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Bivariant algebraic cobordism with bundles

Toni Annala and Shoji Yokura

Abstract

The purpose of this paper is to study an extended version of bivariant derived algebraic cobordism in which the cycles carry a vector bundle on the source as additional data. We show that, over a field of characteristic 0, this extends the analogous homological theory of Lee and Pandharipande. We then proceed to study in detail the restricted theory where only rank 1 vector bundles are allowed, and employ the obtained structural results to prove a weak version of the projective bundle formula for bivariant cobordism. Since the proof of this theorem works very generally, we introduce the notion of precobordism theories for quasi-projective derived schemes over an arbitrary Noetherian ring of finite Krull dimension as a reasonable class of theories where the proof can be carried out, and prove some of their basic properties. These results can be considered as the first steps towards a Levine–Morel style algebraic cobordism over a base ring that is not a field of characteristic 0.

1. Introduction

Complex cobordism is the universal oriented cohomology theory in algebraic topology. In algebraic geometry, a similar role is expected to be played by algebraic cobordism, which first appeared in the pioneering work of Voevodsky on the Milnor conjectures [Voe96, Voe03b, Voe03a] and was later provided a geometric model by Levine and Morel [Lev09, LM07] for smooth varieties over a field of characteristic 0. The cobordism ring of a scheme X contains a lot of information: for example, two other fundamental invariants – the Chow ring of X and the K-theory of vector bundles on X – can be recovered from the algebraic cobordism ring of X. Moreover, the ability to compute the cobordism ring of simple varieties in terms of the geometric presentation has practical implications: for example, it has been used previously to prove conjectures in enumerative geometry (degree 0 Donaldson–Thomas conjectures, see [LP09]). More recently, Haution studied in [Hau20] the fixed loci of involutions of varieties using a ring closely related to the cobordism ring of $\mathrm{Spec}(k)$. For another recent application of algebraic cobordism, although in a slightly different spirit, see [SS21], where Sechin and Semenov study algebraic groups using Morava K-theories, which are cohomology theories constructed from algebraic cobordism by adding relations.

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In [Ann21b], building on the innovative work of Lowrey–Schürg [LS16] and the construction of the universal bivariant theory in [Yok09] (cf. [Yok19]), the first-named author was able to overcome the smoothness assumptions (while staying in characteristic 0) using derived algebraic geometry. The purpose of this paper is to continue this work. More precisely, the initial goal was to perform a similar extension of the algebraic bordism of varieties with vector bundles of Lee–Pandharipande [LP12]. However, quite surprisingly, many of the proofs did not use the characteristic 0 assumption in any essential way, allowing us to get nontrivial results over a more general base ring. Our constructions are in fact much more general than just a construction of a cohomology theory. The cohomology theory is just a small part of a larger bivariant theory, which also contains the corresponding homology theory (for example, Chow groups, K-theory of coherent sheaves, algebraic bordism) and forces these two theories to interact in a well-behaved manner.

Summary of the main results

In Section 3, working over a field of characteristic 0, we construct the bivariant cobordism $\Omega^{*,*}$ with vector bundles, generalizing both the algebraic cobordism of bundles $\omega_{*,*}$ constructed in [LP12] and the bivariant derived algebraic cobordism Ω^* constructed in [Ann21b]. The bivariant group $\Omega^{d,r}(X \to Y)$ is generated by cycles of the form

$$[V \to X, E]\,,$$

where the morphism $V \to X$ is projective, the composition $V \to Y$ is quasi-smooth of relative dimension $-d$, and E is a rank r vector bundle on V. The relations imposed on these cycles are similar to those used to construct Ω^* in [Ann21b]. Our theory extends the homological theory $\omega_{*,*}$ of Lee–Pandharipande to a bivariant theory; indeed, the latter is recovered as the associated homology theory via the natural isomorphism

$$\Omega^{*,*}(X \to \mathrm{pt}) \cong \omega_{-*,*}(tX) \quad \text{(Theorem 3.20)}\,,$$

where tX denotes the truncation of the derived scheme X. Moreover, the bivariant algebraic cobordism Ω^* from [Ann21b] can also be recovered $\Omega^{*,*}$ by the formula

$$\Omega^{*,0}(X \to Y) \cong \Omega^*(X \to Y) \quad \text{(Proposition 3.12)}\,.$$

In Section 4 we study *precobordism theories* $\mathbb{B}^*$ and the associated *theories with line bundles* $\mathbb{B}^{*,1}$. These are bivariant theories satisfying a derived analogue of the derived double point cobordism relations of Levine–Pandharipande (Ω^* and $\Omega^{*,1}$ are examples of such theories). The setting of precobordism theories turns out to be a rather useful one. In the first main result of this section, we completely determine the structure of $\mathbb{B}^{*,1}$ (apart from the bivariant product) in terms of that of $\mathbb{B}^*$, over an arbitrary Noetherian base ring of finite Krull dimension (Theorems 4.13 and 4.14). This is a fundamental computation and leads to another nontrivial result, namely the projective bundle formula for trivial projective bundles

$$\Omega^*(\mathbb{P}^n \times X \to Y) \cong \bigoplus_{i=0}^{n} \Omega^*(X \to Y) \quad \text{(Theorem 4.23)}\,.$$

The framework of precobordism theories and the weak projective bundle formula have been used in much subsequent work: for example, to prove the general projective bundle formula and Conner–Floyd theorem [Ann22] and to show that, in positive characteristic, quasi-smooth derived schemes are cobordant to regular varieties after inverting the characteristic in the coefficients [Ann21a]. Moreover, the weak projective bundle formula provides the starting point in efforts to realize non-$\mathbb{A}^1$-invariant higher algebraic cobordism [AI22].

2. Background

Here, we concisely recall some of the necessary background material.

2.1 Derived algebraic geometry

Throughout the article, we will freely use the languages of ∞-categories [Lur09] and derived algebraic geometry [Lur17, Lur18, TV08]. Derived algebraic geometry is a generalization of algebraic geometry where the geometric objects – *derived schemes* – are modelled locally by (homotopy types of) simplicial commutative rings. In general, this differs from spectral algebraic geometry, where the local models are equivalent to (connective) $\mathbb{E}_\infty$-ring spectra.

The most notable practical difference of derived algebraic geometry to the ordinary one is that maps between derived schemes form a space (or rather, a homotopy type) instead of a set. Consequently, derived schemes form an ∞-category instead of an ordinary category; the category of ordinary schemes embeds as a fully faithful discrete subcategory into the category of derived schemes. A derived scheme X has a *truncation* (also called the *underlying classical scheme*) $tX \hookrightarrow X$. Locally, the inclusion of the truncation is dual to the map of simplicial rings $A \to \pi_0(A)$, where $\pi_0(A)$ is the constant simplicial ring of path components of A. Setting up the theory of derived schemes is a rather involved process, but it is not necessary for the reader to worry much about foundational issues of the subject in order to understand the content of this paper. It suffices to understand how to work with derived schemes, and for this we suggest the reader to take a look at, for example, [Ann21b, Ann20, KST18, Kha20, KR19].

Throughout this work, $X \times_Z Y$ denotes the *fibre product* in the ∞-category of derived schemes. There is potential for confusion here: if X, Y and Z are ordinary schemes, then $X \times_Z Y$ is an ordinary scheme if and only if the corresponding classical fibre square is *Tor-independent*. In particular, $X \times_Z Y$ is in general different from the classical fibre product $X \times^{\mathrm{cl}}_Z Y$. However, it is true in general that $t(X \times_Z Y) = X \times^{\mathrm{cl}}_Z Y$.

2.2 Bivariant theories

We quickly recall the formalism of bivariant theories [FM81, Ful98]. A more comprehensive review of the aspects of the theory that are relevant to this work can be found in [Ann21b, Yok09].

Let $\mathcal{V}$ be a category which has a final object pt and all fibre products. We fix a class of maps, called *confined maps*, which are closed under composition and base change and contain all the identity maps, and a class of fibre squares, called *independent squares*, which satisfy the following:

(i) If the two inside squares in

$$\begin{tikzcd} X'' \arrow[r, "h'"] \arrow[d, "f''"] & X' \arrow[r, "g'"] \arrow[d, "f'"] & X \arrow[d, "f"] \\ Y'' \arrow[r, "h"'] & Y' \arrow[r, "g"'] & Y \end{tikzcd} \quad \text{or} \quad \begin{tikzcd} X' \arrow[r, "h''"'] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ Y' \arrow[r, "h'"'] \arrow[d, "g'"'] & Y \arrow[d, "g"] \\ Z' \arrow[r, "h"'] & Z \end{tikzcd}$$

are independent, then the outside square is also independent.

(ii) Any squares of the following forms are independent:

$$\begin{tikzcd} X \arrow[r, "\mathrm{id}_X"] \arrow[d, "f"'] & X \arrow[d, "f"] \\ Y \arrow[r, "\mathrm{id}_X"'] & Y, \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\mathrm{id}_X"'] & Y \arrow[d, "\mathrm{id}_Y"] \\ X \arrow[r, "f"'] & Y, \end{tikzcd}$$

where $f\colon X \to Y$ is any morphism.

Definition 2.1. A *bivariant theory* $\mathbb{B}^*$ on a category $\mathcal{V}$ with values in the category of graded Abelian groups is an assignment to each morphism $X \xrightarrow{f} Y$ in the category $\mathscr{V}$ of a graded Abelian group $\mathbb{B}^*\big(X \xrightarrow{f} Y\big)$. The morphism f is often suppressed from the notation. The bivariant groups are equipped with the following three basic operations:

(i) *Product*: For morphisms $f\colon X \to Y$ and $g\colon Y \to Z$, the product operation
$$\bullet\colon \mathbb{B}^i\big(X \xrightarrow{f} Y\big) \otimes \mathbb{B}^j\big(Y \xrightarrow{g} Z\big) \longrightarrow \mathbb{B}^{i+j}\big(X \xrightarrow{g\circ f} Z\big)$$
is defined.

(ii) *Pushforward*: For morphisms $f\colon X \to Y$ and $g\colon Y \to Z$ with f *confined*, the pushforward operation
$$f_*\colon \mathbb{B}^i\big(X \xrightarrow{g\circ f} Z\big) \longrightarrow \mathbb{B}^i\big(Y \xrightarrow{g} Z\big)$$
is defined.

(iii) *Pullback*: For an *independent* square
$$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ Y' \arrow[r, "g"'] & Y, \end{tikzcd}$$
the pullback operation
$$g^*\colon \mathbb{B}^i\big(X \xrightarrow{f} Y\big) \longrightarrow \mathbb{B}^i\big(X' \xrightarrow{f'} Y'\big)$$
is defined.

These three operations are required to satisfy seven compatibility axioms; A_1, A_2, A_3, A_{12}, A_{13}, A_{23}, A_{123} [FM81, Part I, § 2.2].

A bivariant theory unifies both a covariant theory and a contravariant theory in the following sense:

- We have the associated ("*Borel–Moore*") *homology groups* $\mathbb{B}_*(X) := \mathbb{B}^{-*}(X \to \mathrm{pt})$ which are covariant for confined morphisms.
- We have the associated *cohomology groups* $\mathbb{B}^*(X) := \mathbb{B}(X \to X)$ which are contravariant for all morphisms.

A structure-preserving transformation $\gamma\colon \mathbb{B}^* \to \mathbb{B}'^*$ is called a *Grothendieck transformation*. It induces natural transformations $\gamma\colon \mathbb{B}_* \to \mathbb{B}'_*$ and $\gamma\colon \mathbb{B}^* \to \mathbb{B}'^*$ between the associated homology and cohomology theories, respectively.

Definition 2.2. Let $\mathscr{S}$ be a class of maps (*specialized morphisms*) in $\mathscr{V}$ which is closed under compositions and contains all identity maps. Suppose that to each $f\colon X \to Y$ in $\mathscr{S}$, there is assigned an element $\theta(f) \in \mathbb{B}^*\big(X \xrightarrow{f} Y\big)$ satisfying

(i) $\theta(g \circ f) = \theta(f) \bullet \theta(g)$;

(ii) $\theta(\mathrm{id}_X) = 1_X$ for all X.

Such an assignment θ is called an ($\mathscr{S}$-)*orientation* of $\mathbb{B}^*$. An ($\mathscr{S}$-)*oriented bivariant theory* is a pair $(\mathbb{B}^*, \theta)$ consisting of a bivariant theory $\mathbb{B}^*$ and an $\mathscr{S}$-orientation θ of $\mathbb{B}^*$. We warn that this terminology differs from that of [Yok09], where an oriented bivariant theory is a bivariant theory that is equipped with operations that behave like multiplication by Chern classes of line bundles.

An orientation makes the covariant functor $\mathbb{B}_*(X)$ a contravariant functor along morphisms in $\mathscr{S}$ and the contravariant functor $\mathbb{B}^*$ a covariant functor along morphisms in $\mathscr{C} \cap \mathscr{S}$: indeed,

(i) for a morphism $f\colon X \to Y \in \mathscr{S}$, the *Gysin pullback homomorphism*

$$f^!\colon \mathbb{B}_*(Y) \longrightarrow \mathbb{B}_*(X) \quad \text{is defined by} \quad f^!(\alpha) := \theta(f) \bullet \alpha\,;$$

(ii) for a morphism $f\colon X \to Y \in \mathscr{C} \cap \mathscr{S}$, the *Gysin pushforward homomorphism*

$$f_!\colon \mathbb{B}^*(X) \longrightarrow \mathbb{B}^*(Y) \quad \text{is defined by} \quad f_!(\alpha) := f_*(\alpha \bullet \theta(f))\,.$$

We recall the definition of a bivariant ideal [Ann21b, Definition 3.5] and a characterization of the bivariant ideal generated by bivariant elements.

