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Subvarieties of geometric genus zero of a very general hypersurface

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Abstract

We show that any k-dimensional subvariety with geometric genus zero of a very general degree d hypersurface $X \subset \mathbb{P}^n$ is a union of lines and conics if $d > \frac{1}{5}(7n + 2(9-k))$. In particular, any rational curve on a very general degree d hypersurface $X \subset \mathbb{P}^n$ is a line or a conic if $d > \frac{1}{5}(7n+16)$.

1. Introduction

In this article, we study k-dimensional subvarieties Z with geometric genus zero of a very general degree d hypersurface $X \subset \mathbb{P}^n$. Here "geometric genus zero" means that if $Y \to Z$ is a desingularization, then $\mathrm{H}^0(Y, K_Y) = 0$. In particular, when $k = 1$, this specializes to the study of rational curves on X.

Ein [Ein88, Ein91] proved that if $d \geqslant 2n - k$, then every k-dimensional subvariety of a very general degree d hypersurface $X \subset \mathbb{P}^n$ has non-zero geometric genus. (Actually, he considered, more generally, complete intersections.) Voisin [Voi96, Voi98] improved Ein's result by 1, that is, she proved that the same result holds if $d \geqslant 2n-k-1$ and $k \leqslant n-3$. When $d = 2n-k-2$, Pacienza [Pac03] proved that any k-dimensional subvariety with geometric genus zero of a very general degree d hypersurface $X \subset \mathbb{P}^n$ is a component of the subvariety covered by the lines on X. This result was generalized by Pacienza [Pac04] and Clemens and Ran [CR04]. Clemens and Ran proved that if $d \geqslant n$, $d \geqslant \left\lfloor \frac{1}{2}(3n-k) \right\rfloor + 1$ and $\frac{1}{2}d(d+1) \geqslant 3n - 1 - k$, then any k-dimensional subvariety with geometric genus zero of a very general degree d hypersurface $X \subset \mathbb{P}^n$ is contained in the union of the lines lying on X. (In fact, they proved a more general result. See [CR04] for the full statement.) Combining Clemens and Ran's result with a result about the Fano scheme of lines on X, Riedl and Yang [RY20] proved that if $d \geqslant \frac{1}{2}(3n+1)$, then any rational curve on a very general degree d hypersurface $X \subset \mathbb{P}^n$ is a line. Recently, Coskun and Riedl [CR20] proved the same result under the weaker inequality $d \geqslant \frac{3}{2}n$.

In this article, we address the case $d \geqslant \frac{7}{5}n + \mathrm{const}$.

Main Theorem (Theorem 6.4). *If $d > \frac{1}{5}(7n + 2(9-k))$, then any k-dimensional subvariety with geometric genus zero of a very general degree d hypersurface $X \subset \mathbb{P}^n$ is a union of lines and conics.*

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As a corollary, it follows that if $d > \frac{1}{5}(7n+16)$, then any rational curve on a very general degree d hypersurface $X \subset \mathbb{P}^n$ is a line or a conic.

Outline of the proof. In proving the main theorem, we follow closely the approach of Ein, Voisin, Pacienza, Clemens and Ran. Let us recall roughly how Clemens and Ran proved their result:

Step 1. For a general point z of a k-dimensional subvariety Z of a very general degree d hypersurface $X \subset \mathbb{P}^n$, find a canonically determined codimension 1 linear subsystem $|K| \subset |\mathcal{I}_z(1)|$ of the linear system $|\mathcal{I}_z(1)|$ consisting of hyperplanes passing through z. The linear system $|K|$ is equal to the linear system $|\mathcal{I}_{l(z)}(1)|$ of hyperplanes containing a line $l(z)$ that passes through z.

Step 2. Using the argument invented by Voisin [Voi98], prove that $l(z)$ intersects X set-theoretically in one or two points.

Step 3. A further argument shows that the line $l(z)$ is contained in X.

A new idea of this paper is that in Step 1 above, we instead find a canonically determined linear subsystem $|K| \subset |\mathcal{I}_z(2)|$ of the linear system $|\mathcal{I}_z(2)|$ consisting of quadratic hypersurfaces passing through z. Examining the base locus $\mathrm{Bs}|K|$, we find a curve $C(z)$ passing through z, which is a line or a possibly degenerate conic. Once we obtain the curve $C(z)$, then implementing Step 2 and Step 3 as above, we can show that $C(z)$ is contained in X. A little further argument shows that $C(z)$ is contained in Z.

We also mention a small technical improvement of an argument appearing in Step 2. If one looks into Clemens and Ran's proof, one realizes that the inequality $d \geqslant \left\lfloor \frac{1}{2}(3n-k) \right\rfloor + 1$ is used not only in Step 1 but also at some point in Step 2. Specifically, the inequality is used in the proof of [CR04, Lemma 6.1]. We refine slightly the argument of this part (cf. Claim 4.4.1) and dispense with the inequality $d \geqslant \left\lfloor \frac{1}{2}(3n-k) \right\rfloor + 1$ in this part of the argument. This improvement is small but crucial for our argument to work.

Organization of the paper. In § 2 we gather several results used further on. We advise the reader to skip this section and come back to it when needed. In §§ 3, 4 and 5, we carry out Steps 1, 2 and 3, respectively. In § 6, we prove the main theorem.

1.1 Notation and convention

We write M^d for $\mathrm{H}^0(\mathbb{P}^n, \mathcal{O}(d))$. For a subscheme $Z \subset \mathbb{P}^n$, we write M^d_Z for $\mathrm{H}^0(\mathbb{P}^n, \mathcal{I}_Z(d))$. If $Z \subset \mathbb{P}^n$ is a subscheme, we denote by $2Z$ the non-reduced subscheme defined by the ideal $\mathcal{I}_Z^2$.

For closed subscheme $W \subset Z \subset \mathbb{P}^n$ of $\mathbb{P}^n$, if $\mathcal{I}_W$ appears as $\mathrm{H}^0(\mathbb{P}^n, \mathcal{I}_W)$, then it is the ideal sheaf of W in $\mathbb{P}^n$, that is, $\mathcal{I}_W \subset \mathcal{O}_{\mathbb{P}^n}$, and if $\mathcal{I}_W$ appears as $\mathrm{H}^0(Z, \mathcal{I}_W)$, then it is the ideal sheaf of W in Z, that is, $\mathcal{I}_W \subset \mathcal{O}_Z$.

