Algebraic Geometry 10 (4) (2023) 394–431
doi:10.14231/AG-2023-013

Canonical models of toric hypersurfaces

Victor V. Batyrev

Abstract

Let Z be a nondegenerate hypersurface in a d-dimensional torus $(\mathbb{C}^*)^d$ defined by a Laurent polynomial f with a d-dimensional Newton polytope P. The subset $F(P) \subset P$ consisting of all points in P having integral distance at least 1 to all integral supporting hyperplanes of P is called the Fine interior of P. If $F(P) \neq \emptyset$, we construct a unique projective model $\widetilde{Z}$ of Z having at worst canonical singularities and obtain minimal models $\widehat{Z}$ of Z by crepant morphisms $\widehat{Z} \to \widetilde{Z}$. We show that the Kodaira dimension $\kappa = \kappa(\widetilde{Z})$ equals $\min\{d-1, \dim F(P)\}$ and the general fibers in the Iitaka fibration of the canonical model $\widetilde{Z}$ are nondegenerate $(d-1-\kappa)$-dimensional toric hypersurfaces of Kodaira dimension 0. Using $F(P)$, we obtain a simple combinatorial formula for the intersection number $(K_{\widetilde{Z}})^{d-1}$.

1. Historical motivations

Let $M \cong \mathbb{Z}^d$ be a d-dimensional lattice that we identify with the character group of a d-dimensional algebraic torus $\mathbb{T}^d \cong (\mathbb{C}^*)^d$ whose affine coordinate ring $\mathbb{C}\big[\mathbb{T}^d\big] \cong \mathbb{C}\big[t_1^{\pm 1}, \ldots, t_d^{\pm 1}\big]$ consists of Laurent polynomials

$$f(\mathbf{t}) = \sum_{\mathbf{m} \in M} c_{\mathbf{m}} \mathbf{t}^{\mathbf{m}}, \quad c_{\mathbf{m}} \in \mathbb{C},$$

where $c_{\mathbf{m}} = 0$ for all but finitely many $\mathbf{m} \in M$. The Newton polytope $P := \operatorname{Newt}(f)$ of $f(\mathbf{t}) \in \mathbb{C}\big[\mathbb{T}^d\big]$ is the convex hull of all $\mathbf{m} \in M$ such that $c_{\mathbf{m}} \neq 0$. We assume $\dim P = d$ and write $f(\mathbf{t}) = \sum_{\mathbf{m} \in A} c_{\mathbf{m}} \mathbf{t}^{\mathbf{m}}$, where $A \subset M$ is a finite subset with $P = \operatorname{Conv}(A)$. Following Askold Khovanskiĭ [Kho77], a Laurent polynomial $f \in \mathbb{C}\big[\mathbb{T}^d\big]$ and an affine hypersurface $Z := \{f(\mathbf{t}) = 0\} \subset \mathbb{T}^d$ are called *nondegenerate* if the Zariski closure $\overline{Z}_P$ of Z in projective toric variety V_P defined by the normal fan Σ_P has transversal intersections with all k-dimensional $\mathbb{T}^d$-orbits $\mathbb{T}_Q \subset V_P$ corresponding to k-dimensional faces $Q \preceq P$ $(k \geqslant 1)$.

The set of coefficients $\{c_{\mathbf{m}}\}_{\mathbf{m} \in A} \in \mathbb{C}^{|A|}$ defining *nondegenerate Laurent polynomials* $f(\mathbf{t}) = \sum_{\mathbf{m} \in A} c_{\mathbf{m}} \mathbf{t}^{\mathbf{m}}$ is a Zariski open dense subset $U_A \subset \mathbb{C}^{|A|}$ that can be explicitly described by the nonvanishing condition for the *principal A-determinant* E_A introduced by Israel Gelfand, Mikhail Kapranov and Andrei Zelevinski [GKZ94].

According to Khovanskiĭ [Kho77, Kho78, Kho84], one can resolve singularities of the projective hypersurface $\overline{Z}_P \subset V_P$ and obtain a smooth projective model $\overline{Z}$ of a nondegenerate hypersurface $Z \subset \mathbb{T}^d$ as Zariski closure $\overline{Z}$ of Z in some smooth projective equivariant torus

Received 30 May 2022, accepted in final form 6 August 2022.
2020 Mathematics Subject Classification 14E30, 14B05. 14M25, 52B20
Keywords: Kodaira dimension, minimal models, nondegenerate hypersurfaces, toric varieties

embedding $\mathbb{T}^d \hookrightarrow V$ corresponding to a regular simplicial refinement Σ of the normal fan Σ_P. Obviously, there exist countably many possibilities to choose such a refinement Σ.

The minimal model program (MMP), or Mori theory, is aimed at constructing the most "economic" projective representatives in each birational class of algebraic varieties. Such representatives are called *minimal models*, and they are main objectives of MMP [KM98, Mat02, Bir18a].

The present paper is inspired by the author's construction of $(d-1)$-dimensional minimal Calabi–Yau varieties $\widehat{Z}$ using projective compactifications of nondegenerate affine hypersurfaces $Z \subset \mathbb{T}^d$ defined by Laurent polynomials whose Newton polytopes P are *reflexive* [Bat94]. These minimal Calabi–Yau varieties allow one to produce many topologically different examples of smooth Calabi–Yau 3-folds and compute their Hodge numbers by combinatorial formulas, and perfectly illustrate the mirror symmetry phenomenon in theoretical physics [CK99, KS98, KS00, AGH+15]. We remark that the normal fan Σ_P of any reflexive polytope P defines a Gorenstein toric Fano variety V_P. One can choose an "economic" simplicial refinement $\widehat{\Sigma}$ of Σ_P such that the corresponding toric variety $\widehat{V}$ has at worst terminal singularities and the corresponding proper birational toric morphism $\widehat{V} \to V_P$ is *crepant*. Such a refinement $\widehat{\Sigma}$ may be not unique. The Zariski closure $\widetilde{Z} := \overline{Z}_P$ of the affine hypersurface $Z \subset \mathbb{T}^d$ in $\widetilde{V} := V_P$ is a Calabi–Yau hypersurface $\widetilde{Z}$ with at worst Gorenstein canonical singularities. The minimal Calabi–Yau models $\widehat{Z}$ of Z are partial crepant desingularizations of $\widetilde{Z}$ obtained as Zariski closures of affine hypersurfaces $Z \subset \mathbb{T}^d$ in the simplicial toric varieties $\widehat{V}$.

A generalization of the above construction to minimal models of arbitrary nondegenerate toric hypersurfaces $Z \subset \mathbb{T}^d$ has been suggested by Shihoko Ishii [Ish99], who showed that minimal models of Z can *always* be obtained as Zariski closures $\widehat{Z}$ of Z in some appropriately chosen simplicial toric varieties $\widehat{V}$ having at worst terminal singularities. Moreover, Ishii proposed two ways for constructing such a toric variety $\widehat{V}$: the traditional method and a shorter one.

The traditional method of MMP begins from a smooth projective model $\overline{Z} \subset V$ obtained in some smooth toric variety V by Khovanskii's method. The traditional MMP-method suggests a sequence of birational modifications of the pair $(V, \overline{Z})$ using the combinatorial framework of toric Mori theory developed by Miles Reid [Rei83] (see also [Fuj03, FS04] and [CLS11, Section 15]). It is useful to keep in mind the short exact sequence describing the canonical sheaf $\mathcal{K}_{\overline{Z}}$ on $\overline{Z}$ by the adjunction formula:

$$0 \to \mathcal{O}_V(K_V) \to \mathcal{O}_V\big(K_V + \overline{Z}\,\big) \to \mathcal{K}_{\overline{Z}} \to 0\,.$$

We remark that the Kodaira dimension $\kappa := \kappa\big(\,\overline{Z}\,\big)$ of the smooth hypersurface $\overline{Z} \subset V$ is nonnegative if and only if the linear systems corresponding to some multiples of the adjoint divisor $K_V + \overline{Z}$ on V are not empty, that is, the adjoint class $\big[K_V + \overline{Z}\,\big]$ belongs to the closed cone $C_{\mathrm{eff}}(V) \subset H^2(V, \mathbb{R})$ of effective divisors of V.

If the adjoint divisor $K_V + \overline{Z}$ is nef, then $\overline{Z}$ is a minimal model. Otherwise, there exists a toric extremal ray $R = \mathbb{R}_{\geqslant 0} v \in H_2(V, \mathbb{R})$ with negative intersection number $\big(v, K_V + \overline{Z}\big) < 0$. Applying an elementary birational toric modification from toric Mori theory, one can obtain another pair $\big(V', \overline{Z}'\big)$ consisting of a simplicial toric variety V' having at worst terminal singularities and another projective model $\overline{Z}' \subset V'$ of Z, etc. Eventually, after finitely many elementary birational toric modifications, one comes to a projective simplicial toric variety $\widehat{V}$ with at worst terminal singularities containing a $\mathbb{Q}$-Cartier divisor $\widehat{Z}$ such that the adjoint divisor $K_{\widehat{V}} + \widehat{Z}$ is nef. Then the projective hypersurface $\widehat{Z} \subset \widehat{V}$ is a minimal model of Z. Unfortunately, the traditional MMP-method does not allow one to say much about the resulting simplicial toric variety $\widehat{V}$ and about the minimal model $\widehat{Z}$ itself.

For this reason, Ishii suggested a more informative and short way for constructing the simplicial toric variety $\widehat{V}$. It uses the rational polytope $F(P) \subset M_{\mathbb{R}}$ corresponding to the adjoint divisor $K_V + \overline{Z}$ on some smooth projective toric variety V obtained by Khovanskiĭ's method using a refinement Σ of the normal fan of P. (Ishii denotes the rational polytope $F(P)$ by $\square_h$; see [Ish99, Section 3.3].) It is easy to see that the rational polytope $F(P)$ is independent of such a refinement Σ. We note that the polytope $F(P)$ was earlier introduced in the doctoral thesis of Jonathan Fine [Fin83] and, following Reid [Rei87, Appendix to Section 4], we call the polytope $F(P)$ the *Fine interior* of P. (Fine uses the name *heart of* P for the rational polytope $F(P)$; see [Fin83, Section 4].) An important observation of Ishii is that all 1-dimensional cones in the simplicial fan $\widehat{\Sigma}$ defining the toric variety $\widehat{V}$ are uniquely determined by P. They are spanned by the lattice vectors from a special finite set

$$S_F(P) := \{\nu_1, \dots, \nu_p\} = \widehat{\Sigma}[1] \subset \Sigma[1]$$

characterized by the property

$$\nu \in S_F(P) \Leftrightarrow \min_{x \in F(P)} \langle x, \nu \rangle = \min_{x \in P} \langle x, \nu \rangle + 1 \,.$$

Ishii called the lattice vectors $\nu \in S_F(P)$ *contributing to the Fine interior* $F(P)$. In the present paper, we call the finite set $S_F(P)$ the *support of the Fine interior* of P.

Ishii suggested to construct the simplicial fan $\widehat{\Sigma}$ with $\widehat{\Sigma}[1] = S_F(P)$ as the normal fan of some simple d-dimensional polytope $\square(\varepsilon)$ whose primitive inward-pointing facet normals are exactly the lattice vectors $\nu_1, \dots, \nu_p \in S_F(P)$. Note that there exists a natural bijection between all projective simplicial fans $\widehat{\Sigma}$ with $\widehat{\Sigma}[1] = S_F(P)$ and maximal $(p-d)$-dimensional GKZ-cones, or Mori chambers, in the moving cone of $\widehat{V}$ (see [OP91], [HKP06, Appendix A], [CLS11, Section 12]). In order to obtain a minimal model $\widehat{Z} \subset \widehat{V}$, one has to choose the simple polytope $\square(\varepsilon)$ in such a way that the corresponding Mori chamber contains the class of the adjoint divisor $\big[K_{\widehat{V}} + \widehat{Z}\big]$. In [Ish99, Section 3.3], the simple polytope $\square(\varepsilon)$ was obtained by so-called "puffing up" the polytope $F(P)$. We note that Ishii's method does not use the toric crepant morphisms that appeared in constructing Calabi–Yau minimal models corresponding to reflexive Newton polytopes P, and her method does not show that in general the minimal model $\widehat{Z}$ can be always obtained as nef divisor on the simplicial toric variety $\widehat{V}$, that is, that the corresponding Mori chamber can be chosen to contain simultaneously two nef divisor classes: the adjoint class $\big[K_{\widehat{V}} + \widehat{Z}\big]$ and the class $\big[\widehat{Z}\big]$. The latter is obtained in this paper using the *canonical model* $\widetilde{Z}$ of Z and crepant morphisms $\widehat{Z} \to \widetilde{Z}$ that allow one the combinatorial computation of the Kodaira dimension $\kappa\big(\widehat{Z}\big)$, the intersection number $\big(K_{\widehat{Z}}\big)^{d-1}$ and all plurigenera of the minimal model $\widehat{Z}$ via the polytope $F(P)$; see [Gie21].

2. Introduction

In this paper, we suggest a new method of "puffing up" the Fine interior $F(P)$ and constructing minimal models of nondegenerate hypersurfaces Z. This method is a direct generalization of the construction of minimal Calabi–Yau models in the case of reflexive Newton polytopes P; see [Bat94]. If P is a d-dimensional reflexive polytope, then its Fine interior $F(P)$ is just the origin $0 \in M$, and the support of the Fine interior $S_F(P)$ is exactly the set of all nonzero lattice points on the boundary of the dual reflexive polytope P^*; see Figure 1.

Recall that the construction of minimal Calabi–Yau compactifications $\widehat{Z}$ of Z consists of the

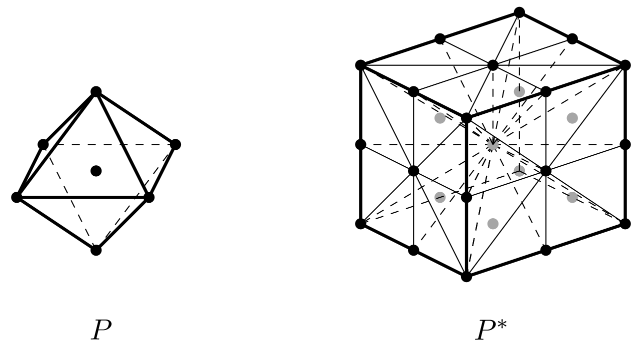

Figure 1. Example of a polytope P and the dual reflexive polytope P^*

following two steps [Bat94]:

- First, one considers the Zariski closure $\widetilde{Z}$ of the affine hypersurface Z in the Gorenstein toric Fano variety $\widetilde{V} := V_P$ corresponding to the normal fan $\widetilde{\Sigma} := \Sigma_P$ of P. The projective hypersurface $\widetilde{Z} \subset \widetilde{V}$ is a Calabi–Yau variety with at worst Gorenstein canonical singularities. From now on, we call the projective hypersurface $\widetilde{Z}$ the *canonical model* of $Z \subset \mathbb{T}^d$.
- Second, one considers the Zariski closure $\widehat{Z}$ of Z in the toric variety $\widehat{V}$ corresponding to a maximal projective simplicial refinement $\widehat{\Sigma}$ of the fan $\widetilde{\Sigma}$ satisfying the condition $\widehat{\Sigma}[1] = P^* \cap \left(\mathbb{Z}^d \setminus \{0\}\right)$. The corresponding toric morphism $\widehat{V} \to \widetilde{V}$ is crepant, and $\widehat{Z}$ is a minimal Calabi–Yau variety with at worst Gorenstein terminal singularities obtained as maximal projective crepant partial desingularization (a MPCP-desingularization) of the canonical model $\widetilde{Z}$. Such a desingularization is determined by choosing triangulations of facets of P^*, and it is not unique in general.

The main idea of the new construction of minimal models $\widehat{Z}$ is based on another "puffing up" of the rational polytope $F(P)$ using the real full-dimensional polytopes $F(\lambda P) \subset M_{\mathbb{R}}$ which are Fine interiors of the real λ-multiples of the lattice polytope P. If $\lambda = 1 + \varepsilon$, then for all sufficiently small $\varepsilon > 0$, the normal fan $\widetilde{\Sigma}$ of the full-dimensional real polytope $F((1+\varepsilon)P)$ is independent of ε. Note that the fan $\widetilde{\Sigma}$ is not simplicial in general, but all primitive lattice vectors $\nu \in \widetilde{\Sigma}[1]$ belong to the support $S_F(P)$. Moreover, one has a Minkowski sum decomposition

$$F((1+\varepsilon)P) = F(P) + \varepsilon C(P)\,, \quad \text{where}$$

$$C(P) := \left\{x \in M_{\mathbb{R}} \mid \langle x, \nu\rangle \geqslant \min_{p \in P}\langle p, \nu\rangle \ \ \forall \nu \in S_F(P)\right\}$$

is a d-dimensional rational polytope containing P. We call the rational polytope $C(P)$ the *canonical hull of* P.

The key step in our construction is to consider the toric variety $\widetilde{V}$ corresponding to the normal fan $\widetilde{\Sigma}$ of the d-dimensional Minkowski sum

$$\widetilde{P} := C(P) + F(P)\,.$$

We show that both the projective toric variety $\widetilde{V} = V_{\widetilde{\Sigma}}$ and the Zariski closure $\widetilde{Z}$ of Z in $\widetilde{V}$ have at worst $\mathbb{Q}$-Gorenstein canonical singularities. Moreover, the canonical class $K_{\widetilde{Z}}$ is a semiample $\mathbb{Q}$-Cartier divisor on $\widetilde{Z}$. We call the projective variety $\widetilde{Z}$ the *canonical model* of Z. If $\widehat{\Sigma}$ is a maximal projective simplicial refinement of the fan $\widetilde{\Sigma}$ with the property $\widehat{\Sigma}[1] = S_F(P)$, then the corresponding toric morphism $\varphi\colon \widehat{V} \to \widetilde{V}$ is crepant, the toric variety $\widehat{V}$ has at worst terminal singularities, and the Zariski closure $\widehat{Z}$ of Z in $\widehat{V}$ is a minimal model together with the crepant morphism $\varphi\colon \widehat{Z} \to \widetilde{Z}$. Moreover, the minimal model $\widehat{Z} \subset \widehat{V}$ is a semiample $\mathbb{Q}$-Cartier divisor of $\widehat{V}$.

The paper is organized as follows. In Section 3, we deal with combinatorial tools from convex geometry of M-lattice polytopes P. It turns out to be very useful to extend the class of M-lattice polytopes and consider a larger class of convex polytopes $P \subset M_{\mathbb{R}}$ which includes all rational polytopes and even some real ones. We call polytopes P in this class *generalized Delzant polytopes* (see Definition 3.1). Using the dual N-lattice in $N_{\mathbb{R}}$ and the natural pairing $\langle *, * \rangle\colon M \times N \to \mathbb{Z}$, we consider the upper-convex piecewise linear function

$$\mathrm{Min}_P\colon N_{\mathbb{R}} \to \mathbb{R}, \quad y \mapsto \mathrm{Min}_P(y) := \min_{x \in P} \langle x, y \rangle \ \ (y \in N_{\mathbb{R}})$$

associated with $P \subset M_{\mathbb{R}}$, and we use it in the definitions of the following three combinatorial objects:

- the *Fine interior of* P:
$$F(P) := \{x \in M_{\mathbb{R}} \mid \langle x, \nu \rangle \geqslant \mathrm{Min}_P(\nu) + 1 \ \ \forall \nu \in N \setminus \{0\}\},$$
- the *support of the Fine interior of* P: (if $F(P) \neq \emptyset$):
$$S_F(P) := \{\nu \in N \mid \mathrm{Min}_{F(P)}(\nu) = \mathrm{Min}_P(\nu) + 1\},$$
- the *canonical hull* $C(P)$ *of* P (if $F(P) \neq \emptyset$):
$$C(P) := \{x \in M_{\mathbb{R}} \mid \langle x, \nu \rangle \geqslant \mathrm{Min}_P(\nu) \ \ \forall \nu \in S_F(P)\}.$$

In Section 4, we concentrate our attention upon the canonical hull $C(P)$ of P. We call full-dimensional generalized Delzant polytopes P such that $P = C(P)$ *canonically closed*. We show that multiples λP of any full-dimensional generalized Delzant polytope P are always canonically closed for sufficiently large $\lambda \gg 1$.

Our main interest in Section 5 is the relation between the canonical refinement Σ_P^{can} of the normal fan Σ_P of P and its Fine interior $F(P)$. In particular, we show that the canonical refinement Σ_P^{can} of Σ_P is the normal fan of the Fine interior $F(\lambda P)$ for sufficiently large $\lambda \gg 1$.

Section 6 is devoted to main combinatorial results that we use in our construction of minimal models of nondegenerate hypersurfaces Z via their Newton polytopes P. We denote by $\widetilde{V}$ the projective toric variety defined by the fan $\widetilde{\Sigma}$ that is the normal fan of the Minkowski sum $\widetilde{P} := C(P) + F(P)$. The toric variety $\widetilde{V}$ will play a central role in our construction. We show that

1) the set $\widetilde{\Sigma}[1]$ of primitive lattice generators of 1-dimensional cones of the fan $\widetilde{\Sigma}$ is contained in $S_F(P)$;
2) the projective toric variety $\widetilde{V}$ is $\mathbb{Q}$-Gorenstein, and it has at worst canonical singularities;
3) for any 2-dimensional cone $\sigma \in \widetilde{\Sigma}(2)$ spanned by two primitive lattice vectors $\nu_i, \nu_j \in S_F(P)$, one has the following two properties:
   - the $(d-2)$-dimensional affine linear subspace $L_\sigma^1 \subset M_{\mathbb{R}}$ defined by two linear equations
   $$\langle x, \nu_i \rangle = \mathrm{Min}_P(\nu_i) + 1, \quad \langle x, \nu_j \rangle = \mathrm{Min}_P(\nu_j) + 1$$
   has nonempty intersection with the Fine interior $F(P)$;

- the $(d-2)$-dimensional affine linear subspace $L^0_\sigma \subset M_{\mathbb{R}}$ defined by two linear equations
$$\langle x, \nu_i \rangle = \mathrm{Min}_P(\nu_i)\,, \quad \langle x, \nu_j \rangle = \mathrm{Min}_P(\nu_j)$$
contains at least one vertex of the Newton polytope P.