Definition 2.3. Let $\mathbb{B}^*$ be a bivariant theory. A (*graded*) *bivariant ideal* $\mathbb{I} \subset \mathbb{B}^*$ consists of graded subgroups $\mathbb{I}\big(X \xrightarrow{f} Y\big) \subset \mathbb{B}^*\big(X \xrightarrow{f} Y\big)$ for each $f\colon X \to Y$ such that

(i) if $\alpha \in \mathbb{I}\big(X \xrightarrow{g\circ f} Z\big)$, then $f_*(\alpha) \in \mathbb{I}\big(Y \xrightarrow{g} Z\big)$ for $f\colon X \to Y$ confined;

(ii) if $\alpha \in \mathbb{I}\big(X \xrightarrow{f} Y\big)$, then $g^*(\alpha) \in \mathbb{I}\big(X' \xrightarrow{f'} Y'\big)$ for all $g\colon Y' \to Y$ such that the Cartesian square

$$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ Y' \arrow[r, "g"'] & Y \end{tikzcd}$$

is independent;

(iii) if $\alpha \in \mathbb{I}(X \to Y)$, then $\beta \bullet \alpha \in \mathbb{I}(X' \to Y)$ for any $\beta \in \mathbb{B}^*(X' \to X)$ and $\alpha \bullet \gamma \in \mathbb{B}^*(X \to Y')$ for any $\gamma \in \mathbb{B}^*(Y \to Y')$.

Bivariant ideals are exactly the kernels of Grothendieck transformations.

Proposition 2.4. (i) *The* (*object-wise*) *kernel of a Grothendieck transformation* $\gamma\colon \mathbb{B}^* \to \mathbb{B}'^*$ *is a bivariant ideal.*

(ii) *Given a bivariant ideal* $\mathbb{I} \subset \mathbb{B}^*$, *one may form the* quotient bivariant theory $\mathbb{B}^*/\mathbb{I}$ *by setting* $(\mathbb{B}^*/\mathbb{I})(X \to Y) := \mathbb{B}^*(X \to Y)/\mathbb{I}(X \to Y)$ *and equipping these groups with the bivariant operations induced by* $\mathbb{B}^*$.

A (*bivariant*) *subset* S of $\mathbb{B}^*$ (denoted by $S \subset \mathbb{B}^*$) is a collection of subsets $S(X \to Y) \subset \mathbb{B}^*(X \to Y)$ – one for each map $X \to Y$ in $\mathcal{V}$. Given a subset $S \subset \mathbb{B}^*$, we denote by $\langle S \rangle_{\mathbb{B}^*}$

the *bivariant ideal of* $\mathbb{B}^*$ *generated by* S, that is, the smallest bivariant ideal of $\mathbb{B}^*$ containing S. When the bivariant theory $\mathbb{B}^*$ is clear from context, the simpler notation $\langle S \rangle$ is often used instead. As in the case of rings, we do have a simple description of the bivariant ideal generated by a subset, at least in good situations (see [Ann21b, Lemma 3.8] for the proof).

Proposition 2.5. *Let* S *be a bivariant subset of* $\mathbb{B}^*$. *Assuming that* all Cartesian squares are independent, $\langle S \rangle \big(X \xrightarrow{h} Y\big)$ *consists of linear combinations of elements of the form*

$$f_*(\alpha \bullet g^*(s) \bullet \beta)\,,$$

where f, g, α, β *and* s *are as in the following diagram and* $s \in S(A \to B)$*:*

$$\begin{tikzcd}
& X \arrow[drr, bend left, "h"] & & \\
A'' \arrow[ur, bend left, "f"] \arrow[r, "\alpha"] & A' \arrow[r] \arrow[d] & B' \arrow[r, "\beta"] \arrow[d, "g"] & Y \\
& A \arrow[r, "s"] & B\,. &
\end{tikzcd}$$

In the above diagram, we assume the bottom square to be Cartesian (*hence independent*) *and* f *to be confined.*

Remark 2.6. Following [FM81], in the above diagram

$$A'' \xrightarrow{\alpha} A'$$

means that $\alpha \in \mathbb{B}^*(A'' \to A')$ is a bivariant element.

3. Bivariant algebraic cobordism with vector bundles

Here, motivated by the algebraic bordism $\omega_{*,*}(X)$ of vector bundles studied by Lee and Pandharipande in [LP12], we consider its bivariant analogue $\Omega^{*,*}\big(X \xrightarrow{f} Y\big)$ and prove that $\Omega^{-*,*}(X \to \mathrm{pt})$ recovers $\omega_{*,*}(X)$. See [Yok22] (cf. [Yok20]) for related work on another, closely related, theory $\Omega^{*,*}(X, Y)$ such that $\Omega^{-*,*}(X, \mathrm{pt})$ also recovers Lee–Pandharipande's $\omega_{*,*}(X)$. Throughout the section, we work over a Noetherian base ring A of finite Krull dimension. All derived schemes are assumed to be quasi-projective over A.

3.1 Construction of $\Omega^{*,*}$

Let us begin with the free modules $\mathcal{M}_+^{i,r}\big(X \xrightarrow{f} Y\big)$ over the Lazard ring $\mathbb{L}$ generated by cobordism cycles of the form

$$\big[V \xrightarrow{h} X, E\big]\,,$$

where V is connected, $h\colon V \to X$ is projective, the composite $f \circ h\colon V \to Y$ is quasi-smooth of virtual relative dimension $-i$, and E is a vector bundle of rank r on V.

Remark 3.1. If we drop the *connectedness* condition on the source variety V in the above definition of $\mathcal{M}_+^{i,r}\big(X \xrightarrow{f} Y\big)$, then the correct definition is as the free $\mathbb{L}$-module generated by cycles of the form $\big[V \xrightarrow{h} X, E\big]$, where $h\colon V \to X$ is projective and the composite $f \circ h\colon V \to Y$ is quasi-smooth, modulo the *additivity relation*

$$\big[V \xrightarrow{h} X, E\big] = \sum_j \big[V_j \xrightarrow{h_j} X, E_j\big]\,,$$

where $V = \sqcup V_j$, h_j is the restriction of h to V_j, and $E_j = E|_{V_j}$ is the restriction of E to V_j.

These groups form bivariant theories by linearly extending the following operations:

(i) *Pushforward*: Take $f\colon X \to X'$ and $g\colon X' \to Y$, where f is projective. We define the pushforward map $f_*\colon \mathcal{M}_+^{i,r}\big(X \xrightarrow{g\circ f} Y\big) \to \mathcal{M}_+^{i,r}(X' \to Y)$ by

$$f_*\big(\big[V \xrightarrow{h} X, E\big]\big) = \big[V \xrightarrow{f\circ h} X', E\big]\,.$$

(ii) *Pullback*: Suppose that we have a map $g\colon Y' \to Y$. Then we define the pullback map

$$g^*\colon \mathcal{M}_+^{i,r}(X \to Y) \longrightarrow \mathcal{M}_+^{i,r}(X \times_Y Y' \to Y')$$

by

$$g^*([V \to X, E]) = [V \times_Y Y' \to X \times_Y Y', E']\,,$$

where E' is the pullback of E to $V \times_Y Y'$.

(iii) *Bivariant products* $\bullet_\oplus$ *and* $\bullet_\otimes$: There are two natural candidates for the bivariant product,

$$\bullet_\oplus\colon \mathcal{M}_+^{i,r}\big(X \xrightarrow{f} Y\big) \times \mathcal{M}_+^{j,s}\big(Y \xrightarrow{g} Z\big) \longrightarrow \mathcal{M}_+^{i+j,r+s}\big(X \xrightarrow{g\circ f} Z\big)\,,$$
$$\bullet_\otimes\colon \mathcal{M}_+^{i,r}\big(X \xrightarrow{f} Y\big) \times \mathcal{M}_+^{j,s}\big(Y \xrightarrow{g} Z\big) \longrightarrow \mathcal{M}_+^{i+j,rs}\big(X \xrightarrow{g\circ f} Z\big)\,,$$

which are defined by bilinearly extending the operations

$$\big[V \xrightarrow{h} X, E\big] \bullet_\oplus \big[W \xrightarrow{k} Y, F\big] = \big[V' \xrightarrow{h\circ k''} X, E' \oplus F'\big]\,,$$
$$\big[V \xrightarrow{h} X, E\big] \bullet_\otimes \big[W \xrightarrow{k} Y, F\big] = \big[V' \xrightarrow{h\circ k''} X, E' \otimes F'\big]\,,$$

where V' is as in the Cartesian diagram

$$\begin{tikzcd} V' \arrow[r, "h'"] \arrow[d, "k''"] & X' \arrow[r, "f'"] \arrow[d, "k'"] & W \arrow[d, "k"] & \\ V \arrow[r, "h"] & X \arrow[r, "f"] & Y \arrow[r, "g"] & Z \end{tikzcd}$$

and E' and F' are the pullbacks of E and F to V', respectively.

Proposition 3.2. *The bivariant theory $\mathcal{M}_+^{*,*}(X \to Y)$ is commutative with respect to both products $\bullet_\oplus$ and $\bullet_\otimes$.*

Proof. The arguments of [Yok09, § 4] go through mutatis mutandis. □

We can also choose natural orientations along quasi-smooth morphisms. Indeed, if $f\colon X \to Y$ is quasi-smooth of relative virtual dimension $-i$, then one may define the two orientations

$$\theta_\oplus(f) := \big[X \xrightarrow{\mathrm{id}} X, 0\big] \in \mathcal{M}_+^{i,0}(X \to Y)\,,$$
$$\theta_\otimes(f) := \big[X \xrightarrow{\mathrm{id}} X, \mathcal{O}_X\big] \in \mathcal{M}_+^{i,1}(X \to Y)$$

for the bivariant theories $(\mathcal{M}_+^{*,*}, \bullet_\oplus)$ and $(\mathcal{M}_+^{*,*}, \bullet_\otimes)$, respectively. Note that the choice of the vector bundle is – in both cases – essentially forced upon us by the requirement that the orientation of the identity morphism $X \to X$ should be the multiplicative identity of the ring $\mathcal{M}_+^{*,*}(X \to X)$.

Like for any other bivariant theory, we have the associated homology and cohomology theories, which we denote by

$$\mathcal{M}^+_{*,*}(X) := \mathcal{M}_+^{-*,*}(X \to \mathrm{pt}) \,,$$
$$\mathcal{M}_+^{*,*}(X) := \mathcal{M}_+^{*,*}\big(X \xrightarrow{\mathrm{id}} X\big) \,.$$

The Gysin maps of the oriented bivariant theories $(\mathcal{M}_+^{*,*}, \bullet_\oplus)$ and $(\mathcal{M}_+^{*,*}, \bullet_\otimes)$ coincide.

Definition 3.3. Given a vector bundle E of rank r on X, we may define its *top Chern class* as

$$c_r(E) = s^* s_!(1_X) \in \mathcal{M}_+^{r,*}(X) \,,$$

where $s\colon X \to E$ is the zero section. Note that the top Chern class depends on the choice of the bivariant product, which determines the unit object 1_X.

From here up to Section 4, where we deal with the bivariant product $\bullet_\otimes$, we deal with the bivariant product $\bullet_\oplus$, and for the sake of simplicity, we just write $\bullet$ without the suffix $\oplus$.

To obtain our bivariant algebraic cobordism of vector bundles $\Omega^{*,*}\big(X \xrightarrow{f} Y\big)$ from the above groups $\mathcal{M}_+^{*,*}\big(X \xrightarrow{f} Y\big)$, we use relations similar to those in [Ann21b], whose origin lies in the paper [LS16] of Lowrey and Schürg. Let us quickly recall the construction of the bivariant derived algebraic cobordism Ω^* from [Ann21b] using the language introduced in this paper. We start with the bivariant theory $\mathcal{M}_+^*$ such that $\mathcal{M}_+^*(X \to Y)$ is the free graded $\mathbb{L}$-module on generators $[V \to X]$, with V connected, $V \to X$ projective and the composition $V \to Y$ quasi-smooth. In other words, the bivariant theory $\mathcal{M}_+^*$ can be naturally identified with the bivariant theory $\mathcal{M}_+^{*,0}$ as there is only one vector bundle of rank 0. Moreover, the algebraic bordism groups $d\Omega_*$ of Lowrey and Schürg are, by construction, expressible as quotients of the homology $\mathbb{L}$-modules $\mathcal{M}^+_*(X) := \mathcal{M}_+^{-*}(X \to \mathrm{pt})$; we denote by $\mathcal{R}^{\mathrm{LS}}$ the bivariant subset of $\mathcal{M}_+^*$ such that $\mathcal{R}^{\mathrm{LS}}(X \to \mathrm{pt})$ is defined to be the kernel of $\mathcal{M}_+^{-*}(X \to \mathrm{pt}) = \mathcal{M}^+_*(X) \to d\Omega_*(X) = \Omega^{-*}(X \to \mathrm{pt})$ and $\mathcal{R}^{\mathrm{LS}}(X \to Y)$ is defined to be empty whenever $Y \neq \mathrm{pt}$. The bivariant algebraic cobordism Ω^* constructed in [Ann21b] is the quotient theory $\mathcal{M}_+^*/\langle \mathcal{R}^{\mathrm{LS}} \rangle_{\mathcal{M}_+^*}$.

Definition 3.4. We define the *bivariant algebraic cobordism with vector bundles* as

$$\Omega^{*,*} := \mathcal{M}_+^{*,*}/\langle \mathcal{R}^{\mathrm{LS}} \rangle_{\mathcal{M}_+^{*,*}} \,,$$

where we regard $\mathcal{R}^{\mathrm{LS}}$ as a bivariant subset of $\mathcal{M}_+^{*,*}$ via the identification $\mathcal{M}_+^* = \mathcal{M}_+^{*,0}$.

