A non-Archimedean analogue
of Campana's notion of specialness

Jackson S. Morrow and Giovanni Rosso

ABSTRACT

Let K be an algebraically closed, complete, non-Archimedean valued field of characteristic zero, and let $\mathscr{X}$ be a K-analytic space (in the sense of Huber). In this work, we pursue a non-Archimedean characterization of Campana's notion of specialness. We say that $\mathscr{X}$ is K-analytically special if there exist a connected, finite-type algebraic group G/K, a dense open subset $\mathscr{U} \subset G^{\mathrm{an}}$ with $\mathrm{codim}(G^{\mathrm{an}} \setminus \mathscr{U}) \geqslant 2$, and an analytic morphism $\mathscr{U} \to \mathscr{X}$ which is Zariski dense.

With this definition, we prove several results which illustrate that this definition correctly captures Campana's notion of specialness in the non-Archimedean setting. These results inspire us to make non-Archimedean counterparts to conjectures of Campana. As preparation for our proofs, we prove auxiliary results concerning the indeterminacy locus of a meromorphic mapping between K-analytic spaces, the notion of pseudo-K-analytically Brody hyperbolic, and extensions of meromorphic maps from smooth, irreducible K-analytic spaces to the analytification of a semi-abelian variety.

1. Introduction

The study of rational points on varieties over number fields is one of the fundamental questions in arithmetic geometry. For curves, we have a good understanding of the behaviour of the rational points, and this behaviour is governed by the genus of the curve. If the curve has genus $g \leqslant 1$, then the rational points on the curve become infinite after a finite extension of the base number field; in this case, we say that the curve is arithmetically special. If the curve has genus $g \geqslant 2$, then a famous result of Faltings [Fal83] asserts that the rational points on the curve are finite over any number field, and in this case, we say that the curve is arithmetically hyperbolic. It is natural to ask if a similar dichotomy holds for higher-dimensional varieties.

The deep and contrasting conjectures of Lang [Lan86, pp. 161–162] and Campana [Cam04, Section 9] posit that such a relationship holds for certain classes of varieties. Lang's conjecture asserts that a variety of general type over a number field is pseudo-arithmetically hyperbolic [Jav20, Definition 7.2], and Campana's conjecture claims that a special variety (Definition 5.2) over a number field is arithmetically special (Definition 5.17). There has been significant progress

on both of these conjectures (see, for example, [Fal91, Fal94, Voj96, Voj99] and [HT00a, HT00b, BT00] and the excellent book [Nic20] for more references). Since understanding the arithmetic properties of such varieties is difficult, we seek other ways to describe being of general type and special. There exist (conjectural) complex analytic characterizations of these notions (see, for example, [Lan86, Kob98] and [Cam04]), and recently, there has been work on providing a (conjectural) non-Archimedean characterization of general type (see, for example, [Che94, Che96, JV21, Mor21, Sun20]).

Main contributions

In this work, we offer a non-Archimedean interpretation of Campana's notion of specialness with the desideratum that our interpretation is equivalent to other characterizations of specialness. Our definition is motivated by the (conjectural) complex analytic analogue of specialness, which goes by the name Brody special (Definition 5.11) and states that a complex manifold is Brody special if there exists a dense entire map from $\mathbb{C}$ to it. For example, the complex analytification of an abelian variety is Brody special by the Riemann uniformization theorem.

A natural first guess for a non-Archimedean analogue of this notion would be to ask for a dense analytic morphism from the non-Archimedean analytification of $\mathbb{A}^1$ or $\mathbb{G}_m$ into our analytic space. However, results of Cherry [Che94] tell us that this definition will not suffice. For example, the non-Archimedean analytification of an abelian variety with good reduction will not admit any non-constant morphism from these spaces, and since these analytic spaces should be special, we see that this naive notion does not suffice.

Instead of testing specialness on non-Archimedean entire curves, we will test on big analytic opens of the non-Archimedean analytification of connected algebraic groups. To set notation for our definition, let K be an algebraically closed, complete, non-Archimedean valued field of characteristic zero, let $\mathscr{X}$ be a K-analytic space (in the sense of Huber), and let X^{an} denote the non-Archimedean analytification of a variety X over K.

DEFINITION 1.1. We say that $\mathscr{X}$ is *K-analytically special* if there exist a connected, finite-type algebraic group G/K, a dense open subset $\mathscr{U} \subset G^{\mathrm{an}}$ with $\mathrm{codim}(G^{\mathrm{an}} \setminus \mathscr{U}) \geqslant 2$, and an analytic morphism $\mathscr{U} \to \mathscr{X}$ which is Zariski dense.

We highlight two points in the above definition. First, it may seem unnatural to test whether a K-analytic space is K-analytically special by asking for the existence of a Zariski dense analytic morphism from a big analytic open of an algebraic group; however, we provide explanations for these conditions in Example 6.3 and Remark 6.6. The second is that we do not require our K-analytic space to be compact. This allows us to avoid the language of logarithmic geometry and orbifolds when working with non-compact K-analytic space, which eliminates some technical difficulties.

We prove several results to illustrate that our definition correctly captures Campana's notion of specialness (Definitions 5.2 and 5.4). Our first result states that a K-analytically special K-analytic space cannot dominate a positive-dimensional pseudo-K-analytically Brody hyperbolic K-analytic space, which can be viewed as a non-Archimedean version of [Cam04, Proposition 9.27]. Roughly speaking, a K-analytic space is pseudo-K-analytically Brody hyperbolic if it does not admit a dominant analytic morphism from $\mathbb{G}_m^{\mathrm{an}}$ or from a dense open subset $\mathscr{U} \subset A^{\mathrm{an}}$ with $\mathrm{codim}(A^{\mathrm{an}} \setminus \mathscr{U}) \geqslant 2$, where A is an abelian variety over K. We refer the reader to Section 4 for the precise definition and a discussion of the notion of pseudo-K-analytically Brody hyperbolic.

Theorem A (Theorem 6.9). *Let K be an algebraically closed, complete, non-Archimedean valued field of characteristic zero, and let $\mathscr{X}$ and $\mathscr{Y}$ be irreducible, reduced, separated K-analytic spaces. If $\mathscr{Y}$ is a K-analytically special K-analytic space and $\mathscr{X}$ is a positive-dimensional pseudo-K-analytically Brody hyperbolic K-analytic space (Definition 4.1), then there is no dominant morphism $\mathscr{Y} \to \mathscr{X}$.*

Using Theorem A, we are able to identify several classes of K-analytically special K-analytic spaces in terms of their intrinsic geometry and certain abelian properties of their fundamental group. We recall that our guiding principle is that the notions of specialness and being of general type contrast each other.

In [Mor21], the first author proved that a closed subvariety X of a semi-abelian variety over K is of logarithmic general type if and only if it is pseudo-K-analytically Brody hyperbolic. This result builds off of the works [Abr94, Nog98], where the authors show that the first condition is equivalent to the special locus of X being properly contained in X, which essentially means that X is not the translate of a semi-abelian subvariety. We also note that results of Iitaka [Iit76, Theorems 2 and 4] (see also Vojta's result in [Voj99, Theorem 5.15]) and Campana [Cam04, Theorem 5.1] imply that if X is a translate of a semi-abelian variety, then X is a special variety (see Definition 5.6 for the notion of special non-proper variety).

With our guiding principle in mind, we use results of Vojta and Theorem A to prove that X being a translate of a semi-abelian subvariety is equivalent to X^{an} being K-analytically special. The techniques of our proof can easily be adapted to the algebraic case, thus giving us an equivalence between being K-analytically special and special in the sense of Campana. We point out that the analogous result for Brody special (that is, for complex analytic varieties) holds due to the works [Abr94, Nog98, Cam04].

Theorem B (Theorem 7.1). *Let K be an algebraically closed, complete, non-Archimedean valued field of characteristic zero. Let G/K be a semi-abelian variety, and let $X \subset G$ be a closed subvariety. Then, the following are equivalent:*

(1) *The variety X is the translate of a semi-abelian subvariety.*
(2) *The space X^{an} is K-analytically special (Definition 1.1).*
(3) *The variety X is special (Definitions 5.2 and 5.4).*

Our final main result is related to the abelianity conjecture of Campana [Cam04, Conjecture 7.1]. This conjecture postulates that for a variety $X/\mathbb{C}$, being special is equivalent to the topological fundamental group of the complex analytification of X being virtually abelian (that is, it contains a finite-index abelian subgroup). For results concerning this abelianity conjecture, we refer the reader to [Cam04, Theorem 7.8], [Yam10, Theorem 1.1], and [JR22, Theorem 1.12].

The development of fundamental groups for a non-Archimedean analytic space has a rich history, which we briefly exposit. Berkovich K-analytic spaces possess nice topological properties. For example, an important result of Berkovich [Ber99, Corollary 9.6] says that a smooth, connected Berkovich K-analytic space admits a topological universal covering which is a simply connected K-analytic space, and hence one can describe the topological fundamental group of a Berkovich K-analytic space via loops modulo homotopy. This illustrates that Berkovich K-analytic spaces have similarities to complex manifolds. However, we note that the topological fundamental group does not detect interesting arithmetic properties of a variety; indeed, the topological fundamental group of the analytification of any variety over K with good reduction is trivial.

While there are too few topological coverings, working directly with the étale fundamental group of a Berkovich K-analytic space is unwieldy (see, for example, [dJo95, Proposition 7.4]). To remedy this, André [And03] introduced the tempered fundamental group of a Berkovich K-analytic space, which sits between the topological and étale fundamental groups and provides us with the correct fundamental group to study non-Archimedean analogues of Campana's abelianity conjectures.

As a first step in this direction, we prove that the Berkovich analytification of a projective surface with negative Kodaira dimension is K-analytically special if and only if its tempered fundamental group is virtually abelian; see [BL00, Theorem 3.1] for the analogous complex analytic statement.

Theorem C (Theorem 8.1). *Let K be an algebraically closed, complete, non-Archimedean valued field of characteristic zero, and let X/K is a smooth, projective surface with negative Kodaira dimension. Then the following are equivalent:*

(1) *The surface X has irregularity $q(X) = h^0\big(X, \Omega_X^1\big)$ less than 2.*
(2) *The space X^{an} is K-analytically special.*
(3) *The tempered fundamental group $\pi_1^{\mathrm{temp}}(X^{\mathrm{an}})$ of X^{an} is virtually abelian.*

The above results inspire us to formulate non-Archimedean counterparts to conjectures of Campana. Below, we present shortened versions of these conjectures as many of the definitions have been omitted from our discussion up to this point. For precise statements of these conjectures, we refer the reader to Section 9.

Conjecture 1.2 (Non-Archimedean Campana's conjectures). *Let X/K be a smooth projective variety. Then, the following are equivalent:*

(1) *The variety X is special (Definitions 5.2 and 5.4);*
(2) *The space X^{an} is K-analytically special.*

Conjecture 1.3 (Non-Archimedean Campana's abelianity conjecture for fundamental groups). *Let X/K be a smooth projective variety. If X^{an} is K-analytically special, then $\pi_1^{\mathrm{temp}}(X^{\mathrm{an}})$ is virtually abelian.*

Preparatory results

In order to prove the above theorems, we need to prove three auxiliary results, which we believe to be of independent interest.

The first one (Theorem 3.1) concerns the indeterminacy locus of a meromorphic mapping between K-analytic spaces and is a non-Archimedean analogue of a result due to Remmert [Rem57]. The proof presented in Section 3 has been provided to us by Brian Conrad. We use this result to show that our notion of K-analytically special is a bi-meromorphic invariant, which is a crucial property (see, for example, the proof of Theorem C). Roughly speaking, a K-analytic space being K-analytically special means that it admits a dominant, meromorphic map from an algebraic group, and so in order for this notion to be a bi-meromorphic invariant, we need to understand the indeterminacy locus of a general meromorphic map. Theorem 3.1 states that when the source is normal and the target is reduced and proper, the domain of definition of a meromorphic map is an open *analytic* subset whose complement has codimension at least 2. Note that this is precisely the condition we require in our definition of K-analytically special.

For the second result, we begin by offering a new, more natural definition of pseudo-K-analytically Brody hyperbolic (Definition 4.1) and deduce an equivalent way of testing this notion (Theorem 4.4), which brings it closer in line with our notion of K-analytically special. To expand on this, the first definition of pseudo-K-analytically Brody hyperbolic appeared in [Mor21, Definition 2.2] and contained a seemingly unnatural condition of studying algebraic maps from big algebraic opens of abelian varieties. Moreover, with this original definition, it is unclear if the statement of Theorem A is true. To fix this issue, we modify this definition to test pseudo-K-analytically Brody hyperbolicity on big *analytic* opens of analytifications of algebraic groups and prove that one can actually test this notion on analytic maps from $\mathbb{G}_{m,K}^{\mathrm{an}}$ and from big *analytic* opens of analytifications of abelian varieties. With this new definition, we immediately arrive at Theorem A.

The final preparatory theorem (Theorem 7.4) is an extension result concerning meromorphic maps from smooth, irreducible adic spaces to the analytification of a semi-abelian variety, which is a non-Archimedean analogue of [BLR90, Section 8.4, Corollary 6] and [Moc12, Lemma A.2]. We use this result and Theorem 4.4 show that [Mor21, Theorem A] remains true with our new definition of pseudo-K-analytically Brody hyperbolic (see Proposition 7.10). Equipped with this fact, the proof of Theorem B follows from utilizing results of Vojta [Voj99] on the Ueno fibration of a closed subvariety of a semi-abelian variety.

Organization

The paper has two parts. Sections 2, 3, and 4 form the first part and consist of background material and auxiliary results. The remaining sections focus on defining and proving results related to our notion of K-analytically special. More precisely, we organize the paper as follows.

In Section 2, we review non-Archimedean analytic spaces, describe several properties preserved by analytification, and study the sheaf of meromorphic functions on a taut locally strongly Noetherian adic space. In Section 3, we prove Theorem 3.1, following the proof given to us by Brian Conrad. This theorem is a non-Archimedean analogue of a theorem of Remmert stating that the indeterminacy locus of a meromorphic morphism between a normal and a proper non-Archimedean analytic space has codimension at least 2. In Section 4, we give a new definition of pseudo-K-analytically Brody hyperbolic and deduce an equivalent way of testing this notion (Theorem 4.4).

We begin our discussion on the various notions of specialness in Section 5 and offer our non-Archimedean characterization of specialness, which we call K-analytically special, in Section 6. In this section, we prove several basic properties of being K-analytically special and our first main theorem (Theorem A). In Section 7, we prove our second main theorem (Theorem B) concerning when a closed subvariety of a semi-abelian variety is K-analytically special. In Section 8, we prove our final main theorem (Theorem C), which characterizes when a projective surface of negative Kodaira dimension is K-analytically special in terms of its tempered fundamental group. To conclude, we make non-Archimedean counterparts to Campana's conjectures in Section 9.

Conventions

We establish the following to be used throughout.

Fields and algebraic geometry. We will let k denote an algebraically closed field of characteristic zero and let K be an algebraically closed, complete, non-Archimedean valued field of characteristic zero. A variety X over a field will be an integral, separated scheme of finite type

over said field. We will use K_X to denote the canonical divisor on X and $q(X) = h^0(X, \Omega_X^1)$ to denote the irregularity of X. For a smooth variety X/k and a line bundle $\mathscr{L}$ on X, we use $\kappa(X, \mathscr{L})$ to denote the Iitaka dimension of $\mathscr{L}$, which we briefly recall. For each $m \geqslant 0$ such that $h^0(X, \mathscr{L}^{\otimes m}) \neq 0$, the linear system $|\mathscr{L}^{\otimes m}|$ induces a rational map from X to a projective space of dimension $h^0(X, \mathscr{L}^{\otimes m}) - 1$. The Iitaka dimension of $\mathscr{L}$ is the maximum over all $m \geqslant 1$ of the dimension of the image for this rational map. The Kodaira dimension $\kappa(X)$ of X is the Iitaka dimension of the canonical bundle.

Analytic spaces. We will make use of various analytifications of a variety X over a field.

When $k = \mathbb{C}$, we will use X^{an} to denote the complex analytification of X, which corresponds to taking the complex-valued points $X(\mathbb{C})$ of X.

When $k = K$, we will denote the adic space associated with X by X^{an} (as in [Hub94]). Sometimes we will need to consider the corresponding Berkovich space, and we denote the Berkovich analytification of X by X^{Ber} (as in [Ber90] or good Berkovich K-analytic spaces in [Ber93]). As these non-Archimedean analytifications are fundamental objects in our study, we devote Section 2 to describing their properties and relationships. When referring to rigid analytic, Berkovich, and adic spaces which may not be algebraic, we will use script notation $\mathscr{X}$, $\mathscr{Y}$, $\mathscr{Z}$. In our work, all rigid analytic spaces over $\mathrm{Sp}(K)$ are taut, all Berkovich K-analytic spaces are strict and Hausdorff, and all adic spaces are taut, locally strongly Noetherian, and locally of finite type over $\mathrm{Spa}(K, K^\circ)$, unless otherwise stated. We will mainly use adic spaces, but sometimes we need to refer to a different type of analytic space and will make it clear as to which category of analytic space we are using in certain instances.

That being said, whenever we refer to a *K-analytic space*, we mean a taut, locally strongly Noetherian, and locally of finite type adic space over $\mathrm{Spa}(K, K^\circ)$.

2. Preliminaries on non-Archimedean analytic spaces

In this section, we provide necessary background on non-Archimedean analytic spaces. In particular, we describe the equivalence between certain subcategories of rigid analytic, Berkovich K-analytic, and adic spaces, recall basic properties of adic spaces, discuss properties of analytifications of algebraic varieties and algebraic morphisms, and finally introduce the sheaf of meromorphic functions on a taut locally strongly Noetherian adic space.

2.1 Comparisons between rigid analytic, Berkovich K-analytic spaces, and adic spaces

THEOREM 2.2 ([Ber93, Theorem 1.6.1], [Hub96, Proposition 8.3.1]). *The category of taut rigid analytic spaces over $\mathrm{Sp}(K)$ is equivalent to the category of Hausdorff strict Berkovich K-analytic spaces.*

THEOREM 2.3 ([Hub96, Proposition 8.3.1], [Hen16, Theorem 0.1]). *The category of taut adic spaces locally of finite type over $\mathrm{Spa}(K, K^\circ)$ is equivalent to the category of Hausdorff strict Berkovich K-analytic spaces. At the level of topological spaces, this equivalence sends an adic space $\mathscr{X}$ to its universal Hausdorff quotient $[\mathscr{X}]$.*

Remark 2.4. We note that the adic space associated with an algebraic variety is an example of a taut adic space locally of finite type, and since it will be relevant in Subsection 7.3, we emphasize that Theorem 2.3 tells us that taut adic spaces locally of finite type are not Hausdorff.

We also recall that the notions of properness and finiteness are equivalent for rigid analytic and Berkovich K-analytic spaces.

LEMMA 2.5. *Let $\mathscr{X}$ and $\mathscr{Y}$ be rigid analytic spaces over K, and let $\mathscr{X}^{\mathrm{Ber}}$ and $\mathscr{Y}^{\mathrm{Ber}}$ denote the associated Berkovich K-analytic spaces. Let $f\colon \mathscr{X} \to \mathscr{Y}$ denote a morphism of rigid analytic spaces, and let $f^{\mathrm{Ber}}\colon \mathscr{X}^{\mathrm{Ber}} \to \mathscr{Y}^{\mathrm{Ber}}$ denote the associated morphism of Berkovich K-analytic spaces. Then,*

(1) *f is proper if and only if f^{Ber} is proper;*
(2) *f is finite if and only if f^{Ber} is finite.*

Proof. This follows from [Ber90, Proposition 3.3.2] and [Ber93, Example 1.5.3]. □

2.6 Basic properties of adic spaces

We now recall the notions of reduced, normal, and irreducible adic spaces following [Man20].

