gassmann triples from a pair of subgroups of G to a pair of G -sets with possibly nontransitive actions. Our Gassmann-equivalent sets come from étale cohomology groups and, in particular, from Néron–Severi groups of geometrically rational surfaces with their Galois group action. We show that Gassmann equivalence provides a cohomological expression of L-equivalence; see Lemma 2.10 and Remark 2.11. Using the fact that Gassmann triples of small order are trivial allows us to significantly limit the number of Sarkisov links we have to consider in the proof of the main result.

As a by-product of our results on Gassmann equivalence, we obtain conditions which forbid nontrivial L-equivalence of zero-dimensional schemes to exist; for example, L-equivalent zero-dimensional connected reduced schemes (respectively, reduced schemes) of degree $d \leq 6$ (respectively, $d \leq 5$) are isomorphic, and L-equivalence always implies isomorphism for fields with procyclic Galois groups such as $k = \mathbb{R}$, $k = \mathbb{F}_q$; see Example 2.12.

Finally for rational surfaces, our main result has the following consequence: if X is a smooth

projective rational surface, then there exists a zero-dimensional scheme depending only on X , which will have to be blown up by any birational isomorphism $\phi: \mathbb{P}^2 \dashrightarrow X$; see Corollary 5.9. We think of these associated schemes as *rationality centers* of X . This is in contrast with the higher-dimensional geometry, where the associated rationality centers are not well defined, even up to stable birational equivalence. For instance, a K3 surface associated with a cubic fourfolds should not be unique up to isomorphism: it should be unique up to derived equivalence [Has16, Remark 27], or possibly up to L-equivalence, as hinted in [KS18, Section 2.6.1].

This text is organized as follows. Section 2 is devoted to Gassmann equivalence of G -sets. We relate it to L-equivalence and provide sufficient conditions for two Gassmann-equivalent G -sets to be isomorphic. In Section 3, we define the invariant $c(\phi)$ which captures the factorization center of a birational map ϕ and formulate the main theorem (Theorem 3.5) of the paper, which we prove in Section 4 and Section 5. We also study rational curves on del Pezzo surfaces in Section 4, which will be used to define and study the virtual Néron–Severi sets in Section 5. At the end of the paper, we explain the concept of rationality center for rational surfaces and illustrate it in the case of del Pezzo surfaces.

Conventions

We work over a perfect field k unless otherwise specified. All schemes are of finite type over k . By a surface we mean a connected smooth projective (but not necessarily geometrically irreducible) surface X over k .

2. Gassmann equivalence

We explain the basics about the Burnside ring of a profinite group G , the Grothendieck ring of varieties and a homomorphism between them when G is the absolute Galois group. Our presentation is similar to that in [Rök11].¹ This homomorphism allows us to relate L-equivalence of reduced zero-dimensional schemes to Gassmann equivalence of G -sets and thus to rule out the possibility of nontrivial L-equivalence in small degree.

2.1 Definition and basic properties

Let G be a profinite group, and let **Gset** be the semiring of isomorphism classes of finite G -sets on which G acts continuously for the profinite topology, where finite sets are considered with the discrete topology. Continuity of the action is equivalent to the requirement that the stabilizer of any point is open (and, in particular, a finite-index subgroup). Here in **Gset**, the addition (respectively, multiplication) is defined by disjoint unions (respectively, Cartesian products). We define the Burnside ring $\text{Burn}(G)$ of G to be the Grothendieck ring associated with **Gset**. When G is a finite group, the definition of $\text{Burn}(G)$ is the classical one. We sometimes refer to elements of $\text{Burn}(G)$, that is, combinations of isomorphism classes of G -sets with integer coefficients, as *virtual G -sets*. We note that the number of elements (respectively, the number of orbits) in a G -set gives rise to a ring homomorphism (respectively, a group homomorphism) $\text{Burn}(G) \rightarrow \mathbb{Z}$.

Let F be a field of characteristic zero, and let $\text{Rep}(G, F)$ be the abelian monoidal category of finite-dimensional G -representations over F . Let $K_0(\text{Rep}(G, F))$ be the Grothendieck ring

¹Note however that [Rök11] makes an erroneous statement on p. 943 (§2) that the homomorphism from the Burnside ring to the representation ring is injective on transitive G -sets (cf. Example 2.7(2)).

of $\text{Rep}(G, F)$. There is a well-defined ring homomorphism

$$\mu_G: \text{Burn}(G) \rightarrow K_0(\text{Rep}(G, F)) \quad (2.1)$$

that sends the class of a continuous finite G -set A to the class of the permutation representation $F[A]$. We are interested in the kernel of this homomorphism.

LEMMA-DEFINITION 2.1. *Let G be a profinite group, and let A, B be continuous finite G -sets. Fix a field F of characteristic zero. The following conditions are equivalent:*

- (1) *We have $\mu_G(A) = \mu_G(B)$ in $K_0(\text{Rep}(G, F))$.*
- (2) *We have $F[A] \simeq F[B]$ as $F[G]$ -modules.*

We say that two continuous finite G -sets A and B are Gassmann equivalent if they satisfy one of the above equivalent conditions.

Proof. It is clear that condition (2) implies condition (1). Conversely, if $\mu_G(A) = \mu_G(B)$, then since $F[A]$ and $F[B]$ are finite-dimensional (as F -vector spaces), both $F[A]$ and $F[B]$ admit composition series in $\text{Rep}(G, F)$ of finite length [EH18, Lemma 3.9]. The Jordan–Hölder theorem [Wei13, Exercise II.6.3] shows that $F[A]$ and $F[B]$ have the same collection of isomorphism classes of (simple) Jordan–Hölder factors. As both G -modules $F[A]$ and $F[B]$ factor through a finite quotient G/H of G (we can take H to be the intersection of the kernels of the two actions, which is an open subgroup of G , hence has finite index), Maschke’s theorem implies that $F[A]$ and $F[B]$ are semisimple G -modules. Hence $F[A] \simeq F[B]$. $\square$

The following lemma reduces Gassmann equivalence to the case of finite group actions. If A is a G -set, we write G^A for the kernel of the action; thus G^A is a normal subgroup, and G/G^A acts on A faithfully.

LEMMA 2.2. *Two continuous finite G -sets A and B are Gassmann equivalent if and only if $G^A = G^B =: H$ and A and B are Gassmann equivalent as G/H -sets.*

Proof. This follows from Lemma 2.1, together with the observation that $G^A = \ker \rho_A$ (and similarly $G^B = \ker(\rho_B)$), where $\rho_A: G \rightarrow \text{GL}(V, F)$ is the homomorphism defining the $F[G]$ -module structure of $F[A]$ and V is the underlying F -vector space of $F[A]$. $\square$

LEMMA 2.3 ([Par13, Proposition 1], cf. [Pra17, Definition 1]). *Let G be a finite group, and let A and B be finite G -sets. Fix a field F of characteristic zero. The following conditions are equivalent:*

- (1) *Every $g \in G$ fixes the same number of elements in A and in B .*
- (2) *There exist subgroups $H_1, \dots, H_r, H'_1, \dots, H'_r$ of G such that*

$$A \simeq \bigsqcup_{i=1}^r G/H_i, \quad B \simeq \bigsqcup_{i=1}^r G/H'_i,$$

and for each conjugacy class $T \subset G$, these subgroups satisfy

$$\sum_{i=1}^r \frac{|T \cap H_i|}{|H_i|} = \sum_{i=1}^r \frac{|T \cap H'_i|}{|H'_i|}. \quad (2.2)$$

- (3) *The G -sets A and B are Gassmann equivalent.*

In particular, the kernel of μ_G is independent of the choice of F .

Proof. The equivalence of conditions (1), (2) and (3) for $F = \mathbb{C}$ is proved in [Par13, Proposition 1]. Finally, condition (3) is independent of the choice of F because the functor $\text{Rep}(G, \mathbb{Q}) \rightarrow \text{Rep}(G, F)$ is injective on isomorphism classes [Ser77, § 14.6]. $\square$

Remark 2.4. The number of G -orbits in Gassmann-equivalent continuous G -sets is the same. This follows from Lemma 2.2 and Lemma 2.3(2), or directly as the number of orbits equals the multiplicity of the trivial representation in the permutation representation.

PROPOSITION 2.5. *Let G be a profinite group, and let A and B be Gassmann-equivalent continuous finite G -sets. If one of the conditions*

- G/G^A is a cyclic group;
- A is transitive, and the stabilizer of a point is normal in G ;
- A is transitive, and $|A| \leq 6$;
- $|A| \leq 5$

is satisfied, then A and B are isomorphic G -sets.

If $\alpha \in \text{Burn}(G)$, then we write $|\alpha|$ for the smallest $\max(|A|, |B|)$ over all possible representations $\alpha = [A] - [B]$ with continuous G -sets A and B . We have the following immediate corollary of Proposition 2.5.

COROLLARY 2.6. *Let $\alpha \in \text{Ker}(\mu_G)$. If $|\alpha| \leq 5$, then $\alpha = 0$.*

Proof of Proposition 2.5. By Lemma 2.2, we can assume that G is a finite group and the G -actions on both A and B are faithful. When G is a finite cyclic group, the result is [Par13, Proposition 4.1]. If A is transitive, then by Remark 2.4, the G -set B is also transitive. Let H (respectively, H') be the stabilizer of an element $a \in A$ (respectively, $b \in B$). Then G/H and G/H' are Gassmann equivalent; in this case one refers to (G, H, H') as a Gassmann triple. It is well known and easy to see (for example using (2.2) with $r = 1$) that if H is normal, then $H' = H$. It is a nontrivial computation that if $[G : H] \leq 6$ in Gassmann triple (G, H, H') , the subgroups H and H' will be conjugate so that A and B are isomorphic; see [Per77, proof of Theorem 3] or [BdS02, § 2, p. 69].

Finally, we need to consider the case where A may not be transitive but $t := |A| \leq 5$. We write $A = \{1, \dots, t\}$ and let S_t be the permutation group of t elements. Since the G -action on A is assumed to be faithful, it realizes G as a subgroup of S_t .

Up to adding trivial G -sets $\{*\}$, we can assume $t = 5$. By Remark 2.4, the G -sets A and B have the same number of G -orbits, and we argue according to the number of orbits m . When $m = 5$, both A and B are trivial G -sets. When $m = 4$, then both A and B are isomorphic to $\{1, 2\} \sqcup \{*\} \sqcup \{*\} \sqcup \{*\}$, and $G \simeq \mathbb{Z}/2\mathbb{Z}$ acts on $\{1, 2\}$ by involution. The case $m = 1$ is already covered by the third case of Proposition 2.5. So it remains to study the case where $m = 2$ or 3.