We note that two Minkowski summands $C(P)$ and $F(P)$ of $\widetilde{P}$ define two nef $\mathbb{Q}$-Cartier divisors on the toric variety $\widetilde{V}$ and two toric morphisms $\varrho\colon \widetilde{V} \to V_{C(P)}$ and $\vartheta\colon \widetilde{V} \to V_{F(P)}$ that have opposite properties with respect to canonical class K:

$$\begin{tikzcd} K>0 & \widetilde{V} \arrow[dl, "\varrho"'] \arrow[dr, "\vartheta"] & K<0 \\ V_{C(P)} \arrow[rr, dashed] & & V_{F(P)}\,. \end{tikzcd}$$

The toric morphism $\varrho\colon \widetilde{V} \to V_{C(P)}$ is birational and K-positive; that is, the $\mathbb{Q}$-Cartier canonical divisor $K_{\widetilde{V}}$ has positive intersection with all 1-dimensional toric strata in $\widetilde{V}$ contracted by ϱ. We show that via the toric morphism ϱ, one can consider $\widetilde{V}$ as a minimal canonical partial resolution of $V_{C(P)}$. The toric morphism $\vartheta\colon \widetilde{V} \to V_{F(P)}$ is either birational if $\dim F(P) = d$, or a toric $\mathbb{Q}$-Fano fibration if $\dim F(P) < d$. A maximal projective simplicial refinement $\widehat{\Sigma}$ of $\widetilde{\Sigma}$ with $\widehat{\Sigma}[1] = S_F(P)$ defines a crepant birational toric morphism $\varphi\colon \widehat{V} \to \widetilde{V}$ such that $\widehat{V}$ is a projective simplicial toric variety with at worst terminal singularities.

In Section 7, we define the *canonical model* $\widetilde{Z}$ of Z as the Zariski closure of Z in the $\mathbb{Q}$-Gorenstein toric variety $\widetilde{V}$. By the construction of $\widetilde{V}$, the canonical model $\widetilde{Z}$ is a big semiample $\mathbb{Q}$-Cartier divisor on $\widetilde{V}$ corresponding to the polytope $C(P)$. The crucial fact is that the canonical model $\widetilde{Z}$ is a normal variety satisfying the adjunction formula, so that the canonical class of $\widetilde{Z}$ is the restriction of the adjoint nef $\mathbb{Q}$-Cartier divisor $K_{\widetilde{V}} + \widetilde{Z}$ on the toric variety $\widetilde{V}$ to the semiample hypersurface $\widetilde{Z}$.

In Section 8, we show that the hypersurface $\widetilde{Z} \subset \widetilde{V}$ has at worst $\mathbb{Q}$-Gorenstein canonical singularities, and we obtain minimal models $\widehat{Z}$ of Z as Zariski closures of $Z \subset \mathbb{T}^d$ in the simplicial terminal toric varieties $\widehat{V}$ defined by the simplicial refinements $\widehat{\Sigma}$ of the normal fan $\widetilde{\Sigma}$ satisfying the condition $\widehat{\Sigma}[1] = S_F(P)$.

In Section 9, using the toric morphism $\vartheta\colon \widetilde{V} \to V_{F(P)}$, we investigate the Iitaka fibration of $\widetilde{Z}$ defined by the pluricanonical ring
$$R := \bigoplus_{m \geqslant 0} H^0\big(\widetilde{Z}, mK_{\widetilde{Z}}\big)\,.$$
In particular, we prove the following formula for the Kodaira dimension:
$$\kappa\big(\widetilde{Z}\big) = \min\{\dim F(P), d-1\}$$
and compute the intersection number $\big(K_{\widetilde{Z}}\big)^{d-1}$:
$$\big(K_{\widetilde{Z}}\big)^{d-1} = \begin{cases} \mathrm{Vol}_d(F(P)) + \sum\limits_{\substack{Q \prec F(P) \\ \dim Q = d-1}} \mathrm{Vol}_{d-1}(Q) & \text{if } \dim F(P) = d\,, \\ 2\mathrm{Vol}_{d-1}(F(P)) & \text{if } \dim F(P) = d-1\,, \\ 0 & \text{if } \dim F(P) < d-1\,, \end{cases}$$
where $\mathrm{Vol}_k(\cdot) = k!\mathrm{vol}(\cdot)$ denotes the k-dimensional volume normalized by the condition $\mathrm{Vol}_k(\Delta_k) = 1$ if Δ_k is the basic k-dimensional lattice simplex.

Recently, Julius Giesler found a nice general combinatorial formula for all plurigenera $g_m(\widetilde{Z})$ ($m \geqslant 0$); see [Gie21]. His formula yields another method of proving the above combinatorial formula for $\left(K_{\widetilde{Z}}\right)^{d-1}$.

It is an important fact that the intersections of the canonical hypersurface $\widetilde{Z} \subset \widetilde{V}$ with the $(d - \dim F(P))$-dimensional generic fibers of the toric morphism $\vartheta\colon \widetilde{V} \to V_{F(P)}$ are nondegenerate toric hypersurfaces of Kodaira dimension 0 embedded into canonical toric $\mathbb{Q}$-Fano varieties of dimension $d - \dim F(P)$. In particular, we obtain that the toric pairs $\left(\widetilde{V}, \widetilde{Z}\right)$ together with the toric morphisms $\vartheta\colon \widetilde{V} \to V_{F(P)}$ corresponding to the adjoint nef $\mathbb{Q}$-Cartier divisor $K_{\widetilde{V}} + \widetilde{Z}$ provide toric examples of the log Calabi–Yau fibrations suggested recently by Caucher Birkar [Bir18b, Bir21].

The last section, Section 10, gives a short overview of some possible further developments of the considered topic and contains open questions.

3. Main combinatorial objects

Let $M \cong \mathbb{Z}^d$ be a lattice of rank d and $N := \operatorname{Hom}(M, \mathbb{Z})$ the dual lattice. We denote by $\langle *, * \rangle\colon M \times N \to \mathbb{Z}$ the natural pairing. We extend this pairing to the real vector spaces $M_{\mathbb{R}}$ and $N_{\mathbb{R}}$. For any convex compact subset $\square \subset M_{\mathbb{R}}$, we consider the upper-convex real-valued function $\operatorname{Min}_{\square}\colon N_{\mathbb{R}} \to \mathbb{R}$ defined as $\operatorname{Min}_{\square}(y) := \min_{x \in \square} \langle x, y \rangle$.

The main combinatorial object in our study is a full-dimensional convex lattice polytope $P \subset M_{\mathbb{R}}$, that is, a d-dimensional convex hull of a finite set $A \subset M$. We denote by Σ_P the normal fan of P. There is a bijection between k-dimensional faces $Q \preceq P$ ($0 \leqslant k \leqslant d$) and $(d-k)$-dimensional cones

$$\sigma_Q := \{ y \in N_{\mathbb{R}} \mid \operatorname{Min}_P(y) = \langle x, y \rangle \ \ \forall x \in Q \} \in \Sigma_P$$

in the dual space $N_{\mathbb{R}}$. The convex function $\operatorname{Min}_P\colon N_{\mathbb{R}} \to \mathbb{R}$ is Σ_P-piecewise linear; it defines an ample Cartier divisor $\mathcal{L}_P$ on the corresponding toric variety $V_P := V_{\Sigma_P}$. Obviously, we have

$$\operatorname{Min}_P(y) = \min_{m \in P \cap M} \langle m, y \rangle = \min_{m \in A} \langle m, y \rangle \,.$$

It turns out that our study of d-dimensional lattice polytopes requires us to consider a larger class of convex polytopes. First of all, we need to work with rational polytopes in $M_{\mathbb{R}}$ having dimension at most d. Moreover, we must allow one to multiply any considered convex polytope by positive real numbers and to shift these polytopes by real vectors $x \in M_{\mathbb{R}}$. At the same time, we want to keep connections of considered convex polytopes with the lattices M and N. All these requirements lead us to a larger class of convex polytopes of dimension at most d that we call *generalized Delzant polytopes*. This notion is inspired by the notion of Delzant polytopes in symplectic toric geometry [Del88, Gui94, Can01].

Definition 3.1. Let $P \subset M_{\mathbb{R}}$ be a compact convex polytope of dimension at most d. We call P a *generalized Delzant polytope* if there exist a finite set of lattice vectors $\nu_1, \ldots, \nu_s \subset N$ and a finite set of real numbers $\delta_1, \ldots, \delta_s \in \mathbb{R}$ such that

$$P = \{ x \in M_{\mathbb{R}} \mid \langle x, \nu_i \rangle \geqslant \delta_i, \, 1 \leqslant i \leqslant s \} \,.$$

The normal fan Σ_P of a generalized Delzant polytope P is a generalized fan in $N_{\mathbb{R}}$ defining a toric variety $V(\Sigma_P)$, see [CLS11, Definition 6.2.2], and the Σ_P-piecewise linear function Min_P defines an ample $\mathbb{R}$-Cartier divisor $\mathcal{L}_P$ on $V(\Sigma_P)$.

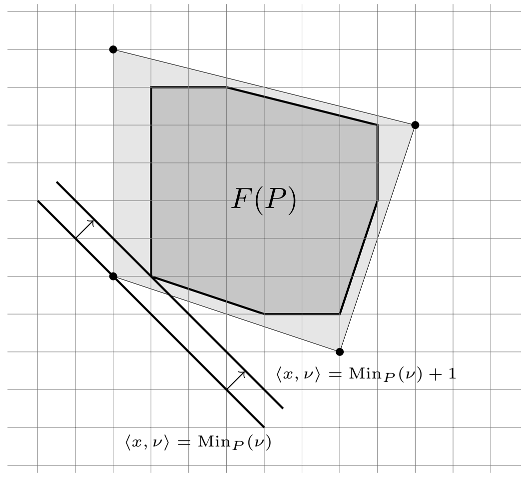

Figure 2. Example of a polytope and its Fine interior

Example 3.2. A 2-dimensional generalized Delzant polytope $P \subset \mathbb{R}^2$ is simply a convex n-gon whose sides have rational slopes.

Definition 3.3. Let $P \subset M_{\mathbb{R}}$ be a full-dimensional generalized Delzant polytope. We define the *Fine interior* $F(P)$ of P as

$$F(P) := \{x \in M_{\mathbb{R}} \mid \langle x, \nu \rangle \geqslant \mathrm{Min}_P(\nu) + 1 \ \ \forall \nu \in N \setminus \{0\}\}\,;$$

see Figure 2.

Remark 3.4. Obviously, the Fine interior $F(P)$ is a compact subset strictly contained in the interior P° of P, and $F(P)$ commutes with shifts by any real vectors; that is, for any generalized Delzant polytope P, one has

$$F(P + x) = F(P) + x \quad \forall x \in M_{\mathbb{R}}\,. \tag{3.1}$$

Remark 3.5. Let P be a full-dimensional lattice polytope. Since $\mathrm{Min}_P(N) \subset \mathbb{Z}$, every interior lattice point $m \in P^\circ \cap M$ is contained in $F(P)$. So we obtain the inclusion $\mathrm{Conv}(P^\circ \cap M) \subseteq F(P)$. In particular, $F(P) \neq \emptyset$ as soon as P contains at least one interior lattice point. If $\dim P = 2$, one has the equality

$$F(P) = \mathrm{Conv}(P^\circ \cap M)\,;$$

see [Bat17, Proposition 2.9]. This equality does not hold in general if $\dim P \geqslant 3$.

Example 3.6. Let P be a 3-dimensional parallelepiped with eight lattice vertices

$$(0,0,0)\,,\ (-1,1,1)\,,\ (1,-1,1)\,,\ (1,1,-1)\,,\ (2,0,0)\,,\ (0,2,0)\,,\ (0,0,2)\,,\ (1,1,1)\,.$$

Then P has no interior lattice points, but $F(P) = \{(1/2,1/2,1/2)\} \neq \emptyset$. Note that the Fine interior of the rational shifted polytope $P' = P - (1/2,1/2,1/2)$ is $\{0\}$, and the polar dual of P' is the lattice polytope

$$\mathrm{Conv}\{\pm(0,1,1), \pm(1,0,1), \pm(1,1,0)\}\,.$$

Remark 3.7. Note that the Fine interior is *monotone*, that is,

$$P \subseteq P' \Rightarrow F(P) \subseteq F(P')\,,$$

because $\mathrm{Min}_P(\nu) \geqslant \mathrm{Min}_{P'}(\nu)$ for all $\nu \in N$. In particular, one has $F(P) \subseteq F(\lambda P)$ for any real $\lambda \geqslant 1$ if P contains the origin $0 \in M$. Using shifts as in (3.1), one easily obtains

$$F(P) \neq \emptyset \Rightarrow F(\lambda P) \neq \emptyset \quad \forall \lambda \geqslant 1\,.$$

Proposition 3.8. *Let $P \subset M_{\mathbb{R}}$ be a full-dimensional generalized Delzant polytope. There exists a positive real number λ_P^0 such that $F(\lambda P) \neq \emptyset$ if and only if $\lambda \geqslant \lambda_P^0$. Moreover, the polytope $F(\lambda P)$ is full-dimensional if and only if $\lambda > \lambda_P^0$.*

Proof. By Remark 3.4, we may assume that P contains the origin $0 \in M$. We set

$$\lambda_P^0 := \inf\{\lambda \in \mathbb{R}_{>0} \mid F(\lambda P) \neq \emptyset\}\,.$$

By Remark 3.7, we have $F(\lambda_P^0 P) \neq \emptyset$ because $M_{\mathbb{R}}$ is a complete Banach space. For all $\lambda > \lambda_P^0$, the polytope $F(\lambda P)$ contains the full-dimensional Minkowski sum $F\big(\lambda_P^0 P\big) + \big(\lambda - \lambda_P^0\big)P$. Thus, $F(\lambda P)$ is full-dimensional for all $\lambda > \lambda_P^0$. On the other hand, $F\big(\lambda_P^0 P\big)$ cannot be full-dimensional because otherwise $F\big(\big(\lambda_P^0 - \varepsilon\big)P\big)$ would be full-dimensional for sufficiently small $\varepsilon > 0$, and the latter contradicts $F\big(\big(\lambda_P^0 - \varepsilon\big)P\big) = \emptyset$. □

Definition 3.9. Let P be a d-dimensional generalized Delzant polytope. Assume $F(P) \neq \emptyset$. A nonzero lattice point $\nu \in N$ is called *essential* for $F(P)$ if

$$\mathrm{Min}_{F(P)}(\nu) = \mathrm{Min}_P(\nu) + 1\,.$$

The set of all essential lattice points $\nu \in N$ for $F(P)$ is called the *support of* $F(P)$; we denote it by $S_F(P)$.

Remark 3.10. Using Remark 3.4, we obtain that $S_F(P + x) = S_F(P)$ for all $x \in M_{\mathbb{R}}$. We call a lattice vector $\nu \in N$ in a cone $\sigma \in \Sigma_P$ of the normal fan Σ_P*-irreducible* if $\nu \neq \nu_1 + \nu_2$ for any $\nu_1, \nu_2 \in \sigma \cap N$. Since the function Min_P is linear on every cone $\sigma \in \Sigma_P$, we obtain that the two inequalities

$$\langle x, \nu_1\rangle \geqslant \mathrm{Min}_P(\nu_1) + 1\,, \quad \langle x, \nu_2\rangle \geqslant \mathrm{Min}_P(\nu_2) + 1$$

for $\nu_1, \nu_2 \in \sigma \cap N$ imply $\langle x, \nu\rangle \geqslant \mathrm{Min}_P(\nu) + 2 > \mathrm{Min}_P(\nu) + 1$, where $\nu = \nu_1 + \nu_2$, that is, $\nu \notin S_F(P)$. Therefore, every $\nu \in S_F(P)$ must be Σ_P-irreducible. By Gordan's lemma, the set of Σ_P-irreducible elements $\nu \in N$ for any $\sigma \in \Sigma_P$ is finite (Hilbert basis). So $S_F(P)$ is a finite set consisting of some Σ_P-irreducible primitive lattice vectors $\nu \in N$, and the Fine interior $F(P)$ is a generalized Delzant polytope of dimension at most d. Moreover, one has

$$F(P) = \{x \in M_{\mathbb{R}} \mid \langle x, \nu\rangle \geqslant \mathrm{Min}_P(\nu) + 1 \;\; \forall \nu \in S_F(P)\}$$

because the compact polytope $F(P)$ is strictly contained in the open halfspace

$$\{x \in M_{\mathbb{R}} \mid \langle x, \nu\rangle > \mathrm{Min}_P(\nu) + 1\} \quad \text{if } \nu \notin S_F(P)\,,$$

and we can ignore the corresponding inequality $\langle x, \nu\rangle \geqslant \mathrm{Min}_P(\nu) + 1$ for $F(P)$.

Note that computer calculations of the Fine interior $F(P)$ and its support $S_F(P)$ that are using the Hilbert basis of lattice semigroups $\sigma \cap N$ can be rather time-consuming. Another method for such a calculation of $F(P)$ and $S_F(P)$ is provided by the next useful statement.

Proposition 3.11. *Let P be a full-dimensional generalized Delzant polytope. Denote by $\Sigma_P[1] \subset N$ the set of all primitive inward-pointing facet normals of P. Assume that $F(P) \neq \emptyset$. Then*

$$S_F(P) \subseteq \mathrm{Conv}(\Sigma_P[1])\,.$$

In particular, the set $S_F(P)$ is finite, and

$$F(P)=\left\{x\in M_{\mathbb{R}}\mid \langle x,\nu\rangle\geqslant \mathrm{Min}_P(\nu)+1\ \ \forall\nu\in \mathrm{Conv}(\Sigma_P[1])\cap\{N\setminus\{0\}\}\right\}.$$

Proof. First, we note that $0\in\mathrm{Conv}(\Sigma_P[1])$ because the lattice vectors in $\Sigma_P[1]$ generate the complete normal fan Σ_P. Take a lattice vector $\nu\in S_F(P)$. Since the normal fan Σ_P is complete, ν is contained in some d-dimensional cone $\sigma\in\Sigma_P(d)$. Consider the set of primitive lattice vectors

$$\{\nu_1,\nu_2,\dots,\nu_k\}:=\sigma\cap\Sigma_P[1]$$

such that $\sigma=\sum_{i=1}^k\mathbb{R}_{\geqslant 0}\nu_i$. It is sufficient to show that $\nu\in\mathrm{Conv}\{0,\nu_1,\dots,\nu_k\}$ because of the inclusion $\mathrm{Conv}(0,\nu_1,\dots,\nu_k)\subset\mathrm{Conv}(\Sigma_P[1])$.

Assume $\nu\notin\mathrm{Conv}(0,\nu_1,\dots,\nu_k)$. It follows from the inclusion $\mathbb{R}_{\geqslant 0}\nu\subset\sigma$ that there exists a positive $\lambda\in\mathbb{R}$ such that $\lambda<1$ and $\lambda\nu\in\mathrm{Conv}(\nu_1,\dots,\nu_k)$. In particular, one can write $\lambda\nu$ as a nonnegative linear combination $\sum_{i=1}^k\lambda_i\nu_i$ with $\sum_{i=1}^k\lambda_i=1$. Take any $x\in F(P)$, and consider the linear combination of k inequalities $\langle x,\nu_i\rangle\geqslant\mathrm{Min}_P(\nu_i)+1$ with the coefficients λ_i ($1\leqslant i\leqslant k$). We obtain $\langle x,\lambda\nu\rangle\geqslant\mathrm{Min}_P(\lambda\nu)+1$ because Min_P is linear on σ. So we get

$$\langle x,\nu\rangle\geqslant\mathrm{Min}_P(\nu)+\lambda^{-1}>\mathrm{Min}_P(\nu)+1\ \ \forall x\in F(P).$$

Hence $\nu\notin S_F(P)$. This gives a contradiction. □

Example 3.12. Let P be the 3-dimensional lattice parallelepiped from Example 3.6. Then $S_F(P)$ consists of six lattice vectors

$$S_F(P)=\{\pm(0,1,1),\pm(1,0,1),\pm(1,1,0)\}.$$

Definition 3.13. Let P be a d-dimensional generalized Delzant polytope with $F(P)\neq\emptyset$. We call the polytope

$$C(P):=\{x\in M_{\mathbb{R}}\mid\langle x,\nu\rangle\geqslant\mathrm{Min}_P(\nu)\ \ \forall\nu\in S_F(P)\}$$

the canonical hull of P. Note that $C(P)$ contains P because the inequality $\langle x,\nu\rangle\geqslant\mathrm{Min}_P(\nu)$ holds for all $x\in P$ and all $\nu\in N$. In particular, $C(P)$ is a generalized Delzant polytope of dimension d. The compactness of $C(P)$ follows from the compactness of $F(P)$ and from the equality

$$F(P)=\{x\in M_{\mathbb{R}}\mid\langle x,\nu\rangle\geqslant\mathrm{Min}_P(\nu)+1\ \ \forall\nu\in S_F(P)\}$$

in Remark 3.10 showing that the cone spanned by $S_F(P)$ equals $N_{\mathbb{R}}$.