DEFINITION 2.7 ([Man20, Definition 1.3]). The adic space $\mathscr{X}$ is *normal* (respectively, *reduced*) if it can be covered by affinoid adic spaces of the form $\mathrm{Spa}(A, A^+)$, where A is normal (respectively, reduced).

DEFINITION 2.8 ([Man20, Definition 1.11]). The adic space $\mathscr{X}$ is *irreducible* if it cannot be written as the disjoint union of two proper closed adic subspaces.

2.9 Properties of analytifications of algebraic morphisms

We now discuss facts concerning the analytification functor from locally of finite type K-schemes to adic spaces over K. The analogous results for rigid analytic spaces over K and Berkovich K-analytic spaces are treated in [Con99, Section 5] and [Ber90, Section 3.4], respectively.

LEMMA 2.10. *Let X be a K-scheme which is locally of finite type, and let X^{an} (respectively, X^{Ber}) denote the adic space (respectively, Berkovich K-analytic space) associated with X. Then,*

(1) *X is reduced if and only if X^{an} (respectively, X^{Ber}) is reduced;*
(2) *X is normal if and only if X^{an} (respectively, X^{Ber}) is normal;*
(3) *X is separated if and only if X^{an} is separated (respectively, $|X^{\mathrm{Ber}}|$ is Hausdorff);*
(4) *X is smooth if and only if X^{an} (respectively, X^{Ber}) is smooth;*
(5) *X is irreducible if and only if X^{an} (respectively, X^{Ber}) is irreducible.*

Proof. The first and second statements follow from [Ber90, Proposition 3.4.3] for Berkovich K-analytic spaces and from [Con99, Theorem 5.1.3(1)] and [Hub96, §1.1.11(c)] for adic spaces. The third statement follows from [Ber90, Theorem 3.4.8(1)] for Berkovich K-analytic spaces and from [Con99, Theorem 5.2.1] and [Hub96, Remark 1.3.19] for adic spaces. The fourth statement follows from [Ber90, Proposition 3.4.6(3)] for Berkovich K-analytic spaces and from [Con99, Theorem 5.2.1] and [Hub96, Proposition 1.7.11] for adic spaces. The fifth statement follows from [Duc18, Proposition 2.7.16] for Berkovich K-analytic spaces and from [Con99, Theorem 2.3.1] and [Hub96, §1.1.11(c)] for adic spaces. We mention that irreducibility of a Berkovich K-analytic space refers to irreducibility in the Zariski analytic topology (see [Duc18, §1.5.1] for the definition). □

LEMMA 2.11. *Let $f\colon X \to Y$ be a locally of finite type morphism of varieties over K, and let $f^{\mathrm{an}}\colon X^{\mathrm{an}} \to Y^{\mathrm{an}}$ (respectively, $f^{\mathrm{Ber}}\colon X^{\mathrm{Ber}} \to Y^{\mathrm{Ber}}$) denote the associated morphism of adic*

spaces (respectively, Berkovich K-analytic spaces). Then, f^{an} (respectively, f^{Ber}) is partially proper (respectively, boundaryless).

Proof. The second statement follows from [Ber93, Proposition 1.5.5(ii)] since X^{Ber} and Y^{Ber} are boundaryless Berkovich K-analytic spaces.

For the first statement, we first note that the first statement and [Ber93, Section 1.6] imply that the associated morphism of rigid analytic spaces is partially proper, and now the result follows from [Hub96, Remark 1.3.19(iii)]. □

LEMMA 2.12. *Let $f\colon X \to Y$ be a locally of finite type morphism of varieties over K, and let $f^{\mathrm{an}}\colon X^{\mathrm{an}} \to Y^{\mathrm{an}}$ (respectively, $f^{\mathrm{Ber}}\colon X^{\mathrm{Ber}} \to Y^{\mathrm{Ber}}$) denote the associated morphism of adic spaces (respectively, Berkovich K-analytic spaces). Then,*

(1) *f is surjective if and only if f^{an} (respectively, f^{Ber}) is surjective,*

(2) *f is flat if and only if f^{an} (respectively, f^{Ber}) is flat.*

Proof. The first (respectively, second) statement for f^{an} follows from Lemma 2.11 and [Hub96, Proof of Proposition 8.3.4] (respectively, [Hub96, Lemma 1.1.10(iii)]). The statements for f^{Ber} can be found in [Ber90, Proposition 3.4.6]. □

LEMMA 2.13. *Let $f\colon X \to Y$ be a locally of finite type morphism of varieties over K, and let $f^{\mathrm{an}}\colon X^{\mathrm{an}} \to Y^{\mathrm{an}}$ (respectively, $f^{\mathrm{Ber}}\colon X^{\mathrm{Ber}} \to Y^{\mathrm{Ber}}$) denote the associated morphism of adic spaces (respectively, Berkovich K-analytic spaces). If f is flat, then f^{an} (respectively, f^{Ber}) is flat and partially proper (respectively, boundaryless) and open.*

Proof. This follows from Lemma 2.11, Lemma 2.12(2), and [Duc18, Theorem 9.2.3 and Remark 9.2.4] and [Hub96, Proof of Proposition 8.3.3]. □

2.14 The sheaf of meromorphic functions on an adic space

To conclude this preliminaries section, we define the sheaf of meromorphic functions on a taut, locally strongly Noetherian adic space over $\mathrm{Spa}(K, K^{\circ})$. We follow the exposition of Bosch [Bos83].

First, we define the sheaf locally for a strongly Noetherian affinoid adic space. For any ring R, denote by $Q(R)$ its total ring of fractions. Let $\mathscr{X} = \mathrm{Spa}(A, A^{+})$ be an affinoid adic space where A is strongly Noetherian. Let $\mathscr{U} \subset \mathscr{X}$ be a rational subset. By [Hub94, Proof of Theorem 2.5, Case II, Point (II.1.iv)], the restriction map $\mathscr{O}_{\mathscr{X}}(\mathscr{X}) \to \mathscr{O}_{\mathscr{X}}(\mathscr{U})$ is flat, and hence it gives us a homomorphism between the corresponding total rings of fractions. Moreover, we can define a presheaf $\mathscr{M}_{\mathscr{X}}$ on the set of rational subset $\mathscr{U}$ of $\mathscr{X}$ via

$$\mathscr{M}_{\mathscr{X}}(\mathscr{U}) = Q(\mathscr{O}_{\mathscr{X}}(\mathscr{U})).$$

LEMMA 2.15. *Let $f = g/h \in Q(\mathscr{O}_{\mathscr{X}}(\mathscr{X}))$, where $g, h \in \mathscr{O}_{\mathscr{X}}(\mathscr{X})$ and h is not a zero-divisor. Consider the presheaf $(\mathscr{O}_{\mathscr{X}} : f)$ which associates with each rational subset $\mathscr{U} \subset \mathscr{X}$ the $\mathscr{O}_{\mathscr{X}}(\mathscr{U})$-ideal*

$$(\mathscr{O}_{\mathscr{X}}(\mathscr{U}) : f) := \{a \in \mathscr{O}_{\mathscr{X}}(\mathscr{U}) : af \in \mathscr{O}_{\mathscr{X}}(\mathscr{U})\}.$$

Then, $(\mathscr{O}_{\mathscr{X}} : f)$ is a coherent $\mathscr{O}_{\mathscr{X}}$-module.

Proof. The proof is identical to that of [Bos83, Lemma 2.1]. □

LEMMA 2.16. *The presheaf $\mathscr{M}_{\mathscr{X}}$ defines a sheaf on $\mathscr{X}$.*

Proof. The proof follows in the same manner as that of [Bos83, Lemma 2.2], and so we will only offer a sketch. Let $\{\mathscr{U}_1, \ldots, \mathscr{U}_n\}$ denote an open cover of $\mathscr{X} = \mathrm{Spa}(A, A^+)$ by rational subsets. Consider the functions $f_i \in \mathscr{M}_\mathscr{X}(\mathscr{U}_i)$ such that $f_{i|\mathscr{U}_i \cap \mathscr{U}_j} = f_{j|\mathscr{U}_i \cap \mathscr{U}_j}$ for all $1 \leqslant i, j \leqslant n$. By Lemma 2.15, we can construct the coherent $\mathscr{O}_\mathscr{U}$-ideal $(\mathscr{O}_{\mathscr{U}_i} : f_i)$. These ideals will coincide on all intersections $\mathscr{U}_i \cap \mathscr{U}_j$ and hence will glue together to form a coherent $\mathscr{O}_\mathscr{X}$-module, which we denote by $\mathscr{J}$. Using the Noether decomposition theorem, we can see that not all elements of $\mathscr{J}(\mathscr{X})$ are zero-divisors in $\mathscr{O}_\mathscr{X}(\mathscr{X})$. Let $h \in \mathscr{J}(\mathscr{X})$ denote such an element. Now $(h_{|\mathscr{U}_i})f_i \in \mathscr{O}_\mathscr{X}(\mathscr{U}_i)$ for all i, and since they coincide on $\mathscr{U}_i \cap \mathscr{U}_j$, they glue to define a function $g \in \mathscr{O}_\mathscr{X}(\mathscr{X})$. Therefore, we can define a global element $f = g/h \in \mathscr{M}_\mathscr{X}(\mathscr{X})$ that restricts to f_i on $\mathscr{U}_i$ and can be shown to be unique. $\qquad\square$

Using the above results, we can globalize this construction.

DEFINITION 2.17. Let $\mathscr{X}$ be a taut, locally strongly Noetherian adic space over $\mathrm{Spa}(K, K^\circ)$. We define the *sheaf of meromorphic functions on $\mathscr{X}$* as follows. Consider an affinoid open cover $\{\mathscr{U}_i\}$ of $\mathscr{X}$ where each $\mathscr{U}_i$ is of the form $\mathrm{Spa}(A_i, A_i^+)$ for A_i a strongly Noetherian ring. By Lemma 2.16, we can define the sheaf $\mathscr{M}_{\mathscr{U}_i}$ for each $\mathscr{U}_i$ and glue these together to get a global sheaf $\mathscr{M}_\mathscr{X}$.

Remark 2.18. The stalk at $x \in \mathscr{X}$ is $Q(\mathscr{O}_{\mathscr{X},x})$. Any non-zero-divisor germ $h \in \mathscr{O}_{\mathscr{X},x}$ can be extended to a function on an affinoid neighborhood $\mathscr{U}$ of x such that h is not a zero-divisor in $\mathscr{O}_\mathscr{X}(\mathscr{U})$.

DEFINITION 2.19. For each global meromorphic function $f \in \mathscr{M}_\mathscr{X}(\mathscr{X})$, the coherent $\mathscr{O}_\mathscr{X}$-ideal $(\mathscr{O}_\mathscr{X} : f)$ can be constructed via Lemma 2.15; we will call this ideal sheaf the *ideal of denominators of f*.

We now define the indeterminacy locus of a global meromorphic function $f \in \mathscr{M}_\mathscr{X}(\mathscr{X})$. To do so, we briefly recall [Hub96, § 1.4.1], which illustrates how one can define a closed adic subspace associated with a coherent $\mathscr{O}_\mathscr{X}$-ideal $\mathscr{I}$. Let

$$V(\mathscr{I}) := \left\{ x \in \mathscr{X} : \mathscr{I}_x \neq \mathscr{O}_{\mathscr{X},x} \right\}$$

and $\mathscr{O}_{V(\mathscr{I})} := (\mathscr{O}_\mathscr{X}/\mathscr{I})_{|V(\mathscr{I})}$. Note that for every $x \in \mathscr{X}$, the support of v_x is equal to the maximal ideal $\mathfrak{m}_x$ of $\mathscr{O}_{\mathscr{X},x}$. By definition, $\mathscr{I}_y \subset \mathfrak{f}_y$ for every $y \in V(\mathscr{I})$, and the valuation v_y induces a valuation v'_y of $\mathscr{O}_{V(\mathscr{I}),y}$. By considering the quotient topology, one has that for every open affinoid subspace $\mathscr{V} \subset \mathscr{X}$, the mapping $\mathscr{O}_\mathscr{X}(\mathscr{V}) \to \mathscr{O}_{V(\mathscr{I})}(V(\mathscr{I}) \cap \mathscr{V})$ is a quotient map. Moreover, the triple

$$V(\mathscr{I}) := \left(V(\mathscr{I}), \mathscr{O}_{V(\mathscr{I})}, (v'_y : y \in V(\mathscr{I}))\right)$$

defines an adic space, which we call the *closed adic space associated with the $\mathscr{O}_\mathscr{X}$-ideal $\mathscr{I}$*.

We now return to our goal of defining the set of poles and zeros of a global meromorphic function.

DEFINITION 2.20. Let $f \in \mathscr{M}_\mathscr{X}(\mathscr{X})$.

- The closed adic space $P_f := V((\mathscr{O}_\mathscr{X} : f))$ associated with the coherent $\mathscr{O}_\mathscr{X}$-ideal $(\mathscr{O}_\mathscr{X} : f)$ is called *the set of poles of f*.
- Analogously to the sheaf of denominators, we can define the sheaf of numerators $(\mathscr{O}_\mathscr{X} : f) \cdot f$, which is a coherent $\mathscr{O}_\mathscr{X}$-ideal. The closed adic space $Z_f := V((\mathscr{O}_\mathscr{X} : f) \cdot f)$ associated with the sheaf of numerators is called *the set of zeros of f*.

- The *indeterminacy locus* of f is defined as the intersection $P_f \cap Z_f$. Note that by Hilbert's nullstellensatz, $f \in \mathscr{O}_{\mathscr{X}}(\mathscr{X})$ if and only if $P_f = \emptyset$.

We want to show that $\mathscr{M}_{\mathscr{X}}(\mathscr{X})$ is a field when $\mathscr{X}$ is irreducible and reduced, and to prove this result, we will need a simple lemma.

LEMMA 2.21. *Let $\mathscr{X}$ be a reduced, taut, locally strongly Noetherian adic space over $\mathrm{Spa}(K, K^\circ)$, and let $f \in \mathscr{M}_{\mathscr{X}}(\mathscr{X})$. Then f is a unit in $\mathscr{M}_{\mathscr{X}}(\mathscr{X})$ if and only if Z_f does not contain a non-empty open subspace.*

Proof. We may assume that $\mathscr{X} = \mathrm{Spa}(A, A^+)$ is affinoid and that A is complete, and so we write $f = g/h$ with $g, h \in A$ such that h is not a zero-divisor in A. We have that $Z_f \subset V(g)$, where $V(g)$ denotes the vanishing locus of g and both Z_f and $V(g)$ coincide on the complement of $V(h)$. Since h is not a zero-divisor, $V(h)$ does not contain a non-empty open subspace, and moreover, we have that Z_f contains an open subspace if and only if $V(g)$ contains an open subspace. However, this only happens if g is a zero-divisor in A, and hence the result follows. $\square$

We now record a useful corollary, which is the adic analogue of [Bos83, Corollary 2.5].

COROLLARY 2.22. *If $\mathscr{X}$ is an irreducible, reduced, taut, locally strongly Noetherian adic space over $\mathrm{Spa}(K, K^\circ)$, then $\mathscr{M}_{\mathscr{X}}(\mathscr{X})$ is a field.*

Proof. This follows from Lemma 2.21 and the fact that a closed adic subspace $\mathscr{Y}$ of an irreducible adic space $\mathscr{X}$ which contains a non-empty open subspace must be the entire space, that is, $\mathscr{Y} = \mathscr{X}$. $\square$

When $\mathscr{X}$ is irreducible and reduced, any morphism $\mathscr{X} \to \mathbb{P}^{1,\mathrm{an}}$ that is not identically equal to ∞ is naturally identified with a meromorphic function $f \in \mathscr{M}_{\mathscr{X}}(\mathscr{X})$. We will need to know when such a morphism will separate points.

LEMMA 2.23. *Suppose that $\mathscr{X}$ is an irreducible, reduced, separated, taut, locally strongly Noetherian adic space over $\mathrm{Spa}(K, K^\circ)$. Let $x, y \in \mathscr{X}$ be two distinct points. Then, there exists a meromorphic function $f \in \mathscr{M}_{\mathscr{X}}(\mathscr{X})$ such that $f(x) \neq f(y)$, where we consider f as a morphism $\mathscr{X} \to \mathbb{P}^{1,\mathrm{an}}$.*

Proof. By the valuative criterion for separatedness [Hub96, Proposition 1.3.7], we have that points of $\mathscr{X}$ are separated, and in particular, we have that $(\mathscr{O}_{\mathscr{X},x}, \mathscr{O}_{\mathscr{X},x}^+) \ncong (\mathscr{O}_{\mathscr{X},y}, \mathscr{O}_{\mathscr{X},y}^+)$. Moreover, we choose a germ $f \in (\mathscr{O}_{\mathscr{X},x}, \mathscr{O}_{\mathscr{X},x}^+)$ which is not in $(\mathscr{O}_{\mathscr{X},y}, \mathscr{O}_{\mathscr{X},y}^+)$, and then $f(x) \neq f(y)$, where we say that $f(y) = \infty$ (as $f \notin (\mathscr{O}_{\mathscr{X},y}, \mathscr{O}_{\mathscr{X},y}^+)$). Since $\mathscr{X}$ is irreducible and reduced, $\mathscr{O}_{\mathscr{X},x}$ is a subring of $\mathscr{M}_{\mathscr{X}}(\mathscr{X})$, and hence f defines an element of $\mathscr{M}_{\mathscr{X}}(\mathscr{X})$. $\square$

3. Meromorphic maps between non-Archimedean analytic spaces

In this section, we prove a non-Archimedean variant of a result of Remmert [Rem57, p. 333] concerning the codimension of the indeterminacy locus of a meromorphic map between certain K-analytic spaces. The proofs presented here were given to us by Brian Conrad.

THEOREM 3.1. *Let K be an algebraically closed, complete, non-Archimedean valued field of characteristic zero. Let $\mathscr{X}/K$ be a normal, taut rigid analytic space, and let $\mathscr{Y}/K$ be a proper, reduced, taut rigid analytic space. The indeterminacy locus of any meromorphic map $\mathscr{X} \dashrightarrow \mathscr{Y}$ is an analytic subset of codimension at least 2.*

First, we recall the definition of meromorphic mapping following Remmert.

DEFINITION 3.2. Let $\mathscr{X}$ and $\mathscr{Y}$ be reduced rigid analytic spaces. We say that a map $\varphi\colon \mathscr{X} \dashrightarrow \mathscr{Y}$ is *meromorphic* if there exists an analytic subset $\mathscr{E} \subset \mathscr{X} \times \mathscr{Y}$ which is mapped properly onto $\mathscr{X}$ by the projection map $\mathrm{pr}_1\colon \mathscr{X} \times \mathscr{Y} \to \mathscr{X}$ of $\mathscr{X} \times \mathscr{Y}$ onto the first factor such that outside a nowhere-dense analytic set $\mathscr{Z} \subset \mathscr{X}$, this map is bi-holomorphic. Moreover, $\mathrm{pr}_1^{-1}(\mathscr{Z})$ is nowhere dense in $\mathscr{E}$. We call the set $\mathscr{E}$ the *graph of φ*.

Remark 3.3. We now explain the reason for calling $\mathscr{E}$ the graph of φ. Let $\mathscr{X}$ be a reduced rigid analytic space, and let $\mathscr{U}$ and $\mathscr{V}$ be open subsets of $\mathscr{X}$. Suppose that we are given two maps $\varphi_1\colon \mathscr{U} \to \mathscr{Y}$ and $\varphi_2\colon \mathscr{V} \to \mathscr{Y}$ that coincide on the intersection $\mathscr{U} \cap \mathscr{V}$. Then the closure of the graph of φ_1 coincides with the closure of the graph of φ_2, and this graph defines a meromorphic map φ as in Definition 3.2.