If $\mathcal{F}$ is a locally free sheaf on a scheme X and x is a point of X, we write $\mathcal{F}|_x$ for the fiber of $\mathcal{F}$ over x. Likewise, if s is a section of $\mathcal{F}$, then $s|_x$ denotes the restriction of s to the fiber over x.

If $\mathcal{F}$ is a locally free sheaf of rank r, then $\det \mathcal{F}$ denotes $\wedge^r \mathcal{F}$. If $\mathcal{F}$ is a coherent sheaf on a smooth variety, then $\det \mathcal{F}$ denotes $\bigotimes_{i=0}^m (\det \mathcal{L}_i)^{(-1)^i}$, where $0 \to \mathcal{L}_m \to \cdots \to \mathcal{L}_0 \to \mathcal{F} \to 0$ is a locally free resolution of $\mathcal{F}$.

For lines bundles $\mathcal{L}_1$ and $\mathcal{L}_2$ on a variety, we write $\mathcal{L}_1 \geqslant \mathcal{L}_2$ to mean that $\mathcal{L}_1 \simeq \mathcal{L}_2 \otimes \mathcal{O}(E)$ for some effective divisor E. For a locally free sheaf $\mathcal{F}$, we write $\mathbf{V}(\mathcal{F})$ for $\mathbf{Spec}\, S(\mathcal{F})$.

For a finite-dimensional vector space V, we write $\mathbb{P}(V)$ for the projective space of 1-dimensional quotients of V and $\mathbb{P}_*(V)$ for the projective space of 1-dimensional subspaces of V; that is, $\mathbb{P}_*(V)=\mathbb{P}\big(V^\vee\big)$.

Throughout the paper, we work over $\mathbb{C}$.

2. Preliminaries

In this section, we collect some facts used in the arguments starting from § 3.

2.1 Linear systems on curves

Lemma 2.1. (1) *Let $e<d$ be positive integers and $V\subset \mathrm{H}^0\big(\mathbb{P}^1,\mathcal{O}(d)\big)$ a vector subspace. Then for general $Q\in \mathrm{H}^0\big(\mathbb{P}^1,\mathcal{O}(d-e)\big)$, the map*

$$\mathrm{H}^0\big(\mathbb{P}^1,\mathcal{O}(e)\big)\xrightarrow{\times Q}\mathrm{H}^0\big(\mathbb{P}^1,\mathcal{O}(d)\big)/V$$

is injective if $\dim \mathrm{H}^0\big(\mathbb{P}^1,\mathcal{O}(e)\big)\leqslant \dim \mathrm{H}^0\big(\mathbb{P}^1,\mathcal{O}(d)\big)/V$ and surjective if $\dim \mathrm{H}^0\big(\mathbb{P}^1,\mathcal{O}(e)\big)>\dim \mathrm{H}^0\big(\mathbb{P}^1,\mathcal{O}(d)\big)/V$.

(2) *Let $e<d$ be positive integers. Let $x,z\in\mathbb{P}^2$ be points and $l\subset\mathbb{P}^2$ a line passing through x. Let $\mathcal{J}\subset\mathcal{O}_{\mathbb{P}^2}$ be the product of the defining ideal of z and that of l. Let $C\subset\mathbb{P}^2$ be the subscheme defined by $\mathcal{J}$. Let $V\subset \mathrm{H}^0(C,\mathcal{I}_x(d))$ be a vector subspace such that $\dim \mathrm{H}^0(C,\mathcal{I}_x(d))/V\geqslant e+1$. The map*

$$\mathrm{H}^0(C,\mathcal{I}_x(e))\xrightarrow{\times Q}\mathrm{H}^0(C,\mathcal{I}_x(d))/V \tag{2.1}$$

is not injective for general $Q\in \mathrm{H}^0(C,\mathcal{O}(d-e))$ if and only if $\mathrm{H}^0(C,\mathcal{I}_l(d))\subset V$. In particular, for general $Q,Q'\in \mathrm{H}^0(C,\mathcal{O}(d-e))$, the map

$$\mathrm{H}^0(C,\mathcal{I}_x(e))\xrightarrow{\times Q}\mathrm{H}^0(C,\mathcal{I}_x(d))/\big(\mathrm{H}^0(C,\mathcal{I}_x(e))\cdot Q'\big) \tag{2.2}$$

is not injective.

Proof. (1) The morphism

$$\mathbb{P}_*\mathrm{H}^0\big(\mathbb{P}^1,\mathcal{O}(e)\big)\times\mathbb{P}_*\mathrm{H}^0\big(\mathbb{P}^1,\mathcal{O}(d-e)\big)\to\mathbb{P}_*\mathrm{H}^0\big(\mathbb{P}^1,\mathcal{O}(d)\big)$$

induced by the product of sections is a finite morphism. If $\dim\big(\mathrm{H}^0\big(\mathbb{P}^1,\mathcal{O}(e)\big)\cdot Q\big)\cap V=k\geqslant 1$ for general $Q\in \mathrm{H}^0\big(\mathbb{P}^1,\mathcal{O}(d-e)\big)$, then $\dim\mathbb{P}_*V\geqslant\dim\mathbb{P}_*\mathrm{H}^0\big(\mathbb{P}^1,\mathcal{O}(d-e)\big)+k-1$, which is equivalent to $\dim \mathrm{H}^0\big(\mathbb{P}^1,\mathcal{O}(e)\big)-k\geqslant \mathrm{H}^0\big(\mathbb{P}^1,\mathcal{O}(d)\big)/V$. The result follows from this.

