



# On the behavior of the Kodaira dimension under smooth morphisms

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## ABSTRACT

We prove several results on the additivity of the Kodaira dimension under smooth morphisms of smooth projective varieties.

## 1. Introduction

The purpose of this paper is to prove some of the conjectures in [Pop21] about the additivity of the Kodaira dimension under smooth morphisms between smooth projective varieties. Our results are unconditional when varieties of general type are involved in some way, or when the base space is either of dimension at most 3 or a good minimal model of Kodaira dimension zero; for other varieties, we need a version of the nonvanishing conjecture due to Campana and Peternell [CP11]. We use the convention that an *algebraic fiber space* is a surjective morphism  $f: X \rightarrow Y$  with connected fibers between two smooth projective varieties  $X$  and  $Y$  (over the field of complex numbers). We usually denote the general fiber by the letter  $F$ . We say that an algebraic fiber space is *smooth* if  $f$  is a smooth morphism.

A particular case of Conjecture 2.1 or 3.1 in [Pop21] predicts that if  $f: X \rightarrow Y$  is a smooth algebraic fiber space and  $X$  is of general type, then  $Y$  is of general type as well. Our first result is the following more general statement.

**THEOREM A.** *Let  $f: X \rightarrow Y$  be a smooth algebraic fiber space whose general fiber  $F$  satisfies  $\kappa(F) \geq 0$ . Then  $Y$  is of general type if and only if  $\kappa(X) = \kappa(F) + \dim Y$ .*

By the easy addition formula [Mor87, Corollary 1.7], we always have the elementary inequality  $\kappa(X) \leq \kappa(F) + \dim Y$ . Also recall that Iitaka's conjecture, to the effect that  $\kappa(X) \geq \kappa(F) + \kappa(Y)$ , is known when  $Y$  is of general type [Vie83, Corollary IV]. One half of the theorem is therefore already known. Our contribution is to prove the other half: the identity  $\kappa(X) = \kappa(F) + \dim Y$  implies that  $Y$  is of general type.

*Note.* Using the same techniques, one can show that if  $f: X \rightarrow Y$  is an algebraic fiber space with  $Y$  not uniruled,  $\kappa(F) \geq 0$ , and  $\kappa(X) = \kappa(F) + \dim Y$ , and if  $V \subseteq Y$  denotes the smooth locus of  $f$ , then  $V$  is of log general type. This answers positively another question suggested by [Pop21] when  $Y$  is not uniruled but is weaker than [Pop21, Conjecture 3.1]. See also Remark 5.

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Our next result deals with smooth algebraic fiber spaces with canonically polarized fibers.

**THEOREM B.** *Let  $f: X \rightarrow Y$  be a smooth algebraic fiber space such that each fiber  $F$  is canonically polarized. If  $\kappa(Y) \geq 0$  or  $Y$  is uniruled, then we have  $\kappa(X) = \kappa(F) + \kappa(Y)$ .*

For technical reasons, the case where  $\kappa(Y) = -\infty$  and  $Y$  is not uniruled is missing; note however that a well-known conjecture in birational geometry, called the *nonvanishing conjecture*, predicts that  $\kappa(Y) = -\infty$  is in fact equivalent to  $Y$  being uniruled. Thus up to this prediction, our theorem implies the additivity of the Kodaira dimension for any smooth algebraic fiber space with fibers of this type. Unlike the other results in this paper, besides hyperbolicity-type techniques, Theorem B also relies on the existence and properties of moduli spaces of varieties of general type. We also observe that essentially the same proof works even when  $X$  and  $Y$  are only assumed to be quasi-projective, with the base of log Kodaira dimension  $\kappa(Y) \geq 0$ ; see Theorem 10.<sup>1</sup>

We also prove a more general result that depends on the following conjecture of Campana–Petersen [CP11, Conjecture 2.4], which can be seen as a generalization of the nonvanishing conjecture.

**CONJECTURE (Campana–Petersen).** *Let  $Y$  be a smooth projective variety, and assume  $K_Y \sim_{\mathbb{Q}} A + B$ , where  $A$  and  $B$  are  $\mathbb{Q}$ -divisors such that  $A$  is effective and  $B$  is pseudo-effective. Then  $\kappa(Y) \geq \kappa(A)$ .*

The nonvanishing conjecture is the case  $A = 0$ . Concretely, it says that if  $K_Y$  is pseudo-effective—which is equivalent to  $Y$  not being uniruled [BDPP13]—then  $\kappa(Y) \geq 0$ . It can be easily checked [CP11, §2.3, pp. 53–54] that the Campana–Petersen conjecture is implied by the existence of good minimal models; it therefore holds if  $\dim Y \leq 3$  since this conjecture in the minimal model program (MMP) is known to hold in this range. On the other hand, the Campana–Petersen conjecture is definitely weaker than the full abundance conjecture; in fact, we believe that it is basically equivalent to the nonvanishing conjecture [Sch22].

**THEOREM C.** *Let  $f: X \rightarrow Y$  be a smooth algebraic fiber space, and assume that the Campana–Petersen conjecture holds. Then  $\kappa(F) + \kappa(Y) \geq \kappa(X)$ .*

*Note.* More precisely, we need the Campana–Petersen conjecture for the fibers of the Iitaka fibration of  $Y$  if  $\kappa(Y) \geq 0$ , and we need the nonvanishing conjecture for  $Y$  itself if  $\kappa(Y) = -\infty$ .