First we show that G uniquely determines the length of the orbits of the G -set A . If $m = 3$, then A is isomorphic to either $A_3 \sqcup \{*\} \sqcup \{*\}$ or $A_2 \sqcup A'_2 \sqcup \{*\}$ with $|A_3| = 3$ and $|A_2| = |A'_2| = 2$. So $G \leq S_3$ in the former case, and $G \leq \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ in the latter case. In the former case, since the G -action is transitive on A_3 , the group G contains $\mathbb{Z}/3\mathbb{Z} \leq S_3$, so we cannot embed G into $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. Therefore, G uniquely determines the length of the orbits. If $m = 2$, then A is isomorphic to either $A_3 \sqcup A_2$ or $A_4 \sqcup \{*\}$ with $|A_i| = i$. So $G \leq S_3 \times \mathbb{Z}/2\mathbb{Z}$ in the former case, and $G \leq S_4$ in the latter case. In the former case, since the G -action is transitive on both A_3 and A_2 , the group G contains $\mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \leq S_3 \times \mathbb{Z}/2\mathbb{Z}$. Since S_4 has no elements of order 6,

we cannot embed G into S_4 . Hence the lengths of the orbits are determined by G in this case as well.

We still assume $m = 2$ or 3 . If $A = A' \sqcup A''$ as G -sets, where A' is transitive and A'' is a disjoint union of trivial G -sets, then we also have the same type of decomposition $B = B' \sqcup B''$ with $|A'| = |B'|$. It follows that $\mu_G(A') = \mu_G(B')$, so $A' \simeq B'$ since both G -sets are transitive. This covers the cases where $A = A_3 \sqcup \{*\} \sqcup \{*\}$ and $A = A_4 \sqcup \{*\}$. If $A = A_2 \sqcup A'_2 \sqcup \{*\}$, then either $G = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, or G is the diagonal $\mathbb{Z}/2\mathbb{Z} \leq \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. In either case we verify that G uniquely determines the G -set structure of A . If $A = A_3 \sqcup A_2$, then either $G \leq \mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, or $G = S_3 \times \mathbb{Z}/2\mathbb{Z}$. Once again, in either case, we verify that G uniquely determines the G -set structure of A . $\square$

The following example shows that the lower bounds on the order of G -sets in Proposition 2.5 are optimal.

Example 2.7. (1) [Par13, 1.1] Let $G = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, and let

$$H_1 = \langle (1, 0) \rangle, \quad H_2 = \langle (0, 1) \rangle, \quad H_3 = \langle (1, 1) \rangle,$$

and

$$H'_1 = \{(0, 0)\}, \quad H'_2 = H'_3 = G.$$

Then H_1, H_2, H_3 and H'_1, H'_2, H'_3 satisfy the conditions of Lemma 2.3(2) and thus give rise to nonisomorphic nontransitive Gassmann-equivalent sets of order 6 with orbit decompositions $2+2+2$ and $4+1+1$, respectively. Indeed, for abelian groups the condition (2.2) can be rewritten as

$$\sum_{H_i \ni g} \frac{1}{|H_i|} = \sum_{H'_i \ni g} \frac{1}{|H'_i|} \quad \text{for all } g \in G,$$

which is immediately verified.

(2) [Per77, § 4, p. 358] Let $G = \text{PSL}_2(\mathbb{F}_7) \simeq \text{PSL}_3(\mathbb{F}_2)$ be the simple group of order 168. Let A be the set of $\mathbb{F}_2$ -points of the projective plane over $\mathbb{F}_2$ and B be the set of $\mathbb{F}_2$ -lines on this plane, that is, points of the dual projective plane. Then A and B are transitive G -sets of order 7. Using simple linear algebra of the $\text{PSL}_3(\mathbb{F}_2)$ -action on $\mathbb{P}_{\mathbb{F}_2}^2$, one shows that A and B satisfy Lemma 2.3(1), so that A and B are Gassmann equivalent, and one can check that they are not isomorphic. See also [BdS02, Theorem 3], which shows that this pair is the only nontrivial Gassmann triple of faithful transitive G -sets of order 7.

2.2 L-equivalence and Gassmann equivalence

Let $K_0(\text{Var}/k)$ denote the Grothendieck ring of varieties with generators given by isomorphism classes $[X]$ of schemes of finite type over k and relations generated by cut-and-paste relations

$$[X] = [Z] + [X \setminus Z] \tag{2.3}$$

for every closed $Z \subset X$. The ring structure on $K_0(\text{Var}/k)$ is induced by products of schemes. We write $\mathbb{L} = [\mathbb{A}^1]$.

It is known that $\mathbb{L}$ is a zero-divisor [Bor18] and that the annihilator of $\mathbb{L}^k$, $k \geq 1$, encodes deep geometric information. Following [KS18] we call two smooth projective connected varieties X and Y *L-equivalent* if for some $k \geq 0$,

$$\mathbb{L}^k \cdot ([X] - [Y]) = 0.$$

We sometimes refer to the $[X] = [Y]$ case as trivial L-equivalence. It is currently unknown if zero-dimensional varieties can be nontrivially L-equivalent. The smallest-dimensional example of nontrivial L-equivalence is that of genus 1 curves over nonclosed fields [SZ20]. See [KS18, SZ20] for some details about a conjectural relationship between L-equivalence and derived equivalence, and the references therein for the currently known examples. Note that the classes in the Grothendieck ring are insensitive to nonreduced structure; hence when studying L-equivalence, we can always assume schemes to be reduced.

Remark 2.8. For fields of positive characteristic, there exist two alternative definitions of the Grothendieck ring of varieties, hence alternative definitions of L-equivalence.

(1) First of all, one can define the Grothendieck ring $K_0^{\text{bl}}(\text{Var}/k)$ generated by classes of smooth projective varieties with blowup relations as in [Bit04, Theorem 3.1(bl)]. We have an obvious homomorphism

$$K_0^{\text{bl}}(\text{Var}/k) \rightarrow K_0(\text{Var}/k),$$

which is known to be an isomorphism if k is a field of characteristic zero [Bit04, Theorem 3.1].

(2) Furthermore, in positive characteristic one can define a modified Grothendieck ring [NS09] by imposing an additional relation of identifying varieties related by universal homeomorphisms (originating from totally inseparable coverings in positive characteristic). It is not known if this additional relation in fact gives rise to a nonisomorphic ring as all the standard invariants which are used to distinguish elements in the Grothendieck ring factor through the modified ring as well [NS09].

Let k be a field, and let $G_k = \text{Gal}(\bar{k}/k)$, where $\bar{k}$ denotes the separable closure of k . For a (not necessarily smooth or projective) variety X/k , we consider its ℓ -adic cohomology $H_{\text{ét},c}^i(X_{\bar{k}}, \mathbb{Q}_{\ell})$ (with $\ell \neq \text{char}(k)$) with compact support as a G_k -module. These groups are finite-dimensional $\mathbb{Q}_{\ell}$ -vector spaces [FK88, Remark I.12.16]; they vanish outside the range $0 \leq i \leq 2 \dim(X)$ and give rise to a group homomorphism

$$\mu_{\text{ét}}: K_0(\text{Var}/k) \rightarrow K_0(\text{Rep}(G_k, \mathbb{Q}_{\ell})) \quad (2.4)$$

defined by

$$\mu_{\text{ét}}([X]) = \sum_{i=0}^{2 \dim(X)} (-1)^i [H_{\text{ét},c}^i(X_{\bar{k}}, \mathbb{Q}_{\ell})].$$

We have

$$\mu_{\text{ét}}(\mathbb{L}) = \mu_{\text{ét}}([\mathbb{P}^1] - 1) = [\mathbb{Q}_{\ell}(-1)], \quad (2.5)$$

and furthermore, the projective bundle formula for étale cohomology implies that $\mu_{\text{ét}}$ is a $\mathbb{Z}[\mathbb{L}]$ -module homomorphism, where $\mathbb{L}$ acts by multiplication by $\mathbb{Q}_{\ell}(-1)$ on the Galois representations. (The map $\mu_{\text{ét}}$ is even a ring homomorphism by the Künneth formula [Mil80, Corollary VI.8.23], but we do not need this fact.)

The étale realization (2.4) is useful for extracting information from a class in the Grothendieck ring of varieties. We record the following example to be used later.

Example 2.9. Let X be a geometrically rational smooth projective surface. Then since all cohomology classes on X are algebraic, $\mu_{\text{ét}}([X]) = [\mathbb{Q}_{\ell}] + [\mathbb{Q}_{\ell}(-2)] + [\text{NS}(X_{\bar{k}}) \otimes \mathbb{Q}_{\ell}(-1)]$, where $\mathbb{Q}_{\ell}$ is considered as a trivial 1-dimensional G_k -representation. In particular, if X and X' are two such surfaces and $[X] = [X']$, then $\text{NS}(X_{\bar{k}})$ and $\text{NS}(X'_{\bar{k}})$ have the same class in $K_0(\text{Rep}(G_k, \mathbb{Q}_{\ell}))$.

We explain how Gassmann equivalence relates to L-equivalence of reduced k -schemes of dimension zero. Let $\mathbf{EtSch}_k$ be the semiring of k -schemes which are étale over $\mathrm{Spec}(k)$. As we assume k to be perfect, $\mathbf{EtSch}_k$ is also the semiring of reduced k -schemes of dimension zero. Here in $\mathbf{EtSch}_k$, the addition (respectively, multiplication) is defined by disjoint unions (respectively, products over $\mathrm{Spec}(k)$). For every $Z \in \mathbf{EtSch}_k$, its base change $Z_{\bar{k}}$ to the separable closure $\bar{k}$ of k is endowed with a G_k -action. By Galois descent, the map $\mathbf{EtSch}_k \rightarrow \mathbf{Gset}$ sending $Z \in \mathbf{EtSch}_k$ to the underlying continuous G_k -set of $Z_{\bar{k}}$ is an isomorphism of semirings. As we assume k to be perfect, this induces a ring isomorphism

$$\mathbb{Z}[\mathrm{Var}^0/k] \simeq \mathrm{Burn}(G_k), \quad (2.6)$$

where Var^0/k denotes the set of (irreducible) k -varieties of dimension zero.

We have a natural ring homomorphism

$$\mathbb{Z}[\mathrm{Var}^0/k] \rightarrow K_0(\mathrm{Var}/k) \quad (2.7)$$

that sends Z to $[Z]$. It follows from the blowup presentation of the Grothendieck ring [Bit04, Theorem 3.1] that over fields of characteristic zero (2.7) admits a splitting given by $X \mapsto \mathrm{Spec}(H^0(X, \mathcal{O}_X))$ (X smooth projective); hence for characteristic zero fields, (2.7) is injective.

Throughout this text, the same notation $Z_{\bar{k}}$ (and also Z itself when it does not lead to any confusion) denotes the underlying G_k -set of $Z_{\bar{k}}$ for every étale k -scheme Z .

We work with Gassmann equivalence of such schemes considered as sets with Galois group action.

LEMMA 2.10. *Let Z and Z' be étale k -schemes. If Z and Z' are L-equivalent in the Grothendieck ring or in any of the modifications explained in Remark 2.8, then $Z_{\bar{k}}$ and $Z'_{\bar{k}}$ are Gassmann equivalent.*

Proof. We use the étale realization (2.4), which is defined by precomposition on $K_0^{\mathrm{bl}}(\mathrm{Var}/k)$ and also factors through the modified Grothendieck ring from Remark 2.8(2) [NS09, Proposition 4.1(3)].