Another way of phrasing Definition 3.4, after recalling the details of how $d\Omega_*$ is obtained from $\mathcal{M}^+_*$ in [LS16], consists of the following three steps:

(i) *Naive cobordism relation.* Let $\mathcal{R}^{\mathrm{fib}}$ be the bivariant ideal of $\mathcal{M}_+^{*,*}$ generated by the elements

$$[W_0 \to X, 0] - [W_\infty \to X, 0] \in \mathcal{M}_+^{*,0}(X \to \mathrm{pt}) \,,$$

where $W_0 \to X$ and $W_\infty \to X$ are the fibres of a projective map $W \to \mathbb{P}^1 \times X$, with quasi-smooth domain W over $0 \times X$ and $\infty \times X$, respectively. We denote the bivariant theory $\mathcal{M}_+^{*,*}/\mathcal{R}^{\mathrm{fib}}$ by $\Omega^{*,*}_{\mathrm{naive}}$.

(ii) *Formal group law relation.* Let $\mathcal{R}^{\mathrm{fgl}}$ be the bivariant ideal of $\Omega^{*,*}_{\mathrm{naive}}$ generated by

$$c_1(\mathscr{L}_1 \otimes \mathscr{L}_2)1_X - F(c_1(\mathscr{L}_1), c_1(\mathscr{L}_2))1_X \in \Omega^{*,*}_{\mathrm{naive}}(X \to \mathrm{pt}),$$

where X is a smooth scheme, $\mathscr{L}_1$ and $\mathscr{L}_2$ are globally generated line bundles on X, and F is the universal formal group law of the Lazard ring. The careful reader should be worried at this point, as a formal power series F gives an infinite sum. However, the generating relations above are well-defined elements, as the first Chern classes of globally generated line bundles act nilpotently on $\Omega^{\mathrm{naive}}_{*,*}$ (for a proof, see [LS16, Proposition 3.15]). We denote the bivariant theory $\Omega^{*,*}_{\mathrm{naive}}/\mathcal{R}^{\mathrm{fgl}}$ by $\underline{\Omega}^{*,*}$

(iii) *Strict normal crossing relation.* Suppose that D is a strict normal crossing divisor on a smooth scheme W with prime divisors $D_1, \dots, D_r$ with multiplicities $n_1, \dots, n_r$. The formal group law relation allows us to express $[D \to W, 0]$ as a $\mathbb{L}$-linear combination of inclusions of the prime divisors D_i and their intersections. The latter expression has an obvious lift $\xi_{D,W} \in \underline{\Omega}_{*,*}(D)$. Denote by $\mathcal{R}^{\mathrm{snc}}$ the bivariant ideal of $\underline{\Omega}^{*,*}$ generated by

$$1_D - \xi_{D,W} \in \underline{\Omega}^{*,*}(D \to \mathrm{pt})$$

for all strict normal crossing divisor inclusions $D \subset W$ with W smooth. The kernel $\mathcal{R}'$ of the natural Grothendieck transformation $\mathcal{M}^{*,*}_+ \to \underline{\Omega}^{*,*}/\mathcal{R}^{\mathrm{snc}}$ is by construction $\langle \mathcal{R}^{\mathrm{LS}} \rangle$, and therefore we obtain $\Omega^{*,*}$ in three steps. We denote the Grothendieck transformation $\mathcal{M}^{*,*}_+ \to \Omega^{*,*}$ (which is the quotient homomorphism $\mathcal{M}^{*,*}_+(X \to Y) \to \Omega^{*,*}(X \to Y)$ for each $X \to Y$) by Θ for later use.

Example 3.5. For every X, we have $\big[\{0\} \times X \to \mathbb{P}^1 \times X, 0\big] - \big[\{\infty\} \times X \to \mathbb{P}^1 \times X, 0\big] = 0 \in \Omega^{0,0}\big(\mathbb{P}^1 \times X\big)$ (cf. [Ann21b, Proposition 3.7]).

Example 3.6. Let $X \to Y$ be a morphism, let $f \colon W \to \mathbb{P}^1 \times X$ be such a projective map that the composition $W \to Y$ is quasi-smooth, and let W be a vector bundle on E. Then

$$\begin{aligned}
&[W_0 \to X; E|_{W_0}] - [W_\infty \to X; E|_{W_\infty}] \\
&\quad = \mathrm{pr}_{2*}\Big(\Big(\big[\{0\} \times X \to \mathbb{P}^1 \times X, 0\big] - \big[\{\infty\} \times X \to \mathbb{P}^1 \times X, 0\big]\Big) \bullet \big[W \to \mathbb{P}^1 \times X, E\big]\Big) \\
&\quad = 0 \in \Omega^{*,*}(X \to Y),
\end{aligned}$$

where W_i is the fibre of f over $\{i\} \times X$ and $\mathrm{pr}_2 \colon \mathbb{P}^1 \times X \to X$ is the projection to the second factor.

Next, we study in greater detail the bivariant ideal $\langle \mathcal{R}^{\mathrm{LS}} \rangle$ of $\mathcal{M}^{*,*}_+$ defining $\Omega^{*,*}$. Denote by $[E]$ the class $[X \to X, E]$ in $\mathcal{M}^{*,*}_+(X \to X)$. We begin with the following simple observations.

Lemma 3.7. *Let E be a vector bundle on Y, and let $\alpha \in \mathcal{M}^{*,*}_+(X \to Y)$. Then $\alpha \bullet [E] = [f^*E] \bullet \alpha$.* □

Lemma 3.8. *The theory $\mathcal{M}^{*,*}_+$ consists of linear combinations of elements of the form $f_*([E] \bullet \alpha)$, where $\alpha \in \mathcal{M}^{*,0}_+$.*

Proof. Indeed,

$$\big[X \xrightarrow{f} Y, E\big] = f_*\big([E] \bullet \big[X \xrightarrow{\mathrm{id}} X, 0\big]\big) \in \mathcal{M}^{*,*}_+\big(Y \xrightarrow{g} Z\big),$$

where $[E] \in \mathcal{M}^{*,*}_+\big(X \xrightarrow{\mathrm{id}} X\big)$. □

It is now possible to give an explicit generating set for the ideal $\langle \mathcal{R}^{\mathrm{LS}} \rangle$.

Proposition 3.9 (cf. [Ann21b, Lemma 3.8]). *The kernel of the Grothendieck transformation $\mathcal{M}_+^{*,*} \to \Omega^{*,*}$ consists of linear combinations of elements of the form*

$$f_*([E] \bullet \alpha_0 \bullet g^*(s) \bullet \beta_0)\,,$$

where $f, g, \alpha_0 \in \mathcal{M}_+^{*,0}$, $\beta_0 \in \mathcal{M}_+^{*,0}$ and $s \in \mathcal{R}^{\mathrm{LS}}$ are as in the diagram

$$\begin{tikzcd}
 & X \arrow[drr, bend left, "h"] & & \\
A'' \arrow[ur, bend left, "f"] \arrow[r, "\alpha_0"] & A' \arrow[r] \arrow[d] & B' \arrow[r, "\beta_0"] \arrow[d, "g"] & Y \\
 & A \arrow[r, "s"] & B = \mathrm{pt} &
\end{tikzcd}$$

and E is a vector bundle on A''. In the above diagram, we assume the bottom square to be Cartesian and f to be confined.

Proof. It is clear that elements of the above form lie in $\langle \mathcal{R}^{\mathrm{LS}} \rangle_{\mathcal{M}_+^{*,*}}$; therefore, it is enough to show that these elements form a bivariant ideal, that is, they satisfy the three conditions (i), (ii) and (iii) of Definition 2.3. For the first two, see [Ann21b, proof of Lemma 3.8].

We are left to check that elements of the above form are closed under left and right multiplication.

(a) *Right multiplication.* Consider $\gamma \in \mathcal{M}_+^{*,*}\big(Y \xrightarrow{h} Y'\big)$. Using the bivariant axiom (A_{12}), see [FM81], one observes that

$$f_*([E] \bullet \alpha_0 \bullet g^*(s) \bullet \beta_0) \bullet \gamma = f_*\big(([E] \bullet \alpha_0 \bullet g^*(s) \bullet \beta_0) \bullet \gamma\big)\,.$$

Moreover, by Lemma 3.8, we may assume γ to be of the form $i_*([F] \bullet \gamma_0)$, where $i\colon W \to Y$ is projective, F is a vector bundle on W, and $\gamma_0 \in \mathcal{M}_+^{*,0}\big(W \xrightarrow{h \circ i} Y'\big)$. Considering the Cartesian diagram

$$\begin{tikzcd}
C'' \arrow[r] \arrow[d, "i'''"] & C' \arrow[r] \arrow[d, "i''"] & D' \arrow[r] \arrow[d, "i'"] & W \arrow[d, "i"] & \\
A'' \arrow[r] & A' \arrow[r] & B' \arrow[r] & Y \arrow[r] & Y'\,,
\end{tikzcd}$$

we compute, employing the bivariant axioms (A_{123}) and (A_{13}) as well as Lemma 3.7, that

$$f_*\big(([E] \bullet \alpha_0 \bullet g^*(s) \bullet \beta_0) \bullet i_*([F] \bullet \gamma_0)\big) = f_*\big(i'''_*\big([E' \oplus F'] \bullet i''^*(\alpha_0) \bullet i'^*(g^*(s)) \bullet i^*(\beta_0) \bullet \gamma_0\big)\big)\,,$$

where E' and F' are the pullbacks of E and F to C'', respectively. The right-hand side of the above equation is of the desired form, so we are done.

(b) *Left multiplication.* Now consider $\gamma \in \mathcal{M}_+^{*,*}\big(X' \xrightarrow{h} X\big)$. We apply the bivariant projection formula (A_{123}) to the fibre diagram

$$\begin{tikzcd}
C'' \arrow[r, "f'"] \arrow[d, "h'"] & X' \arrow[d, "h"] & \\
A'' \arrow[r, "f"] & X \arrow[r] & Y
\end{tikzcd}$$

to compute that

$$\gamma \bullet f_*([E] \bullet \alpha_0 \bullet g^*(s) \bullet \beta_0) = f'_*(f^*(\gamma) \bullet [E] \bullet \alpha_0 \bullet g^*(s) \bullet \beta_0)\,.$$

By Lemma 3.8, we may assume $f^*(\gamma)$ to be of the form $i_*([F]\bullet\gamma_0)$, where $i\colon D \to C''$ is projective, F is a vector bundle on D, and $\gamma_0 \in \mathcal{M}_+^{*,0}\big(D \xrightarrow{h' \circ i} A''\big)$. Then, by the bivariant axiom (A_{12}) and Lemma 3.7,

$$f'_*\big(i_*([F] \bullet \gamma_0) \bullet [E] \bullet \alpha_0 \bullet g^*(s) \bullet \beta_0\big) = f'_*\big(i_*\big([F \oplus E'] \bullet \gamma_0 \bullet \alpha_0 \bullet g^*(s) \bullet \beta_0\big)\big)\,,$$

where E' is the pullback of E to D, which is of the desired form. □

3.2 Basic properties

The following proposition is a direct consequence of the construction of $\Omega^{*,*}$.

Proposition 3.10. *The bivariant theory $\Omega^{*,*}$ is commutative and has strong orientations along smooth morphisms.*

Proof. The first claim is trivial, and the second claim follows from [Ann21b, Proposition 2.9]. □

Over a base field k of characteristic 0, we may compare $\Omega^{*,*}$ with the bivariant derived algebraic cobordism Ω^* constructed in [Ann21b].

Definition 3.11. We define the following two Grothendieck transformations:

- attaching the zero bundle: $\mathscr{Z}\colon \Omega^*(X \to Y) \to \Omega^{*,0}(X \to Y)$ defined by $\mathscr{Z}([V \to X]) := [V \to X, 0]$;
- the *forgetful Grothendieck transformation*: $\mathscr{F}\colon \Omega^{*,*}(X \to Y) \to \Omega^*(X \to Y)$ defined by $\mathscr{F}([V \to X, E]) = [V \to X]$.

Proposition 3.12. *The above Grothendieck transformations restrict to give an isomorphism $\Omega^* \cong \Omega^{*,0}$.*

We can also extend the theory of Chern classes from Ω^* to $\Omega^{*,*}$.

Proposition 3.13 (cf. [Ann21b, Theorem 3.17]). *Let k be a field of characteristic 0, and let E be a vector bundle of rank r over a quasi-projective derived k-scheme X. Then we have the total* Chern class

$$c(E) = 1_X + c_1(E) + \cdots + c_r(E) \in \Omega^{*,0}(X) = \Omega^{*,0}\big(X \xrightarrow{\mathrm{id}} X\big)\,,$$

where $c_i(E) \in \Omega^{i,0}(X)$. These classes satisfy the following basic properties:

(i) *Naturality in pullbacks: Given $f\colon X' \to X$, we have $f^*c(E) = c(f^*E)$.*

(ii) *Whitney sum formula: Given a short exact sequence of vector bundles*

$$0 \longrightarrow E' \longrightarrow E \longrightarrow E'' \longrightarrow 0\,,$$

we have $c(E) = c(E') \bullet c(E'')$.

(iii) *Normalization: The top Chern class of Definition 3.3 agrees with $c_r(E)$. Moreover,*

$$c_r(E) = [V(s) \to X, 0]\,,$$

where $V(s) \hookrightarrow X$ is the inclusion of the derived vanishing locus of a global section s of E.

(iv) *Formal group law: Given* any *derived scheme X and* any *two line bundles $\mathscr{L}$ and $\mathscr{L}'$ on X, we have*

$$c_1(\mathscr{L} \otimes \mathscr{L}') = F(c_1(\mathscr{L}), c_1(\mathscr{L}')),$$

where F is the universal formal group law of $\mathbb{L}$.