Iitaka’s conjecture of course predicts the opposite inequality  $\kappa(X) \geq \kappa(F) + \kappa(Y)$  for an arbitrary morphism; it is known to be implied by the existence of a good minimal model for the geometric generic fiber of  $f$  by a result of Kawamata [Kaw85]. Therefore, we obtain the following loosely formulated consequence.

**COROLLARY D.** *The conjectures of the MMP imply [Pop21, Conjecture 2.5], stating that if  $f: X \rightarrow Y$  is a smooth algebraic fiber space, then  $\kappa(X) = \kappa(F) + \kappa(Y)$ .*

In any event, we obtain unconditional results—and hence the solution to [Pop21, Conjecture 2.1]—for algebraic fiber spaces over surfaces and threefolds.

**COROLLARY E.** *If  $f: X \rightarrow S$  is a smooth algebraic fiber space, with  $S$  a smooth projective surface, then  $\kappa(X) = \kappa(F) + \kappa(S)$ .*

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<sup>1</sup>Since this paper was written, Campana [Cam22] has extended this result to morphisms whose fibers have semiample canonical bundle, with no conditions on the base.

COROLLARY F. *If  $f: X \rightarrow Y$  is a smooth algebraic fiber space, with  $Y$  a smooth projective threefold, then  $\kappa(F) + \kappa(Y) \geq \kappa(X)$ .*

In Corollary E, we have equality due to the fact that Iitaka’s conjecture is known over surfaces; see [Cao18] and the references therein. This is not yet known when  $Y$  is an arbitrary threefold.

Theorem C can be easily reduced to the two cases  $\kappa(Y) = -\infty$  and  $\kappa(Y) = 0$ , and is therefore a consequence of the following two results.

PROPOSITION G. *Let  $f: X \rightarrow Y$  be a smooth morphism between smooth projective varieties, with  $Y$  uniruled. Then  $\kappa(X) = -\infty$ .*

Thus the additivity of the Kodaira dimension for smooth morphisms over varieties  $Y$  with  $\kappa(Y) = -\infty$  holds if one assumes the nonvanishing conjecture. The proof of Proposition G is the first step in the proof of Theorem B. It works by reduction to the case  $Y = \mathbb{P}^1$ , where a stronger version is a deep result of Viehweg–Zuo [VZ01]. Therefore, the main new input is the following.

THEOREM H. *Let  $f: X \rightarrow Y$  be a smooth algebraic fiber space with  $\kappa(Y) = 0$ .*

- (i) *If the Campana–Petersen conjecture holds on  $Y$ , so for instance if  $Y$  has a good minimal model, then  $\kappa(F) \geq \kappa(X)$ .*
- (ii) *If  $Y$  is actually a good minimal model, in other words if  $K_Y \sim_{\mathbb{Q}} 0$ , then  $\kappa(X) = \kappa(F)$ . Moreover, we have  $P_m(F) \geq P_m(X)$  for all  $m \geq 1$ , with equality if  $Y$  is simply connected (for example, Calabi–Yau).*

Thus [Pop21, Conjecture 2.5] holds unconditionally when  $Y$  is a good minimal model with  $\kappa(Y) = 0$ . When  $Y$  is an abelian variety, this is a special case of [MP21, Theorem A (2)]. Note for completeness that the conjecture also holds when  $X$  is a good minimal model with  $\kappa(X) = 0$  since then so is  $Y$  by [TZ20].

The proofs of the results in this paper rely on a Hodge module construction from [PS14, PS17], which was in turn heavily inspired by techniques of Viehweg–Zuo in their work on hyperbolicity. Because of the smoothness assumption, semistable reduction and the technical passage to logarithmic Higgs bundles are however not needed. The proof of Theorem H is somewhat more delicate (than for instance that of Theorem A), needing both an additional technical ingredient and important analytic results from [PT18, CP17] (see also [HPS18]).

*Remark.* The disadvantage of this method is the lack of symmetry between the assumptions and the conclusion. Say  $f: X \rightarrow Y$  is a smooth algebraic fiber space and  $D$  a divisor on  $Y$ . Roughly speaking, we start from the assumption that  $mK_X - f^*D$  is *effective* for some  $m \geq 1$ , and the conclusion is that  $m'K_Y - D$  is *pseudo-effective* for  $m' \gg m$ . The Campana–Petersen conjecture removes this asymmetry and allows us to put “pseudo-effective” or “effective” in both places; but it would of course be desirable to have unconditional results.

We conclude by observing that the Campana–Petersen conjecture, together with the techniques above, also leads to the following characterization of  $\kappa(Y)$  that can be seen as a generalization of Theorem A.