For the remainder of this section, let $\mathscr{X}/K$ be a normal rigid analytic space, let $\mathscr{Y}/K$ be a proper, reduced rigid analytic space, let $\varphi\colon \mathscr{X} \dashrightarrow \mathscr{Y}$ be a meromorphic map, and let $\mathscr{E}$ denote the graph of φ. By Definition 3.2, the morphism

$$\mathscr{E} \setminus \mathrm{pr}_1^{-1}(\mathscr{Z}) \longrightarrow \mathscr{X} \setminus \mathscr{Z} \tag{3.1}$$

is an isomorphism. We first note that there is a unique minimal $\mathscr{Z}$ with the above properties and that its formation is Tate-local on $\mathscr{X}$. We may and do assume that $\mathscr{Z}$ is minimal. Our goal is to show that $\mathscr{Z}$ in $\mathscr{X}$ has codimension at most 2 everywhere along $\mathscr{Z}$. Observe that this assertion is vacuously true when $\mathscr{Z}$ is empty. Since the formation of $\mathscr{Z}$ is Tate-local, we may assume that $\mathscr{X}$ is affinoid, then connected, and hence irreducible, so $\mathscr{X} \setminus \mathscr{Z}$ is also irreducible. Therefore, by (3.1) and the nowhere density of $\mathrm{pr}_1^{-1}(\mathscr{Z})$ in $\mathscr{E}$, we have that $\mathscr{E}$ is irreducible. The image of $\mathrm{pr}_1\colon \mathscr{E} \to \mathscr{X}$ is an analytic set which contains the non-empty, Zariski open $\mathscr{X} \setminus \mathscr{Z}$, so it has non-empty interior in $\mathscr{X}$, and since $\mathscr{X}$ is irreducible, we have that this image must coincide with $\mathscr{X}$, and hence the fibers of pr_1 are non-empty.

Since $\mathscr{E}$ is equidimensional (by irreducibility), its pure dimension is the same as that of the irreducible affinoid $\mathscr{X}$ because we may determine the dimension using $\mathscr{X} \setminus \mathscr{Z}$ and $\mathscr{E} \setminus \mathrm{pr}_1^{-1}(\mathscr{Z})$. If there exists a section $s\colon \mathscr{X} \to \mathscr{E}$, then $\mathscr{X} \to \mathscr{E} \to \mathscr{Y}$ is an actual map which agrees on $\mathscr{X} \setminus \mathscr{Z}$ with the given meromorphic map φ, and then the minimality of $\mathscr{Z}$ will imply that $\mathscr{Z}$ is empty.

We now study the fibers of the surjective morphism $\mathrm{pr}_1\colon \mathscr{E} \to \mathscr{X}$ over $\mathscr{Z}$. Before doing so, we recall a result concerning proper morphisms.

LEMMA 3.4. *Let $f\colon \mathscr{X} \to \mathscr{S}$ be a proper map of rigid analytic spaces. Then the locus of points in $\mathscr{S}$ over which the fiber of f is finite is an admissible open.*

Proof. We may first assume that $\mathscr{S} = \mathrm{Sp}(A)$ is affinoid, so $\mathscr{X}$ is quasi-compact and separated. Now consider the associated map $f^{\mathrm{Ber}}\colon \mathscr{X}^{\mathrm{Ber}} \to \mathscr{S}^{\mathrm{Ber}} = M(A)$ of Berkovich spaces. By Lemma 2.5(1), the notions of properness are equivalent, and so we have that f^{Ber} is proper.

If $s \in \mathscr{S}$ has fiber $\mathscr{X}_s$ that is finite, then likewise f^{Ber} has analytic fiber that is finite, hence a finite set; see Lemma 2.5(2). By [Ber90, Corollary 3.3.11], we get an open $\mathscr{U}$ in $\mathscr{S}^{\mathrm{Ber}}$ over which f^{Ber} has finite fibers. In $M(A)$, a base of neighborhoods around any point is provided by rational affinoid domains, and we can arrange it to be strict since $M(A)$ is strict. Thus, we can find a rational affinoid $\mathrm{Sp}(B)$ in $\mathrm{Sp}(A)$ so that $M(B)$ is contained in $\mathscr{U}$ and contains s. We have that $\mathrm{Sp}(B)$ is an admissible open in $\mathrm{Sp}(A)$ around s over which f has finite fibers (since every fiber $\mathscr{X}_t$ is a quasi-compact and separated $\kappa(t)$-analytic space with $(\mathscr{X}_t)^{\mathrm{Ber}}$ over $M(\kappa(t))$

identified with the fiber of f^{Ber} over t in $M(A)$, and $\mathscr{X}_t \to \mathrm{Sp}(\kappa(t))$ is finite if and only if the Berkovich space over $M(\kappa(t))$ is finite (in the analytic sense); see again Lemma 2.5(2). □

LEMMA 3.5. *All fibers of the surjection* $\mathrm{pr}_1 \colon \mathscr{E} \to \mathscr{X}$ *over* $\mathscr{Z}$ *have positive dimension.*

Proof. Suppose for the sake of contradiction that there is some $z \in \mathscr{Z} \subset \mathscr{X}$ which has a finite fiber. By Lemma 3.4, we have that the locus of points in $\mathscr{X}$ over which the fiber is finite is an admissible open. By passing to a connected affinoid neighborhood of such a z, we may and do assume that all of the fibers of pr_1 are finite and $\mathscr{X} = \mathrm{Sp}(A)$ for a normal domain A (since connected and normal implies irreducible and reduced). However, since pr_1 is proper and quasi-finite, it is also finite, and hence $\mathscr{E}$ is also affinoid, say $\mathscr{E} = \mathrm{Sp}(B)$. Therefore, pr_1 is a finite surjection $\mathrm{Sp}(B) \to \mathrm{Sp}(A)$ that restricts to an isomorphism over the complement of $\mathscr{Z}$.

Recall that $\mathscr{Z}$ in $\mathrm{Sp}(A)$ and $\mathrm{pr}_1^{-1}(\mathscr{Z})$ in $\mathrm{Sp}(B)$ are nowhere dense, so the irreducibility of $\mathscr{E} \setminus \mathrm{pr}_1^{-1}(\mathscr{Z})$ in $\mathscr{E}$ forces $\mathscr{E}$ to be irreducible. As such, we have that B_{red} is a domain and the surjective map $\mathrm{Sp}(B_{\mathrm{red}}) \to \mathrm{Sp}(A)$ is an isomorphism over $\mathrm{Sp}(A) \setminus \mathscr{Z}$. We will show that the induced map on affinoid algebras $A \to B_{\mathrm{red}}$ is an isomorphism.

Once we have shown this, the inverse map $B_{\mathrm{red}} \to A$ will define a morphism $\mathrm{Sp}(A) \to \mathrm{Sp}(B_{\mathrm{red}}) \to \mathrm{Sp}(B)$ that is a section to $\mathrm{pr}_1 \colon \mathrm{Sp}(B) \to \mathrm{Sp}(A)$ because the composition of these two maps agrees with the identity away from $\mathscr{Z}$ and hence is the identity as $\mathscr{Z}$ is nowhere dense in the reduced $\mathrm{Sp}(A)$. However, we noted before Lemma 3.4 that the existence of a section allows us to show that $\mathscr{Z}$ is empty, which contradicts the existence of $z \in \mathscr{Z}$.

We now return to showing that the module finite map $A \to B_{\mathrm{red}}$ between Noetherian domains is an isomorphism. Note that this map is injective, and so by the normality of A, the map is an isomorphism if the induced map of fraction fields has degree 1. Let d denote the degree of the map of fraction fields. We know that $\mathrm{Spec}(B_{\mathrm{red}}) \to \mathrm{Spec}(A)$ is finite flat of degree d over some nonempty Zariski open V, and by the Jacobson property of A, see [BGR84, §6.1.1, Proposition 3], there is a closed point t which is in V and also away from the proper closed set corresponding to the ideal of $\mathscr{Z}$. Moreover, $A \to B_{\mathrm{red}}$ induces a finite flat map of degree d after completion at the maximal ideal of t, but that map coincides with the map upon completion at t arising from the finite analytic map $\mathrm{Sp}(B_{\mathrm{red}}) \to \mathrm{Sp}(A)$. Recall that this last map is an isomorphism over the complement of $\mathscr{Z}$ and hence upon the completion at t, and therefore, we have that $d = 1$. □

To complete the proof, we need to show that the irreducible components of $\mathscr{Z}$ have codimension at least 2. We will need the following lemma.

LEMMA 3.6. *If* $q \colon \mathscr{T} \to \mathscr{S}$ *is a proper surjective map between K-analytic spaces that are equidimensional with the same dimension $d \geqslant 0$, then every non-empty admissible open in $\mathscr{S}$ contains a point over which the fiber is finite.*

Proof. The case where $d = 0$ is trivial, and we proceed by induction on d, so we assume $d > 0$.

The first step is to reduce to when $\mathscr{T}$ and $\mathscr{S}$ are each irreducible. We can precompose with the normalization of $\mathscr{T}$ so that $\mathscr{T}$ is normal with connected components $\mathscr{T}_1, \ldots, \mathscr{T}_n$. Each $\mathscr{T}_j$ has image that is an analytic set in an irreducible component of $\mathscr{S}$. Each irreducible component of $\mathscr{S}$ must then be the image of some $\mathscr{T}_j$ since q is surjective, and the connected components $\mathscr{T}_j$ which map onto an irreducible component of $\mathscr{S}$ must factor through the normalization of that irreducible component (cf. [Con99, Theorem 2.2.4]).

With this, we can express q as a "disjoint union" of two types of maps: some of the $\mathscr{T}_j$ map onto a connected component of the normalization of $\mathscr{S}$, and some $\mathscr{T}_j$ map onto a proper analytic set in an irreducible component of $\mathscr{S}$. If we can prove our result for the first type, then by working over the Zariski open in $\mathscr{S}$ away from the images of the maps of the second type, we can conclude our result for the original q. Since the normalization of $\mathscr{S}$ is finite (and surjective), we can thereby reduce to the case where $\mathscr{T}$ and $\mathscr{S}$ are connected and normal.

Let $\mathscr{U} = \mathrm{Sp}(A)$ be a connected affinoid in $\mathscr{S}$, so it is irreducible. It is enough to find one fiber over $\mathscr{U}$ that is finite. The connected components of $\mathscr{V} = q^{-1}(\mathscr{U})$ are its irreducible components, and there are finitely many by the quasi-compactness of q. Note that at least one of these irreducible components maps onto $\mathscr{U}$. By an argument similar to above with the $\mathscr{T}_j$, we can find a point $u \in \mathscr{U}$ that is only contained in the image of those components of $\mathscr{V}$ which map onto $\mathscr{U}$. Let $\mathscr{W}$ be the union of those components.

Pick a non-zero, non-unit $f \in A$ that vanishes at u. The pullback $f' = q^*(f)$ on $\mathscr{W}$ defines an analytic set $\mathscr{W}'$ mapping onto the zero locus $\mathscr{U}'$ of f in $\mathscr{U}$. Endow $\mathscr{W}'$ and $\mathscr{U}'$ with reduced structures. Each of $\mathscr{W}'$ and $\mathscr{U}'$ is equidimensional of dimension $d-1$ since $\mathscr{U}$ is irreducible and every irreducible (and hence connected) component of $\mathscr{W}$ maps onto $\mathscr{U}$. The map $q \colon \mathscr{W}' \to \mathscr{U}'$ satisfies the original hypotheses but with dimension $d-1$. Therefore, by induction, every non-empty admissible open in $\mathscr{U}'$ has finite fiber in $\mathscr{W}'$. Now, pick an open around u which avoids the analytic images of the components of $\mathscr{V}$ not part of $\mathscr{W}$, so we get a finite fiber for $\mathscr{W}' \to \mathscr{U}'$ which is also a fiber for $q^{-1}(\mathscr{U}') \to \mathscr{U}'$, and so is a fiber of $\mathscr{V} = q^{-1}(\mathscr{U}) \to \mathscr{U}$. $\square$

PROPOSITION 3.7. *Every irreducible component of $\mathscr{Z}$ has codimension at least 2.*

Proof. Suppose for the sake of contradiction that $\mathscr{Z}$ has an irreducible component $\mathscr{Z}'$ of codimension 1 in the connected normal affinoid $X = \mathrm{Sp}(A)$; in particular, we assume that $\mathscr{Z}'$ has pure codimension 1.

First, we reduce to the case where $\mathscr{Z} = \mathscr{Z}'$. Pick $z' \in \mathscr{Z}'$ not in any other irreducible component of $\mathscr{Z}$, and let $\mathscr{W}$ be a connected affinoid open in $\mathscr{X}$ around z' that is inside the Zariski open complement of the finite union of the other irreducible components of $\mathscr{Z}$. We can replace $\mathscr{X}$ with such a $\mathscr{W}$ (since the formation of $\mathscr{Z}$ is local in $\mathscr{X}$), and so we may and do assume that $\mathscr{Z}$ is irreducible of pure codimension 1.

Next, we reduce to the case where $\mathscr{Z}$ is defined by a principal ideal. We may assume that $\mathscr{Z}$ is reduced, so $\mathscr{Z} = \mathrm{Sp}(A/P)$, where P is some prime ideal of A with height 1. Since A is normal, we have that A_P is a DVR, and so there is an affine open $\mathrm{Spec}(A_a)$ in $\mathrm{Spec}(A)$ containing P for which P_a is principal with generator $f \in A$. By looking at the map $\mathrm{Sp}(A) \to \mathrm{Spec}(A)$, we see that for the Zariski open $\mathscr{U} = \{a \neq 0\}$ in $\mathscr{X}$, the intersection $\mathscr{Z} \cap \mathscr{U}$ is defined by the ideal generated by f. Note that a is non-zero on $\mathrm{Sp}(A/P)$ because a is not in P since $\{P\} \in \mathrm{Spec}(A_a)$. As such, we have that its sup-norm on $\mathrm{Sp}(A/P)$ is positive, and by replacing a with some ca^n for $c \in K^\times$ and n a positive integer, we can arrange it so that this sup-norm is 1. With this, we have that $\mathscr{V} = \mathrm{Sp}(A\langle a\rangle)$ is an affinoid open in $\mathscr{X} = \mathrm{Sp}(A)$ that meets $\mathscr{Z} = \mathrm{Sp}(A/P)$, and $\mathscr{V} \cap \mathscr{Z}$ cannot equal $\mathscr{V}$ since $\mathscr{Z}$ has pure codimension 1 in the irreducible $\mathscr{X}$. We can replace $\mathscr{X}$ with a connected component of $\mathscr{V}$ that touches $\mathscr{Z}$, so we retain all preceding properties and gain that the radical ideal of $\mathscr{Z}$ in $\mathscr{X}$ is principal, say fA.

We know that $\mathscr{E}$ is irreducible with the same pure dimension d as the irreducible $\mathscr{X}$ (so $d > 0$ since $\mathscr{X}$ has the irreducible subspace $\mathscr{Z}$ with positive codimension), and the map $\mathrm{pr}_1 \colon \mathscr{E} \to \mathscr{X}$ is surjective. Thus, the analytic function $f' := \mathrm{pr}_1^*(f)$ on $\mathscr{E}$ determines an analytic set in $\mathscr{E}$ that does not exhaust the space (because its image in $\mathscr{X}$ is $\mathscr{Z}$ rather than $\mathscr{X}$), and so it has pure

dimension $d - 1$. Moreover, the map of vanishing loci

$$q \colon V(f')_{\mathrm{red}} \longrightarrow V(f) = \mathscr{X}$$

is a proper surjection between analytic spaces each of pure dimension $d - 1$. Now, Lemma 3.5 tells us that *all* of the fibers of q have position dimension; however, Lemma 3.6 asserts that there are fibers which are finite. Therefore, we have reached a contradiction to our original assumption that $\mathscr{X}$ has an irreducible component $\mathscr{X}'$ of codimension 1, and hence our result follows. $\square$

Proof of Theorem 3.1. This follows from Definition 3.2 and Proposition 3.7. $\square$

To conclude this section, we show that the statement of Theorem 3.1 carries over from rigid analytic to adic spaces. We note that Definition 3.2 works for adic spaces mutatis mutandis.

PROPOSITION 3.8. *Let K be an algebraically closed, complete, non-Archimedean valued field of characteristic zero. Let $\mathscr{X}$ be a normal, taut, locally of finite type adic space over $\mathrm{Spa}(K, K^\circ)$, and let $\mathscr{Y}$ be a proper, reduced, taut, locally of finite type adic space over $\mathrm{Spa}(K, K^\circ)$. The indeterminacy locus of any meromorphic map $\mathscr{X} \dashrightarrow \mathscr{Y}$ is an analytic subset of codimension at least* 2.

Proof. First, we recall that there is a functor $\mathfrak{r}$ from the category of rigid analytic spaces to the category of adic spaces which induces an equivalence on certain subcategories (cf. Theorems 2.2 and 2.3). We note that the image of a normal, taut (respectively, a proper, reduced, taut) rigid analytic space over $\mathrm{Sp}(K)$ via $\mathfrak{r}$ will be a normal, taut, locally of finite type (respectively, proper, reduced, taut, locally of finite type) adic space over $\mathrm{Spa}(K, K^\circ)$ via [Hub96, §1.1.11 and Remark 1.3.9(iv)] and Definition 2.7. Next, we have that $\mathfrak{r}$ will map an open immersion of rigid analytic spaces to an open immersion of adic spaces (op. cit., §1.1.11(b)), that $\mathfrak{r}$ is fully faithful (op. cit., §1.1.11(d)), and that $\mathfrak{r}$ preserves dimensions (op. cit., Proposition 1.8.11(i)). The result now follows from Theorem 3.1 and by applying the functor $\mathfrak{r}$. $\square$

4. On pseudo-K-analytically Brody hyperbolic varieties

In this section, we offer a new definition of pseudo-K-analytically Brody hyperbolic which differs slightly from [Mor21] and prove a result describing how one can test this notion.

To begin, we offer our new definition.

DEFINITION 4.1. *Let $\mathscr{X}$ be a K-analytic space, and let $\mathscr{D} \subset \mathscr{X}$ be a closed subset. Then $\mathscr{X}$ is K-analytically Brody hyperbolic modulo $\mathscr{D}$ (or: the pair $(\mathscr{X}, \mathscr{D})$ is K-analytically Brody hyperbolic) if*

- *every non-constant analytic morphism $\mathbb{G}_{m,K}^{\mathrm{an}} \to \mathscr{X}$ factors over $\mathscr{D}$, and*
- *for every abelian variety A over K and every dense open subset $\mathscr{U} \subset A^{\mathrm{an}}$ satisfying $\mathrm{codim}(A^{\mathrm{an}} \setminus \mathscr{U}) \geqslant 2$, every non-constant analytic morphism $\mathscr{U} \to \mathscr{X}$ factors over $\mathscr{D}$.*

DEFINITION 4.2. *A K-analytic space $\mathscr{X}$ over K is pseudo-K-analytically Brody hyperbolic if there is a proper closed subset $\mathscr{D} \subsetneq \mathscr{X}$ of $\mathscr{X}$ such that $(\mathscr{X}, \mathscr{D})$ is K-analytically Brody hyperbolic.*

Remark 4.3. Definition 4.1 differs from [Mor21, Definition 2.2] in that we require every non-constant *analytic* morphism from a big *analytic* open of the analytification of an abelian variety to factor over a proper closed subset, whereas [Mor21, Definition 2.2] only requires every non-constant algebraic morphism from a big algebraic open of an abelian variety to factor over a proper closed subset. In Section 7.3, we show that the results from [Mor21] still hold with this new definition.

The goal of this section is to prove the following.

