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Dense entire curves in rationally connected manifolds

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With an appendix by János Kollár

Abstract

We show the existence of metrically dense entire curves (of growth order 0) in rationally connected complex projective manifolds, confirming for this case a conjecture formulated by the first-named author, according to which such entire curves on projective manifolds exist if and only if these are special in the sense of birational geometry.

We next show the existence of dense entire curves, avoiding the singular locus, in certain log-terminal normal rational surfaces. This implies via results of Grassi and Oguiso the existence of dense entire curves into any Calabi–Yau threefold fibered in abelian surfaces or elliptic curves.

We then show that a dense entire curve may be chosen on any rationally connected manifold in such a way that it does not lift to a given ramified cover, answering in this case a question of Corvaja and Zannier about the Nevanlinna analog of the 'weak Hilbert property' of arithmetic geometry. As shown in the appendix by Kollár, this entire curve may be chosen independently of the ramified cover.

1. Introduction

Our main result is the existence of dense entire curves (that is, a holomorphic map from $\mathbb{C}$ with metrically dense image) in rationally connected manifolds (see Theorems 4.2 and 8.11, and Section 9).

Theorem A. *Let X be a rationally connected complex projective manifold with a countable subset M, a reduced hypersurface D and a non-trivially ramified cover $Y \to X$.*

Then there exists a holomorphic map $c\colon \mathbb{C} \to X$ ('entire curve') such that the following hold:

(i) *The image $c(\mathbb{C})$ is dense in X.*

(ii) *The countable set M is contained in the image $c(\mathbb{C})$,*

(iii) *The image $c(\mathbb{C})$ meets D transversally somewhere.*

(iv) *The map c cannot be lifted to an entire curve in Y.*

(v) *The order ρ_c (in the sense of Nevanlinna theory) equals zero.*

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The non-liftability confirms (for rationally connected manifolds) a conjecture of Corvaja and Zannier.

We also prove some results in these directions for normal projective surfaces and for compact Kähler manifolds with $c_1 = 0$.

The existence of a dense entire curve for any rationally connected manifolds confirms, for this case, a conjecture, which motivated the present text, that for compact Kähler manifolds, the existence of a dense entire curve is equivalent to being special (the definition is recalled below). Special manifolds are in a precise sense 'opposite' to manifolds of general type. While, following a conjecture of Lang, the non-existence of Zariski-dense entire curves on manifolds of general type has been investigated since decades, our result seems to be the first one in the opposite direction, beyond the classical case of unirational or abelian manifolds.

Although such dense curves exist on any unirational manifold for simple function-theoretic interpolation reasons, it is presently unknown whether rationally connected manifolds are all unirational. In fact, the contrary is expected to be true quite generally. Moreover, our proof, which is mainly algebraic, based on the 'comb smoothing' technique of [KMM92], provides a (to our knowledge) new approach to these interpolation properties and plays a role similar to the power series expansion of entire functions in this broader context. We do not know, however, if any entire curve in a rationally connected manifold can be approximated by algebraic maps, as in our construction.

The conjectural links formulated by Lang between arithmetic and hyperbolicity properties lead to conjecture that the existence of a Zariski-dense entire curve should occur on X defined over a number field k if and only if $X(k')$ is Zariski dense for some finite extension k' of k. As this arithmetic property is unknown for X rationally connected, our result give some support to this arithmetic conjecture.

We show, finally, that the entire curves we construct in any rationally connected manifolds may be chosen so as to have growth order $\rho_f = 0$. In particular, any rationally connected complex projective manifold X admits dense entire curves $h\colon \mathbb{C} \to X$ with $\rho_h = 0$. In the opposite direction, we showed in [CW16] that any complex projective manifold X of dimension n admitting a non-degenerate holomorphic map $F\colon \mathbb{C}^n \to X$ of order $\rho_F < 2$ is rationally connected.

For normal projective surfaces X with only quotient singularities, we show the following: if there is an effective non-zero $\mathbb{Q}$-divisor Δ such that $K + \Delta$ is $\mathbb{Q}$-trivial and such that the pair (X, Δ) is log terminal, then there exists a dense entire curve avoiding the singular locus. Here the 2-dimensional minimal model program (MMP) plays a crucial role in the proof.

2. Review of rationally connected manifolds

Recall that a projective manifold X is said to be

1. *rational* if it is bimeromorphic to $\mathbb{P}^n$,
2. *unirational* if it is dominated by $\mathbb{P}^n$ (which means that there exists a non-degenerate (that is, dominant) meromorphic map $\Phi\colon \mathbb{P}^n \dashrightarrow X$),
3. *rationally connected* if any two generic points of X are connected by a rational curve in X,
4. *Fano* if its anticanonical bundle is ample.

Properties 1, 2 and 3 are invariant under birational transformations.

One has the implications

$$\begin{array}{ccccc} & & & & \text{Fano} \\ & & & & \Downarrow \\ \text{rational} & \Longrightarrow & \text{unirational} & \Longrightarrow & \text{rationally connected}\,. \end{array}$$

For curves (respectively, surfaces), these properties (respectively, the first three properties) are equivalent. But, starting in dimension 3, many unirational manifolds are known to be non-rational. On the other hand, although it is expected that most rationally connected threefolds are not unirational, there is presently no known invariant to distinguish them.

Examples of rationally connected threefolds not known to be unirational, and possibly non-unirational, are 'general' quartics in $\mathbb{P}^4$, double covers of $\mathbb{P}^3$ branched over a smooth sextic and standard conic bundles over $\mathbb{P}^2$ with smooth discriminant of large degree.

3. Brief review of special manifolds

Let X be an n-dimensional smooth connected compact complex manifold (either projective or compact Kähler). Its Kodaira dimension is denoted by $\kappa(X)$.

Definition 3.1 ([Cam04, Definition 2.24 and Theorem 2.27]). A compact Kähler manifold X is said to be *special* if $\kappa(X, L) < p$ for any rank 1 coherent sheaf $L \subset \Omega^p_X$ and any $p > 0$.

These manifolds are higher-dimensional generalizations of rational and elliptic curves. In particular, manifolds X which are either rationally connected or with $\kappa(X) = 0$ are special.[1]

We refer to [Cam04] for details on this class of manifolds and some of the reasons to introduce them, the main one being the existence of the 'core map' which splits any X into its opposite parts: special vs general type. In fact, the decomposition $c = (j \circ r)^n$ of the 'core map' (see [Cam04] for details) shows that the 'building blocks' of special manifolds should be (smooth) 'orbifold pairs' either 'rationally connected,'[2] or with $\kappa = 0$. Moreover [Cam04, Corollary 8.11], such a manifold X is special if it is $\mathbb{C}^n$-dominable (that is, if there exists a meromorphic map from $\mathbb{C}^n$ to X that is regular and submersive at some point). The converse is not expected to hold in general (but partially weaker versions are conjectured below).

Below we summarize what we expect to be true for special manifolds. Many more variants can be formulated; we state only the simplest ones.

Conjecture 3.2 ([Cam04, § 9.2, 9.8, 9.5 and 9.20]). *Let X be a compact connected Kähler manifold X. Then the following statements are equivalent:*

(i) *The manifold X is special.*

(ii) *The Kobayashi pseudo-metric of X vanishes identically.*

(iii) *There is a entire curve $h\colon \mathbb{C} \to X$ with Zariski-dense image.*

(iv) *There is a entire curve $h\colon \mathbb{C} \to X$ with metrically dense image.*

(v) *Any two generic points of X are joined by some entire curve.*

(vi) *Any countable subset of X is contained in the image of some entire curve.*

[1]However, among n-dimensional manifolds with $\kappa = k$, there are both special and non-special examples, for any $k \neq 0, n$.

[2]Here, rationally connected is in the sense of having $\kappa^+ = -\infty$. When there is no orbifold structure, a classical conjecture claims that this is equivalent to rational connectedness.

This conjecture is motivated by the above-mentioned decomposition $c = (j \circ r)^n$ of the core map, which essentially reduces these conjectures to the two cases of orbifold pairs either rationally connected in the above sense or with $\kappa = 0$.

An arithmetic analog is the following.

Conjecture 3.3. *Let X be a projective variety defined over a number field K. Then the following statements are equivalent:*

(i) *The variety X is special* (*as a complex variety*).

(α) *The variety X is potentially dense* (*that is, $X(k)$ is Zariski dense for some finite field extension k/K*).

Moreover, we conjecture the following.

Conjecture 3.4 ([Cam04, Conjecture 7.1]). *The group $\pi_1(X)$ is almost abelian if X is a special compact Kähler manifold.*

• Property (v) means that special manifolds are expected to be analogs of rationally connected manifolds, replacing algebraic maps $\mathbb{C} \to X$ by their transcendental version, entire curves $h\colon \mathbb{C} \to X$ with h holomorphic, non-constant.

• In general, we have the obvious implications

$$\begin{array}{ccccc} (\text{vi}) & \Rightarrow & (\text{iv}) & \Rightarrow & (\text{iii}) \\ \Downarrow & & \Downarrow & & \\ (\text{v}) & \Rightarrow & (\text{ii}) . & & \end{array}$$

• If X is unirational, all the properties of the above three conjectures are satisfied.

• If there exists a surjective holomorphic map $p\colon \mathbb{C}^n \to X$, then all the properties of Conjecture 3.2 are satisfied.

• Properties (ii) and (v) are obvious for rationally connected manifolds.

The main purpose of this article is to show that every rationally connected manifold admits a dense entire curve. In fact, we prove that properties (iii), (iv) and (vi) hold for rationally connected manifolds (Theorem 4.2).

Thus all the properties of Conjecture 3.2 hold for every rationally connected manifold. In particular, we confirm that property (vi) holds for them.

We also deduce a Nevanlinna analog of the Hilbert irreducibility property introduced in [CZ17]. Similar results hold more classically when X is a complex torus.

We give some brief remarks on the much more challenging class of manifolds with zero first Chern class in Section 7 below.

• It is proved in [BL00] that a projective surface is special if and only if it is $\mathbb{C}^2$-dominable (with the possible exception of non-elliptic and non-Kummer $K3$ surfaces). This might however be a low-dimensional phenomenon, and it is not expected to remain true in dimension 3.

Remark 3.5. Vojta [Voj87] has introduced a dictionary between entire curves and infinite sequences of rational points. The conjectures above imply that if X is defined over a number field, the set of k'-rational points is infinite for some finite extension k'/k if and only if X contains an entire curve. Vojta's analogy thus becomes an equivalence between entire curves and arithmetic geometry. The Nevanlinna version of the Hilbert property discussed in Section 9 is another illustration of this statement.

4. Entire curves in rationally connected manifolds

We recall here classical results used in the next subsections.

4.1 Comb smoothing

We recall the technique of *comb smoothing* introduced in [KMM92]; see also [Deb01]. From here on, we always view the projective line as $\mathbb{P}_1 = \mathbb{C} \cup \{\infty\}$, that is, with a distinguished point at infinity.

Let C be the reducible curve given by gluing two rational curves together transversally in one point. Let V be the variety obtained from blowing up the point $(\infty, 0)$ in $\mathbb{P}_1 \times \mathbb{C}$. Using $[y_0 : y_1]$, t as coordinates on $\mathbb{P}_1 \times \mathbb{C}$, we may embed C into V as the total transform of $t = 0$. With L denoting an affine line, the function t realizes V as a flat projective L-scheme, with a special fiber isomorphic to C and all other fibers isomorphic to smooth rational curves. By abuse of notation, we also denote this special fiber by C.

In explicit coordinates, we have

$$V = \{([y_0 : y_1]; t; [s_0 : s_1]) \in \mathbb{P}_1 \times \mathbb{C} \times \mathbb{P}_1 \colon s_1 y_0 = s_0 y_1 t\} \mapsto t \in L\,.$$

For a given morphism $f \colon C \to X$ with values in a projective manifold X, the morphism f can be extended to a holomorphic map F defined on an open neighborhood of C in V if the following condition is fulfilled [Kol03, Theorem 7.6]:

$$\boxed{f^*TX \text{ is spanned by global sections on } C \text{ and } H^1(C, f^*TX) = \{0\}} \tag{$*$}$$

Let $r \geqslant 0$ be an integer. Recall [Deb01, Definition 4.5], that a (parameterized) rational curve $f \colon \mathbb{P}_1 \to X$ is said to be *r-free* if $f^*(TX) \otimes \mathcal{O}_{\mathbb{P}_1}(-(r-1))$ is ample on $\mathbb{P}_1$ or, equivalently, if $H^1(\mathbb{P}_1, f^*(TX) \otimes \mathcal{O}_{\mathbb{P}_1}(-(r+1))) = 0$. This is an open condition on the space of such maps.

The condition 1-free is frequently called *very free*.

If $f \colon \mathbb{P}_1 \to X$ is r-free and its image goes through $r+1$ distinct points, there are small deformations of f going through any r of these points and through any given point sufficiently close to the last $(r+1)$st one. Instead of fixing r points, one may also fix jets of orders $m_1, \dots, m_s$ provided $\sum_{j=1}^{s}(m_j + 1) \leqslant r$.

If $f \colon C \to X$ is a morphism whose restrictions to the two irreducible components of C are r_1-free, respectively r_2-free, then condition $(*)$ is fulfilled. Hence f extends to a holomorphic map F defined on some neighborhood of C in V. For each fiber of $V \to L$ contained in this neighborhood, we obtain a rational curve f_t. Moreover, in this case, the curves f_t are $(r_1 + r_2)$-free rational curves.

Summarizing, we have the following.

Proposition 4.1. *Let X be a projective manifold, equipped with a distance function d induced by a Hermitian metric on X, and let $g_i \colon \mathbb{P}_1 = \mathbb{C} \cup \{\infty\} \to X$ be r_i-free rational curves ($i = 1, 2$, $r_i \geqslant 0$) with $g_1(\infty) = g_2(0)$. Then for every $R, R', \epsilon > 0$, there exist an $(r_1 + r_2)$-free rational curve $c \colon \mathbb{P}_1 = \mathbb{C} \cup \{\infty\} \to X$ and a parameter $\lambda > 0$ such that $d(c(z), g_1(z)) < \epsilon$ for all z with $|z| < R$ and $d(c(\lambda z), g_2(z)) < \epsilon$ for all z with $|z| > R'$.*

Moreover, given $p_0, p_1, \dots, p_{r_1} \in \mathbb{C}$ and $q_1, \dots, q_{r_2} \in \mathbb{C}^*$, the curve c may be chosen in such a way that there are points $\tilde{p}_i$, $\tilde{q}_i$ on $\mathbb{C}$ with $c(\tilde{p}_i) = g_1(p_i)$, $|\tilde{p}_i - p_i| < \epsilon$ and $c(\tilde{q}_i) = g_2(q_i)$.