(2) If $\mathrm{H}^0(C,\mathcal{I}_l(d))\subset V$, then $\mathrm{H}^0(C,\mathcal{I}_l(e))$ is in the kernel of the map (2.1). To show the "only if" part, put

$$\begin{aligned}U&=\mathbb{P}_*\mathrm{H}^0(C,\mathcal{I}_x(e))\setminus\mathbb{P}_*\mathrm{H}^0(C,\mathcal{I}_l(e))\,,\\ W&=\mathbb{P}_*\mathrm{H}^0(C,\mathcal{O}(d-e))\setminus\big(\mathbb{P}_*\mathrm{H}^0(C,\mathcal{I}_z(d-e))\cup\mathbb{P}_*\mathrm{H}^0(C,\mathcal{I}_l(d-e))\big)\,.\end{aligned}$$

Let $\mu\colon U\times W\to\mathbb{P}_*\mathrm{H}^0(C,\mathcal{I}_x(d))$ be the morphism induced by the product of sections. One can check that the dimension of fibers of μ is at most 1. (The reason that μ is not necessarily quasi-finite is that $\dim\mathbb{P}_*\mathrm{H}^0(C,\mathcal{O})=2$.) If the kernel of (2.1) contains $\mathrm{H}^0(C,\mathcal{I}_l(e))$ for general $Q\in \mathrm{H}^0(C,\mathcal{O}(d-e))$, then $\mathrm{H}^0(C,\mathcal{I}_l(d))\subset V$, and we are done. If not, then letting $k\geqslant 1$ be the dimension of the kernel of (2.1) for general $Q\in \mathrm{H}^0(C,\mathcal{O}(d-e))$, we have $k-1+\dim W-1\leqslant\dim V-1$, which is equivalent to $\dim \mathrm{H}^0(C,\mathcal{I}_x(d))/V\leqslant e+1-k$. This contradicts the assumption that $\dim \mathrm{H}^0(C,\mathcal{I}_x(d))/V\geqslant e+1$.

The last sentence of the lemma is clear because $\mathrm{H}^0(C,\mathcal{I}_x(e))\cdot Q'$ contains $\mathrm{H}^0(C,\mathcal{I}_l(d))$. $\square$

2.2 Grassmannians

In this section, we introduce Grassmann varieties and their variants used further on.

2.2.1 *The varieties $\mathbb{L}_1$ and $\mathbb{G}_1$.* Let $\mathbb{G}_1$ be the Grassmann variety parametrizing lines in $\mathbb{P}^n$. The tangent space $T_l\mathbb{G}_1$ of $\mathbb{G}_1$ at l is naturally isomorphic to $\operatorname{Hom}\left(M_l^1, M^1/M_l^1\right)$. (For the notation of M_l^1, see § 1.1.) There are a natural isomorphism

$$M_l^d/M_{2l}^d \simeq M_l^1 \otimes M^{d-1}/M_l^{d-1}$$

and a natural multiplication map

$$\mu\colon M^1/M_l^1 \otimes M^{d-1}/M_l^{d-1} \to M^d/M_l^d\,.$$

For a map $\varphi \in \operatorname{Hom}\left(M_l^1, M^1/M_l^1\right)$, consider the composition $\mu\circ\left(\varphi\otimes\mathrm{id}_{M^{d-1}/M_l^{d-1}}\right)\colon M_l^d/M_{2l}^d \to M^d/M_l^d$. We define a pairing

$$\bullet\colon T_l\mathbb{G}_1 \otimes M_l^d/M_{2l}^d \to M^d/M_l^d \tag{2.3}$$

by $\varphi\bullet v = -\mu\circ\left(\varphi\otimes\mathrm{id}_{M^{d-1}/M_l^{d-1}}\right)(v)$ for $\varphi\in T_l\mathbb{G}_1 = \operatorname{Hom}\left(M_l^1, M^1/M_l^1\right)$ and $v\in M_l^d/M_{2l}^d$.

Let $\mathbb{L}_1$ be the subvariety of $\mathbb{P}^n\times\mathbb{G}_1$ consisting of pairs (x,l) with $x\in l$, and let $q_1\colon \mathbb{L}_1\to\mathbb{P}^n$ and $r_1\colon\mathbb{L}_1\to\mathbb{G}_1$ be the projections. For $x\in\mathbb{P}^n$, put $\mathbb{G}_1^x := q_1^{-1}(x)$. We consider $\mathbb{G}_1^x$ as a subvariety of $\mathbb{G}_1$ by the closed immersion $r_1|_{\mathbb{G}_1^x}$; that is, $\mathbb{G}_1^x$ consists of lines passing through x.

There is a natural isomorphism

$$T_l\mathbb{G}_1^x \simeq \operatorname{Hom}\left(M_l^1, M_x^1/M_l^1\right),$$

and the pairing (2.3) restricts to

$$\bullet\colon T_l\mathbb{G}_1^x \otimes M_l^d/M_{2l}^d \to M_x^d/M_l^d\,. \tag{2.4}$$

For later use, we record the following facts as a lemma.

Lemma 2.2. (1) *For $0\neq\varphi\in\operatorname{Hom}\left(M_l^1, M^1/M_l^1\right)$, the map $\varphi\bullet\colon M_l^d/M_{2l}^d \to M^d/M_l^d$ given by the pairing (2.3) has rank at least d.*

(2) *For $0\neq\varphi\in\operatorname{Hom}\left(M_l^1, M_x^1/M_l^1\right)$, the map $\varphi\bullet\colon M_l^d/M_{2l}^d\to M_x^d/M_l^d$ given by the pairing (2.4) is surjective.*

2.2.2 *The spaces $\mathbb{L}_2$ and $\mathbb{G}_2$.* We denote by $\mathbb{G}_2$ the moduli space of (possibly degenerate) conics in $\mathbb{P}^n = \mathbb{P}(V)$. Precisely, $\mathbb{G}_2$ is the moduli space of pairs (C,H), where H is a projective plane in $\mathbb{P}^n$ and C is a possibly degenerate conic on H, that is, a closed subscheme of H defined by a quadratic equation. (Of course, H is determined by C, and we might sometimes omit H to indicate the pair (C,H).) Denote by $\mathbb{G}_2^\circ$ the open subscheme of $\mathbb{G}_2$ consisting of pairs (C,H) with C a smooth conic.

Let $\mathbb{L}_2$ be the moduli space of triples (x,C,H), where H is a projective plane in $\mathbb{P}^n$, x is a point of H and C is a possibly degenerate conic on H passing through x. We have projections $q_2\colon\mathbb{L}_2\to\mathbb{P}^n$ and $r_2\colon\mathbb{L}_2\to\mathbb{G}_2$. Put $\mathbb{L}_2^\circ := r_2^{-1}(\mathbb{G}_2^\circ)$.

For a point $x\in\mathbb{P}^n$, put $\mathbb{G}_2^x := q_2^{-1}(x)$. We regard $\mathbb{G}_2^x$ as a subvariety of $\mathbb{G}_2$ by the immersion $r_2|_{\mathbb{G}_2^x}$, so a point (x,C,H) of $\mathbb{G}_2^x$ will be denoted by (C,H).