Remark 3.14. If P is a d-dimensional lattice polytope with $F(P)\neq\emptyset$, then $C(P)$ is a rational polytope. In general, $C(P)$ is not a lattice polytope.

Example 3.15. Let $P\subset\mathbb{R}^5$ be the 5-dimensional lattice polytope defined by the inequalities

$$x_i\geqslant 0\ (1\leqslant i\leqslant 5),\quad x_1+x_2+x_3+x_4+2x_5\leqslant 7,\quad x_5\leqslant 3.$$

Then $F(P)=\{(1,1,1,1,1)\}$ and

$$S_F(P)=\{e_1,e_2,e_3,e_4,e_5,-e_1-e_2-e_3-e_4-2e_5\}.$$

The canonical hull $C(P)$ is a rational 5-simplex containing the rational vertex $(0,0,0,0,7/2)$. Note that the primitive inward-pointing lattice normal vector to the facet $\{x_5=3\}$ of P is not an element of $S_F(P)$.

Proposition 3.16. *Let $P\subset M_{\mathbb{R}}$ and $P'\subset M'_R$ be two full-dimensional generalized Delzant polytopes with $F(P)\neq\emptyset$ and $F(P')\neq\emptyset$. Then the Cartesian product*

$$P\times P'\subset M_{\mathbb{R}}\times M'_{\mathbb{R}}$$

is a full-dimensional generalized Delzant polytope with $F(P \times P') \neq \emptyset$, and the following statements hold:

(a) $F(P \times P') = F(P) \times F(P')$;

(b) $S_F(P \times P') = \{(\nu, 0) \mid \nu \in S_F(P)\} \cup \{(0, \nu') \mid \nu' \in S_F(P')\}$;

(c) $C(P \times P') = C(P) \times C(P')$.

Proof. If

$$P = \{x \in M_{\mathbb{R}} \mid \langle x, \nu_i \rangle \geqslant \delta_i,\ 1 \leqslant i \leqslant k\} \quad \text{and} \quad P' = \{x \in M_{\mathbb{R}} \mid \langle x, \nu'_j \rangle \geqslant \delta',\ 1 \leqslant j \leqslant l\},$$

then the product $P \times P'$ is defined by

$$P \times P' = \{(x, x') \in M_{\mathbb{R}} \times M'_{\mathbb{R}} \mid \langle (x, x'), (\nu_i, 0) \rangle \geqslant \delta_i \ \ \forall i,\ \langle (x, x'), (0, \nu'_j) \rangle \geqslant \delta'_j \ \ \forall j\}.$$

So $P \times P'$ is also a generalized Delzant polytope. For any $(\nu, \nu') \in N \times N'$, we have

$$\mathrm{Min}_{P \times P'}(\nu, \nu') = \mathrm{Min}_P(\nu) + \mathrm{Min}_{P'}(\nu').$$

(a) If $(x, x') \in F(P \times P')$, then

$$\langle (x, x'), (\nu_i, 0) \rangle \geqslant \mathrm{Min}_{P \times P'}(\nu, 0) \geqslant \mathrm{Min}_{P \times P'}((\nu, 0)) + 1 \quad \forall \nu \in N \setminus \{0\},$$
$$\langle (x, x'), (\nu_i, 0) \rangle \geqslant \mathrm{Min}_{P \times P'}(0, \nu') \geqslant \mathrm{Min}_{P \times P'}((0, \nu')) + 1 \quad \forall \nu' \in N' \setminus \{0\};$$

that is, $(x, x') \in F(P) \times F(P')$. In follows that $F(P \times P') \subseteq F(P) \times F(P')$. On the other hand, if $(x, x') \in F(P) \times F(P')$ and $(\nu, \nu') \neq (0, 0)$, then

$$\langle (x, x'), (\nu, \nu') \rangle = \langle x, \nu \rangle + \langle x', \nu' \rangle \geqslant \mathrm{Min}_P(\nu) + \mathrm{Min}_{P'}(\nu') + 1 \geqslant \mathrm{Min}_{P \times P'}(\nu, \nu') + 1.$$

(b) The equality

$$\mathrm{Min}_{P \times P'}(\nu, \nu') + 1 = \mathrm{Min}_{F(P) \times F(P')}(\nu, \nu')$$

can hold only if $\nu = 0$ or $\nu' = 0$, which implies that we have $\nu' \in S_F(P')$ or $\nu \in S_F(P)$.

(c) This follows from item (b). □

Proposition 3.17. *Let $C(P)$ be the canonical hull of a generalized Delzant polytope P with $F(P) \neq \emptyset$. Then*

(a) $\mathrm{Min}_P(\nu) = \mathrm{Min}_{C(P)}(\nu)$ *for all* $\nu \in S_F(P)$;

(b) $F(P) = F(C(P))$; *that is, the polytopes P and $C(P)$ have the same Fine interior.*

Proof. (a) It follows from

$$C(P) := \{x \in M_{\mathbb{R}} \mid \langle x, \nu \rangle \geqslant \mathrm{Min}_P(\nu) \ \ \forall \nu \in S_F(P)\}$$

that $\mathrm{Min}_{C(P)}(\nu) \geqslant \mathrm{Min}_P(\nu)$ for all $\nu \in S_F(P)$. The opposite inequality $\mathrm{Min}_P(\nu) \geqslant \mathrm{Min}_{C(P)}(\nu)$ holds for all $\nu \in N$ because $P \subseteq C(P)$.

(b) By Remark 3.10,

$$F(P) = \{x \in M_{\mathbb{R}} \mid \langle x, \nu \rangle \geqslant \mathrm{Min}_P(\nu) + 1 \ \ \forall \nu \in S_F(P)\}.$$

On the other hand, we have the inclusion

$$F(C(P)) \subseteq \{x \in M_{\mathbb{R}} \mid \langle x, \nu \rangle \geqslant \mathrm{Min}_{C(P)}(\nu) + 1 \ \ \forall \nu \in S_F(P)\}.$$

By item (a), we have $\mathrm{Min}_P(\nu) = \mathrm{Min}_{C(P)}(\nu)$ for all $\nu \in S_F(P)$. This implies the inclusion $F(C(P)) \subseteq F(P)$.

Since P is contained in $C(P)$, we obtain the opposite inclusion $F(P) \subseteq F(C(P))$. □

Corollary 3.18. *The support of the Fine interior of $C(P)$ equals $S_F(P)$.*

Proof. All lattice vectors $\nu \in S_F(P)$ are contained in the support of the Fine interior of $C(P)$ because

$$\mathrm{Min}_{F(P)}(\nu) - \mathrm{Min}_{C(P)}(\nu) = \mathrm{Min}_{F(P)}(\nu) - \mathrm{Min}_P(\nu) = 1 \quad \forall \nu \in S_F(P)\,.$$

The inclusion $P \subseteq C(P)$ implies that

$$\mathrm{Min}_{F(P)}(\nu) - \mathrm{Min}_{C(P)}(\nu) \geqslant \mathrm{Min}_{F(P)}(\nu) - \mathrm{Min}_P(\nu) \quad \forall \nu \in N\,.$$

Therefore, it follows from $\mathrm{Min}_{F(P)}(\nu) - \mathrm{Min}_{C(P)}(\nu) = 1$ that $\mathrm{Min}_{F(P)}(\nu) - \mathrm{Min}_P(\nu) \leqslant 1$. On the other hand, $\mathrm{Min}_{F(P)}(\nu) - \mathrm{Min}_P(\nu) \geqslant 1$ holds for all $\nu \in N \setminus \{0\}$. So, we obtain $\mathrm{Min}_{F(P)}(\nu) - \mathrm{Min}_P(\nu) = 1$, and the support of the Fine interior of $C(P)$ is contained in $S_F(P)$. ☐

Corollary 3.19. *Let P be a full-dimensional generalized Delzant polytope with $F(P) \neq \emptyset$. Then $C(C(P)) = C(P)$.*

Proof. The statement follows straightforwardly from Corollary 3.18 and Proposition 3.17. ☐

4. Canonically closed polytopes

Definition 4.1. Let P be a full-dimensional generalized Delzant polytope $P \subset M_{\mathbb{R}}$ with $F(P) \neq \emptyset$. We call P *canonically closed* if $C(P) = P$.

Remark 4.2. It follows from Corollary 3.19 that the canonical hull $C(P)$ of a full-dimensional Delzant polytope P is always canonically closed.

Proposition 4.3. *Let P be a full-dimensional generalized Delzant polytope with $F(P) \neq \emptyset$. Then P is canonically closed if and only if for any facet $Q \prec P$, the primitive inward-pointing facet normal ν_Q belongs to $S_F(P)$, that is, satisfies the equation* $\mathrm{Min}_{F(P)}(\nu_Q) = \mathrm{Min}_P(\nu_Q) + 1$.

Proof. Assume that P is canonically closed. Then

$$P = C(P) = \{x \in M_{\mathbb{R}} \mid \langle x, v\rangle \geqslant \mathrm{Min}_P(\nu) \;\; \forall \nu \in S_F(P)\}\,.$$

Therefore, any facet Q of P is defined by the equation $\langle x, \nu\rangle = \mathrm{Min}_P(\nu)$ for some $\nu \in S_F(P)$. The primitive inward-pointing lattice normal vector ν_Q to a given facet Q is uniquely determined by Q. So $\nu_Q \in S_F(P)$.

On the other hand, one can always write a convex full-dimensional polytope P as intersection of all half-spaces $\langle x, \nu_Q\rangle \geqslant \mathrm{Min}_P(\nu_Q)$ corresponding to its facet $Q \prec P$:

$$P = \bigcap_{\substack{Q \prec P \\ \dim Q = d-1}} \{x \in M_{\mathbb{R}} \mid \langle x, \nu_Q\rangle \geqslant \mathrm{Min}_P(\nu_Q)\}\,.$$

If all ν_Q belong to $S_F(P)$, then $C(P) \subseteq P$. The opposite inclusion $P \subseteq C(P)$ holds for all P. Thus, we obtain $P = C(P)$. ☐

Proposition 4.4. *Let P be a 2-dimensional lattice polytope with $F(P) \neq \emptyset$. Then $C(P) = P$.*

Proof. By Proposition 4.3, it suffices to show that the primitive inward-pointing lattice normal ν_Q to any 1-dimensional side $Q \prec P$ belongs to $S_F(P)$. Take a 1-dimensional side $Q \prec P$. By Remark 3.5, we have $F(P) = \mathrm{Conv}(P^\circ \cap M)$. In particular, $\{P^\circ \cap M\} \neq \emptyset$, and we can find a lattice point $m \in F(P)$ such that $\mathrm{Min}_{F(P)}(\nu_Q) = \langle m, \nu_Q\rangle$. Take two lattice points $m_1, m_2 \in Q$ such that $m_1 - m_2$ is a primitive lattice vector; see Figure 3.

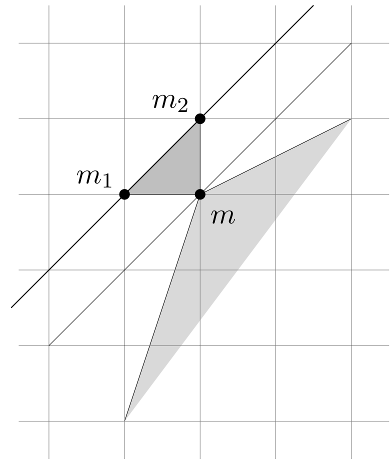

Figure 3. Construction in the proof of Proposition 4.4

Then the lattice triangle $\tau := \mathrm{Conv}(m, m_1, m_2)$ is a standard basic triangle because it follows from

$$\langle m, \nu_Q \rangle = \min_{a \in F(P) \cap M} \langle a, \nu_Q \rangle$$

that all lattice points in τ are only its vertices m, m_1, m_2. Therefore, the lattice distance between the segment $[m_1, m_2]$ and m is 1, and we obtain

$$\langle m, \nu_Q \rangle = \langle m_1, \nu_Q \rangle + 1 = \langle m_2, \nu_Q \rangle + 1 \,.$$

Thus, $\nu_Q \in S_F(P)$. □

Remark 4.5. The statement in Proposition 4.4 does not hold true in general for lattice polytopes P of dimension $d \geqslant 3$ or for 2-dimensional rational polytopes.

Example 4.6. Let $P := \mathrm{Conv}\{e_0, e_1, e_2, e_3\} \subset M_{\mathbb{R}}$ be a 3-dimensional lattice simplex with vertices

$$e_0 := (-1,-1,-1)\,, \quad e_1 := (1,1,0)\,, \quad e_2 := (1,0,1)\,, \quad e_3 := (0,1,1)\,.$$

Then $F(P) = \{0\}$ and $C(P) = \mathrm{Conv}\{e_0, e_1, e_2, e_3, e_4\}$, where $e_4 := (1,1,1)$; that is, $C(P) \neq P$. Note that the inward-pointing normal $\nu_Q = (-1,-1,-1) \in N$ to the facet $Q := \mathrm{Conv}\{e_1, e_2, e_3\} \prec P$ is not an element of $S_F(P)$ because Q has lattice distance 2 from the origin $0 = F(P)$.

Example 4.7. Let $P \subset \mathbb{R}^2$ be the triangle with vertices $(-1,0)$, $(0,3/2)$, $(4,-5/2)$. Then the Fine interior $F(P)$ is the triangle with vertices $(0,0)$, $(0,1/2)$, $(1,-1/2)$. The canonical hull $C(P)$ is strictly larger than the triangle $P \subsetneq C(P)$; that is, P is not canonically closed. The integral distance between the side $Q := [(-1,0),(0,3/2)]$ and the rational vertex $(0,1/2) \in F(P)$ is 2. The canonical hull $C(P)$ is a quadrilateral containing one more vertex $(-1,1/2)$; see Figure 4.

Definition 4.8 ([Bat94]). Let $P \subset M_{\mathbb{R}}$ be a d-dimensional lattice polytope containing the origin $0 \in M$ in its interior. The polytope P is called *reflexive* if the dual polytope

$$P^* := \{y \in N_{\mathbb{R}} \mid \langle x, y \rangle \geqslant -1 \ \ \forall x \in P\}$$

is a lattice polytope.

Proposition 4.9. *A d-dimensional lattice polytope P containing 0 in its interior is reflexive if and only if $F(P) = \{0\}$ and P is canonically closed, that is, $C(P) = P$.*

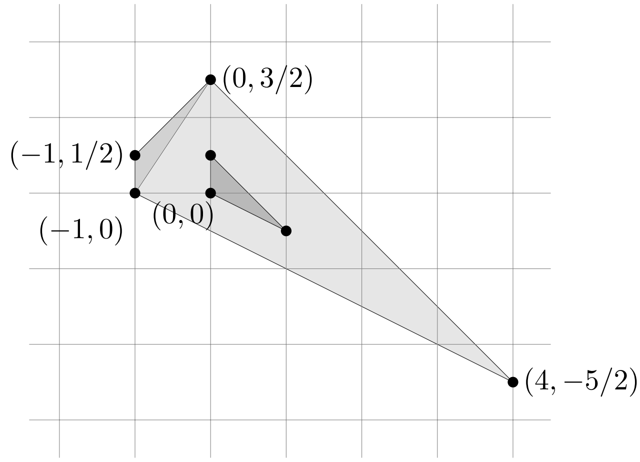

Figure 4. Example 4.7

Proof. Assume that P is reflexive. Then the interior lattice point $0 \in P$ belongs to $F(P)$. On the other hand, for every vertex $\nu \in P^*$, one has $\mathrm{Min}_P(\nu) = -1$. Therefore, the inequalities $\langle x, \nu\rangle \geqslant \mathrm{Min}_P(\nu) + 1$ defining $F(P)$ are $\langle x, \nu\rangle \geqslant 0$ for all vertices $\nu \in P^*$. There exists a positive linear combination $\sum_i \lambda_i \nu_i$ of vertices of P^* which is equal to 0. Therefore, $0 \in M_{\mathbb{R}}$ is the single solution of the inequalities defining the set

$$\{x \in M_{\mathbb{R}} \mid \langle x, \nu\rangle \geqslant 0 \ \ \forall \nu \in P^* \cap (N \setminus \{0\})\}\,.$$

It follows that $F(P) = \{0\}$ and that $S_F(P)$ contains all vertices of P^*. By Proposition 4.3, we obtain $C(P) = P$.

Now assume $F(P) = \{0\}$ and $C(P) = P$. By Proposition 4.3, all primitive inward-pointing normals ν_Q to facets $Q \prec P$ belong to $S_F(P)$. So we obtain $\mathrm{Min}_P(\nu_Q) = \mathrm{Min}_{F(P)}(\nu_Q) - 1 = -1$. Therefore, P^* is the convex hull of $S_F(P)$; that is, P^* is a lattice polytope. $\square$

Remark 4.10. Using the classification due to Alexander Kasprzyk [Kas10], it was shown in [BKS22] that up to a unimodular isomorphism there are $661\,280 = 665\,599 - 4\,319$ 3-dimensional lattice polytopes P with $F(P) = \{0\}$ which are not canonically closed, that is, which are not reflexive. However, the canonical hulls $C(P)$ of these 3-dimensional polytopes P are always reflexive.

Proposition 4.11. *Let P be a full-dimensional generalized Delzant polytope with $F(P) \neq \emptyset$, and let $Q \prec P$ be a facet of P. Assume that Q has nonempty Fine interior; that is, $F(Q) \neq \emptyset$. Then the primitive inward-pointing facet normal ν_Q belongs to $S_F(P)$. In particular, a d-dimensional lattice polytope P is canonically closed if every facet $Q \prec P$ contains at least one lattice point in its relative interior.*

Remark 4.12. The converse is not true in general. The lattice triangle with vertices $(1, 0)$, $(0, 1)$, $(-1, -1)$ is canonically closed, but all its facets (sides) have no interior lattice points.

Example 4.13. There are exactly nine examples of 3-dimensional lattice polytopes P with $F(P)$ nonempty and without interior lattice points [BKS22]. All these nine polytopes are canonically closed because their facets contain at least one lattice point in their relative interiors.

Proof of Proposition 4.11. Take a point $q \in F(Q)$, and consider an arbitrary lattice vector $\nu \in S_F(P)$ such that $\nu \notin \mathbb{R}\nu_Q$. Then ν defines a nonzero element in the lattice $N/\mathbb{Z}\nu_Q$, and we obviously have $\mathrm{Min}_Q(\nu) \geqslant \mathrm{Min}_P(\nu)$. As $q \in F(Q)$, we obtain

$$\langle q, \nu\rangle \geqslant \mathrm{Min}_Q(\nu) + 1 \geqslant \mathrm{Min}_P(\nu) + 1\,.$$

On the other hand, q cannot belong to $F(P)$ because q is in the relative interior of the facet $Q \prec P$. Hence q cannot satisfy the inequality $\langle q, v_Q\rangle \geqslant \mathrm{Min}_P(\nu_Q) + 1$. Therefore, the lattice primitive normal ν_Q must be essential for $F(P)$; that is, $\nu_Q \in S_F(P)$. □

Corollary 4.14. *Let P be a full-dimensional lattice polytope. Then for any integer $k \geqslant d$, the lattice polytope kP is canonically closed.*

Proof. The statement follows from Proposition 4.11 because for any $(d-1)$-dimensional lattice face $Q \prec P$ and for any $k \geqslant \dim Q + 1 = d$, the lattice polytope kQ always has lattice points in its relative interior. □

Corollary 4.15. *Let P be a full-dimensional generalized Delzant polytope. Then for a sufficiently large positive real number λ, the polytope λP is canonically closed.*

Proof. By Proposition 4.11, it is enough to choose the real number $\lambda \in \mathbb{R}$ such that λQ has nonempty Fine interior for any facet $Q \prec P$. □

Proposition 4.16. *Let P be a full-dimensional generalized Delzant polytope with $F(P) \neq \emptyset$. If P is canonically closed, then λP is canonically closed for all $\lambda \geqslant 1$.*

Proof. By Remark 3.7, $F(\lambda P) \neq \emptyset$ for any $\lambda \geqslant 1$.

Consider $\lambda > 1$. We claim that $F(P) + (\lambda - 1)P \subseteq F(\lambda P)$. Indeed, take $x \in F(P)$ and $y = (\lambda - 1)y'$ with $y' \in P$. Then for any $\nu \in N \setminus \{0\}$, we get

$$\langle x, \nu\rangle \geqslant \mathrm{Min}_P(\nu) + 1\,, \quad \langle y', \nu\rangle \geqslant \mathrm{Min}_P(\nu)\,, \quad \langle y, \nu\rangle \geqslant (\lambda - 1)\,\mathrm{Min}_P(\nu)\,.$$

Thus, $\langle x, \nu\rangle \geqslant \lambda\,\mathrm{Min}_P(\nu) + 1 = \mathrm{Min}_{\lambda P}(\nu) + 1$; that is, $x + y \in F(\lambda P)$.