Proof. Let V be as above. We embed the first rational curve into V as $\mathbb{C} \cup \{\infty\} \ni x \mapsto ([1 : x], 0, [1 : 0])$ and the second one as $\mathbb{C} \cup \{\infty\} \ni x \mapsto ([0 : 1], 0, [1 : x])$. Now the pair

(g_1, g_2) defines a morphism from C to X which extends to a holomorphic map F defined on an open neighborhood. Thus for small enough t, we obtain a rational curve $f_t\colon \mathbb{C}\cup\{\infty\}\to X$ as

$$\mathbb{C}\cup\{\infty\}\ni x\mapsto F\left([1:x],t,[1:xt]\right).$$

We define $\lambda = 1/t$ and obtain

$$\mathbb{C}\cup\{\infty\}\ni \lambda x\mapsto F\left([1:\lambda x],t,[1:\lambda xt]\right)=F\left([t:x],t,[1:x]\right).$$

It is now easy to check explicitly that

$$d(c(z),g_1(z))<\epsilon \quad \forall |z|<R \quad \text{and} \quad d(c(\lambda z),g_2(z))<\epsilon \quad \forall |z|>R'.$$

Since the g_i are r_i-free, c is (r_1+r_2)-free, and we may choose F in such a way that the points p_i and q_j are preserved; that is, for a small neighborhood W of 0 in $\mathbb{C}$, we have maps $\sigma_i,\tau_j\colon W\to V$ with $F(\sigma_i(t))\equiv f(p_i)$ and $F(\tau_j(t))\equiv f(q_j)$ (for $1\leqslant i\leqslant r_1$, $1\leqslant j\leqslant r_2$). We observe that

$$\lim_{t\to 0} f_t(p_i)=g_1(p_i) \quad \text{and} \quad \lim_{t\to\infty}\sigma_i(t)=p_i \quad \forall i\,.$$

Hence for t small enough, $\tilde{p}_i$ defined via $f_t(\tilde{p}_i)=\sigma_i(t)$ satisfies $|\tilde{p}_i-p_i|<\epsilon$. □

4.2 Dense entire curves in rationally connected manifolds

We present our main result on the existence of dense entire curves in rationally connected manifolds. Later we deduce from it stronger forms by applying it to projective jet bundles of X (see Corollary 4.7).

Theorem 4.2. *Let X be a rationally connected complex projective manifold, let A be a closed analytic subset of codimension at least 2, and let M be a countable subset of $X\setminus A$. Then there exists an entire curve $h\colon\mathbb{C}\to X$ such that*

(i) *$M\subset h(\mathbb{C})$,*

(ii) *$h(\mathbb{C})\subset X\setminus A$.*

In particular, if M is dense in X, so is $h(\mathbb{C})$.

Corollary 4.3. *Rationally connected manifolds have all the properties of Conjecture 3.2.*

Assumptions 4.4. From here on, X always denotes a rationally connected complex projective manifold, A a closed analytic subset of X of codimension at least 2, and $d(\,,\,)$ denotes a distance function on X induced by some smooth Hermitian metric on X.

Lemma 4.5. *Let X, A, d be as defined above and $r\in\mathbb{N}$. Let $f\colon\mathbb{P}_1\to X$ be an $(r-1)$-free rational curve such that $f(\mathbb{P}_1)\cap A=\emptyset$. Let $a_i\in\mathbb{C}$, $i=1,\dots,r$, be given, and let $R>0$ and $\varepsilon>0$ be given with $R\geqslant|a_i|+\varepsilon$ for all i. Let $q\in X\setminus A$. Then, there exist a rational curve $h\colon\mathbb{P}_1\to X$ and $a_i'\in\mathbb{C}$, $i=1,\dots,r+1$, such that*

(i) *h is r-free;*

(ii) *$h(\mathbb{P}_1)\cap A=\emptyset$;*

(iii) *$d(h(z),f(z))\leqslant\varepsilon$ if $|z|\leqslant R$;*

(iv) *$h(a_i')=f(a_i)$, $i=1,\dots,r$;*

(v) *$|a_i'-a_i|\leqslant\varepsilon$, $i=1,\dots,r$;*

(vi) *$h(a_{r+1}')=q$.*

Proof. We first prove the existence of a 1-free curve $g\colon \mathbb{P}_1 \to X$ avoiding A and containing q and $f(\infty)$. Indeed, from [Deb01, Corollary 4.28] we get the existence of a 1-free curve $g\colon \mathbb{P}_1 \to X$ such that $g(0) = f(\infty)$ and $g(1) = q$. Since the deformations of g are unobstructed, there is a holomorphic map $G\colon T \times \mathbb{P}_1 \to X$, where T is an open neighborhood of 0 in the vector space $H^0(\mathbb{P}_1, f^*(TX) \otimes \mathcal{O}_{\mathbb{P}_1}(-\{0,1\}))$ inducing deformations $G(t,\cdot) = g_t$ of g fixing $g(0)$ and $g(1)$, with $dG(0,z)(t) = t(z)$, for $z \in \mathbb{P}_1$ and $t \in H^0(\mathbb{P}_1, f^*(TX))$. Now $g^*(TX)$ is generated outside of 0 and 1 by its sections, and so G has maximal rank near $\{0\} \times (\mathbb{P}_1 \setminus \{0,1\})$, so that $G^{-1}(A)$ has codimension at least 2 in $T \times \mathbb{P}_1$. Its projection in T thus has codimension at least 1, and hence the generic g_t avoids A.

We now apply the comb smoothing technique (see Proposition 4.1) to the comb defined by the rational curves f and g. We obtain an r-free rational curve h having properties (i), (iii), (iv), (v), (vi). Since $f(\mathbb{P}_1) \cup g(\mathbb{P}_1)$ does not intersect A, we see that $C \cap F^{-1}(A)$ (in the notation of Proposition 4.1) is empty. It follows that $F^{-1}(A)$ does not intersect an open neighborhood of C in V. Therefore, we may choose h such that $h(\mathbb{P}_1) \subset X \setminus A$ (that is, such that property (ii) holds). □

Proof of Theorem 4.2. Let $M := \{x_1, x_2, \dots, x_n, \dots\}$ be the given sequence in $X \setminus A$. We construct inductively a sequence of rational curves $h_n\colon \mathbb{P}_1 = \mathbb{C} \cup \{\infty\} \to X$, $R_n > 0$, $\varepsilon_n > 0$, $a_n^k \in \mathbb{C}$ for all $k \geqslant n \geqslant 1$ such that

0. $\varepsilon_n \to 0$ when $n \to +\infty$, the sequence ε_n being decreasing;
1. h_n is n-free;
2. $d(h_n(\mathbb{P}_1), A) \geqslant \varepsilon_n > 0$; in particular, $h_n(\mathbb{P}_1) \cap A = \emptyset$;
3. $d(h_{n+1}(z), h_n(z)) < \varepsilon_n/2^{n+1}$ for $|z| \leqslant R_n$;
4. $R_{n+1} \geqslant R_n + 1$;
5. $h_k\bigl(a_n^k\bigr) = x_n$ for all $k \geqslant n \geqslant 1$;
6. $\bigl|a_n^{k+1} - a_n^k\bigr| \leqslant \varepsilon_n/2^{k+1}$ for all $k \geqslant n \geqslant 1$;
7. $|a_n^n| \leqslant R_n$.

Assume that these data are given. From properties 0, 3 and 4, we see that the maps h_n converge uniformly on compact subsets of $\mathbb{C}$ to a holomorphic map $h\colon \mathbb{C} \to X$. For a fixed number n, the sequence a_n^k is a Cauchy sequence by property 6, hence convergent. We define $a_n := \lim_{k\to+\infty} a_n^k$. Property 5 implies that $h(a_n) = x_n$ for all n. Thus $M \subset h(\mathbb{C})$.

From property 3, we obtain that $d(h(z), h_n(z)) \leqslant \epsilon_n/2^n$ for all $z \in \mathbb{C}$ with $|z| \leqslant R_n$. In combination with property 2, this yields $d(h(z), A) \geqslant \epsilon_n(1 - 2^{-n}) > 0$ for all $z \in \mathbb{C}$ with $|z| \leqslant R_n$. Since this holds for all n, we obtain $d(h(z), A) > 0$ for all $z \in \mathbb{C}$; that is, $h(\mathbb{C}) \cap A = \emptyset$.

We now construct the sequences $h_n, R_n, \varepsilon_n, a_n^k$ enjoying the above properties 1–7 by induction on n. This will prove Theorem 4.2.

For $n = 1$, just take for h_1 any 1-free rational curve avoiding A and with $h_1(1) = x_1$, and choose $a_1^1 = R_1 = 1$. Choose $\varepsilon_1 = \frac{1}{2} d(h_1(\mathbb{P}_1), A)$.

Fix n. Assume h_m, ε_m, R_m, a_m^k already constructed for all k, m with $m \leqslant k \leqslant n$ and having properties 1–7 above. We apply Lemma 4.5 with $f = h_n$, $a_i = a_i^n$, $R = R_n$ and $\epsilon = \epsilon_n/2^{n+1}$. This yields an $(n+1)$-free rational curve $h_{n+1} = h$. We set $a_k^{n+1} = a'_k$ (in the notation of Lemma 4.5). We finally choose $R_{n+1} > \max\bigl\{R_n + 1, \bigl|a_{n+1}^{n+1}\bigr|\bigr\}$ and $\varepsilon_{n+1} < \min\bigl\{\varepsilon_n/2^{n+1}, d(h_{n+1}(\mathbb{P}_1), A)\bigr\}$. □

A slight modification of the proof gives the following.

Corollary 4.6. *Let X be a rationally connected projective manifold with a free rational curve $c\colon \mathbb{P}_1 \to X$ and a distance function $d(\, , \,)$ induced by some Hermitian metric. Then for all $R > 0$ and $\epsilon > 0$, there exists an entire curve $h\colon \mathbb{C} \to X$ with dense image and such that $d(h(z), c(z)) < \epsilon$ for all $z \in \mathbb{C}$ with $|z| < R$.*

Theorem 4.2 may be improved by prescribing jets instead of mere values.

Corollary 4.7. *Let X, A, M be as in Theorem 4.2. Let $M = \{x_n : n \in \mathbb{N}\}$ be as in the proof of Theorem 4.2. For each $n \in \mathbb{N}$, choose a number $m_n \in \mathbb{N} \cup \{0\}$ and fix an m_n-jet j_n of a map from the unit disc in $\mathbb{C}$ to X at x_n. Then the map h of Theorem 4.2 which sends $a_n \in \mathbb{C}$ to x_n can be chosen to have its m_n-jet at a_n coincide with j_n for every n.*

Proof. The proof is the same as before with the following modifications.

As starting point we need r-free rational curves with prescribed jet at one point (see Lemma 4.8). Then we use comb smoothing as described, noting that r-free curves may be deformed, keeping m_i-jets $j_1, \dots, j_i$ as long as $\sum_i m_i < r$ (instead of just keeping $r-1$ points). □

Lemma 4.8. *Let X be a rationally connected manifold, $p, q \in X$ $(p \neq q)$ and $r, k \in \mathbb{N}$, and let γ be a k-jet at p. Then there exists an r-free rational curve $c\colon \mathbb{P}_1 \to X$ with $c(\infty) = q$ and $c(0) = p$ such that the k-jet at $c(0) = p$ agrees with the prescribed k-jet γ.*

Proof. Since X is rationally connected, there is a 1-free rational curve $c_0\colon \mathbb{P}_1 = \mathbb{C} \cup \{\infty\} \to X$ with $c_0(0) = p$ and $c_0(\infty) = q$. The curve c_0 being 1-free means that $\deg(L_i) > 0$ for all L_i if $c_0^*TX \simeq \oplus L_i$. Choose $N > \max\{k, r\}$, and define $c_1(z) = c_0(z^N)$. Now c_1 is N-free and maps 0 to p and ∞ to q, and the k-jet at p is the zero jet. The vector bundle $c_1^*TX \otimes \mathcal{O}(N-1)$ is ample. It follows that the deformations of the curve c_1 are unobstructed. Sections of $\mathcal{O}(N-1)$ on $\mathbb{P}_1$ correspond to homogeneous polynomials of degree $N-1$. Any k-jet can be realized by a polynomial of degree k. Therefore, there exists an open neighborhood Ω of 0 in the space $J_{p,k}X$ of k-jets at p such that for every $\tilde{\gamma} \in \Omega$, we may deform c_1 to a curve $c_2\colon \mathbb{P}_1 \to X$ with $c_2(0) = p$ and k-jet $\tilde{\gamma}$ at p while preserving $c_2(\infty) = q$. The deformed curve is still N-free. Finally, we use scaling; that is, we define $c(z) = c_2(\lambda z)$ for an appropriately chosen $\lambda \in \mathbb{C}^*$. In this way, we may achieve that the k-jet of c equals γ. □

Remark. This gives an entire curve analog of the weak approximation property in arithmetic geometry on rationally connected manifolds. This connection with the Nevanlinna–Hilbert property treated below was pointed out to us by P. Corvaja.

One can refine Corollary 4.7 by imposing that h meet a given divisor D transversally at each of its regular intersection points.

Theorem 4.9. *In the set-up of Theorem 4.2, assume that in addition a reduced divisor (not necessarily irreducible or connected) D in X is given. Then we may choose the entire curve $h\colon \mathbb{C} \to X$ provided by Theorem 4.2 in such a way that* every *intersection point of $h(\mathbb{C})$ and D is transversal.*

Remark. The transversality condition implies in particular that $h(\mathbb{C})$ avoids the singular locus of D.

Before proving the theorem, we need some auxiliary lemmas.