THEOREM 4.4. *Let $\mathscr{X}/K$ be an irreducible, reduced, separated K-analytic space, and let $\mathscr{D} \subset \mathscr{X}$ be a closed subset. Then, $\mathscr{X}$ is K-analytically Brody hyperbolic modulo $\mathscr{D}$ if and only if for every connected, algebraic group G/K and every dense open subset $\mathscr{U} \subset G^{\mathrm{an}}$ with $\operatorname{codim}(G^{\mathrm{an}} \setminus \mathscr{U}) \geqslant 2$, every non-constant morphism $\mathscr{U} \to \mathscr{X}$ factors over $\mathscr{D}$.*

To prove Theorem 4.4, we follow the line of reasoning from [JX20, Section 3.2]; however, we need to transport several of the scheme-theoretic arguments to the category of adic spaces.

LEMMA 4.5. *Let $\operatorname{Spa}(R,R^{\circ})$ and $\operatorname{Spa}(S,S^{\circ})$ be affinoid K-analytic spaces, and suppose that $\operatorname{Spa}(R,R^{\circ})$ is irreducible and reduced. Let $\pi\colon \operatorname{Spa}(S,S^{\circ}) \to \operatorname{Spa}(R,R^{\circ})$ be a faithfully flat morphism. Then, there exist a dense affinoid subspace $\operatorname{Spa}\big(\tilde{R},\tilde{R}^{\circ}\big) \subset \operatorname{Spa}(R,R^{\circ})$ and an affinoid subspace $\operatorname{Spa}\big(\tilde{S},\tilde{S}^{\circ}\big) \subset \operatorname{Spa}(S,S^{\circ})$ such that the restricted map $\pi\colon \operatorname{Spa}\big(\tilde{S},\tilde{S}^{\circ}\big) \to \operatorname{Spa}\big(\tilde{R},\tilde{R}^{\circ}\big)$ is finite and flat.*

Proof. We will prove the result using Berkovich spaces and then transfer it over to adic spaces. As such, let $M(R)$ and $M(S)$ denote the associated Berkovich affinoid spaces, and let $\pi^{\mathrm{Ber}}\colon M(S) \to M(R)$ denote the associated map of Berkovich affinoid spaces.
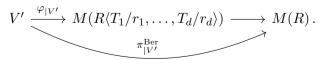
First assume that the morphism $\pi^{\mathrm{Ber}}\colon M(S) \to M(R)$ is quasi-finite; that is, for every $y \in M(S)$, we have that $\dim_y \pi^{\mathrm{Ber}} = 0$. If we let $\mathscr{Z}$ denote the relative interior of π^{Ber}, then the restriction $\pi^{\mathrm{Ber}}_{|\mathscr{Z}}$ is a finite and flat morphism, and hence open by [Ber93, Proposition 3.2.7]. Using [Ber93, Lemma 3.1.2], we may find small affinoid opens $M(S') \subset M(S)$ and $M(R') \subset M(R)$ such that the morphism $\pi^{\mathrm{Ber}}_{|M(S')}\colon M(S') \to M(R')$ is finite and flat. To translate the result to affinoid K-analytic spaces, we use Lemma 2.5(2) and [Hub96, Lemma 1.4.5(iv) and Proof of Proposition 8.3.3].

Now suppose that π^{Ber} is not quasi-finite; that is, there exists a point $y \in M(S)$ such that $\dim_y \pi^{\mathrm{Ber}} = d \geqslant 1$. By [Duc07, Theorems 4.6 and 3.2], there exists an affinoid neighborhood $V = M(S')$ of y such that there exists a quasi-finite morphism

$$\varphi\colon V \to \mathbb{A}^d_{M(R)}\,.$$

This means that φ is topologically proper and quasi-finite at every point of V. Let $y' \in V$, and let $x' = \varphi(y')$. Suppose that $M(R\langle T_1/r_1, \ldots, T_d/r_d\rangle) \subset \mathbb{A}^d_{M(R)}$ is a relative closed disk such that x' lies in the relative interior of $M(R\langle T_1/r_1, \ldots, T_d/r_d\rangle)$; that is, it lies in the topological interior of $M(R\langle T_1/r_1, \ldots, T_d/r_d\rangle)$. Let $V' = \varphi^{-1}(M(R\langle T_1/r_1, \ldots, T_d/r_d\rangle))$. We claim that $\varphi_{|V'}$ is finite at y'.

First, we note that $\varphi_{|V'}$ is quasi-finite as $M(R\langle T_1/r_1, \ldots, T_d/r_d\rangle)$ is a compact, closed subset of $\mathbb{A}^d_{M(R)}$, and hence it suffices to show that $\varphi_{|V'}$ is boundaryless at y'; that is, y' is in the relative interior $\operatorname{Int}(V'/M(R\langle T_1/r_1, \ldots, T_d/r_d\rangle))$. From the above and [Duc07, Theorem 4.6], we have the following morphisms:

$$V' \xrightarrow{\varphi_{|V'}} M(R\langle T_1/r_1, \ldots, T_d/r_d\rangle) \xrightarrow{} M(R)\,.$$
$$\underset{\pi^{\mathrm{Ber}}_{|V'}}{\underbrace{}}$$

By the choice of $M(R\langle T_1/r_1, \ldots, T_d/r_d\rangle)$, we have that y' is in the relative interior $\operatorname{Int}(V'/M(R))$. Now the claim follows from [Ber93, Proposition 1.5.5(ii)] as

$$\operatorname{Int}(V'/M(R)) = \operatorname{Int}(V'/M(R\langle T_1/r_1, \ldots, T_d/r_d\rangle)) \cap \varphi_{|V'}^{-1}(\operatorname{Int}(M(R\langle T_1/r_1, \ldots, T_d/r_d\rangle)/M(R)))\,,$$

and hence $y' \in \mathrm{Int}(V'/M(R\langle T_1/r_1, \ldots, T_d/r_d \rangle))$, and therefore $\varphi_{|V'}$ is finite at y'. By [Ber93, Proposition 3.1.4], we can find affinoid neighborhoods $M(R'\langle T_1/r_1', \ldots, T_d/r_d' \rangle)$ of x' and $V'' = M(S'')$ of y' such that φ induces a finite morphism

$$M(S'') \longrightarrow M(R'\langle T_1/r_1', \ldots, T_d/r_d' \rangle),$$

and so we have that S'' is finite over $R'\langle T_1/r_1', \ldots, T_d/r_d' \rangle$. If we consider the ideal $I = (T_1, \ldots, T_n)$ in S'' and take quotients, we have that S''/I is finite over R', and hence there exists an affinoid subset $M(S''/I)$ of V'' such that the morphism $M(S''/I) \to M(R')$ is finite. Moreover, we may assume, by [Ber93, Lemma 3.1.2], that $M(R') \subset M(R)$ is an affinoid open as we may take $M(R'\langle T_1/r_1', \ldots, T_d/r_d' \rangle)$ to be arbitrarily small.

Translating this result back to adic spaces using Lemma 2.5(2) and [Hub96, Lemma 1.4.5(iv) and p. 425], we have a finite morphism $\mathrm{Spa}\left(S''/I, \left(S''^\circ/I \cap S''^\circ\right)^c\right) \to \mathrm{Spa}\left(R', R'^\circ\right)$, where $\left(S''^\circ/I \cap S''^\circ\right)^c$ is the integral closure of $S''^\circ/I \cap S''^\circ$ in S''/I and where $\mathrm{Spa}\left(R', R'^\circ\right)$ is open in $\mathrm{Spa}(R, R^\circ)$. Since $\mathrm{Spa}\left(R', R'^\circ\right)$ is reduced, [BH22, Theorem 2.21] tells us that the flat locus is a dense open subset of $\mathrm{Spa}\left(R', R'^\circ\right)$. Consider a smaller affinoid open of the flat locus, call it $\mathrm{Spa}\left(R'', R''^\circ\right)$, and let $\mathrm{Spa}\left(S''', S'''^\circ\right)$ denote its preimage in $\mathrm{Spa}\left(S''/I, \left(S''^\circ/I \cap S''^\circ\right)^c\right)$. Then, we have that the restricted morphism $\pi\colon \mathrm{Spa}\left(S''', S'''^\circ\right) \to \mathrm{Spa}\left(R'', R''^\circ\right)$ is finite and flat, and $\mathrm{Spa}\left(R'', R''^\circ\right)$ is a dense affinoid subspace of $\mathrm{Spa}(R, R^\circ)$, as desired. $\qquad\square$

LEMMA 4.6. Let $W_1 = \mathrm{Spa}(R, R^\circ)$, $W_2 = \mathrm{Spa}(S, S^\circ)$, and $W_3 = \mathrm{Spa}(T, T^\circ)$ be affinoid K-analytic spaces. Let $\pi\colon W_2 \to W_1$ be a non-constant, faithfully flat morphism, and let $f\colon W_2 \to W_3$ be a non-constant morphism. Assume that W_1 and W_2 are irreducible and normal and that there exists a dense subset $E \subset W_1$ such that $f_{|\pi^{-1}(x)}$ is constant for every $x \in E$. Then there exists a unique $h\colon W_1 \to W_3$ such that $f = h \circ \pi$.

Proof. The proof follows closely the proof of [JX20, Lemma 3.11]. We will prove the result using rigid analytic spaces, and we then transfer it over to adic spaces. To ease notation, we will simply identify W_1, W_2, and W_3 and the morphisms π and f with their associated images under the quasi-inverse of the functor $\mathfrak{r}$ from [Hub94, Proposition 4.5].

We want to complete the diagram

$$\begin{array}{ccc} W_1 & \xleftarrow{\quad \pi \quad} & W_2 \\ {\scriptstyle h}\Big\downarrow & \swarrow {\scriptstyle f} & \\ W_3\,, & & \end{array}$$

which is equivalent to completing the diagram

$$\begin{array}{ccc} R & \xrightarrow{\quad \pi^* \quad} & S \\ {\scriptstyle h^*}\Big\uparrow & \nearrow {\scriptstyle f^*} & \\ T\,. & & \end{array}$$

To construct h^*, it is enough to show that $f^*(T) \subset \pi^*(R)$. First, we consider $g \in f^*(T)$, and then we will construct $r \in R$ such that $g = \pi^*(r)$.

By Lemma 4.5, shrinking further to ensure that the degree is constant, we may find an affinoid subspace $W_2' = \mathrm{Sp}(Q) \subset W_2$ and an admissible open affinoid subspace $W_1' = \mathrm{Sp}(R_1) \subset W_1$ such that $\pi_{|W_2'}\colon W_2' \to W_1'$ is finite and flat of degree d:

$$\begin{array}{ccc}
R_1 \xrightarrow{\;\pi^*_{|W_2'}\;} Q & \quad & W_1' \xleftarrow{\;\pi_{|W_2'}\;} W_2' \\
\uparrow \qquad \uparrow & & \downarrow \qquad\quad \downarrow \\
R \xrightarrow{\;\pi^*\;} S\,, & & W_1 \xleftarrow{\;\pi\;} W_2\,.
\end{array}$$

Let $\operatorname{tr}(\pi^*_{|W_2'}) \colon Q \to R_1$ be the trace as defined in [Sta22, Section 0BSY]. For our chosen $g \in f^*(T)$, we denote by $g_{|W_2'}$ the restriction to W_2'; that is, its image in Q. We now consider $\tilde{g} := (1/d)\operatorname{tr}(g_{|W_2'}) \in R_1$; this $\tilde{g}$ will be the element r such that $\pi^*(r) = g$.

Returning to the proof, we first show that g and $\pi^*(\tilde{g})$ coincide as elements in $\pi^*(R_1) \subset \pi^*(\operatorname{Frac}(R_1)) = \pi^*(\operatorname{Frac}(R))$ inside $\operatorname{Frac}(S)$. Let $x \in W_2'(K)$ be an element belonging to $\pi^{-1}(W_1')$ and $\pi^{-1}(E)$; then we have

$$\pi^*(\tilde{g})(x) = \tilde{g}(\pi(x)) = \frac{1}{d}\left(\sum_{y \in W_2' \mid \pi(y)=\pi(x)} g(y)\right) = g(x)\,,$$

where for the last equality we used that $g \in f^*(T)$ and that f is constant on $\pi^{-1}(x)$. As $\pi^{-1}(W_1')$ is dense in W_2' and $\pi^{-1}(E)$ is dense in W_2', then $g = \pi^*(\tilde{g})$ as elements of $\pi^*(\operatorname{Frac}(R))$.

As $\pi^* \colon R \to S$ is faithfully flat, then we can use [JX20, Lemma 3.10(2)] to see that

$$\pi^*(R) = S \cap \pi^*(\operatorname{Frac}(R))\,,$$

which gives $f^*(T) \subset \pi^*(R)$. Note that this gives a homomorphism $h^* \colon T \to R$ of affinoid algebras, and since a homomorphism of affinoid algebras is continuous and bounded [BGR84, §6.1.3, Theorem 1; §6.2.2, Proposition 1; §6.2.3, Proposition 1], this gives a morphism $h \colon \operatorname{Sp}(R) \to \operatorname{Sp}(T)$ of affinoid rigid analytic spaces. Using [Hub94, Proposition 4.5(iv)], we arrive at our desired morphism of affinoid K-analytic spaces

$$h \colon \operatorname{Spa}(R, R^\circ) \to \operatorname{Spa}(T, T^\circ)\,.$$

To conclude, we note that the construction of $\pi^* r$ is independent of W_2'. If we choose a different open subset W_2'' which is finite and flat over an open of W_1 and we define $\tilde{g}' := (1/d)\operatorname{tr}(g_{|W_2''})$, we get that for every x in $\pi^{-1}(E)$ for which both functions are defined,

$$\pi^*(\tilde{g}')(x) = \tilde{g}(\pi(x)) = \frac{1}{d}\left(\sum_{y' \in W_2'' \mid \pi(y')=\pi(x)} g(y')\right) = g(x)\,;$$

hence $\pi^*(\tilde{g}')$ and $\pi^*(\tilde{g})$ coincide on the dense set $\pi^{-1}(E)$. $\qquad\qquad\square$

Lemma 4.7. *Let $\pi \colon \mathscr{Y} \to \mathscr{B}$ be a non-constant, surjective, flat morphism between normal, irreducible, separated, locally of finite type adic spaces over $\operatorname{Spa}(K, K^\circ)$. Let $f \colon \mathscr{Y} \to \mathscr{Z}$ be a non-constant morphism of adic spaces, where $\mathscr{Z}$ is a K-analytic space. Let $E \subset \mathscr{B}(K)$ be a dense subset. Assume that for every $x \in E$, the restriction $f_{|\pi^{-1}(x)}$ is constant. Then there exists a morphism $h \colon \mathscr{B} \to \mathscr{Z}$ such that $f = h \circ \pi$.*

Proof. We want to complete the diagram

$$\begin{array}{ccc}
\mathscr{B} & \xleftarrow{\;\pi\;} & \mathscr{Y} \\
\;\downarrow{\scriptstyle h} & \swarrow{\scriptstyle f} & \\
\mathscr{Z}\,. & &
\end{array}$$

We cover $\mathscr{Y}$ by open affinoids; as π is surjective, it is enough to prove the theorem for any of these opens. We can further shrink these opens so that the image of each of them in $\mathscr{Z}$ is contained in an affinoid set, so we can assume that both $\mathscr{Y}$ and $\mathscr{Z}$ are affinoid. Since $\mathscr{Y}$ is affinoid and $\mathscr{B}$ is separated, the morphism $\pi\colon \mathscr{Y} \to \mathscr{B}$ is affinoid; that is, the preimage of an open affinoid of $\mathscr{B}$ is an open affinoid in $\mathscr{Y}$. Indeed, this fact can be proved using the same argument as in [Sta22, Tag 01SG]. Now if we cover $\mathscr{B}$ by open affinoids $\mathscr{V}_i$, then the preimages $\mathscr{W}_i = \pi^{-1}(\mathscr{V}_i)$ are open affinoids. We note that we may assume that all of the affinoids will have rings of integral elements isomorphic to the power bounded elements as our assumptions imply that the adic spaces in question come from rigid analytic spaces by [Hub94, Proposition 4.5(iii)]. Therefore, we are reduced to the situation of Lemma 4.6. The independence from the choices in the construction of the element r in loc. cit. ensures that we can glue together these locally defined maps into a global map h. $\qquad\square$

LEMMA 4.8. *Let $\mathscr{X}/K$ be an irreducible, reduced, separated K-analytic space. Let $\mathscr{D} \subset \mathscr{X}$ be a closed subset such that every non-constant analytic morphism $\mathbb{G}_{m,K}^{\mathrm{an}}$ factors over $\mathscr{D}$. Let G/K be a connected, finite-type affine group scheme, and let $\mathscr{U}$ be a dense open of G^{an} with $\mathrm{codim}(G^{\mathrm{an}} \setminus \mathscr{U}) \geqslant 2$. If $\varphi\colon \mathscr{U} \to \mathscr{X}$ is an analytic morphism such that $\varphi(\mathscr{U}) \nsubseteq \mathscr{D}$, then φ is constant.*

Proof. We proceed by induction on $\dim G$. When $\dim G \leqslant 1$, the result is clear because such a finite-type, connected, affine group scheme contains a dense $\mathbb{G}_{m,K}$.

To show that $\varphi\colon \mathscr{U} \to \mathscr{X}$ is constant, it suffices to show that the meromorphic map $G^{\mathrm{an}} \dashrightarrow \mathscr{X}$ is constant. Let $\mathscr{Z} = \overline{\varphi(\mathscr{U})}$ denote the analytic closure of the image of φ. Since G^{an} is irreducible (by Lemma 2.10(5)) and hence $\mathscr{U}$ is irreducible, the closure $\mathscr{Z} \subset \mathscr{X}$ is irreducible. Moreover, to show that φ is constant, we may and do assume $\mathscr{Z} = \mathscr{X}$.

Since φ is dominant, it will induce an embedding of fields of meromorphic function $\varphi^*(\mathscr{M}_{\mathscr{X}}(\mathscr{X})) \subseteq \mathscr{M}_{G^{\mathrm{an}}}(G^{\mathrm{an}})$. This follows because Corollary 2.22 implies that $\mathscr{M}_{\mathscr{X}}(\mathscr{X})$ and $\mathscr{M}_{G^{\mathrm{an}}}(G^{\mathrm{an}})$ are fields (for this latter fact, we use that G^{an} is smooth, which follows from [Sta22, Tag 047N] and Lemma 2.10(4)), and the dominant morphism will induce a map of fields of meromorphic functions, which must be an injection.

We now claim that if $\varphi^*(\mathscr{M}_{\mathscr{X}}(\mathscr{X})) = K$, then $\varphi\colon G^{\mathrm{an}} \dashrightarrow \mathscr{X}$ is constant. Indeed, if φ was not constant, then the image of φ contains at least two distinct values x and y in $\mathscr{X}$. Since $\mathscr{X}$ is separated, Lemma 2.23 tells us that we can find a meromorphic function f on $\mathscr{X}$ that separates these points, but then the composition $f \circ \varphi$ is a non-constant element of $\varphi^*(\mathscr{M}_{\mathscr{X}}(\mathscr{X}))$, which yields a contradiction. Thus, it suffices to prove that $\varphi^*(\mathscr{M}_{\mathscr{X}}(\mathscr{X})) = K$. Below, we will suppress the subscript in the notation for the sheaf of meromorphic functions.

To prove this statement, we first note that for an irreducible closed subgroup $H \subset G$, we may form the quotient

$$\pi_H\colon G \longrightarrow G/H\,,$$

where G/H is a smooth, quasi-projective scheme by [DG70, Exposé VI$_{\mathrm{A}}$, Section 3]. By [Uli17, Example 2.21], the analytification functor commutes with taking (stack) quotients, and hence we may identify $(G/H)^{\mathrm{an}} \cong G^{\mathrm{an}}/H^{\mathrm{an}}$. We note that the result from loc. cit. is only for Berkovich K-analytic spaces; however, using Theorem 2.3, we can transfer this identification to adic spaces.