If $\nu \in S_F(P)$ is an inward-pointing primitive lattice normal to a facet $Q \prec P$ such that $\langle x, \nu\rangle = \langle y', \nu\rangle + 1$ for some $x \in F(P)$ and $y' \in P$, then $\lambda y' \in \lambda P$, $x + (\lambda - 1)y' \in F(\lambda P)$, and

$$\langle x - y', \nu\rangle = \langle (x + (\lambda - 1)y') - \lambda y', \nu\rangle = 1\,,$$

that is, $\nu \in S_F(\lambda P)$. By Proposition 4.3, the convex polytope λP is canonically closed. □

Corollary 4.17. *Let P be a full-dimensional generalized Delzant polytope. Then there exists a positive real number $\lambda_P \geqslant \lambda_P^0$ such that λP is canonically closed if and only if $\lambda \geqslant \lambda_P$.*

Proof. Define

$$\lambda_P := \inf\{\lambda \in \mathbb{R}_{>0} \mid F(P) \neq \emptyset,\, C(\lambda P) = \lambda P\}\,.$$

Then $\lambda_P \geqslant \lambda_P^0$.

By Proposition 4.16, λP is canonically closed for any $\lambda > \lambda_P$. We claim that $\lambda_P P$ also is canonically closed. The polytopes λP and $F(\lambda P)$ continuously depend on λ. By Proposition 4.3, we have to show that for any primitive inward-pointing facet normal ν_Q of P, one has $\nu_Q \in S_F(\lambda_P P)$. Indeed, by taking the limit from above, we obtain

$$1 = \lim_{\lambda \to \lambda_P + 0} \big(\mathrm{Min}_{F(\lambda P)}(\nu_Q) - \mathrm{Min}_{\lambda P}(\nu_Q)\big) = \mathrm{Min}_{F(\lambda_P P)}(\nu_Q) - \mathrm{Min}_{\lambda_P P}(\nu_Q)\,;$$

that is, $\nu_Q \in S_F$. By the definition of λ_P, the convex polytope λP is not canonically closed for $\lambda_P^0 \leqslant \lambda < \lambda_P$. □

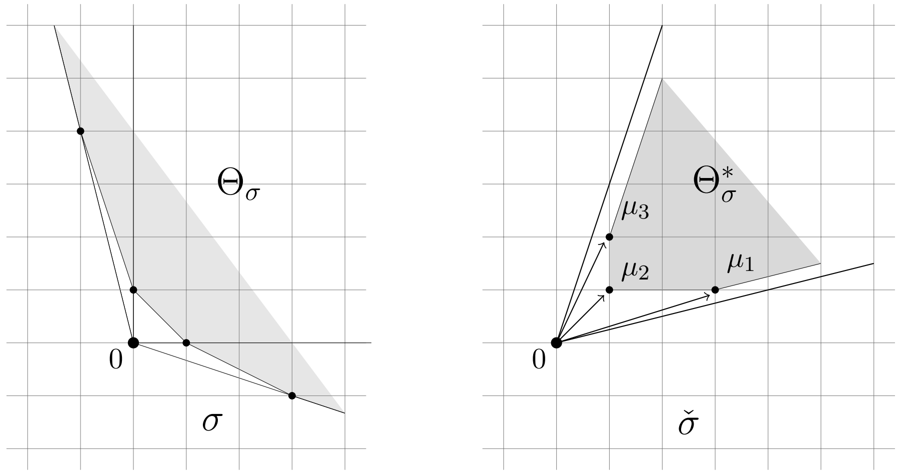

Figure 5. Rational finite polyhedral cone σ and its dual $\check{\sigma}$

Definition 4.18. Let P be a d-dimensional lattice polytope $P \subset M_{\mathbb{R}}$ with $F(P) \neq \emptyset$. We call P *integrally closed* if $\operatorname{Conv}(C(P) \cap M) = P$. It is clear that every canonically closed lattice polytope P is integrally closed.

Remark 4.19. By Proposition 4.4, every 2-dimensional lattice polytope is simultaneously integrally closed and canonically closed.

Remark 4.20. The 5-dimensional lattice polytope P in Example 3.15 is integrally closed, but it is not canonically closed because $F(P) = \{0\}$ while P is not reflexive (cf. Proposition 4.9).

Remark 4.21. Integrally closed lattice polytopes P with $F(P) = \{0\}$ are called *pseudoreflexive* in [Bat17]. They satisfy a combinatorial duality that generalizes the polar duality for reflexive polytopes. Anvar Mavlyutov suggested to use this generalized duality in the mirror symmetry for Calabi–Yau complete intersections in toric varieties [Mav11].

5. The Fine interior and the canonical refinement

In this section, we consider toric varieties associated with noncompact polyhedra [CLS11, Section 7.1].

Let $\sigma \subset N_{\mathbb{R}}$ be a d-dimensional rational finite polyhedral cone with vertex $0 = \sigma \cap -\sigma$. Consider the d-dimensional convex lattice polyhedron

$$\Theta_\sigma := \operatorname{Conv}(\sigma \cap (N \setminus \{0\})) \subset \sigma$$

with recession cone σ. Then the polar dual d-dimensional rational polyhedron

$$\Theta^*_\sigma := \{x \in M_{\mathbb{R}} \mid \langle x, y \rangle \geqslant 1 \ \ \forall y \in \Theta_\sigma\}$$

is contained in the dual cone $\check{\sigma} := \{x \in M_{\mathbb{R}} \mid \langle x, y \rangle \geqslant 0 \ \ \forall y \in \sigma\}$ which is the recession cone of Θ^*_σ; see Figure 5.

One has a natural bijection between compact facets $\theta_i \prec \Theta_\sigma$ and rational vertices $\mu_i \in \Theta^*_\sigma$ such that

$$\theta_i = \{y \in \Theta_\sigma \mid \langle \mu_i, y \rangle = 1\}\,.$$

Denote by $\mathcal{M}(\sigma) := \{\mu_1, \dots, \mu_s\}$ the set of all (rational) vertices of the polyhedron Θ^*_σ. Then $\mathcal{M}(\sigma)$ belongs to the interior of the recession cone $\check{\sigma}$, and we have

$$\Theta_\sigma = \{y \in \sigma \,|\, \langle \mu, y\rangle \geqslant 1 \ \ \forall \mu_i \in \mathcal{M}(\sigma)\}\,, \quad \Theta^*_\sigma = \mathrm{Conv}(\mathcal{M}(\sigma)) + \check{\sigma}\,.$$

Definition 5.1. Consider the upper-convex piecewise linear function on σ

$$\gamma_\sigma(y) := \mathrm{Min}_{\Theta^*_\sigma}(y) = \min_{x \in \Theta^*_\sigma} \langle x, y\rangle = \min_{\mu_i \in \mathcal{M}(\sigma)} \langle \mu_i, y\rangle \quad \forall y \in \sigma\,.$$

The domains of linearity of the function γ_σ correspond 1-to-1 to rational points $\mu_i \in \mathcal{M}(\sigma)$, and they define a fan σ_{can} which is called the *canonical refinement* of the full-dimensional cone σ. The canonical refinement σ_{can} is the normal fan of the convex polyhedron Θ^*_σ; see [CLS11, Definition 7.1.3].

Proposition 5.2 ([CLS11, Proposition 11.4.15]). *Let X_σ be the affine toric variety of a d-dimensional cone $\sigma \subset N_{\mathbb{R}}$. Then the canonical refinement σ_{can} of σ defines a quasi-projective toric variety $X_{\sigma_{\mathrm{can}}}$ together with a proper toric morphism*

$$\varphi_\sigma \colon X_{\sigma_{\mathrm{can}}} \to X_\sigma$$

such that the toric variety $X_{\sigma_{\mathrm{can}}}$ has $\mathbb{Q}$-Gorenstein canonical singularities and the canonical class $K_{X_{\sigma_{\mathrm{can}}}}$ is an ample $\mathbb{Q}$-Cartier divisor defined by the upper-convex σ_{can}-piecewise linear function $\gamma_\sigma = \mathrm{Min}_{\Theta^*_\sigma}$. In particular, Θ^*_σ is the supporting polyhedron of the canonical class on $X_{\sigma_{\mathrm{can}}}$.

Our interest in the canonical refinements is motivated by the following statement.

Theorem 5.3. *Let Σ_P be the normal fan of a d-dimensional generalized Delzant polytope $P \subset M_{\mathbb{R}}$; that is,*

$$P = \bigcap_{\sigma \in \Sigma_P(d)} (p_\sigma + \check{\sigma})\,,$$

where $p_\sigma \in P$ denotes the vertex of P corresponding to a d-dimensional cone $\sigma \in \Sigma_P(d)$. Then

$$F(P) = \bigcap_{\sigma \in \Sigma_P(d)} (p_\sigma + \Theta^*_\sigma)\,.$$

Proof. Take a d-dimensional cone $\sigma \in \Sigma_P(d)$. Then for any nonzero lattice vector $\nu \in \sigma \cap N$, we have $\mathrm{Min}_P(\nu) = \langle p_\sigma, \nu\rangle$, where p_σ is the vertex of P corresponding to σ. Hence, we obtain

$$p_\sigma + \Theta^*_\sigma = \{x \in M_{\mathbb{R}} \mid \langle x, \nu\rangle \geqslant \mathrm{Min}_P(\nu) + 1 \ \ \forall \nu \in \sigma \cap N \setminus \{0\}\}$$

because $\langle x, \nu\rangle \geqslant \mathrm{Min}_P(\nu) + 1$ is equivalent to $\langle x - p_\sigma, \nu\rangle \geqslant 1$. Since Σ_P is a complete fan, every lattice vector $\nu \in N$ is contained in some d-dimensional cone σ. This implies the statement. $\square$

Corollary 5.4. *Let $\sigma \in \Sigma_P[d]$ be a full-dimensional cone of the normal fan Σ_P. We denote by $\sigma_{\mathrm{can}}[1]$ be the set of all primitive lattice generators of 1-dimensional cones in the fan σ_{can}. Then*

$$F(P) = \Big\{x \in M_{\mathbb{R}} \mid \langle x, \nu\rangle \geqslant \mathrm{Min}_P(\nu) + 1 \ \ \forall \nu \in \bigcup_{\sigma \in \Sigma_P(d)} \sigma_{\mathrm{can}}[1]\Big\}\,.$$

One can extend the notion of the canonical refinement from "local" to "global" by gluing the canonical refinements σ_{can} of d-dimensional cones $\sigma \in \Sigma(d)$ of a complete fan Σ. Using the equalities

$$\Theta_\sigma \cap \Theta_{\sigma'} = \Theta_{\sigma \cap \sigma'} \quad \forall \sigma, \sigma' \in \Sigma(d)\,,$$

we can glue local canonical refinements $\sigma_{\rm can}$ of cones $\sigma \in \Sigma(d)$ and obtain a global canonical refinement $\Sigma^{\rm can}$ of the complete fan Σ.

In the next section, we will prove the following general statement about the canonical refinement $\Sigma_P^{\rm can}$ of the normal fan Σ_P of a canonically closed full-dimensional generalized Delzant polytopes P.

Theorem 5.5. *Let P be a canonically closed full-dimensional generalized Delzant polytope with $F(P) \neq \emptyset$. Then for any $\lambda > 1$, the normal fan $\Sigma_{F(\lambda P)}$ of the full-dimensional polytope $F(\lambda P)$ is the canonical refinement $\Sigma_P^{\rm can}$ of the normal fan Σ_P of P, and one has the Minkowski sum decomposition*

$$F(\lambda P) = F(P) + (\lambda - 1)P\,.$$

Corollary 5.6. *Let P be a canonically closed full-dimensional generalized Delzant polytope with $F(P) \neq \emptyset$. Then one has the Minkowski sum decomposition*

$$F(2P) = F(P) + P\,.$$

Proof. The statement follows from Theorem 5.5 if one takes $\lambda = 2$. □

Corollary 5.7. *Let P be an arbitrary d-dimensional generalized Delzant polytope, and let $\Sigma_P^{\rm can}$ be the canonical refinement of its normal fan Σ_P. Then $\Sigma_P^{\rm can}$ is the normal fan $\Sigma_{F(\lambda P)}$ of the Fine interior $F(\lambda P)$ for all $\lambda \gg 1$.*

Proof. For $\lambda \gg 1$, the polytope $F(\lambda P)$ is full-dimensional and the polytope λP is canonically closed (see Corollary 4.15 and Proposition 4.16). □

Proposition 5.8. *Let P be a canonically closed d-dimensional generalized Delzant polytope with $F(P) \neq \emptyset$. Then for any $\lambda > 1$, the combinatorial type of the polytope $F(\lambda P)$ is independent on λ, and the set of vertices of $F(\lambda P)$ equals*

$$\bigsqcup_{\sigma \in \Sigma_P(d)} \{\lambda p_\sigma + \mu_i \mid \mu_i \in \mathcal{M}(\sigma)\}\,.$$

Proof. By Theorem 5.5, for $\lambda > 1$, the normal fan of $F(\lambda P)$ is just the canonical refinement of Σ_P which is independent of $\lambda > 1$. The combinatorial structure of the polytope $F(\lambda)$ is dual to that of its normal fan. So it is also independent of $\lambda > 1$. Using the Minkowski sum decomposition

$$F(\lambda P) = (\lambda - 1)P + F(P)\,, \quad \lambda > 1\,,$$

we see that every vertex $q \in F(\lambda P)$ has a unique representation as a sum

$$q = (\lambda - 1)p_\sigma + r = \lambda p_\sigma + (r - p_\sigma)\,,$$

where p_σ is a vertex of P corresponding to some d-dimensional cone $\sigma \in \Sigma_P(d)$ and r is a vertex of $F(P)$ such that $r - p_\sigma \in \check{\sigma}$. Note that the vertex q of $F(\lambda P)$ corresponds to a full-dimensional cone of the canonical refinement $\Sigma_P^{\rm can}$ which is contained in σ and spanned by some compact facet $\theta_i \prec \Theta_\sigma$. Therefore, the vertex $r \in F(P) \subset P$ determines the facet $\theta_i \prec \Theta$ by the condition

$$\theta_i = \{y \in \Theta_\sigma \mid \langle r - p_\sigma, y\rangle = 1\}\,;$$

that is, $r - p_\sigma = \mu_i \in \mathcal{M}(\sigma)$ and $q = \lambda p_\sigma + \mu_i$. □

Corollary 5.9. *Let P be any 2-dimensional lattice polytope with $F(P) \neq \emptyset$. Then $F(P)$ is the convex hull of the lattice points*

$$\bigcup_{\sigma \in \Sigma_P(2)} \{p_\sigma + \mu_i \,|\, \mu_i \in \mathcal{M}(\sigma)\}\,.$$

Proof. We know that if P is a 2-dimensional lattice polytope with $F(P) \neq \emptyset$, then P is canonically closed and $F(P) = \mathrm{Conv}(\mathrm{Int}(P) \cap M)$. For any 2-dimensional cone $\sigma \in \Sigma_P(2)$, the set $\mathcal{M}(\sigma)$ consists of lattice vectors because the canonical refinement Σ_P^{can} of Σ_P defines a 2-dimensional Gorenstein toric variety. By Proposition 5.8, for all $\lambda > 1$, the set of vertices of the Fine interior $F(\lambda P)$ equals

$$\bigsqcup_{\sigma \in \Sigma_P(2)} \{\lambda p_\sigma + \mu_i \mid \mu_i \in \mathcal{M}(\sigma)\}\,.$$

Taking the limit $\lambda \to 1$, we obtain that all vertices of $F(P)$ are contained in the finite set of the limiting lattice points

$$\bigcup_{\sigma \in \Sigma_P(2)} \{p_\sigma + \mu_i \mid \mu_i \in \mathcal{M}(\sigma)\} \subset \mathrm{Int}(P) \cap M\,.$$

Note that after taking the limit $t \to 1$, two disjoint finite subsets

$$\{p_\sigma + \mu_i \mid \mu \in \mathcal{M}(\sigma)\}\,, \quad \{p_{\sigma'} + \mu_j \mid \mu_j \in \mathcal{M}(\sigma')\}$$

corresponding to different 2-dimensional cones $\sigma, \sigma' \in \Sigma_P(2)$ may get common elements. This happens, for example, for all 2-dimensional reflexive polygons. □

6. The canonical toric variety $\widetilde{V}$

Definition 6.1. Let P be a full-dimensional generalized Delzant polytope with $F(P) \neq \emptyset$. Consider the continuous real function $\delta_P \colon N_{\mathbb{R}} \to \mathbb{R}$,

$$\delta_P(y) := \mathrm{Min}_{F(P)}(y) - \mathrm{Min}_P(y)\,, \quad y \in N_{\mathbb{R}}\,.$$

Proposition 6.2. *The function δ_P is linear on cones of the normal fan of the Minkowski sum $P + F(P)$, and it has the following properties:*

(a) *$\delta_P(y) \geqslant 0$ for all $y \in N_{\mathbb{R}}$;*

(b) *$\delta_P(y) = 0$ if and only if $y = 0$;*

(c) *$\delta_P(\lambda y) = \lambda \delta_P(y)$ for all $\lambda \in \mathbb{R}_{\geqslant 0}$;*

(d) *$\{\nu \in N \mid \delta_P(\nu) = 1\} = S_F(P)$;*

(e) *$\{\nu \in N \mid 0 < \delta_P(\nu) < 1\} = \emptyset$.*

In particular,

$$\Delta_P := \{y \in N_{\mathbb{R}} \mid \delta_P(y) \leqslant 1\}$$

is a full-dimensional compact subset in $N_{\mathbb{R}}$ containing $S_F(P)$ on its boundary.

Proof. The normal fan Σ' of the Minkowski sum $P + F(P)$ is the coarsest common refinement of the normal fans Σ_P and $\Sigma_{F(P)}$; see [CLS11, Proposition 6.2.13]. Therefore, the upper-convex functions Min_P and $\mathrm{Min}_{F(P)}$ are linear on each cone of the normal fan Σ'. This implies item (c). Since the polytope $F(P)$ is strictly contained in the interior of P, we have $\mathrm{Min}_{F(P)}(y) > \mathrm{Min}_P(y)$ for all $y \in N_{\mathbb{R}} \setminus \{0\}$. This implies items (a) and (b). The statement (d) follows from the definition of $S_F(P)$, and item (e) follows from the inequality $\mathrm{Min}_{F(P)}(\nu) - \mathrm{Min}_P(\nu) \geqslant 1$ for all nonzero $\nu \in N$. □

We note that the compact set Δ_P is usually not convex (see Figures 6 and 8).

Theorem 6.3. *Let P be a full-dimensional generalized Delzant polytope ($F(P) \neq \emptyset$). Denote by $\widetilde{\Sigma}$ the normal fan of the Minkowski sum $\widetilde{P} := F(P) + C(P)$. Then any primitive inward-pointing facet normal $\nu \in \widetilde{\Sigma}[1]$ is contained in the support of the Fine interior $S_F(P)$.*

Proof. Since all generalized Delzant polytopes λP ($\lambda \in \mathbb{R}_{>0}$) have the same normal fan, by Proposition 3.11, there exists a finite subset $S \subset N$ such that $S_F(\lambda P) \subseteq S$ for all $\lambda \geqslant 1$. Recall that for any $\lambda > 1$, the Fine interior $F(\lambda P)$ is full-dimensional because it contains the full-dimensional Minkowski sum $F(P) + (\lambda - 1)P$ (see the proof of Proposition 4.16). On the other hand, any primitive inward-pointing facet normal ν_Q of the polytope $F(\lambda P)$ belongs to $S_F(\lambda P)$; that is, $\nu_Q \in S$. For sufficiently small $\varepsilon > 0$, the set $\{\nu_1, \ldots, \nu_k\} = \Sigma[1] := \Sigma_{F((1+\varepsilon)P)}[1] \subset N$ of all primitive inward-pointing normals to facets $Q_1(\varepsilon), \ldots, Q_k(\varepsilon)$ of the full-dimensional polytope $F((1+\varepsilon)P)$ does not depend on ε, and the corresponding supporting affine hyperplanes $H_i(\varepsilon)$ spanned by the facets $Q_i(\varepsilon)$ contribute to the Fine interior $F((1+\varepsilon)P)$ of the polytope $(1+\varepsilon)P$; that is, $\Sigma[1] = \{\nu_1, \ldots, \nu_k\} \subseteq S_F((1+\varepsilon)P)$ and

$$\mathrm{Min}_{F((1+\varepsilon)P)}(\nu_i) - \mathrm{Min}_{(1+\varepsilon)P}(\nu_i) = 1 \quad \forall i \in \{1, \ldots, k\}\,.$$

The polytopes $(1+\varepsilon)P$ and $F((1+\varepsilon)P)$ depend continuously on ε. Hence, by taking the limit $\varepsilon \to 0$, we obtain that

$$\mathrm{Min}_{F(P)}(\nu_i) - \mathrm{Min}_{P}(\nu_i) = 1 \quad \forall i \in \{1, \ldots, k\}\,;$$

that is, $\{\nu_1, \ldots, \nu_k\} = \Sigma[1] \subseteq S_F(P)$.