For a point $(C,H)\in\mathbb{G}_2^x$, we describe the tangent spaces $T_{(C,H)}\mathbb{G}_2$ and $T_{(C,H)}\mathbb{G}_2^x$. The pair (C,H) corresponds to a 3-dimensional quotient $\pi\colon V\twoheadrightarrow U$ of V and a 1-dimensional subspace $\iota\colon R\hookrightarrow S^2U$ of S^2U (ι being the inclusion map), where $H=\mathbb{P}(U)$ and R determines the conic C.

For linear maps $\varphi\colon V \to U$ and $\psi\colon R \to S^2U$, the first-order deformations of π and ι

$$\begin{aligned} \pi + \epsilon\varphi &\colon V \otimes \mathbb{C}[\epsilon] \to U \otimes \mathbb{C}[\epsilon]\,, \\ \iota + \epsilon\psi &\colon R \otimes \mathbb{C}[\epsilon] \to S^2(U \otimes \mathbb{C}[\epsilon]) \end{aligned} \tag{2.5}$$

give a tangent vector of $\mathbb{G}_2$ at (C,H), where $\epsilon^2 = 0$. Let $\varphi'\colon V \to U$ and $\psi'\colon R \to S^2U$ be another pair of linear maps, and consider the first-order deformations $\pi + \epsilon\varphi'$ and $\iota + \epsilon\psi'$ determined by them. Then the two first-order deformations $(\pi + \epsilon\varphi, \iota + \epsilon\psi)$ and $(\pi + \epsilon\varphi', \iota + \epsilon\psi')$ are equivalent (here the equivalence is defined obviously), thus giving the same tangent vector, if and only if $(\varphi' - \varphi, \psi' - \psi)$ lies in the image of the map

$$\Delta\colon\ \operatorname{End}(U) \oplus \operatorname{End}(R) \to \operatorname{Hom}(V,U) \oplus \operatorname{Hom}\bigl(R, S^2U\bigr)$$

defined by $\Delta(\alpha,\beta) = (\alpha \circ \pi, (\mathrm{id}\cdot\alpha)\circ\iota - \iota\circ\beta)$, where the map $\mathrm{id}\cdot\alpha\colon S^2U \to S^2U$ is defined by $(\mathrm{id}\cdot\alpha)(uv) = u\alpha(v) + \alpha(u)v$ for $u, v \in U$. Hence we obtain an isomorphism $T_{(C,H)}\mathbb{G}_2 \simeq \operatorname{Coker}\Delta$.

Next we consider when the tangent vector of $\mathbb{G}_2$ at (C,H) determined by (2.5) is a tangent vector of $\mathbb{G}_2^x$. The point x corresponds to a 1-dimensional quotient $\tau\colon V \to A$. Since H and C pass through x, there is a surjection $\pi'\colon U \to A$ with $\tau = \pi'\circ\pi$, and the composition $S^2(\pi')\circ\iota\colon R \to A^2$ is a zero map. We denote by $\operatorname{Hom}(V \xrightarrow{\pi} U, U \xrightarrow{\pi'} A)$ the subspace of $\operatorname{Hom}(V,U)$ consisting of maps $\varphi\colon V \to U$ such that $\varphi(\operatorname{Ker}\pi) \subset \operatorname{Ker}\pi'$. For $\varphi \in \operatorname{Hom}\bigl(V \xrightarrow{\pi} U, U \xrightarrow{\pi'} A\bigr)$, we denote by $\bar\varphi$ the unique map $\bar\varphi\colon U \to A$ such that $\pi'\circ\varphi = \bar\varphi\circ\pi$. Define a map

$$\Gamma\colon\ \operatorname{Hom}(V \xrightarrow{\pi} U, U \xrightarrow{\pi'} A) \oplus \operatorname{Hom}\bigl(R, S^2U\bigr) \to \operatorname{Hom}\bigl(R, A^2\bigr)$$

by $\Gamma(\varphi,\psi) = S^2(\pi')\circ\psi - (\pi'\cdot\bar\varphi)\circ\iota$, where $\pi'\cdot\bar\varphi\colon S^2U \to A^2$ is defined by $(\pi'\cdot\bar\varphi)(uv) = \pi'(u)\bar\varphi(v) + \bar\varphi(u)\pi'(v)$ for $u, v \in U$. Then one can see that $\operatorname{Ker}\Gamma \supset \operatorname{Im}\Lambda$ and that the tangent vector of $\mathbb{G}_2$ at (C,H) determined by (2.5) is tangent to $\mathbb{G}_2^x$ if and only if $(\varphi,\psi) \in \operatorname{Ker}\Gamma$. Therefore, we have $T_{(C,H)}\mathbb{G}_2^x \simeq \operatorname{Ker}\Gamma/\operatorname{Im}\Delta$.

Next we define a map $\delta_v\colon M_C^d \to M^d/M_C^d = \mathrm{H}^0(C,\mathcal{O}(d))$ for $v \in T_{(C,H)}\mathbb{G}_2$. We choose a lift $\tilde\iota\colon R \to S^2V$ of the inclusion map $\iota\colon R \to S^2U$; that is, $S^2(\pi)\circ\tilde\iota = \iota$. Then we have $M_C^d = \operatorname{Ker}(\pi)\cdot S^{d-1}V + \tilde\iota(R)\cdot S^{d-2}V$. For $(\varphi,\psi) \in \operatorname{Hom}(V,U) \oplus \operatorname{Hom}\bigl(R, S^2U\bigr)$, define a map

$$\delta_{(\varphi,\psi)}\colon M_C^d \to S^dU/R\cdot S^{d-2}U = \mathrm{H}^0(C,\mathcal{O}(d))$$

by

$$\delta_{(\varphi,\psi)}(F + \lambda\cdot D) = -\bigl(\varphi\cdot\pi^{d-1}\bigr)(F) + \bigl(\bigl(\psi\circ S^2\pi\bigr)(\lambda) - (\varphi\cdot\pi)(\lambda)\bigr)\cdot S^{d-2}\pi(D)\,, \tag{2.6}$$

where $F \in \operatorname{Ker}(\pi)\cdot S^{d-1}V$, $\lambda \in \tilde\iota(R)$ and $D \in S^{d-2}V$, and the map $\varphi\cdot\pi^{k-1}\colon S^kV \to S^kU$ is defined by

$$\bigl(\varphi\cdot\pi^{k-1}\bigr)(v_1\cdots v_k) = \sum_{i=1}^{k} \pi(v_1)\cdots\varphi(v_i)\cdots\pi(v_k)\,.$$