For any étale k -scheme Z , we have

$$\mu_{\mathrm{ét}}([Z]) = [H_{\mathrm{ét}}^0(Z_{\bar{k}}, \mathbb{Q}_{\ell})] = [\mathbb{Q}_{\ell}[Z_{\bar{k}}]] = \mu_{G_k}([Z_{\bar{k}}]). \quad (2.8)$$

For each $j \geq 0$, we consider the composition

$$\mathrm{Burn}(G_k) \simeq \mathbb{Z}[\mathrm{Var}^0/k] \xrightarrow{\times \mathbb{L}^j} K_0(\mathrm{Var}/k) \xrightarrow{\mu_{\mathrm{ét}}} K_0(\mathrm{Rep}(G_k, \mathbb{Q}_{\ell})) \xrightarrow{\otimes \mathbb{Q}_{\ell}(j)} K_0(\mathrm{Rep}(G_k, \mathbb{Q}_{\ell})),$$

which is equal to μ_{G_k} from (2.1) by (2.5) and (2.8). Thus if Z and Z' are L-equivalent, then $Z_{\bar{k}}$ and $Z'_{\bar{k}}$ are Gassmann equivalent. $\square$

Remark 2.11. Properties of classes $[X]$ of smooth projective varieties in the Grothendieck rings often go in parallel with properties of their Chow motives [And04]. In our situation, by [And04, Exemple 4.1.6.1], if k has characteristic zero, then two étale k -schemes Z and Z' are Gassmann equivalent if and only if the Chow motives of Z and Z' with rational coefficients are isomorphic. Thus in this setting Lemma 2.10 says that L-equivalence implies isomorphism of Chow motives. The same result is expected for all smooth projective varieties, and it follows from the conjectural uniqueness of direct sum decompositions for Chow motives; see for example [Göt01, Conjectures 2.5 and 2.6].

Example 2.12. Let k be a field such that all continuous G_k -actions on finite sets factor through a finite cyclic quotient of G_k , for example $k = \mathbb{R}$ or $k = \mathbb{F}_q$. By Proposition 2.5 cyclic groups do not allow nontrivial Gassmann equivalence; hence by Lemma 2.10 L-equivalence of étale k -schemes implies their isomorphism.

COROLLARY 2.13. *Let Z and Z' be étale k -schemes. If Z and Z' are L-equivalent in the Grothendieck ring or any of the modifications explained in Remark 2.8, then $Z \simeq Z'$ as soon as one of the following conditions is satisfied:*

- *The scheme Z is k -irreducible and Galois over $\mathrm{Spec}(k)$.*
- *The scheme Z is k -irreducible with $\deg_k Z \leq 6$.*
- *The scheme Z satisfies $\deg_k Z \leq 5$.*

Proof. By Lemma 2.10 the $G_{\bar{k}}$ -sets $Z_{\bar{k}}$ and $Z'_{\bar{k}}$ are Gassmann equivalent, and by Proposition 2.5 they are isomorphic. $\square$

Example 2.14. Translating Example 2.7(1) into the language of étale schemes, we obtain the following. Let $k = \mathbb{Q}$, and define degree 6 schemes

$$\begin{aligned} Z &= \mathrm{Spec}(\mathbb{Q}(\sqrt{\alpha})) \sqcup \mathrm{Spec}(\mathbb{Q}(\sqrt{\beta})) \sqcup \mathrm{Spec}(\mathbb{Q}(\sqrt{\alpha\beta})), \\ Z' &= \mathrm{Spec}(\mathbb{Q}(\sqrt{\alpha}, \sqrt{\beta})) \sqcup \mathrm{Spec}(\mathbb{Q}) \sqcup \mathrm{Spec}(\mathbb{Q}), \end{aligned}$$

where we choose any $\alpha, \beta \in \mathbb{Q}^*$ which are nontrivial and distinct in $\mathbb{Q}^*/(\mathbb{Q}^*)^2$. Then Z and Z' are Gassmann equivalent and have isomorphic Chow motives (see Remark 2.11), but we do not know how to check if they are L-equivalent or not.

Embedding Z and Z' into $\mathbb{P}^2$ so that both images are in general position (for instance, if $\alpha + \beta \neq 1$, we can send Z onto $[\pm\sqrt{\alpha} : 1 : 0]$, $[0 : \pm\sqrt{\beta} : 1]$, $[1 : 0 : \pm\sqrt{\alpha\beta}]$ and Z' onto $[\pm\sqrt{\alpha} : \pm\sqrt{\beta} : 1]$, $[1 : 0 : 1]$, $[0 : 1 : 1]$), we obtain two del Pezzo surfaces of degree 3: $X = \mathrm{Bl}_Z(\mathbb{P}^2)$ and $X' = \mathrm{Bl}_{Z'}(\mathbb{P}^2)$. These two surfaces are not isomorphic as the first one has three k -rational lines and the second one has five. However, we have an isomorphism

$$\mathrm{NS}(X_{\bar{\mathbb{Q}}}) \otimes \mathbb{Q} \simeq \mathrm{NS}(X'_{\bar{\mathbb{Q}}}) \otimes \mathbb{Q}$$

of permutation Galois representations, which shows that the associated Galois set is not uniquely defined. One can make a similar example integrally, using integral Gassmann triples as in [Sco93, Pra17]. See Example 5.12, where we explain that using our techniques we *can* recover Z (respectively, Z') from X (respectively, X'), providing another proof that they are not isomorphic.

Finally note that we do not know if $[X] = [X']$ as by the blowup relation in the Grothendieck ring of varieties, $[X] = [\mathbb{P}^2] + \mathbb{L}[Z]$ and $[X'] = [\mathbb{P}^2] + \mathbb{L}[Z']$, so $[X] = [X']$ would imply the L-equivalence of Z and Z' , which is unknown.

3. Factorization centers

In this section, we introduce the invariant $c(\phi)$ keeping track of the factorization centers in any decomposition of a birational isomorphism of ϕ between surfaces into a sequence of blowups and blowdowns, formulate the main theorem and interpret it in terms of the truncated Grothendieck ring of varieties.

3.1 Formulation of the main result

Fix a perfect field k . Let $\phi: X \dashrightarrow Y$ be a birational isomorphism of smooth projective k -surfaces. By the strong factorization theorem [Man86, Corollary of Lemma III.4.4] (or [Sta18, Lemma 54.17.2]), we have a decomposition

$$\begin{array}{ccc} & \tilde{X} & \\ \alpha \swarrow & & \searrow \beta \\ X & \dashrightarrow \phi \dashrightarrow & Y \end{array} \quad (3.1)$$

with both α and β being compositions of blowups with smooth centers $Z_1, \dots, Z_r$ and $Z'_1, \dots, Z'_s$, respectively, which are zero-dimensional smooth schemes. The *factorization center* of ϕ is defined as

$$c(\phi) = \sum_{i=1}^r [Z_i] - \sum_{i=1}^s [Z'_i] \in \mathbb{Z}[\mathrm{Var}^0/k] \quad (3.2)$$

(see Section 2.2 for the definition of $\mathbb{Z}[\mathrm{Var}^0/k]$).

We explain the well-definedness of c and its basic properties. To do that, it is most convenient to consider the groupoid Bir_2/k of birational types of surfaces, whose objects are smooth projective surfaces, and whose morphisms are birational isomorphisms.

Recall that if $\mathcal{C}$ is a groupoid and G a group, a homomorphism from $\mathcal{C}$ to G is a functor from $\mathcal{C}$ to G , where G is considered as a groupoid with one object.

LEMMA 3.1. *The factorization center $c(\phi)$ does not depend on the choice of factorization of ϕ and defines a homomorphism $c: \mathrm{Bir}_2/k \rightarrow \mathbb{Z}[\mathrm{Var}^0/k]$. Explicitly, for any two birational isomorphisms of surfaces $\phi: X \dashrightarrow X'$ and $\psi: X' \dashrightarrow X''$, we have*

$$c(\psi \circ \phi) = c(\psi) + c(\phi).$$

In particular, for any surface X , we have a homomorphism

$$c: \mathrm{Bir}(X) \rightarrow \mathbb{Z}[\mathrm{Var}^0/k].$$

Proof. Consider the diagram (3.1). Let $E_1, \dots, E_m \subset X$ (respectively, $E'_1, \dots, E'_n \subset Y$) be the irreducible components of the exceptional divisor of ϕ (respectively, ϕ^{-1}). Let $D_1, \dots, D_t \subset \tilde{X}$ be the irreducible divisors which are contracted by both α and β . This way the centers Z_i , $i = 1, \dots, r$ (respectively, Z'_i , $i = 1, \dots, s'$), are in one-to-one correspondence with the collection of divisors $\{E_i\}_{i=1, \dots, m} \cup \{D_j\}_{j=1, \dots, t}$ (respectively, $\{E'_i\}_{i=1, \dots, n} \cup \{D_j\}_{j=1, \dots, t}$).

Note that each prime divisor D contracted by α or β is birational to $\mathbb{P}^1 \times Z$, where Z is the center of the corresponding blowup, which can be recovered from D as $Z \simeq \mathrm{Spec}(H^0(D, \mathcal{O}_D))$. Canceling out the centers corresponding to $D_1, \dots, D_t$, we obtain

$$c(\phi) = \sum_{i=1}^r [Z_i] - \sum_{i=1}^s [Z'_i] = \sum_{i=1}^m [\mathrm{Spec}(H^0(E_i, \mathcal{O}))] - \sum_{i=1}^n [\mathrm{Spec}(H^0(E'_i, \mathcal{O}))].$$

Thus the expression (3.2) only depends on the exceptional divisors of ϕ and ϕ^{-1} ; hence $c(\phi)$ is independent of the choice of strong factorization (3.1).

To show that $c(\psi \circ \phi) = c(\psi) + c(\phi)$, consider the diagram

$$\begin{array}{ccccc}
 & & X_3 & & \\
 & \swarrow \mu & & \searrow \mu' & \\
 X_1 & & & & X_2 \\
 \swarrow \tau & & & & \searrow \nu' \\
 X & \xrightarrow{\phi} & X' & \xrightarrow{\psi} & X''
 \end{array}$$

where τ, τ' (respectively, ν, ν') provide a strong factorization for ϕ (respectively, ψ) and μ, μ' provide a strong factorization for $\nu^{-1}\tau'$. It is clear that c is additive on regular birational isomorphisms; hence

$$\begin{aligned}
 c(\psi \circ \phi) &= c(\nu' \circ \mu') - c(\tau \circ \mu) \\
 &= c(\nu' \circ \mu') - c(\nu \circ \mu') + c(\tau' \circ \mu) - c(\tau \circ \mu) \\
 &= c(\nu') - c(\nu) + c(\tau') - c(\tau) = c(\psi) + c(\phi).
 \end{aligned}$$

□

COROLLARY 3.2. *There is a unique assignment $\phi \mapsto c(\phi) \in \mathbb{Z}[\text{Var}^0/k]$ defined for all birational isomorphisms ϕ between smooth projective k -surfaces, satisfying the following axioms:*

- (1) *For any (biregular) isomorphism ϕ , we have $c(\phi) = 0$.*
- (2) *If $\tilde{X}$ is a blowup of X with smooth connected center Z and $\phi: \tilde{X} \rightarrow X$ the contraction map, then $c(\phi) = -[Z]$ and $c(\phi^{-1}) = [Z]$.*
- (3) *For composable birational isomorphisms, $c(\psi \circ \phi) = c(\psi) + c(\phi)$.*

Proof. The uniqueness follows from the strong factorization (3.1), and the existence is the content of Lemma 3.1. □

Remark 3.3. The invariant $c(\phi)$ is generalized to higher-dimensional varieties in [LS22], to the equivariant setting in [KT22] and to varieties with a logarithmic form in [CKT23]. In all these contexts, properties of $c(\phi)$ are analogous to those stated in Corollary 3.2.