Lemma 4.10. *Let D be a reduced hypersurface in a projective manifold X, and let $u\colon \mathbb{P}_1 \to X$ be a 2-free rational curve. Then there exist arbitrarily small deformations $\tilde{u}$ of u such that every intersection point of $\tilde{u}(\mathbb{P}_1)$ and D is transversal.*

Proof. A generic deformation of u will avoid any given analytic subset of codimension at least 2. Therefore, a generic deformation of u avoids $\mathrm{Sing}(D)$. Now let $p \in \mathbb{P}_1$ be a point with $u(p) \in D\setminus\mathrm{Sing}(D)$. By assumption, $u^*TX \otimes I(2\{p\})$ is an ample vector bundle on $\mathbb{P}_1$. By Grothendieck's theorem, u^*TX is a direct sum of line bundles. We choose a direct summand L of u^*TX which is complementary to u_p^*TD at p. Now we take a section of u^*TX which vanishes at p but has non-zero derivative in the direction of L. Then the corresponding deformations yield rational curves which intersect D with multiplicity 1 in p. Since $D \cap u(\mathbb{P}_1)$ is finite (in fact, its cardinality is bounded by $\deg(u^*\mathcal{L}(D))$), it follows that a generic deformation will have transversality at every point of intersection. □

Lemma 4.11. *Let X be a complex manifold, D a (reduced) hypersurface on X and $h\colon \mathbb{C} \to X$ an entire curve. Let G be a relatively compact open subset of $\mathbb{C}$ with $h(\partial G) \cap D = \emptyset$. Let d be a distance function on X induced by some Hermitian metric. Then there exists a number $\delta > 0$ such that the following two properties hold for every entire curve $\tilde{h}\colon \mathbb{C} \to X$ with $d\big(\tilde{h}(z), h(z)\big) < \delta$ for all $z \in \bar{G}$:*

(i) *$\tilde{h}(\partial G) \cap D = \emptyset$.*

(ii) *$\deg_G(h^*D) = \deg_G\big(\tilde{h}^*D\big)$.*

Proof. This follows easily using Rouché's theorem. □

Lemma 4.12. *Let X be a complex manifold, D a (reduced) hypersurface on X and $h\colon \mathbb{C} \to X$ an entire curve. Let G be a relatively compact open subset of $\mathbb{C}$ with $h(\partial G) \cap D = \emptyset$. Let d be a distance function on X induced by some Hermitian metric. Assume that h^*D is reduced (that is, all the multiplicities are 1). Then there exists a number $\delta > 0$ such that the following two properties hold for every entire curve $\tilde{h}\colon \mathbb{C} \to X$ with $d\big(\tilde{h}(z), h(z)\big) < \delta$ for all $z \in \bar{G}$:*

(i) *$\tilde{h}(\partial G) \cap D = \emptyset$.*

(ii) *$\tilde{h}^*D$ is reduced on G.*

Proof. Let $p_1, \dots, p_s$ be the points of h^*D. We choose small balls B_i such that $p_i \in B_i \subset G$ and such that $p_j \notin \overline{B_i}$ for $i \neq j$. Then we choose $\delta_i > 1$ such that $\deg_{B_i}\big(\tilde{h}^*D\big) = 1$ for every entire curve $\tilde{h}\colon \mathbb{C} \to X$ with $d\big(\tilde{h}(z), h(z)\big) < \delta_i$ for all $z \in B_i$ (We can do this due to Lemma 4.11). Finally, we choose $\delta = \min_i \delta_i$. □

Proof of Theorem 4.9. We proceed as in the proof of Theorem 4.2 with the following modifications. We choose the rational curve h_n such that h_n^*D is reduced on $\{z \in \mathbb{C}\colon |z| \leqslant R_n\}$. (This is possible due to Lemma 4.10.) Thanks to Lemma 4.12, there exists a number $\delta_n > 0$ such that any entire curve $\tilde{h}\colon \mathbb{C} \to X$ with $d\big(h_n(z), \tilde{h}(z)\big) < \delta_n$ for all $|z| \leqslant R_n$ preserves this property (that is, $\tilde{h}^*D$ is reduced on the disc with radius R_n). We choose ϵ_n such that $\epsilon_n < \delta_n$. In this way, we finally arrive at an entire curve h with h^*D reduced everywhere. □

Remark 4.13. The arithmetic version (that is, the potential density of rationally connected manifolds X defined over a number field) of Theorem 4.2 is open when X is not known to be unirational, even for conic bundles over $\mathbb{P}_n$, $n \geqslant 2$. There are (at least) two known cases: smooth quartics in $\mathbb{P}_4$ (see [HT00]) and some conic bundles over $\mathbb{P}_2$ (see [BMS14], where $X(\mathbb{Q})$ is shown to already be Zariski dense).

4.3 An orbifold situation

In some situations, we can impose tangency conditions as well. These are simple instances of an extension of the preceding results to the 'orbifold pair' situation, for which the appropriate techniques have presently not been developed.

Corollary 4.14. *Let $S \subset \mathbb{P}_3$ be a smooth sextic surface. There exist entire curves $g\colon \mathbb{C} \to \mathbb{P}_3$ with metrically dense image C such that C is tangent to S at each point where it meets S.*

Proof. Let $\pi\colon X \to \mathbb{P}_3$ be the double cover ramified along S; then X is rationally connected since it is Fano. Let $h\colon \mathbb{C} \to X$ be a dense entire curve. Then $g := \pi \circ h\colon \mathbb{C} \to \mathbb{P}_3$ is a dense entire curve with the claimed property. □

We do not know how to prove Corollary 4.14 directly on $(\mathbb{P}_3, S)$ without going to the double cover.

Remark. Observe that in the situation of Corollary 4.14, the degree of $K_{\mathbb{P}_3} + \left(1 - \frac{1}{2}\right) \cdot S$ is negative (that is, the smooth 'orbifold pair' $\left(\mathbb{P}_3, \left(1 - \frac{1}{2}\right) \cdot S\right)$ is Fano). The statement of Corollary 4.14 thus means that this orbifold pair admits metrically dense 'orbifold entire curves' (in the 'divisible' sense). This is conjectured in [Cam11] to be the case for arbitrary smooth orbifolds which are Fano (or, more generally, rationally connected in the orbifold sense).

4.4 Brody curves

An entire curve $f\colon \mathbb{C} \to X$ is a *Brody curve* if its derivative is uniformly bounded with respect to the Euclidean metric on $\mathbb{C}$ and an arbitrary Hermitian metric on X. Brody's theorem states that on a compact complex manifold X, there is a non-constant Brody curve if and only if there is a non-constant entire curve.

However, there is no such statement concerning *dense* curves. Indeed, an abelian threefold blown up along a curve is constructed in [Win07] which admits (lots of) dense entire curves although every Brody curve is contained in the exceptional divisor. The existence of dense Brody curves is thus not a bimeromorphic property.

Our results give no information about the existence of (Zariski-)dense Brody curves on arbitrary rationally connected manifolds (although they obviously exist on some of them).

5. Normal rational surfaces

If X is a singular complex algebraic variety with desingularization $\pi\colon \tilde{X} \to X$, then there exists a (Zariski-)dense entire curve $h\colon \mathbb{C} \to X$ if and only if there exists a (Zariski-)dense entire curve $\tilde{h}\colon \mathbb{C} \to \tilde{X}$. (This is clear because we can lift every holomorphic map $h\colon \mathbb{C} \to X$ whose image $h(\mathbb{C})$ is not contained in the singular locus X^{sing}.)

However, it is a much more delicate question whether there exists a (Zariski-)dense entire curve $h\colon \mathbb{C} \to X$ which avoids the singular locus X^{sing} of X.

In this section, we prove the existence of such curves for certain normal rational surfaces (Theorem 5.1). Note that most normal rational surfaces have no Zariski-dense entire curves in their smooth locus. See Examples 5.11 and 5.12 at the end of this section.

We consider the following situation: S is a normal projective surface with only quotient singularities, such that $-K_S = \Delta$, an effective non-zero $\mathbb{Q}$-divisor on S with $\mathrm{Coeff}(\Delta) \leqslant 1$, where $\mathrm{Coeff}(\Delta)$ is the largest of the coefficients of the irreducible components of the support of Δ.

This permits us to apply to S the MMP, which preserves the condition on $\mathrm{Coeff}(\Delta)$ and ends up giving, after finitely many K-negative contractions, a pair (S', Δ') such that $K_{S'}$ is nef, or $K_{S'}$ is anti-ample with Picard number 1, or the pair admits a Fano contraction $f\colon S' \to B$ on a smooth projective curve B of relative Picard number 1. This allows us to characterize the existence of dense entire curves in the smooth locus of S.

Theorem 5.1. *Let S be a normal rational surface with only quotient singularities and $F \subset S$ be a finite subset containing the singular points of S. Let Δ be an effective non-zero $\mathbb{Q}$-divisor such that $(K_S + \Delta)$ is $\mathbb{Q}$-trivial. Assume* $\mathrm{Coeff}(\Delta) < 1$.

(i) *For any two generic points of $S \setminus F$, there is a 1-free rational curve through these two points avoiding F.*

(ii) *There is an entire curve $h\colon \mathbb{C} \to S \setminus F$ with dense image.*

Remark. The proof shows that the conclusions still hold true when $S \setminus F$ is rationally connected, if $\mathrm{Coeff}(\Delta) \leqslant 1$.

Proof. Applying the MMP for surfaces (see [KM98, Theorems 3.47, 4.11 and 4.18]), we obtain a birational map $\mu\colon S \to S'$ to a normal surface S' with only quotient singularities and a $\mathbb{Q}$-divisor $\Delta' := \mu_*(\Delta)$ on S' with $\mathrm{Coeff}(\Delta') < 1$ and $K_{S'} + \Delta'$ $\mathbb{Q}$-trivial, such that one of the following properties holds:

1. The canonical divisor $K_{S'}$ is nef.
2. The anticanonical divisor $-K_{S'}$ is ample.
3. There is a Fano fibration $f\colon S' \to B = \mathbb{P}_1$ of relative Picard number 1.

In our situation, the first case does not occur. Indeed, in this case we have $0 \leqslant \kappa(S') = \kappa(S) = \kappa(S, -\Delta) = -\infty$ since Δ is $\mathbb{Q}$-effective and non-zero, the MMP preserves the Kodaira dimension, and $\kappa(S') \geqslant 0$ if $K_{S'}$ is nef.

In the second case, the pair $(S', 0)$ satisfies the assumptions of [Xu12, Theorem 1.4] since $K_{S'}$ is ample. This theorem implies the existence of a 1-free rational curve in $S \setminus F$ through any two generic points, and our first claim has been proved in [Xu12]. Based on the existence of these free rational curves, claim (ii) follows as in the smooth case, proved in Theorem 4.2.

In the third case, the existence of a dense entire curve and of rational curves with the said properties follows from Proposition 5.10. $\square$

Remark. When we have $\Delta = 0$ above, one also expects the existence of dense entire curves, but this is an open question. See Example 5.11 for an example in which this conclusion does not follow from the arguments given here but instead follows from an explicit description.

It would also be interesting to get a criterion to recognize from the initial data (S, Δ) itself whether one ends up, for some suitable MMP, at the $\mathbb{Q}$-Fano case with $\rho = 1$ or at the fibered case for any MMP-sequence of contractions.

For the proof of Proposition 5.10 we need some preparatory lemmas.

Assumptions 5.2. We write S for a projective normal surface with only quotient singularities, and there exists a surjective Fano fibration $f\colon S \to B$ of relative Picard number 1. This implies that all fibers $S_b = f^{-1}\{b\}$ are irreducible. The $\mathbb{Q}$-divisor $\Delta_f := \sum_b (1 - 1/m_b)\{b\}$ (with m_b the multiplicity of the fiber $f^{-1}(b)$ for each $b \in B$) denotes the *orbifold base* of f.

Lemma 5.3. *Let S be a normal surface with a holomorphic map $f\colon S \to B$ with irreducible fibers. If there exists an entire curve $h\colon \mathbb{C} \to S^{\mathrm{reg}}$ such that $f \circ h\colon \mathbb{C} \to B$ is non-constant, then* $\deg(K_B + \Delta_f) \leqslant 0$. *Thus, either B is elliptic and $\Delta_f = 0$, or $(B, \Delta_f) = (\mathbb{P}_1, \Delta_f)$ is in the short classical 'platonic' list of orbifolds on $\mathbb{P}_1$ with* (*semi-*)*negative canonical bundle.*

Proof. This follows from [CW09] since $f \circ h$ is then an orbifold morphism to (B, Δ_f). □

We will next use the following properties:

1. If $m_b = 1$, then $S_b \cong \mathbb{P}_1$ is smooth [KM99, Notation 11.5.1], and f is thus a $\mathbb{P}_1$-bundle over B if and only if $\Delta_f = 0$.

2. Assume[3] that there exists a finite ramified cover $\beta\colon B' \to B$ which ramifies at order exactly m_b over b for every $b \in B$. Let $\sigma\colon S' \to S$ and $f'\colon S' \to B'$ be deduced by base changing f by β and normalizing the base change $S \times_B B'$. Then f' has generically reduced fibers, S' still has quotient singularities, with $K_{S'} + \Delta'$ $\mathbb{Q}$-trivial [KM98, Propositions 5.20 and 4.18], and $\mathrm{Coeff}(\Delta') \leqslant 1$. We can still apply the MMP to S' and now arrive at a fibration $f''\colon S'' \to B'$ which has only reduced fibers and is thus a $\mathbb{P}_1$-bundle.

Lemma 5.4. *Let $f\colon S \to B$ be as in Assumptions 5.2. Let $f'\colon S' \to B'$ be deduced from f by a finite base change $B' \to B$. Assume that all fibers of $f'\colon S' \to B'$ have a reduced component. Let $F \subset S^{\mathrm{reg}}$ be a finite set. Then $S^{\mathrm{reg}} \setminus F$ contains a dense entire curve* (*respectively, a* 1*-free rational curve through any two of its generic points*) *if B' is elliptic* (*respectively, rational*).

Proof. It is sufficient to show this for $(S')^{\mathrm{reg}} \setminus F'$, where F' contains the inverse image in S' of S^{sing} and of F. But S' dominates a $\mathbb{P}_1$-bundle over B', as follows from the preceding observations, which implies the claims. □

Corollary 5.5 now follows from the existence of a suitable 'orbifold-étale' base change $B' \to B$, with B' either elliptic or rational, when (B, Δ_f) is in this list of 'platonic' orbifolds.

Corollary 5.5. *Let $f\colon S \to B$ be a Fano fibration of relative Picard number* 1 *and S a normal projective surface with only quotient singularities. The following properties are then equivalent:*

(i) *We have* $\deg(K_B + \Delta_f) < 0$ (*respectively,* $\deg(K_B + \Delta_f) \leqslant 0$), *where (B, Δ_f) is the orbifold base of f.*

(ii) *There exists a rational curve* (*respectively, an entire curve*) *on S^{reg} which is not f-vertical (that is, not contained in some fiber of f).*

(iii) *There exists a* 1*-free rational curve through any two generic points of S* (*respectively, a dense entire curve*) *which is contained in S^{reg}.*

We will now relate K_S to $K_B + \Delta_f$ by means of a 'canonical bundle' formula for K_S.

Let $r\colon \tilde{S} \to S$ be a minimal resolution and $t\colon \tilde{S} \to S_0$ be a relative minimal model of $\tilde{S}$ over B, so that $f_0\colon S_0 \to B$ is a $\mathbb{P}_1$-bundle, with $f_0 \circ t = f \circ r$. Then $S_0 = \mathbb{F}_m$, the Hirzebruch surface of index m for some $m \in \mathbb{N}_0$. In this case, we denote by G (respectively, G') a section of f_0 with $G \cdot G = -m$ (respectively, a section of f_0 with $G \cdot G' = 0$, or equivalently disjoint from G). We have $K_{S_0} = -(G + G') + (f_0)^*(K_B)$. We will then denote by C and C' the strict transforms in S of G and G', respectively.

[3] This is true unless $B = \mathbb{P}_1$ and the support of Δ_f consists of one or two points with different multiplicities. However, in these two cases there is still such a dominant finite algebraic map $\beta\colon \mathbb{C} \to B$, which suffices for our purposes.