COROLLARY I. *Let  $f: X \rightarrow Y$  be a smooth algebraic fiber space with general fiber  $F$  satisfying  $\kappa(F) \geq 0$ . If the Campana–Petersen conjecture holds on  $X$  and  $Y$ , then*

$$\kappa(Y) = \max \{ \kappa(L) \mid mK_X - f^*L \text{ is pseudo-effective for some } m \geq 1 \},$$

*where the maximum is taken over all line bundles on  $Y$ .*

## 2. Proofs

### 2.1 Proof of Theorem A

We start by considering a general construction. Let  $A$  be an ample line bundle on  $Y$ . According to [Mor87, Proposition 1.14], the condition  $\kappa(X) = \kappa(F) + \dim Y$  is equivalent to having

$$H^0(X, \omega_X^{\otimes m} \otimes f^* A^{-1}) \neq 0$$

for some  $m \geq 1$ . Using the projection formula, this is equivalent to the existence of a nontrivial (hence injective) morphism

$$L := A \otimes (\omega_Y^{-1})^{\otimes m} \hookrightarrow f_* \omega_{X/Y}^{\otimes m}. \quad (1)$$

Note that by the invariance of plurigena, the sheaf  $f_* \omega_{X/Y}^{\otimes m}$  is locally free of rank  $P_m(F)$  on all of  $Y$ .

We now use Viehweg's well-known fiber product trick, in the following form: let  $X^{(m)}$  denote the  $m$ -fold fiber product  $X \times_Y \cdots \times_Y X$ , with its induced morphism  $f^{(m)}: X^{(m)} \rightarrow Y$ . Note that this is still a smooth algebraic fiber space, with general fiber isomorphic to the  $m$ -fold product  $F \times \cdots \times F$ . Moreover,

$$f_*^{(m)} \omega_{X^{(m)}/Y}^{\otimes m} \simeq (f_* \omega_{X/Y}^{\otimes m})^{\otimes m}, \quad (2)$$

for example by [Mor87, Corollary 4.11]. This gives us an inclusion  $L^{\otimes m} \hookrightarrow f_*^{(m)} \omega_{X^{(m)}/Y}^{\otimes m}$ . By replacing  $X$  with  $X^{(m)}$  and the fiber space  $f$  with  $f^{(m)}$ , we are therefore allowed to assume from the beginning that we are working with a fiber space  $f$  such that

$$H^0(X, B^{\otimes m}) \neq 0, \quad \text{where } B = \omega_{X/Y} \otimes f^* L^{-1}. \quad (3)$$

Under this assumption, one can bring into play the machinery used in [PS14, PS17], strongly inspired in turn by the celebrated method developed by Viehweg–Zuo [VZ01, VZ02], based on the existence of so-called Viehweg–Zuo sheaves. Note however that, unlike in the works above, the line bundle  $L$  we consider here is usually not positive anymore.

First, using [PS17, Theorem 2.2] and writing  $\mathcal{A}_Y = \text{Sym } T_Y$ , we see that (3) implies that there exists a graded  $\mathcal{A}_Y$ -module  $\mathcal{G}_\bullet$  that is coherent over  $\mathcal{A}_Y$  and has the following among its special properties:

- (i) One has  $\mathcal{G}_0 \simeq L$ .
- (ii) Each  $\mathcal{G}_k$  is torsion-free.
- (iii) There exist a regular holonomic  $\mathcal{D}_Y$ -module  $\mathcal{M}$  with good filtration  $F_\bullet \mathcal{M}$  and an inclusion of graded  $\mathcal{A}_Y$ -modules  $\mathcal{G}_\bullet \subseteq \text{gr}_\bullet^F \mathcal{M}$ .
- (iv) The filtered  $\mathcal{D}$ -module  $(\mathcal{M}, F_\bullet \mathcal{M})$  underlies a polarizable Hodge module  $M$  on  $Y$ , with strict support  $Y$ , and  $F_k \mathcal{M} = 0$  for  $k < 0$ .

We next follow closely [PS17, § 3.2]. The argument is based on considering the chain of  $\mathcal{O}_Y$ -module homomorphisms

$$0 \longrightarrow \mathcal{G}_0 \xrightarrow{\theta_0} \mathcal{G}_1 \otimes \Omega_Y^1 \xrightarrow{\theta_1 \circ \text{id}} \mathcal{G}_2 \otimes (\Omega_Y^1)^{\otimes 2} \longrightarrow \cdots$$

obtained from the graded  $\mathcal{A}_Y$ -module structure on  $\mathcal{G}_\bullet$  and the generalized Kodaira–Spencer maps

$$\theta_k: \mathcal{G}_k \longrightarrow \mathcal{G}_{k+1} \otimes \Omega_Y^1,$$

and originated in work of Kovács and Viehweg–Zuo. Using the identification  $\mathcal{G}_0 \simeq L$  and the fact

that  $\text{Ker}(\theta_k)^\vee$  is a weakly positive sheaf for each  $k$ ,<sup>2</sup> the exact same argument as in the proof of [PS17, Theorem 3.5] shows that at least one of the following two cases must happen:

- (i) The line bundle  $L^{-1} = A^{-1} \otimes \omega_Y^{\otimes m}$  is pseudo-effective.
- (ii) There exists a weakly positive sheaf  $\mathcal{W}$  on  $Y$  and an inclusion

$$\mathcal{W} \otimes L \hookrightarrow (\Omega_Y^1)^{\otimes N}$$

for some positive integer  $N > 0$ .