One can see that it is well defined independently of the expression of an element of M_C^d as a sum $F + \lambda\cdot D$ and that the map $\delta_{(\varphi,\psi)}$ is independent of the choice of the lift $\tilde\iota$. Moreover, if (φ,ψ) and (φ',ψ') are equivalent, that is, $(\varphi'-\varphi, \psi'-\psi) \in \operatorname{Im}\Delta$, then $\delta_{(\varphi,\psi)} = \delta_{(\varphi',\psi')}$. This allows us to define δ_v for $v \in T_{(C,H)}\mathbb{G}_2$; that is, $\delta_v = \delta_{(\varphi,\psi)}$ if v is represented by (φ,ψ). One can also see that $\delta_v = \delta_{(\varphi,\psi)}$ vanishes on

$$M_{2C}^d = \operatorname{Ker}(\pi)^2\cdot S^{d-2}V + \operatorname{Ker}(\pi)\cdot\tilde\iota(R)S^{d-2}V + \tilde\iota(R)^2\cdot S^{d-4}V\,,$$

so it induces a map

$$\delta_v \colon M^d_C/M^d_{2C} \to S^dU/R\cdot S^{d-2}U = \mathrm{H}^0(C,\mathcal{O}(d))\,, \tag{2.7}$$

which we denote by the same letter δ_v (or $\delta_{(\varphi,\psi)}$).

One can see that if $v \in T_{(C,H)}\mathbb{G}^x_2$, then the image of the map δ_v lies in $\mathrm{H}^0(C,\mathcal{I}_x(d))$, so we consider δ_v as a map with target space $\mathrm{H}^0(C,\mathcal{I}_x(d))$:

$$\delta_v \colon M^d_C/M^d_{2C} \to \mathrm{H}^0(C,\mathcal{I}_x(d))\,. \tag{2.8}$$

Lemma 2.3. *Assume that the conic C is smooth. For a non-zero tangent vector $v \in T_{(C,H)}\mathbb{G}^x_2$, the codimension of $\delta_v(M^d_C/M^d_{2C})$ in $\mathrm{H}^0(C,\mathcal{I}_x(d))$ is at most 3.*

Proof. Choose $(\varphi,\psi) \in \operatorname{Ker}\Gamma$ representing v. We use the formula (2.6). If $\varphi(\operatorname{Ker}\pi) \neq 0$, then the subspace of $\mathrm{H}^0(C,\mathcal{I}_x(d))$ spanned by $-\big(\varphi\cdot\pi^{d-1}\big)(F)$, where $F \in \operatorname{Ker}(\pi)\cdot S^{d-1}V$, is of codimension at most 1. When $\varphi(\operatorname{Ker}\pi) = 0$, by replacing (φ,ψ) with an equivalent pair if necessary, we may assume $\varphi = 0$. Since $v \neq 0$, we have $\psi(R) \not\subset R$. Then the subspace of $\mathrm{H}^0(C,\mathcal{I}_x(d))$ spanned by $\big(\psi\circ S^2\pi\big)(\lambda)\cdot S^{d-2}\pi(D)$, where $\lambda \in \tilde{\imath}(R)$ and $D \in S^{d-2}V$, is of codimension at most 3. ☐

2.2.3 *The spaces $\mathbb{L}_3$ and $\mathbb{G}_3$.* Let $\mathbb{G}_3$ be the moduli space of pairs (C,H), where H is a projective plane in $\mathbb{P}^n = \mathbb{P}(V)$ and C is a degenerate conic on H; that is, C is either a union of distinct two lines or a double line. Clearly, $\mathbb{G}_3$ is a closed subvariety of $\mathbb{G}_2$. Denote by $\mathbb{G}^\circ_3$ the open subscheme of $\mathbb{G}_3$ consisting of pairs (C,H) with C a union of distinct two lines.

Let $\mathbb{L}_3$ be the moduli space of triples (x,C,H), where $(C,H) \in \mathbb{G}_3$ and x is a singular point of C; that is, if C is a union of two distinct lines, then x is their intersection point, and if C is a double line, then x is any point of C. We have projections $q_3 \colon \mathbb{L}_3 \to \mathbb{P}^n$ and $r_3 \colon \mathbb{L}_3 \to \mathbb{G}_3$. Put $\mathbb{L}^\circ_3 := r_3^{-1}(\mathbb{G}^\circ_3)$.

For a point $x \in \mathbb{P}^n$, put $\mathbb{G}^x_3 := q_3^{-1}(x)$ and $\mathbb{G}^{x\circ}_3 =: \mathbb{G}^x_3 \cap \mathbb{L}^\circ_3$. We regard $\mathbb{G}^x_3$ as a subvariety of $\mathbb{G}_3$ by the immersion $r_3|_{\mathbb{G}^x_3}$, so a point (x,C,H) of $\mathbb{G}^x_3$ will be denoted by (C,H).

Since $\mathbb{G}^x_3$ is a subvariety of $\mathbb{G}^x_2$, we have the inclusion $T_{(C,H)}\mathbb{G}^x_3 \subset T_{(C,H)}\mathbb{G}^x_2$ of tangent spaces, so we can consider the map δ_v in (2.8) for $v \in T_{(C,H)}\mathbb{G}^x_3$.

To state a lemma, we need to prepare notation. Let $\widetilde{\mathbb{G}}^{x\circ}_3$ be the moduli space of triples (l_1,l_2,H), where H is a projective plane passing through x in $\mathbb{P}^n$ and l_1 and l_2 are distinct lines on H such that $l_1 \cap l_2 = \{x\}$. By associating $(l_1\cup l_2,H)$ with (l_1,l_2,H), we obtain a degree 2 étale morphism $q \colon \widetilde{\mathbb{G}}^{x\circ}_3 \to \mathbb{G}^{x\circ}_3$. The map $q_* \colon T_{(l_1,l_2,H)}\widetilde{\mathbb{G}}^{x\circ}_3 \to T_{(l_1\cup l_2,H)}\mathbb{G}^{x\circ}_3$ between tangent spaces is an isomorphism.