Lemma 5.6. *Under Assumptions 5.2, we have $K_S \sim f^*(K_B + \Delta_f) - (C + C')$. Moreover, C and C' are still disjoint.*

Proof. Each singular fiber of the fibration $f \circ r\colon \tilde{S} \to B$ has two reduced components, each of which is met once by either $\tilde{G}$ or $\tilde{G}'$, the strict transforms of G and G' in $\tilde{S}$. The contraction $r\colon \tilde{S} \to S$ then sends each of these components to two distinct singular points of the corresponding fiber of f, by [KM99, Lemma 11.5.5], which implies that C and C' are disjoint, too. The first claim then follows from the fact that $F_b := (S_b)_{\mathrm{red}}$ is irreducible for each $b \in B$, so that $K_S = a \cdot F_b - C - C'$ locally near F_b, with $a = (m_b - 1)$ by adjunction since F_b has multiplicity m_b in f. □

Lemma 5.7. *Under Assumptions 5.2, we have $C \cdot C' = 0$ and $C' \sim C + k\Phi$ for some $k \geqslant 0$ (after possibly exchanging C, C'). Moreover, for each f-horizontal irreducible curve $H \subset S$ different from C and C', we have $H \sim d(C' + \ell\Phi)$ with $d := H \cdot \Phi > 0$ and $\ell = H \cdot C/d \geqslant 0$.*

Proof. The surface S is $\mathbb{Q}$-factorial (because it has only quotient singularities), and its Picard number is 2 since $f\colon S \to B$ has relative Picard number 1. Hence every divisor is numerically equivalent to a linear combination of C' and Φ. The assertions of the lemma are now obvious. □

Assumptions 5.8. Let $f\colon S \to B$ be a Fano fibration with S normal and having only quotient singularities. Now assume that $K_S + \Delta$ is $\mathbb{Q}$-trivial, where Δ is a non-zero effective $\mathbb{Q}$-divisor on S. We write $\Delta = \Delta^h + \Delta^v$, where Δ^h (respectively, Δ^v) is the f-horizontal (respectively, f-vertical) part of Δ.

We write $\mathrm{Coeff}^h(\Delta)$ for the largest coefficient in Δ of the irreducible components of Δ^h.

Lemma 5.9. *Under Assumptions 5.2 and 5.8, we have $\deg(K_B + \Delta_f) \leqslant 0$. Moreover, we have $\deg(K_B + \Delta_f) < 0$ if $\mathrm{Coeff}^h(\Delta) < 1$.*

Proof. We have $\Delta = \Delta^h + \Delta^v \sim K_S \sim (C + C') - f^*(K_B + \Delta_f)$, $\sim$ denoting linear equivalence. Thus we have $-f^*(K_B + \Delta_f) \sim \Delta^v + \Delta^h - (C + C')$, and it is sufficient to show that $\Delta^h - (C + C')$ is $\mathbb{Q}$-effective. We may write $\Delta^h = aC + a'C' + R$, where $R = \sum_j a_j C_j / d_j$, where the C_j are irreducible multisections of f distinct from C and C', each of degree $d_j > 0$ over B, and the $a_j > 0$ are rational numbers.

For each j, we have $C_j = d_j(C' + k_j\Phi)$ for some $k_j \in \mathbb{Q}$. Thus $R = \sum_j a_j(C' + k_j\Phi)$. Calculating intersection numbers with Φ gives $2 = (C + C') \cdot \Phi = \Delta^h \cdot \Phi = a + a' + \sum_j a_j$. Therefore,

$$\begin{aligned} \Delta^h - (C + C') &\sim aC + a'C' + (2 - a - a') \sum\nolimits_j (C' + k_j\Phi) - (C + C') \\ &= (1 - a)(C' - C) + (2 - a - a') \left(\sum\nolimits_j k_j\right) \Phi\,. \end{aligned}$$

Recall that $k_j \geqslant 0$. Furthermore, $(C' - C)$ is effective (Lemma 5.7). Hence the above equation implies that $\left(\Delta^h - C - C'\right)$ is $\mathbb{Q}$-effective, and so is $(\Delta - C - C')$, which implies $\deg(\Delta_f + K_B) \leqslant 0$.

If $\mathrm{Coeff}(\Delta) < 1$, we have $a, a' < 1$. Consequently, $1 - a > 0$ which implies that $\left(\Delta^h - C - C'\right)$ is non-zero, and so $\deg(\Delta_f + K_B) < 0$. □

We have the following criterion for the existence of 1-free rational curves (respectively, dense entire curves) in the regular part of such Fano fibrations. It immediately implies Theorem 5.1.

Proposition 5.10. *Let S be a normal projective surface with only quotient singularities, with a non-zero effective $\mathbb{Q}$-divisor Δ on S such that $K_S + \Delta$ is $\mathbb{Q}$-trivial. Let $f\colon S \to B$ be a Fano* *fibration to a smooth curve B of relative Picard number* 1. *Assume* $\mathrm{Coeff}^h(\Delta) < 1$ (*respectively,* $\mathrm{Coeff}^h(\Delta) \leqslant 1$). *The following three properties then hold:*

(i) *We have* $\deg(K_B + \Delta_f) < 0$ (*respectively,* $\deg(K_B + \Delta_f) \leqslant 0$), *where* (B, Δ_f) *is the orbifold base of* f.

(ii) *There exists a rational curve* (*respectively, an entire curve*) *on* S^{reg} *which is not* f*-vertical*

(iii) *There exists a* 1*-free rational curve through any two generic points of* S (*respectively, a dense entire curve*) *which is contained in* S^{reg}.

Proof. The equivalence of properties (i), (ii) and (iii) is due to Corollary 5.5. Property (i) holds because of Lemma 5.9. □

5.1 Examples

Example 5.11. Let $S := A/\mathbb{Z}_4$, where $A = E \times E$ and $E := \mathbb{C}/\mathbb{Z}[\sqrt{-1}]$, and $\mathbb{Z}_4$ is generated by the multiplication by $\sqrt{-1}$ on the two factors simultaneously. Then S (known as the Ueno surface) has quotient non-canonical singularities and a $\mathbb{Q}$-trivial canonical bundle. If $\pi \colon S' \to S$ is its minimal resolution, then $-2K_{S'} = \sum E_j$, where the E_j are the -4-curves lying over the non-canonical singularities. Proposition 5.10 applies to S' but not to S (since all rational curves on S meet some of the non-canonical singularities of S). See [Cam11] for a description of the MMP on S'. Also observe that S^{reg} contains many dense entire curves (coming from A) although it contains no rational curves.

On most normal rational surfaces, every entire curve meets the singular locus, as shown (for $m \geqslant 5$) in the simple example below.

Example 5.12. Let $f \colon S \to \mathbb{P}_1$ be a Fano fibration from a normal surface S with Picard number 2 and with only $2m$ ordinary double points as singularities, lying in pairs on $m \geqslant 0$ double fibers of f. Such surfaces are obtained by applying the following steps on m distinct fibers, starting from any Hirzebruch surface S_0. Blow up one point, then the intersection point of the two -1-curves on the first blow-up. One thus gets a fiber with three components: one double -1-curve meeting two reduced -1-curves. Then blow down these last two curves to ordinary double points.

From Theorem 5.1 and [CW09] (applied as in Lemma 5.3), we get that S^{reg} contains 1-free rational curves (respectively, dense entire curves) if and only if $m \leqslant 3$ (respectively, $m \leqslant 4$).

6. Dense entire curves transversal to all divisors

One can strengthen the results of Section 5 to deal with all divisors D simultaneously. (This is motivated by Theorem 8.6 below.)

Theorem 6.1. *Let X be a rationally connected complex projective manifold, X an analytic subset of codimension at least* 2 *and M a countable subset of $X \setminus A$. Then there exists an entire curve $h \colon \mathbb{C} \to X \setminus A$ with $M \subset h(\mathbb{C})$ such that the image $h(\mathbb{C})$ intersects every irreducible divisor $D \subset X$ transversally at some of the smooth points of D.*

Proof. Due to Theorem 4.2, there exists an entire curve $h \colon \mathbb{C} \to X \setminus A$ with $M \subset h(\mathbb{C})$. We have to show that this map h may be chosen in such a way that its image meets any irreducible divisor $D \subset X$ transversally at some of its smooth points.

Let $S \subset \mathrm{Chow}(X)$ be the subset parametrizing irreducible divisors: it is a countable union of irreducible quasi-projective varieties.[4] Thus S is the union of an increasing sequence of compact subsets S_k, $k \geqslant 1$, closures of their interiors, with S_k included in the interior of S_{k+1} for each k. For each $s \in S$ and $g\colon \mathbb{P}_1 = \mathbb{C} \cup \{\infty\} \to X$, we consider the following property T:

- There is a complex number $z \in D\big(1, \frac{1}{2}\big)$ such that $g(\mathbb{P}_1)$ intersects D_s transversally in $g(z)$ (that is, $g'(z) \notin T_{g(z)} D_s$; in particular, $g(z) \in D_s^{\mathrm{reg}}$).
- There is no other point w in $\overline{D\big(1, \frac{1}{2}\big)}$ with $g(w) \in D_s$.

This property T is open in g and s. We may choose a 1-free rational curve $g_s\colon \mathbb{P}_1 \to X$ such that $g_s(\mathbb{P}_1)$ is transverse to D_s at $q_s =: g_s(1) \in D_s^{\mathrm{reg}}$ and has no other zero on the closed disc $\overline{D\big(1, \frac{1}{2}\big)}$. In particular, g_s has property T with respect to s. By the openness of property T, there exist an open neighborhood $s \in B_s \subset S$ and a positive number $\alpha_s > 0$ such that property T still holds for all $(s', \tilde{g})$, where $s' \in B_s$ and $\tilde{g}\colon \overline{D\big(1, \frac{1}{2}\big)} \to X$ is holomorphic such that $d(\tilde{g}(z), g_{s_m}(z)) \leqslant \alpha_m$ for all $z \in \overline{D\big(1, \frac{1}{2}\big)}$. In other words, $\tilde{g}\big(\overline{D\big(1, \frac{1}{2}\big)}\big)$ meets each $D_{s'}$ ($s' \in B_s$) transversally at a single point which is in the image of the interior of the annulus.

For each $k > 0$, there thus exist finitely many $s \in S_{k+1} \setminus S_k$ such that the open sets B_s cover the (compact) closure $\overline{S_{k+1} \setminus S_k}$ of $S_{k+1} \setminus S_k$. We may therefore choose a sequence s_m (together with a sequence $g_m = g_{s_m}$) such that $S = \cup_m B_{s_m}$.

Denote by $(x_n)_{n\in\mathbb{N}}$ an enumeration of the elements of M. We now consider the following sequence of points $y_n, n \geqslant 1$: $y_{3m} := x_m$, $y_{3m+1} := q'_m$, where $q'_m := g_m(0) \in g_m(\mathbb{C}) \subset g_m(\mathbb{P}_1)$. Finally, let $y_{3m+2} := q_m = g_m(1)$, for every $m > 0$. We now construct inductively on m the sequence of rational curves h_n according to this sequence of points y_n as in the proof of Theorem 4.2, with the only restriction that for each $m > 0$, the curve h_{3m+2} is obtained from h_{3m+1} using the comb defined by h_{3m+1} and g_m, that is, by smoothing this comb. Indeed, observe that $h_{m+1}(\mathbb{P}_1)$ contains the point $q'_m \in g_m(\mathbb{P}_1))$ in addition to the points $x_1, \ldots, x_m, q'_1, \ldots, q'_m, q_1, \ldots, q_{m-1}$.

Choose h_{3m+2} such that $d(g'(z), g_m(z)) \leqslant \alpha_m / 2^{3m+3}$ for all $z \in \overline{D\big(1, \frac{1}{2}\big)}$, where $g'(z) = h_{3m+2}(y)$ with $\sigma \cdot y = z$ and $h_{3m+2} := H(\sigma, \cdot)$ in the notation of the proof of Lemma 4.5. The transversality of h_{3m+2} to D_s then holds for $s \in B(s, r_m/2)$, and also for the subsequent $h_n, n > 3m+2$, provided the sequence ε_n introduced in the proof of Theorem 4.2 is sufficiently small (it can be chosen inductively). The fact that h_n converges uniformly on compacts of $\mathbb{C}$ to an entire map h satisfying the claims of Theorem 6.1 is checked as in the proof of Theorem 4.2. ☐

6.1 An analog for $\mathbb{C}^n$

Here we discuss dense entire curves in $\mathbb{C}^n$, proving a strong existence result which will be useful later on.

Theorem 6.2. *Let E, D be closed analytic subsets in $\mathbb{C}^n$. Assume* $\operatorname{codim}(E) \geqslant 2$, $\operatorname{codim}(D) \geqslant 1$. *Let S be a countable subset of $\mathbb{C}^n \setminus E$. Then there exists a holomorphic map $F\colon \mathbb{C} \to \mathbb{C}^n$ such that*

(i) *$F(\mathbb{C})$ is dense in $\mathbb{C}^n$;*

(ii) *$S \subset F(\mathbb{C})$;*

(iii) *$F(\mathbb{C})$ does not intersect E;*

(iv) *$F(\mathbb{C})$ and D are transversal in every point of intersection.*

[4]Since $h^1(X, \mathcal{O}_X) = 0$ here, these quasi-projective varieties are Zariski-open subsets of projective spaces and, in particular, smooth.

Proof. By enlarging S, we may and do assume that S is a dense subset of $\mathbb{C}^n \setminus E$. We assume that E contains the singular locus of D. Let $g\colon \mathbb{C} \to \mathbb{C}^n$ be a holomorphic map with $g(\mathbb{Z}) = S$. Let H denote the Fréchet vector space of holomorphic maps from $\mathbb{C}$ to $\mathbb{C}^n$. For every $v \in H$, we define

$$F_v(z) = g(z) + \left(e^{2\pi i z} - 1\right)v(z)$$

and observe that $F_v(\mathbb{Z}) = S$.

For each compact subset $K \subset \mathbb{C}$ we define a subset W_K of H by choosing all those $v \in H$ for which the following properties hold:

1. The intersection $F_v(\partial K) \cap D$ is empty.
2. The intersection $F_v(K) \cap E$ is empty.
3. The set $F_v(K)$ intersects D transversally; that is, we have $F_v'(z) \notin T_{F_v(z)}D$ for $z \in K$ with $F_v(z) \in D$.

We claim: *If ∂K is a smooth real curve, then W_K is open and dense in H.*

First, we explain how the claim implies the theorem: As a Fréchet space, H has the Baire property. We cover $\mathbb{C}$ with countably many balls B_i and set $W_i = W_{\overline{B_i}}$. Then each W_i is open and dense in H, and due to the Baire property, $\cap_i W_i$ is not empty. Hence we may choose an element $v \in \cap W_i$ and define $F = F_v$.