Since π_H is flat and surjective, we have that the morphism

$$\pi_{H^{\mathrm{an}}}\colon G^{\mathrm{an}} \longrightarrow G^{\mathrm{an}}/H^{\mathrm{an}}$$

is flat and surjective by Lemma 2.12. Since $G^{\mathrm{an}}/H^{\mathrm{an}}$ is smooth and irreducible (as it is the image

of an irreducible space), $\mathscr{M}(G^{\mathrm{an}}/H^{\mathrm{an}})$ is a field by Corollary 2.22, and moreover, we have that $\pi_{H^{\mathrm{an}}}^*(\mathscr{M}(G^{\mathrm{an}}/H^{\mathrm{an}})) \cong \mathscr{M}(G^{\mathrm{an}})^{H^{\mathrm{an}}}$.

Since $\pi_{H^{\mathrm{an}}}$ is partially proper, flat, and surjective, Lemma 2.13 tells us that $\pi_{H^{\mathrm{an}}}(\mathscr{U})$ is a big open subset of $G^{\mathrm{an}}/H^{\mathrm{an}}$; that is, the codimension of the complement is greater than or equal to 2. Moreover, there exists a dense subspace $\mathscr{V} \subset \pi_{H^{\mathrm{an}}}(\mathscr{U})$ such that for every point $x \in \mathscr{V}$, the open subset $\mathscr{U} \cap \pi_{H^{\mathrm{an}}}^{-1}(x) \subset \pi_{H^{\mathrm{an}}}^{-1}(x)$ is big and satisfies $\varphi(\mathscr{U} \cap \pi_{H^{\mathrm{an}}}^{-1}(x)) \not\subseteq \mathscr{D}$. As an adic space, $\pi_{H^{\mathrm{an}}}^{-1}(x)$ is isomorphic to H^{an}, and the induction hypothesis says that $\varphi_{|\pi_{H^{\mathrm{an}}}^{-1}(x)}$ is constant for every $x \in \mathscr{V}$.

We want to apply Lemma 4.7 to the situation where $\mathscr{Y} = \mathscr{U}$, $\pi = \pi_{H^{\mathrm{an}}}$, $\mathscr{B} = \pi_{H^{\mathrm{an}}}(\mathscr{U})$, and $\mathscr{Z} = \mathscr{X}$. Therefore, we need to verify that $\mathscr{U}$ and $\pi_{H^{\mathrm{an}}}(\mathscr{U})$ are normal, irreducible, separated, locally of finite type adic spaces over $\mathrm{Spa}(K, K^\circ)$. As G^{an} is smooth, irreducible, separated, and locally of finite type (see Lemma 2.10) and $\mathscr{U}$ is an open subset of G^{an}, we have that $\mathscr{U}$ satisfies all of these properties. To show these properties for $\pi_{H^{\mathrm{an}}}(\mathscr{U})$, we first note that a quasi-projective scheme is separated [Sta22, Tag 01VX], and hence $G^{\mathrm{an}}/H^{\mathrm{an}}$ is smooth, irreducible, separated, and locally of finite type by Lemma 2.10. By Lemma 2.13, the set $\pi_{H^{\mathrm{an}}}(\mathscr{U})$ is an open subset of $G^{\mathrm{an}}/H^{\mathrm{an}}$ and hence has the desired properties.

Now, Lemma 4.7 tells us that there exists a map $h_{H^{\mathrm{an}}} \colon \pi_{H^{\mathrm{an}}}(\mathscr{U}) \to \mathscr{X}$ such that $\varphi = h_{H^{\mathrm{an}}} \circ \pi_{H^{\mathrm{an}}}$. In particular,

$$\varphi^*(\mathscr{M}(\mathscr{X})) \subset \pi_{H^{\mathrm{an}}}^*(\mathscr{M}(G^{\mathrm{an}}/H^{\mathrm{an}})) \subset \mathscr{M}(G^{\mathrm{an}})^{H^{\mathrm{an}}}.$$

By [JX20, Lemma 3.14], we have that $G^{\mathrm{an}} = \langle H_1^{\mathrm{an}}, \ldots, H_s^{\mathrm{an}} \rangle$, where the H_i are proper connected closed subgroups of G. Therefore, we conclude that

$$\varphi^*(\mathscr{M}(\mathscr{X})) \subset \bigcap_{i=1}^s \pi_{H_i^{\mathrm{an}}}^*(\mathscr{M}(G^{\mathrm{an}}/H_i^{\mathrm{an}})) \subset \bigcap_{i=1}^s \mathscr{M}(G^{\mathrm{an}})^{H_i^{\mathrm{an}}} = \mathscr{M}(G^{\mathrm{an}})^{\langle H_1^{\mathrm{an}}, \ldots, H_s^{\mathrm{an}} \rangle} = \mathscr{M}(G^{\mathrm{an}})^{G^{\mathrm{an}}} = K,$$

as desired. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \square$

PROPOSITION 4.9. *Let $\mathscr{X}/K$ be an irreducible, reduced, separated K-analytic space. Let $\mathscr{D} \subset \mathscr{X}$ be a closed subset such that every non-constant analytic morphism $\mathbb{G}_{m,K}^{\mathrm{an}}$ factors over $\mathscr{D}$ and for every abelian variety A over K and every dense open $\mathscr{U}$ of A^{an} with $\mathrm{codim}(A^{\mathrm{an}} \setminus \mathscr{U}) \geqslant 2$, every non-constant analytic morphism $\mathscr{U} \to \mathscr{X}$ factors over $\mathscr{D}$. Then, for every connected, finite-type algebraic group G/K and every dense open $\mathscr{U} \subset G^{\mathrm{an}}$ with $\mathrm{codim}(G^{\mathrm{an}} \setminus \mathscr{U}) \geqslant 2$, every analytic morphism $\varphi \colon \mathscr{U} \to \mathscr{X}$ factors over $\mathscr{D}$.*

Proof. Let G/K be a connected, finite-type algebraic group, and let $\mathscr{U} \subset G^{\mathrm{an}}$ be a dense open with $\mathrm{codim}(G^{\mathrm{an}} \setminus \mathscr{U}) \geqslant 2$. Let $\varphi \colon \mathscr{U} \to \mathscr{X}$ be an analytic morphism such that $\varphi(\mathscr{U}) \not\subseteq \mathscr{D}$. We will show that φ is constant.

Let $H \subset G$ be the (unique) normal affine connected subgroup of G such that $A := G/H$ is an abelian variety over K (see [Con02, Theorem 1.1]). Denote by $\pi \colon G \to A$ the quotient map. Since π is flat and surjective, we have that $\mathscr{V} := \pi^{\mathrm{an}}(\mathscr{U})$ is open in A^{an} by Lemma 2.13 with $\mathrm{codim}(A^{\mathrm{an}} \setminus \mathscr{V}) \geqslant 2$. For every $x \in A^{\mathrm{an}}$, set $G_x^{\mathrm{an}} := \pi^{\mathrm{an}^{-1}}(x)$ and $\mathscr{U}_x := \mathscr{U} \cap G_x^{\mathrm{an}}$. Since $G^{\mathrm{an}} \setminus \mathscr{U}$ has codimension greater than or equal to 2, there exists a dense open $\mathscr{V}_1 \subset \mathscr{V}$ such that for every $x \in \mathscr{V}_1$, the closed subset $G_x^{\mathrm{an}} \setminus \mathscr{U}_x$ has codimension greater than or equal to 2.

Let η denote the generic point of A^{an}; recall that A^{an} is irreducible by Lemma 2.10(5). If $\varphi(\mathscr{U}_\eta)$ is contained in $\mathscr{D}$, as $\mathscr{U}_\eta$ is dense in $\mathscr{U}$, we have that $\varphi(\mathscr{U})$ is also contained in $\mathscr{D}$, but this contradicts our original assumption. Thus, $\varphi(\mathscr{U}_\eta)$ is not contained in $\mathscr{D}$, and hence $\mathscr{U}_1 := \varphi^{-1}(\mathscr{X} \setminus \mathscr{D})$ is a non-empty open of $\mathscr{U}$. Since π^{an} is flat and surjective, $\mathscr{V}_2 := \pi^{\mathrm{an}}(\pi^{\mathrm{an}^{-1}}(\mathscr{V}_1) \cap \mathscr{U}_1)$

is open in $\mathscr{V}_1$. Note that for every point $x \in \mathscr{V}_2$, the open subset $\mathscr{U}_x$ of G_x^{an} is big and $\varphi(\mathscr{U}_x)$ is not contained in $\mathscr{D}$.

Therefore, by our first assumption and Lemma 4.8, we have that $\varphi_{|\mathscr{U}_x}$ is constant for every $x \in \mathscr{V}_2$. Since $\mathscr{V}_2$ is dense in A^{an}, it follows from Lemma 4.7 that there is a morphism $h \colon \mathscr{V} \to \mathscr{X}$ such that $\varphi = h \circ \pi_{|\mathscr{U}}^{\mathrm{an}}$. Since $\varphi(\mathscr{U}) \nsubseteq \mathscr{D}$, we have that $h(\mathscr{V}) \nsubseteq \mathscr{D}$. Moreover, since $\mathscr{V}$ is a big open of A^{an}, our second assumption implies that h is constant on $\mathscr{V}$, and hence φ is constant, as desired. $\qquad\square$

Proof of Theorem 4.4. This follows from Definition 4.1 and Proposition 4.9. $\qquad\square$

5. Brody special, special, arithmetically special, and geometrically special varieties

In this section, we recall various notions of special.

5.1 Special varieties in the sense of Campana

To begin, we describe Campana's notion of special from two perspectives. The first notion concerns the non-existence of certain sheaves, while the second notion characterizes specialness in terms of the non-existence of certain fibrations.

5.1.1 *Special varieties via Bogomolov sheaves.* First, we recall the definition of specialness in terms of Bogomolov sheaves. To define these sheaves, let p be a positive integer, and let $\mathscr{L} \subset \Omega_X^p$ be a saturated, rank 1, coherent sheaf. We say that $\mathscr{L}$ is a *Bogomolov sheaf* for X if the Iitaka dimension $\kappa(X, \mathscr{L})$ of $\mathscr{L}$ equals $p > 0$.

DEFINITION 5.2 ([Cam04]). A proper variety X is *special* if it has no Bogomolov sheaves.

EXAMPLE 5.3. From this definition, we have several fundamental examples of special varieties. It is immediate that a curve C/k is special if and only if the genus of C is less than 2 and that tori and abelian varieties are special. Also, rationally connected varieties are special since $\mathrm{Sym}^m(\Omega^p) = 0$ for any $p, m > 0$. Finally, a variety with either first Chern number equal to zero or Kodaira dimension equal to zero is special (see [Cam04] for details).

5.3.1 *Special varieties via fibrations of general type.* Another way to think of special varieties is via fibrations to varieties of general type. Consider a fibration $f \colon X \to Y$, with X and Y smooth and projective. Recall that a fibration $f \colon X \to Y$ is *of general type* if $K_Y(\Delta_f)$ is big and $\dim Y \geqslant 1$, where Δ_f is an effective $\mathbb{Q}$-Cartier divisor encoding the multiple fibers of f. More precisely, for every irreducible divisor $E \subset Y$, let $f^*(E)$ be the scheme-theoretic inverse image of E in X, and write

$$f^*(E) = R + \sum_i m_f(E_i)E_i \,,$$

where the E_i are irreducible divisors in X, the $t_i \geqslant 0$ are integers, and the codimension of R is at least 2. Define $m_f(E) := \inf_i(t_i)$. The $\mathbb{Q}$-Cartier divisor Δ_f above is defined as

$$\Delta_f := \sum_E \left(1 - \frac{1}{m_f(E)} \right) E$$

and is called the *multiplicity divisor of f*, which encodes the defect of smoothness of the fibers.

Definition 5.4. A proper variety X is *special* if it does not admit a fibration of general type.

It is a result of Campana [Cam04, Theorem 2.27] that these two notions are equivalent. As a consequence of Definition 5.4, a special variety does not dominate a positive-dimensional variety of general type. Using this result, combined with some non-trivial arguments, allows us to deduce the specialness of the last two families of examples of Example 5.3.

5.4.1 *Special varieties that are not proper.* Most of the aforementioned work of Campana works in much larger generality. Indeed, there is a way to define the notion of special for orbifolds, which roughly speaking are proper varieties $\overline{X}$ equipped with an orbifold divisor

$$\Delta := \sum_{E_i} \left(1 - \frac{1}{m_i} \right) E_i \, ,$$

where the E_i are Cartier divisors of $\overline{X}$ and the $m_i \in [1, 2, \dots, +\infty]$ are almost all 1 (so that it is a finite sum).

Whenever all the m_i that are not 1 are $+\infty$, we recover the classical case of logarithmic geometry, that we recall briefly for successive applications to semi-abelian schemes. Let X/K be a smooth variety, and consider a smooth compactification $\overline{X}$ of X, with normal crossing divisor complement Δ; we call $(\overline{X}, \Delta)$ a logarithmic pair. We define the logarithmic canonical divisor $K_{\overline{X}, \Delta} := K_{\overline{X}} + \Delta$, and we have a notion of logarithmic general type for X in terms of the logarithmic Kodaira dimension of $K_{\overline{X}, \Delta}$ (see [Voj99, Definition 5.3]). There is also a notion of logarithmic Bogomolov sheaf for a logarithmic pair [Cam04, Section 2.9, p. 542], where one replaces $\mathscr{L} \subset \Omega_X^p$ with $\mathscr{L} \subset \Omega_{\overline{X}}^p(\log \Delta)$. The logarithmic analogues of Definitions 5.2 and 5.4, both due to Campana in the more general settings of orbifolds, are the following.

Definition 5.5. A variety X is *special* if $(\overline{X}, \Delta)$ does not admit any logarithmic Bogomolov sheaves.

Definition 5.6. A variety X is *special* if $(\overline{X}, \Delta)$ does not admit a fibration of logarithmic general type.

Campana [Cam04, Section 2.9, p. 542] notes that Definitions 5.2 and 5.4 are equivalent via the same proof as in the setting where $\Delta = \emptyset$. We also note that the logarithmic Kodaira dimension is a birational invariant [Cam20, Example 7.2.6]; hence the two definitions are independent of $\overline{X}$.

5.7 Intermezzo on weakly special varieties

There is a weaker notion of special, called weakly special, which closely resembles Definition 5.4.

Definition 5.8. We say that X/k is *weakly special* if there are no finite k-étale covers $X' \to X$ admitting a dominant rational map $X' \to Z'$ to a positive-dimensional variety Z' of general type.

Remark 5.9. Every special variety is weakly special [Cam04, Proposition 9.29]. Note that weakly special curves and surfaces are special [Cam20, Example 7.2(6)], but already for threefolds, the converse is not true; there is an example by Bogomolov and Tschinkel (cf. [Cam20, Section 8.7]).

5.10 Brody special

Let $k = \mathbb{C}$. For a complex manifold $X^{\mathrm{an}}/\mathbb{C}$, there is a (conjectural) complex analytic characterization of specialness.

DEFINITION 5.11. A complex manifold X^{an} is *Brody special* if there is a Zariski dense holomorphic map $\mathbb{C} \to X^{\mathrm{an}}$.

EXAMPLE 5.12. From this definition, it is immediate that for a curve C over $\mathbb{C}$, the space C^{an} is Brody special if and only if the genus of C is less than 2. Also, a torus $\mathbb{C}^g$ is Brody special, and hence an abelian variety $A/\mathbb{C}$ is Brody special via the Riemann uniformization theorem. Recently, Campana–Winkelmann [CW20] proved that the complex analytification of a rationally connected variety is Brody special.

5.13 Geometrically special varieties

Recently, Javanpeykar–Rousseau [JR22] defined a new notion of special, called geometrically special, which is conjecturally equivalent to Campana's original notion.

DEFINITION 5.14 ([JR22]). A variety X/k is *geometrically special over k* if for every dense open subset $U \subset X$, there exist a smooth, quasi-projective, connected curve C/k, a point in $c \in C(k)$, a point $u \in U(k)$, and a sequence of morphisms $f_i \colon C \to X$ with $f_i(c) = u$ for $i = 1, 2, \ldots$ such that $C \times X$ is covered by the graphs $\Gamma_{f_i} \subset C \times X$ of these maps.

EXAMPLE 5.15. In [JR22, Propositions 2.14 and 3.1], the authors prove that a rationally connected variety and an abelian variety are geometrically special over k, which recovers the examples from Examples 5.3 and 5.12.

5.16 Arithmetically special varieties

The notion of special also has a (conjectural) arithmetic counterpart, that aims to capture when the rational points of a variety are potentially dense.

DEFINITION 5.17. A proper variety X/k is *arithmetically special over k* if there exist a finitely generated subfield $k' \subset k$ and a model $\mathscr{X}$ for X over k' such that $\mathscr{X}(k')$ is dense in $\mathscr{X}$.

EXAMPLE 5.18. As with previous examples, a curve C/k is arithmetically special if and only if the genus of C is less than 2 and C can be defined over a number field. Indeed, it is easy to see that a genus zero curve is arithmetically special, and it is a well-known but not obvious fact that an elliptic curve is arithmetically special. Furthermore, Faltings's theorem [Fal83] asserts that a curve of genus $g \geqslant 2$ is not arithmetically special. By [HT00b, Section 3], any abelian variety is arithmetically special. For further examples of arithmetically special varieties, we refer the reader to [HT00a, BT00, LN22].

Remark 5.19. All of the above notions of special are stable under birational morphisms, finite étale covers, and products. We refer the reader to [JR22, Section 2] for details.

6. A non-Archimedean analytic characterization of special and weakly special

In this section, we offer our definition of special for a K-analytic space and describe some basic properties of these special K-analytic spaces.

DEFINITION 6.1. We say that $\mathscr{X}$ is *K-analytically special* if there exist a connected, finite-type algebraic group G/K, a dense open subset $\mathscr{U} \subset G^{\mathrm{an}}$ with $\mathrm{codim}(G^{\mathrm{an}} \setminus \mathscr{U}) \geqslant 2$, and an analytic morphism $\mathscr{U} \to \mathscr{X}$ which is Zariski dense.

Example 6.2. From Definition 6.1, we can immediately find several examples of K-analytically special varieties. First, a curve C/K is K-analytically special if and only if the genus of C is less than 2, and second, a connected, finite-type algebraic group is K-analytically special.

While checking K-analytic specialness with big analytic opens of algebraic groups may seem unnatural, we do so in order to incorporate the following types of examples.

Example 6.3. Let A/K be a simple abelian surface with good reduction. From Definition 6.1, it is clear that $A \setminus \{0\}$ is K-analytically special, but if we did not test specialness on big analytic opens of algebraic groups, then $A \setminus \{0\}$ would not be K-analytically special. To see this, we note that if G/K is an algebraic group and $G^{\mathrm{an}} \to (A \setminus \{0\})^{\mathrm{an}}$ a non-constant, analytic morphism, then the composition $G^{\mathrm{an}} \to (A \setminus \{0\})^{\mathrm{an}} \to A^{\mathrm{an}}$ will be the translate of a group homomorphism, and hence constant by our assumption that A is simple.

Indeed, using Chevalley's decomposition theorem [Con02, Theorem 1.1], we may write G as an extension of an abelian variety B by a normal, affine group H. Note that each pair of points in H will be connected by a $\mathbb{G}_m$, and hence the image of H^{an} in A^{an} will be a point due to work of Cherry [Che94, Theorem 3.2]. Moreover, the morphism $G^{\mathrm{an}} \to A^{\mathrm{an}}$ will factor through the analytification of the abelian variety B^{an}, but since B is proper, rigid analytic GAGA, [Köp74] implies that the morphism $B^{\mathrm{an}} \to A^{\mathrm{an}}$ is algebraic; that is, it is the analytification of an algebraic morphism $B \to A$. It is well known that any morphism $B \to A$ is the translate of a group homomorphism, and therefore the image of $G^{\mathrm{an}} \to A^{\mathrm{an}}$ will be the translate of an abelian subvariety. Since A was assumed to be simple, the image of G^{an} in A^{an} will be a point, and, moreover, $(A \setminus \{0\})^{\mathrm{an}}$ would not be K-analytically special.