For $i = 1,2$, we define the morphism $d_i \colon \widetilde{\mathbb{G}}^{x\circ}_3 \to \mathbb{G}^x_1$ by $d_i(l_1,l_2,H) = l_i$. We have the induced maps

$$d_{i*} \colon T_{(l_1,l_2,H)}\widetilde{\mathbb{G}}^{x\circ}_3 \to T_{l_i}\mathbb{G}^x_1$$

between tangent spaces. Since the morphism $(d_1,d_2) \colon \widetilde{\mathbb{G}}^{x\circ}_3 \to \mathbb{G}^x_1 \times \mathbb{G}^x_1$ is an immersion, we have $\operatorname{Ker} d_{1*} \cap \operatorname{Ker} d_{2*} = \{0\}$.

We denote by E_{3,l_i} the subspace $q_*\left(\operatorname{Ker} d_{i*}\right)$ of $T_{(l_1\cup l_2,H)}\mathbb{G}^{x\circ}_3$.

Lemma 2.4. *Let $v \in T_{(l_1,\cup l_2,H)}\mathbb{G}^{x\circ}_3$.*

(1) *If $v \notin E_{3,l_1} \cup E_{3,l_2}$, then the image of the map*

$$\delta_v \colon M^d_C/M^d_{2C} \to \mathrm{H}^0(C,\mathcal{I}_x(d))$$

has codimension at most 3 in $\mathrm{H}^0(C,\mathcal{I}_x(d))$, where $C = l_1 \cup l_2$.

(2) *For $i=1,2$, if $v \in E_{3,l_i} \setminus \{0\}$, then $\operatorname{Im}\delta_v \subset \mathrm{H}^0(C,\mathcal{I}_{l_i}(d))$ and the codimension of $\operatorname{Im}\delta_v$ in $\mathrm{H}^0(C,\mathcal{I}_{l_i}(d))$ is at most 1.*

Proof. The point x corresponds to a 1-dimensional quotient $\tau\colon V \to A$. The triple (l_1,l_2,H) corresponds to a 3-dimensional quotient $\pi\colon V \to U$ such that there exists a $\pi'\colon U \to A$ with $\pi' \circ \pi = \tau$ and to 1-dimensional subspaces $\iota_i\colon \Lambda_i \hookrightarrow U$, $i=1,2$, such that $\Lambda_1 \neq \Lambda_2$ and $\pi'(\Lambda_i)=0$ for $i=1,2$, where $H=\mathbb{P}(U)$ and Λ_i determines the line l_i. (Here ι_i denotes the inclusion map.) Define

$$\Delta_3\colon \operatorname{End}(U)\oplus\operatorname{End}(\Lambda_1)\oplus\operatorname{End}(\Lambda_2)\to\operatorname{Hom}(V,U)\oplus\operatorname{Hom}(\Lambda_1,U)\oplus\operatorname{Hom}(\Lambda_2,U)$$

by

$$\Delta_3(\alpha,\beta_1,\beta_2)=(\alpha\circ\pi,\alpha\circ\iota_1-\iota_1\circ\beta_1,\alpha\circ\iota_2-\iota_2\circ\beta_2)\,,$$

and

$$\Gamma_3\colon \operatorname{Hom}(V\xrightarrow{\pi}U,U\xrightarrow{\pi'}A)\oplus\operatorname{Hom}(\Lambda_1,U)\oplus\operatorname{Hom}(\Lambda_2,U)\to\oplus\operatorname{Hom}(\Lambda_1,A)\oplus\operatorname{Hom}(\Lambda_2,A)$$

by

$$\Gamma_3(\varphi,\psi_1,\psi_2)=(\pi'\circ\psi_1-\bar\varphi\circ\iota_1,\pi'\circ\psi_2-\bar\varphi\circ\iota_2)\,.$$

Then as in § 2.2.2, we can show that $T_{(l_1,l_2,H)}\widetilde{\mathbb{G}}_3^{x\circ}\simeq\operatorname{Ker}\Gamma_3/\operatorname{Im}\Delta_3$. Choose maps $\tilde\iota_i\colon\Lambda_i\to V$, $i=1,2$, such that $\pi\circ\tilde\iota_i=\iota_i$. Then $M_C^d=\operatorname{Ker}(\pi)\cdot S^{d-1}V+\tilde\iota_1(\Lambda_1)\tilde\iota_2(\Lambda_2)S^{d-2}V$. If $\tilde v\in T_{(l_1,l_2,H)}\widetilde{\mathbb{G}}_3^{x\circ}$ is represented by $(\varphi,\psi_1,\psi_2)\in\operatorname{Ker}\Gamma_3$, then we can also show that for the map

$$\delta_v\colon M_C^d/M_{2C}^d\to\mathrm{H}^0(C,\mathcal{I}_x(d))\subset\mathrm{H}^0(C,\mathcal{O}(d))=S^dU/\Lambda_1\Lambda_2\cdot S^{d-2}U\,,$$

where $v=q_*(\tilde v)$, we have that $\delta_v\big(F+\tilde\lambda_1\tilde\lambda_2D\big)$ equals

$$-\big(\varphi\cdot\pi^{d-1}\big)(F)+\big\{\big(\psi_2\big(\pi\big(\tilde\lambda_2\big)\big)-\varphi\big(\tilde\lambda_2\big)\big)\pi\big(\tilde\lambda_1\big)+\big(\psi_1\big(\pi\big(\tilde\lambda_1\big)\big)-\varphi\big(\tilde\lambda_1\big)\big)\pi\big(\tilde\lambda_2\big)\big\}\cdot\big(S^{d-2}\pi\big)(D)\,,\quad(2.9)$$

where $F\in\operatorname{Ker}(\pi)\cdot S^{d-1}V$, $\tilde\lambda_i\in\tilde\iota_i(\Lambda_i)$ and $D\in S^{d-2}V$. One can also show that $\tilde v$ is in $\operatorname{Ker}d_{i*}$ if and only if $\varphi(\operatorname{Ker}\pi)\subset\Lambda_i$ and $\varphi\big(\tilde\lambda\big)-\psi_i\big(\pi\big(\tilde\lambda\big)\big)\in\Lambda_i$ for all $\tilde\lambda\in\pi^{-1}(\Lambda_i)$.