Second, we prove the claim. The topology on H is defined by the sup-norms on an exhaustive families of discs in $\mathbb{C}$. Hence properties 1 and 2 are obviously open. Since we discuss holomorphic functions, locally uniform convergence of functions implies locally uniform convergence of their derivatives. Therefore, it is clear that property 3 is likewise an open property.

We still have to show density. We fix a finite-dimensional vector subspace V of H such that the induced map $V \times \mathbb{C} \to \mathbb{C}^n$ is everywhere submersive. Then the map

$$\Phi\colon V \times (\mathbb{C} \setminus \mathbb{Z}) \to \mathbb{C}^n$$

given by $(v, z) \mapsto F_v(Z)$ is also everywhere submersive. It follows that $\Phi^{-1}(D)$ (respectively, $\Phi^{-1}(E)$) have codimension at least 1 (respectively, 2) in $V \times (\mathbb{C} \setminus \mathbb{Z})$. Since ∂K has real dimension 1 and K has real dimension 2, it follows that the projection map onto V maps both $\Phi^{-1}(D) \cap (V \times \partial K)$ and $\Phi^{-1}(E) \cap (V \times K)$ to zero-measure subsets of V. This implies density for properties 1 and 2.

Finally, we have to show the density claim for property 3. We have already seen that the set of all $v \in H$ for which $F_v(\partial K) \cap D = \emptyset$ holds is dense. Fix an element $w \in H$ with $F_w(\partial K) \cap D = \emptyset$. Using Rouché's theorem, we know that the degree $\deg(F_v^* D|_K$ is constant in a neighborhood of w in H. If D is locally defined by a holomorphic function h, then

$$v \mapsto \Pi_{x \in K \cap F_v^{-1}(D)} dh(F_v'(x)$$

is holomorphic on H, which implies that the complement of its zero set is dense. This completes the proof of the claim and as we have seen before, the claim implies the theorem. □

7. Manifolds with $c_1 = 0$

We now consider the second main class for testing Conjecture 3.2: compact Kähler manifolds X with $c_1(X) = 0$. For them, Kobayashi already conjectured that their Kobayashi pseudo-distance is identically zero.

Bogomolov decomposition. A compact Kähler manifold with vanishing c_1 is (up to a finite étale cover) a direct product of (see [Bea83, Bog74])

- compact complex tori,
- hyperkähler manifolds (equivalently, compact Kähler manifolds which are holomorphically symplectic),
- Calabi–Yau manifolds which are not hyperkähler (they admit a nowhere vanishing holomorphic n-form, where n is the dimension of the manifold, but no other non-zero holomorphic differential form).

7.1 Abelian varieties

Our result on entire curves in rationally connected manifolds easily extends to abelian varieties and other compact complex tori.

Theorem 7.1. *Let A be a compact complex torus, let Z be a closed analytic subset of codimension at least 2, and let M be a countable subset of $A \setminus Z$. Then there exists an entire curve $h\colon \mathbb{C} \to A \setminus Z$ with dense image and $M \subset h(\mathbb{C})$. Moreover, every hypersurface D intersects $h(\mathbb{C})$ transversally in some point, that is, there is a point $p = h(t) \in h(\mathbb{C}) \cap D$ with $h'(t) \notin T_pD$.*

Proof. We use the universal cover $u\colon \mathbb{C}^n \to A$. Let $E = u^{-1}(Z)$. Let S be a dense countable subset of $\mathbb{C}^n \setminus E$ with $M \subset u(S)$. Due to Theorem 6.2, we obtain a holomorphic map $F\colon \mathbb{C} \to \mathbb{C}^n$ with $S \subset F(\mathbb{C})$ and $F(\mathbb{C}) \cap E = \emptyset$. Define $h := u \circ F$. Now $h\colon \mathbb{C} \to A$ is a dense entire curve which avoids Z and contains M in its image. Finally, observe that due to a result from value distribution theory (see [NW14, Theorem 6.5.1(iii)]), for every hypersurface D in A and every Zariski-dense entire curve h, there is a point in which D and $h(\mathbb{C})$ intersect transversally. □

Remark. Theorem 6.5.1 of [NW14] in combination with the First Main Theorem (F.M.T.) yields that the *truncated* counting function N_1 grows essentially as fast as the *untruncated* counting functions. This means that 'most' intersection points are transversal.

Remark. Recall that, by [Yau78], a compact Kähler manifold admits a finite étale cover which is a complex torus if and only if $c_1(X) = c_2(X) = 0$.

7.2 Manifolds without non-trivial analytic subvarieties

The following trivial remark nevertheless leads to interesting examples.

Proposition 7.2. *Let X be a connected normal compact complex space. Assume that X does not contain any non-trivial irreducible complex subvariety (except for X itself and points). If X is not Kobayashi-hyperbolic, then it contains a Zariski-dense entire curve.*

Proof. We assume that X is hyperbolic. By Brody's lemma (see [Bro78] or [Kob76, Theorem 3.6.3]), there is a non-constant entire curve $f\colon \mathbb{C} \to X$. This curve has Zariski-dense image since by assumption, X does not admit any positive-dimensional analytic subset except X itself. □

These manifolds are connected to those with $c_1 = 0$ by means of the following.

Remark. A compact Kähler manifold X without non-trivial subvarieties is conjectured to be either a simple compact complex torus or hyperkähler; in particular, $c_1(X) = 0$. (See [Cam06, Question 1.4] and [CDV14, Conjecture 1.1].) This conjecture is proved in dimension 2 by Kodaira's classification, and in dimension 3 in [CDV14], which proves that a smooth compact Kähler threefold without subvarieties is a simple torus.

Examples 7.3. 1. The general (that is, outside the countable union of strict Zariski-closed subsets of the relevant moduli space) deformation of the Hilbert scheme $\mathrm{Hilb}^m(K3)$ of m points on a $K3$ surface has no non-trivial subvarieties, by [Ver98], and is not Kobayashi-hyperbolic, by [Ver15]. Hence Proposition 7.2 applies, and such a manifold does admit a Zariski-dense entire curve.

2. If X is a compact Kähler threefold without subvarieties, it is biholomorphic to a compact torus (see [CDV14]) and therefore contains a dense entire curve.

Remark 7.4. Unfortunately, our argument gives no information on the 'size' (measured, say, by its Hausdorff dimension) of the closure of the entire curve obtained from Brody's theorem. Hence we cannot deduce property (i) or (ii) of Theorem A by this method.

Remark 7.5. The result of [Ver15] is based on the following construction [Cam92] of entire curves in some hyperkähler manifolds: if X is a compact Kähler hyperkähler manifold with twistor space Z associated with the Ricci-flat Kähler metric on X of a given class ω, then some member X_s of this twistor family contains an entire curve. This entire curve is obtained by deforming the twistor fibers (which are copies of $\mathbb{P}_1$ with direct sums of $\mathcal{O}(1)$) as normal bundles and taking suitable limits. Verbitsky's proof then shows that this in fact holds for all members of such a family, using the ergodicity of the action of the mapping class group on the Teichmüller space and period domain.

It is quite interesting (but much more difficult) to extend the preceding remarks to the case of 'simple' compact Kähler manifolds,[5] which are those which are not covered by subvarieties of intermediate dimension, or equivalently such that their general point (that is, outside a countable union of strict subvarieties) is not contained in a strict irreducible compact subvariety. In [Cam06], it is conjectured that a 'simple' compact Kähler manifold either is bimeromorphic to a quotient of torus by a finite group of automorphisms or is of even complex dimension and carries a holomorphic 2-form which is symplectic on a non-empty Zariski-open subset. In particular, 'simple' compact Kähler manifolds should have $\kappa = 0$ and are of algebraic dimension zero (that is, have no non-constant meromorphic functions, or equivalently only finitely many irreducible divisors) and are thus special. We thus expect them to have dense entire curves. For surfaces, the existence of Zariski-dense entire curves is known by classification: every surface of algebraic dimension zero is simple and bimeromorphic to either a torus or a $K3$ surface, so far confirming the conjecture, as we see by the proposition below.

Proposition 7.6. *Let S be a compact Kähler surface of algebraic dimension zero. Then S contains a Zariski-dense entire curve.*

Proof. If S is bimeromorphic to a torus, this is clear. If S is bimeromorphic to a $K3$ surface, we can assume that it is a $K3$ surface. In this case, by [BPvdV84, Proposition VIII.3.6], the only connected curves on S are chains of -2-curves,[6] and there exists a holomorphic bimeromorphic map $f\colon S \to S'$ which contracts all curves of S to (singular, normal, Du Val) points of S'. Since

[5] These are the main 'building blocks' in the construction of arbitrary (non-projective) compact Kähler manifolds.

[6] Indeed, all line bundles on S have negative self-intersections, so all nodal classes are classes of -2-curves, and by an easy computation, these have to meet transversally if they are not disjoint, never with triple intersections. Although this is certainly well known, we do not know a reference.

$d_S \equiv 0$, we also have $d_{S'} \equiv 0$. Now, S' does not contain non-trivial subvarieties (that is, curves), and so the conclusion follows from Proposition 7.2. □

We conclude with the observation that the same holds true in dimension 3 by the classification of non-projective compact Kähler threefolds given in [CHP16].

Proposition 7.7. *Let X be a connected compact Kähler threefold of algebraic dimension zero. Then X contains a Zariski-dense entire curve.*

Proof. The claim is obvious, taking Proposition 7.6 into account, since by [CHP16, § 9], we get that X is either

1. 'simple' (hence meromorphically covered by a torus) or
2. bimeromorphically a $\mathbb{P}_1$-fibration with empty discriminant over a surface of algebraic dimension zero (the surface may be singular, without curves, if $K3$). □

The obstruction to extending the above result to all special compact Kähler threefolds lies in the cases of 'general' projective $K3$ surfaces, normal rational surfaces with quotient singularities and torsion canonical bundle (as the one in Example 5.11), and Calabi–Yau threefolds.

7.3 Some Calabi–Yau manifolds

Calabi–Yau manifolds are even more difficult to handle than hyperkähler manifolds. In general, we do not know whether the Kobayashi pseudo-distance vanishes. Thus for an arbitrary Calabi–Yau manifold, none of the properties of Conjecture 3.2 is known to be true (except property (iii), which states that π_1 should be almost abelian).

Let us just mention two peculiar families for which we at least know that the Kobayashi pseudo-distance degenerates. However, in these cases too, the other properties (ii)–(v) are not known.

1. *'General' quintics in $\mathbb{P}_4$*: they contain $(1, 1)$ rational curves of arbitrarily large degree by [Cle83]. (This still works in higher dimensions for general smooth hypersurfaces of degree $(n+2)$ in $\mathbb{P}_{n+1}, n \geqslant 3$.)

2. *Double covers of $\mathbb{P}_3$ ramified over a smooth octic*: they are covered by elliptic curves by [Voi03, Example 2.17]. (This still works for double covers of $\mathbb{P}_n$ ramified over a smooth hypersurface of degree $2(n+1)$.)

7.4 Elliptic fibrations and elliptic Calabi–Yau threefolds

Proposition 7.8. *Let $f\colon X \to B$ be an elliptic fibration from a compact Kähler manifold over a rationally connected manifold B (that is, f is proper, flat, and the generic fibers are elliptic curves). Assume that f has no multiple fibers in codimension 1 over B. Then X contains dense entire curves.*

Remark. By a multiple fiber, we mean a fiber such that *every* irreducible component of it has multiplicity at least 2.

Proof. Let $h\colon \mathbb{C} \to B$ be any dense entire curve avoiding the subset over which f has multiple fibers. By Theorem 4.2, many such entire curves exist. Let

$$Y = X \times_B \mathbb{C} = \{(p, t) \in X \times \mathbb{C}\colon h(t) = f(p)\}\,.$$

Now we may consider the natural projection $Y \to \mathbb{C}$ which is again an elliptic fibration. Moreover, there are no multiple fibers as h avoids the subset of B over which f has multiple fibers. Following the arguments of [BL00], we obtain the existence of a holomorphic map $G\colon \mathbb{C}^2 \to Y$ with dense image. Because there exists a dense entire curve $\zeta\colon \mathbb{C} \to \mathbb{C}^2$ (see, for example, Theorem 6.2), we are now in a position to define a dense entire curve $g\colon \mathbb{C} \to X$ as

$$g(z) \stackrel{\text{def}}{=\!=} \pi(G(\zeta(z)))\,,$$

where $\pi\colon Y \to X$ is the natural projection map. □

Theorem 7.9. *Let X be a simply connected Calabi–Yau threefold with terminal singularities and $c_2(X) \neq 0$. Assume that X admits an elliptic fibration $f\colon X \to B$ over a normal surface B. Then X contains a dense entire curve.*

Proof. It follows from [Gra91, Ogu93] that B is a normal rational surface with only quotient singularities and with $-(K_B + D)$ effective for some effective divisor D on B such that (B, D) is log terminal, the multiple fibers of f lying over a finite set F. From Theorem 5.1, we obtain the existence of a dense entire curve inside $(B^{\mathrm{reg}} \setminus F)$. Now the existence of a dense entire curve in X follows in the same way as for Proposition 7.8. □

Remark 7.10. When $c_2(X) = 0$ and X is smooth of any dimension, the conclusion still holds since, by [Yau78], the threefold X is then covered by a complex torus. We thank S. Diverio for this observation.

The conclusion and method of proof of Theorem 7.9 should still apply to Calabi–Yau manifolds fibered over $\mathbb{P}_1$ (possibly even an elliptic curve), using the results of [DFM19].

8. The Nevanlinna version of the Hilbert property

8.1 Statement

Definition 8.1 ([CZ17, § 2.2]). Let X be a (smooth) projective variety defined over a number field k. Then X is said to have the *weak Hilbert property* over k (WHP for short)[7] if $(X(k) - \cup_j \pi_j(Y_j(k)))$ is Zariski dense in X, for any finite set of covers $\pi_j\colon Y_j \to X$ defined over k, each ramified over a non-empty divisor D_j of X.

Note that $X(k)$ being Zariski dense, Conjecture 3.3 implies that X should be special, and its fundamental group should be almost abelian (Conjecture 3.4). In [CZ17], Corvaja–Zannier propose a Nevanlinna version of the weak Hilbert property in the following form [CZ17, Section 2.4].

Question-Conjecture 8.2. *Let X be a simply connected smooth projective manifold. Assume that there exists a Zariski-dense entire curve $g\colon \mathbb{C} \to X$. For any finite cover $\pi\colon Y \to X$ ramified over a non-empty divisor, with Y irreducible, there exists an entire curve $h\colon \mathbb{C} \to X$ which does not lift to an entire curve $h'\colon \mathbb{C} \to Y$ (that is, such that $\pi \circ h' = h$).*

Let us extend their question to the case of special manifolds.