In the first case, we immediately get that  $\omega_Y$  is big and hence that  $Y$  is of general type. In the second case, we argue as follows. We rewrite the above inclusion as

$$\mathcal{W} \otimes A \hookrightarrow (\Omega_Y^1)^{\otimes N} \otimes \omega_Y^{\otimes m} \quad (4)$$

and observe that  $\mathcal{W} \otimes A$  is big in the sense of Viehweg [PS17, Lemma 3.2]. Now  $\det \Omega_Y^1 = \omega_Y$ ; hence there also exists a (split) inclusion  $\omega_Y^{\otimes m} \hookrightarrow (\Omega_Y^1)^{\otimes m \cdot \dim Y}$ . Combining everything, we deduce the existence of an inclusion  $\mathcal{H} \hookrightarrow (\Omega_Y^1)^{\otimes M}$  for some  $M > 0$ , where  $\mathcal{H}$  is a big torsion-free sheaf. The result of Campana–Păun [CP19, Theorem 7.11] then implies that  $Y$  is of general type. (Campana and Păun only state the result with  $\mathcal{H}$  being a line bundle, but their proof still works when  $\mathcal{H}$  has higher rank; see also [Sch17] for a streamlined proof of this important result.)

*Remark 5* (The nonsmooth case). As mentioned in the introduction, when  $Y$  is not uniruled (which we recall implies that  $\omega_Y$  is pseudo-effective by [BDPP13]), the proof can be extended without much effort to the general case, where  $f$  is only smooth over an open set  $V \subseteq Y$ , to deduce that  $V$  is of log general type. By blowing up, it is immediate to reduce to the case where  $Y \setminus V = D$ , a simple normal crossing divisor. The only difference in the argument is that (2) is now an isomorphism only away from  $D$ , which in practice means that in the proof above, we have to replace  $L$  with  $L(-rD)$  for some integer  $r > 0$ . The same steps now lead to the bigness of a line bundle of the form

$$\omega_Y^{\otimes a} \otimes \mathcal{O}_Y(bD)$$

for some  $a, b > 0$ , and the pseudo-effectivity of  $\omega_Y$  is then needed to deduce the bigness of  $\omega_Y(D)$  by multiplying by a suitable multiple of either  $\omega_Y$  or  $\mathcal{O}_Y(D)$ . (Compare with the proof of [PS17, Theorem 4.1].)

Note that if  $\kappa(Y) \geq 0$  and the complement of  $V$  in  $Y$  has codimension at least 2, it is not hard to check that  $\kappa(V) = \kappa(Y)$ ; see, for example, [MP21, Lemma 2.6]. Thus under this assumption, the conclusion is still that  $Y$  is of general type.

## 2.2 Proof of Theorem B

We divide the proof into a few steps.

*Step 1.* In this step, we deal with the case where  $Y$  is uniruled,<sup>3</sup> so in particular  $\kappa(Y) = -\infty$ , by an easy reduction to the case  $Y = \mathbb{P}^1$ , in which case a theorem of Viehweg–Zuo applies. We are in fact proving Proposition G; no assumption on the fibers of  $f$  is necessary. See also Remark 7 below for an alternative proof based on the techniques used in Theorem A.

<sup>2</sup>This requires the negativity of the kernels of generalized Kodaira–Spencer maps for pure Hodge modules with strict support  $Y$ , proved in [PW16] using results by Zuo and Brunebarbe; see the references therein.

<sup>3</sup>As mentioned in the introduction, conjecturally this is equivalent to  $\kappa(Y) = -\infty$ .

The hypothesis means that there exist a variety  $Z$ , which can be assumed to be smooth and projective, and a dominant rational map

$$Z \times \mathbb{P}^1 \cdots \rightarrow Y.$$

By resolving the indeterminacies of this map, we obtain a generically finite surjective morphism  $\varphi: W \rightarrow Y$  such that  $W$  admits a morphism  $g: W \rightarrow Z$  with general fiber  $\mathbb{P}^1$ . We consider the diagram

$$\begin{array}{ccc} \tilde{X} & \xrightarrow{\psi} & X \\ \downarrow \tilde{f} & & \downarrow f \\ h \left( \begin{array}{ccc} & & \\ & W & \xrightarrow{\varphi} Y \\ & \downarrow g & \\ & Z & \end{array} \right. & & \end{array}$$

where  $h = g \circ \tilde{f}$ . Since  $\psi$  is generically finite, we have  $\kappa(\tilde{X}) \geq \kappa(X)$  (see [Mor87, Corollary 2.3(i)]), and therefore it suffices to show that  $\kappa(\tilde{X}) = -\infty$ . Now note that the general fiber  $H$  of  $h$  admits a smooth morphism (with fiber  $F$ ) to the general fiber of  $g$ , that is, to  $\mathbb{P}^1$ . By [VZ01, Theorem 2], it follows that  $\kappa(H) = -\infty$ , and therefore by easy addition (see [Mor87, Corollary 1.7]), we also have  $\kappa(\tilde{X}) = -\infty$ . As promised, this proves Proposition G.

*Step 2.* In this step, we assume  $\kappa(Y) \geq 0$  and show that we can reduce to the case  $\kappa(Y) = 0$ . First, since the fibers of  $f$  are of general type, the subadditivity

$$\kappa(X) \geq \kappa(F) + \kappa(Y)$$

conjectured by Iitaka holds by [Kol87]. To prove Theorem B, it therefore suffices to show

$$\kappa(F) + \kappa(Y) \geq \kappa(X). \quad (6)$$

We show the reduction for this inequality.