(1) Suppose $\tilde v\notin\operatorname{Ker}d_{1*}\cup\operatorname{Ker}d_{2*}$. If $\varphi(\operatorname{Ker}\pi)\not\subset\Lambda_1\cup\Lambda_2$, then the subspace of $\mathrm{H}^0(C,\mathcal{I}_x(d))$ spanned by $-\big(\varphi\cdot\pi^{d-1}\big)(F)$, where $F\in\operatorname{Ker}(\pi)\cdot S^{d-1}V$, is of codimension at most 1. If, say, $\varphi(\operatorname{Ker}\pi)=\Lambda_1$, then the image of δ_v is $\big(\Lambda_1S^{d-1}U+\Lambda_2^2S^{d-2}U+\Lambda_1\Lambda_2S^{d-2}U\big)/\Lambda_1\Lambda_2S^{d-2}U$, which is of codimension 1 in $\mathrm{H}^0(C,\mathcal{I}_x(d))$. If $\varphi(\operatorname{Ker}\pi)=0$, then replacing (φ,ψ_1,ψ_2) with an equivalent triple, we may assume $\varphi=0$. Since $\tilde v\notin\operatorname{Ker}d_{1*}\cup\operatorname{Ker}d_{2*}$, we have $\psi_i(\Lambda_i)\not\subset\Lambda_i$ for $i=1,2$. Then the subspace spanned by $(\psi_2(\lambda_2)\lambda_1+\psi_1(\lambda_1)\lambda_2)\cdot S^{d-2}D$, where $\lambda_i\in\Lambda_i$ and $D\in S^{d-2}V$, is of codimension 3 in $\mathrm{H}^0(C,\mathcal{I}_x(d))$.

(2) Suppose $\tilde v\in\operatorname{Ker}d_{1*}\setminus\{0\}$. If $\varphi(\operatorname{Ker}\pi)=\Lambda_1$, then $\operatorname{Im}\delta_v=\mathrm{H}^0(C,\mathcal{I}_{l_1}(d))$. If $\varphi(\operatorname{Ker}\pi)=0$, then we may assume $\varphi=0$ as above. Since $\psi_1(\Lambda_1)\subset\Lambda_1$ and $\psi_2(\Lambda_2)\not\subset\Lambda_2$, the image of δ_v is $\big(\Lambda_1^2S^{d-2}U+\Lambda_1\Lambda_2S^{d-2}U\big)/\Lambda_1\Lambda_2S^{d-2}U$, which is of codimension 1 in $\mathrm{H}^0(C,\mathcal{I}_{l_1}(d))$. □

2.2.4 *The spaces $\mathbb{L}_4$ and $\mathbb{G}_4$.* Let $\mathbb{G}_4$ be the moduli space of pairs (C,H), where H is a projective plane in $\mathbb{P}^n=\mathbb{P}(V)$ and C is a double line on H; that is, C is a degenerate conic on H such that its reduced induced closed subscheme C_{red} is a line. Then $\mathbb{G}_4$ is a closed subvariety of $\mathbb{G}_2$.

Let $\mathbb{L}_4$ the moduli space of triples (x,C,H), where x is a point of $\mathbb{P}^n$ and $(C,H)\in\mathbb{G}_4$ with $x\in C$. We have projections $q_4\colon\mathbb{L}_4\to\mathbb{P}^n$ and $r_4\colon\mathbb{L}_4\to\mathbb{G}_4$.

For a point $x\in\mathbb{P}^n$, put $\mathbb{G}_4^x:=q_4^{-1}(x)$. We regard $\mathbb{G}_4^x$ as a subvariety of $\mathbb{G}_4$ by the immersion $r_4|_{\mathbb{G}^x}$, so a point (x,C,H) of $\mathbb{G}_4^x$ will be denoted by (C,H).

Since $\mathbb{G}_4^x$ is a subvariety of $\mathbb{G}_2^x$, we have the inclusion $T_{(C,H)}\mathbb{G}_4^x \subset T_{(C,H)}\mathbb{G}_2^x$ of tangent spaces, so we can consider the map δ_v in (2.8) for $v \in T_{(C,H)}\mathbb{G}_4^x$.

In order to state a lemma below, we prepare some notation. Define a morphism $d\colon \mathbb{G}_4^x \to \mathbb{G}_1^x$ by $d(C,H) = l$, where $l = C_{\mathrm{red}}$. Let $\mathrm{Gr}(2,\mathbb{P}(V))^x$ be the Grassmann variety parametrizing projective planes in $\mathbb{P}(V)$ passing through x. Define $d'\colon \mathbb{G}_4^x \to \mathrm{Gr}(2,\mathbb{P}(V))^x$ by $d'(C,H) = H$. There are induced maps $d_*\colon T_{(C,H)}\mathbb{G}_4^x \to T_l\mathbb{G}_1^x$ and $d'_*\colon T_{(C,H)}\mathbb{G}_4^x \to T_H \mathrm{Gr}(2,\mathbb{P}(V))^x$. Since the morphism $(d,d')\colon \mathbb{G}_4^x \to \mathbb{G}_1^x \times \mathrm{Gr}(2,\mathbb{P}(V))^x$ is an immersion, we have $\operatorname{Ker} d_* \cap \operatorname{Ker} d'_* = \{0\}$.

We denote by $E_{4,l}$ the subspace $\operatorname{Ker} d_*$ of $T_{(C,H)}\mathbb{G}_4^x$, and by $E_{4,H}$ the subspace $\operatorname{Ker} d'_*$.

Lemma 2.5. *Let $v \in T_{(C,H)}\mathbb{G}_4^x$.*

(1) *If $v \notin E_{4,l} \cup E_{4,H}$, then the image of the map*

$$\delta_v\colon M_C^d/M_{2C}^d \to \mathrm{H}^0(C,\mathcal{I}_x(d))$$

has codimension at most 1 in $\mathrm{H}^0(C,\mathcal{I}_x(d))$.

(2) *If $v \in E_{4,l} \cup E_{4,H}$, then $\operatorname{Im}\delta_v \subset \mathrm{H}^0(C,\mathcal{I}_l(d))$ and the codimension of $\operatorname{Im}\delta_v$ in $\mathrm{H}^0(C,\mathcal{I}_l(d))$ is at most 1.*

Proof. The proof is similar to that of Lemma 2.4. Details are left to the reader. □

2.2.5 *The spaces $\mathbb{L}_5$ and $\mathbb{G}_5^x$.* Let $\mathbb{L}_5$ the moduli space of quadruples (x,l_1,l_2,H), where H is a plane in $\mathbb{P}^n$, x is a point of H and l_1 and l_2 are lines on H such that $x \in l_1$. We have projections $q_5\colon \mathbb{L}_5 \to \mathbb{P}^n$ and a morphism $r_5\colon \mathbb{L}_5 \to \mathbb{G}_3$ given by $(x,l_1,l_2,H) \mapsto (x, l_1\cup l_2, H)$, where if $l_1 = l_2$, we understand that $l_1 \cup l_2$ is a double line. Denote by $\mathbb{L}_5^\circ$ the open subscheme of $\mathbb{L}_5$ consisting of (x,l_1,l_2,H) with $x \notin l_2$. For a point $x \in \mathbb{P}^n$, put $\mathbb{G}_5^x := q_5^{-1}(x)$ and $\mathbb{G}_5^{x\circ} := \mathbb{G}_5^x \cap \mathbb{L}_5^\circ$. We consider $\mathbb{G}_5^{x\circ}$ as a locally closed subscheme of $\mathbb{G}_3^\circ$ by the immersion $r_5|_{\mathbb{G}_5^{x\circ}}$, so a point (x,l_1,l_2,H) of $\mathbb{G}_5^{x\circ}$ will be denoted by (C,H) with $C = l_1 \cup l_2$.