Next we want to show that for sufficiently small $\varepsilon > 0$, one has the following Minkowski sum decomposition:

$$F((1+\varepsilon)P) = F(P) + \varepsilon C(P)\,.$$

Note that the full-dimensional polytope $F((1+\varepsilon)P)$ is the intersection of the k half-spaces defined by primitive lattice facet normals $\nu_1, \ldots, \nu_k$. So we have

$$\begin{aligned} F((1+\varepsilon)P) &= \{x \in M_{\mathbb{R}} \mid \langle x, \nu\rangle \geqslant \mathrm{Min}_{(1+\varepsilon)P}(\nu) + 1 \;\; \forall \nu \in \Sigma[1]\} \\ &= \{x \in M_{\mathbb{R}} \mid \langle x, \nu\rangle \geqslant (1+\varepsilon)\,\mathrm{Min}_{P}(\nu) + 1 \;\; \forall \nu \in \Sigma[1]\} \\ &= \{x \in M_{\mathbb{R}} \mid \langle x, \nu\rangle \geqslant (\mathrm{Min}_{P}(\nu) + 1) + \varepsilon\,\mathrm{Min}_{P}(\nu) \;\; \forall \nu \in \Sigma[1]\}\,. \end{aligned}$$

This implies that

$$F(P) + \varepsilon C(P) \subseteq F((1+\varepsilon)P$$

and that the full-dimensional polytope $F((1+\varepsilon)P)$ defines an ample $\mathbb{R}$-Cartier divisor

$$D(\varepsilon) := \sum_{i=1}^{k} -(1+\varepsilon)\,\mathrm{Min}_{P}(\nu_i) D_i - \sum_{i=1}^{k} D_i$$

on the projective toric variety $V := V(\Sigma)$ of the fan Σ which is independent of the small $\varepsilon > 0$. In particular, $D(\varepsilon) - D(\varepsilon') = (\varepsilon' - \varepsilon)\sum_{i=1}^{k} \mathrm{Min}_{P}(\nu_i) D_i$ is an $\mathbb{R}$-Cartier divisor on V. Therefore,

$$D := -\sum_{i=1}^{k} \mathrm{Min}_{P}(\nu_i) D_i$$

is an $\mathbb{R}$-Cartier divisor on V.

We claim that any primitive inward-pointing facet normal $\nu \in S_F(P)$ of $C(P)$ is contained in $\Sigma[1]$. Indeed, let $\Gamma_\nu \subset C(P)$ be a facet of $C(P)$ defined by the equation $\langle x, \nu\rangle = \mathrm{Min}_{P}(\nu)$. Then there exist a point $x' \in F(P)$ such that $\langle x', \nu\rangle = \mathrm{Min}_{P}(\nu) + 1$. The $(d-1)$-dimensional polytope $x' + \varepsilon\Gamma_\nu$ is contained in $F(P) + \varepsilon C(P) \subseteq F((1+\varepsilon)P)$, and

$$\langle x'', \nu\rangle = (1+\varepsilon)\operatorname{Min}_P + 1 = \operatorname{Min}_{(1+\varepsilon)P}(\nu) + 1 \quad \forall x'' \in x' + \varepsilon\Gamma_\nu\,.$$

Hence, $x' + \varepsilon\Gamma$ is a facet of $F((1+\varepsilon)P)$ and $\nu \in \Sigma[1]$.

Since the full-dimensional polytope $C(P)$ equals the intersection of all half-spaces $\langle *, \nu\rangle \geqslant \operatorname{Min}_P(\nu)$ defined by primitive inward-pointing facet normals $\nu \in S_F(P)$, and $\operatorname{Min}_{C(P)}(\nu) = \operatorname{Min}_P(\nu)$ for all $\nu \in S_F(P)$ (see Proposition 3.17(a)), we obtain

$$C(P) = \{x \in M_{\mathbb{R}} \mid \langle x, \nu\rangle \geqslant \operatorname{Min}_P(\nu) \ \ \forall \nu \in \Sigma[1]\}\,,$$

that is, that the Σ-piecewise linear function corresponding to the $\mathbb{R}$-Cartier divisor D equals the upper-convex Σ-piecewise linear function $\operatorname{Min}_{C(P)}$.

On the other hand, $\operatorname{Min}_{F(P)}$ is the limit $\varepsilon \to 0$ of convex Σ-piecewise linear functions $\operatorname{Min}_{F((1+\varepsilon)P)}$. Therefore, $\operatorname{Min}_{F(P)}$ is also an upper-convex Σ-piecewise linear function. Now we obtain that for sufficiently small $\varepsilon > 0$, the strictly upper-convex Σ-piecewise linear function $\operatorname{Min}_{F((1+\varepsilon)P)}$ equals the sum of two upper-convex Σ-piecewise linear functions:

$$\operatorname{Min}_{F((1+\varepsilon)P)} = \operatorname{Min}_{F(P)} + \varepsilon \operatorname{Min}_{C(P)}\,.$$

Therefore, $F((1+\varepsilon)P) = F(P) + \varepsilon C(P)$, and Σ is the coarsest common refinement of the normal fans $\Sigma_{F(P)}$ and $\Sigma_{C(P)}$ (see [CLS11, Proposition 6.2.13]). In particular, the projective fan Σ with $\Sigma[1] \subset S_F(P)$ equals the normal fan $\widetilde{\Sigma}$ of the Minkowski sum $\widetilde{P} := C(P) + F(P)$; that is, $\widetilde{\Sigma}[1] \subset S_F(P)$. □

Corollary 6.4. *Let $P \subset M_{\mathbb{R}}$ be a full-dimensional lattice polytope with $F(P) \neq \emptyset$, and let $\widetilde{P} := F(P) + C(P)$. Then the normal fan of $\widetilde{P}$ equals the normal fan of $F((1+\varepsilon)P)$ for sufficiently small positive ε.*

Proof. The statement immediately follows from the proof of Theorem 6.3. □

Corollary 6.5. *Let $P \subset M_{\mathbb{R}}$ be a full-dimensional lattice polytope with $F(P) \neq \emptyset$, and let $\widetilde{P} := F(P) + C(P)$. Denote by $\widetilde{V}$ the projective toric variety of the normal fan $\widetilde{\Sigma}$ of $\widetilde{P}$. Then $\widetilde{V}$ is a $\mathbb{Q}$-Gorenstein toric variety with at worst canonical singularities.*

Proof. Since $\widetilde{\Sigma}$ is a common refinement of the normal fans $\Sigma_{C(P)}$ and $\Sigma_{F(P)}$, both functions $\operatorname{Min}_{C(P)}$ and $\operatorname{Min}_{F(P)}$ are $\widetilde{\Sigma}$-piecewise linear. It follows that

$$\delta_{C(P)} := \operatorname{Min}_{F(P)} - \operatorname{Min}_{C(P)}$$

is also $\widetilde{\Sigma}$-piecewise linear. By Proposition 3.17(a), the $\widetilde{\Sigma}$-piecewise linear function $\delta_{C(P)}$ has value 1 on every primitive lattice vector $\nu \in \widetilde{\Sigma}[1] \subseteq S_F(P)$. Therefore, the canonical class of $\widetilde{V}$ is a $\mathbb{Q}$-Cartier divisor defined by the $\widetilde{\Sigma}$-piecewise linear function $\delta_{C(P)}$. By Proposition 3.17 and Corollary 3.18, we can apply Proposition 6.2 to the rational polytope $C(P)$ and obtain

$$\delta_{C(P)}(\nu) = \operatorname{Min}_{F(P)}(\nu) - \operatorname{Min}_{C(P)}(\nu) \geqslant 1 \quad \forall \nu \in N \setminus \{0\}\,.$$

Therefore, all singularities of the toric variety $\widetilde{V}$ are at worst canonical. □

Corollary 6.6. *Let $\widehat{\Sigma}$ be a projective refinement of the fan $\widetilde{\Sigma}$ such that $\widehat{\Sigma}[1] = S_F(P)$. Then the corresponding projective toric morphism $\varphi\colon \widehat{V} \to \widetilde{V}$ is crepant. Furthermore, if $\psi\colon V \to \widetilde{V}$ is a toric desingularization of $\widetilde{V}$ defined by a regular refinement Σ of $\widetilde{\Sigma}$, then*

$$K_V = \psi^* K_{\widetilde{V}} + \sum_{\nu \in \Sigma[1] \setminus \widetilde{\Sigma}[1]} \widetilde{a}(\nu) D_\nu\,,$$

where $\widetilde{a}(\nu) := \delta_{C(P)}(\nu) - 1 = \mathrm{Min}_{F(P)}(\nu) - \mathrm{Min}_{C(P)}(\nu) - 1$. *In particular,* $\varphi\colon \widehat{V} \to \widetilde{V}$ *is a maximal projective partial crepant desingularization of* $\widetilde{V}$.

Proof. By Proposition 3.17(a), the $\widetilde{\Sigma}$-piecewise linear function $\delta_{C(P)} := \mathrm{Min}_{F(P)} - \mathrm{Min}_{C(P)}$ representing the canonical class $K_{\widetilde{V}}$ takes value 1 on every lattice vector $\nu \in S_F(P) = \widehat{\Sigma}[1]$. Therefore, $\varphi^* K_{\widetilde{V}} = K_{\widehat{V}}$; that is, φ is crepant.

If $\psi\colon V \to \widetilde{V}$ is an arbitrary toric desingularization of $\widetilde{V}$ defined by a regular refinement Σ of $\widetilde{\Sigma}$, then the discrepancy $\widetilde{a}(\nu)$ for a divisorial valuation $\nu \in N$ is the difference between the values of the Σ-piecewise linear functions corresponding to K_V and $K_{\widetilde{V}}$; that is,

$$\widetilde{a}(\nu) = \delta_{C(P)}(\nu) - 1 = (\mathrm{Min}_{F(P)}(\nu) - \mathrm{Min}_{C(P)}(\nu)) - 1\,.$$

In particular, $\widetilde{a}(\nu) = 0$ if and only if $\nu \in S_F(P)$; that is, φ is a maximal projective partial crepant desingularization of $\widetilde{V}$. $\square$

Theorem 6.7. *Let* $P \subset M_{\mathbb{R}}$ *be a full-dimensional lattice polytope with* $F(P) \neq \emptyset$, *let* $\widetilde{P} = C(P) + F(P)$, *and let* $\widetilde{\Sigma}$ *be the normal fan of* $\widetilde{P}$. *Consider an arbitrary 2-dimensional cone* $\sigma \in \widetilde{\Sigma}$, *and denote by* $\nu_i, \nu_j \in \widetilde{\Sigma}[1] \subseteq S_F(P)$ *its spanning primitive lattice vectors. Then*

(a) *the* $(d-2)$*-dimensional affine subspace* $L^1_\sigma \subset M_{\mathbb{R}}$ *defined by the equations*

$$\langle x, \nu_i\rangle = \mathrm{Min}_P(\nu_i) + 1\,, \quad \langle x, \nu_j\rangle = \mathrm{Min}_P(\nu_j) + 1$$

has nonempty intersection with $F(P)$*;*

(b) *the* $(d-2)$*-dimensional affine subspace* $L^0_\sigma \subset M_{\mathbb{R}}$ *defined by the equations*

$$\langle x, \nu_i\rangle = \mathrm{Min}_P(\nu_i)\,, \quad \langle x, \nu_j\rangle = \mathrm{Min}_P(\nu_j)$$

contains at least one vertex of P.

Proof. (a) By Corollary 6.4, the normal fan of $\widetilde{P}$ equals the normal fan of the Fine interior of $(1+\varepsilon)P$ for small $\varepsilon > 0$. Therefore, for these ε, the d-dimensional polytope $F((1+\varepsilon)P)$ has a $(d-2)$-dimensional face $\Gamma_{\sigma,\varepsilon}$ whose affine span is the intersection $L^1_{\sigma,\varepsilon}$ of two affine hyperplanes defined by the equations

$$\langle x, \nu_i\rangle = \mathrm{Min}_{(1+\varepsilon)P}(\nu_i) + 1\,, \quad \langle x, \nu_j\rangle = \mathrm{Min}_{(1+\varepsilon)P}(\nu_j) + 1\,.$$

Assume that the intersection $L^1_\sigma \cap F(P)$ is empty. This means that the distance between the compact convex set $F(P)$ and the affine linear subspace L^1_σ is positive. On the other hand, we have

$$L^1_\sigma = \lim_{\varepsilon\to 0} L^1_{\sigma,\varepsilon}\,, \quad F(P) = \lim_{\varepsilon \to 0} F((1+\varepsilon)P)\,.$$

This implies that for sufficiently small $\varepsilon > 0$, the intersection $L_{\sigma,\varepsilon} \cap F((1+\varepsilon)P)$ must also be empty. The latter contradicts the fact that the $(d-2)$-dimensional face $\Gamma_{\sigma,\varepsilon}$ of $F((1+\varepsilon)P)$ is contained in $L^1_{\sigma,\varepsilon}$ for all sufficiently small $\varepsilon > 0$.

(b) Consider the 2-dimensional sublattice $N_\sigma \subset N$ spanned by $N \cap \sigma$. Then $N_\sigma \cong \mathbb{Z}^2$ is a direct summand of N. Let e_1, e_2 be a $\mathbb{Z}$-basis of N_σ. Consider the lattice projection

$$\pi_\sigma\colon M \to \mathbb{Z}^2\,, \quad m \mapsto (\langle m, e_1\rangle, \langle m, e_2\rangle)\,.$$

Since $\widetilde{V}$ has at worst canonical singularities, the lattice triangle with vertices 0, ν_i, ν_j has no interior lattice points, and one can choose a $\mathbb{Z}$-basis $e_1, e_2 \in N_\sigma$ in such a way that $\nu_i = e_1$ and $\nu_j = e_1 + ke_2$ for some positive integer k. We claim that all $k+1$ lattice vectors

$$e_1, e_1 + e_2, e_1 + 2e_2, \ldots, e_1 + (k-1)e_2, e_1 + ke_2$$

belong to $S_F(P)$. For the lattice vectors $e_1 = \nu_i$ and $e_1 + ke_2 = \nu_j$, this follows from Theorem 6.3. By Proposition 3.17(a), we have

$$\begin{aligned}\mathrm{Min}_{C(P)}(\nu_i) + 1 = \mathrm{Min}_P(\nu_i) + 1 = \mathrm{Min}_{F(P)}(\nu_i)\,,\\ \mathrm{Min}_{C(P)}(\nu_j) + 1 = \mathrm{Min}_P(\nu_j) + 1 = \mathrm{Min}_{F(P)}(\nu_j)\,.\end{aligned}$$

The functions $\mathrm{Min}_{F(P)}$ and $\mathrm{Min}_{C(P)}$ are linear on the 2-dimensional cone σ. This implies

$$\mathrm{Min}_{C(P)}(\nu) + 1 = \mathrm{Min}_{F(P)}(\nu)$$

for any lattice point ν of the segment $[\nu_i, \nu_j]$. By Corollary 3.18, all these points belong to $S_F(P) = S_F(C(P))$. In particular, we have

$$\mathrm{Min}_P(\nu) + 1 = \mathrm{Min}_{F(P)}(\nu) \quad \forall \nu \in N_\sigma \cap [\nu_i, \nu_j]\,;$$

that is, $e_1 + re_2 \in S_F(P)$ for all $r \in \{0, 1, \dots, k\}$.

Now apply item (a) and take a point $p_0 \in F(P) \cap L^1_\sigma$. Then

$$\langle p_0, \nu_i\rangle = \mathrm{Min}_P(\nu_i) + 1 = \mathrm{Min}_{F(P)}(\nu_i)\,, \quad \langle p_0, \nu_j\rangle = \mathrm{Min}_P(\nu_j) + 1 = \mathrm{Min}_{F(P)}(\nu_j)\,.$$

By the linearity of $\mathrm{Min}_{F(P)}$ on σ, we obtain

$$\langle p_0, \nu\rangle = \mathrm{Min}_P(\nu) + 1 \quad \forall \nu \in N_\sigma \cap [\nu_i, \nu_j]\,.$$

In particular, this implies

$$\langle p_0, \nu\rangle \in \mathbb{Z} \quad \forall \nu \in N_\sigma \cap [\nu_i, \nu_j]\,.$$

Since the set $N_\sigma \cap [\nu_i, \nu_j]$ generates the sublattice $N_\sigma \subseteq N$, we obtain that the π_σ-projection of $p_0 \in M_{\mathbb{Q}}$ to $\mathbb{Q}^2$ is a lattice point $\overline{p_0} \in \mathbb{Z}^2$.

Obviously, the projection $P_\sigma := \pi_\sigma(P) \subset \mathbb{R}^2$ is a lattice polygon contained in the angle A_σ defined by two inequalities $\langle x, \nu_i\rangle \geqslant \mathrm{Min}_P(\nu_i)$ and $\langle x, \nu_j\rangle \geqslant \mathrm{Min}_P(\nu_j)$:

$$P_\sigma \subset A_\sigma = \big\{(x_1, x_2) \in \mathbb{R}^2 \mid x_1 \geqslant \mathrm{Min}_P(\nu_i),\ x_1 + kx_2 \geqslant \mathrm{Min}_P(\nu_j)\big\}\,,$$

and both sides of A_σ contain lattice vertices of P_σ. Moreover, the lattice point $\overline{p_0} \in \mathbb{Z}^2 \subset \mathbb{R}^2$ is the intersection of two lines with equations

$$x_1 = \mathrm{Min}_P(\nu_i) + 1\,, \quad x_1 + kx_2 = \mathrm{Min}_P(\nu_j) + 1\,.$$

Therefore, the intersection of two another lines with equations

$$x_1 = \mathrm{Min}_P(\nu_i)\,, \quad x_1 + kx_2 = \mathrm{Min}_P(\nu_j)$$

is also a lattice point $q \in \mathbb{Z}^2$ which is the vertex of the angle $A_\sigma \subset \mathbb{R}^2$. By shifting the lattice polytope P, we can now assume without loss of generality that $\mathrm{Min}_P(\nu_i) = \mathrm{Min}_P(\nu_j) = 0$. Then we obtain $q = (0, 0)$ and $\overline{p_0} = (1, 0)$, the 2-dimensional lattice polygon $P_\sigma = \pi_\sigma(P)$ is contained in the angle

$$A_\sigma = \big\{(x_1, x_2) \in \mathbb{R}^2 \mid x_1 \geqslant 0, x_1 + kx_2 \geqslant 0\big\}\,,$$

and P_σ has lattice vertices on both sides of A_σ. Moreover, P_σ contains the lattice point $(1, 0) = \overline{p_0} = \pi_\sigma(p_0) \in \pi_\sigma(F(P))$ in its interior. Elementary convexity considerations show that all above conditions for P_σ together with the condition $(0, 0) \notin P_\sigma$ can be satisfied only if $k = 1$ and $(0, 1)$, $(1, -1)$ are vertices of P_σ. In the last situation, the lattice vector $2e_1 + e_2 = e_1 + (e_1 + e_2) = \nu_i + \nu_j$ must belongs to $S_F(P)$ because the line $1 = 2x_1 + x_2$ defines a side of P_σ and the line $2 = 2x_1 + x_2$ contains $(1, 0) = \pi_\sigma(p_0) \in \pi_\sigma(F(P))$. The conclusion that $n_i, n_j, \nu_i + \nu_j \in S_F(P)$ leads to contradiction since $S_F(P) = S_F(C(P))$, $F(P) = F(C(P))$ and the two functions $\mathrm{Min}_{F(P)}$,

$\operatorname{Min}_{C(P)}$ are linear on the cone $\sigma \in \widetilde{\Sigma}$ spanned by n_i and n_j (see Remark 3.10). So we obtain that $(0,0)$ must be a vertex of P_σ, and there exists a vertex $v \in P \cap L^0_\sigma$ such that $\pi_\sigma(v) = (0,0)$. □

Proof of Theorem 5.5. Let P be a full-dimensional canonically closed Delzant polytope. By Proposition 4.16, we have $C(\lambda P) = \lambda P$ for all $\lambda \geqslant 1$. In the proof of Theorem 6.3, we have shown that for sufficiently small $\varepsilon = \lambda - 1 > 0$, we have $F(\lambda P) = F(P) + (\lambda - 1)P$ and the normal fan Σ of $F((1+\varepsilon)P)$ equals the normal fan $\widetilde{\Sigma}$ of $F(P) + P$. In other words, $\widetilde{\Sigma}$ is a refinement of the normal fan Σ_P such that the corresponding refinement of any full-dimensional cone $\sigma \in \Sigma_P(d)$ is determined by the full-dimensional domains of linearity in σ of the $\widetilde{\Sigma}$-piecewise linear function

$$\delta_P(y) = \operatorname{Min}_{F(P)}(y) - \operatorname{Min}_P(y) = \operatorname{Min}_{F(P)}(y) - \langle p_\sigma, y\rangle \quad \forall y \in \sigma\,,$$

where p_σ is a vertex of P corresponding to $\sigma \in \Sigma_P(d)$. Since $F(P) \subset p_\sigma + \Theta^*_\sigma$ (see Theorem 5.3), we obtain that $\gamma_\sigma(y) \leqslant \operatorname{Min}_{F(P)}(y) - \langle p_\sigma, y\rangle$ for all $y \in \sigma$. This implies

$$\Theta_\sigma = \{y \in \sigma \mid \gamma_\sigma(y) \geqslant 1\} \subseteq \{y \in \sigma \mid \operatorname{Min}_{F(P)}(y) - \langle p_\sigma, y\rangle \geqslant 1\}\,.$$

Since $\widetilde{\Sigma}[1] \subset S_F(P)$ and $\operatorname{Min}_{F(P)}(\nu) - \langle p_\sigma, \nu\rangle = 1$ for all $\nu \in S_F(P)$, we obtain the opposite inclusion

$$\{y \in \sigma \mid \operatorname{Min}_{F(P)}(y) - \langle p_\sigma, y\rangle \geqslant 1\} \subset \Theta_\sigma\,.$$

Therefore, we obtain the equality of the convex sets

$$\{y \in \sigma \mid \operatorname{Min}_{F(P)}(y) - \langle p_\sigma, y\rangle \geqslant 1\} = \Theta_\sigma$$

for any $\sigma \in \Sigma_P(d)$; that is, the refinement of Σ_P defined by the fan $\widetilde{\Sigma}$ is canonical. The equality $\{y \in \sigma \mid \operatorname{Min}_{F(P)}(y) - \langle p_\sigma, y\rangle \geqslant 1\} = \Theta_\sigma$ shows that full-dimensional subcones in the canonical refinement of a cone $\sigma \in \Sigma_P$ correspond 1-to-1 to rational vectors $\mu = f - p_\sigma \in \mathcal{M}(\sigma)$, where f is a vertex of $F(P)$.