To this end, we consider the Iitaka fibration  $g: Y \rightarrow Z$ , which after base change can be assumed to be a morphism, with  $Z$  smooth and projective. We again write  $h = g \circ f$ , with general fiber  $H$ , and we also denote the general fiber of  $g$  by  $G$ . Thus we have a smooth morphism  $H \rightarrow G$ , with fiber  $F$ , and since  $\kappa(G) = 0$ , we may assume  $\kappa(F) \geq \kappa(H)$ . On the other hand, by definition we have  $\dim Z = \kappa(Y)$ , and easy addition applied to  $h$  gives

$$\kappa(H) + \dim Z \geq \kappa(X).$$

Combining the two inequalities, we obtain (6).

*Step 3.* Under the assumption that the fibers of  $f$  are canonically polarized, by Taji [Taj16, Theorem 1.5] we also know the validity of Campana's isotriviality conjecture, and hence the Kebekus–Kovács conjecture, stating that if  $\kappa(Y) \geq 0$ , then  $\kappa(Y) \geq \text{Var}(f)$ . Using this in combination with Step 2, we may therefore assume  $\text{Var}(f) = 0$ , that is, that  $f$  is birationally isotrivial.<sup>4</sup>

*Step 4.* In this final step, we show the assertion in the theorem under the assumption that  $\text{Var}(f) = 0$  (with no assumptions on  $Y$ ). This is a consequence of the existence and properties of the moduli space of canonically polarized varieties, as suggested to us by Ziquan Zhuang.

Concretely, the family  $f$  induces a morphism

$$\varphi: Y \longrightarrow \mathbf{M}$$

<sup>4</sup>It is worth noting that in this proof, the hyperbolicity-type techniques present in the other results in this paper are hidden in the results from [VZ01] and [Taj16].



to the appropriate coarse moduli space of canonically polarized varieties [Vie95, Kol23]. Since  $f$  is birationally isotrivial, it follows that  $\varphi$  is constant on an open set and therefore constant everywhere; in other words, the family  $f$  is isotrivial. Since varieties of general type have finite automorphism group ( $\mathbf{M}$  is the coarse moduli space of a Deligne–Mumford stack), it is therefore well known that there is a finite étale base change  $\tilde{Y} \rightarrow Y$  such that

$$\tilde{X} := X \times_Y \tilde{Y} \simeq F \times \tilde{Y},$$

where  $F$  is any fiber of  $f$ . We thus have

$$\kappa(X) = \kappa(\tilde{X}) = \kappa(F) + \kappa(\tilde{Y}) = \kappa(F) + \kappa(Y).$$

This concludes the proof.

*Remark 7.* Proposition G can be rephrased as saying that if  $f: X \rightarrow Y$  is smooth and  $\kappa(X) \geq 0$ , then  $K_Y$  is pseudo-effective. Here is an alternative proof of this fact, resembling that of Theorem A. We consider the construction in the proof of that theorem, only now we take  $A = \mathcal{O}_Y$ , which is possible because  $\kappa(X) \geq 0$ . The exact same proof then shows either that we have directly that  $K_Y$  is pseudo-effective (case (i)) or that there is an inclusion

$$\mathcal{W} \hookrightarrow (\Omega_Y^1)^{\otimes N} \otimes \omega_Y^{\otimes m},$$

where  $\mathcal{W}$  has pseudo-effective determinant (case (ii)). But [CP19, Theorem 7.6] says precisely that in the latter case  $K_Y$  is again pseudo-effective.

*Remark 8.* In fact, recent work in symplectic geometry can be used to prove the following strengthening of Proposition G: *Let  $f: X \rightarrow Y$  be a smooth algebraic fiber space. If  $K_X$  is pseudo-effective, then  $K_Y$  is also pseudo-effective.* Here is a sketch of the proof. If  $K_Y$  is not pseudo-effective, then  $Y$  is covered by rational curves. After restricting to a smooth rational curve whose normal bundle has nonnegative degree, we obtain a smooth algebraic fiber space  $g: Z \rightarrow \mathbb{P}^1$  such that  $K_Z$  is pseudo-effective. But a very recent theorem by Pieloch [Pie21, Theorem 1.1]—answering a question by Starr [Sta17]—says that  $Z$  is covered by sections of  $g$  and therefore uniruled. This gives a contradiction.

*Remark 9* (The quasi-projective case). A very similar proof to that of Theorem B in fact leads to a statement for smooth morphisms between quasi-projective varieties, addressing an analogous case of the more general [Pop21, Conjecture 3.1]. We thank Sung Gi Park for pointing this out.

**THEOREM 10.** *Let  $f: U \rightarrow V$  be a smooth projective algebraic fiber space with canonically polarized fibers. If  $\kappa(V) \geq 0$ , then we have  $\kappa(U) = \kappa(F) + \kappa(V)$ .*

*Proof.* This is the analog of Steps 2–4 in the proof of Theorem B.

Since the fibers are of general type, the analog of Iitaka’s conjecture, namely

$$\kappa(U) \geq \kappa(F) + \kappa(V),$$

holds by [KP17, Theorem 1.3]. Therefore, we only need to focus on the opposite inequality. We can then reduce to the case  $\kappa(V) = 0$  by using the analog of the Iitaka fibration and the easy addition theorem in the case of open varieties; see [Lit77, Theorems 4 and 5].