Question-Conjecture 8.3. *Let X be a special compact Kähler manifold. For any finite cover $\pi\colon Y \to X$ ramified over a non-empty divisor, with Y irreducible, there exists a Zariski-dense entire curve $h\colon \mathbb{C} \to X$ which does not lift to an entire curve $h'\colon \mathbb{C} \to Y$ (that is, such that $\pi \circ h' = h$).*

[7] The classical Hilbert property does not require the covers $Y_j \to X$ to be ramified; it thus implies, by the Chevalley–Weil theorem, that X is algebraically simply connected.

We say that $\mathrm{NHP}(X)$ holds if X possesses this property, and we then say that X has the NHP (for Nevanlinna–Hilbert property).

A stronger version is the question of whether there is a single entire curve h on X such that for any finite cover $\pi\colon Y \to X$ ramified over a non-empty divisor of X, the curve h does not lift to Y. We will denote this property by $\mathrm{NHP}^+(X)$.

These properties are preserved by finite étale covers and smooth blow-ups.

Lemma 8.4. *The* NHP *and* NHP *for Galois covers are preserved by finite étale covers and bimeromorphic equivalence.*

Proof. Let $f\colon X' \to X$ be a surjective holomorphic map between connected compact complex manifolds.

First assume that f is finite étale. Let $\pi\colon Y \to X$ be actually ramified and $\pi'\colon Y' \to X'$ be deduced from π by the base change f. If $h\colon \mathbb{C} \to X$ is Zariski dense and lifts to Y, it lifts to Y', which is étale over Y. This then lifts some lifting of h to X'. This gives contradiction if $\mathrm{NHP}(X')$ holds, and so $\mathrm{NHP}(X')$ implies $\mathrm{NHP}(X)$. Conversely, let $\pi'\colon Y' \to X'$ be given and actually ramified. If $h'\colon \mathbb{C} \to X'$ is Zariski dense and liftable to Y', then $h := f \circ h'\colon \mathbb{C} \to X$ is Zariski dense and liftable to Y' (which is actually ramified over X); thus $\mathrm{NHP}(X)$ is violated as well.

Let us now assume that f is bimeromorphic. Since the fundamental groups of X and X' coincide, so do (up to bimeromorphic equivalence) the covers of X and X' which actually ramify. This (easily) implies the equivalence of $\mathrm{NHP}(X)$ and $\mathrm{NHP}(X')$ since the existence of lifts of Zariski-dense entire curves is also a bimeromorphic property.

The proof for Galois covers is similar. $\square$

A simple tool in finding non-liftable curves is the following.

Proposition 8.5. *Let $h\colon \mathbb{C} \to X$ be an entire curve and H a hypersurface of X such that there exists a regular point $a \in H$ in which $h(\mathbb{C})$ and H intersect with order of contact t. Let $\pi\colon X_1 \to X$ be a finite Galois cover with branch locus containing H such that π ramifies at order $s \geqslant 2$ over H at a. Then h cannot be lifted to an entire curve $\tilde{h}\colon \mathbb{C} \to X_1$ if t does not divide s. In particular, if $h(\mathbb{C})$ and H meet transversally at a, then h does not lift to Y.*

Proof. Since π is Galois, it ramifies at order s at any point of Y over $a \in H$. Since $h(\mathbb{C})$ intersect at order s at a, if it lifted to Y, its order of contact with H would be a multiple of s. $\square$

8.2 Rationally connected and abelian manifolds

We have the following stronger form for rationally connected manifolds, in which a *fixed* entire curve h does not lift to *any* Galois ramified cover $\pi\colon Y \to \mathbb{P}^n$.

Theorem 8.6. *Let X be a rationally connected complex projective manifold or a complex compact torus. Let Z be a closed analytic subset of codimension at least 2. Then there exists an entire curve $f\colon \mathbb{C} \to X$ such that the following hold:*

(i) *The image $f(\mathbb{C})$ is dense.*

(ii) *The curve f cannot be lifted to any ramified Galois cover $\tau\colon X' \to X$.*

(iii) *We have $f(\mathbb{C}) \cap Z = \{\}$.*

Proof. Combine Theorem 6.1, respectively Theorem 7.1, with Proposition 8.5. $\square$

8.3 Special surfaces

Proposition 8.7 ([Cam04]). *Let S be a compact Kähler surface. We denote by S' an arbitrary finite étale cover of S. The following are equivalent:*

(i) *The surface S is special.*

(ii) *No S' maps meromorphically onto a variety of general type.*

(iii) *We have $\kappa(S) \leqslant 1$ and $q(S') \leqslant 2$ for all S'.*

(iv) *We have $\kappa(S) \leqslant 1$, and $\pi_1(S)$ is virtually abelian.*

(v) *The surface S is one of the following:*

  (a) *rational,*

  (b) *birational to $\mathbb{P}_1 \times E$, with E elliptic,*

  (c) *some S' birational to an Enriques, bielliptic or $K3$ surface or a torus,*

  (d) *some S' that admits an elliptic fibration $f\colon S' \to B$ with either B elliptic and no multiple fiber, or $B = \mathbb{P}_1$ and at most two multiple fibers.*

Theorem 8.8. *Let S be a special compact Kähler surface. Then S satisfies the Nevanlinna–Hilbert property for Galois covers, except (maybe) if S is bimeromorphic to a $K3$ surface both non-elliptic and non-Kummer. If we assume the Green–Griffiths Lang conjecture, a non-special surface does not fulfill this property.*

Proof. We will first check the property in the cases (v.a)–(v.c) of Proposition 8.7, case (v.d) being treated in Lemma 8.9 below. If S is rational, it is rationally connected and so satisfies the NHP, by Theorem 8.6. If S is in the class (v.b), some birational model has an elliptic fibration, and so the conclusion follows from Lemma 8.9. Now if S is in the class (v.c), it has a finite étale cover bimeromorphic to either a $K3$ surface or a compact torus. In the case of a compact torus, the conclusion follows from Theorem 8.6. If S is bimeromorphic to an elliptic surface, we invoke Lemma 8.9.

Finally, if S is bimeromorphic to a Kummer surface S_1, let A be the associated abelian surface such that S_1 is the desingularization $\tau\colon S_1 \to S_0 = A/{\sim}$ of the quotient of A by the involution $x \mapsto -x$. Let Z be the set of sixteen points fixed by $x \mapsto -x$.

Let us first discuss the case of a cover $S' \to S_0$ whose ramification locus is not contained in the total transform of Z. Due to Theorem 8.6, there is a dense entire curve $f_0\colon \mathbb{C} \to A \setminus Z$ which cannot be lifted to any Galois cover of A. Let $f\colon \mathbb{C} \to S$ be the induced entire curve in S. It has the desired property: Every ramified cover over S with ramification not contained in Z induces a ramified cover over A, and any lifting of f would induce a lifting of f_0 for the associated ramified cover over A.

It remains to deal with the covers $\pi\colon S' \to S$ with non-empty ramification contained in the total transform of Z. Such a cover induces a cover $\pi_0\colon S'' \to S_0$ unramified outside $\tau(Z)$. Note that Z is a subgroup of the group variety A and acts transitively on Z. Moreover, the Z-action on A descends to an action on S_0. Hence, without loss of generality, we may assume that the given cover π_0 is ramified at $\tau(0)$ (where 0 is the neutral element in the abelian surface A).

Let P be a neighborhood of 0 in A which is biholomorphic to the polydisc $\Delta \times \Delta$, and $P_0 = \tau(P) \subset S_0$. Observe that $\pi\colon A \to S_0$ restricts to an unramified $2:1$ Galois cover $P \setminus \{0\} \to P_0 \setminus \{\tau(0)\}$. Since $P \setminus \{0\}$ is simply connected, every cover $P' \to P_0$ unramified outside 0 is either everywhere unramified or, after restricting to a connected component of P', isomorphic to $P \to P_0$. We assumed that there is ramification at 0; hence locally, $\pi_0\colon S'' \to S_0$ is isomorphic to $P \to P_0$.

Let $\rho\colon \mathbb{C}^2 \to A$ be the universal cover of the abelian surface. Then P lifts to the bidisc in $\mathbb{C}^2$, and the projection $\pi\colon A \to S_0$ is induced by the global map $\tilde{\tau}(x, y) = (u, v, w) \in \mathbb{C}^3$ with $u = x^2, v = y^2, w = x \cdot y$ satisfying $w^2 = u \cdot v$, the equations defining the quotient B of $\mathbb{C}^2$ by the involution $(x, y) \mapsto (-x, -y)$. This global quotient B maps onto S_0. We choose holomorphic functions $g, h\colon \mathbb{C} \to \mathbb{C}$ such that $f(z) = (z \cdot g(z), z \cdot h(z))\colon \mathbb{C} \to \mathbb{C}^2$ has dense image in $\mathbb{C}^2$. Now define a holomorphic map $F\colon \mathbb{C} \to B$ by the equations

$$F(z) := \big(z \cdot (g(z))^2, z \cdot (h(z))^2, z \cdot g(z) \cdot h(z)\big)\,.$$

This map does not lift to $\mathbb{C}^2$, not even locally near $(0, 0, 0))$, although $z \mapsto F(z^2)$ does. The map F composed with the projection $B \to S_0$ defines a map $F_0\colon \mathbb{C} \to S_0$ which locally near $\tau(0)$ does not lift to A. Since locally near $\tau(0)$, the cover $\pi_0\colon S'' \to S_0$ is isomorphic to $\tau\colon A \to S_0$, neither $F\colon \mathbb{C} \to S_0$ nor the induced entire curve $F_0\colon \mathbb{C} \to S$ lifts to S'', respectively S'.

Thus any Kummer $K3$ surface has the NHP. □

Lemma 8.9. *A special surface S admitting an elliptic fibration $f\colon S \to B$ has the* NHP *for Galois covers.*

Proof. We check that S satisfies the condition given by Proposition 8.5. Let $\pi\colon S' \to S$ be a Galois cover, actually ramified over an irreducible divisor $R \subset S$. Let (B, D_f), with $D_f := \sum_j (1 - 1/m_j) \cdot \{b_j\}$, be the orbifold base of the fibration f, where $b_j \in B$ for all j and m_j is the multiplicity[8] of the fiber S_{b_j} of f over b_j.

Since S is special, $\deg(K_B + D_f) \leqslant 0$. For any $b \in B$, there thus exists an entire map $h\colon \mathbb{C} \to (B, D_f)$, which goes through b and which is an orbifold entire curve (that is, such that its order of contact with any $b' \in B$ is equal to the multiplicity of D_f at b', which is equal to 1 if b' is not any one of the b_j).

Let $f_h\colon S_h := S \times_B \mathbb{C} \to \mathbb{C}$ be deduced from f by the base change $h\colon \mathbb{C} \to B$. Then f_h has no multiple fibers, and (by [BL00]) admits a holomorphic section $s\colon \mathbb{C} \to S_h$ going through any given reduced component F of any of the fibers of f_h.

The main result of [BL00] then implies the following.

Lemma 8.10. *Assume that either R is a reduced component of some fiber of f_h, or $f(R) = B$. Then there exists a holomorphic map $H\colon \mathbb{C}^2 \to X$ which is unramified over its image, which is a Zariski-open subset of S meeting R. Compose H with a suitable injection $j\colon \mathbb{C} \to \mathbb{C}^2$ with dense image and meeting transversally $H^{-1}(R)$ at some point. Proposition 8.5 and Lemma 8.4 imply* NHP(S) *for Galois covers.*

We still need to deal with the case when R is a non-reduced component of some fiber $F := f_h^{-1}(0)$ of f_h. Let $m > 1$ be the multiplicity of R in F. Let $\mu\colon \mathbb{C} \to \mathbb{C}$ be defined by $\mu(z) = z^m$, let $k := h \circ \mu\colon \mathbb{C} \to B$, and let $f_k\colon S'_k \to \mathbb{C}$ be deduced from f_h by the base change $\mu\colon \mathbb{C} \to C$ after taking a smooth bimeromorphic model S'_k of $S_k := S \times_B \mathbb{C}$.

The natural generically finite map $\sigma_k\colon S'_k \to S_h$ is not étale, but it is étale over the generic point of R and also outside the fiber F. Applying [BL00] again, to the fibration $f_k\colon S'_k \to \mathbb{C}$, we obtain a holomorphic map $H_k\colon \mathbb{C}^2 \to S'_k$ which is unramified over the generic point of the inverse image R_k of R, and thus a Zariski-dense holomorphic map $h_k\colon \mathbb{C} \to S'_k$ which is transversal to R_k. The map $\sigma_k \circ h_k\colon \mathbb{C} \to S_h$ is thus transversal to R at some of its generic points, establishing NHP(S) for Galois covers. □

[8]The 'classical' and 'non-classical' versions coincide for elliptic fibrations.

8.4 Removing the Galois condition

Theorem 8.11. *Let X be a complex projective manifold. Assume that X is rationally connected or that there exists a surjective and submersive holomorphic map $\rho\colon \mathbb{C}^N \to X$. Let $\pi\colon Y \to X$ be a finite map with non-empty ramification. Then there exists a dense entire curve $h\colon \mathbb{C} \to X$ which cannot be lifted to Y (that is, there are no entire curves $\tilde{h}\colon \mathbb{C} \to Y$ with $h = \pi \circ \tilde{h}$).*

This result differs from Theorem 8.6 in two points: First, the ramified cover π is no longer required to be Galois. Second (this is the price we pay for dropping the Galois condition), here the non-liftable dense entire curve h may depend on π.

Proof. First, we discuss the case $\dim_{\mathbb{C}} X = 1$. Then X is a rational or elliptic curve. If X is an elliptic curve, Y is of genus at least 2 and therefore hyperbolic; that is, there are no non-constant holomorphic maps from $\mathbb{C}$ to Y. If X is a rational curve, we choose the natural injection $\mathbb{C} \subset \mathbb{P}_1$ as $h\colon \mathbb{C} \to \mathbb{P}_1$. Any ramified cover $Y \to \mathbb{P}_1$ ramifies at at least two points, hence is ramified at at least one point in the image $h(\mathbb{C})$. Since h is immersive, it follows that h cannot be lifted to an entire curve in Y.