PROPOSITION 3.4. (1) *For any field extension L/k we have a commutative diagram*

$$\begin{array}{ccccc}
 \text{Bir}(X, Y) & \xrightarrow{c} & \mathbb{Z}[\text{Var}^0/k] & \xrightarrow[\text{(2.6)}]{\simeq} & \text{Burn}(G_k) \\
 \downarrow & & \downarrow & & \downarrow \\
 \text{Bir}(X_L, Y_L) & \xrightarrow{c} & \mathbb{Z}[\text{Var}^0/L] & \xrightarrow[\text{(2.6)}]{\simeq} & \text{Burn}(G_L),
 \end{array}$$

where the left vertical arrow is an extension of scalars for birational maps, the middle vertical arrow is defined by extension of scalars, that is, it maps an étale k -scheme Z to the sum of the connected components of $Z \times_k L$, and the right vertical arrow is the restriction of the group action defined through the map $\text{Gal}(\bar{L}/L) \rightarrow \text{Gal}(\bar{k}/k)$ induced by any choice of embedding $\bar{k} \hookrightarrow \bar{L}$ compatible with $k \hookrightarrow L$. In particular, applying the diagram to an automorphism $\sigma \in \text{Aut}(k)$ (with $L = k$), the left square gives $c(\sigma(\phi)) = \sigma(c(\phi))$; that is, c commutes with the group action by the automorphisms of the field.

- (2) *For any finite field extension L/k and surfaces X, Y over L , let $X|_k, Y|_k$ denote the under-*

lying k -surfaces. We have the following commutative diagram:

$$\begin{array}{ccccc} \mathrm{Bir}(X, Y) & \xrightarrow{c} & \mathbb{Z}[\mathrm{Var}^0/L] & \xrightarrow[\text{(2.6)}]{\simeq} & \mathrm{Burn}(G_L) \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{Bir}(X|_k, Y|_k) & \xrightarrow{c} & \mathbb{Z}[\mathrm{Var}^0/k] & \xrightarrow[\text{(2.6)}]{\simeq} & \mathrm{Burn}(G_k), \end{array}$$

where the middle vertical map is the restriction of scalars.

Let us clarify the notation $X|_k$ in the proposition. By $X|_k$ we mean the k -surface given by the composition $X \rightarrow \mathrm{Spec}(L) \rightarrow \mathrm{Spec}(k)$ (this is not the Weil restriction of scalars). If L/k is nontrivial, the surface $X|_k$ is not geometrically connected: $X|_k \times_k L$ is isomorphic to a disjoint union of $[L : k]$ copies of X .

Proof. By Lemma 3.1, in items (1) and (2), it is sufficient to check a single blowup, where the statements are clear. $\square$

The main result of the paper is the following.

THEOREM 3.5. *For any two smooth projective k -surfaces X, Y and any two birational isomorphisms $\phi, \psi: X \dashrightarrow Y$, we have $c(\phi) = c(\psi)$. In particular, $c: \mathrm{Bir}(X) \rightarrow \mathbb{Z}[\mathrm{Var}^0/k]$ is the zero map.*

We prove Theorem 3.5 at the end of Section 5. The result is straightforward when k is algebraically closed because in this case the invariant takes values in $\mathbb{Z} \cdot [\mathrm{Spec}(k)] \simeq \mathbb{Z}$ and measures the difference of the ranks of two Néron–Severi groups (cf. Proposition 4.4). However, Theorem 3.5 becomes a nontrivial statement when k is an arbitrary perfect field, and its proof depends on the 2-dimensional minimal model program. This result is also specific for surfaces and fails to be true in higher dimension, even over algebraically closed fields [LS22].

Example 3.6. Let $k = \mathbb{R}$. In this case Theorem 3.5 says that birational automorphisms of surfaces blow up the same number of rational points as the number of rational divisors they blow down. It also says that they blow up the same number of pairs of complex conjugate points as the number of pairs of complex conjugate divisors they blow down. This can be proved directly by considering the Galois action on the Néron–Severi group (see also Example 2.12).

Remark 3.7. A different proof of Theorem 3.5 in the case when the surfaces X and Y are rational can be deduced from a more recent result by Lamy and Schneider [LS21], who analyze relations between Sarkisov links to prove that $\mathrm{Cr}_2(k)$ is generated by involutions. This implies that every homomorphism from $\mathrm{Cr}_2(k)$ to a free abelian group is trivial (for $k = \mathbb{R}$ one can also deduce this from [Zim18, Theorem 1.1]). On the other hand, this is not true for other types of del Pezzo surfaces: Blanc, Schneider and Yasinsky have constructed nontrivial homomorphisms $\mathrm{Bir}(X) \rightarrow \mathbb{Z}$ for Severi–Brauer surfaces X ; see [BSY22].

Example 3.8. Consider the following composition ϕ of type II links (see Definition 4.2) between del Pezzo surfaces dP_d (where $d = K^2$ stands for the degree)

$$\begin{array}{ccccccc} & \mathrm{dP}_7 & & \mathrm{dP}_3 & & \mathrm{dP}_4 & \\ & \swarrow \mathrm{Bl}_2 & \searrow \mathrm{Bl}_1 & \swarrow \mathrm{Bl}_5 & \searrow \mathrm{Bl}_2 & \swarrow \mathrm{Bl}_1 & \searrow \mathrm{Bl}_5 \\ \phi: \mathbb{P}^2 & \dashrightarrow & \mathrm{dP}_8 & \dashrightarrow & \mathrm{dP}_5 & \dashrightarrow & \mathbb{P}^2, \end{array}$$

where each map Bl_i is the blowup (respectively, blowdown) along a Galois orbit Z_i (respectively, Z'_i) of degree i . See [Isk96] for the general results on links, or see the explicit constructions explained below for the existence of the three links above. We have

$$c(\phi) = [Z_2] - [k] + [Z_5] - [Z'_2] + [k] - [Z'_5] = [Z_2] - [Z'_2] + [Z_5] - [Z'_5],$$

so that $c(\phi) = 0$ is equivalent to $Z_2 \simeq Z'_2$ and $Z_5 \simeq Z'_5$.

One way to understand this is to read the diagram as a composition of two different well-known rationality constructions for the del Pezzo surface dP_5 : one blows up a point P and contracts a Galois orbit of five (-1) -curves which are proper preimages of conics through P (connecting dP_5 with $\mathbb{P}^2$ moving right), or one blows up a Galois orbit of two points Q_1 and Q_2 and contracts five (-1) -curves obtained as proper preimages of cubics passing through Q_1 and Q_2 onto a rational quadric dP_8 and then transforms it to $\mathbb{P}^2$ (connecting dP_5 with $\mathbb{P}^2$ moving left). This way dP_5 has two potentially different rationality centers that can be blown up by $\mathbb{P}^2 \dashrightarrow \text{dP}_5$, namely Z_5 and Z'_5 , and the result is that these centers are in fact isomorphic (see Corollary 5.9 and Example 5.10).

3.2 Interpretation in terms of the Grothendieck ring of varieties

From the perspective of the Grothendieck ring of varieties, we have the following interpretation of $c(\phi)$.

LEMMA 3.9. *For any birational isomorphism $\phi: X \dashrightarrow Y$ between smooth projective surfaces, we have the following identity in $K_0(\text{Var}/k)$:*

$$[Y] = [X] + \mathbb{L} \cdot \bar{c}(\phi), \quad (3.3)$$

where $\bar{c}$ is the composition of c with the natural homomorphism $\mathbb{Z}[\text{Var}^0/k] \rightarrow K_0(\text{Var}/k)$ from (2.7).

Proof. Using Lemma 3.1 we see that (3.3) is preserved under compositions of birational isomorphisms. Since birational isomorphisms are decomposed into blowups and blowdowns along k -étale subschemes, it suffices to check Lemma 3.9 for a single blowup, where the result is clear by Corollary 3.2(2) and the blowup formula in the Grothendieck ring. $\square$

Remark 3.10. We see that $\mathbb{L} \cdot \bar{c}(\phi) \in K_0(\text{Var}/k)$ only depends on $[X]$ and $[Y]$; however, it is known that $\mathbb{L}$ is a zero-divisor [Bor18], so a priori we cannot divide $\mathbb{L} \cdot \bar{c}(\phi)$ by $\mathbb{L}$ and deduce that $\bar{c}(\phi)$ (or $c(\phi)$) depends only on X and Y but not on ϕ .

We can informally reformulate Theorem 3.5 as the statement that zero-dimensional k -varieties cannot be L -equivalent via surfaces. Let us explain this. For each $n \geq 0$, consider $K_0(\text{Var}^{\leq n}/k)$, the abelian group generated by isomorphism classes of varieties of dimension at most n , modulo cut-and-paste relations (2.3). For each $n \leq m$, there is a (generally noninjective) group homomorphism

$$K_0(\text{Var}^{\leq n}/k) \rightarrow K_0(\text{Var}^{\leq m}/k),$$

and $K_0(\text{Var}/k)$, as an abelian group, is the colimit of this system. For every n , there is a surjective homomorphism

$$K_0(\text{Var}^{\leq n}/k) \rightarrow \mathbb{Z}[\text{Bir}_n/k],$$

where Bir_n/k is the set of k -birational classes of dimension n . The kernel of this homomorphism is spanned by varieties of dimension at most $n-1$, but it may not be isomorphic to $K_0(\text{Var}^{\leq n-1}/k)$; see [Zak17] for an interpretation of this in terms of an algebraic K-theory spectral sequence.

For $n = 0$, the group $K_0(\text{Var}^{\leq 0}/k)$ is canonically isomorphic to $\mathbb{Z}[\text{Var}^0/k]$, and $K_0(\text{Var}^{\leq 1}/k)$ fits into a split exact sequence

$$0 \rightarrow \mathbb{Z}[\text{Var}^0/k] \rightarrow K_0(\text{Var}^{\leq 1}/k) \rightarrow \mathbb{Z}[\text{Bir}_1/k] \rightarrow 0. \quad (3.4)$$

Our main results can be reformulated as results about $K_0(\text{Var}^{\leq 2}/k)$.

COROLLARY 3.11 (of Theorem 3.5). (1) We have a short exact sequence

$$0 \rightarrow \mathbb{Z}[\text{Var}^0/k] \oplus \mathbb{Z}[\text{Bir}_1/k] \rightarrow K_0(\text{Var}^{\leq 2}/k) \rightarrow \mathbb{Z}[\text{Bir}_2/k] \rightarrow 0, \quad (3.5)$$

where the last map sends a combination of varieties to the birational classes of its 2-dimensional components, and the first map sends a k -variety Z of dimension zero to $[Z]$ and a birational class of a curve to the class of its unique smooth projective model.