Now again by [Taj16, Theorem 1.5], which is also valid in the quasi-projective case, we may therefore assume  $\text{Var}(f) = 0$ . Then the exact same argument as in Step 4 of the proof of Theorem B, based on the properties of the moduli space of varieties of general type, shows that the relative canonical model  $U'$  of  $U$  over  $V$  has an étale cover  $\tilde{U}$  which is birational to  $F \times V$ .

The statement then follows since on one hand  $\kappa(U) = \kappa(U')$  as  $U$  and  $U'$  admit proper birational morphisms from a fixed smooth quasi-projective variety  $W$  (see [Lit77, §3]), and on the other hand the log Kodaira dimension is preserved under étale covers (see [Lit77, Theorem 3]).  $\square$

Note, however, that a more general statement, removing the condition on the base and allowing for fibers with semiample canonical bundle, appeared very recently in [Cam22].

*Remark 11.* If confirmed, the results of [WW23] would lead to the validity of the statement of Theorem 10 in the case where the fibers of  $f$  are only assumed to be of general type. One would only need to replace  $f$  with its relative canonical model and apply the same argument.

### 2.3 Proof of Theorem C

If  $\kappa(Y) = -\infty$ , the result follows from the nonvanishing conjecture together with Proposition G, while if  $\kappa(Y) = 0$ , it follows from Theorem H(i). If  $\kappa(Y) > 0$ , we can reduce to the case  $\kappa = 0$  using the standard argument involving the Iitaka fibration: we may assume that this is a morphism  $g: Y \rightarrow Z$ , with general fiber  $G$  satisfying  $\kappa(G) = 0$ . Therefore, the general fiber  $H$  of  $h = g \circ f$  has a smooth morphism  $H \rightarrow G$  with fiber  $F$ , and consequently  $\kappa(F) \geq \kappa(H)$ . On the other hand, easy addition for  $h$  gives

$$\kappa(H) + \kappa(Y) = \kappa(H) + \dim Z \geq \kappa(X).$$

Hence the remaining point is to prove Theorem H.

### 2.4 Proof of Theorem H

If  $\kappa(F) = -\infty$ , then we also have  $\kappa(X) = -\infty$  by easy addition. We may therefore assume  $\kappa(F) \geq 0$  throughout. The key players are again the vector bundles

$$\mathcal{F}_m := f_* \omega_{X/Y}^{\otimes m} \quad \text{for } m \geq 1.$$

We also consider their twists

$$\mathcal{E}_m := \mathcal{F}_m \otimes \omega_Y^{\otimes m} \simeq f_* \omega_X^{\otimes m} \quad \text{for } m \geq 1,$$

for which we have

$$r_m := \text{rk}(\mathcal{E}_m) = \text{rk}(\mathcal{F}_m) = P_m(F) \quad \text{and} \quad h^0(Y, \mathcal{E}_m) = P_m(X).$$

We have for each  $m$  a (split) inclusion

$$\det \mathcal{F}_m \hookrightarrow \mathcal{F}_m^{\otimes r_m},$$

and therefore using the Viehweg fiber product trick just as in the proof of Theorem A, we obtain an inclusion

$$(\det \mathcal{F}_m)^{\otimes m} \hookrightarrow f_*^{(mr_m)} \omega_{X^{(mr_m)}/Y}^{\otimes m}.$$

Proceeding precisely as in the proof of that theorem (where the role of  $L$  there is now played by  $\det \mathcal{F}_m$ ), the arguments from [PS17] then lead to one of the following two possibilities:

- (i) The line bundle  $(\det \mathcal{F}_m)^{-1}$  is pseudo-effective.
- (ii) There exist a weakly positive sheaf  $\mathcal{W}$  on  $Y$  and an inclusion

$$\mathcal{W} \otimes \det \mathcal{F}_m \hookrightarrow (\Omega_Y^1)^{\otimes N}$$

for some positive integer  $N > 0$ .



Recall on the other hand that  $\mathcal{F}_m$  is a weakly positive sheaf by [Vie83, Theorem III], and therefore  $\det \mathcal{F}_m$  is a pseudo-effective line bundle, for example by [Vie83, Lemma 1.6(6)]. If statement (i) holds, then both  $\det \mathcal{F}_m$  and  $(\det \mathcal{F}_m)^{-1}$  are pseudo-effective, and so  $\det \mathcal{F}_m \equiv 0$  (that is, it is numerically trivial).

On the other hand, if statement (ii) holds, by taking the saturation of the left-hand side inside  $(\Omega_Y^1)^{\otimes N}$ , we obtain an exact sequence

$$0 \longrightarrow \mathcal{W}' \otimes \det \mathcal{F}_m \longrightarrow (\Omega_Y^1)^{\otimes N} \longrightarrow \mathcal{Q} \longrightarrow 0,$$

where  $\mathcal{W}'$  is a weakly positive sheaf and  $\mathcal{Q}$  is torsion-free. We also know that  $\det \mathcal{Q}$  is pseudo-effective; this follows from [CP19, Theorem 1.3], which says that if  $K_Y$  is pseudo-effective, then any quotient of any tensor power of  $\Omega_Y^1$  has pseudo-effective determinant. Passing to determinants, we then obtain an isomorphism of the form

$$\omega_Y^{\otimes a} \simeq (\det \mathcal{F}_m)^{\otimes b} \otimes P, \quad (12)$$

where  $a, b > 0$  and  $P$  is a pseudo-effective line bundle.