Now we start the proof of the general case, that is, $\dim_{\mathbb{C}} X \geqslant 2$. Let $R \subset Y$ denote the ramification divisor, and let $B = \pi(R) \subset X$ be the branching locus. Fix $p \in X \setminus B$. For every $q \in \pi^{-1}(p)$, we choose a real continuous curve $c_q\colon [0,1] \to Y$ with $c_q(0) = q$, $c_q(1) \in R$ and $\pi(c_q(t)) \notin B$ for $0 \leqslant t < 1$. Since $\dim_{\mathbb{C}} X \geqslant 2$, we may, by slightly perturbing the curves c_q if necessary, assume that for $q \neq q'$, we have $\pi(c_q([0,1])) \cap \pi(c'_q([0,1])) = \{p\}$. Thus $T = \cup_q \pi(c_q([0,1]))$ is a tree, that is, a simply connected real 1-dimensional simplicial complex which may be visualized as $d = \#\pi^{-1}(p)$ line segments glued together in one point. Due to Proposition 8.13, respectively Proposition 8.14, we obtain an entire curve $h_n\colon \mathbb{C} \to X$ with embeddings $\zeta_n\colon T \to \mathbb{C}$ such that $\lim_{n\to\infty} h_n \circ \zeta_n = i$, where $i\colon T \to X$ is the inclusion map. For sufficiently large n, we may now replace T with $(h_n \circ \zeta_n)(T)$ and obtain an obstruction to the lifting of the entire curve h_n by using Proposition 8.12.

Finally, it is a consequence of Corollary 4.6 that the entire curve may be chosen in such a way that it has dense image. $\square$

Proposition 8.12. *Let $\pi\colon Y \to X$ be a finite ramified cover of complex manifolds with ramification divisor $R \subset Y$ and branching locus $B = \pi(R)$. Let $p \in X \setminus B$. For every $q \in \pi^{-1}(p)$, let $c_q\colon [0,1] \to Y$ be a continuous real curve with $c_q(0) = q$ and $c_q(1) \in R$ and $\pi(c_q(t)) \notin B$ for all $0 \leqslant t < 1$. Let $h\colon \mathbb{C} \to X$ be an entire curve such that for every $q \in \pi^{-1}(p)$, there is a smooth real curve $\gamma_q\colon [0,1] \to \mathbb{C}$ with $\gamma_q(0) = 0$, $h \circ \gamma_q = \pi \circ c_q$ and $h'(\gamma_q(1)) \notin TB$. Then there do not exist any lifts $\tilde{h}\colon \mathbb{C} \to Y$; that is, there is no holomorphic map $\tilde{h}\colon \mathbb{C} \to Y$ with $\pi \circ \tilde{h} = h$.*

Proof. Assume the converse. Since $\gamma_q(0) = 0$ for all q, we have $h(0) = h(\gamma_q(0)) = \pi(c_q(0)) = \pi(q) = p$. Fix $q \in \pi^{-1}(p)$ such that $\tilde{h}(0) = q$. Observe that

- $\pi \circ \tilde{h} \circ \gamma_q = h \circ \gamma_q = \pi \circ c_q$;
- $\tilde{h} \circ \gamma_q(0) = q = c_q(0)$;
- $\pi \circ c_q(t) \notin B$ for all $t < 1$.

It follows that $\tilde{h} \circ \gamma_q = c_q$ and therefore $(\tilde{h} \circ \gamma_q)(1) \in R$. In combination with the relation $h'(\gamma_q(1)) \notin TB$, this yields a contradiction. $\square$

Proposition 8.13. *Let T be a (real) tree, that is, a simply connected finite 1-dimensional simplicial complex. Let X be a complex projective manifold with a surjective and submersive* *holomorphic map $\rho\colon \mathbb{C}^N \to X$. Let $c\colon T \to X$ be a continuous map. Let $\epsilon > 0$, and let $d(\ ,\ )$ be a distance function on X induced by a Hermitian metric. Then there exist an entire curve $h\colon \mathbb{C} \to X$ and a continuous map $\tilde{c}\colon T \to \mathbb{C}$ such that $h(\mathbb{C})$ is dense in X and $d(c(x), h(\tilde{c}(x))) < \epsilon$ for all $x \in T$.*

Proof. We choose a closed embedding $\tilde{c}\colon T \to \mathbb{C}$ and an infinite discrete subset $S \subset \mathbb{C}$ with $S \cap \tilde{c}(T) = \{\ \}$. Upon replacing c with a small deformation, we may assume that $c(T)$ contains no critical values of ρ. Since ρ is surjective, we may lift $c\colon T \to X$ to a continuous map $\hat{c}\colon T \to \mathbb{C}^N$. The theorem of Arakelyan–Nersesyan [Ner84] implies that continuous maps from $\tilde{c}(T) \cup S$ to $\mathbb{C}^N$ may be approximated uniformly by holomorphic maps from $\mathbb{C}$ to $\mathbb{C}^N$. Hence the continuous map $\hat{c} \circ (\tilde{c})^{-1}\colon \tilde{c}(T) \to \mathbb{C}^N$ may be approximated uniformly by holomorphic maps $\tilde{h}$ from $\mathbb{C}$ to $\mathbb{C}^N$ in such a way that $\tilde{h}(\mathbb{C})$ is dense in $\mathbb{C}^N$. This implies the assertion by taking $h := \rho \circ \tilde{h}$. □

Proposition 8.14. *Let T be a (real) tree, that is, a simply connected finite 1-dimensional simplicial complex. Let X be a rationally connected complex projective manifold. Let $c\colon T \to X$ be a continuous map. Let $\epsilon > 0$, and let $d(\ ,\ )$ be a distance function on X induced by a Hermitian metric. Then there exist a free rational curve $h\colon \mathbb{P}_1 \to X$ and a continuous map $\tilde{c}\colon T \to \mathbb{P}_1$ such that $d(c(x), h(\tilde{c}(x))) < \epsilon$ for all $x \in T$.*

Proof. We decompose the tree into arcs. Locally, each arc admits an approximation by free rational curves; see Lemma 8.15. Due to compactness, we obtain a decomposition into finitely many arcs such that each arc admits an approximation by a free rational curve. Using comb smoothing (in the form given by Corollary 8.18), we obtain an approximation by one free rational curve. □

Lemma 8.15. *Let X be a rationally connected projective manifold with a distance function $d(\ ,\ )$ (induced by some Hermitian metric). Let $\epsilon > 0$, $r \in \mathbb{N}$, and let $\gamma\colon [-1,1]$ be a regular real curve (that is, γ is C^1 with $\gamma'(t) \neq 0$ for all t). Then there exists a positive number $0 < \delta < 1$ such that for every $-\delta < a < b < \delta$, there is an r-free rational curve $c\colon \mathbb{P}_1 \to X$ satisfying*

(i) *$d(c(t), \gamma(t)) < \epsilon$ for every $t \in [a,b]$;*
(ii) *$c(a) = \gamma(a)$, $c(b) = \gamma(b)$;*
(iii) *$c'(a) = \gamma'(a)$, $c'(b) = \gamma'(b)$.*

Proof. Without loss of generality, we assume $r \geqslant 2$. We start by choosing an r-free rational curve $h\colon \mathbb{P}_1 \to X$ with $h(0) = \gamma(0)$ and $h'(0) = \gamma'(0)$. We consider the deformations of h. Since h is free, these deformations are unobstructed. Thus we may fix a ball $B \subset V = H^0(\mathbb{P}_1, h^*TX)$ parametrizing deformations of h by a map $\Phi\colon B \times \mathbb{P}_1 \to X$.

Choosing the ball small enough, we may assume $d(\Phi(p,t), h(t)) < \epsilon/2$ for all $p \in B$ and $t \in [-1,+1]$. We may furthermore assume that Φ extends to $\bar{B} \times \mathbb{P}_1$, where $\bar{B}$ is the closure in V and compact.

Because h is 1-free, for every pair (a,b) with $-1 \leqslant a < b \leqslant +1$, the evaluation map

$$B \ni p \mapsto \Phi(p,a) \times \Phi(p,b)$$

contains an open neighborhood of (a,b) in $X \times X$ in its image. Using compactness, we choose $\eta > 0$ such that for all $a,b \in [-1,1]$ and all $a',b' \in X$ with $d(\gamma(a),a') < \eta$ and $d(\gamma(b),b') < \eta$, there exists a parameter $p \in B$ with $\Phi(p,a) = a'$ and $\Phi(p,b) = b'$.

Next we choose $\delta > 0$ in such a way that

$$d(c(t), \gamma(t)) < \min\{\eta, \epsilon/2\} \quad \forall t,\ |t| < \delta\,.$$

Given any two numbers a, b satisfying $-\delta < a < b < \delta$, we have $d(c(a), \gamma(a)) < \eta$ and $d(c(b), \gamma(b)) < \eta$ and therefore may choose $p \in B$ with $\Phi(p, a) = \gamma(a)$ and $\Phi(p, b) = \gamma(b)$. Then we define $c(t) := \Phi(p, t)$. We obtain a r-free rational curve c. Evidently c has property (ii). Furthermore, c has property (i) because

$$d(c(t), \gamma(t)) \leqslant d(c(t), h(t)) + d(h(t), \gamma(t)) \leqslant \epsilon/2 + \epsilon/2 = \epsilon\,.$$

Finally, we note that property (iii) may be shown in the same way as in Corollary 4.7. □

Remark 8.16. As pointed out to us by F. Forstneric, there is a related result by A. Gournay, who proved—under a certain additional 'regularity' hypothesis—a Runge-type approximation theorem (see [Gou12]) on domains in Riemann surfaces mapped to compact almost complex manifolds containing 'free' rational curves. Where applicable, the statement of Lemma 8.15 can be deduced from Gournay's result as follows: First approximate a given map $\gamma\colon [0,1] \to X$ by a real-analytic map; then extend it to a holomorphic map g on some simply connected open neighborhood of $[0,1]$. If a certain 'regularity' condition is fulfilled, Gournay's theorem then implies that g may be approximated by a rational curve, implying the statement of Lemma 8.15. However, the 'regularity' assumption, although satisfied for 'generic' almost complex structures, seems difficult to check on a given complex structure.

Lemma 8.17. *Let T be a tree, and let $c\colon T \to C$ be a continuous closed embedding with $C = \{(z, w) \in \mathbb{C}^2\colon zw = 0\}$. Then there exist a sequence of non-zero complex numbers ϵ_n with $\lim_{n\to\infty} \epsilon_n = 0$ and a sequence of continuous maps $c_n\colon T \to C_{\epsilon_n}$ such that $\lim c_n = c$ uniformly on T.*

Proof. The assertion is easily verified if $c(T)$ is contained in one of the two irreducible components of C. Hence we assume that $c(T)$ is not contained in one of the irreducible components, implying that $c(T)$ contains $(0, 0)$ since this is the only point in which the two components intersect.

Let $0 \in T$ denote the point with $c(0) = (0, 0)$. Let T_i° denote the connected components of $T \setminus \{(0)\}$, and let $T_i = T_i^\circ \cup \{(0)\}$ be the closure of T_i°. We choose an injective continuous curve $\gamma\colon [0,1] \to \mathbb{C}$ with $\gamma(0) = 0$ such that $\gamma(t) + x^2 \neq 0$ for all $t \in]0,1]$ and $(0, x)$, $(x, 0) \in T$.

Since T is a tree, we know that $T_i \times [0,1]$ is simply connected. The choice of γ thus allows us to choose a branch of $\sqrt{\gamma(t) + x^2}$ (with $(x, 0)$, respectively $(0, x)$, in T_i and $t \in [0,1]$). We choose the branch $h_i(x, t)$ with $\sqrt{0 + x^2} = x$ for some point $(x, 0)$, respectively $(0, x)$, in T_i. By continuity it follows that $\sqrt{0 + x^2} = x$ for all x. Consequently, $\lim_{t\to 0} h_i(t, x) = x$ for all $(x, 0)$, respectively $(0, x)$, in T_i. We define $c_{t,i}\colon T_i \to \mathbb{C}^2$ as

$$c_{t,i}(x) = \begin{cases} \frac{1}{2}(h_i(t,x) + x, h_i(t,x) - x) & \text{if } T_i \subset \mathbb{C} \times \{(0)\},\\ \frac{1}{2}(h_i(t,x) - x, h_i(t,x) + x) & \text{if } T_i \subset \{(0)\} \times \mathbb{C}.\end{cases}$$

We observe that $\lim_{t\to 0} c_{t,i} = c$. Now $h_i(t, 0)^2 = \gamma(t)$. We fix a branch of the square root along γ:

$$\xi(t) \stackrel{\text{def}}{=\!=} \sqrt{\gamma(t)}\,, \quad t \in [0,1]\,.$$

Then for each i, there is an element $\sigma_i \in \{+1, 1\}$ such that $h_{t,i}(0) = \sqrt{\gamma(t) + 0} = \sigma_i \xi(t)$ and consequently $c_{t,i}(0) = (\sigma_i \xi(t), \sigma_i \xi(t))$. The problem is that for distinct i, j, we may have $\sigma_i \neq \sigma_j$.

We enumerate the T_i as $T_1, \ldots, T_r$. Since T is a tree and hence contains no closed loops, each subtree T_i contains a unique edge ending at $(0, 0)$. Now we define $c_t\colon T \to \mathbb{C}^2$ recursively as follows: If c_t is defined on $\cup_{i\leqslant k} T_i$, then we choose a small curve inside $C_{\gamma^2(t)}$ starting at $c_t(0)$

and ending at $c_{t,k+1}(0)$. Because

$$||(\xi(t),\xi(t)-(-\xi(t),-\xi(t))|| = 2\sqrt{2\gamma(t)}$$

and $\lim_{t\to 0}\gamma(t)=0$, this small curve can be chosen as small as desired if t is small enough. Thus we may attach this curve to $c_{t,k+1}(T_{k+1})$ without changing this subtree too much.

In this way, we can construct a family of continuous maps $c_t\colon T\to C_{\gamma^2(t)}$ with $\lim_{t\to 0} c_t = c$ (uniformly on T). □

Corollary 8.18. *Let C be a comb in a rationally connected projective manifold, let T be a tree, and let $c\colon T\to C$ be a closed continuous embedding. Then c may be approximated by $h_n\circ c_n$, where the h_n are free rational curves and the $c_n\colon T\to\mathbb{P}_1$ are closed continuous embeddings.*

9. Dense entire curves of order zero

For an entire curve $f\colon\mathbb{C}\to X$ with values in a Kähler manifold X, the characteristic function (in its Ahlfors–Shimizu form) is defined as

$$T_f(r)=\int_0^r\left(\int_{\Delta_t} f^*\omega\right)\frac{dt}{t}\,,$$

where ω is a Kähler form on X and $\Delta_t=\{z\in\mathbb{C}\colon |z|<t\}$.

The *order* ρ_f of f is defined as

$$\rho_f=\limsup_{r\to\infty}\frac{\log T_f(r)}{\log r}\,.$$

We will show that the entire curves in rationally connected manifolds constructed previously in this article may be constructed in such a way that the order ρ_f equals zero.

This is interesting in view of the following facts:

- For every rational map $f\colon\mathbb{C}\to X$, we have $T_f(r)=O(\log r)$ and consequently $\rho_f=0$.
- For an abelian variety A and a non-constant entire curve $f\colon\mathbb{C}\to A$, we have $\rho_f\geqslant 2$.
- The notion of the order ρ_f may generalized to non-degenerate holomorphic maps from $\mathbb{C}^n$ to X. If (for $n=\dim(X)$) there exists a non-degenerate holomorphic map $f\colon\mathbb{C}^n\to X$ with $\rho_f<2$, then X is rationally connected (see [CW16, NW12]).