(2) We have a (noncanonical) splitting

$$K_0(\text{Var}^{\leq 2}/k) \simeq \mathbb{Z}[\text{Var}^0/k] \oplus \mathbb{Z}[\text{Bir}_1/k] \oplus \mathbb{Z}[\text{Bir}_2/k],$$

and in particular $K_0(\text{Var}^{\leq 2}/k)$ is a (torsion-)free abelian group.

(3) If Z and Z' are étale k -schemes and

$$\mathbb{L}^i \cdot ([Z] - [Z']) = 0$$

in $K_0(\text{Var}^{\leq 2}/k)$ with $i \in \{0, 1, 2\}$ (where $\mathbb{L}^i \cdot ([Z] - [Z'])$ is represented by $[\mathbb{A}^i \times_k Z] - [\mathbb{A}^i \times_k Z']$), then Z and Z' are isomorphic.

Proof. (1) Because resolution of singularities and weak factorization (even strong factorization) are known for surfaces over arbitrary perfect fields, the proof of [Bit04] goes through to show that we have an isomorphism

$$K_0^{\text{bl}}(\text{Var}^{\leq 2}/k) \rightarrow K_0(\text{Var}^{\leq 2}/k),$$

where the first group is defined by smooth projective varieties of dimension at most 2 and Bittner's blowup relations. These relations can be equivalently presented as

$$[Y] = [X] + \mathbb{L} \cdot \bar{c}(\phi) \quad (3.6)$$

for all birational isomorphisms $\phi: X \dashrightarrow Y$ between smooth projective surfaces. Indeed, the relations (3.6) include the blowup relations if $\phi: Y \rightarrow X$ is a smooth blowup as a particular case, and conversely, the relations (3.6) follow as soon as we impose the blowup relations as in the proof of Lemma 3.9. Since curves admit unique smooth projective models, we have the canonical splitting of (3.4), giving

$$K_0(\text{Var}^{\leq 1}/k) = K_0^{\text{bl}}(\text{Var}^{\leq 1}/k) \simeq \mathbb{Z}[\text{Var}^0/k] \oplus \mathbb{Z}[\text{Bir}_1/k].$$

Furthermore, we have an obvious short exact sequence

$$K_0^{\text{bl}}(\text{Var}^{\leq 1}/k) \rightarrow K_0^{\text{bl}}(\text{Var}^{\leq 2}/k) \rightarrow \mathbb{Z}[\text{Bir}_2/k] \rightarrow 0,$$

and to prove (3.5) it suffices to show that the first map in the sequence is split-injective.

The splitting is based on Theorem 3.5 and is not canonical. First of all, we choose a smooth projective representative for each birational class of surfaces. If L is a 2-dimensional function field, we write M_L for the chosen model. We define the splitting

$$\epsilon: K_0^{\text{bl}}(\text{Var}^{\leq 2}/k) \rightarrow K_0^{\text{bl}}(\text{Var}^{\leq 1}/k)$$

by the identity on classes of smooth projective curves and zero-dimensional schemes, and if X is a smooth projective surface, we define

$$\epsilon([X]) = \mathbb{L} \cdot \bar{c}(M_{k(X)} \xrightarrow{\psi} X)$$

for any choice of a birational isomorphism ψ between X and its model $M_{k(X)}$. The fact that this is independent of ψ is the content of Theorem 3.5, and the fact that ϵ is well defined, that is, preserves the relations (3.6), follows immediately from the property that c is additive on compositions (Lemma 3.1) applied to a composition of birational isomorphisms $M \xrightarrow{\psi} X \xrightarrow{\phi} Y$ (where $M := M_{k(X)} = M_{k(Y)}$).

(2) This part follows from (3.5); explicit splittings are constructed in the proof of part (1).

(3) The case $i = 2$ is clear because if $\mathbb{A}^i \times_k Z$ and $\mathbb{A}^i \times_k Z'$ are k -birational, then Z and Z' are isomorphic. For $i = 1$, the element $\mathbb{L} \cdot ([Z] - [Z'])$ is the image of $(-[Z] + [Z'], [\mathbb{P}^1 \times_k Z] - [\mathbb{P}^1 \times_k Z'])$ under the first map in (3.5), and since this map is injective, $[Z] = [Z']$; that is, Z and Z' are isomorphic. Finally, the case $i = 0$ follows again from (3.5). $\square$

4. Birational geometry of surfaces

In this section, we recall the minimal model program and the Sarkisov link decomposition for surfaces, which is used in our proof of Theorem 3.5. In Proposition 4.4, we prove Theorem 3.5 for birational types with particularly simple links. In Section 4.2, we investigate linear systems of rational curves on del Pezzo surfaces, which will be needed to finish the proof of Theorem 3.5 in Section 5.

4.1 Birational classification of surfaces and links

Let X/k be a geometrically irreducible surface. Recall that X is called minimal if it does not have a Galois orbit of disjoint (-1) -curves, or equivalently, every regular birational map $X \rightarrow X'$ from X to a smooth surface is an isomorphism. We say that X is rational if it is birational to $\mathbb{P}^2$ over k and that X is geometrically rational if it is birational to $\mathbb{P}^2$ over $\bar{k}$. A del Pezzo surface is a smooth projective geometrically irreducible surface with ample anticanonical class.

A conic bundle is a fibered surface $\pi: X \rightarrow C$, with C a smooth projective curve, such that the generic fiber of π is a smooth rational curve and X has Picard rank 2. For a conic bundle $\pi: X \rightarrow C$, the sheaf $\pi_* \omega_{X/C}^\vee$ is locally free of rank 3, and we have an embedding $X \hookrightarrow \mathbb{P}(\pi_* \omega_{X/C}^\vee)$ over C realizing each fiber of $\pi: X \rightarrow C$ as a plane conic (see [Has09, Corollary 3.7]). The number m of geometric singular fibers of a conic bundle over a curve of genus zero equals

$$m = 8 - K_X^2, \tag{4.1}$$

see for example [Isk79, Theorem 3]; in particular, $K_X^2 \leq 8$.

We write $\text{NS}(X)$ for the Néron–Severi group of divisors modulo algebraic equivalence; by the theorem of Néron–Severi, it is a finitely generated abelian group. Each contraction of a Galois orbit of (-1) -curves decreases the rank of the Néron–Severi group; hence a sequence of contractions always terminates to produce a minimal surface birationally equivalent to the given one.

We have the following classification result going back to Enriques, Manin and Iskovskikh; see [Kol96, Theorem III.2.2].

THEOREM 4.1 (Minimal model program in dimension 2). *Any geometrically irreducible minimal surface X/k is isomorphic to exactly one of the following:*

- *Non-geometrically rational case:*
 - (1) *Surface with nef K_X*
 - (2) *Conic bundle $\pi: X \rightarrow C$ with $g(C) > 0$*
- *Geometrically rational case:*
 - (3) *Conic bundle $X \rightarrow C$ with $g(C) = 0$*
 - (4) *Minimal del Pezzo surface of Picard rank 1*

In cases (2) and (3), $\text{NS}(X)$ is of rank 2.

Note that if k is algebraically closed, any conic bundle in case (2) or (3) has no singular fibers (as components of singular fibers are (-1) -curves); thus case (2) consists of ruled surfaces, and case (3) consists of Hirzebruch surfaces $\mathbb{F}_n$.

DEFINITION 4.2 (Sarkisov links, see [Isk96, § 2.2]). Let $X \rightarrow B$ and $X' \rightarrow B'$ be Mori fiber spaces with $\dim X = \dim X' = 2$. Here each Mori fiber space is either a del Pezzo surface of Picard rank 1 or a conic bundle of Picard rank 2 over a smooth curve. A Sarkisov link between X and X' is a birational map $\nu: X \dashrightarrow X'$ satisfying one of the following descriptions:

- (Type I) The map ν^{-1} is the blowup of X at a smooth closed point with $B = \text{Spec}(k)$ and $X' \rightarrow B'$ a conic bundle.
- (Type II) We have a commutative diagram

$$\begin{array}{ccccc} X & \xleftarrow{\alpha} & Y & \xrightarrow{\beta} & X' \\ \downarrow & & & & \downarrow \\ B & \xrightarrow{\sim} & & & B' \end{array}, \quad (4.2)$$

and $\nu = \beta \circ \alpha^{-1}$, where both α and β are blowups at a smooth closed point. In this case,

- (Type IIC) either both $X \rightarrow B$ and $X' \rightarrow B'$ are conic bundles,
- (Type IID) or $B = B' = \text{Spec}(k)$.
- (Type III) This is the inverse of link (I).
- (Type IV) The Mori fiber spaces $X \rightarrow B$ and $X' \rightarrow B'$ are conic bundles, and ν is an isomorphism not respecting the conic bundle structures.

We rely on decomposing birational isomorphisms of surfaces into Sarkisov links [Isk96]. Some of the links are easier to deal with.

LEMMA 4.3. *If $\nu: X \dashrightarrow X'$ is a Sarkisov link of type IIC, then $K_X^2 = K_{X'}^2$, $\text{rk NS}(X) = \text{rk NS}(X')$, and $c(\nu) = 0$.*

Proof. Let $\phi: X \rightarrow B$ be the conic bundle as in (4.2). By [Isk96, Theorem 2.6], the map ν is an elementary transformation of $X \rightarrow B$. More precisely, α in (4.2) is the blowup at a smooth closed point $p \in X$ lying in a smooth fiber C of ϕ , and β in (4.2) is the contraction of the proper transformation $\tilde{C}$ of C under α . Since $\tilde{C} \simeq C \simeq \mathbb{P}^1 \times_{\text{Spec}(k)} \{p\}$, the blowup center of β is isomorphic to p . Hence $\text{rk NS}(X) = \text{rk NS}(X')$ and $c(\nu) = 0$. The equality $K_X^2 = K_{X'}^2$ follows from the fact that $X \rightarrow B$ and $X' \rightarrow B'$ have the same number of geometric singular fibers, as can be seen using (4.1). $\square$

Recall that the degree of a geometrically rational surface X is defined to be K_X^2 . The next result is a step toward Theorem 3.5.

PROPOSITION 4.4. *Let X, X' be a pair of birational minimal geometrically irreducible surfaces. Assume that X is either (1) geometrically irrational or (2) geometrically rational and of degree at most 4; then the same holds for X' , and for any birational isomorphism $\phi: X \dashrightarrow X'$, we have*

$$c(\phi) = (\mathrm{rk} \mathrm{NS}(X') - \mathrm{rk} \mathrm{NS}(X)) \cdot [\mathrm{Spec}(k)]. \quad (4.3)$$

In particular, $c(\phi) = 0$ for birational automorphisms of such surfaces.

Proof. (1) It is clear that X' is geometrically irrational and, moreover, X and X' have the same type in Theorem 4.1. If X, X' have nef canonical class, then every birational isomorphism is a biregular isomorphism by [IS96, § II.7.3, Corollary 1], so that $X \simeq X'$ and $c(\phi) = 0$ as required.

If X, X' are conic bundles over a curve of positive genus, then by [Sch22, Corollary 3.2], birational maps between them can be decomposed into Sarkisov links of type IIC, in which case $\mathrm{rk} \mathrm{NS}(X) = \mathrm{rk} \mathrm{NS}(X')$ and $c(\phi) = 0$ by Lemma 4.3.