*Remark 13.* Up to here we only used that  $\kappa(F) \geq 0$  and  $K_Y$  is pseudo-effective.

If we assume  $K_Y \sim_{\mathbb{Q}} 0$ , from (12) we again deduce that  $(\det \mathcal{F}_m)^{-1}$  is pseudo-effective, and therefore as above  $\det \mathcal{F}_m \equiv 0$ . It follows from [CP17, Theorem 5.2] (see also [HPS18, Corollary 27.2]) that  $\mathcal{F}_m$  is a hermitian flat vector bundle and therefore comes from a unitary representation of the fundamental group. If we assume in addition that  $Y$  is simply connected, we then obtain  $\mathcal{F}_m \simeq \mathcal{O}_Y^{\oplus r_m}$ , or equivalently

$$\mathcal{E}_m \simeq (\omega_Y^{\otimes m})^{\oplus r_m}.$$

Note that by hypothesis  $\omega_Y^{\otimes m} \simeq \mathcal{O}_X$  when  $m$  is sufficiently divisible, and otherwise  $\omega_Y^{\otimes m}$  vanishes. Consequently,  $P_m(X) = P_m(F)$  when  $P_m(Y) = 1$ , and  $P_m(X) = 0$  otherwise. In general, we only have  $r_m \geq h^0(Y, \mathcal{E}_m)$  (because unitary representations are semisimple), and therefore by the same argument, we obtain

$$P_m(F) \geq P_m(X) \quad \text{for all } m \geq 1,$$

hence also  $\kappa(F) \geq \kappa(X)$ . On the other hand, a result of Cao–Păun [CP17, Theorem 5.6] states, for any algebraic fiber space  $f$ , that if  $c_1(\det \mathcal{F}_m) = 0 \in H^2(X, \mathbb{R})$  for some  $m \geq 2$ , then Iitaka's conjecture holds for  $f$ . Cao and Păun prove this result under the assumption that  $c_1(\det \mathcal{F}_m) = 0 \in H^2(X, \mathbb{Z})$ , but because of [HPS18, Theorem 27.2], the weaker assumption is sufficient to conclude that  $\mathcal{F}_m$  is a hermitian flat bundle. The rest of the argument then proceeds as in [CP17]. In our case, this gives the opposite inequality  $\kappa(X) \geq \kappa(F)$  and concludes the proof of item (ii).

We are thus left with proving item (i). We need the following lemma.

**LEMMA 14.** *Let  $E$  be a nonzero vector bundle on a projective variety  $X$ . If  $h^0(X, E) \geq \operatorname{rk} E + 1$ , then there is a line bundle  $L$  with  $h^0(X, L) \geq 2$  and an inclusion  $L \hookrightarrow E^{\otimes r}$  for some  $1 \leq r \leq \operatorname{rk} E$ .*

*Proof.* Let  $\mathcal{F} \subseteq E$  be the torsion-free subsheaf generated by the global sections of  $E$ , and let  $1 \leq r \leq \operatorname{rk} E$  be the generic rank of  $\mathcal{F}$ . Choose  $r$  global sections  $s_1, \dots, s_r \in H^0(X, E)$  that generate the stalk of  $\mathcal{F}$  at a general point of  $X$ . The resulting morphism  $\mathcal{O}_X^{\oplus r} \rightarrow \mathcal{F}$  is injective; it cannot be surjective, for otherwise it would be an isomorphism, and then  $h^0(X, E) = h^0(X, \mathcal{F}) = r \leq \operatorname{rk} E$ , contradicting the assumption that  $h^0(X, E) \geq \operatorname{rk} E + 1$ . Let  $x \in X$  be one of the points where the morphism fails to be surjective on stalks. Since  $\mathcal{F}$  is globally generated, depending on the situation, we can do one of the following two things:

- If  $s_1, \dots, s_r$  remain linearly independent at  $x$ , we can find an additional global section  $s \in H^0(X, E)$  such that  $s(x)$  does not lie in the linear span of  $s_1(x), \dots, s_r(x)$  inside the stalk  $E_x$ . It follows that, among the  $r + 1$  sections

$$s_1 \wedge \dots \wedge s_r, \quad s_1 \wedge \dots \wedge s_{i-1} \wedge s \wedge s_{i+1} \wedge \dots \wedge s_r \quad \text{for } i = 1, \dots, r$$

of the bundle  $\bigwedge^r E$ , at least two are linearly independent.

- If  $s_1, \dots, s_r$  do not remain linearly independent at  $x$ , we can in any case find other global sections  $t_1, \dots, t_r \in H^0(X, E)$  such that  $t_1(x), \dots, t_r(x) \in E_x$  form part of a minimal system of generators of  $\mathcal{F}_x$ . Then

$$s_1 \wedge \dots \wedge s_r \quad \text{and} \quad t_1 \wedge \dots \wedge t_r$$

are linearly independent global sections of the bundle  $\bigwedge^r E$ .