(v) *Nilpotency: The Chern classes $c_i(E)$ are nilpotent with respect to the bivariant product.*

3.3 Relation to Lee–Pandharipande's algebraic cobordism with vector bundles $\omega_{*,*}(X)$

Here we show that for all quasi-projective derived schemes X over a field k of characteristic 0,

$$\Omega_{*,*}(X) \cong \omega_{*,*}(tX),$$

showing that the bivariant theory $\Omega^{*,*}$ we have constructed in the beginning of this section is an extension of the homological theory $\omega_{*,*}$ of Lee–Pandharipande. We first show that there is a well-defined morphism $\iota\colon \omega_{*,*}(tX) \to \Omega_{*,*}(X)$, which can be described on the level of cycles as $[V \to X, E] \mapsto [V \to X, E]$. That this map is well defined follows from the next lemma (after applying Poincaré duality [Ann21b, Proposition 2.12] to identify homology and cohomology of smooth schemes).

LEMMA 3.14 (Derived double point cobordism, cf. [LP09, § 3.3]). *Let A be a Noetherian ring of finite Krull dimension, $X \to Y$ be a morphism of quasi-projective derived A-schemes, $W \to \mathbb{P}^1 \times X$ be a projective morphism such that the composition $W \to \mathbb{P}^1 \times Y$ is quasi-smooth, and let E be a vector bundle on W. Denote by W_0 the fibre over 0, and suppose that the fibre over ∞ is the sum of two virtual Cartier divisors A and B. Then*

$$[W_0 \to X, E|_{W_0}] = [A \to X, E|_A] + [B \to X, E|_B] - [\mathbb{P}_{A\cap B}(\mathcal{O}_W(A) \oplus \mathcal{O}) \to X, E|_{A\cap B}] \quad (3.1)$$

in $\Omega^{*,*}(X \to Y)$.

Proof. Let us start with immediate reductions. It is enough to show that

$$\begin{aligned}[][W_0 \to W, E|_{W_0}] &= [A \to W, E|_A] + [B \to W, E|_B] \\ &\quad - [\mathbb{P}_{A\cap B}(\mathcal{O}_W(A) \oplus \mathcal{O}) \to W, E|_{A\cap B}] \in \Omega^{*,*}(W).\end{aligned} \quad (3.2)$$

The formula (3.1) then follows in a standard way using bivariant operations. But of course, (3.2) can be obtained from the formula

$$[W_0 \to W] = [A \to W] + [B \to W] - [\mathbb{P}_{A\cap B}(\mathcal{O}_W(A) \oplus \mathcal{O}) \to W] \in \Omega^*(W) \quad (3.3)$$

by multiplying with $[W \to W, E]$, so it is enough to look at the theory Ω^*. Finally, we can use the formal group law relation to $W_\infty = A + B$ in order to deduce that

$$\begin{aligned}[][W_0 \to W] = [W_\infty \to W] &= \sum_{i,j} a_{ij} c_1(\mathcal{O}_W(A))^i c_1(\mathcal{O}_W(B))^j \\ &= [A \to W] + [B \to W] + [A \cap B \to W] \bullet \sum_{i,j \geqslant 1} a_{ij} c_1(\mathcal{O}_W(A))^{i-1} \bullet c_1(\mathcal{O}_W(B))^{j-1} \\ &= [A \to W] + [B \to W] + i_!\Bigg(\sum_{i,j\geqslant 1} a_{ij} c_1(\mathcal{O}_W(A)|_{A\cap B})^{i-1} \bullet c_1(\mathcal{O}_W(B)|_{A\cap B})^{j-1}\Bigg),\end{aligned}$$

where i is the inclusion $A \cap B \hookrightarrow W$ and the last equality follows from the projection formula. Since the line bundles $\mathcal{O}_W(A)|_{A\cap B}$ and $\mathcal{O}_W(B)|_{A\cap B}$ are duals of each other [LP09, § 0.2], we have reduced the proof of the theorem to proving the following claim: if Z is a derived scheme and $\mathscr{L}$ is a line bundle on Z, then

$$[\mathbb{P}_Z(\mathscr{L} \oplus \mathcal{O}) \to Z] = -\sum_{i,j \geqslant 1} a_{ij} c_1(\mathscr{L})^{i-1} \bullet c_1\big(\mathscr{L}^\vee\big)^{j-1} \in \Omega^*(Z)\,. \tag{3.4}$$

In order to prove (3.4), choose a smooth scheme T together with a line bundle $\mathscr{L}'$ and a morphism $f\colon Z \to T$ such that $f^*\mathscr{L}' \simeq \mathscr{L}$. Since $\Omega^*(T)$ is known to be isomorphic to the algebraic bordism group $\Omega_{\dim(T)-*}(T)$ of Levine–Morel, we can compute that

$$\begin{aligned}
[\mathbb{P}_Z(\mathscr{L} \oplus \mathcal{O}) \to Z] &= f^*([\mathbb{P}_T(\mathscr{L}' \oplus \mathcal{O}) \to T]) \\
&= -f^*\Bigg(\sum_{i,j \geqslant 1} a_{ij} c_1(\mathscr{L}')^{i-1} \bullet c_1\big(\mathscr{L}'^\vee\big)^{j-1}\Bigg) \quad \text{([LP09, Lemma 3.3])} \\
&= \sum_{i,j \geqslant 1} a_{ij} c_1(\mathscr{L})^{i-1} \bullet c_1\big(\mathscr{L}^\vee\big)^{j-1},
\end{aligned}$$

proving the claim. □

From now on, unless stated otherwise, we will work over a field k of characteristic 0. We begin by proving the surjectivity of ι.

Lemma 3.15. *The map $\iota\colon \omega_{*,*}(tX) \to \Omega_{*,*}(X)$ is surjective.*

Proof. Consider the cobordism cycle $[V \to V, E]$ in $\Omega_{*,*}(V)$. We have already remarked that $\Omega_{*,0}(V)$ agrees with the Lowrey–Schürg algebraic bordism group of V, and as this is equivalent to the Levine–Morel (Levine–Pandharipande) algebraic bordism of the truncation tV, we know that the cycle $[V \to V, 0] \in \Omega_{*,0}(V)$ is equivalent to a cycle α which is a linear combination of smooth schemes mapping to V. Therefore, $[V \to V, E] = [V \to V, E] \bullet \alpha$, and the right-hand side is clearly contained in the image of $\omega_{*,*}(tX) \to \Omega_{*,*}(X)$. □

Next, we show the injectivity of ι. The argument is similar to the one used in [LP12]. Let us begin with a definition.

Definition 3.16. Let $\Psi_{r,d}$ be the space of homogeneous degree d integral coefficient polynomials in variables $c_1, \dots, c_r$, where c_i has degree i. Let $\Phi \in \Psi_{r,d}$ be a polynomial. Consider the *differentiation morphism*

$$\partial_\Phi\colon \mathcal{M}_+^{*,r}(X \to Y) \longrightarrow \Omega^{*+d,r}(X \to Y)$$

defined by

$$\partial_\Phi([V \to X, E]) := h_*\big(\Phi(E) \bullet [V \to V, E]\big)\,,$$

where $\Phi(E) := \Phi(c_1(E), \dots, c_r(E))$. We define a bilinear pairing

$$\rho\colon \Psi_{r,d} \times \mathcal{M}_+^{*,r}(X \to Y) \longrightarrow \Omega^{*+d}(X \to Y)$$

by the formula $\rho(\Phi, \alpha) := \mathscr{F}\big(\partial_\Phi(\alpha)\big)$.

We claim that ρ induces a well-defined pairing $\rho\colon \Psi_{r,d} \times \Omega^{*,r}(X \to Y) \to \Omega^{*+d}(X \to Y)$. This, and a lot more, follows from the next result.

Proposition 3.17. *The differential operations*

$$\partial_\Phi\colon \Omega^{*,*} \longrightarrow \Omega^{*,*}$$

are well defined, commute with pushforwards and pullbacks, and are linear over the subtheory $\Omega^{*,0} = \Omega^*$. *Moreover,* $\partial_{\Phi_1}\partial_{\Phi_2} = \partial_{\Phi_1\Phi_2}$, *and the generating differentials* ∂_{c_n} *satisfy a generalized Leibniz rule*

$$\partial_{c_n}(\alpha \bullet \beta) = \sum_{i+j=n} \partial_{c_i}(\alpha) \bullet \partial_{c_j}(\beta)\,.$$

Proof. (i) *Well-definedness of* $\partial_\Phi \colon \Omega^{*,*} \to \Omega^{*,*}$. Using the characterization of the relations imposed on $\mathcal{M}_+^{*,*}$, we need only to show that the kernel of the Grothendieck transformation $\Theta \colon \mathcal{M}_+^{*,*} \to \Omega^{*,*}$ is sent to zero by ∂_Φ (the differential operations ∂_Φ are clearly $\mathbb{L}$-linear). By Proposition 3.9, the kernel of Θ consists of linear combinations of elements of the form $g_*([E]\bullet r)$, where r is in the kernel of $\mathcal{M}_+^{*,0} \to \Omega^{*,0}$. As $r = 0 \in \Omega^{*,*}$,

$$\partial_\Phi(g_*([E]\bullet r)) = g_*\big(\Phi(E) \bullet [E] \bullet r\big)$$

is zero in $\Omega^{*,*}$, and therefore $\partial_\Phi \colon \Omega^{*,*} \to \Omega^{*,*}$ is well defined.

(ii) ∂_Φ *commutes with* f_*. It suffices to check this for a generating element $g_*([E]\bullet\alpha)$. We compute that

$$\begin{aligned}\partial_\Phi\big(f_*(g_*([E]\bullet\alpha))\big) &= \partial_\Phi\big((f\circ g)_*([E]\bullet\alpha)\big)\\ &= (f\circ g)_*(\phi(E)\bullet[E]\bullet\alpha)\\ &= f_*\big(\partial_\Phi(g_*([E]\bullet\alpha))\big)\,,\end{aligned}$$

proving the claim.

(iii) ∂_Φ *commutes with* g^*. The proof is similar to the above.

(iv) $\partial_{\Phi_1}\partial_{\Phi_2} = \partial_{\Phi_1\Phi_2}$. This is trivial.

(v) *Generalized Leibniz rule and linearity over* $\Omega^{*,0}$. Consider the Cartesian diagram

$$\begin{tikzcd} V' \arrow[r] \arrow[d, "g'"] & X' \arrow[r] \arrow[d] & W \arrow[d, "g"] & \\ V \arrow[r, "f"] & X \arrow[r] & Y \arrow[r] & Z\,, \end{tikzcd}$$

where f and g are projective. Then

$$f_*([E]\bullet\alpha)\bullet g_*([F]\bullet\beta) = f_*\big(g'_*\big([E'\oplus F']\bullet g^*(\alpha)\bullet\beta\big)\big)\,,$$

where E' and F' are the pullbacks of E and F to V', respectively. Hence

$$\begin{aligned}\partial_{c_n}\big(f_*([E]\bullet\alpha)\bullet g_*([F]\bullet\beta)\big) &= \partial_{c_n}\big((f\circ g')_*\big([E'\oplus F']\bullet g^*(\alpha)\bullet\beta\big)\big)\\ &= (f\circ g')_*\big(c_n(E'\oplus F')\bullet[E'\oplus F']\bullet g^*(\alpha)\bullet\beta\big)\\ &= \sum_{i+j=n}(f\circ g')_*\big(c_i(E')\bullet c_j(F')\bullet[E'\oplus F']\bullet g^*(\alpha)\bullet\beta\big)\\ &= \sum_{i+j=n} f_*(c_i(E)\bullet[E]\bullet\alpha)\bullet g_*(c_j(F)\bullet[F]\bullet\beta)\\ &= \sum_{i+j=n}\partial_{c_i}\big(f_*([E]\bullet\alpha)\big)\bullet\partial_{c_j}\big(g_*([F]\bullet\beta)\big)\,,\end{aligned}$$

proving that the generalized Leibniz rule holds. This also shows that if $\partial_{c_i}(\alpha) = 0$ for all $i > 0$, then $\partial_\Phi(\alpha\bullet\beta) = \alpha\bullet\partial_\Phi(\beta)$ and $\partial_\Phi(\beta\bullet\alpha) = \partial_\Phi(\beta)\bullet\alpha$ for all Φ. Hence the ∂_Φ are linear over the subtheory Ω^*. $\square$

Now that we have shown that the pairing $\rho\colon \Psi_{r,*} \times \Omega^{*,r} \to \Omega^*$ is well defined, we would like to make sure that it is compatible with the pairing defined by Lee and Pandharipande.

Lemma 3.18. *Let X be a quasi-projective derived scheme. Then the natural map $\iota\colon \omega_{*,r}(tX) \to \Omega_{*,r}(X)$ induces a commutative square*

$$\begin{tikzcd} \Psi_{r,d} \times \omega_{*,r}(tX) \arrow[r] \arrow[d] & \omega_{*-d}(tX) \arrow[d, "\cong"] \\ \Psi_{r,d} \times \Omega_{*,r}(X) \arrow[r] & \Omega_{*-d}(X)\,, \end{tikzcd}$$

where the upper horizontal map is the bilinear pairing ρ^X defined in [LP12, § 4.2], *and where the lower horizontal map is ρ.*

Proof. As both pairings are defined in essentially the same way, the claim follows from the fact that multiplication by the bivariant Chern classes recovers the Chern class operators of Levine–Morel [Ann21b, Proposition 3.19]. □

We are finally ready to prove the following injectivity result.

Lemma 3.19. *The natural (cross product) map $\mathscr{P}\colon \omega_{*,*}(\mathrm{pt}) \otimes_{\mathbb{L}} \Omega^*(X \to Y) \to \Omega^{*,*}(X \to Y)$ defined by*

$$[V \to \mathrm{pt}, E] \otimes [W \to X] \longmapsto [V \times W \to X, E']\,,$$

where E' is the pullback of E to $V \times W$, is injective.