We remark that the idea of testing specialness and hyperbolicity on big open of algebraic groups goes back to Lang [Lan86] and was studied by Vojta in [Voj15]. In this latter work, Vojta showed that for A an abelian variety over $\mathbb{C}$ and U a dense open subset of A with $\operatorname{codim}(A \setminus U) \geqslant 2$, U^{an} is Brody special by [Voj15, Section 4].

6.4 Basic properties: Birational invariance, products, and ascending along finite étale covers

For the remainder of this section, we prove that our notion of K-analytically special is preserved under birational morphisms, products, and finite étale covers.

Lemma 6.5. Let $\mathscr{X} \dashrightarrow \mathscr{Y}$ be a bi-meromorphic morphism between proper, integral K-analytic spaces. Then, $\mathscr{X}$ is K-analytically special if and only if $\mathscr{Y}$ is K-analytically special.

Proof. Suppose that $\mathscr{X}$ is K-analytically special, so there exist a connected, finite-type algebraic group G/K, an open dense subset $\mathscr{U} \subset G^{\mathrm{an}}$ with $\operatorname{codim}(G^{\mathrm{an}} \setminus \mathscr{U}) \geqslant 2$, and a Zariski dense, analytic morphism $\varphi \colon \mathscr{U} \to \mathscr{X}$. Since φ is Zariski dense, the composition $\mathscr{U} \to \mathscr{X} \dashrightarrow \mathscr{Y}$ defines a meromorphic map $G^{\mathrm{an}} \dashrightarrow \mathscr{Y}$. By [Sta22, Tag 047N] and Lemma 2.10(4), the space G^{an} is smooth, and so by Proposition 3.8, this meromorphic map is defined on some dense open $\mathscr{U}'$ of G^{an} with $\operatorname{codim}(G^{\mathrm{an}} \setminus \mathscr{U}') \geqslant 2$. Therefore, $\mathscr{Y}$ is K-analytically special. To conclude, we note that the proof of the converse statement follows in the exact same manner. $\qquad\square$

Remark 6.6. We remark that if we tested K-analytic specialness on big *algebraic* opens of connected, finite-type algebraic groups, then bi-meromorphic invariance would not follow. Indeed, the proof of Lemma 6.5 boils down to showing that a meromorphic map between K-analytic spaces is defined on an open subset whose complement has codimension at least 2. However, we

do not know of a way to guarantee that this open subset is algebraic, and therefore, it is crucial that we test K-analytic specialness on big *analytic* opens of connected, finite-type algebraic groups.

LEMMA 6.7. *Let $\mathscr{X}/K$ and $\mathscr{Y}/K$ be K-analytic spaces which are both K-analytically special. Then the product $\mathscr{X} \times \mathscr{Y}$ is K-analytically special.*

Proof. Let G (respectively, G') be a connected, finite-type algebraic group over K, let $\mathscr{U} \subset G^{\mathrm{an}}$ (respectively, $\mathscr{U}' \subset G'^{\mathrm{an}}$) be a dense open subset with $\mathrm{codim}(G^{\mathrm{an}} \setminus \mathscr{U}) \geqslant 2$ (respectively, $\mathrm{codim}(G'^{\mathrm{an}} \setminus \mathscr{U}') \geqslant 2$) such that there exists a Zariski dense analytic morphism $\varphi \colon \mathscr{U} \to \mathscr{X}$ (respectively, $\varphi' \colon \mathscr{U}' \to \mathscr{Y}$). We have that $G \times G'$ is a connected, finite-type algebraic group over K and that the fibered products $G^{\mathrm{an}} \times G'^{\mathrm{an}}$ and $\mathscr{U} \times \mathscr{U}'$ exist in the category of locally of finite type adic spaces over $\mathrm{Spa}(K, K^\circ)$ by [Hub96, Proposition 1.2.2(a)]. The same argument from [Sta22, Tag 01JR] tells us that $\mathscr{U} \times \mathscr{U}'$ is a dense open subset of $G^{\mathrm{an}} \times G'^{\mathrm{an}}$. Moreover, we claim that $\mathrm{codim}((G^{\mathrm{an}} \times G'^{\mathrm{an}}) \setminus (\mathscr{U} \times \mathscr{U}')) \geqslant 2$. Indeed, using [Con99, Theorem 5.1.3.1] and [Hub96, Proposition 1.8.11(i)], we deduce that $\dim(G^{\mathrm{an}} \times G'^{\mathrm{an}}) = \dim(G) + \dim(G')$; the claim now follows from [Hub96, Corollary 1.8.8]. To conclude, we have that the morphism $(\varphi \times \varphi') \colon \mathscr{U} \times \mathscr{U}' \to \mathscr{X} \times \mathscr{Y}$ is Zariski dense, and so $\mathscr{X} \times \mathscr{Y}$ is K-analytically special. $\square$

LEMMA 6.8. *Let $\mathscr{X} \to \mathscr{Y}$ be a finite étale morphism between proper K-analytic spaces. Then $\mathscr{X}$ is K-analytically special if and only if $\mathscr{Y}$ is K-analytically special.*

Proof. If $\mathscr{X}$ is K-analytically special, then it follows that $\mathscr{Y}$ is K-analytically special since $\mathscr{X} \to \mathscr{Y}$ is finite étale.

Now suppose that $\mathscr{Y}$ is K-analytically special. Let G/K be a connected, finite-type algebraic group, let $\mathscr{U} \subset G^{\mathrm{an}}$ be open dense such that $\mathrm{codim}(G^{\mathrm{an}} \setminus \mathscr{U}) \geqslant 2$, and let $\mathscr{U} \to \mathscr{Y}$ be a Zariski dense, analytic morphism. By [Hub96, Proposition 1.2.2(a)], we can construct the fibered product diagram

$$\begin{array}{ccc} \mathscr{V} & \longrightarrow & \mathscr{X} \\ \downarrow & & \downarrow \\ \mathscr{U} & \longrightarrow & \mathscr{Y}, \end{array}$$

and since finite étale covers are stable under base change [Hub96, Lemma 1.4.5(i) and Proposition 1.6.7(iv)], we have that $\mathscr{V} \to \mathscr{U}$ is finite étale. By [Sta22, Tag 047N] and Lemma 2.10(4), the space G^{an} is smooth over K, and so purity of the branch locus (see, for example, [And03, Theorem III.2.1.11] or [Han20, Corollary 2.15]) implies that the finite étale morphism $\mathscr{V} \to \mathscr{U}$ extends to a finite étale morphism $\mathscr{G}' \to G^{\mathrm{an}}$. By the non-Archimedean analogue of Riemann's existence theorem [Lüt93, Theorem 3.1], the finite étale morphism $\mathscr{G}' \to G^{\mathrm{an}}$ algebraizes; in particular, there is a finite étale morphism of locally of finite type schemes $G' \to G$ whose analytification coincides with $\mathscr{G}' \to G^{\mathrm{an}}$. Note that every connected component G'' of G' has the structure of a connected, finite-type group scheme over K, and with this structure, the morphism $G'' \to G$ is a homomorphism. Since smooth morphisms preserve codimension, we have that $\mathrm{codim}(G''^{\mathrm{an}} \setminus \mathscr{V}) \geqslant 2$; note that $\mathscr{V} \to \mathscr{U} \to \mathscr{Y}$ (and hence $\mathscr{V} \to \mathscr{X} \to \mathscr{Y}$) is Zariski dense.

To conclude, we need to show that the image of $\mathscr{V} \to \mathscr{X}$ is Zariski dense. Suppose to the contrary that it is not. Let $\mathscr{Z}$ denote the complement of the Zariski closure of the image of $\mathscr{V}$ inside of $\mathscr{X}$. By assumption, $\mathscr{Z}$ is a non-empty open subset of $\mathscr{X}$. Since $\mathscr{X} \to \mathscr{Y}$ is étale, the image of $\mathscr{Z}$ in $\mathscr{Y}$ is open [Hub96, Proposition 1.7.8], and so we have an open subset of

$\mathscr{Y}$ which is not in the image of $\mathscr{V}$. However, this contradicts the fact that $\mathscr{V} \to \mathscr{X} \to \mathscr{Y}$ is Zariski dense, and therefore, we have that $\mathscr{V} \to \mathscr{X}$ is Zariski dense, and so $\mathscr{X}$ is K-analytically special. □

Recall that one can characterize special varieties as those which do not admit a fibration of general type. In the non-Archimedean setting, Theorem 4.4 allows us to show that a K-analytically special variety cannot dominate a pseudo-K-analytically Brody hyperbolic variety.

THEOREM 6.9. *Let $\mathscr{X}$ and $\mathscr{Y}$ be irreducible, reduced, separated K-analytic spaces over K. If $\mathscr{Y}$ is K-analytically special and $\mathscr{X}$ is a positive-dimensional pseudo-K-analytically Brody hyperbolic variety, then there is no dominant morphism $\mathscr{Y} \to \mathscr{X}$.*

Proof. Suppose that there existed a dominant morphism $\mathscr{Y} \to \mathscr{X}$. Then there exist a connected, finite-type algebraic group G, a dense open subset $\mathscr{U}$ of G^{an} with $\mathrm{codim}(G^{\mathrm{an}} \setminus \mathscr{U}) \geqslant 2$, and an analytic morphism $\mathscr{U} \to \mathscr{Y} \to \mathscr{X}$, which is Zariski dense since the composition of dominant morphisms is dominant. However, this contradicts Theorem 4.4, and therefore, there cannot exist a dominant morphism $\mathscr{Y} \to \mathscr{X}$. □

To conclude this section, we define the notion of K-analytically weakly special.

DEFINITION 6.10. We say that $\mathscr{X}$ is K-*analytically weakly special* if there are no finite K-étale covers $\mathscr{X}' \to \mathscr{X}$ admitting a dominant meromorphic map $\mathscr{X}' \to \mathscr{Z}'$ to a positive-dimensional K-analytic space $\mathscr{Z}'$ which is pseudo-K-analytically Brody hyperbolic.

With this definition, we have the following corollary to Lemma 6.8 and Theorem 6.9.

COROLLARY 6.11. *A K-analytically special K-analytic space is K-analytically weakly special.*

7. K-analytically special subvarieties of semi-abelian varieties and properties of quasi-Albanese maps

In this section, we will prove Theorem B and use these results to deduce some properties of the quasi-Albanese map of a K-analytically special variety.

To begin, we recall the statement of Theorem B.

THEOREM 7.1. *Let X be a closed subvariety of a semi-abelian variety G over K. Then, the following are equivalent:*

(1) *The variety X is the translate of a semi-abelian subvariety.*
(2) *The space X^{an} is K-analytically special (Definition 6.1).*
(3) *The variety X is special (Definitions 5.2 and 5.4).*

To prove Theorem 7.1, we will use some deep theorems about closed subvarieties of semi-abelian varieties. First, we recall progress on the Green–Griffiths–Lang–Vojta conjecture for closed subvarieties of semi-abelian varieties.

THEOREM 7.2 ([Abr94, Nog98, Mor21]). *Let X be a closed subvariety of a semi-abelian variety G over K. Then, X is of logarithmic general type if and only if X^{an} is pseudo-K-analytically Brody hyperbolic.*

We note that Theorem 7.2 was proved using the definition of pseudo-K-analytically Brody hyperbolic from [Mor21], and we will need to know that this result holds with our new definition (Definition 4.1). The proof of this fact will occupy the next subsection.

7.3 Extending meromorphic maps to analytifications of semi-abelian varieties

The goal of this subsection is to prove the following extension result concerning meromorphic maps from smooth, irreducible analytic spaces to analytifications of semi-abelian varieties.

THEOREM 7.4. *Let $\mathscr{Z}$ be a smooth, irreducible K-analytic space, and let $\mathscr{U} \subset \mathscr{Z}$ be a dense open with $\mathrm{codim}(\mathscr{Z} \setminus \mathscr{U}) \geqslant 2$. Let G/K be a semi-abelian variety. Then any analytic morphism $\mathscr{U} \to G^{\mathrm{an}}$ uniquely extends to an analytic morphism $\mathscr{Z} \to G^{\mathrm{an}}$.*

The algebraic variant of Theorem 7.4 is well known (see, for example, [BLR90, Theorem 4.4.1] and [Moc12, Lemma A.2]). To prove our results, we begin by studying the case where G is proper (that is, where G is an abelian variety). We first show that the analytic Picard group of $\mathscr{U}$ is in bijection with the analytic Picard group of $\mathscr{Z}$ (Lemma 7.5), and then the result follows using similar reasoning as in the proof of [BLR90, Corollary 8.4.6]. Next, we prove the claim in the setting where G is a (split) torus using the non-Archimedean variant of the second Hebbarkeitssatz. Finally, we conclude the proof using these two pieces and a descent argument.

To start, we prove our result on analytic Picard groups.

LEMMA 7.5. *Let $\mathscr{Z}$ be a smooth, irreducible K-analytic space, and let $\mathscr{U} \subset \mathscr{Z}$ be a dense open with $\mathrm{codim}(\mathscr{Z} \setminus \mathscr{U}) \geqslant 2$. Then, $\mathrm{Pic}(\mathscr{U})$ is in bijection with $\mathrm{Pic}(\mathscr{Z})$.*

Proof. By the non-Archimedean version of the Remmert–Stein theorem [Lüt74], the group of Weil divisors on $\mathscr{U}$ is isomorphic to the group of Weil divisors on $\mathscr{Z}$. Since $\mathscr{Z}$ (respectively, $\mathscr{U}$) is smooth, the group of Weil divisors on $\mathscr{Z}$ (respectively, $\mathscr{U}$) is isomorphic to the group of Cartier divisors on $\mathscr{Z}$ (respectively, $\mathscr{U}$) by [Mit11, Theorem 8.9]. Now since $\mathscr{Z}$ (respectively, $\mathscr{U}$) is irreducible, the classical argument (see, for example, [Har77, Proposition II.6.15]) tells us that Cartier divisors on $\mathscr{Z}$ (respectively, $\mathscr{U}$) are in bijection with line bundles on $\mathscr{Z}$ (respectively, $\mathscr{U}$). Therefore, line bundles on $\mathscr{U}$ are in bijection with line bundles on $\mathscr{Z}$. $\qquad\square$

Remark 7.6. The algebraic variant of Lemma 7.5 is well known, and the result is actually not true for complex analytic manifolds (see [Huy05, Section 2.3] for a discussion). Indeed, in the complex analytic setting, it is not true that Cartier divisors on a smooth complex manifold $X(\mathbb{C})$ are in bijection with line bundles on $X(\mathbb{C})$; instead, they are in bijection with line bundles $\mathscr{L}$ on $X(\mathbb{C})$ such that $H^0(X(\mathbb{C}), \mathscr{L}) \neq 0$. Moreover, there are many examples of line bundles that do not come from Cartier divisors on complex manifolds and line bundles on big, dense opens which do not extend to the entire space (see, for example, [Huy05, Remark 2.3.21]).

The issue in the complex analytic setting is that when $X/\mathbb{C}$ is separated, $X(\mathbb{C})$ is Hausdorff, and hence if X has positive dimension, it cannot also be irreducible in the complex analytic topology. The lack of irreducibility prevents the scheme-theoretic argument from going through as we need to know that if the restriction of a sheaf to each open covering is constant, then the sheaf is constant. Therefore, it is essential that we work with adic spaces and not Berkovich spaces (cf. Remark 2.4).

LEMMA 7.7. *Let $\mathscr{Z}$ be a smooth, irreducible K-analytic space, and let $\mathscr{U} \subset \mathscr{Z}$ be a dense open with $\mathrm{codim}(\mathscr{Z} \setminus \mathscr{U}) \geqslant 2$. Let A/K be an abelian variety. Then any analytic morphism $\varphi\colon \mathscr{U} \to A^{\mathrm{an}}$ uniquely extends to an analytic morphism $\widetilde{\varphi}\colon \mathscr{Z} \to A^{\mathrm{an}}$.*

Proof. We will closely follow the proof of [BLR90, Corollary 8.4.6]. First, we base change A/K to an abelian variety $A_{\mathscr{Z}} := A \times_K \mathscr{Z}$. Recall that $A \cong A^{**}$, see [BLR90, Theorem 8.4.5], and so by rigid analytic GAGA [Köp74], we have $A^{\mathrm{an}} \cong A^{**\mathrm{an}}$, where $A^{*\mathrm{an}} \cong A^*$ represents the adic

Picard functor of A^{an} (cf. [BL84, Section 1]). Now a morphism $\varphi \colon \mathscr{U} \to A_{\mathscr{Z}}^{\mathrm{an}}$ corresponds to an analytic line bundle on $A^{*\,\mathrm{an}} \times_{\mathscr{Z}} \mathscr{U}$. Since $\mathscr{Z}/K$ is smooth, we have that $A^{*\,\mathrm{an}} \to \mathscr{Z}$ is smooth, and so Lemma 7.5 implies that an analytic line bundle on $A^{*\,\mathrm{an}} \times_{\mathscr{Z}} \mathscr{U}$ uniquely extends to a line bundle on $A_{\mathscr{Z}}^{*\,\mathrm{an}}$ and hence gives rise to an extension $\widetilde{\varphi} \colon \mathscr{Z} \to A_{\mathscr{Z}}^{**\mathrm{an}}$. $\qquad\square$

LEMMA 7.8. *Let $\mathscr{Z}/K$ be a smooth K-analytic space, and let $\mathscr{U} \subset \mathscr{Z}$ be an open subset with $\mathrm{codim}(\mathscr{Z} \setminus \mathscr{U}) \geqslant 2$. For T/K a split torus, any analytic morphism $\mathscr{U} \to T^{\mathrm{an}}$ uniquely extends to an analytic morphism $\mathscr{Z} \to T^{\mathrm{an}}$.*

Proof. Note that it suffices to show that if $\mathscr{L}$ is any line bundle on $\mathscr{Z}$ that admits a generating section $s_{\mathscr{U}} \in H^0(\mathscr{U}, \mathscr{L})$, then $s_{\mathscr{U}}$ extends to a generating section of $\mathscr{L}$ over $\mathscr{Z}$. This follows from the non-Archimedean variant of the second Hebbarkeitssatz [Lüt74]. $\qquad\square$

Proof of Theorem 7.4. Let G/K denote a semi-abelian variety, and suppose that we have the presentation

$$0 \to T \to G \to A \to 0 \,,$$

where T is a split torus over K and A is an abelian variety over K. First we note that the quotient map $G \to A$ is faithfully flat with smooth fibers and hence smooth by [Sta22, Tag 01V8]. In fact, we have that $G \times_A G \cong T \times G$ via $(g,h) \to (g-h, g)$, and similarly $G \times_A G \times_A G \cong T \times T \times G$. We note that since analytification commutes with fiber products, we have that $G^{\mathrm{an}} \times_{A^{\mathrm{an}}} G^{\mathrm{an}} \cong T^{\mathrm{an}} \times G^{\mathrm{an}}$ and $G^{\mathrm{an}} \times_{A^{\mathrm{an}}} G^{\mathrm{an}} \times_{A^{\mathrm{an}}} G^{\mathrm{an}} \cong T^{\mathrm{an}} \times T^{\mathrm{an}} \times G^{\mathrm{an}}$.
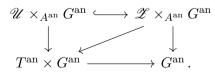
Let $\mathscr{Z}$ be a smooth, irreducible K-analytic space, and let $\mathscr{U} \subset \mathscr{Z}$ be a dense open with $\mathrm{codim}(\mathscr{Z} \setminus \mathscr{U}) \geqslant 2$. By Lemma 7.7, we can uniquely extend the composition $\mathscr{U} \to G^{\mathrm{an}} \to A^{\mathrm{an}}$ to $\mathscr{Z} \to A^{\mathrm{an}}$, which gives us the commutative diagram

$$
\begin{array}{ccc}
\mathscr{U} & \lhook\joinrel\longrightarrow & \mathscr{Z} \\
\downarrow & & \downarrow \\
G^{\mathrm{an}} & \longrightarrow & A^{\mathrm{an}} \,.
\end{array}
$$

Pulling back along the map $G^{\mathrm{an}} \to A^{\mathrm{an}}$, we have the commutative diagram

$$
\begin{array}{ccc}
\mathscr{U} \times_{A^{\mathrm{an}}} G^{\mathrm{an}} & \lhook\joinrel\longrightarrow & \mathscr{Z} \times_{A^{\mathrm{an}}} G^{\mathrm{an}} \\
\downarrow & & \downarrow \\
G^{\mathrm{an}} \times_{A^{\mathrm{an}}} G^{\mathrm{an}} & \longrightarrow & G^{\mathrm{an}} \,.
\end{array}
$$

Recalling that $G^{\mathrm{an}} \times_{A^{\mathrm{an}}} G^{\mathrm{an}} \cong T^{\mathrm{an}} \times G^{\mathrm{an}}$, we get a map $\mathscr{U} \times_{A^{\mathrm{an}}} G^{\mathrm{an}} \to T^{\mathrm{an}} \times G^{\mathrm{an}} \to T^{\mathrm{an}}$ via the first projection. By Lemma 7.8, this uniquely extends to a map $\mathscr{Z} \times_{A^{\mathrm{an}}} G^{\mathrm{an}} \to T^{\mathrm{an}}$ since $\mathscr{U} \times_{A^{\mathrm{an}}} G^{\mathrm{an}} \subset \mathscr{Z} \times_{A^{\mathrm{an}}} G^{\mathrm{an}}$ is an open immersion of smooth K-analytic spaces. This gives us the following diagonal arrow in the above diagram:

$$
\begin{array}{ccc}
\mathscr{U} \times_{A^{\mathrm{an}}} G^{\mathrm{an}} & \lhook\joinrel\longrightarrow & \mathscr{Z} \times_{A^{\mathrm{an}}} G^{\mathrm{an}} \\
\downarrow & \nearrow & \downarrow \\
T^{\mathrm{an}} \times G^{\mathrm{an}} & \longrightarrow & G^{\mathrm{an}} \,.
\end{array}
$$

Note that the top triangle commutes since it does so after composing with the first and second projections $T^{\mathrm{an}} \times G^{\mathrm{an}} \to T^{\mathrm{an}}$ (by the extension property) and $T^{\mathrm{an}} \times G^{\mathrm{an}} \to G^{\mathrm{an}}$ (which was given). Moreover, the bottom triangle commutes because everything is a morphism over G^{an}.