(2) It follows from the classification of links in [Isk96, Theorem 2.6] that elementary links will only connect X to minimal surfaces of degree at most 4 or conic bundles of degree 3 and Picard rank 2, and that birational isomorphisms between such surfaces will be decomposed into

- Biregular isomorphisms
- Type IIC links
- Bertini and Geiser involutions
- Blowups of a rational point (between degree 4 and degree 3 surfaces)

For each of these types, the claim of Proposition 4.4 is true; namely, in the first three cases $\mathrm{rk} \mathrm{NS}(X) = \mathrm{rk} \mathrm{NS}(X')$ and $c(\phi) = 0$ (using Lemma 4.3 for the second case), while in the last case the ranks differ by 1 and the result is true by the definition of c . Equality (4.3) follows by the additivity of c under compositions (Lemma 3.1). $\square$

Remark 4.5. From the proof of Proposition 4.4, we note that in (4.3), we have actually $c(\phi) = 0$ or $c(\phi) = \pm[\mathrm{Spec}(k)]$. Moreover, $c(\phi) = \pm[\mathrm{Spec}(k)]$ only happens when ϕ is a map between a del Pezzo surface of degree 4 and a conic bundle of degree 3.

4.2 Rational curves on del Pezzo surfaces

Linear systems of rational curves on del Pezzo surfaces are closely related to factorization centers: for instance, to create a rational 2-dimensional quadric X , one needs to blow up $\mathbb{P}^2$ in Z_2 , where Z_2 is a degree 2 étale k -scheme, and to contract a line through the center. This way the original scheme Z_2 can be recovered as the scheme parametrizing rulings on X (cf. Definition 5.2).

DEFINITION 4.6. Let $j \geq 1$. We call a complete linear system $|L|$ of curves on a del Pezzo surface X a *linear system of degree j rational curves* if a general member $C \in |L|$ is a smooth rational curve and $(-K_X) \cdot L = j$.

For each $j \geq 1$, we consider $\mathcal{H}^j(X)$, the Hilbert scheme of curves on X with general members of each component being smooth rational curves of degree j . By an easy computation (see Lemma 4.7(1)), we have $H^1(X, L) = 0$; hence $\mathcal{H}^j(X)$ is smooth [Kol96, Theorem 2.8]. In fact, over the algebraic closure $\bar{k}$, each $\mathcal{H}^j(X_{\bar{k}})$ is a disjoint union of projective spaces parametrizing effective divisors in the corresponding linear systems. We refer to points of $\mathcal{H}^1(X)$ as lines on

X , points of $\mathcal{H}^2(X)$ as conics and so on. When X is a twisted form of $\mathbb{P}^1 \times \mathbb{P}^1$ (a minimal del Pezzo surface of degree 8), all curves have even degree, and families of conics on X are also called rulings.

We note that by the adjunction formula, a linear system L of rational curves of degree j satisfies

$$L^2 = j - 2, \quad (-K_X) \cdot L = j. \quad (4.4)$$

However, the latter numerical conditions are not sufficient to deduce that a linear system consists of rational curves. Indeed, the linear system $3H + E$ on $\text{Bl}_P(\mathbb{P}^2)$, where H is the class of a line on $\mathbb{P}^2$ and E is the exceptional divisor, satisfies (4.4) with $j = 10$ but has a fixed component and contains no rational curves.

We investigate how to find all linear systems of rational curves of a given degree following [Man86]. The first step is to solve (4.4). Let X be a del Pezzo surface of degree $d = K_X^2$. Assume that $X_{\bar{k}}$ is not isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$, so that $X_{\bar{k}}$ is a blowup of $\mathbb{P}^2$ in $r := 9 - d$ points by [Man86, Theorem IV.2.5], [Isk79, § 3]. Write $D = aH - \sum_{i=1}^r b_i E_i$, where H is the pullback of the hyperplane class and the E_i are the irreducible components of the exceptional divisor. Since $-K_X = 3H - \sum_{i=1}^r E_i$, relation (4.4) translates into

$$\begin{cases} \sum_{i=1}^r b_i = 3a - j, \\ \sum_{i=1}^r b_i^2 = a^2 - j + 2. \end{cases} \quad (4.5)$$

The case $j = 1$ is that of lines on del Pezzo surfaces [Man86].

LEMMA 4.7. *Let X be a del Pezzo surface of degree d over an algebraically closed field k .*

- (1) *A linear system of rational curves of degree j satisfies $h^0(X, L) = j$, and higher cohomology groups of L vanish.*
- (2) *Any linear system satisfying (4.4) with $j \geq 1$ is nonempty.*
- (3) *A linear system satisfying (4.4) with $1 \leq j \leq d - 1$ is a linear system of rational curves.*
- (4) *Let $1 \leq j \leq d - 1$. The assignment $|D| \mapsto |-K_X - D|$ establishes a bijection between linear systems of rational curves of degrees j and $d - j$.*

Proof. (1) Let $C \in |L|$ be a smooth rational curve. Taking cohomology for the short exact sequence

$$0 \rightarrow \mathcal{O}_X \rightarrow \mathcal{O}_X(C) \rightarrow \mathcal{O}_C(C) \rightarrow 0,$$

we get $H^1(X, \mathcal{O}_X(C)) = H^2(X, \mathcal{O}_X(C)) = 0$, and hence $h^0(X, \mathcal{O}_X(C)) = j$ by the Riemann–Roch theorem.

(2) We have $h^2(X, L) = h^0(X, L^\vee \otimes \omega_X) = 0$ (because $L^\vee \otimes \omega_X$ has negative anticanonical degree), and the Riemann–Roch theorem implies

$$h^0(X, L) \geq j.$$

(3) If $X \simeq \mathbb{P}^1 \times \mathbb{P}^1$, then it is easy to see that solutions of (4.4) are precisely divisors $(a, 1)$, $a \geq 0$, and $(1, b)$, $b \geq 0$, and these are linear systems of rational curves (no upper bound on the degree needed). Now assume that X is a blowup of $\mathbb{P}^2$ in $0 \leq r \leq 8$ points. Since D satisfies (4.4), it follows that $-K_X - D$ satisfies (4.4) with degree $d - j$. By part (2), both $|D|$ and $|-K_X - D|$ are nonempty. Therefore, if $D = aH - \sum_{i=1}^r b_i E_i$, then $0 \leq a \leq 3$. Solving equations (4.5)

with $b_i \in \mathbb{Z}$ under the assumption $1 \leq j \leq d-1$ gives rise to the following divisors, which we list up to reordering of the exceptional divisors:

- $a = 0$: E_1
- $a = 1$: $H - \sum_{i=1}^t E_i$
- $a = 2$: $2H - \sum_{i=1}^{r-t} E_i$
- $a = 3$: $3H - 2E_1 - \sum_{i=2}^r E_i$

Indeed, the cases $a = 0, 1$ are straightforward, and the cases $a = 2, 3$ follow via the substitution $D \mapsto -K_X - D$. The $a = 1, 2$ cases are only possible when $r \leq 7$ ($d \geq 2$). We must have $0 \leq t \leq 2$, and for $r = 6$ (respectively, $r = 7$), only allow $t = 1, 2$ (respectively, $t = 2$). Under these conditions each of the divisors in the list is linearly equivalent to a smooth rational curve.

(4) This follows from part (3). $\square$

PROPOSITION 4.8. *Let X/k be a del Pezzo surface of degree d .*

- (1) *For each $j \geq 1$, the scheme $\mathcal{H}^j(X)$ is either empty or a smooth Severi–Brauer fibration of relative dimension $j-1$ over a smooth zero-dimensional scheme $\mathcal{M}^j(X)$. For each $1 \leq j \leq d-1$, we have a natural isomorphism $\mathcal{M}^j(X) \simeq \mathcal{M}^{d-j}(X)$.*
- (2) *Assume that $\mathcal{H}^j(X)$ is nonempty, and let $Z \subset X$ be a zero-dimensional subscheme of degree $j-1$. Let $\mathcal{H}^j(X, Z)$ be the closed subvariety of $\mathcal{H}^j(X)$ parametrizing curves containing Z . If $\mathcal{H}^j(X, Z)$ is nonempty and zero-dimensional, then it is isomorphic to $\mathcal{M}^j(X)$.*
- (3) *Suppose that $X_{\bar{k}}$ is obtained by blowing up $r \leq 5$ points on $\mathbb{P}^2$ with exceptional divisors $E_1, \dots, E_r$. Let $H \in \text{NS}(X_{\bar{k}})$ be the pullback of the hyperplane class. Then the classes of conics (respectively, cubics) on $X_{\bar{k}}$ are $H - E_i$ and $2H - \sum_{i=1}^4 E_i$ (respectively, H and $2H - \sum_{i=1}^3 E_i$) for some reordering of the exceptional divisors.*
- (4) *Consider the following cases:*
 - (dP₈) *Let X be a minimal del Pezzo surface of degree 8. Then $\mathcal{M}^4(X) \simeq \text{Spec}(k)$ and $\mathcal{M}^2(X) \simeq \mathcal{M}^6(X) \simeq Z_2$, where Z_2 is an étale k -scheme of degree 2.*
 - (dP₆) *Let X be a del Pezzo surface of degree 6. Then*

$$\mathcal{M}^2(X) \simeq \mathcal{M}^4(X) \simeq Z_3, \quad \mathcal{M}^3(X) \simeq Z_2$$

for degree 2 (respectively, degree 3) étale k -schemes Z_2, Z_3 .

- (dP₅) *Let X be a del Pezzo surface of degree 5. Then $\mathcal{M}^2(X) \simeq \mathcal{M}^3(X) \simeq Z_5$ for a degree 5 étale k -scheme Z_5 .*

Proof. (1) Since $H^1(X, \mathcal{O}_X) = 0$, see [Kol96, Lemma III.3.2.1], deforming an effective divisor D as a subscheme is equivalent to deforming it in its linear system, so $\mathcal{H}^j(X)_{\bar{k}}$ is either empty or a disjoint union of complete linear systems (see [Kol96, Exercise I.1.14.2]). Since the Hilbert polynomial of a rational curve C on a del Pezzo surface is determined by its degree $C \cdot (-K_X)$, and since the Hilbert scheme is projective [Kol96, Theorem I.1.4], in particular of finite type, it follows that $\mathcal{H}^j(X)_{\bar{k}}$ is a finite disjoint union of projective spaces. (Finiteness also follows from the fact there are finitely many solutions for (4.4) with fixed j .) By Lemma 4.7(1), these projective spaces have dimension $j-1$.

Thus we have shown that $\mathcal{H}^j(X)$ is a smooth scheme of finite type over k isomorphic over $\bar{k}$ to a finite disjoint union of projective spaces. Let $\mathcal{M}^j(X) = \text{Spec}(H^0(\mathcal{H}^j(X), \mathcal{O}))$; it is a smooth zero-dimensional scheme, and $\mathcal{H}^j(X) \rightarrow \mathcal{M}^j(X)$ is a Severi–Brauer fibration. Finally, the isomorphism $\mathcal{M}^j(X) \simeq \mathcal{M}^{d-j}(X)$ is given by $|C| \mapsto |-K_X - C|$, using Lemma 4.7(4).