Proof. The group $\omega_{*,r}(\mathrm{pt})$ is free as an $\mathbb{L}$-module [LP12, Theorem 2]; let $e_1, e_2, \dots, e_N$ be a basis. Then one can find a dual basis $e_1^\vee, e_2^\vee, \dots, e_N^\vee$ for $\mathbb{L} \otimes_{\mathbb{Z}} \Psi_{r,*}$ satisfying $\rho\bigl(e_j^\vee, e_i\bigr) = \delta_{ji}$; see [LP12, Proposition 8]. Hence, given

$$\alpha = \sum_{i=1}^{N} e_i \otimes \alpha_i \in \omega_{*,*}(\mathrm{pt}) \otimes_{\mathbb{L}} \Omega^*(X \to Y)\,,$$

we compute that $\rho\bigl(e_j^\vee, \mathscr{P}(\alpha)\bigr) = \alpha_j$. Hence $\alpha = 0$ if and only if all the α_i are zero, proving the injectivity. □

Combining Lemmas 3.15 and 3.19, we obtain the following.

Theorem 3.20. *Let k be a field of characteristic* 0. *Then the map*

$$\omega_{*,*}(tX) \longrightarrow \Omega_{*,*}(X)$$

is an isomorphism for all quasi-projective derived k-schemes X.

Proof. As a special case of Lemma 3.19, we know that the natural map $\omega_{*,*}(\mathrm{pt}) \otimes_{\mathbb{L}} \Omega_*(X) \to \Omega_{*,*}(X)$ is injective. On the other hand, as $\Omega_*(X)$ is naturally equivalent to $\Omega_*(tX)$, which in turn is naturally equivalent to $\omega_*(tX)$, we conclude that the map $\omega_{*,*}(\mathrm{pt}) \otimes_{\mathbb{L}} \omega_*(tX) \to \Omega_{*,*}(X)$ is injective. Moreover, by the results of Lee–Pandharipande, this map is naturally equivalent to the map $\omega_{*,*}(tX) \to \Omega_{*,*}(X)$, which we already know to be a surjection by Lemma 3.15, and the claim follows. □

4. Bivariant precobordism with line bundles and the weak projective bundle formula

The cobordism with bundles $\Omega^{*,*}(X \to Y)$ is a bivariant theory with respect to the product $\bullet_\otimes$. It has an interesting subtheory: $\Omega^{*,1}(X \to Y)$, *the bivariant cobordism theory with line bundles.* This theory comes with a natural inclusion $(\Omega^*, \bullet) \to (\Omega^{*,1}, \bullet_\otimes)$ which equips a cycle $[V \to X]$ with the trivial line bundle $\mathcal{O}_V$. This also induces the orientation $\theta_\otimes$ on $\Omega^{*,1}$.

Here, we study the structure of bivariant theories of line bundles and use the gained knowledge to compute the cobordism group of $\mathbb{P}^n \times X \to Y$ in terms of that of $X \to Y$ (weak projective bundle formula, see Theorem 4.23). In Section 4.1 we introduce the notion of a *bivariant precobordism theory* $\mathbb{B}^*$ and the *associated theory with line bundles* $\mathbb{B}^{*,1}$, which form a natural class of bivariant theories for which the results of this section hold. We note that they are quite general: our results hold over an arbitrary Noetherian base ring A of finite Krull dimension. In Section 4.2 we show that, additively (that is, disregarding the bivariant product), $\mathbb{B}^{*,1}$ is just a direct sum of copies of $\mathbb{B}^*$. In Section 4.3 we connect the structure of $\mathbb{B}^{*,1}(X \to Y)$ to the structure of $\mathbb{B}^*(\mathbb{P}^n \times X \to Y)$.

Throughout the section, unless otherwise specified, A will be a Noetherian ring of finite Krull dimension.

4.1 Bivariant precobordism theories

In this subsection we are going to introduce the theories for which the results of this section apply. Let us denote by $\mathcal{M}^*_+$ the universal bivariant theory in [Yok09] applied to quasi-projective derived A-schemes with projective morphisms as confined morphisms, quasi-smooth morphisms as specialized morphisms and all Cartesian squares as independent squares (see also [Ann21b, § 2.1]). We recall that $\mathcal{M}^d_+(X \xrightarrow{f} Y)$ is the free Abelian group on equivalence classes of projective maps $h\colon V \to X$ such that the composition $f \circ h\colon V \to Y$ is a quasi-smooth morphism of relative virtual dimension $-d$, modulo the relation identifying disjoint union with summation. Also recall that a quasi-smooth morphism $f\colon X \to Y$ of relative virtual dimension $-d$ has a canonical orientation

$$\theta(f) := \left[X \xrightarrow{\mathrm{id}} X\right] \in \mathcal{M}^d_+(X \to Y),$$

and these are stable under pullback.

Definition 4.1. Let $\mathbb{B}^*$ be a quotient theory of $\mathcal{M}^*_+$.

(i) We say that $\mathbb{B}^*$ is a *naive cobordism theory* if, given a projective morphism $W \to \mathbb{P}^1 \times X$ such that the composition

$$W \to \mathbb{P}^1 \times X \xrightarrow{\mathrm{id} \times f} \mathbb{P}^1 \times Y$$

is quasi-smooth of relative dimension d, we have

$$[W_0 \to X] = [W_\infty \to X] \in \mathbb{B}^{-d}\big(X \xrightarrow{f} Y\big),$$

where W_0 and W_∞ are the fibres of $W \to \mathbb{P}^1 \times X$ over $0 \times X$ and $\infty \times X$, respectively.

(ii) We say that $\mathbb{B}^*$ is a *precobordism theory* if it is a naive cobordism theory, and if given line bundles $\mathscr{L}_1$ and $\mathscr{L}_2$ on X, we have

$$\begin{aligned} c_1(\mathscr{L}_1 \otimes \mathscr{L}_2) = {} & c_1(\mathscr{L}_1) + c_1(\mathscr{L}_2) - c_1(\mathscr{L}_1) \bullet c_1(\mathscr{L}_2) \bullet [\mathbb{P}_1 \to X] \\ & - c_1(\mathscr{L}_1) \bullet c_1(\mathscr{L}_2) \bullet c_1(\mathscr{L}_1 \otimes \mathscr{L}_2) \bullet ([\mathbb{P}_2 \to X] - [\mathbb{P}_3 \to X]) \end{aligned} \tag{4.1}$$

in $\mathbb{B}^1(X) := \mathbb{B}^1\big(X \xrightarrow{\mathrm{id}} X\big)$, where

$$\mathbb{P}_1 := \mathbb{P}_X(\mathscr{L}_1 \oplus \mathcal{O})\,, \quad \mathbb{P}_2 := \mathbb{P}_X(\mathscr{L}_1 \oplus (\mathscr{L}_1 \otimes \mathscr{L}_2) \oplus \mathcal{O})\,,$$
$$\mathbb{P}_3 := \mathbb{P}_{\mathbb{P}_X(\mathscr{L}_1 \oplus (\mathscr{L}_1 \otimes \mathscr{L}_2))}(\mathcal{O}(-1) \oplus \mathcal{O})\,.$$

Lemma 4.2. *Suppose that $\mathbb{B}^*$ is a bivariant precobordism theory and $\mathscr{L}$ is a line bundle on a quasi-projective derived A-scheme X. Then the first Chern class $c_1(\mathscr{L}) \in \mathbb{B}^1(X)$ is nilpotent (with respect to $\bullet$).*

Proof. The proof splits to three parts.

(i) Suppose that $\mathscr{L}$ is globally generated. By the Noetherianity hypothesis on A, we can find finitely many global sections $s_1, \dots, s_n$ of $\mathscr{L}$ that generate it. These sections correspond to virtual Cartier divisors $D_1, \dots, D_n$ whose total derived intersection is empty. As $c_1(\mathscr{L}) = [D_i \to X] \in \mathbb{B}^1(X)$ for any i, see [LS16, proof of Lemma 3.9], we see that $c_1(\mathscr{L})^n = 0$.

(ii) Suppose that the dual bundle $\mathscr{L}^\vee$ is globally generated. We can now use (4.1) with $\mathscr{L}_1 = \mathscr{L}$ and $\mathscr{L}_2 = \mathscr{L}^\vee$ to conclude that

$$c_1(\mathscr{L}) + c_1\big(\mathscr{L}^\vee\big) - c_1(\mathscr{L}) \bullet c_1\big(\mathscr{L}^\vee\big) \bullet [\mathbb{P}_X(\mathscr{L} \oplus \mathcal{O}) \to X] = 0$$

(recall that $c_1(\mathcal{O}_X) = 0$). It follows that

$$c_1\big(\mathscr{L}^\vee\big) = -\frac{c_1(\mathscr{L})}{1 - c_1(\mathscr{L}) \bullet [\mathbb{P}_X(\mathscr{L} \oplus \mathcal{O}) \to X]}\,,$$

which is clearly nilpotent (and well defined) since $c_1(\mathscr{L})$ is nilpotent by the first part.

(iii) In general, as X is quasi-projective, any line bundle $\mathscr{L}$ is equivalent to $\mathscr{L}_1 \otimes \mathscr{L}_2^\vee$ with $\mathscr{L}_1$ and $\mathscr{L}_2$ globally generated. As both $c_1(\mathscr{L}_1)$ and $c_1\big(\mathscr{L}_2^\vee\big)$ are nilpotent, the nilpotency of $c_1(\mathscr{L})$ follows from the formula (4.1). ☐

Any theory satisfying the naive cobordism relations and analogues of the double point cobordism relations of [LP09] is a bivariant precobordism theory.

Proposition 4.3 (cf. [LP09, § 0.3]). *Suppose that $\mathbb{B}^*$ is a naive cobordism theory in the sense of Definition 4.1. Then $\mathbb{B}^*$ is a precobordism theory if and only if for any quasi-smooth $W \to \mathbb{P}^1 \times X$ with fibres W_0 over $\{0\} \times X$ and W_∞ over $\{\infty\} \times X$ such that W_∞ equivalent to the sum of divisors $A \to W$ and $B \to W$, the double point cobordism formula*

$$[W_0 \to X] = [A \to X] + [B \to X] - [\mathbb{P} \to X] \tag{4.2}$$

holds in $\mathbb{B}^1(X)$, where $\mathbb{P} := \mathbb{P}_Z(\mathcal{O}(A) \oplus \mathcal{O})$ with Z the derived intersection of A and B in W.

Proof. Suppose that $\mathscr{L}_1$ and $\mathscr{L}_2$ are line bundles on a quasi-projective derived A-scheme X, and let A, B and C be divisors in the linear systems of $\mathscr{L}_1$, $\mathscr{L}_2$ and $\mathscr{L}_1 \otimes \mathscr{L}_2$, respectively. Note that if $A, B, C = W_0$ are as in the statement, then (4.2) is a special case of (4.1) as by assumption $c_1(\mathscr{L}_2) \bullet c_1(\mathscr{L}_1 \otimes \mathscr{L}_2)$ vanishes (B and W_0 do not meet). Therefore, the only nontrivial part is to show that the double point cobordism formula implies (4.1).

Let X_1 be the derived blow-up of X at $A \cap C$, and let $A_1, C_1 \to X_1$ be the strict transforms of A and C, respectively. Note that A_1 and C_1 do not meet inside X_1; see [Ann21b, Lemma 4.5]. Let $B_1 \to X_1$ be the natural map

$$\mathrm{Bl}_{A \cap B \cap C}(B) \to \mathrm{Bl}_{A \cap C}(X)\,,$$

which fits by the contravariant naturality of blow-ups (see [KR19, Theorem 4.1.5(ii)]) in a derived Cartesian square

$$\begin{tikzcd} B_1 \arrow[r] \arrow[d] & X_1 \arrow[d] \\ B \arrow[r] & X \end{tikzcd} \tag{4.3}$$

and is therefore a virtual Cartier divisor. Note that the divisors $A_1 + B_1$ and C_1 are rationally equivalent inside X_1.

Next, blow up X_1 at the intersection $B_1 \cap C_1$ to obtain X_2 and the strict transformations $A_2, B_2, C_2 \to X_2$, which are naturally equivalent to A_1, B_1 and C_1, respectively. As C_2 does not meet either A_2 or B_2 inside X, and as $A_2 + B_2$ is rationally equivalent to C_2, we get two sections s_1 and s_2 of $\mathscr{L} := \mathcal{O}_{X_2}(A_2 + B_2) \simeq \mathcal{O}_{X_2}(C_2)$ generating $\mathscr{L}$. Together with the natural map $X_2 \to X$, this induces a morphism

$$\pi\colon X_2 \longrightarrow \mathbb{P}^1 \times X\,,$$

whose fibre over $\{0\} \times X$ is equivalent to C, and whose fibre over $\{\infty\} \times X$ is sum of the divisors $A_2 \simeq A$ and B_2 in X_2. As the intersection of A_2 and B_2 inside X_2 is equivalent to $A \cap B =: Z$, the double point cobordism formula (4.2) implies that

$$\begin{aligned} c_1(\mathscr{L}_1 \otimes \mathscr{L}_2) &= [C \to X] \\ &= [A \to X] + [B_2 \to X] - [\mathbb{P}_Z(\mathscr{L} \oplus \mathcal{O}) \to X] \\ &= c_1(\mathscr{L}_1) + c_1(\mathscr{L}_2) \bullet [X_1 \to X] \\ &\quad - c_1(\mathscr{L}_1) \bullet c_1(\mathscr{L}_2) \bullet [\mathbb{P}_X(\mathscr{L} \oplus \mathcal{O}) \to X] \quad (B_1 \text{ and } B_2 \text{ are equivalent, } (4.3))\,. \end{aligned} \tag{4.4}$$

By the blow-up relation, we have

$$\begin{aligned} [X_1 \to X] = [X \to X] &- c_1(\mathscr{L}_1) \bullet c_1(\mathscr{L}_1 \otimes \mathscr{L}_2) \bullet [\mathbb{P}(\mathscr{L}_1 \oplus (\mathscr{L}_1 \otimes \mathscr{L}_2) \oplus \mathcal{O})] \\ &+ c_1(\mathscr{L}_1) \bullet c_1(\mathscr{L}_1 \otimes \mathscr{L}_2) \bullet [\mathbb{P}_{\mathbb{P}(\mathscr{L}_1 \oplus (\mathscr{L}_1 \otimes \mathscr{L}_2))}(\mathcal{O}(-1) \otimes \mathcal{O}) \to X] \end{aligned} \tag{4.5}$$

(see [LP09, Lemma 5.1], the proof in the derived case goes through word-for-word). Combining (4.4) with (4.5), we get (4.1), finishing the proof. □

Example 4.4. The bivariant algebraic cobordism Ω^* constructed in [Ann21b] over a field k of characteristic 0 is a precobordism theory in the above sense. Indeed, Ω^* is a naive cobordism theory by construction, and the double point cobordism relation holds for $\Omega^* = \Omega^{*,0}$ by Lemma 3.14.