Either way, by construction, all these sections come from global sections of the sheaf  $\bigwedge^r \mathcal{F}$ . Let  $L$  be the reflexive hull of  $\bigwedge^r \mathcal{F}$ ; then  $L$  is a line bundle with  $h^0(X, L) \geq 2$ . We also have inclusions  $L \hookrightarrow \bigwedge^r E \hookrightarrow E^{\otimes r}$  because  $E$  is locally free.  $\square$

Let us now suppose for the sake of contradiction that we have  $P_m(X) \geq P_m(F) + 1$ . The lemma, applied to the vector bundle  $\mathcal{E}_m$ , then gives us an inclusion

$$L \hookrightarrow \mathcal{E}_m^{\otimes r} \simeq \mathcal{F}_m^{\otimes r} \otimes \omega_Y^{\otimes mr}$$

for some integer  $1 \leq r \leq r_m$ , where  $L$  is a line bundle with  $h^0(X, L) \geq 2$ , and so  $\kappa(L) \geq 1$ . Arguing as above, using the fiber product trick, we obtain one of the following two possibilities:

- (i) The line bundle  $L^{-1} \otimes \omega_Y^{\otimes mr}$  is pseudo-effective.
- (ii) There exist a weakly positive sheaf  $\mathcal{W}$  on  $Y$  and a short exact sequence

$$0 \longrightarrow \mathcal{W} \otimes L \longrightarrow (\Omega_Y^1)^{\otimes N} \otimes \omega_Y^{\otimes mr} \longrightarrow \mathcal{Q} \longrightarrow 0$$

for some positive integer  $N > 0$ .

In case (i), since  $\kappa(L) \geq 1$ , the Campana–Peternell conjecture implies that  $\kappa(Y) \geq 1$ , contradicting our assumption that  $\kappa(Y) = 0$ .

In case (ii), as above,  $\det \mathcal{Q}$  must be a pseudo-effective line bundle by [CP19, Theorem 1.3]. Passing to determinants in the short exact sequence in case (ii), we now obtain an isomorphism

$$\omega_Y^{\otimes a} \simeq L^{\otimes b} \otimes P,$$

where  $a, b > 0$  and  $P$  is a pseudo-effective line bundle. We then conclude just as in case (i).

## 2.5 Proof of Corollary I

Let  $f: X \rightarrow Y$  be a smooth algebraic fiber space whose general fiber  $F$  satisfies  $\kappa(F) \geq 0$ . We use the notation

$$\kappa_f(Y) := \max \{ \kappa(Y, L) \mid mK_X - f^*L \text{ is pseudo-effective for some } m \geq 1 \}$$

and show that  $\kappa(Y) = \kappa_f(Y)$  provided that  $\kappa(F) \geq 0$ . One inequality is easy. Our assumption that  $\kappa(F) \geq 0$  implies that the relative canonical divisor  $K_{X/Y}$  is pseudo-effective; the reason is that  $\omega_{X/Y}$  has a canonical singular hermitian metric with semipositive curvature [PT18, Theorem 4.2.2] (see also [HPS18, Theorem 27.1]). Since  $K_X - f^*K_Y \equiv K_{X/Y}$ , we get  $\kappa_f(Y) \geq \kappa(Y)$ .

To prove the other inequality, let  $L$  be a line bundle on  $Y$  such that  $mK_X - f^*L$  is pseudo-effective and such that  $\kappa_f(Y) = \kappa(Y, L)$ . We may assume  $\kappa(Y, L) \geq 0$  and consider an integer

$n \geq 1$  such that the linear system  $|nL|$  gives the Iitaka fibration of  $L$ . After replacing  $Y$  with a log resolution of the base locus of  $|nL|$ , and  $f: X \rightarrow Y$  with the fiber product, we can assume  $nL \sim B + E$  with  $B$  semiample and  $E$  effective; consequently,  $nmK_X - f^*B \sim n(mK_X - f^*L) + E$  is still pseudo-effective. Moreover, because of the assumption on  $n$ , we have that  $\kappa(Y, B) = \kappa(Y, L)$ ; hence after replacing  $L$  with  $B$ , we may assume from the beginning that  $L$  is semiample. The Campana–Petrernell conjecture (on  $X$ ) implies that  $mK_X - f^*L$  is effective for  $m \gg 0$ ; this is proved in [Sch22]. We can now apply exactly the same argument as in the proof of Theorem A, up to the line containing (4), and obtain a short exact sequence

$$0 \longrightarrow \mathcal{W} \otimes L \longrightarrow (\Omega_Y^1)^{\otimes N} \otimes \omega_Y^{\otimes m} \longrightarrow \mathcal{Q} \longrightarrow 0$$

for some  $N \geq 1$ , with  $\mathcal{W}$  weakly positive. (Concretely,  $\mathcal{W}$  is a torsion-free sheaf with a semipositively curved singular hermitian metric.) Since  $m \geq 1$ , both  $K_Y$  and  $\det \mathcal{Q}$  must therefore be pseudo-effective: this follows by combining [CP19, Theorem 7.6] (to conclude that  $K_Y$  is pseudo-effective) and [CP19, Theorem 1.3] (to conclude that any quotient of any tensor power of  $\Omega_Y^1$  has pseudo-effective determinant). Passing to determinants in the short exact sequence above, we see that there are two integers  $a, b \geq 1$  such that

$$aK_Y \sim bL + \det \mathcal{Q} + \det \mathcal{W}.$$

As both  $\det \mathcal{Q}$  and  $\det \mathcal{W}$  are pseudo-effective, the Campana–Petrernell conjecture (on  $Y$ ) now gives us the desired inequality  $\kappa(Y) \geq \kappa(Y, L) = \kappa_f(Y)$ .

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