(2) We claim that the projection $\mathcal{H}^j(X, Z) \rightarrow \mathcal{M}^j(X)$ is an isomorphism. By Galois descent it is sufficient to verify this over the algebraic closure. Each fiber of $\mathcal{H}^j(X)_{\bar{k}} \rightarrow \mathcal{M}^j(X)_{\bar{k}}$ is equal to $|L|$ for some line bundle L . It follows from the assumption that the linear system defined by $H^0(X_{\bar{k}}, L \otimes I_Z) \subset H^0(X_{\bar{k}}, L)$ is a point in $|L|$, where I_Z is the ideal sheaf of $Z_{\bar{k}} \subset X_{\bar{k}}$. Hence $\mathcal{H}^j(X, Z)_{\bar{k}} \rightarrow \mathcal{M}^j(X)_{\bar{k}}$ is an isomorphism.

(3) This is proven by solving (4.5).

(4) Galois descent identifies a smooth zero-dimensional scheme with a set with Galois action. Thus it suffices to verify the numbers of conic bundles and nets of cubics over the algebraic closure in each case. For case (dP₈), the linear system of quartic rational curves on $\mathbb{P}^1 \times \mathbb{P}^1$ is defined by $\mathcal{O}(1, 1)$ and the two pencils of conics $\mathcal{O}(1, 0)$ and $\mathcal{O}(0, 1)$. The cases (dP₆) and (dP₅) follow from part (3). $\square$

5. Models of large degree

In this section, we deal with surfaces admitting the most interesting birational automorphisms. These are geometrically rational surfaces with minimal models of degree at least 5. At the end of this section, we prove Theorem 3.5 and discuss rationality centers for rational surfaces.

5.1 Virtual Néron–Severi sets

We introduce a type of surfaces X/k which we call *models of large degree*:

(dP₉) $\mathbb{P}^2$ or a Severi–Brauer surface

(dP₈) Minimal del Pezzo surface of degree 8 (in this case, $X_{\bar{k}} \simeq \mathbb{P}^1 \times \mathbb{P}^1$)

(C₈) Smooth conic bundle over a conic (in this case, $X_{\bar{k}} \simeq \mathbb{F}_n$, and we assume $n \geq 1$ to avoid the overlap with type (dP₈))

(dP₆) Del Pezzo surface of degree 6

(dP₅) Del Pezzo surface of degree 5

Note that we do not make any assumption on the Picard rank of the surfaces in the list above.

PROPOSITION 5.1. *All minimal geometrically rational surfaces X with $K_X^2 \geq 5$ and all conic bundles over a curve of genus 0 with $K_X^2 \geq 5$ are among the models of large degree.*

Proof. By the work of Iskovskih [Isk79] explained in Theorem 4.1(3), (4), such a surface X is a del Pezzo surface of Picard rank 1 or a conic bundle of Picard rank 2. Minimal del Pezzo surfaces of degree $K_X^2 \geq 5$ are all among (dP₉), (dP₈), (dP₆), (dP₅) (a del Pezzo surface of degree 7 is not minimal; see for example [Isk79, end of §3, Corollary]).

Assume that X is a conic bundle over a curve of genus zero; using (4.1), we have $5 \leq K_X^2 \leq 8$. By [Isk79, Theorem 5], the conic bundle X is a model of large degree. $\square$

DEFINITION 5.2. For each model of large degree, we define its *virtual Néron–Severi Galois set* $A_X \in \text{Burn}(G_k) \simeq \mathbb{Z}[\text{Var}^0/k]$ as follows. In each case, Z_i refers to one of the finite étale k -schemes of degree i introduced in Proposition 4.8(4).

(dP₉) $A_X = [\text{Spec}(k)]$.

(dP₈) $A_X = [Z_2]$, where Z_2 parametrizes the rulings (that is, the conic bundle structures) on $X_{\bar{k}}$.

(C₈) $A_X = 2[\text{Spec}(k)]$.

- (dP₆) $A_X = [Z_2] + [Z_3] - [\text{Spec}(k)]$, where Z_3 parametrizes three pencils of conics on $X_{\bar{k}}$ and Z_2 parametrizes two families of cubics.
- (dP₅) $A_X = [Z_5]$, where Z_5 parametrizes five pencils of conics on $X_{\bar{k}}$.

Note that in all cases except for (dP₆), the virtual Néron–Severi set is realized as a set. The following lemma explains the name Néron–Severi set. See (2.1) for the definition of μ_{G_k} .

LEMMA 5.3. *If $A_X \in \text{Burn}(G_k)$ is the virtual Néron–Severi set of X and F is any field of characteristic zero, then $\mu_{G_k}(A_X) = [\text{NS}(X_{\bar{k}}) \otimes F] \in K_0(\text{Rep}(G_k, F))$.*

Proof.

- (dP₉) In this case, A_X is a set consisting of one element, and $\text{NS}(X_{\bar{k}})$ is the 1-dimensional trivial G_k -representation.
- (dP₈) In this case, A_X is the set of rulings on X , and $\text{NS}(X_{\bar{k}})$ is freely generated by these rulings.
- (C₈) In this case, A_X is a set with two elements and trivial action. Since K_X and the class of a fiber form a basis of $\text{NS}(X) \otimes F$ and $\text{rk NS}(X_{\bar{k}}) = \text{rk NS}(\mathbb{F}_n) = 2$, necessarily $\text{NS}(X_{\bar{k}})$ is of rank 2 with trivial Galois action.
- (dP₆) By Proposition 4.8(3), the vector space $\text{NS}(X_{\bar{k}}) \otimes F$ is generated by three classes of conics and two classes of cubics, modulo a 1-dimensional space with a trivial G_k -action; this implies the result.
- (dP₅) By Proposition 4.8(3), the group $\text{NS}(X_{\bar{k}}) \otimes F$ is freely generated by classes of conics A_X . $\square$

It follows from Lemma 5.3 that the rank of $\text{NS}(X)$ equals the number of orbits of the virtual Néron–Severi set A_X . In particular, X has Picard rank 1 if and only if A_X has only one (virtual) orbit. Since μ_{G_k} is not injective, we cannot simply define the virtual Galois set of a surface using $\mu_{G_k}(A_X) = [\text{NS}(X_{\bar{k}}) \otimes F]$; see Example 2.14 for an illustration. This is why we define it case by case in Definition 5.2.

Remark 5.4. The Galois sets forming A_X naturally appear in the Chow motive of X , see [Gil15, § 3.3, (9) and Remarks following it], and as semiorthogonal components in the derived category of the corresponding surface [BSS11], [AB18, Propositions 9.8, 10.1]. Furthermore, singular versions of those k -algebras show up in the study of the derived categories of the corresponding singular del Pezzo surfaces [Kuz21, KKS22, Xie21].

PROPOSITION 5.5. *Let X be a model of large degree, and let $\phi: X \dashrightarrow X'$ be a birational isomorphism to another minimal surface; then X' is also a model of large degree, and*

$$c(\phi) = A_{X'} - A_X. \quad (5.1)$$

First we prove that if we apply μ_{G_k} to both sides of (5.1), we have equality.

LEMMA 5.6. *Let X, X' be models of large degree. Let $\phi: X \leftarrow Y \rightarrow X'$ be a composition of a blowup and a blowdown (the centers of the blowups can be disconnected or empty). We have*

$$\mu_{G_k}(A'_X) - \mu_{G_k}(A_X) = \mu_{G_k}(c(\phi)).$$

Proof. Let Z, Z' be the centers of the two blowups. By the blowup formula, we have $[X] + \mathbb{L} \cdot [Z] = [Y] = [X'] + \mathbb{L} \cdot [Z']$ in the Grothendieck ring. Applying the étale realization (2.4) to this equality (cf. Example 2.9) and using Lemma 5.3 with $F = \mathbb{Q}_\ell$, we get

$$\mu_{G_k}(A_X) + [\mathbb{Q}_\ell[Z_{\bar{k}}]] = [\text{NS}(Y_{\bar{k}}) \otimes \mathbb{Q}_\ell] = \mu_{G_k}(A_{X'}) + [\mathbb{Q}_\ell[Z'_{\bar{k}}]] \in K_0(\text{Rep}(G_k, \mathbb{Q}_\ell)). \quad (5.2)$$

Hence $\mu_{G_k}(A'_X) - \mu_{G_k}(A_X) = \mu_{G_k}(c(\phi))$. $\square$

Before proving Proposition 5.5, we establish some particular cases, which rely on the Gassmann equivalence being trivial for Galois sets of order at most 5.

LEMMA 5.7. *Let X, X' be models of large degree. Let $\phi: X \leftarrow Y \rightarrow X'$ be a composition of a blowup and a blowdown. Assume that either (1) the two blowups have isomorphic centers, or (2) $K_Y^2 \geq 5$. Then (5.1) holds for ϕ .*

Proof. Let Z, Z' be the centers of the two blowups. We need to show that

$$\alpha := (A_X + [Z]) - (A_{X'} + [Z']) = 0.$$

By Lemma 5.6 and Corollary 2.6, it is sufficient to show that $|\alpha| \leq 5$, that is, to represent α as a difference of two G_k -sets, each having order at most 5. Recall that the virtual G_k -set A_X is represented by a G_k -set of order at most 5, except possibly when $K_X^2 = 6$, in which case $A_X + [\text{Spec}(k)]$ is a G_k -set, of order 5.

(1) Since $[Z] = [Z']$, we have $\alpha = A_X - A_{X'}$. If $K_X^2 = K_{X'}^2 \neq 6$, then $|\alpha| \leq 5$ as $|A_X| = |A_{X'}| \leq 5$. If $K_X^2 = K_{X'}^2 = 6$, then $\alpha = (A_X + [\text{Spec}(k)]) - (A_{X'} + [\text{Spec}(k)])$ also shows that $|\alpha| \leq 5$.

(2) Assume that both $K_X^2, K_{X'}^2$ are not equal to 6. In this case, by (5.2) both $A_X + [Z]$ and $A_{X'} + [Z']$ have order equal to $\text{rk NS}(Y_{\bar{k}}) = 10 - K_Y^2 \leq 5$, and the original representation of α shows that $|\alpha| \leq 5$. The cases when $K_X^2 = 6$ or $K_{X'}^2 = 6$ are analogous and are left to the reader. $\square$

Proof of Proposition 5.5. By [Isk96, Theorem 2.5], any birational isomorphism between minimal geometrically rational surfaces is a composition of Sarkisov links presented in Definition 4.2. Since c is a homomorphism and sends isomorphisms to zero, it suffices to prove Proposition 5.5 for every link of type I, II or III.

For type I links, we write $a \leftarrow b$ for a link $\phi: X \leftarrow X'$ with $K_X^2 = a, K_{X'}^2 = b$. We have the following possibilities according to [Isk96, Theorem 2.6(i)]: $9 \leftarrow 8, 9 \leftarrow 5, 8 \leftarrow 6$. Here $X' \rightarrow B'$ is a conic bundle of degree at least 5; hence X' is a model of large degree by Proposition 5.1, and (5.1) follows from Lemma 5.7(2) (with one of the centers empty). Exactly the same argument proves the claim for links of type III.