Construction 4.5 (Universal precobordism theory $\underline{\Omega}^*$). It is clear that there is a universal precobordism theory $\underline{\Omega}^*$ over A constructed from $\mathcal{M}^*_+$ by enforcing the naive cobordism relation and either of the formulas (4.1) or (4.2) for all line bundles satisfying restrictions of each formula (compare to the construction of Section 3.1). In fact, it is enough to enforce only the double point cobordism relations (4.2) since the naive cobordism relation is a special case of it. Clearly, any other precobordism theory is a quotient of $\underline{\Omega}^*$.

Next, we are going to construct the associated bivariant theory with line bundles. Let $\mathcal{M}^{*,1}_+$ be as in Section 3.1; that is, $\mathcal{M}^{d,1}_+\big(X \xrightarrow{f} Y\big)$ is the free Abelian group on equivalence classes

$$\big[V \xrightarrow{h} X, \mathscr{L}\big]\,,$$

where $h\colon V \to X$ is projective, the composition $f \circ h\colon V \to Y$ is quasi-smooth of virtual relative dimension $-d$, and $\mathscr{L}$ is a line bundle on X. The bivariant product $\bullet = \bullet_\otimes$ makes $\mathcal{M}^{*,1}_+$ into a bivariant theory. Note that we can identify $\mathcal{M}_+^*$ with the subtheory of $\mathcal{M}_+^{*,1}$ consisting of cycles where the line bundle $\mathscr{L}$ is trivial.

Definition 4.6 (Bivariant precobordism with line bundles $\mathbb{B}^{*,1}$). Let $\mathbb{B}^* = \mathcal{M}_+^*/\mathbb{I}$ be a precobordism theory. We define the associated *precobordism with line bundles* $\mathbb{B}^{*,1}$ as

$$\mathbb{B}^{*,1} := \mathcal{M}_+^{*,1}/\langle \mathbb{I} \rangle_{\mathcal{M}_+^{*,1}} .$$

Remark 4.7 (Double point cobordism with a line bundle). Let us record the following observation here, which will be the basis for most of our arguments. Suppose that we have a morphism $X \to Y$ of derived schemes. Given a projective map $W \to \mathbb{P}^1 \times X$ such that the composition $W \to \mathbb{P}^1 \times Y$ is quasi-smooth and a line bundle $\mathscr{L}$ on W, let W_0 and W_∞ be the fibres over 0 and ∞, respectively. Suppose that $W_\infty \hookrightarrow W$ is the sum of two divisors A and B in W. Then

$$[W_0 \to X, \mathscr{L}|_{W_0}] = [A \to X, \mathscr{L}|_A] + [B \to X, \mathscr{L}|_B] - [\mathbb{P} \to X, \mathscr{L}|_{A\cap B}] \in \mathbb{B}^{*,1}(X \to Y) ,$$

where $\mathbb{P}$ is defined as in Proposition 4.3. Similarly, if $\mathscr{L}$ is a globally generated line bundle on W, such relations are called *double point cobordisms with globally generated line bundles*, and they will play a role in Section 4.3.

Let us record the following convenient characterization for the elements of the ideal $\langle \mathbb{I} \rangle_{\mathcal{M}_+^{*,1}}$.

Proposition 4.8. *Given a line bundle $\mathscr{L}$ on X, denote by $[\mathscr{L}]$ the element*

$$\big[X \xrightarrow{\mathrm{id}} X, \mathscr{L}\big] \in \mathcal{M}_+^{0,1}(X) .$$

The bivariant ideal $\langle \mathbb{I} \rangle_{\mathcal{M}_+^{*,1}}$ as in Definition 4.6 consists of linear combinations of elements of the form $f_*([\mathscr{L}] \bullet \alpha)$, where $\alpha \in \mathbb{I}(X \to Y)$, and the map $X \to Y$ factors through a projective morphism $f\colon X \to X'$.

Proof. The proof is the same as that of Proposition 3.9, which does not use any assumptions on the base ring. □

Proposition 4.8 has several immediate consequences.

Definition 4.9. By Proposition 4.8, the forgetful Grothendieck transformation $\mathscr{F}\colon \mathcal{M}_+^{*,1} \to \mathcal{M}_+^*$ induces a well-defined *forgetful transformation* $\mathscr{F}\colon \mathbb{B}^{*,1} \to \mathbb{B}^*$ for any precobordism theory $\mathbb{B}^*$.

Proposition 4.10 (cf. Proposition 3.17). *The differential operator $\partial_{c_1}\colon \mathcal{M}_+^{*,1} \to \mathbb{B}^{*+1,1}$ defined by the formula*

$$\big[V \xrightarrow{f} X, \mathscr{L}\big] \longmapsto f_*(c_1(\mathscr{L}) \bullet [V \to V])$$

induces a well-defined operator

$$\partial_{c_1}\colon \mathbb{B}^{*,1} \longrightarrow \mathbb{B}^{*+1,1} .$$

Proof. The well-definedness follows as in the proof of Proposition 3.17. □

Example 4.11. The associated precobordism with line bundles $\underline{\Omega}^{*,1}$ for the universal precobordism $\underline{\Omega}^*$ is obtained by enforcing the double point cobordism relations with line bundles (Remark 4.7) to $\mathcal{M}_+^{*,1}$.

Example 4.12. The associated precobordism with line bundles for the bivariant algebraic cobordism Ω^* of [Ann21b] is $\big(\Omega^{*,1}, \bullet_\otimes\big)$ as constructed in Section 3.1. This follows from Proposition

4.8 above and from Proposition 3.9, after noting that an element of the form $[\mathscr{L}] \bullet_{\oplus} \alpha$, with $\alpha \in \mathcal{M}^*_+$, where $\mathcal{M}^*_+$ is identified with $\mathcal{M}^{*,0}_+$, is the same as the element of the form $[\mathscr{L}] \bullet_{\otimes} \alpha$, where $\mathcal{M}^*_+$ is now identified with the subtheory of $\mathcal{M}^{*,1}_+$ consisting of cycles with trivial line bundles.

4.2 Structure of $\mathbb{B}^{*,1}$

The purpose of this section is to prove the following two theorems.

Theorem 4.13. *Let $\mathbb{B}^*$ be a precobordism theory over a Noetherian base ring A of finite Krull dimension. Then the cobordism group $\mathbb{B}^{*,1}(\mathrm{pt})$ admits a $\mathbb{B}^*(\mathrm{pt})$-linear basis $([\mathbb{P}^i \to \mathrm{pt}, \mathcal{O}(1)])_{i=0}^{\infty}$.*

The following natural map is similar to that in Lemma 3.19.

Theorem 4.14. *Let $\mathbb{B}^*$ be a precobordism theory over a Noetherian base ring A of finite Krull dimension. The natural (cross product) map*

$$\mathbb{B}^{*,1}(\mathrm{pt} \to \mathrm{pt}) \otimes_{\mathbb{B}^*(\mathrm{pt})} \mathbb{B}^*(X \to Y) \longrightarrow \mathbb{B}^{*,1}(X \to Y)$$

defined by

$$([V \to \mathrm{pt}, \mathscr{L}], [W \to X]) \longmapsto [V \to \mathrm{pt}, \mathscr{L}] \times [W \to X] = [V \times W \to X, \mathscr{L}'],$$

where $\mathscr{L}'$ is the pullback of $\mathscr{L}$ to $V \times W$, is an isomorphism.

For the rest of the subsection, $\mathbb{B}^*$ will be a fixed precobordism theory (for example, bivariant algebraic cobordism Ω^*), and $\mathbb{B}^{*,1}$ will denote the associated precobordism of line bundles defined in the previous subsection.

Our strategy is to first prove the surjectivity part of Theorem 4.14 (see Proposition 4.20), which is done by explicitly constructing algebraic cobordisms realizing desirable relations (see Lemma 4.19). The rest of Theorems 4.14 and 4.13 then follows with a relatively little amount of effort. Until the end of this subsection, X will be a quasi-projective derived scheme, $\mathscr{L}$ a line bundle on X and $D \hookrightarrow X$ a virtual Cartier divisor in the linear system of $\mathscr{L}$. Note that at least one such D always exists, as the derived vanishing locus of the zero section is a virtual Cartier divisor.

We begin with the following construction.

Construction 4.15. Let X, $\mathscr{L}$ and D be as above. Let $W = W(X, \mathscr{L}, D)$ be the blow-up $\mathrm{Bl}_{\infty \times D}\left(\mathbb{P}^1 \times X\right)$, and let $\widetilde{\mathscr{L}}$ be the line bundle $\mathscr{L}(-E)$ on W, where E denotes the exceptional divisor of the blow-up. The pair $\left(W \to \mathbb{P}^1, \widetilde{\mathscr{L}}\right)$ satisfies the following properties:

(i) The fibre of W over 0 is equivalent to X, and the restriction of $\widetilde{\mathscr{L}}$ to the fibre is equivalent to $\mathscr{L}$.

(ii) The exceptional divisor E is equivalent to $\mathbb{P}_D(\mathscr{L} \oplus \mathcal{O}_D)$, and $\widetilde{\mathscr{L}}|_E$ is equivalent to $\mathscr{L}(1)$. This is true because the conormal bundle of E inside the blow-up, which can be identified with $\mathcal{O}(-E)|_E$, is equivalent to $\mathcal{O}(1)$; see [KR19, § 4.3.4].

(iii) The strict transform $\widetilde{\infty \times X}$ of $\infty \times X$ is equivalent to X, and the restriction of $\widetilde{\mathscr{L}}$ to it is the trivial line bundle $\mathcal{O}_X$. Indeed, the restriction of the divisor E to the strict transform is D, and therefore the line bundle $\mathcal{O}_W(E)$ restricts to $\mathscr{L}$. It follows that $\widetilde{\mathscr{L}} = \mathscr{L}(-E)$ restricts to $\mathcal{O}_X$.

We note that W from the above construction can be understood as an algebraic cobordism over X, witnessing the equivalence between the fibres over $0 \times X$ and $\infty \times X$. Next, we are going to build a tower of projective bundles on W.

Construction 4.16. Let X, $\mathscr{L}$ and D be as in Construction 4.15. Let us moreover denote by $\big(W_0, \widetilde{\mathscr{L}_0}\big)$ the pair $\big(W, \widetilde{\mathscr{L}}\big)$ in Construction 4.15.

We define $\big(W_{i+1}, \widetilde{\mathscr{L}_{i+1}}\big)$ recursively as

$$W_{i+1} := \mathbb{P}_{W_i}\big(\widetilde{\mathscr{L}_i} \oplus \mathcal{O}_{W_i}\big) \quad \text{and} \quad \widetilde{\mathscr{L}_{i+1}} := \widetilde{\mathscr{L}_i}(1)\,.$$

Moreover, we shall denote by π_i the composition of the natural maps

$$W_i \longrightarrow W_{i-1} \longrightarrow \cdots \longrightarrow W \longrightarrow \mathbb{P}^1 \times X$$

and by π_i' the composition

$$W_i \longrightarrow W_{i-1} \longrightarrow \cdots \longrightarrow W\,.$$

We record the following immediate observations.

Lemma 4.17. *Let X, $\mathscr{L}$ and D be as above, and define pairs $(T_i = T_i(X,\mathscr{L}),\ \mathscr{L}_i = \mathscr{L}_i(X,\mathscr{L}))$ recursively by*

$$(T_0, \mathscr{L}_0) := (X, \mathscr{L}) \quad \textit{and} \quad (T_{i+1}, \mathscr{L}_{i+1}) := (\mathbb{P}_{T_i}(\mathscr{L}_i \oplus \mathcal{O}_{T_i}), \mathscr{L}_i(1))\,.$$

Let π_i, π_i' be as in Construction 4.16. Then

(i) *the fibre of π_i over $0 \times X$ is equivalent to T_i, and the restriction of $\widetilde{\mathscr{L}_i}$ is equivalent to $\mathscr{L}_i$;*

(ii) *the fibre of π_i' over the exceptional divisor E is equivalent to the fibre product $D \times_X T_{i+1}$, and the restriction of the line bundle $\widetilde{\mathscr{L}_i}$ to it is equivalent to $\mathscr{L}_{i+1}$.*

Proof. Both claims are true by inspection by focusing on the fibres of interest in Construction 4.16. □

Lemma 4.18. *Let us define recursively*

$$(P_0, \mathscr{M}_0) := (\mathrm{pt}, \mathcal{O}_{\mathrm{pt}}) \quad \textit{and} \quad (P_{i+1}, \mathscr{M}_{i+1}) := (\mathbb{P}_{P_i}(\mathscr{M}_i \oplus \mathcal{O}_{P_i}), \mathscr{M}_i(1))\,,$$

and let X, $\mathscr{L}$, W_i, π_i' be as in Construction 4.16. Then

(i) *the fibre of $\pi_i'\colon W_i \to W_0 = W$ over $\widetilde{\infty \times X}$ is equivalent to $P_i \times X$, and the restriction of the line bundle $\widetilde{\mathscr{L}_i}$ to it is equivalent to $\mathscr{M}_i = \mathscr{M}_i \boxtimes \mathcal{O}_X$;*

(ii) *the fibre of $\pi_i'\colon W_i \to W_0$ over the intersection of the exceptional divisor E and the strict transform $\widetilde{\infty \times X}$ is equivalent to $P_i \times D$, and the restriction of the line bundle $\widetilde{\mathscr{L}_i}$ to it is equivalent to $\mathscr{M}_i = \mathscr{M}_i \boxtimes \mathcal{O}_D$.*

Proof. Again, this is clear by construction by focusing on the fibre of interest. Moreover, the second claim follows trivially from the first. □

We can now use Lemmas 4.17 and 4.18 to prove the following.