It remains to show $F(\lambda P) = F(P) + (\lambda - 1)P$ for all $\lambda > 1$. By Corollary 5.4, the 1-dimensional primitive lattice generators of the normal fan of $F(\lambda P)$ are contained in $\Sigma_{\text{can}}[1] = \{\nu_1, \ldots, \nu_k\}$, where $\Sigma_{\text{can}} = \widetilde{\Sigma}$ is the canonical refinement of the normal fan of λP. Therefore, we can obtain $F(\lambda P)$ as

$$F(\lambda P) = \left\{x \in M_{\mathbb{R}} \mid \langle x, \nu\rangle \geqslant \operatorname{Min}_{\lambda P}(\nu) + 1 \ \ \forall \nu \in \widetilde{\Sigma}[1]\right\}.$$

On the other hand, the Minkowski sum $F(P) + (\lambda - 1)P$ corresponds to the sum of two nef $\mathbb{R}$-Cartier divisors on $\widetilde{V}$:

$$-\sum_{i=1}^{k}(1 + \operatorname{Min}_P(\nu_i))D_i\,, \quad -\sum_{i=1}^{k}(\lambda - 1)\operatorname{Min}_P(\nu_i))D_i\,.$$

By adding these divisors, we obtain the ample $\mathbb{R}$-Cartier divisor on $\widetilde{V}$

$$-\sum_{i=1}^{k}(1 + \lambda \operatorname{Min}_P(\nu_i))D_i = -\sum_{i=1}^{k}(1 + \operatorname{Min}_{\lambda P}(\nu_i))D_i\,.$$

This implies $F(\lambda P) = F(P) + (\lambda - 1)P$. □

Corollary 6.8. *Let P be a full-dimensional generalized Delzant polytope ($F(P) \neq \emptyset$). Then*

$$\widetilde{P} := F(P) + C(P) = F(2C(P))\,.$$

Proof. It is sufficient to apply Theorem 5.5 to the canonically closed polytope $C(P)$ and put $\lambda = 2$, because $F(C(P)) = F(P)$ (see Proposition3.17(b)). □

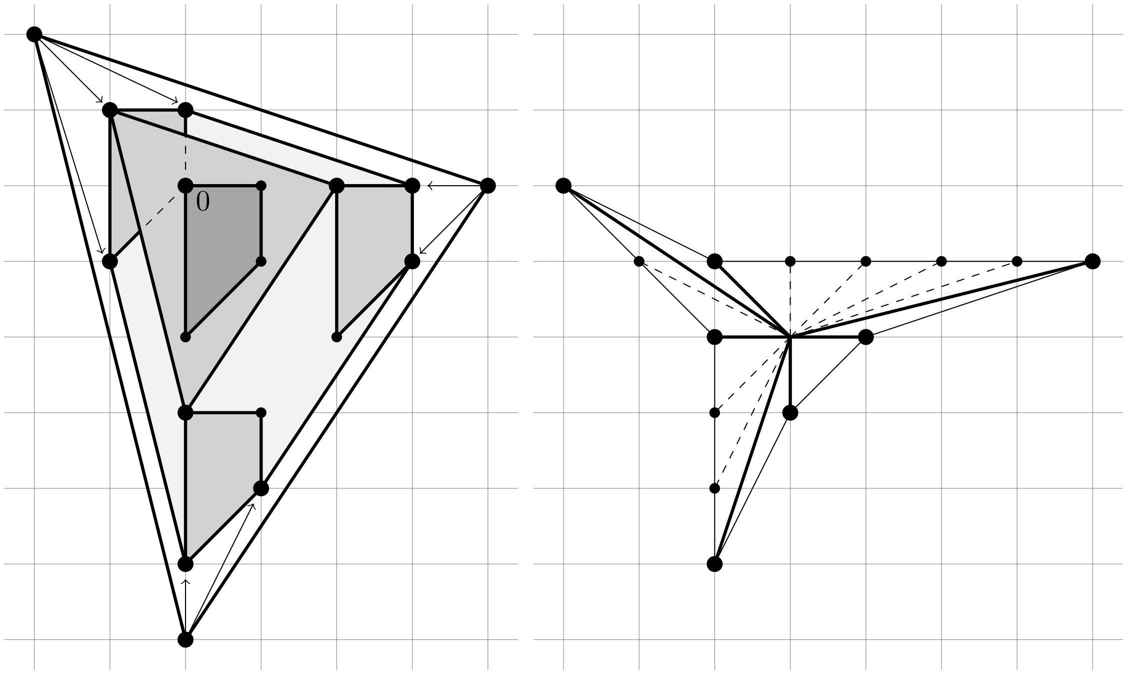

Figure 6. A 2-dimensional lattice polytope and the normal fan of the Minkowski sum $P + I(P)$

The following result of Robert Koelman was pointed out to the author by Dimitrios Dais; see Figure 6 for an example.

Corollary 6.9 ([Koe91, Corollary 2.4.3, p. 52]). *Let P be a 2-dimensional lattice polytope. Assume that $I(P) := \mathrm{Conv}(P^\circ \cap M)$ is nonempty. Then the normal fan of the Minkowski sum $P + I(P)$ defines a projective toric surface $\widetilde{V}$ with at worst Gorenstein quotient A_n-singularities.*

Proof. By Remark 3.5 and Proposition 4.4, we have $F(P) = I(P)$ and $C(P) = P$. Therefore,

$$\widetilde{P} = C(P) + F(P) = P + I(P)\,.$$

Now the statement follows from Theorem 6.3 and Corollary 6.5 because 2-dimensional canonical toric singularities are exactly A_n-singularities; see [CLS11, Proposition 11.2.8]. □

7. Canonical models of hypersurfaces and the adjunction

Let $V = V_\Sigma$ be any d-dimensional normal toric variety defined by a fan $\Sigma \subset N_\mathbb{R}$. We consider the Zariski closure $\overline{Z}$ in V of an affine hypersurface $Z \subset \mathbb{T}^d$ defined by zeros of a Laurent polynomial f. We need the following general statement.

Proposition 7.1. *Let $Z \subset \mathbb{T}^d$ be an arbitrary (not necessarily nondegenerate) affine hypersurface defined by a Laurent polynomial $f(\mathbf{t})$ with Newton polytope P. Then the Zariski closure $\overline{Z}$ of Z in a normal toric variety V_Σ is linearly equivalent to the torus-invariant Weyl divisor*

$$D_f := -\sum_{\nu \in \Sigma[1]} \mathrm{Min}_P(\nu) D_\nu\,.$$

Proof. The Weyl divisor class group of a normal toric variety V_Σ does not depend on torus orbits of codimension greater than or equal to 2. Therefore, we can assume that the fan Σ consists only of cones of dimension 0 and 1, where 1-dimensional cones $\sigma = \mathbb{R}_{\geqslant 0}\nu \in \Sigma[1]$ are spanned by primitive lattice vectors $\nu \in \Sigma[1]$. The toric variety V_Σ is smooth because V_Σ is normal. Therefore, Weyl divisors D on V_Σ can be identified with the invertible sheaves $\mathcal{O}(D)$ using the local equations of D in V_Σ. We may consider the Zariski closure $\overline{Z}$ as the zero set of a natural global section $s \in H^0\big(V, \mathcal{O}(\overline{Z})\big)$ and use the open affine covering

$$V_\Sigma = \bigcup_{\sigma \in \Sigma(1)} U_\sigma\,,$$

where every open subset U_σ consists of two torus orbits: the open torus $\mathbb{T}^d$ and the closed torus orbit $D_\nu \subset U_\sigma$ isomorphic to the divisor

$$\{t_1 = 0\} \subset \mathbb{C} \times (\mathbb{C}^*)^{d-1} = \operatorname{Spec} \mathbb{C}\big[t_1, t_2^{\pm}, \ldots, t_d^{\pm}\big] \cong U_\sigma\,.$$

The maximal ideal (t_ν) in the local ring of $D_\nu \subset U_\sigma \subseteq V_\Sigma$ defines a discrete valuation of $\mathbb{C}(t_1, t_2, \ldots, t_d)$ which has value 1 on $t_\nu := t_1$ and value $\operatorname{Min}_P(\nu)$ on the Laurent polynomial $f(\mathbf{t})$. In particular, the Laurent polynomial $t_1^{-\operatorname{Min}_P(\nu)} f(\mathbf{t})$ defines the local equation of the Zariski closure $\overline{Z}$ in U_σ. Therefore, the rational function $t_\nu^{\operatorname{Min}_P(\nu)}/f(\mathbf{t})$ is the generator of the invertible sheaf $\mathcal{O}\big(\overline{Z}\big)$ over U_σ. On the other hand, the invertible sheaf $\mathcal{O}(D_f)$ corresponding to the torus-invariant divisor $D_f := -\sum_{\nu\in\Sigma[1]} \operatorname{Min}_P(\nu) D_\nu$ is generated over U_σ by $t_\nu^{\operatorname{Min}_P(\nu)}$, where t_ν is the generator of the vanishing ideal of the closed torus orbit $D_\nu \subset U_\sigma$. The multiplication by the Laurent polynomial f defines an isomorphism $\mathcal{O}\big(\overline{Z}\big) \cong \mathcal{O}(D_f)$. Therefore, D_f is linearly equivalent to $\overline{Z}$. □

Corollary 7.2. *The divisor class*

$$\big[\,\overline{Z}\,\big] \in \bigoplus_{\nu \in \Sigma[1]} \mathbb{Z} D_\nu \cong \mathbb{Z}^{\Sigma[1]}$$

of the Zariski closure $\overline{Z}$ of Z in V_Σ depends only on the Newton polytope P of the defining Laurent polynomial of f. If one chooses another Laurent polynomial $f(\mathbf{t})\mathbf{t}^m$ $(m \in M)$ defining the same affine hypersurface $Z \subset \mathbb{T}^d$, then P will be replaced by the shifted Newton polytope $P + m$ and one obtains

$$\big[\,\overline{Z}_{f(\mathbf{t})\mathbf{t}^m}\big] = \sum_{\nu\in\Sigma[1]} (\operatorname{Min}_P(\nu) + \langle m, \nu\rangle) D_\nu\,.$$

Definition 7.3. Let $Z \subset \mathbb{T}^d$ be a nondegenerate affine hypersurface defined by a Laurent polynomial $f(\mathbf{t})$ with Newton polytope P. We assume $F(P) \neq \emptyset$ and call the Zariski closure $\widetilde{Z} \subset \widetilde{V}$ in the projective toric variety associated with the rational polytope $\widetilde{P} := F(P) + C(P)$ the *canonical model* of Z.

Proposition 7.4. *The canonical model $\widetilde{Z}$ is a semiample big $\mathbb{Q}$-Cartier divisor on the toric variety $\widetilde{V}$.*

Proof. Since the full-dimensional polytope $C(P)$ is a Minkowski summand of $\widetilde{P}$, we obtain a natural birational toric morphism $\varrho\colon \widetilde{V} \to V_{C(P)}$, and the canonical model $\widetilde{Z}$ is the pull-back under ϱ of the ample $\mathbb{Q}$-Cartier divisor $\overline{Z}_{C(P)} \subset V_{C(P)}$. This implies the statement. □

Remark 7.5. Note that if $P = C(P)$, then the toric morphism $\varrho\colon \widetilde{V} \to V_{C(P)} = V_P$ is determined by the canonical refinement Σ_P^{can} of the normal fan Σ_P, and the singularities of $\widetilde{Z}$ are toroidal.

In general, we have $P \neq C(P)$ if P contains facets Q whose normals do not belong to $S_F(P)$. These "bad facets" of P define divisors on V_P and on $\overline{Z}$ that must disappear on $\widetilde{V}$ and on the canonical (respectively, minimal) models $\widetilde{Z}$ (respectively, $\widehat{Z}$). Therefore, in general, there is no birational toric morphism

$$\rho' \colon \widetilde{V} \to V_P \,.$$

This is the reason why the singularities of canonical and minimal models $\widetilde{Z}$ and $\widehat{Z}$ are not toroidal in general. We illustrate this fact in Example 8.3 (Case 1).

The next theorem is very important.

Theorem 7.6. *The canonical model $\widetilde{Z}$ is a normal variety, and it satisfies the adjunction formula*

$$K_{\widetilde{Z}} = \left(K_{\widetilde{V}} + \widetilde{Z}\right)|_{\widetilde{Z}} \ . \tag{7.1}$$

The main technical difficulty in the proof of Theorem 7.6 is the fact that the hypersurface $\widetilde{Z} \subset \widetilde{V}$ may contain some torus orbits in the toric variety $\widetilde{V}$. This could make problems for the adjunction as can be seen from the next example.

Example 7.7. Consider a surface $S \subset \mathbb{P}(1, 1, a, a) = V$ $(a \geqslant 1)$ defined by weighted homogeneous equation of degree $a + 1$

$$z_0 z_2 - z_1 z_3 = 0 \,.$$

Then the surface S is smooth and isomorphic to the ruled surface $\mathbb{F}_{a-1}$. However, if $a \geqslant 3$, the adjunction formula for S does not hold because $\mathbb{F}_{a-1}$ is not a del Pezzo surface. The adjunction in this case fails because the singularities of the weighted projective space V include a 1-dimensional torus orbit $\Theta \subset V$ contained in S. This is an example of a quasi-smooth divisor S in the weighted projective space that is not well formed in sense of Fletcher [Ian00].

Our proof of Theorem 7.6 is based on the following statement.

Proposition 7.8. *None of the $(d-2)$-dimensional torus orbits $T_\sigma \subset \widetilde{V}$ corresponding to 2-dimensional cones $\sigma \in \widetilde{\Sigma}$ is contained in the canonical model $\widetilde{Z}$.*

Proof. Let $\nu_i, \nu_j \in N$ be primitive lattice generators such that $\sigma = \mathbb{R}_{\geqslant 0}\nu_i + \mathbb{R}_{\geqslant 0}\nu_j$. Consider two supporting hyperplanes

$$L_i \colon \langle x, \nu_i \rangle = \mathrm{Min}_P(\nu_i) \quad \text{and} \quad L_j \colon \langle x, \nu_j \rangle = \mathrm{Min}_P(\nu_j) \,.$$

Then $L_i \cap P \neq \emptyset$ and $L_j \cap P \neq \emptyset$. Moreover, the Zariski closure Z_σ of Z in the open affine toric subvariety $U_\sigma \subset \widetilde{V}$ corresponding to the 2-dimensional cone σ contains the unique closed torus orbit $T_\sigma \subset U_\sigma$ if and only if $L_i \cap L_j \cap P \neq \emptyset$. It remains to apply Theorem 6.7(b). □

Proof of Theorem 7.6. Since a normal toric variety $\widetilde{V}$ is Cohen–Macaulay and $\widetilde{Z}$ is a hypersurface in $\widetilde{V}$, by Serre's criterion for normality, it is sufficient to show that $\widetilde{Z}$ is smooth in codimension 1. It is enough to check that smoothness on the open subset

$$\widetilde{U}^{(2)} := \bigcup_{\substack{\sigma \in \widetilde{\Sigma} \\ \dim \sigma = 2}} U_\sigma$$

because the complement $W := \widetilde{V} \setminus \widetilde{U}^{(2)}$ has codimension 3 in $\widetilde{V}$. By Proposition 7.8, for every 2-dimensional cone $\sigma \in \widetilde{\Sigma}$, the closed torus orbit T_σ is not contained in the Zariski closure $\widetilde{Z}$. Let $Q := L_i \cap L_j \cap P$. Then Q is a face of P. If $\dim Q = 0$, then $T_\sigma \cap \widetilde{Z} = \emptyset$, and therefore $U_\sigma \cap \widetilde{Z}$ is smooth. If $\dim Q > 0$, then it follows from the nondegeneracy of Z that the intersection $T_\sigma \cap \widetilde{Z}$ is transversal along a smooth reduced divisor in T_σ, and singularities of $U_\sigma \cap \widetilde{Z}$ must be contained in $T_\sigma \cap \widetilde{Z}$; that is, they have codimension at least 2.

Finally, the adjunction holds for the canonical model $\widetilde{Z} \subset \widetilde{V}$ because $\widetilde{Z}$ is transversal to all torus orbit in $U^{(2)}$ and the adjunction holds on the open subset $U^{(2)} \subset \widetilde{Z}$. □

8. Minimal models

Theorem 8.1. *The projective hypersurface $\widetilde{Z} \subset \widetilde{V}$ is a $\mathbb{Q}$-Gorenstein variety with nef $\mathbb{Q}$-Cartier canonical divisor $K_{\widetilde{Z}}$ and with at worst canonical singularities.*

Proof. We take a common regular refinement Σ of the normal fan Σ_P and of the fan $\widehat{\Sigma}$. Denote by $\overline{Z}$ the Zariski closure of Z in the smooth toric variety $V := V_\Sigma$. By Proposition 7.1, we have $\overline{Z} = \sum_{\nu\in\Sigma[1]} - \mathrm{Min}_P(\nu) D_\nu$. Combining this with $K_V = \sum_{\nu\in\Sigma[1]} -D_\nu$, we obtain

$$K_V + \overline{Z} = \sum_{\nu\in\Sigma[1]} (-1 - \mathrm{Min}_P(\nu)) \cdot D_\nu \,.$$

We may consider Σ as a regular refinement of $\widetilde{\Sigma}$ and obtain the birational toric morphism $\varphi\colon V \to \widetilde{V}$. Then

$$K_V + \overline{Z} = \varphi^*\big(K_{\widetilde{V}} + \widetilde{Z}\big) + \sum_{\nu\in\Sigma[1]\setminus\widetilde{\Sigma}[1]} a(\nu) D_\nu \,. \tag{8.1}$$

In order to compute the discrepancies $a(\nu)$, we write the right-hand side in the last equality as a linear combination of the toric divisors D_ν ($\nu \in \Sigma[1]$) using

$$\begin{aligned}
\varphi^*\big(K_{\widetilde{V}} + \widetilde{Z}\big) &= \sum_{\nu\in\Sigma[1]} - \mathrm{Min}_{F(P)}(\nu) D_\nu \\
&= \sum_{\nu\in\widetilde{\Sigma}[1]} - \mathrm{Min}_{F(P)}(\nu) D_\nu + \sum_{\nu\in\Sigma[1]\setminus\widetilde{\Sigma}[1]} - \mathrm{Min}_{F(P)}(\nu) D_\nu \\
&= \sum_{\nu\in\widetilde{\Sigma}[1]} (-1 - \mathrm{Min}_P(\nu)) D_\nu + \sum_{\nu\in\Sigma[1]\setminus\widetilde{\Sigma}[1]} - \mathrm{Min}_{F(P)}(\nu) D_\nu \,,
\end{aligned}$$

$$\varphi^*\big(K_{\widetilde{V}} + \widetilde{Z}\big) + \sum_{\nu\in\Sigma[1]\setminus\widetilde{\Sigma}[1]} a(\nu) D_\nu = \sum_{\nu\in\widetilde{\Sigma}[1]} (-1 - \mathrm{Min}_P) D_\nu + \sum_{\nu\in\Sigma[1]\setminus\widetilde{\Sigma}[1]} (- \mathrm{Min}_{F(P)} + a(\nu)) D_\nu \,.$$

This implies

$$-1 - \mathrm{Min}_P(\nu) = - \mathrm{Min}_{F(P)} + a(\nu) \quad \forall \nu \in \Sigma[1] \setminus \widetilde{\Sigma}[1] \,.$$

Now we restrict equation (8.1) to $\overline{Z}$ and obtain

$$K_{\overline{Z}} = \varphi^*\big(K_{\widetilde{Z}}\big) + \sum_{\nu\in\Sigma[1]\setminus\widetilde{\Sigma}[1]} a(\nu) \big(D_\nu \cap \overline{Z}\big) \,. \tag{8.2}$$

Since

$$a(\nu) = \mathrm{Min}_{F(P)}(\nu) - \mathrm{Min}_P(\nu) - 1 \geqslant 0 \quad \forall \nu \in N \setminus \{0\} \,, \tag{8.3}$$

the singularities of $\widetilde{Z}$ are at worst canonical. □

Theorem 8.2. *Let $\widehat{\Sigma}$ be a maximal projective simplicial refinement of the normal fan $\widetilde{\Sigma} = \Sigma_{\widetilde{P}}$ with $\widehat{\Sigma}[1] = S_F(P)$. Then the Zariski closure $\widehat{Z}$ of the affine hypersurface $Z \subset \mathbb{T}^d$ in $\widehat{V} := V\big(\widehat{\Sigma}\big)$*

is a projective minimal model of Z; that is, $\widehat{\Sigma}$ is a $\mathbb{Q}$-factorial projective algebraic variety with at worst terminal singularities and with semiample canonical class $K_{\widehat{Z}}$. Moreover, the crepant birational toric morphism $\varphi\colon \widehat{V} \to \widetilde{V}$ induces a birational crepant morphism $\varphi\colon \widehat{Z} \to \widetilde{Z}$.