We observe that from the definition of the characteristic function given above, the following may be easily deduced:

If f_n is a sequence of holomorphic maps from $\mathbb{C}$ to some Kähler manifold (X,ω) which converges locally uniformly on $\mathbb{C}$ to a holomorphic map $f\colon\mathbb{C}\to X$, then the sequence of characteristic functions $T_{f_n}\colon[1,\infty[\to\mathbb{R}^+$ converges locally uniformly on $[1,\infty[$ to $T_f\colon[1,\infty[\to\mathbb{R}^+$.

Let X be a projective rationally connected manifold. We fix a Kähler class ω. For any rational curve $f\colon\mathbb{P}_1\to X$, the degree $\deg(f)$ is defined as the volume of its image with respect to the Kähler form, that is,

$$\deg(f)=\int_{\mathbb{P}_1} f^*\omega\,.$$

For any $c>0$, let F_c denote the set of all free rational curves of degree at most c. Due to the Kähler assumption, the degree is invariant under deformations. Hence there is a constant $U>0$ such that for any two given points p, q, there is a free rational curve f in the family F_U connecting p with q.

Let $g\colon [1,\infty[\to [1,\infty[$ be a continuous function with the following properties:

1. The map $r \mapsto g(r)/\log r$ is monotone increasing on $[2,\infty[$ and unbounded.
2. We have $\lim_{r\to\infty} \log g(r)/\log r = 0$.
3. We have $g(2) \geqslant 2U\log 2$.

For example, we may take

$$g(r) = \min\left\{(\log r)^2, 2U\log r\right\} \quad \text{or} \quad g(r) = \min\{(\log(\log r))\cdot \log r, 2U\log r\}\,.$$

We choose recursively a sequence f_n of rational curves in X with $\deg(f_n) = nU$.

We start with a rational curve f_1 of degree $d \leqslant U$. For $r \geqslant 1$, we have

$$T_{f_1}(r) \leqslant \int_1^r \left(\int_{D_t} f_1^*\omega\right)\frac{dt}{t} \leqslant U\log r\,.$$

Since $g(2) \geqslant 2U\log 2$, we have $T_{f_1}(2) \leqslant \frac{1}{2}g(2)$. By the monotonicity of $g(r)/\log r$, it follows that $T_{f_1}(r) \leqslant \frac{1}{2}g(r)$ for all $r \geqslant 2$.

We claim that the recursive construction of the rational curves f_n as made above may be carried out in such a way that

$$T_{f_n}(r) \leqslant \left(1 - \frac{1}{2^{n+1}}\right) g(r) \quad \forall r \geqslant 2\,,\ \forall n\,.$$

Given f_n with $T_{f_n}(r) \leqslant \left(1 - 1/2^{n+1}\right) g(r)$ for all $r \geqslant 2$, we first choose $R_n > 2$ such that $g(R_n)/\log R_n > 2(n+1)U$. Now we choose f_{n+1} such that on $\{z\colon |z| \leqslant R_n\}$, it is close enough to f_n in order to ensure

$$T_{f_{n+1}}(r) \leqslant \left(1 - \frac{1}{2^{n+2}}\right) g(r) \quad \forall r \leqslant R_n\,.$$

Let us now consider $T_f(r)$ for $r > R_n$:

$$\begin{aligned}
T_{f_{n+1}}(r) &= T_{f_{n+1}}(R_n) + \int_{R_n}^r \left(\int_{D_t} f_{n+1}^*\omega\right)\frac{dt}{t}\\
&\leqslant T_{f_{n+1}}(R_n) + \int_{R_n}^r (n+1)U\frac{dt}{t}\\
&= T_{f_{n+1}}(R_n) + (n+1)U\left(\log r - \log R_n\right)\,.
\end{aligned}$$

By the choice of R_n, the condition $r \geqslant R_n$ and the monotonicity of $g(r)/\log r$, we have $2U(n+1) < g(r)/\log r$. Therefore,

$$T_{f_{n+1}}(r) \leqslant T_{f_{n+1}}(R_n) + \tfrac{1}{2}g(r) - (n+1)U\log R_n\,.$$

Because of the relation $\deg(f_{n+1}) = (n+1)U$, we have

$$T_{f_{n+1}}(R_n) \leqslant \int_1^{R_n} \left(\int_{D_t} f_{n+1}^*\omega\right)\frac{dt}{t} \leqslant (n+1)U\log R_n\,.$$

Hence $T_{f_{n+1}}(R_n) - (n+1)U\log R_n < 0$ and consequently $T_{f_{n+1}}(r) \leqslant \frac{1}{2}g(r)$ for all $r \geqslant R_n$. It follows that

$$T_{f_{n+1}}(r) \leqslant \left(1 - \frac{1}{2^{n+2}}\right) g(r) \quad \forall r \geqslant 2\,.$$

Finally, we obtain an entire curve $f\colon \mathbb{C} \to X$ as $f = \lim f_n$, and our construction implies $T_f(r) \leqslant g(r)$, which implies that f is of order zero:

$$\rho_f = \limsup_{r\to\infty} \frac{\log T_f(r)}{\log r} \leqslant \limsup_{r\to\infty} \frac{\log g(r)}{\log r} = 0\,.$$

The entire curves constructed in this way thus have growth order zero.

Appendix

János Kollár

We explain how modifications of the methods of the above article and [FW05] give the following.

Theorem A.1. *Let X be a smooth, proper variety over $\mathbb{C}$. Then there is a holomorphic map $\phi\colon \mathbb{D} \to X$ such that, for every finite cover $p\colon Y \to X$, the map ϕ lifts to $\phi_Y\colon \mathbb{D} \to Y$ if and only if p is étale. If X is rationally connected, then there is also such a map $\phi\colon \mathbb{C} \to X$.*

By a *finite cover*, we mean a dominant, finite morphism from an irreducible variety Y to X; thus ramification is allowed. If X is rationally connected, then it is simply connected; hence a non-trivial cover must have ramification.

The open (respectively, closed) complex unit disc is denoted by $\mathbb{D}$ (respectively, $\overline{\mathbb{D}}$). A map $g\colon \overline{\mathbb{D}} \to X$ is holomorphic if it extends holomorphically to a larger disc.

The $\phi\colon \mathbb{C} \to X$ variant of Theorem A.1 is called the *Nevanlinna version of the Hilbert property*, following [CZ17, § 2.4]. It is proved in Theorem 8.6 for Galois covers. Also, Theorem 8.11 shows that for every non-trivial $p\colon Y \to X$, there is a $\phi_Y\colon \mathbb{C} \to X$ that cannot be lifted to Y, but the ϕ_Y constructed there does depend on Y. See Example A.8 for the difference between Galois covers and arbitrary covers in this context.

Background results

The first ingredients of the proof are the following. They are not stated but essentially proved in [FW05] and Section 5 above; see especially Proposition 4.1.

Proposition A.2. *Let X be a smooth, proper variety over $\mathbb{C}$, and fix any Riemannian metric on X. Let $\psi_i\colon \overline{\mathbb{D}} \to X$ be countably many holomorphic maps and $\epsilon_i > 0$. Then there exist a holomorphic map $\phi\colon \mathbb{D} \to X$ and embeddings $\tau_i\colon \overline{\mathbb{D}} \hookrightarrow \mathbb{D}$ such that*

$$\operatorname{dist}\bigl(\phi(\tau_i(z)), \psi_i(z)\bigr) < \epsilon_i$$

for every $z \in \overline{\mathbb{D}}$ and every i.

In the rationally connected case, we get a similar result for discs that come from rational curves.

Proposition A.3. *Let X be a smooth, proper, rationally connected variety over $\mathbb{C}$, and fix any Riemannian metric on X. Let $g_i\colon \mathbb{P}^1 \to X$ be countably many free rational curves, $\sigma_i\colon \overline{\mathbb{D}} \hookrightarrow \mathbb{P}^1$ closed discs and $\epsilon_i > 0$. Set $\psi_i := g_i \circ \sigma_i$. Then there exist a holomorphic map $\phi\colon \mathbb{C} \to X$ and embeddings $\tau_i\colon \overline{\mathbb{D}} \hookrightarrow \mathbb{C}$ such that*

$$\operatorname{dist}\bigl(\phi(\tau_i(z)), \psi_i(z)\bigr) < \epsilon_i$$

for every $z \in \overline{\mathbb{D}}$ and every i.

We also use the following Lefschetz-type result for surjectivity of π_1.

Proposition A.4 ([HL85, Kol03]). *Let X be a smooth, proper variety over $\mathbb{C}$. Then there is a family of smooth curves $W \leftarrow C_W \stackrel{u}{\to} X$ with the following property:*

For every finite cover $p\colon Y \to X$, there is a dense, open $W_Y \subset W$ such that for every $w \in W_Y$,

(i) *the fiber product $C_w \times_X Y$ is irreducible and smooth, and*

(ii) *$C_w \times_X Y \to C_w$ is étale if and only if p is étale.*

Furthermore, if X is rationally connected, we can choose $C_W = W \times \mathbb{P}^1$.

Proof. If X is projective, then the universal family of smooth curve sections with linear spaces works by [HL85]. In the proper case, we use the images of linear space sections of a projective modification $X' \to X$.

The rationally connected case is proved in [Kol03, Corollary 7]. $\square$

We also need some lemmas that first show that countably many maps suffice in Theorem A.1 if $\dim X = 1$.

Lemma A.5. *Let S be a Riemann surface with universal cover $\pi\colon \mathbb{D} \to S$. Let $p\colon S' \to S$ be a finite cover such that π lifts to $\pi'\colon \mathbb{D} \to S'$. Then p is unramified.*

Proof. After base change to $\mathbb{D}$, we have $\tilde{\pi}\colon \mathbb{D} \times_S \mathbb{D} \to \mathbb{D} \times_S S'$. Here $\mathbb{D} \times_S \mathbb{D}$ is a disjoint union of countably many copies of $\mathbb{D}$, and each one gives a section of the coordinate projection $\tilde{p}\colon \mathbb{D} \times_S S' \to \mathbb{D}$. Since the group of deck transformations acts transitively on the connected components of $\mathbb{D} \times_S S'$, we see that $\tilde{p}$ is unramified, and so is p. $\square$

Corollary A.6. *Using the notation of Lemma A.5, let $\pi_m\colon \overline{\mathbb{D}}(1-1/m) \to S$ denote the restriction of π for $m \in \mathbb{N}$. If infinitely many of the π_m lift to $\pi'_m\colon \overline{\mathbb{D}}(1-1/m) \to S'$, then p is unramified.*

Proof. Fix $z_0 \in \mathbb{D}$ such that p is unramified over $\pi(z_0)$. Then $\pi'_m(z_0)$ is the same point $z'_0 \in S'$ for infinitely many m. These define a lifting $\pi'\colon \mathbb{D} \to S'$. So p is unramified by Lemma A.5. $\square$

Lemma A.7. *There are countably many discs $\sigma_j\colon \overline{\mathbb{D}} \hookrightarrow \mathbb{CP}^1$ such that a cover $p\colon S' \to \mathbb{CP}^1$ is trivial if and only if every σ_j lifts to $\sigma'_j\colon \overline{\mathbb{D}} \hookrightarrow S'$.*

Proof. Choose discs σ_j such that every finite subset of $\mathbb{CP}^1$ is contained in the image of some $\sigma_j(\mathbb{D})$. For example, we can use the interiors and exteriors of all circles $|z| = r$, where either r or r^{-1} is a natural number.

For any $p\colon S' \to \mathbb{CP}^1$ there is a σ_j whose image contains all branch points. Then $p^{-1}(\sigma_j(\mathbb{D}))$ is irreducible. So σ_j lifts if and only if p is an isomorphism over $\sigma_j(\mathbb{D})$, hence everywhere. $\square$

Proof of Theorem A.1. We choose curves $C_{w_*} \to X$, discs $\tau_*\colon \overline{\mathbb{D}} \to C_{w_*}$ and $\epsilon_* > 0$ as follows:

1. First, we pick a dense set of points $w_i \in W$ (Zariski dense is sufficient.)
2. Next, we choose discs $\tau_{ij}\colon \overline{\mathbb{D}} \to C_{w_i}$ as in Corollary A.6 (respectively, Lemma A.7).
3. Set $\epsilon_k = 2^{-k}$.

We apply Propositions A.2 and A.3 to the triply infinite family of all $\tau_{ijk} := \tau_{ij}$, $\epsilon_{ijk} := \epsilon_k$. We claim that any resulting $\phi\colon \mathbb{D} \to X$ (respectively, $\phi\colon \mathbb{C} \to X$) works for Theorem A.1. To see this, choose any finite, non-étale cover $p\colon Y \to X$. By Proposition A.4 and step 1, there is a w_i such that $C_{w_i} \times_X Y$ is smooth, irreducible, and $C_{w_i} \times_X Y \to C_{w_i}$ is non-étale.

Next, by step 2 there is a $\tau_{ij}\colon \overline{\mathbb{D}} \to C_{w_i}$ that does not lift to $C_{w_i} \times_X Y$. If ϕ lifts to $\phi_Y\colon \mathbb{D} \to Y$ (respectively, $\phi_Y\colon \mathbb{C} \to Y$) then, fixing i, j and letting $k \to \infty$, a subsequence of the holomorphic maps $\phi_Y \circ \tau_{ijk}\colon \overline{\mathbb{D}} \to Y$ converges to a lifting of $\tau_{ij}\colon \overline{\mathbb{D}} \to X$. Thus we get a lifting of τ_{ij} to $C_{w_i} \times_X Y$. This is impossible by Corollary A.6 (respectively, Lemma A.7). □

The difference between lifting to Galois covers and arbitrary covers is illustrated by the following.

Example A.8. Let $\phi\colon \mathbb{D} \to S$ be a holomorphic map to a Riemann surface that is not simply connected.

1. If ϕ is surjective and étale, then it cannot be lifted to any non-étale, Galois cover $S' \to S$.
2. If $\phi^{-1}(s)$ is finite for some $s \in S$, then ϕ can be lifted to some non-étale, non-Galois cover $S' \to S$.

For example, let $u\colon \mathbb{D} \to S$ be the universal cover, and consider the smaller discs $\overline{\mathbb{D}}(1-c) \subset \mathbb{D}$. We get $\phi_c\colon \overline{\mathbb{D}}(1-c) \to S$. Then every ϕ_c can be lifted to some non-étale, non-Galois cover $S' \to S$, but if $c \ll 1$, then ϕ_c cannot be lifted to any non-étale, Galois cover $S' \to S$.

Claim 2 shows that a feature of the proof of Theorem A.1—that the image of ϕ returns to some neighborhoods infinitely many times—is necessary.

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