For type IIC links, the result is true by Lemma 4.3 and Proposition 5.1. For a type IID link $X \leftarrow Y \rightarrow X'$, by the first statement of Proposition 4.4(2), we have that $K_X^2 \geq 5$ if and only if $K_{X'}^2 \geq 5$. Hence X is a model of large degree if and only if X' is.

It remains to show (5.1) for each link of type IID. We write $a \leftarrow d \rightarrow b$ for a type IID link $X \leftarrow Y \rightarrow X'$ between surfaces of degree a, d, b . Since c takes values in a torsion-free abelian group, it vanishes on involutions and in particular on the Bertini and Geiser involutions; these are links with $d = 1$ and $d = 2$, respectively, in the list of links in [Isk96, Theorem 2.6(ii)]. On the other hand, links with $d \geq 5$ are covered by Lemma 5.7(2).

Thus we only have to consider links with $d = 3$ or $d = 4$.

CLAIM 5.8. *Let $X \xleftarrow{\phi} Y \xrightarrow{\phi'} X'$ be a link of type IID such that $K_Y^2 \geq 3$. Let $Z \subset X$ be the blowup center of ϕ and $D \subset X$ the divisor contracted by $\phi' \circ \phi^{-1}$. Then each irreducible component of $D_{\bar{k}}$ is a smooth rational curve of degree δ (according to Definition 4.6) containing exactly $\delta - 1$ points of $Z_{\bar{k}}$. For the type IID links with $K_Y^2 \in \{3, 4\}$ listed below, δ has the following description:*

- $9 \leftarrow 4 \rightarrow 5, 9 \leftarrow 3 \rightarrow 9$: $\delta = 6$ (which are conics in $\mathbb{P}^2$ in the classical sense)
- $8 \leftarrow 4 \rightarrow 8$: $\delta = 4$

- $8 \leftarrow 3 \rightarrow 5$: $\delta = 6$
- $6 \leftarrow 4 \rightarrow 6$: $\delta = 3$
- $6 \leftarrow 3 \rightarrow 6$: $\delta = 4$
- $5 \leftarrow 4 \rightarrow 9$: $\delta = 2$
- $5 \leftarrow 3 \rightarrow 8$: $\delta = 3$

Proof. Since $K_Y^2 \geq 3$, the surface $Y_{\bar{k}}$ is obtained by blowing up $r \leq 6$ points on $\mathbb{P}^2$ with exceptional divisors $E_1, \dots, E_r$. Let $H \in \text{NS}(Y_{\bar{k}})$ be the pullback of the hyperplane class of $\mathbb{P}^2$. By solving (4.5), we see that the (-1) -classes on $Y_{\bar{k}}$ are one of the following:

- E_i with $i \in \{1, \dots, r\}$
- $H - E_i - E_j$ with $i, j \in \{1, \dots, r\}$ and $i \neq j$
- (only when $r = 6$) $-K_Y - H + E_i$ with $i \in \{1, \dots, 6\}$

From the above description, any pair of (-1) -curves E and E' on $Y_{\bar{k}}$ satisfies $E \cdot E' \leq 1$. Since $X_{\bar{k}}$ is a simultaneous contraction of disjoint (-1) -curves on $Y_{\bar{k}}$, it follows that each irreducible component C of $D_{\bar{k}}$ is smooth. It also follows that if C has degree δ , then C contains $\delta - 1$ points of $Z_{\bar{k}}$.

The value of δ follows from the matrix description in [Isk96, Theorem 2.6] of the action of each link in the classification on the Picard–Manin space. For instance for $6 \leftarrow 3 \rightarrow 6$, the description in [Isk96, Theorem 2.6(ii), $K_X^2 = 6$, (c)] implies that $D = -2K_X$, which shows that $\delta = (-2K_X^2)/3 = 4$. The same argument works for other links. $\square$

We first consider symmetric links where the centers of the blowup and the blowdown are isomorphic, so that they are covered by Lemma 5.7(1):

- $9 \leftarrow 3 \rightarrow 9$: The first map blows up a Galois orbit of six points, and the second one contracts the Galois orbit of the proper preimages of six conics (in the classical sense: they have degree 6 according to Definition 4.6) passing through five of the points by Claim 5.8. These two Galois orbits are isomorphic.
- $8 \leftarrow 4 \rightarrow 8$: We blow up a Galois orbit of four general points on X and contract the Galois orbit of the proper preimages of quartic curves passing through three of the four points by Claim 5.8. These two Galois orbits are isomorphic.

Finally, we need to deal with the following links (we list them up to inverses):

- $8 \leftarrow 3 \rightarrow 5$: Here X is a quadric with $A_X = [Z_2]$, and X' is a degree 5 del Pezzo surface with $A_{X'} = [Z'_5]$. We need to show that the first map has center Z'_5 and the second map has center Z_2 . By Claim 5.8, the center of the second map parametrizes smooth rational curves of degree 6 on X passing through an orbit of five points. By Proposition 4.8(2) and (4)(dP₈), the center of the second map is a subscheme isomorphic to Z_2 . The center of the first map parametrizes cubics passing through an orbit of two points on X' , and thus by Proposition 4.8(2) and (4)(dP₅) is a scheme isomorphic to Z'_5 .
- $9 \leftarrow 4 \rightarrow 5$: It suffices to show that if Z_5 is the center of the first map and $A_{X'} = [Z'_5]$, then $Z_5 \simeq Z'_5$. One can verify this directly, as in the previous link, or apply Corollary 2.13 as in the proof of Lemma 5.7(2).
- $6 \leftarrow 3 \rightarrow 6$: We have $A_X = [Z_3] + [Z_2] - 1$ and $A_{X'} = [Z'_3] + [Z'_2] - 1$. We claim that $c(\phi) = [Z'_3] - [Z_3]$ and $Z_2 \simeq Z'_2$. Indeed, by Claim 5.8 the second arrow contracts the proper preimages of smooth rational curves of degree 4 passing through the Galois orbit of the

points blown up by the second arrow, and the latter scheme of conics is isomorphic to Z_3 by Proposition 4.8(2) and (4)(dP₆). The same argument with the roles of X and X' reversed implies that the first arrow blows up Z'_3 . Thus $c(\phi) = [Z'_3] - [Z_3]$. By Lemma 5.6, we have $\mu_{G_k}([Z_2]) = \mu_{G_k}([Z'_2])$, hence $Z_2 \simeq Z'_2$ by Proposition 2.5.

- $6 \leftarrow 4 \rightarrow 6$: We have $Z_3 \simeq Z'_3$ and $c(\phi) = [Z'_2] - [Z_2]$, similarly to the previous case, using Proposition 4.8(2), (4)(dP₆) and Proposition 2.5 again. $\square$

Proof of Theorem 3.5. We first assume that X and Y are geometrically irreducible. Composing with contractions to minimal models, and using the additivity of c under composition, we may assume that X and Y are minimal, hence belong to one of the classes from Theorem 4.1.

In the non-geometrically rational case and geometrically rational case with $K_X^2 \leq 4$, the result follows from Proposition 4.4. Finally, in the geometrically rational case with $K_X^2 \geq 5$, the result is Proposition 5.5.

In general, that is, if the surfaces X and Y are not geometrically irreducible, write L_X and L_Y for the fields of regular functions of X and Y ; these are finite field extensions of k . Then ϕ induces a k -isomorphism $\sigma: L_X \rightarrow L_Y$, which allows us to consider both X and Y as smooth projective geometrically irreducible surfaces over $L := L_X$, and ϕ becomes a L -birational isomorphism. This way X and Y are restrictions of scalars of geometrically irreducible surfaces over L (this can be thought of as the Stein factorizations for X and Y over $\text{Spec}(k)$), and the result follows from the geometrically irreducible case considered above and Proposition 3.4(2). $\square$

5.2 Rationality centers

The following corollary tells us that the rationality center of a rational surface X is well defined; that is, for any sequence of blow ups and blowdowns connecting $\mathbb{P}^2$ to X , the virtual Galois set of blowup centers minus the blowdown centers is independent of the choice of the birational isomorphism between them. In the higher-dimensional case, such rationality centers are not well defined (however, see [GS14, Definition 2.3] for a similar class in the localized Grothendieck ring).

COROLLARY 5.9 (of Theorem 3.5). *There exists a unique map*

$$\{\text{Isomorphism classes of rational smooth projective surfaces}\} \xrightarrow{\mathcal{M}} \mathbb{Z}[\text{Var}^0/k]$$

with the following properties:

- (1) We have $\mathcal{M}(X) = A_X$ as in Definition 5.2 for models of large degree.
- (2) For any birational isomorphism $\phi: \mathbb{P}^2 \dashrightarrow X$, we have $c(\phi) = [\mathcal{M}(X)] - 1$.

Proof. For any rational surface, we define $\mathcal{M}(X)$ as $c(\phi) + 1$; by Theorem 3.5 this is independent of the choice of $\phi: \mathbb{P}^2 \dashrightarrow X$. As $\mathcal{M}$ is required to satisfy $[\mathcal{M}(X)] = c(\phi) + 1$, this shows the uniqueness of $\mathcal{M}$. By Proposition 5.5, this is consistent with Definition 5.2. $\square$

Example 5.10. If X is a del Pezzo surface of degree 5, by Definition 5.2, we have $A_X = [Z_5]$. Assume that X has Picard rank 1, which by Lemma 5.3 is equivalent to Z_5 being irreducible. Then for any rationality construction of X (see Example 3.8), that is, a sequence of blowups and blow downs starting with $\mathbb{P}^2$ and ending with X , one of the blowups will have Z_5 as its center.

Example 5.11. Consider the rationality question for a del Pezzo surface X of degree $d \leq 4$. If X is rational, then it is a result of Iskovskikh deduced from his classification of links [Isk96, Theorem 2.6] that such a surface X cannot be minimal; thus it admits a Galois orbit of disjoint (-1) -curves which we can contract via some morphism $\phi: X \rightarrow X'$ to obtain a minimal rational

del Pezzo surface of degree 5, 6, 8 or 9. In each case the rationality center A_X is given in the table. As usual we write Z_j or Z'_j for étale schemes of degree j .

$\deg(X')$	$c(\phi^{-1})$	$A_{X'}$	A_X
5	$[Z_{5-d}]$	$[Z'_5]$	$[Z_{5-d}] + [Z'_5]$
6	$[Z_{6-d}]$	$[Z'_2] + [Z'_3] - 1$	$[Z_{6-d}] + [Z'_3] + [Z'_2] - 1$
8	$[Z_{8-d}]$	$[Z'_2v]$	$[Z_{8-d}] + [Z'_2]$
9	$[Z_{9-d}]$	1	$[Z_{9-d}] + 1$

Example 5.12. Consider the two rational cubic surfaces X, X' introduced in Example 2.14; they have isomorphic rational permutation Néron–Severi representations, and the associated Galois sets cannot be read off from them. However, the construction of $\mathcal{M}_X$ does determine these Galois sets from X itself, as by Corollary 5.9 we have $[Z] = \mathcal{M}_X - 1$ and $[Z'] = \mathcal{M}_{X'} - 1$.

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