Proof. The same arguments as above show that the singularities of $\widehat{Z}$ are at worst terminal, and the induced birational morphism $\varphi\colon \widehat{Z} \to \widetilde{Z}$ is crepant because the equality

$$a(\nu) = \mathrm{Min}_{F(P)}(\nu) - \mathrm{Min}_P(\nu) - 1 = 0$$

holds if and only if $\nu \in S_F(P)$. □

Let us consider some of the simplest examples in arbitrary dimension d showing the roles of $F(P)$, $S_F(P)$ and $C(P)$ in constructing minimal models of nondegenerate toric hypersurfaces $Z \subset \mathbb{T}^d$.

Example 8.3. Let $P \subset \mathbb{R}^d$ be the d-dimensional lattice polytope defined as

$$P := \left\{ x \in \mathbb{R}^d_{\geqslant 0} \mid a \leqslant \sum_{i=1}^d x_i \leqslant b \right\}$$

for some integers $0 \leqslant a < b$. Then

$$F(P) = \left\{ x \in \mathbb{R}^d_{\geqslant 1} \mid a+1 \leqslant \sum_{i=1}^d x_i \leqslant b-1 \right\}.$$

So $F(P) \neq \emptyset$ if and only if $b \geqslant \max\{d+1, a+2\}$.

Let $e_1, \dots, e_d$ be the standard basis of $\mathbb{Z}^d \subset \mathbb{R}^d$. There are two possibilities for $S_F(P)$ if $F(P) \neq \emptyset$:

Case 1. $S_F(P) = \left\{e_1, \dots, e_d, -\sum_{i=1}^d e_i\right\}$. This happens if and only if $b \geqslant d+1$ and $a+2 \leqslant d$. In this case, we have

$$C(P) = \left\{ x \in \mathbb{R}^d_{\geqslant 0} \mid \sum_{i=1}^d x_i \leqslant b \right\},$$

and the minimal model $\widehat{Z} = \widetilde{Z}$ is a projective hypersurface of degree b in $\mathbb{P}^d$ that may have an isolated terminal singularity at the origin $0 \in \mathbb{C}^d \subset \mathbb{P}^d$ if $a \geqslant 2$. If $a \geqslant 2$, then $C(P) \neq P$, and the isolated singularity of the minimal model is not toroidal as soon as $a \geqslant 3$.

Case 2. $S_F(P) = \left\{e_1, \dots, e_d, -\sum_{i=1}^d e_i, \sum_{i=1}^d e_i\right\}$. This happens if and only if $b \geqslant a+2 \geqslant d+1$. If these inequalities hold, we have $C(P) = P$, and the minimal model $\widehat{Z} = \widetilde{Z} = \overline{Z}$ is a smooth projective hypersurface, namely the Zariski closure of Z in the smooth toric variety $\widehat{V} = \widetilde{V} = V_P$ obtained from $\mathbb{P}^d$ by the blow-up of the origin $0 \in \mathbb{C}^d \subset \mathbb{P}^d$.

9. The Kodaira dimension of toric hypersurfaces

We begin this section with simple illustrating examples showing some properties of the Kodaira dimension and of the Iitaka fibration for $(d-1)$-dimensional smooth projective hypersurfaces $X_{a,b}$ of bi-degree (a,b) in the product $\mathbb{P}^{d_1} \times \mathbb{P}^{d_2}$ of two projective spaces ($d = d_1 + d_2$). We use these examples to illustrate general properties of the Kodaira dimension and the Iitaka fibration for canonical models of nondegenerate affine hypersurfaces $Z \subset \mathbb{T}^d$ defined by Laurent polynomials with d-dimensional Newton polytopes P.

The hypersurfaces $X_{a,b} \subset \mathbb{P}^{d_1} \times \mathbb{P}^{d_2}$ have nonnegative Kodaira dimension $\kappa(X_{a,b})$ if and only if $a \geqslant d_1 + 1$ and $b \geqslant d_2 + 1$.

- If $a > d_1 + 1$ and $b > d_2 + 1$, then the canonical class of $X_{a,b}$ is ample and $\kappa(X_{a,b}) = \dim X_{a,b} = d - 1$; that is, $X_{a,b}$ is a smooth projective variety of general type.
- If $(a, b) = (d_1 + 1, d_2 + 1)$, then $X_{a,b}$ has trivial canonical class and X_{d_1+1,d_2+1} is a $(d-1)$-dimensional Calabi–Yau hypersurface; that is, $\kappa(X_{d_1+1,d_2+1}) = 0$.

A more interesting situation appears if, for example, $a = d_1 + 1$ but $b > d_2 + 1$. In this case, the canonical invertible sheaf on $X_{a,b}$ is the restriction to $X_{a,b}$ of the semiample sheaf $\mathcal{K} := \mathcal{O}(0, b-d_2-1)$ on $\mathbb{P}^{d_1} \times \mathbb{P}^{d_2}$. The global sections of $\mathcal{K}$ define the natural surjective projective morphism $\vartheta\colon \mathbb{P}^{d_1} \times \mathbb{P}^{d_2} \to \mathbb{P}^{d_2}$ which is a trivial $\mathbb{P}^{d_1}$-fibration, a Fano fibration, over $\mathbb{P}^{d_2}$. The restriction of ϑ to the hypersurface $X_{a,b}$ defines a proper surjective morphism $X_{a,b} \to \mathbb{P}^{d_2}$ whose general fibers are anticanonical hypersurfaces of degree $d_1 + 1$ in $\mathbb{P}^{d_1}$.

There are two cases:

- If $d_1 = 1$, then the anticanonical hypersurface in $\mathbb{P}^{d_1}$ consists of two points in $\mathbb{P}^1$ and the morphism $\vartheta\colon X_{2,b} \to \mathbb{P}^{d_2}$ is a double covering of $\mathbb{P}^{d_2}$, a higher-dimensional analog of hyperelliptic curves. In particular, the Kodaira dimension of $X_{2,b}$ is $d - 1$.
- If $d_1 > 1$, then for a general point $p \in \mathbb{P}^{d_2}$, the fiber $\vartheta^{-1}(p) \cap X_{d_1+1,b}$ is an irreducible (d_1-1)-dimensional Calabi–Yau hypersurfaces of degree d_1+1 in $\mathbb{P}^{d_1}$. By Proposition 3.16, the Newton polytope P of $X_{d_1+1,b}$ is a product of two simplices of dimensions d_1 and d_2. The Fine interior $F(P)$ is a d_2-dimensional simplex. It is easy to see that the pluricanonical ring of $X_{d_1+1,b}$ is the $(b - d_2 - 1)$-Veronese ring of $\mathbb{P}^{d_2}$ and that the Kodaira dimension of $X_{d_1+1,b}$ equals d_2, that is,
$$\kappa(X_{d_1+1,b}) = \dim F(P) = d_2 < d - 1\,.$$

Now we consider a general case of the toric fibration $\vartheta\colon \widetilde{V} \to V_{F(P)}$ corresponding to the Minkowski summand $F(P)$ of the full-dimensional polytope $\widetilde{P} = F(P) + C(P)$, where $k := \dim F(P) < d$. Define the $(d-k)$-dimensional sublattice
$$N^F := \{n \in N \mid \langle x, n\rangle = \mathrm{Min}_{F(P)}(n)\ \ \forall x \in F(P)\} \subset N\,.$$
The sublattice $N^F \subset N$ is a direct summand of N of rank $d-k$; that is, there exists a complementary sublattice N_F of rank k such that $N = N_F \oplus N^F$ and a natural surjective homomorphism
$$\pi_F\colon N \to N/N^F \cong N_F\,.$$
Denote by $M_F \subset M$ the k-dimensional orthogonal complement of N^F in M. We obtain the natural surjective homomorphism
$$\pi^F\colon M \to M^F := M/M_F$$
and denote by P^F the $(d-k)$-dimensional lattice polytope $\pi^F(P) \subset M^F_{\mathbb{R}}$, that is, the π^F-projection of the lattice polytope P onto the $(d-k)$-dimensional lattice polytope $P^F \subset M^F_{\mathbb{R}}$.

Proposition 9.1. *The $(d-k)$-dimensional lattice polytope $P^F \subset M^F_{\mathbb{R}}$ has the following properties:*

(a) *We have $F(P^F) = \pi^F(F(P))$; that is, the Fine interior of the lattice polytope P^F is the rational point $\pi^F(F(P)$.*

(b) $S_F\big(P^F\big) = N^F \cap S_F(P) \subset N^F$.

(c) *The convex hull* $\Phi^F := \mathrm{Conv}\big(N^F \cap S_F(P)\big)$ *is a* $(d-k)$*-dimensional canonical Fano polytope, that is, a* $(d-k)$*-dimensional lattice polytope containing only the origin* $0 \in N^F$ *as* N^F*-lattice point in the relative interior of* Φ^F.

Proof. The lattice M^F is dual to N^F. Therefore,

$$F\big(P^F\big) = \big\{x \in M^F_{\mathbb{R}} \mid \langle x, \nu \rangle \geqslant \mathrm{Min}_{P^F}(\nu) + 1 \ \ \forall \nu \in N^F \setminus \{0\}\big\}\,.$$

Since P^F is the π^F-projection of P and any linear function $\langle *, \nu\rangle$ with $\nu \in N^F$ is constant on fibers of the π^F-projection, we obtain that $\mathrm{Min}_P(\nu) = \mathrm{Min}_{P^F}(\nu)$ for all $\nu \in N^F$. This implies that $p^F := \pi^F(F(P))$ is the Fine interior of P^F. □

Theorem 9.2. *Let* $Z \subset \mathbb{T}^d$ *be a nondegenerate affine toric hypersurface defined by a Laurent polynomial* f *with a Newton polytope* P. *If* $k := \dim F(P) \geqslant 0$, *then the Kodaira dimension* $\kappa\big(\widetilde{Z}\big)$ *of the canonical model* $\widetilde{Z}$ *equals*

$$\kappa\big(\widetilde{Z}\big) = \min\{k, d-1\}\,,$$

and we have the following three cases:

(a) *If* $k = d$, *then the Iitaka fibration* $\widetilde{Z} \to V_{F(P)}$ *is birational on its image.*

(b) *If* $k = d-1$, *then the minimal model* $\widetilde{Z}$ *is birational to a double cover of a* $(d-1)$*-dimensional toric variety* $V_{F(P)}$; *that is,* $\widetilde{Z}$ *is a higher-dimensional analog of hyperelliptic curves of genus* $g \geqslant 2$.

(c) *If* $0 \leqslant k < d-1$, *then the Iitaka fibration*

$$\widetilde{Z} \to V_{F(P)}$$

is induced by the canonical toric $\mathbb{Q}$*-Fano fibration* $\vartheta \colon \widehat{V} \to V_{F(P)}$ *whose generic fiber is isomorphic to a nondegenerate irreducible* $(d-1-k)$*-dimensional hypersurface of Kodaira dimension* 0 *in some toric* $\mathbb{Q}$*-Fano variety defined by the* $(d-k)$*-dimensional lattice polytope* P^F *with* 0*-dimensional Fine interior.*

Proof. By the adjunction formula, $\big(K_{\widetilde{V}} + \widetilde{Z}\big)|_{\widetilde{Z}}$ is the canonical class $K_{\widetilde{Z}}$. So we obtain the linear maps

$$\Psi_m \colon H^0\big(\widetilde{V}, \mathcal{O}\big(m\big(K_{\widetilde{V}} + \widetilde{Z}\big)\big)\big) \to H^0\big(\widetilde{Z}, \mathcal{O}\big(mK_{\widetilde{Z}}\big)\big)\,, \quad m \geqslant 0\,.$$

Consider the toric morphism $\vartheta \colon \widetilde{V} \to V_{F(P)}$. Then $K_{\widetilde{V}} + \widetilde{Z} = \vartheta^* L$ for some ample divisor L on the toric variety $V_{F(P)}$. Therefore, the dimensions $H^0\big(\widetilde{V}, \mathcal{O}\big(m\big(K_{\widetilde{V}} + \widetilde{Z}\big)\big)\big)$ grow as a degree $\dim F(P)$ polynomial in m.

We consider two cases:

Case 1. $k := \dim F(P) = d$. Then the dimensions $H^0\big(\widetilde{V}, \mathcal{O}\big(m(K_{\widetilde{V}} + \widetilde{Z}\big)\big)\big)$ grow as a degree d polynomial in m. This implies that the restrictions of these global sections to the hypersurface $\widetilde{Z}$ grow as a degree at least $d-1$ polynomial in m, which implies that only $\kappa\big(\widetilde{Z}\big) = \dim \widetilde{Z} = d-1$ is possible.

Case 2. $k := \dim F(P) < d$. We claim that in this case all maps Ψ_m $(m \geqslant 0)$ are injective (which implies $\kappa(Z) \geqslant k$).

Indeed, any global section $s \in H^0\big(\widetilde{V}, \mathcal{O}\big(m\big(K_{\widetilde{V}} + \widetilde{Z}\big)\big)\big)$ is represented by a Laurent polynomial $g(\mathbf{t})$ whose Newton polytope $P(g)$ is contained in the k-dimensional polytope $mF(P)$. If the restriction of such a g to Z is 0, then the Laurent polynomial f divides g and the Newton polytope $P = \mathrm{Newt}(f)$ can be embedded into the Newton polytope of $\mathrm{Newt}(g)$. The latter is impossible by dimension reasons unless $g = 0$.

In order to get the opposite inequality $\kappa(\widetilde{Z}) \leqslant k$, we remark that on the toric variety $\widetilde{V}$, the nef divisor $K_{\widetilde{V}} + \widetilde{Z}$ defines a toric morphism

$$\varphi\colon \widetilde{V} \to V_{F(P)} := \mathrm{Proj} \bigoplus_{m \geqslant 0} H^0\big(\widetilde{V}, \mathcal{O}\big(m\big(K_{\widetilde{V}} + \widetilde{Z}\big)\big)\big)\,,$$

and $V_{F(P)}$ is a toric variety of dimension k.

The fibers of φ over the dense torus orbit $U \subset V_{F(P)}$ are $(d-k)$-dimensional canonical $\mathbb{Q}$-Fano toric varieties, and the restrictions of the global sections of $\mathcal{O}\big(m\big(K_{\widetilde{V}} + \widetilde{Z}\big)\big)$ are trivial. Therefore, the φ-images of the generic intersections of $\widetilde{Z}$ with these fibers are $(d-k-1)$-dimensional. These irreducible $(d-k-1)$-dimensional subvarieties of $\widetilde{Z}$ are mapped to points in $V_{F(P)}$. Consequently, the dimension of the pluricanonical image of $\widetilde{Z}$ can be at most $(d-1) - (d-k-1) = k$. So we obtain $\kappa(\widetilde{Z}) \leqslant k$. □

Consider an illustrative example.

Example 9.3. Let $P = C(P)$ be the canonically closed 3-dimensional lattice simplex with vertices

$$(0,0,0), (3,0,0), (1,3,0), (2,0,3) \in M_{\mathbb{R}} = \mathbb{R}^3\,;$$

see Figure 7. The simplex P contains no lattice points in its interior, but the Fine interior $F(P)$ is a 1-dimensional rational segment of length $1/3$:

$$F(P) = [(4/3,1,1), (5/3,1,1)] \subset M_{\mathbb{R}}\,.$$

The 1-dimensional sublattice $M_F \subset M$ is spanned by $(1,0,0)$. The projection $P^F\colon M \to M^F = M/M_F \cong \mathbb{Z}^2$ is the map $(m_1,m_2,m_3) \mapsto (m_2,m_3)$. The image $P^F(P)$ of P under this projection is the reflexive lattice triangle with vertices $(0,0)$, $(0,3)$, $(3,0)$ having the 0-dimensional Fine interior $\{(1,1)\}$. One can consider the nondegenerate affine surface $Z \subset \mathbb{T}_3 \cong (\mathbb{C}^*)^3$ defined by the equation

$$f(\mathbf{t}) = 1 + t_1^3 + t_1 t_2^3 + t_1^2 t_3^3 = 0$$

with Newton polytope P. The canonical model $\widetilde{Z}$ is an elliptic surface of Kodaira dimension 1 having the natural surjective morphism $\widetilde{Z} \to \mathbb{P}^1$ if we take t_1 as local affine coordinate on $\mathbb{P}^1$. Thus we obtain a 1-parameter t_1-family of plane elliptic cubic curves determined by the affine (t_2,t_3)-equations

$$\big(1 + t_1^3\big) + t_1 t_2^3 + t_1^2 t_3^3 = 0\,, \quad t_1 \in \mathbb{C}$$

having the reflexive Newton polygon with three vertices $(0,0), (3,0), (3,0)$.

The fan $\widetilde{\Sigma} = \widehat{\Sigma}$ defining the 3-dimensional toric variety $\widetilde{V}$ with at worst canonical singularities is generated by five lattice vectors (see Figure 8):

$$S_F(P) = \widetilde{\Sigma}[1] = \widehat{\Sigma}[1] = \{(0,1,0), (0,0,1), (-1,-1), (3,-1,-2), (-3,-2,-1) \subset N_{\mathbb{R}}\,.$$

The terminal toric variety $\widehat{V}$ equals $\widetilde{V}$. It has six isolated terminal μ_3-quotient singularities, and it is generically a toric $\mathbb{P}^2$-fibration over $\mathbb{C}^*$. The 2-dimensional sublattice $N^F = \mathrm{Span}((0,1,0),(0,0,1)) \subset N$ contains altogether three lattice vectors from $S_F(P)$ spanning the fan of $\mathbb{P}^2$:

$$\{(0,1,0), (0,0,1), (0,-1,-1)\} \subset N^F\,.$$

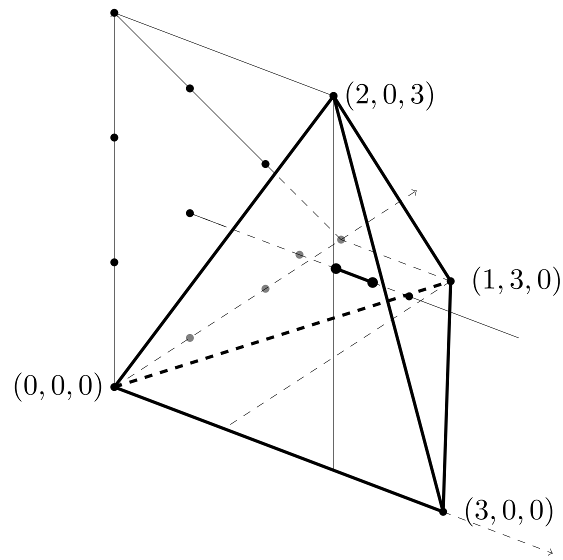

Figure 7. The lattice simplex P

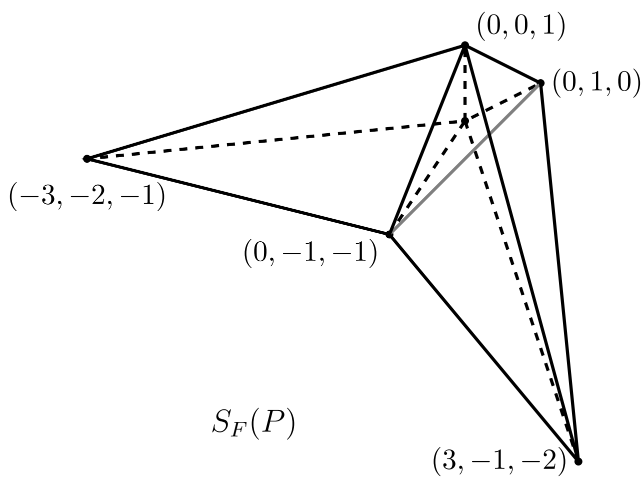

Figure 8. The generating set of the fan $\widetilde{\Sigma}$

We end this section with a combinatorial formula for the top intersection number $\big(K_{\widehat{Z}}\big)^{d-1}$.

Theorem 9.4. *Let P be a d-dimensional lattice polytope with $F(P) \neq \emptyset$. Denote by $\widehat{Z}$ a minimal model of a nondegenerate hypersurface $Z \subset \mathbb{T}^d$ with the Newton polytope P which is obtained from the canonical models $\widetilde{Z}$ by a crepant morphism $\varphi\colon \widehat{Z} \to \widetilde{Z}$. Then*

$$\big(K_{\widehat{Z}}\big)^{d-1} = \big(K_{\widetilde{Z}}\big)^{d-1} = \begin{cases} \mathrm{Vol}_d(F(P)) + \sum_{\substack{Q \prec F(P) \\ \dim Q = d-1}} \mathrm{Vol}_{d-1}(Q) & \text{if } \dim F(P) = d\,, \\ 2\mathrm{Vol}_{d-1}(F(P)) & \text{if } \dim F(P) = d-1\,, \\ 0 & \text{if } \dim F(P) < d-1\,. \end{cases}$$

Proof. The crepant morphism $\varphi\colon \widehat{Z} \to \widetilde{Z}$ implies the equalities

$$\big(K_{\widehat{Z}}\big)^{d-1} = \big(\varphi^*(\widetilde{Z})\big)^{d-1} = \big(K_{\widetilde{Z}}\big)^{d-1}.$$

Therefore, it is sufficient to compute the top intersection number for the canonical divisor on the canonical model $\widetilde{Z} \subset \widetilde{V}$. If the Kodaira dimension is not maximal, this number is 0. The maximal Kodaira dimension $d-1$ of $\widetilde{Z}$ appears only in two cases: $\dim F(P) = d$ and $\dim F(P) = d-1$.