We need to show that the morphism $\mathscr{Z} \times_{A^{\mathrm{an}}} G^{\mathrm{an}} \to G^{\mathrm{an}} \times_{A^{\mathrm{an}}} G^{\mathrm{an}}$ descends to $\mathscr{Z} \to G^{\mathrm{an}}$. To do this, we use the theory of faithfully flat descent in the category of rigid analytic varieties,

which was developed in [Con06, Section 4.2]. First, we note that the morphism of K-analytic spaces $G^{\mathrm{an}} \to A^{\mathrm{an}}$ admits local fpqc quasi-sections via [Con06, Theorem 4.2.2] as it is the analytification of the faithfully flat morphism of K-schemes $G \to A$. Moreover, we may appeal to [Con06, Theorem 4.2.3]. We note that while these results are for rigid analytic spaces, they carry over for the K-analytic spaces we consider due to Theorems 2.2 and 2.3.

In order to utilize [Con06, Theorem 4.2.3], we need to check the cocycle condition that the two pullbacks along $(\cdot) \times_{A^{\mathrm{an}}} G^{\mathrm{an}} \times_{A^{\mathrm{an}}} G^{\mathrm{an}} \rightrightarrows (\cdot) \times_{A^{\mathrm{an}}} G^{\mathrm{an}}$ agree. These pullbacks agree above the dense open $\mathscr{U} \to \mathscr{Z}$, and we have $G^{\mathrm{an}} \times_{A^{\mathrm{an}}} G^{\mathrm{an}} \times_{A^{\mathrm{an}}} G^{\mathrm{an}} \cong T^{\mathrm{an}} \times T^{\mathrm{an}} \times G^{\mathrm{an}}$ over G^{an} (via projection onto the last component), so the uniqueness statement for morphisms to $T^{\mathrm{an}} \times T^{\mathrm{an}}$ shows that they agree everywhere. Therefore, $\mathscr{Z} \times_{A^{\mathrm{an}}} G^{\mathrm{an}} \to G^{\mathrm{an}} \times_{A^{\mathrm{an}}} G^{\mathrm{an}}$ descends to $\mathscr{Z} \to G^{\mathrm{an}}$, and by construction, the restriction to $\mathscr{U}$ is the initial morphism $\mathscr{U} \to G^{\mathrm{an}}$. $\square$

We now use Theorem 7.4 to deduce that an analytic map from a big open subset of an abelian variety to a semi-abelian variety uniquely extends to an algebraic morphism between the abelian variety and the semi-abelian variety.

PROPOSITION 7.9. *Let A/K be an abelian variety, and let $\mathscr{U} \subset A^{\mathrm{an}}$ be a dense open with $\mathrm{codim}(A^{\mathrm{an}} \setminus \mathscr{U}) \geqslant 2$. Let G/K be a semi-abelian variety. Then any analytic morphism $\varphi \colon \mathscr{U} \to G^{\mathrm{an}}$ uniquely extends to an algebraic morphism $\widetilde{\varphi} \colon A \to G$.*

Proof. By Theorem 7.4, the analytic morphism $\varphi \colon \mathscr{U} \to G^{\mathrm{an}}$ extends to an analytic map $\widetilde{\varphi} \colon A^{\mathrm{an}} \to G^{\mathrm{an}}$. The result now follows from [JV21, Lemma 2.15]. $\square$

To conclude this subsection, we show that the results from [Mor21] remain valid with our new definition of K-analytically Brody hyperbolic modulo $\mathscr{D}$ (Definition 4.1).

PROPOSITION 7.10. *Let X/K be a quasi-projective variety which is a closed subvariety of a semi-abelian variety, and let $\Delta \subset X$ be a closed subset. Then, X^{an} is K-analytically Brody hyperbolic modulo Δ^{an} if and only if every analytic morphism $\mathbb{G}_{m,K}^{\mathrm{an}} \to X^{\mathrm{an}}$ factors over Δ^{an} and for every abelian variety A/K and every dense open subset $U \subset A$ with $\mathrm{codim}(A \setminus U) \geqslant 2$, every morphism $U \to X$ factors over Δ.*

Proof. It suffices to show the second implication, namely that if every analytic morphism $\mathbb{G}_{m,K}^{\mathrm{an}} \to X^{\mathrm{an}}$ factors over Δ^{an} and for every abelian variety A/K and every dense open subset $U \subset A$ with $\mathrm{codim}(A \setminus U) \geqslant 2$, every morphism $U \to X$ factors over Δ, then X^{an} is K-analytically Brody hyperbolic modulo Δ^{an}. By Theorem 4.4, it suffices to show that for every abelian variety A/K and every dense open subset $\mathscr{U} \subset A^{\mathrm{an}}$ with $\mathrm{codim}(A^{\mathrm{an}} \setminus \mathscr{U}) \geqslant 2$, every morphism $\mathscr{U} \to X^{\mathrm{an}}$ factors over Δ^{an}. Proposition 7.9 tells us that the morphism $\mathscr{U} \to X^{\mathrm{an}}$ uniquely extends to an algebraic morphism of K-schemes $A \to X$. The desired result follows because [Moc12, Lemma A.2] asserts that the morphism $U \to X$ uniquely extends to a morphism $A \to X$, and the image of the map $A \to X$ must factor through Δ. $\square$

Remark 7.11. We can use Theorem 7.4 and Proposition 7.10 to deduce that for X/K a quasi-projective variety which is a closed subvariety of a semi-abelian variety, X^{an} is K-analytically Brody hyperbolic modulo $\emptyset$ if and only if X^{an} is K-analytically Brody hyperbolic, in the sense of [JV21, Definition 2.3]. Indeed, this follows immediately from the argument from [Mor21, Remark 2.6] and the previously mentioned results.

7.12 Proof of Theorem 7.1

We now return to the proof of Theorem 7.1 and begin by recalling the construction of the Ueno fibration for closed subvarieties of semi-abelian varieties.

DEFINITION 7.13. Let X be a closed subvariety of a semi-abelian variety G.

(1) We define the *stabilizer of X in G* as the maximal closed subgroup $\mathrm{Stab}(X, G)$ of G such that $\mathrm{Stab}(X, G) + X = X$.

(2) We denote the identity component of the closed subgroup $\mathrm{Stab}(X, G)$ by $B(X, G)$.

LEMMA 7.14. *Let X be a closed subvariety of a semi-abelian variety G. The closed subgroup $B(X, G)$ is a semi-abelian subvariety of G.*

Proof. By definition, $B(X, G)$ is connected, and since $B(X, G)$ is a closed subgroup of G, we have that $B(X, G)$ is an algebraic group. Moreover, $B(X, G)$ is a connected and smooth subgroup of G (see [Sta22, Tag 047N] for the statement on smoothness). The result now follows from [Bri17, Corollary 5.4.6(1)]. □

DEFINITION 7.15 ([Voj99, Definition 1.2]). Let X be a closed subvariety of a semi-abelian variety G. Consider the quotient $G/B(X, G)$, which is a semi-abelian variety by [Bri17, Corollary 5.4.6(1)]. The restriction to X of the quotient map $G \to G/B(X, G)$ exhibits X as a fiber bundle with fiber $B(X, G)$. This map $X \to X/B(X, G)$ is called the *Ueno fibration* of X.

To summarize, we have the diagram

$$
\begin{array}{ccc}
G & \longrightarrow & G/B(X, G) \\
\uparrow & & \uparrow \\
X & \longrightarrow & X/B(X, G) \,,
\end{array}
$$

where $X/B(X, G) \subset G/B(X, G)$ is a closed subvariety.

By the following result, the Ueno fibration of X allows us to identify closed subvarieties of a semi-abelian variety which are of logarithmic general type.

THEOREM 7.16 ([Voj99, Theorem 5.16]). *Let X be a closed subvariety of a semi-abelian variety G/K, and let D be an effective Weil divisor on X. Then, $B(X \setminus D, G) = 0$ if and only if $X \setminus D$ is of logarithmic general type.*

We can now prove Theorem 7.1.

Proof of Theorem 7.1. It is clear that if X is the translate of a semi-abelian subvariety, then X^{an} is K-analytically special, so we need to prove the converse; that is, if X^{an} is K-analytically special, then it is the translate of a semi-abelian subvariety.

Let $B(X, G)$ be the semi-abelian subvariety from Definition 7.13 and Lemma 7.14. By considering the Ueno fibration, we have a fibration $X \to X/B(X, G)$, where $X/B(X, G) \subset G/B(X, G)$ is a closed subvariety of a semi-abelian variety. Now Theorem 7.16 asserts that $X/B(X, G)$ is of logarithmic general type, and by Theorem 7.2, we have that $(X/B(X, G))^{\mathrm{an}}$ is pseudo-K-analytically Brody hyperbolic. Moreover, Theorem 6.9 implies that $X/B(X, G)$ must be zero-dimensional, and hence X is isomorphic to $B(X, G)$, which is the translate of a semi-abelian subvariety.

To conclude, we need to show that X is special, in the sense of Definition 5.6, if and only if X is a translate of a semi-abelian variety. This fact can be proved in exactly the same way as above, using Ueno fibration and Vojta's result. $\square$

Using Theorem 7.1, we deduce that the quasi-Albanese map associated with a K-analytically special variety is surjective. First, we recall the definition of the quasi-Albanese variety associated with a smooth, quasi-projective variety.

DEFINITION 7.17 ([Ser59, Iit77]). Let V/K be a smooth variety. The *quasi-Albanese map*

$$\alpha\colon V \longrightarrow \mathrm{QAlb}(V)$$

is a morphism to a semi-abelian variety $\mathrm{QAlb}(V)$ such that:

(1) for any other morphism $\beta\colon V \to B$ to a semi-abelian variety B, there is a morphism $f\colon \mathrm{QAlb}(V) \to B$ such that $\beta = f \circ \alpha$, and

(2) the morphism f is uniquely determined.

PROPOSITION 7.18. *Let X/K be a smooth variety, and suppose that X^{an}/K is K-analytically special. Then, the quasi-Albanese map is surjective.*

Proof. The image of X in $\mathrm{QAlb}(X)$ is K-analytically special, as we can simply compose $\mathscr{U} \to X^{\mathrm{an}}$ with the analytification of α. Moreover, by Theorem 7.1, any K-analytically closed subvariety of $\mathrm{QAlb}(X)$ is the translate of a semi-abelian subvariety, which implies that the image of $X \to \mathrm{QAlb}(X)$ is the translate of a semi-abelian subvariety. The universal property of the quasi-Albanese (Definition 7.17) asserts that the image must equal $\mathrm{QAlb}(X)$. $\square$

8. K-analytically special surfaces of negative Kodaira dimension

In this section, we characterize K-analytically special surfaces of Kodaira dimension $-\infty$ in terms of their irregularity and their tempered fundamental group. In particular, we will prove the following theorem.

THEOREM 8.1. *Let K be an algebraically closed, complete, non-Archimedean valued field of characteristic zero. If X/K is a smooth, projective surface with $\kappa(X) = -\infty$, then the following are equivalent:*

(1) *The surface X has irregularity $q(X)$ less than 2.*

(2) *The surface X is K-analytically special.*

(3) *The tempered fundamental group $\pi_1^{\mathrm{temp}}(X^{\mathrm{Ber}})$ of X^{Ber} is virtually abelian.*

In this section, we will be using Berkovich spaces rather than adic spaces, and we will write X^{Ber} to denote the Berkovich analytification of X. Thanks to Theorem 2.3, there is not much harm in doing so, and we can use all the previously proven results. The choice of Berkovich spaces over adic spaces is due to the fact that we will be using the tempered fundamental group, a notion first introduced by André [And03] and whose definition we will recall below, and Berkovich spaces are better when one discusses topological coverings.

8.2 The tempered fundamental group

For this subsection, we say that a *K-manifold* is a connected, smooth, paracompact, strict Berkovich K-analytic space. We note that the analytification of a smooth variety X/K will be a

K-manifold by Lemma 2.10 and [Ber90, Theorem 3.4.8(ii)]. Let $f\colon \mathscr{X}' \to \mathscr{X}$ be a morphism of K-manifolds.
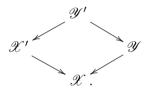
First, we recall the notion of an étale covering from [dJo95].

DEFINITION 8.3. We say that f is an *étale covering* if $\mathscr{X}$ is covered by open subsets $\mathscr{U}$ such that $f^{-1}(\mathscr{U}) = \sqcup \mathscr{V}_j$ and $\mathscr{V}_j \to \mathscr{U}$ is finite étale.

We now distinguish between several types of étale coverings.

DEFINITION 8.4. Let $f\colon \mathscr{X}' \to \mathscr{X}$ be an étale covering, and keep the notation from Definition 8.3.

- We say that f is a *topological covering* if we can choose $\mathscr{U}$ and $\mathscr{V}_j$ such that all the maps $\mathscr{V}_j \to \mathscr{U}$ are isomorphisms.
- We say that f is a *finite étale covering* if f is also finite.
- We say that f is *tempered* if it is the quotient of the composition of a topological covering $\mathscr{Y}' \to \mathscr{Y}$ and a finite étale covering $\mathscr{Y} \to \mathscr{X}$. Here quotient means that we have a diagram

$$
\begin{array}{ccc}
 & \mathscr{Y}' & \\
\swarrow & & \searrow \\
\mathscr{X}' & & \mathscr{Y} \\
\searrow & & \swarrow \\
 & \mathscr{X}\ . &
\end{array}
$$

Equivalently, a tempered covering is an étale covering which becomes a topological covering after pullback by some finite étale covering.

Using the language of fiber functors, we can define the topological, algebraic, tempered, and étale fundamental group of a K-manifold (see [dJo95, Section 2] and [And03, Section III.2]). For a K-manifold $\mathscr{X}$, we will let $\pi_1^{\mathrm{top}}(\mathscr{X})$, $\pi_1^{\mathrm{alg}}(\mathscr{X})$, $\pi_1^{\mathrm{temp}}(\mathscr{X})$, and $\pi_1^{\mathrm{\acute{e}t}}(\mathscr{X})$ denote each of these respective fundamental groups. We now give an important set of examples of tempered fundamental groups as well as a result concerning the birational invariance of the tempered fundamental group.

EXAMPLE 8.5. The tempered fundamental group of the analytification of a smooth, projective curve C/K of genus $g \leqslant 1$ is completely understood. Indeed, if $C \cong \mathbb{P}^1$, then $\pi_1^{\mathrm{temp}}(C^{\mathrm{Ber}}) = \{e\}$, and if C is an elliptic curve, then $\pi_1^{\mathrm{temp}}(C^{\mathrm{Ber}})$ is isomorphic to either $\widehat{\mathbb{Z}}^2$ or $\mathbb{Z} \times \widehat{\mathbb{Z}}$, depending on the reduction type of E (see [And03, §III.2.3.2]).

PROPOSITION 8.6. *Let $\mathscr{X}$ and $\mathscr{Y}$ be a smooth, proper K-manifolds over K, and let $\mathscr{X} \dashrightarrow \mathscr{Y}$ be a bi-meromorphic morphism. Then $\pi_1^{\mathrm{temp}}(\mathscr{X}) \cong \pi_1^{\mathrm{temp}}(\mathscr{Y})$.*

Proof. This is [Lep10, Proposition 1.5]. □

For our proof of Theorem 8.1, it will be key for us to understand when the tempered fundamental group of a curve is virtually abelian.

LEMMA 8.7. *Let C/K be a smooth, projective curve. The tempered fundamental group $\pi_1^{\mathrm{temp}}(C^{\mathrm{Ber}})$ of C^{Ber} is virtually abelian if and only if the genus of C is less than 2.*

Proof. When the genus of C is less than 1, $\pi_1^{\mathrm{temp}}(C^{\mathrm{Ber}})$ is virtually abelian by Example 8.5.

Conversely, we need to show that when C has genus greater than 1, $\pi_1^{\mathrm{temp}}(C^{\mathrm{Ber}})$ is not virtually abelian. First, we note that $\pi_1^{\mathrm{temp}}(C^{\mathrm{Ber}})$ is centerless, and hence non-abelian. This

follows from noting that the profinite completion of $\pi_1^{\mathrm{temp}}(C^{\mathrm{Ber}})$ is the algebraic fundamental group $\pi_1^{\mathrm{alg}}(C)$ (see [And03, §III.2.1.4, p. 128, paragraph 2]), and the algebraic fundamental group of a curve is well known to be centerless (see, for example, [Fal98, Lemma 1]). To conclude, we note that if $\pi_1^{\mathrm{temp}}(C^{\mathrm{Ber}})$ was virtually abelian, then there would exist a finite étale covering $\mathscr{C}' \to C^{\mathrm{Ber}}$ whose tempered fundamental group is abelian. Note that $\mathscr{C}'$ is algebraic by the non-Archimedean Riemann existence theorem [Lüt93], and so there exists some smooth, projective curve C' such that $C'^{\mathrm{Ber}} \cong \mathscr{C}'$. The genus of C' is strictly larger than the genus of C, and hence the tempered fundamental group of C'^{Ber} cannot be abelian by the above discussion. Therefore, if C has genus greater than one, $\pi_1^{\mathrm{temp}}(C^{\mathrm{Ber}})$ is not virtually abelian. $\qquad\square$

Proof of Theorem 8.1. By Lemmas 6.5 and 8.6, all of the proposed equivalent conditions in the statement of Theorem 8.1 are birational invariants, so we are free to make birational modifications. By the Enriques–Kodaira classification theorem (see, for example, [Har77, Theorem V.6.1]), we know that a smooth, projective surface X of Kodaira dimension $-\infty$ is birational to a $\mathbb{P}^1$-bundle over a curve C over K.