Lemma 4.19. *Let X, $\mathscr{L}$, W_i and $\widetilde{\mathscr{L}_i}$ be as in Construction 4.16, T_i and $\mathscr{L}_i$ as in Lemma 4.17 and P_i, $\mathscr{M}_i$ as in Lemma 4.18. Then $\big(\pi_i\colon W_i \to \mathbb{P}^1 \times X, \widetilde{\mathscr{L}_i}\big)$ realizes the relation*

$$\begin{aligned}[][T_i \to X, \mathscr{L}_i] = [P_i \to \mathrm{pt}, \mathscr{M}_i] \times 1_X + c_1(\mathscr{L}) \bullet [T_{i+1} \to X, \mathscr{L}_{i+1}]\\ - c_1(\mathscr{L}) \bullet [P_i \to \mathrm{pt}, \mathscr{M}_i] \times [\mathbb{P}_X(\mathscr{L} \oplus \mathcal{O}_X) \to X, \mathcal{O}_X]\end{aligned}$$

in $\mathbb{B}^{*,1}(X)$. Above, $[P_i \to \mathrm{pt}, \mathscr{M}_i] \in \mathbb{B}^{*,1}(\mathrm{pt})$.

Proof. The proof is just an easy application of the double point cobordism formula (Proposition 4.3) to the map $\pi_i \colon W_i \to \mathbb{P}^1 \times X$ together with the help of Lemmas 4.17 and 4.18. For the convenience of the reader, we are going to give detailed explanations for where the various terms come from. We first note that the left-hand side of the equation comes from the fibre of π_i over $0 \times X$ by Lemma 4.17(i).

The fibre of π_i over $\infty \times X$ is a sum of two divisors: the pre-images of the exceptional divisor E and the strict transform of $\infty \times X$ under the map $\pi_i' \colon W_i \to W_0 = W$. It is to these that we are going to apply Lemma 3.14. The first term on the right-hand side comes from the divisor over $\infty \times X$ by Lemma 4.18(i) as

$$[P_i \to \mathrm{pt}, \mathscr{M}_i] \times 1_X = [P_i \times X \to X, \mathscr{M}_i \boxtimes \mathcal{O}_X] \in \mathbb{B}^{*,1}(X)\,.$$

The second term on the right-hand side corresponds by Lemma 4.17(ii) to the divisor lying over E as

$$c_1(\mathscr{L}) = [D \to X, \mathcal{O}_D]$$

and as the bivariant product is given by the derived fibre product over X.

Finally, we need to identify the third term on the right-hand side as the third term in the double point cobordism. First note that the intersection of the two divisors of last paragraph in W_i is equivalent to $P_i \times D$ by Lemma 4.18(ii). Moreover, as the inclusion $P_i \times D \hookrightarrow P_i \times X$ is merely the pullback of $D \to X$ along the projection, the corresponding normal bundle is just $\mathcal{O}_{P_i} \boxtimes \mathscr{L}|_D$. Therefore, the projective bundle over the intersection (as in Lemma 3.14) is equivalent to $P_i \times \mathbb{P}_D(\mathscr{L} \oplus \mathcal{O}_D)$, and the corresponding line bundle over it is just $\mathscr{M}_i \boxtimes \mathcal{O}_{\mathbb{P}_D(\mathscr{L} \oplus \mathcal{O}_D)}$. That this corresponds to the final term of the equation now follows by recalling the definition of the bivariant exterior product $\times$ and the first Chern class $c_1(\mathscr{L})$ as in the previous paragraph. ☐

Using the previous lemma, it is now easy to prove the desired surjectivity.

Proposition 4.20. *The natural map*

$$\mathbb{B}^{*,1}(\mathrm{pt} \to \mathrm{pt}) \otimes_{\mathbb{B}^*(\mathrm{pt})} \mathbb{B}^*(X \to Y) \longrightarrow \mathbb{B}^{*,1}(X \to Y)$$

is a surjection.

Proof. As the theory is generated under bivariant operations by elements of the form

$$[V \to V, \mathscr{L}] \in \mathbb{B}^*(V)\,,$$

it is enough to show that such an element lies in the image of the map. This follows from Lemma 4.19: modulo the image, we have

$$\begin{aligned} [V \to V, \mathscr{L}] &= [T_0 \to V, \mathscr{L}_0] \\ &\equiv c_1(\mathscr{L}) \bullet [T_1 \to V, \mathscr{L}_1] \\ &\equiv c_1(\mathscr{L})^2 \bullet [T_2 \to V, \mathscr{L}_2] \\ &\;\;\vdots \end{aligned}$$

and as the first Chern class $c_1(\mathscr{L})$ is nilpotent (by the definition of a precobordism theory), we see that $[V \to V, \mathscr{L}]$ lies in the image. ☐

Next, we are going to prove Theorem 4.13. We will need the following lemmas.

Lemma 4.21. *Let $\pi\colon \mathbb{P}^n \to \mathrm{pt}$ be the structure map, and let $T_i(\mathbb{P}^n) = T_i(\mathbb{P}^n, \mathcal{O}(1))$ and $\mathscr{L}_i(\mathbb{P}^n) = \mathscr{L}_i(\mathbb{P}^n, \mathcal{O}(1))$ be defined as in Lemma 4.17. Then*

$$\pi_*\big(c_1(\mathcal{O}(1))^n \bullet [T_n(\mathbb{P}^n) \to \mathbb{P}^n, \mathscr{L}_n(\mathbb{P}^n)]\big) = [P_n \to \mathrm{pt}, \mathscr{M}_n]\,,$$

where P_n and $\mathscr{M}_n$ are defined as in Lemma 4.18.

Proof. Recall that $c_1(\mathcal{O}(1))^n \in \mathbb{B}^{*,1}(X)$ is represented by the cycle associated with a morphism $\mathrm{pt} \to \mathbb{P}^n$ such that $\mathcal{O}(1)|_{\mathrm{pt}}$ is trivial. Moreover, it is clear that the fibre over pt of any $(T_i(\mathbb{P}^n), \mathscr{L}_i(\mathbb{P}^n))$ is $(P_i, \mathscr{M}_i)$, from which the claim follows in the case $i = n$. □

Lemma 4.22. *The group $\mathbb{B}^{*,1}(\mathrm{pt})$ is generated as a $\mathbb{B}^*(\mathrm{pt})$-module by the elements of the form $[\mathbb{P}^i \to \mathrm{pt}, \mathcal{O}(1)]$, where $i = 0, 1, 2, \ldots$.*

Proof. Suppose $[V \to \mathrm{pt}, \mathscr{L}] \in \mathbb{B}^{*,1}(\mathrm{pt})$. By applying Lemma 4.19 as in the proof of Proposition 4.20 to

$$[V \to V, \mathscr{L}] \in \mathbb{B}^{*,1}(V)\,,$$

and then applying the Gysin pushforward morphism $\mathbb{B}^{*,1}(V) \to \mathbb{B}^{*,1}(\mathrm{pt})$, we see that it is enough to show that $[P_i \to \mathrm{pt}, \mathscr{M}_i]$ is expressible in the desired form.

We will proceed by induction on i, the case $i = 0$ being trivial. Suppose that the claim is known to hold up to i. Consider the elements in the $\mathbb{B}^*\big(\mathbb{P}^{i+1}\big)$-linear span of

$$[P_0 \to \mathrm{pt}, \mathscr{M}_0] \times 1_{\mathbb{P}^{i+1}}, [P_1 \to \mathrm{pt}, \mathscr{M}_1] \times 1_{\mathbb{P}^{i+1}}, \ldots, [P_i \to \mathrm{pt}, \mathscr{M}_i] \times 1_{\mathbb{P}^{i+1}}$$

in $\mathbb{B}^{*,1}\big(\mathbb{P}^{i+1}\big)$, which we are going to call *easy elements* for the rest of the proof. Note that any easy element pushes forward to an element expressible in the desired form by the inductive assumption.

Next, we will apply Lemma 4.19 to

$$\big[\mathbb{P}^{i+1} \to \mathbb{P}^{i+1}, \mathcal{O}(1)\big] \in \mathbb{B}^{*,1}\big(\mathbb{P}^{i+1}\big)\,.$$

It follows that, modulo easy elements, the above element is equivalent to

$$\begin{aligned}
\big[\mathbb{P}^{i+1} \to \mathbb{P}^{i+1}, \mathcal{O}(1)\big] &= \big[T_0\big(\mathbb{P}^{i+1}\big) \to \mathbb{P}^{i+1}, \mathscr{L}_0\big(\mathbb{P}^{i+1}\big)\big]\\
&\equiv c_1(\mathcal{O}(1)) \bullet \big[T_1\big(\mathbb{P}^{i+1}\big) \to \mathbb{P}^{i+1}, \mathscr{L}_1\big(\mathbb{P}^{i+1}\big)\big]\\
&\ \ \vdots\\
&\equiv c_1(\mathcal{O}(1))^{i+1} \bullet \big[T_{i+1}\big(\mathbb{P}^{i+1}\big) \to \mathbb{P}^{i+1}, \mathscr{L}_{i+1}\big(\mathbb{P}^{i+1}\big)\big]
\end{aligned}$$

in the notation of Lemma 4.21. Note that the first element is pushed forward to $\big[\mathbb{P}^{i+1} \to \mathrm{pt}, \mathcal{O}(1)\big]$, and, by Lemma 4.21, the last element above is pushed forward to $[P_{i+1} \to \mathrm{pt}, \mathscr{M}_{i+1}] \in \mathbb{B}^{*,1}(\mathrm{pt})$. Hence

$$[P_{i+1} \to \mathrm{pt}, \mathscr{M}_{i+1}] = \big[\mathbb{P}^{i+1} \to \mathrm{pt}, \mathcal{O}(1)\big] + \alpha_i \bullet \big[\mathbb{P}^i \to \mathrm{pt}, \mathcal{O}(1)\big] + \cdots + \alpha_0 \bullet 1 \in \mathbb{B}^{*,1}(\mathrm{pt})\,,$$

where $\alpha_i \in \mathbb{B}^*(\mathrm{pt})$, proving the claim. Above, $\mathbb{B}^*(\mathrm{pt})$ is identified with $\mathbb{B}^{*,0}(\mathrm{pt})$ when we consider the above bivariant product $\alpha_i \bullet \big[\mathbb{P}^i \to \mathrm{pt}, \mathcal{O}(1)\big]$ for each i. □

We are now ready to prove the main theorems.

Proof of Theorem 4.13. By Lemma 4.22, the group $\mathbb{B}^{*,1}(\mathrm{pt})$ is generated as a $\mathbb{B}^*(\mathrm{pt})$-module by the $\beta_i := \big[\mathbb{P}^i \to \mathrm{pt}, \mathcal{O}(1)\big]$; we need to verify their linear independence. If $\alpha_0, \ldots, \alpha_n \in \mathbb{B}^*(\mathrm{pt})$ and $\alpha_n \neq 0$, then $\mathscr{F}\big(\partial_{c_1}^n\big(\alpha_0 + \alpha_1 \bullet \beta_1 + \cdots + \alpha_n \bullet \beta_n\big)\big) = \alpha_n$, proving the claim. □

Proof of Theorem 4.14. The idea is essentially the same as above. □

4.3 Precobordism of trivial projective bundles

Here, we study the structure of the groups $\mathbb{B}^*(\mathbb{P}^n \times X \to Y)$, where $\mathbb{B}^*$ is a precobordism theory of quasi-projective derived schemes over a Noetherian ring A. The morphisms

$$c_1\big(\mathrm{pr}_1^*\mathcal{O}(1)\big)^i \bullet \theta(\mathrm{pr}_2)\bullet \colon \mathbb{B}^*(X \to Y) \longrightarrow \mathbb{B}^{*-n+i}(\mathbb{P}^n \times X \to Y)$$

for $i = 0, \dots, n$ give rise to a morphism

$$\mathbb{P}\colon \bigoplus_{i=0}^{n} \mathbb{B}^{*+n-i}(X \to Y) \longrightarrow \mathbb{B}^*(\mathbb{P}^n \times X \to Y)\,.$$

The main theorem of the subsection is the following.

Theorem 4.23 (Weak projective bundle formula). *Let $\mathbb{B}^*$ be a precobordism theory over a Noetherian base ring A of finite Krull dimension. Then the above map*

$$\mathscr{P}\mathrm{roj}\colon \bigoplus_{i=0}^{n} \mathbb{B}^{*+n-i}(X \to Y) \to \mathbb{B}^*(\mathbb{P}^n \times X \to Y)$$

is an isomorphism of $\mathbb{B}^*(\mathrm{pt})$-modules.

Before embarking on the proof, let us list some consequences. Using Theorem 4.23 and the fact that $\mathbb{B}^*$ has strong orientations along smooth morphisms (cf. Proposition 3.10), we have the following easy corollary.