Case 1. $\dim F(P) = d$. In this case, by the adjunction formula, we have

$$\left[K_{\widetilde{Z}}\right]^{d-1} = \left(\left[K_{\widetilde{V}}\right] + \left[\widetilde{Z}\right]\right)^{d-1} \cdot \left[\widetilde{Z}\right] = \left(\left[K_{\widetilde{V}}\right] + \left[\widetilde{Z}\right]\right)^{d} - \left[K_{\widetilde{V}}\right] \cdot \left(\left[K_{\widetilde{V}}\right] + \left[\widetilde{Z}\right]\right)^{d-1}.$$

The intersection number $\left(\left[K_{\widetilde{V}}\right] + \left[\widetilde{Z}\right]\right)^{d}$ is the degree of the semiample adjoint $\mathbb{Q}$-divisor associated with the polytope $F(P)$. Therefore, we have

$$\left(\left[K_{\widetilde{V}}\right] + \left[\widetilde{Z}\right]\right)^{d} = \mathrm{Vol}_d(F(P)).$$

In the computation of the intersection number $-\left[K_{\widetilde{V}}\right] \cdot \left(\left[K_{\widetilde{V}}\right] + \left[\widetilde{Z}\right]\right)^{d-1}$, we use the formula

$$-\left[K_{\widetilde{V}}\right] = \sum_{\nu \in \widetilde{\Sigma}[1]} [D_\nu]$$

and the fact that the degree of the restriction on a toric divisor $D_\nu \subset \widetilde{V}$ of the semiample $\mathbb{Q}$-divisor $\left[K_{\widetilde{V}}\right] + \left[\widetilde{Z}\right]$ associated with a polytope $F(P)$ equals $\mathrm{Vol}_{d-1}(Q)$ if the supporting hyperplane with the normal vector ν defines a $(d-1)$-dimensional face of $F(P)$ and 0 otherwise. It remains to take the sum over all $\nu \in \widetilde{\Sigma}[1]$.

Case 2. $\dim F(P) = d-1$. In this case, the semiample adjoint $\mathbb{Q}$-divisor $\left[K_{\widetilde{V}}\right] + \left[\widetilde{Z}\right]$ on $\widetilde{V}$ associated with the polytope $F(P)$ defines a toric morphism $\vartheta\colon \widetilde{V} \to V_{F(P)}$ onto a $(d-1)$-dimensional toric variety. Since the restriction of ϑ on the canonical model $\widetilde{Z}$ defines a double covering $\widetilde{Z} \to V_{F(P)}$, we obtain

$$\left[K_{\widetilde{Z}}\right]^{d-1} = \left[\widetilde{Z}\right] \cdot \left(\left[K_{\widetilde{V}}\right] + \left[\widetilde{Z}\right]\right)^{d-1} = 2\mathrm{Vol}_{d-1}(F(P)). \qquad \square$$

Example 9.5. Let P be the 3-dimensional simplex which is the Newton polytope of the equation of the Godeaux surface S in $\mathbb{P}^3/\mu_5$ obtained as quotient of the Fermat quintic surface $z_1^5 + z_2^5 + z_3^5 + z_4^5 = 0$ modulo the action of the 5-order cyclic group $\mu_5 = \langle \zeta \rangle$:

$$(z_1 : z_2 : z_3 : z_4) \mapsto \left(\zeta z_1 : \zeta^2 z_2 : \zeta^3 z_3 : \zeta^4 z_4\right).$$

Then the Fine interior $F(P)$ is a 3-dimensional rational simplex with $\mathrm{Vol}_3(F(P)) = 1/5$, and all its four facets Q are rational triangles with $\mathrm{Vol}_2(Q) = 1/5$. Thus, the above formula yields $K_S^2 = 5 \times 1/5 = 1$.

10. Further developments

10.1 Minimal surfaces

It is natural to study minimal surfaces arising from lattice polytopes P of dimension 3. If $P \subset \mathbb{R}^3$ is a 3-dimensional lattice polytope with $F(P) \neq \emptyset$, then the Chern numbers c_1^2 and c_2 of minimal compactifications $\widehat{S}$ of nondegenerate surfaces $S \subset (\mathbb{C}^*)$ with Newton polytope P are completely determined by the Fine interior $F(P)$ because the number $c_1^2(\widehat{S}) = K_{\widehat{S}}^2$ is one of three integers

$$0, \quad 2\mathrm{Vol}_2(F(P)), \quad \mathrm{Vol}_3(F(P)) + \sum_{\substack{Q \prec F(P) \\ \dim Q = 2}} \mathrm{Vol}_2(Q)$$

and

$$c_2(\widehat{S}) = 12\chi(\mathcal{O}_{\widehat{S}}) - c_1^2(\widehat{S}), \quad \text{where } \chi(\mathcal{O}_{\widehat{S}}) = 1 + \left|F(P) \cap \mathbb{Z}^3\right|.$$

The general method for finding minimal models proposed in this paper was applied to all 674 688 3-dimensional Newton polytopes with only one interior lattice point 0. These polytopes were classified by Kasprzyk [Kas10]. There exist 665 599 almost reflexive 3-dimensional lattice polytopes P including 4 319 reflexive ones characterized by the condition $F(P) = \{0\}$. (A d-dimensional lattice polytope P is called *almost reflexive* if $F(P) = \{0\}$ and $C(P)$ is reflexive.) These polytopes define families of $K3$-surfaces. The remaining 9 089 polytopes give rise to minimal surfaces $\widehat{Z}$ of positive Kodaira dimension $\kappa > 0$ with $p_g = 1$: elliptic surfaces ($\kappa = 1$), Todorov and Kanev surfaces ($\kappa = 2$) [BKS22].

It would be interesting in general to investigate the geography of minimal surfaces $\widehat{S}$ arising from arbitrary 3-dimensional lattice polytopes P with two or more interior lattice points.

10.2 Minimal 3-folds

As for 3-folds, the complete classification of the 473 800 776 4-dimensional reflexive polytopes obtained by Kreuzer and Skarke [KS00] gives rise to a lot of topologically different examples of smooth 3-dimensional Calabi–Yau varieties [AGH$^+$15]. Note that the complete list of 4-dimensional almost reflexive polytopes is still unknown; it is expected to be huge. Therefore, it would be better to investigate some qualitative properties of the corresponding 3-dimensional minimal Calabi–Yau models, that is, about possible isolated terminal cDV-singularities and their description in terms of the corresponding almost reflexive 4-dimensional Newton polytope $P \subset \mathbb{R}^4$. For example, the 4-dimensional almost reflexive lattice polytope $P \subset \mathbb{R}^4$ given by $x_i \geqslant -1$ $(1 \leqslant i \leqslant 4)$, $x_1 + x_2 + x_3 + x_4 \leqslant 1$, $x_1 \leqslant 2$ provides some of the simplest examples of nondegenerate 3-dimensional hypersurfaces whose minimal Calabi–Yau models are not smooth. These minimal models are 3-dimensional Calabi–Yau quintics in $\mathbb{P}^4$ with an isolated terminal cDV-singularity analytically isomorphic to $z_1^2 + z_2^2 + z_3^2 + z_4^2 = 0$; see [Bat17, Example 4.14] and [BKS22, Example 2.9].

It was shown in [BKS22] that up to unimodular equivalence there exist exactly five 3-dimensional Newton polytopes defining Enriques surfaces. It looks reasonable to extend this classification in dimension 4 and to obtain a complete list of all 4-dimensional lattice polytopes P with $\dim F(P) = 0$ and $F(P) \cap M = \emptyset$. This would yield interesting examples of minimal 3-folds with $\kappa = 0$ that are 3-dimensional analogs of Enriques surfaces (see Example 3.6). A particular example of such a 4-dimensional lattice polytope P with $F(P) = \{(1/2, 1/2, 1/2, 1/2)\}$ was proposed to the author by Harald Skarke as the convex hull of the following ten lattice points in $\mathbb{R}^4$:

$$(0,0,0,0)\,,\quad (1,1,1,1)\,,\quad (2,0,0,0)\,,\quad (0,2,0,0)\,,\quad (0,0,2,0)\,,\quad (0,0,0,2)\,,$$
$$(-1,1,1,1)\,,\quad (1,-1,1,1)\,,\quad (1,1,-1,1)\,,\quad (1,1,1,-1)\,.$$

The normal fan Σ_P of P is generated by the twelve primitive lattice vectors

$$\pm(1,1,0,0)\,,\quad \pm(1,0,0,1)\,,\quad \pm(0,1,1,0)\,,\quad \pm(0,1,0,1)\,,\quad \pm(1,0,1,0)\,,\quad \pm(0,0,1,1)\,.$$

10.3 Mirror symmetry beyond reflexive polytopes

The proposed method significantly extends our possibilities and allows one to construct minimal Calabi–Yau models of toric hypersurfaces in the case where the Newton polytope P is not reflexive but only satisfies the condition $F(P) = \{0\}$; see [Bat17]. Many examples of such lattice polytopes P are contained among almost reflexive lattice polytopes P of dimension 3 and 4; see [BKS22, Section 2]. If $F(P) = \{0\}$, then the canonical Calabi–Yau model $\widetilde{Z}$ is the Zariski closure of Z in the $\mathbb{Q}$-Gorenstein toric Fano variety $\widetilde{V}$ corresponding to the canonical hull $C(P)$ of P. It is still an open problem to find a criterion for d-dimensional lattice polytopes P with $F(P) = \{0\}$ such that Calabi–Yau compactifications of nondegenerate affine toric hypersurfaces $Z \subset (\mathbb{C}^*)^d$ with Newton polytope P admit mirrors.

10.4 Minimal models and homogeneous coordinates

Let P be a d-dimensional Newton polytope with $F(P) \neq \emptyset$. We set $n := |S_F(P)|$. Then the simplicial terminal toric varieties $\widehat{V}$ defined above for constructing minimal models of nondegenerate hypersurfaces can be obtained as GIT-quotients of $\mathbb{C}^n$ by the Néron–Severi quasi-torus of dimension $n-d$. It is natural to investigate the combinatorial construction proposed above for canonical and minimal models of toric hypersurfaces using the homogeneous coordinates $\{z_1, \ldots, z_n\}$ of $\widehat{V}$; see [Cox95]. This point of view possibly allows one to obtain an alternative interpretation of the Fine interior $F(P)$ via Newton polytopes of partial derivatives of the defining multi-homogeneous polynomial $h \in \mathbb{C}[z_1, \ldots, z_n]$ that defines the projective toric hypersurface $\{h = 0\} \subset \widehat{V}$ using n homogeneous coordinates [BC94]. The homogeneous coordinates could allow one to construct minimal models $\widehat{Z}$ under some weaker nondegeneracy condition for the coefficients of h using the nonvanishing of the A-discriminant instead of the nonvanishing of the principal A-determinant [GKZ94].

10.5 Complete intersections

The genus formula of Khovanskiĭ for nondegenerate complete intersections [Kho78] and the combinatorial construction of minimal models of Calabi–Yau complete intersections [BB96b, BB96a, BL15] motivate natural generalizations of the combinatorial construction of canonical and minimal models of nondegenerate toric hypersurfaces to the case of complete intersections defined by r Newton polytopes $P_1, \ldots, P_r \subset M_{\mathbb{R}}$. The crucial combinatorial condition $F(P) \neq \emptyset$ has to be applied to the Minkowski sum $P := P_1 + \cdots + P_r$. The canonical model of the affine complete intersection $Z = \bigcap_{i=1}^r Z_i \subset \mathbb{T}^d$ should be obtained as the Zariski closure of Z in the $\mathbb{Q}$-Gorenstein canonical toric variety $\widetilde{V}$ defined by the normal fan of the Minkowski sum

$$\widetilde{P} = C(P) + F(P) = C(P_1) + \cdots + C(P_r) + F(P)\,, \quad \text{where}$$

$$C(P_i) := \{x \in M_{\mathbb{R}} \colon \langle x, \nu\rangle \geqslant \mathrm{Min}_{P_i}(\nu) \ \ \forall \nu \in S_F(P)\}\,, \quad i = 1, \ldots, r\,.$$

This would provide an alternative point of view on the work of Fletcher [Ian00].

Acknowledgements

I would like to express my thanks for helpful discussions to Martin Bohnert, Dimitios Dais, Jonatan Fine, Julius Giesler, Jürgen Hausen, Shihoko Ishii, Alexander Kasprzyk, Ivo Radloff, Miles Reid, Karin Schaller, Harald Skarke, and especially to Jaron Treutlein for his constant interest towards interactions between MMP and lattice polytopes [Tre10].

References

AGH+15 R. Altman, J. Gray, Y.-H. He, V. Jejjala and B. D. Nelson, *A Calabi–Yau database: threefolds constructed from the Kreuzer–Skarke list*, J. High Energy Phys. **2015** (2015), no. 2, 158; doi:10.1007/JHEP02(2015)158.

Bat94 V. V. Batyrev, *Dual polyhedra and mirror symmetry for Calabi–Yau hypersurfaces in toric varieties*, J. Algebraic Geom. **3** (1994), no. 3, 493–535.

Bat17 ______, *The stringy Euler number of Calabi–Yau hypersurfaces in toric varieties and the Mavlyutov duality*, Pure Appl. Math. Q. **13** (2017), no. 1, 1–47; doi:10.4310/PAMQ.2017.v13.n1.a1.

BB96a V. V. Batyrev and L. A. Borisov, *Mirror duality and string-theoretic Hodge numbers*, Invent. Math. **126** (1996), no. 1, 183–203; doi:10.1007/s002220050093.

BB96b ______, *On Calabi–Yau complete intersections in toric varieties*, Higher-Dimensional Complex Varieties (Trento, 1994) (de Gruyter, Berlin, 1996), 39–65; doi:10.1515/9783110814736.39.

BC94 V. V. Batyrev and D. A. Cox, *On the Hodge structure of projective hypersurfaces in toric varieties*, Duke Math. J. **75** (1994), no. 2, 293–338; doi:10.1215/S0012-7094-94-07509-1.

BKS22 V. V. Batyrev, A. M. Kasprzyk and K. Schaller, *On the fine interior of three-dimensional canonical Fano polytopes*, in *Interactions with Lattice Polytopes*, Springer Proc. Math. Stat., vol. 386 (Springer, Cham, 2022), 11–47; doi:10.1007/978-3-030-98327-7_2.

Bir18a C. Birkar, *Birational geometry of algebraic varieties*, Proc. Intern. congress of Math., vol 2 (Rio de Janeiro 2018) (World Sci. Publ., Hackensack, NJ, 2018), 565–590; doi:10.1142/9789813272880_0068.

Bir18b ______, *Log Calabi–Yau fibrations*, 2018, arXiv:1811.10709.

Bir21 ______, *Generalised pairs in birational geometry*, EMS Surv. Math. Sci. **8** (2021), no. 1–2, 5–24; doi:10.4171/emss/42.

BL15 L. A. Borisov and Z. Li, *On complete intersections with trivial canonical class*, Adv. Math. **268** (2015), 339–349; doi:10.1016/j.aim.2014.08.013.

Can01 A. Cannas da Silva, *Lectures on symplectic geometry*, Lect. Notes in Math., vol. 1764 (Springer, Berlin – Heidelberg, 2001); doi:10.1007/978-3-540-45330-7.

Cox95 D. A. Cox, *The homogeneous coordinate ring of a toric variety*, J. Algebraic Geom. **4** (1995), no. 1, 17–50.

CK99 D. A. Cox and S. Katz, *Mirror symmetry and algebraic geometry*, Math. Surveys Monogr., vol. 68 (Amer. Math. Soc., Providence, RI, 1999); doi:10.1090/surv/068.

CLS11 D. A. Cox, J. B. Little and H. K. Schenck, *Toric varieties*, Grad. Stud. Math., vol. 124 (Amer. Math. Soc., Providence, RI, 2011); doi:10.1090/gsm/124.

Del88 T. Delzant, *Hamiltoniens périodiques et images convexes de l'application moment*, Bull. Soc. Math. France **116** (1988), no. 3, 315–339; doi:10.24033/bsmf.2100.

Fin83 J. Fine, *Resolution and completion of algebraic varieties*, Ph.D. Thesis, University of Warwick, 1983, available at http://wrap.warwick.ac.uk/114676/.

Fuj03 O. Fujino, *Notes on toric varieties from Mori theoretic viewpoint*, Tohoku Math. J. **55** (2003), no. 4, 551–564; doi:10.2748/tmj/1113247130.

FS04 O. Fujino and H. Sato, *Introduction to the toric Mori theory*, Michigan Math. J. **52** (2004), no. 3, 649–665; doi:10.1307/mmj/1100623418.

GKZ94 I. M. Gelfand, M. M. Kapranov and A. Zelevinsky, *Discriminants, resultants, and multidimensional determinants*, Math. Theory Appl. (Birkhäuser Boston, Inc., Boston, MA, 1994); doi:10.1007/978-0-8176-4771-1.

Gie21 J. Giesler, *The plurigenera and birational models of nondegenerate toric hypersurfaces*, 2021, arXiv:2112.10523.

Gui94 V. Guillemin, *Moment maps and combinatorial invariants of Hamiltonian T^n-spaces*, Progr. Math., vol. 122 (Birkhäuser Boston, Inc., Boston, MA, 1994); doi:10.1007/978-1-4612-0269-1.

HKP06 M. Hering, A. Küronya and S. Payne, *Asymptotic cohomological functions of toric divisors*, Adv. Math. **207** (2006), no. 2, 634–645; doi:10.1016/j.aim.2005.12.007.

Ian00 A. R. Iano-Fletcher, *Working with weighted complete intersections*, in *Explicit Birational Geometry of 3-Folds*, London Math. Soc. Lecture Note Ser., vol. 281 (Cambridge Univ. Press, Cambridge, 2000), 101–174; doi:10.1017/CBO9780511758942.005.

Ish99 S. Ishii, *The minimal model theorem for divisors of toric varieties*, Tohoku Math. J. **51** (1999), no. 2, 213–226; doi:10.2748/tmj/1178224813.

Kas10 A. M. Kasprzyk, *Canonical toric Fano threefolds*, Canad. J. Math. **62** (2010), no. 6, 1293–1309; doi:10.4153/CJM-2010-070-3.

Kho77 A. G. Khovanskiĭ, *Newton polyhedra, and toroidal varieties*, Funct. Anal. Appl. **11** (1977), no. 4, 289–296; doi:10.1007/BF01077143.

Kho78 ———, *Newton polyhedra, and the genus of complete intersections*, Funct. Anal. Appl. **12** (1978), no. 1, 38–46; doi:10.1007/BF01077562.

Kho84 ———, *Newton polyhedra (resolution of singularities)*, J. Soviet Math. (1984), 2811–2830; doi:10.1007/BF01084822.

Koe91 R. J. Koelman, *The number of moduli of families of curves on toric surfaces*, Ph.D. Thesis, University of Nijmegen, 1991, available at https://repository.ubn.ru.nl/handle/2066/113957.

KM98 J. Kollár and S. Mori, *Birational geometry of algebraic varieties*, Cambridge Tracts in Math., vol. 134 (Cambridge Univ. Press, Cambridge, 1998); doi:10.1017/CBO9780511662560.

KS98 M. Kreuzer and H. Skarke, *Classification of reflexive polyhedra in three dimensions*, Adv. Theor. Math. Phys. **2** (1998), no. 4, 853–871; doi:10.4310/ATMP.1998.v2.n4.a5.

KS00 ———, *Complete classification of reflexive polyhedra in four dimensions*, Adv. Theor. Math. Phys. **4** (2000), no. 6, 1209–1230; doi:10.4310/ATMP.2000.v4.n6.a2.

Mat02 K. Matsuki, *Introduction to the Mori program*, Universitext (Springer-Verlag, New York, 2002); doi:10.1007/978-1-4757-5602-9.

Mav11 A. A. Mavlytov, *Mirror symmetry for Calabi–Yau complete intersections in Fano toric varieties*, 2011, arXiv:1103.20.93.

OP91 T. Oda and H. S. Park, *Linear Gale transforms and Gelfand–Kapranov–Zelevinskij decompositions*, Tohoku Math. J. **43** (1991), no. 3, 375–399; doi:10.2748/tmj/1178227461.

Rei83 M. Reid, *Decomposition of toric morphisms*, in *Arithmetic and Geometry, Vol. II*, Progr. Math., vol. 36 (Birkhäuser Boston, Boston, MA, 1983), 395–418; doi:10.1007/978-1-4757-9286-7_15.

Rei87 ———, *Young person's guide to canonical singularities*, Algebraic Geometry, Bowdoin, 1985 (Brunswick, Maine, 1985), Proc. Sympos. Pure Math., vol. 46 (Amer. Math. Soc., Providence, RI, 1987), 345–414.

Tre10 J. Treutlein, *Birationale Eigenschaften generischer Hyperflächen in algebraischen Tori*, Ph.D. Thesis, Eberhart-Karls-Universität Tübingen, 2010, available at http://nbn-resolving.de/urn:nbn:de:bsz:21-opus-48975.

Victor V. Batyrev batyrev@math.uni-tuebingen.de
Mathematisches Institut, Universität Tübingen, Auf der Morgenstelle 10, 72076 Tübingen, Germany