Since $\mathbb{P}^1$ admits no analytic differentials and is simply connected in either the topological or the tempered sense (see Example 8.5), we have that

$$q(X) = q(C), \quad \pi_1^{\mathrm{top}}(X^{\mathrm{Ber}}) \cong \pi_1^{\mathrm{top}}(C^{\mathrm{Ber}}), \quad \text{and} \quad \pi_1^{\mathrm{temp}}(X^{\mathrm{Ber}}) \cong \pi_1^{\mathrm{temp}}(C^{\mathrm{Ber}}).$$

To prove our result, we will show that having $q(X) > 1$ is equivalent to $\pi_1^{\mathrm{temp}}(X^{\mathrm{Ber}})$ not being virtually abelian and to X not being K-analytically special. By the above, we have that $q(X) > 1$ if and only if C has genus greater than 1, and so the first statement follows from Lemma 8.7.

The second statement follows because a curve C of genus greater than 1 is K-analytically Brody hyperbolic [JV21, Proposition 3.15]. Moreover, any analytic morphism from a big analytic open of the analytification of a connected, finite-type algebraic group will be constant in C^{Ber} by Remark 7.11 and hence cannot be Zariski dense in X^{Ber} as it is contained in a fiber of the morphism $X \to C$. Note that if C has genus less than 1, then X is birational to either $\mathbb{P}^1 \times E$, where E/K is an elliptic curve, or $\mathbb{P}^1 \times \mathbb{P}^1$, which are both clearly K-analytically special. $\qquad\square$

Remark 8.8. The proof of Theorem 8.1 tells us that one cannot replace the tempered fundamental group with the topological fundamental group. To see this, it suffices to note that a curve C/K of genus $g \geqslant 2$ with good reduction will have trivial topological fundamental group, which of course is abelian. We will use this observation when formulating our non-Archimedean counterparts to Campana's conjectures in Section 9.

9. Non-Archimedean variants of Campana's conjectures

In this final section, based on our results above, we formulate two non-Archimedean variants of a series of conjectures of Campana concerning various notions of specialness and their relationship to fundamental groups.

CONJECTURE 9.1 (Campana's conjectures, extended to include k-analytically special). Let X/k be a smooth projective variety. The conditions (1)–(4_∞) and (1)–(4_p) are equivalent:

   (1) The variety X is special (Definitions 5.2 and 5.4).

   (2) The variety X is geometrically special (Definition 5.14).

   (3) The variety X is arithmetically special (Definition 5.17).

(4_∞) If $k = \mathbb{C}$, then X is Brody special (Definition 5.11).

(4_p) If k is a complete, non-Archimedean valued field, then X is k-analytically special (Definition 6.1).

CONJECTURE 9.2 (Campana's abelianity conjecture for fundamental groups, extended to include k-analytically special). Let X/k be a smooth projective variety. The conditions (1)–(4_∞) and (1)–(4_p) are equivalent:

(1) If $k = \mathbb{C}$ and X is special, then $\pi_1(X^{\mathrm{an}})$ is virtually abelian.

(2) If $k = \mathbb{C}$ and X is Brody special, then $\pi_1(X^{\mathrm{an}})$ is virtually abelian.

(3) If $k = \mathbb{C}$ and X is geometrically special, then $\pi_1(X^{\mathrm{an}})$ is virtually abelian.

(4_∞) If $k = \mathbb{C}$ and X is arithmetically special, then $\pi_1(X^{\mathrm{an}})$ is virtually abelian.

(4_p) If k is a complete, non-Archimedean valued field and X^{Ber} is k-analytically special, then $\pi_1^{\mathrm{temp}}(X^{\mathrm{Ber}})$ is virtually abelian.

Remark 9.3. The complex analytic part of Conjecture 9.1, namely the equivalence of conditions (1) and (4_∞), has equivalent incarnation via the Kobayashi pseudo-metric. In [Cam04, Conjecture 9.2], Campana asked if for a proper variety $X/\mathbb{C}$, the variety X being special is equivalent to the Kobayashi pseudo-metric d_X vanishing identically on X^{an}. For a discussion on the Kobayashi pseudo-metric and its relation to hyperbolicity, we refer the reader to [Kob98]. Cherry [Che96] offered a non-Archimedean variant of the Kobayashi pseudo-metric; however, his notion seems to not correctly capture hyperbolic or special properties of a variety. For example, he proves ([Che96, Theorem 4.6]) that his non-Archimedean Kobayashi pseudo-metric is a genuine metric on an abelian variety.

In forthcoming work, the first author provides a new definition of a non-Archimedean Kobayashi pseudo-metric, which does seem to correctly capture hyperbolic and special properties of a variety. As an illustration of this claim, the author shows that for K an algebraically closed, complete, non-Archimedean valued field of characteristic zero that contains a countable dense subset and $\mathcal{X}$ a good, connected Berkovich K-analytic space, if this new non-Archimedean Kobayashi pseudo-metric $d_{\mathcal{X}}$ is in fact a metric, then $\mathcal{X}$ is K-analytically Brody hyperbolic in the sense of [JV21, Definition 2.3]. With this result and others, it seems natural to conjecture that for such a K, a Berkovich K-analytic space is K-analytically special if and only if this new non-Archimedean Kobayashi pseudo-metric $d_{\mathcal{X}}$ is identically zero on $\mathcal{X}$.

ACKNOWLEDGEMENTS

REFERENCES

Abr94   D. Abramovich, *Subvarieties of semiabelian varieties*, Compos. Math. **90** (1994), no. 1, 37–52, http://www.numdam.org/item?id=CM_1994__90_1_37_0.

And03   Y. André, *Period mappings and differential equations. From $\mathbb{C}$ to $\mathbb{C}_p$ (with appendices by F. Kato and N. Tsuzuki)*, MSJ Mem., vol. 12 (Math. Soc. Japan, Tokyo, 2003).

Ber90   V. G. Berkovich, *Spectral theory and analytic geometry over non-Archimedean fields*, Math. Surveys Monogr., vol. 33 (Amer. Math. Soc., Providence, RI, 1990); `doi:10.1090/surv/033`.

Ber93   _____, *Étale cohomology for non-Archimedean analytic spaces*, Inst. Hautes Études Sci. Publ. Math. (1993), no. 78, 5–161; `doi:10.1007/BF02712916`.

Ber99   _____, *Smooth p-adic analytic spaces are locally contractible*, Invent. Math. **137** (1999), no. 1, 1–84; `doi:10.1007/s002220050323`.

BH22   B. Bhatt and D. Hansen, *The six functors for Zariski-constructible sheaves in rigid geometry*, Compos. Math. **158** (2022), no. 2, 437–482; `doi:10.1112/s0010437x22007291`.

BT00   F. A. Bogomolov and Y. Tschinkel, *Density of rational points on elliptic K3 surfaces*, Asian J. Math. **4** (2000), no. 2, 351–368; `doi:10.4310/AJM.2000.v4.n2.a6`.

Bos83   S. Bosch, *Meromorphic functions on proper rigid analytic varieties*, Seminar on Number Theory, 1982–1983 (Talence, 1982/1983), Exp. No. 34 (University Bordeaux I, Talence, 1983).

BGR84   S. Bosch, U. Güntzer, and R. Remmert, *Non-Archimedean analysis. A systematic approach to rigid analytic geometry*, Grundlehren math. Wiss., vol. 261 (Springer-Verlag, Berlin, 1984).

BL84   S. Bosch and W. Lütkebohmert, *Stable reduction and uniformization of abelian varieties. II*, Invent. Math. **78** (1984), no. 2, 257–297; `doi:10.1007/BF01388596`.

BLR90   S. Bosch, W. Lütkebohmert, and M. Raynaud, *Néron models*, Ergeb. Math. Grenzgeb. (3), vol. 21 (Springer-Verlag, Berlin, 1990); `doi:10.1007/978-3-642-51438-8`.

Bri17   M. Brion, *Some structure theorems for algebraic groups*, Algebraic Groups: Structure and Actions, Proc. Sympos. Pure Math., vol. 94 (Amer. Math. Soc., Providence, RI, 2017), 53–126; `doi:10.1090/pspum/094`.

BL00   G. T. Buzzard and S. S. Y. Lu, *Algebraic surfaces holomorphically dominable by $\mathbb{C}^2$*, Invent. Math. **139** (2000), no. 3, 617–659; `doi:10.1007/s002220050021`.

Cam04   F. Campana, *Orbifolds, special varieties and classification theory*, Ann. Inst. Fourier (Grenoble) **54** (2004), no. 3, 499–630; `doi:10.5802/aif.2027`.

Cam20   _____, *Arithmetic aspects of orbifold pairs*, Arithmetic Geometry of Logarithmic Pairs and Hyperbolicity of Moduli Spaces – Hyperbolicity in Montréal, CRM Short Courses (Springer, Cham, 2020), 69–133; `doi:10.1007/978-3-030-49864-1`.

CW20   F. Campana and J. Winkelmann, *Dense entire curves in rationally connected manifolds*, 2020, `arXiv:1905.01104v3`.

Che94   W. Cherry, *Non-Archimedean analytic curves in abelian varieties*, Math. Ann. **300** (1994), no. 3, 393–404; `doi:10.1007/BF01450493`.

Che96   _____, *A non-Archimedean analogue of the Kobayashi semi-distance and its non-degeneracy on abelian varieties*, Illinois J. Math. **40** (1996), no. 1, 123–140; `doi:10.1215/ijm/1255986193`.

Con99   B. Conrad, *Irreducible components of rigid spaces*, Ann. Inst. Fourier (Grenoble) **49** (1999), no. 2, 473–541; `doi:10.5802/aif.1681`.

Con02   _____, *A modern proof of Chevalley's theorem on algebraic groups*, J. Ramanujan Math. Soc. **17** (2002), no. 1, 1–18.

Con06   _____, *Relative ampleness in rigid geometry*, Ann. Inst. Fourier (Grenoble) **56** (2006), no. 4, 1049–1126; `doi:10.5802/aif.2207`.

DG70   M. Demazure and A. Grothendieck, *Schémas en groupes. I: Propriétés générales des schémas en groupes*, Séminaire de Géométrie Algébrique du Bois Marie 1962/64 (SGA 3-I), Lecture Notes in Math., vol. 151 (Springer-Verlag, Berlin – New York, 1970); `doi:10.1007/BFb0058993`.

Duc07   A. Ducros, *Variation de la dimension relative en géométrie analytique p-adique*, Compos. Math. **143** (2007), no. 6, 1511–1532; `doi:10.1112/S0010437X07003193`.

Duc18   _____, *Families of Berkovich spaces*, Astérisque **400** (2018).

Fal83   G. Faltings, *Endlichkeitssätze für abelsche Varietäten über Zahlkörpern*, Invent. Math. **73** (1983), no. 3, 349–366; `doi:10.1007/BF01388432`.

Fal91 _____, *Diophantine approximation on abelian varieties*, Ann. of Math. **133** (1991), no. 3, 549–576; `doi:10.2307/2944319`.

Fal94 _____, *The general case of S. Lang's conjecture*, Barsotti Symposium in Algebraic Geometry (Abano Terme, 1991), Perspect. Math., vol. 15 (Academic Press, San Diego, CA, 1994), 175–182.

Fal98 _____, *Curves and their fundamental groups (following Grothendieck, Tamagawa and Mochizuki)*, Astérisque **252** (1998), Exp. no. 840, 131–150; `http://www.numdam.org/item/?id=SB_1997-1998__40__131_0`.

Han20 D. Hansen, *Vanishing and comparison theorems in rigid analytic geometry*, Compos. Math. **156** (2020), no. 2, 299–324; `doi:10.1112/s0010437x19007371`.

HT00a J. Harris and Y. Tschinkel, *Rational points on quartics*, Duke Math. J. **104** (2000), no. 3, 477–500; `doi:10.1215/S0012-7094-00-10436-X`.

Har77 R. Hartshorne, *Algebraic geometry*, Grad. Texts in Math., vol. 52 (Springer-Verlag, New York – Heidelberg, 1977); `doi:10.1007/978-1-4757-3849-0`.

HT00b B. Hassett and Y. Tschinkel, *Abelian fibrations and rational points on symmetric products*, Internat. J. Math. **11** (2000), no. 9, 1163–1176; `doi:10.1142/S0129167X00000544`.

Hen16 T. Henkel, *A comparison of adic spaces and Berkovich spaces*, 2016, `arXiv:1610.04117`.

Hub94 R. Huber, *A generalization of formal schemes and rigid analytic varieties*, Math. Z. **217** (1994), no. 4, 513–551; `doi:10.1007/BF02571959`.

Hub96 _____, *Étale cohomology of rigid analytic varieties and adic spaces*, Aspects Math., vol. E30 (Friedr. Vieweg & Sohn, Braunschweig, 1996); `doi:10.1007/978-3-663-09991-8`.

Huy05 D. Huybrechts, *Complex geometry. an introduction*, Universitext (Springer-Verlag, Berlin, 2005); `doi:10.1007/b137952`.

Iit76 S. Iitaka, *Logarithmic forms of algebraic varieties*, J. Fac. Sci. Univ. Tokyo Sect. IA Math. **23** (1976), no. 3, 525–544.

Iit77 _____, *On logarithmic Kodaira dimension of algebraic varieties*, in *Complex Analysis and Algebraic Geometry* (Iwanami Shoten, Tokyo, 1977), 175–189.

Jav20 A. Javanpeykar, *The Lang–Vojta conjectures on projective pseudo-hyperbolic varieties*, in *Arithmetic Geometry of Logarithmic Pairs and Hyperbolicity of Moduli Spaces – Hyperbolicity in Montréal*, CRM Short Courses (Springer, Cham, 2020), 135–196; `doi:10.1007/978-3-030-49864-1`.

JR22 A. Javanpeykar and E. Rousseau, *Albanese maps and fundamental groups of varieties with many rational points over function fields*, Int. Math. Res. Not. IMRN **2022** (2022), no 24, 19354—19398; `doi:10.1093/imrn/rnab255`.

JV21 A. Javanpeykar and A. Vezzani, *Non-archimedean hyperbolicity and applications*, J. reine angew. Math. **778** (2021), 1–29; `doi:10.1515/crelle-2021-0032`.

JX20 A. Javanpeykar and J. Xie, *Finiteness properties of pseudo-hyperbolic varieties*, Int. Math. Res. Not. **2020** (2020), no. 3, 1601–1643; `doi:10.1093/imrn/rnaa168`.

dJo95 A. J. de Jong, *Étale fundamental groups of non-Archimedean analytic spaces*, Compos. Math. **97** (1995), no. 1-2, 89–118; `http://www.numdam.org/item?id=CM_1995__97_1-2_89_0`.

Kob98 S. Kobayashi, *Hyperbolic complex spaces*, Grundlehren math. Wiss., vol. 318 (Springer-Verlag, Berlin, 1998); `doi:10.1007/978-3-662-03582-5`.

Köp74 U. Köpf, *Über eigentliche Familien algebraischer Varietäten über affinoiden Räumen*, Schr. Math. Inst. Univ. Münster (2) (1974), Heft 7.

LN22 K.-W. Lai and M. Nakahara, *Uniform potential density for rational points on algebraic groups and elliptic K3 surfaces*, Int. Math. Res. Not. IMRN **2022** (2022), no. 23, 18541-–18588; `doi:10.1093/imrn/rnab237`.

Lan86 S. Lang, *Hyperbolic and Diophantine analysis*, Bull. Amer. Math. Soc. (N.S.) **14** (1986), 159–205; `doi:10.1090/S0273-0979-1986-15426-1`.

Lep10   E. Lepage, *Tempered fundamental group and metric graph of a Mumford curve*, Publ. Res. Inst. Math. Sci. **46** (2010), no. 4, 849–897; `doi:10.2977/PRIMS/28`.

Lüt74   W. Lütkebohmert, *Der Satz von Remmert–Stein in der nichtarchimedischen Funktionentheorie*, Math. Z. **139** (1974), 69–84; `doi:10.1007/BF01194146`.

Lüt93   _____, *Riemann's existence problem for a p-adic field*, Invent. Math. **111** (1993), no. 2, 309–330; `doi:10.1007/BF01231290`.

Man20   L. Mann, *Normal and irreducible adic spaces, the openness of finite morphisms and a Stein factorization*, 2020, `arXiv:2005.06859`.

Mit11   K. Mitsui, *Bimeromorphic geometry of rigid analytic surfaces*, Internat. J. Math. **22** (2011), no. 04, 483–513; `doi:10.1142/S0129167X11006878`.

Moc12   S. Mochizuki, *Topics in absolute anabelian geometry I: generalities*, J. Math. Sci. Univ. Tokyo **19** (2012), no. 2, 139–242.

Mor21   J. S. Morrow, *Non-archimedean entire curves in closed subvarieties of semi-abelian varieties*, Math. Ann. **379** (2021), no. 3, 1003–1010; `doi:10.1007/s00208-020-02051-z`.

Nic20   M.-H. Nicole (ed.), *Arithmetic geometry of logarithmic pairs and hyperbolicity of moduli spaces – Hyperbolicity in Montréal*, CRM Short Courses (Springer, Cham, 2020); `doi:10.1007/978-3-030-49864-1`.

Nog98   J. Noguchi, *On holomorphic curves in semi-abelian varieties*, Math. Z. **228** (1998), no. 4, 713–721; `doi:10.1007/PL00004640`.

Rem57   R. Remmert, *Holomorphe und meromorphe Abbildungen komplexer Räume*, Math. Ann. **133** (1957), 328–370; `doi:10.1007/BF01342886`.

Ser59   J.-P. Serre, *Morphismes universels et variété d'albanese*, Séminaire Claude Chevalley **4** (1958–1959), no. 10, Exp. no. 10; `http://www.numdam.org/item/?id=SCC_1958-1959__4__A10_0`.

Sta22   The Stacks Project Authors, *Stacks Project*, 2022; `http://stacks.math.columbia.edu`.

Sun20   R. Sun, *Non-archimedean hyperbolicity of the moduli space of curves*, 2020, `arXiv:2009.13096`.

Uli17   M. Ulirsch, *Tropicalization is a non-archimedean analytic stack quotient*, Math. Res. Lett. **24** (2017), no. 4, 1205–1237; `doi:10.4310/MRL.2017.v24.n4.a12`.

Voj96   P. Vojta, *Integral points on subvarieties of semiabelian varieties. I*, Invent. Math. **126** (1996), no. 1, 133–181; `doi:10.1007/s002220050092`.

Voj99   _____, *Integral points on subvarieties of semiabelian varieties. II*, Amer. J. Math. **121** (1999), no. 2, 283–313; `doi:10.1007/978-3-030-49864-1`.

Voj15   _____, *A Lang exceptional set for integral points*, in *Geometry and analysis on manifolds*, Progr. Math., vol. 308 (Birkhäuser/Springer, Cham, 2015), 177–207; `doi:10.1007/978-3-319-11523-8_8`.

Yam10   K. Yamanoi, *On fundamental groups of algebraic varieties and value distribution theory*, Ann. Inst. Fourier (Grenoble) **60** (2010), no. 2, 551–563; `doi:10.5802/aif.2532`.

Jackson S. Morrow   jmorrow4692@gmail.com
Centre de Recherche de Mathématiques, Université de Montréal, Montréal, Québec H3T 1J4, Canada

Giovanni Rosso   giovanni.rosso@concordia.ca
Concordia University, Department of Mathematics and Statistics, Montréal, Québec H3G 1M8, Canada