Corollary 4.24. *Let $\mathbb{B}^*$ be a precobordism theory over a Noetherian base ring A of finite Krull dimension. Then we have natural isomorphisms of rings*

$$\mathbb{B}^*(\mathbb{P}^n \times X) \cong \mathbb{B}^*(X)[t]/\big(t^{n+1}\big)\,,$$

where $t := c_1\big(\mathrm{pr}_1{}^*\,\mathcal{O}(1)\big) \in \mathbb{B}^1(\mathbb{P}^n \times X)$ is the pullback of the class $c_1(\mathcal{O}(1)) \in \mathbb{B}^1(\mathbb{P}^n)$ of a hyperplane.

Proof. Consider the derived Cartesian square

$$\begin{array}{ccc} \mathbb{P}^n \times X & \xrightarrow{\mathrm{id}} & \mathbb{P}^n \times X \\ \big\downarrow{\scriptstyle \mathrm{pr}_2} & & \big\downarrow{\scriptstyle \mathrm{pr}_2} \\ X & \xrightarrow{\mathrm{id}} & X\,. \end{array}$$

The commutativity of the bivariant theory $\mathbb{B}^*$ implies that for all $\alpha \in \mathbb{B}^*(X)$, we have $\theta(\mathrm{pr}_2) \bullet \alpha = \mathrm{pr}_2^*(\alpha) \bullet \theta(\mathrm{pr}_2)$. This observation, together with Theorem 4.23, implies that the morphism $\bigoplus_{i=0}^n \mathbb{B}^{*-i}(X) \to \mathbb{B}^*(\mathbb{P}^n \times X)$ defined using the maps $t^i \bullet \mathrm{pr}_2^*\colon \mathbb{B}^*(X) \to \mathbb{B}^{*+i}(\mathbb{P}^n \times X)$ for $i = 0, \dots, n$ is an isomorphism of $\mathbb{B}^*(\mathrm{pt})$-modules. Since $t^{n+1} = 0$, the claim follows. $\square$

We can use Corollary 4.24 to show that the first Chern classes of line bundles are controlled by a formal group law. Indeed, consider the varieties $\mathbb{P}^n \times \mathbb{P}^m$ for various n and m, and consider the class $c_1(\mathcal{O}(1,1)) \in \mathbb{B}^*(\mathbb{P}^n \times \mathbb{P}^m) \cong \mathbb{B}^*(\mathrm{pt})[x,y]/\big(x^{n+1}, y^{m+1}\big)$. The class is uniquely expressible as a sum

$$c_1(\mathcal{O}(1,1)) = \sum_{i,j} a_{i,j}^{n,m} x^i y^j\,,$$

where $a_{i,j}^{n,m} \in \mathbb{B}^*(\mathrm{pt})$, and moreover, by naturality of Chern classes in pullbacks, $a_{i,j}^{n,m}$ does not depend on n and m as long as $i \leqslant n$ and $j \leqslant m$ (denote this by a_{ij}). We then have the following standard result.

Theorem 4.25. *Let $\mathbb{B}^*$ be a precobordism theory over a Noetherian base ring A of finite Krull dimension. Then the series*

$$F_{\mathbb{B}^*}(x,y) := \sum_{i,j} a_{ij} x^i y^j \in \mathbb{B}^*(\mathrm{pt})[[x,y]] \tag{4.6}$$

is a formal group law. Moreover, given a quasi-projective derived A-scheme X, we have

$$c_1(\mathscr{L}_1 \otimes \mathscr{L}_2) = F_{\mathbb{B}^*}(c_1(\mathscr{L}_1), c_1(\mathscr{L}_2)) \in \mathbb{B}^*(X) \tag{4.7}$$

for globally generated line bundles $\mathscr{L}_1$ and $\mathscr{L}_2$ on X.

As an immediate corollary, we get the following.

Corollary 4.26. *Let $\mathbb{B}^*$ be a precobordism theory over a Noetherian base ring A of finite Krull dimension. Then the formal group law of Theorem 4.25 induces a homomorphism of rings $\mathbb{L} \to \mathbb{B}^*(\mathrm{pt})$, where $\mathbb{L}$ is the Lazard ring.*

Proof of Theorem 4.23

We start by verifying the injectivity of $\mathscr{P}\mathrm{roj}$.

Lemma 4.27. *The map*

$$\mathscr{P}\mathrm{roj}\colon \bigoplus_{i=0}^{n} \mathbb{B}^{*+n-i}(X \to Y) \longrightarrow \mathbb{B}^*(\mathbb{P}^n \times X \to Y)$$

is injective.

Proof. Let $\alpha_i, \ldots, \alpha_n \in \mathbb{B}^*(X \to Y)$, $\alpha_i \neq 0$. Then

$$\mathrm{pr}_{2*}\big(c_1(\mathrm{pr}_1^*\mathcal{O}(1))^{n-i} \bullet \mathscr{P}\mathrm{roj}(0,\ldots,0,\alpha_i,\ldots,\alpha_n)\big) = \alpha_i\,,$$

proving that $\mathscr{P}\mathrm{roj}$ has trivial kernel. □

To investigate surjectivity, we make the following definition.

Definition 4.28. Let $X \to Y$ be a map of derived schemes. Define

$$\mathbb{B}^*\big(\mathbb{P}^\infty \times X \to Y\big) := \mathrm{colim}_n \mathbb{B}^*\big(\mathbb{P}^n \times X \to Y\big)\,,$$

where the structure morphisms are given by bivariant pushforwards

$$\mathbb{B}^*\big(X \xrightarrow{f} Y\big) \xrightarrow{\iota_{0*}} \mathbb{B}^*\big(\mathbb{P}^1 \times X \xrightarrow{f\circ \mathrm{pr}_2} Y\big) \xrightarrow{\iota_{1*}} \mathbb{B}^*\big(\mathbb{P}^2 \times X \xrightarrow{f\circ \mathrm{pr}_2} Y\big) \xrightarrow{\iota_{2*}} \mathbb{B}^*\big(\mathbb{P}^3 \times X \xrightarrow{f\circ \mathrm{pr}_2} Y\big) \longrightarrow \cdots$$

along the sequence $\mathrm{pt} = \mathbb{P}^0 \xrightarrow{\iota_0} \mathbb{P}^1 \xrightarrow{\iota_1} \mathbb{P}^2 \xrightarrow{\iota_2} \mathbb{P}^3 \xrightarrow{\iota_3} \cdots$ of standard linear embeddings (notice the slight abuse of notation above: we have denoted $(\iota_k \times \mathrm{id}_X)_*$ simply by ι_{k*}). Above, $\mathrm{pr}_2\colon \mathbb{P}^k \times X \to X$ for $k = 0,1,2,\ldots$ is the projection to the second factor.

The end result of applying the above procedure to the universal precobordism theory $\underline{\Omega}^*$ can be described in very explicit terms.

Lemma 4.29. *The group $\underline{\Omega}^*(\mathbb{P}^\infty \times X \to Y)$ is isomorphic to the group generated by cobordism cycles $[V \to X; \mathscr{L}]$, where $V \to X$ is a projective morphism such that the composition $V \to Y$ is quasi-smooth and $\mathscr{L}$ is a globally generated line bundle on V, modulo the double point cobordism relations with globally generated line bundles (Remark 4.7).*

Proof. A map $V \to \mathbb{P}^n$ is equivalent to the data of a line bundle $\mathscr{L}$ on V and a sequence $s_1, \dots, s_n$ of global sections of $\mathscr{L}$ generating it. However, the choice of the sequence of generating global sections does not have an effect on the class in $\underline{\Omega}^*(\mathbb{P}^\infty \times X \to Y)$ because the sections $x_0 s_1, \dots, x_0 s_n, x_1 s'_1, \dots, x_1 s'_m$ of $\mathscr{L}(1)$ on $\mathbb{P}^1 \times X$ provide a naive cobordism between the two maps $V \to \mathbb{P}^N \times X$ for $N \gg 0$. The rest of the claim follows by interpreting the double point cobordism relations in terms of globally generated line bundles. $\square$

Hence $\underline{\Omega}^*(\mathbb{P}^\infty \times X \to Y)$ can be described as the cobordism group of schemes equipped with a globally generated line bundle. This group can be analyzed by the arguments of Section 4.2, leading to the following result.

Lemma 4.30. *The map*

$$\bigoplus_{i=0}^{\infty} \underline{\Omega}^*(X \to Y) \longrightarrow \underline{\Omega}^*\big(\mathbb{P}^\infty \times X \to Y\big),$$

where the ith map is given by $[V \to X] \mapsto \big[\mathbb{P}^i \times V \to X, \mathcal{O}(1)\big]$, is an isomorphism.

Proof. The proof is essentially the same as the proofs of Theorems 4.13 and 4.14 with the following modification: the line bundle $\widetilde{\mathscr{L}}$ of Construction 4.15 should be replaced by $\mathscr{L}(1-E)$, which is globally generated whenever $\mathscr{L}$ is (see Lemma 4.31 below). Note that if a line bundle $\mathscr{L}$ is globally generated, then so is $\mathscr{L}(1)$ of $\mathbb{P}(\mathscr{L} \oplus \mathcal{O})$. Indeed, by definition, there exists a surjection $\mathscr{L}^\vee \oplus \mathcal{O} \to \mathcal{O}(1)$, which twists to a surjection $\mathcal{O} \oplus \mathscr{L} \to \mathscr{L}(1)$ from a globally generated bundle. $\square$

In the process of proving the above result, we used the following lemma.

Lemma 4.31. *Let $\mathscr{L}$ be a globally generated line bundle on X, let $D \hookrightarrow X$ be a virtual Cartier divisor in the linear system of $\mathscr{L}$, and let W be the blow-up $\mathrm{Bl}_{\infty \times D}\big(\mathbb{P}^1 \times X\big)$. Then the line bundle $\mathscr{L}(1-E)$ is globally generated on W, where E is the exceptional divisor.*

Proof. By [Ann22, Proposition 2.7], there exists a canonical surjection $\mathcal{O}(-1) \oplus \mathscr{L}^\vee \to \mathcal{O}(-E)$ on W. By twisting, we obtain a surjection $\mathscr{L} \oplus \mathcal{O}(1) \to \mathscr{L}(1-E)$, which proves that $\mathscr{L}(1-E)$ is globally generated as a quotient of a globally generated vector bundle. $\square$

To deduce Theorem 4.23 from the above results, we need to know that $\mathbb{B}^*(\mathbb{P}^n \times X \to Y)$ is a subgroup of $\mathbb{B}^*(\mathbb{P}^\infty \times X \to Y)$. This is established by the following result.

Lemma 4.32. *Let $i\colon \mathbb{P}^n \to \mathbb{P}^{n+1}$ be a linear embedding. Then the induced pushforward map*

$$i_*\colon \mathbb{B}^*\big(\mathbb{P}^n \times X \to Y\big) \longrightarrow \mathbb{B}^*\big(\mathbb{P}^{n+1} \times X \to Y\big)$$

is injective.

Proof. Let $p\colon P \to \mathbb{P}^{n+1}$ be the blow-up of $\mathbb{P}^{n+1}$ at a point not meeting the image of i, and consider the Cartesian diagram

$$\begin{array}{ccc} \mathbb{P}^n \times X & \xrightarrow{i'} & P \times X \\ \big\downarrow{=} & & \big\downarrow{p} \\ \mathbb{P}^n \times X & \xrightarrow{i} & \mathbb{P}^{n+1} \times X . \end{array}$$

As i' admits a retraction, i'_* is injective, and as $i'_* = p^* \circ i_*$, so is i_*. $\square$

Proof of Theorem 4.23. By Lemma 4.27, it is enough to verify the surjectivity of $\mathscr{P}$roj. Clearly, it is enough to verify this in the universal case $\mathbb{B}^* = \underline{\Omega}^*$ since any precobordism theory $\mathbb{B}^*$ is a quotient of $\underline{\Omega}^*$. By Lemma 4.32, the map $\iota_n \colon \underline{\Omega}^*(\mathbb{P}^n \times X \to Y) \to \underline{\Omega}^*(\mathbb{P}^\infty \times X \to Y)$ is an injection, so it is enough to verify that $\iota_n \circ \mathscr{P}$roj surjects onto the image of ι_n.

Rephrasing Lemma 4.30, the map

$$\iota_i\big(\theta(\pi_i) \bullet -\big) \colon \bigoplus_{i=0}^{\infty} \underline{\Omega}^*(X \to Y) \longrightarrow \underline{\Omega}^*\big(\mathbb{P}^\infty \times X \to Y\big)$$

is an isomorphism, where π_i is the projection $\mathbb{P}^i \times X \to X$. Multiplication by $c_1(\mathrm{pr}_1^*\mathcal{O}(1))$ gives a well-defined operator $c_1(\mathrm{pr}_1^*\mathcal{O}(1))\bullet$ on $\underline{\Omega}^*(\mathbb{P}^\infty \times X \to Y)$, and clearly $c_1(\mathrm{pr}_1^*\mathcal{O}(1))^{n+1}\bullet$ annihilates the image of ι_n. Hence the image of ι_n is contained in the image of

$$\iota_i\big(\theta(\pi_i) \bullet -\big) \colon \bigoplus_{i=0}^{n} \underline{\Omega}^*(X \to Y) \longrightarrow \underline{\Omega}^*\big(\mathbb{P}^\infty \times X \to Y\big)\,,$$

and therefore $\iota_n \circ \mathscr{P}$roj surjects onto the image of ι_n, proving the claim. □

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Toni Annala tannala@math.ias.edu
Department of Mathematics, University of British Columbia, Room 121, 1984 Mathematics Road, Vancouver, BC, Canada V6T 1Z2
Current address: School of Mathematics, Institute for Advanced Study, Einstein Drive, Princeton, NJ 08540, USA

Shoji Yokura yokura@sci.kagoshima-u.ac.jp
Graduate School of Science and Engineering, Kagoshima University, 21-35 Korimoto 1-Chome, Kagoshima 890-0065, Japan