Since $\mathbb{G}_5^{x\circ}$ is a subvariety of $\mathbb{G}_2^x$, we have the inclusion $T_{(C,H)}\mathbb{G}_5^x \subset T_{(C,H)}\mathbb{G}_2^x$ of tangent spaces, so we can consider the map δ_v in (2.8) for $v \in T_{(C,H)}\mathbb{G}_5^x$.

In order to state a lemma below, we prepare some notation. We define morphisms $e_1\colon \mathbb{G}_5^{x\circ} \to \mathbb{G}_1^x$ and $e_2\colon \mathbb{G}_5^{x\circ} \to \mathbb{G}_1$ by $e_i(l_1\cup l_2, H) = l_i$. We have the induced maps

$$e_{1*}\colon T_{(C,H)}\mathbb{G}_5^{x\circ} \to T_{l_1}\mathbb{G}_1^x \quad \text{and} \quad e_{2*}\colon T_{(C,H)}\mathbb{G}_5^{x\circ} \to T_{l_2}\mathbb{G}_1$$

between tangent spaces. Since the morphism $(e_1,e_2)\colon \mathbb{G}_5^{x\circ} \to \mathbb{G}_1^x \times \mathbb{G}_1$ is an immersion, we have $\operatorname{Ker} e_{1*} \cap \operatorname{Ker} e_{2*} = \{0\}$.

We denote by E_{5,l_i} the subspace $\operatorname{Ker} e_{i*}$ of $T_{(C,H)}\mathbb{G}_5^{x\circ}$, $i = 1,2$.

Lemma 2.6. *Let $v \in T_{(C,H)}\mathbb{G}_5^{x\circ}$.*

(1) *If $v \notin E_{5,l_1} \cup E_{5,l_2}$, then the image of the map*

$$\delta_v\colon M_C^d/M_{2C}^d \to \mathrm{H}^0(C,\mathcal{I}_x(d))$$

has codimension at most 3 in $\mathrm{H}^0(C,\mathcal{I}_x(d))$.

(2) *If $v \in E_{5,l_1} \setminus \{0\}$, then $\operatorname{Im}\delta_v \subset \mathrm{H}^0(C,\mathcal{I}_{l_1}(d))$ and the codimension of $\operatorname{Im}\delta_v$ in $\mathrm{H}^0(C,\mathcal{I}_{l_1}(d))$ is at most 1.*

(3) *If $v \in E_{5,l_2} \setminus \{0\}$, then $\operatorname{Im}\delta_v = \mathrm{H}^0(C,\mathcal{I}_{x\cup l_2}(d))$.*

Proof. The proof is similar to that of Lemma 2.4. Details are left to the reader. □

2.3 Positivity of vector bundles

2.3.1 *The vector bundle $\mathcal{M}^d_{\mathbb{G}(k,\mathbb{P}^n)}$ on $\mathbb{G}(k,\mathbb{P}^n)$.* For $k \geqslant 0$, let $\mathbb{G}(k,\mathbb{P}^n)$ be the Grassmann variety parametrizing k-dimensional projective subspaces of $\mathbb{P}^n = \mathbb{P}(V)$. (So $\mathbb{G}(0,\mathbb{P}^n) = \mathbb{P}^n$.) Giving a point of $\mathbb{G}(k,\mathbb{P}^n)$ is the same as giving a $(k+1)$-dimensional quotient $V \to U$. Let $\pi\colon V \otimes \mathcal{O}_{\mathbb{G}(k,\mathbb{P}^n)} \to \mathcal{U}$ be the universal quotient. Denote by $\mathcal{M}^d_{\mathbb{G}(k,\mathbb{P}^n)}$ the kernel of $S^d(\pi)\colon S^dV \otimes \mathcal{O}_{\mathbb{G}(k,\mathbb{P}^n)} \to S^d\mathcal{U}$. For the proof of the following proposition, refer to [CR04, § 2].

Proposition 2.7. (1) *For $i \geqslant 0$, the vector bundle $\big(\wedge^i \mathcal{M}^d_{\mathbb{G}(k,\mathbb{P}^n)}\big) \otimes \det S^d\mathcal{U}$ is globally generated. In particular, for $k = 0$, the vector bundle $\big(\wedge^i \mathcal{M}^d_{\mathbb{P}^n}\big) \otimes \mathcal{O}_{\mathbb{P}^n}(d)$ is globally generated.*
(2) *The vector bundle $\mathcal{M}^d_{\mathbb{G}(k,\mathbb{P}^n)} \otimes \det \mathcal{U}$ is globally generated.*

2.3.2 *The vector bundles $\mathcal{R}_e$, $\mathcal{V}_e$ on $\mathbb{L}_6$.* The result of this subsection is used only in the proof of Proposition 4.1(1).

Let $\mathbb{L}_6$ be the moduli space of triples (x, l, H), where H is a plane in $\mathbb{P}^n$, x is a point of H and l is a line on H. Denote by $\mathbb{L}_6^\circ$ the open subscheme of $\mathbb{L}_6$ consisting of (x, l, H) with $x \notin l$. Let $q_6\colon \mathbb{L}_6 \to \mathbb{P}^n$ be the projection.

For $(x, l, H) \in \mathbb{L}_6$, we write $x \cup l$ for the closed subscheme of H defined by the product ideal of the defining ideal of x in H and that of l in H. We write $\mathcal{I}_{x\cup l} \subset \mathcal{O}_{\mathbb{P}^n}$ for the defining ideal of $x \cup l$ in $\mathbb{P}^n$. Note that the scheme structure of $x \cup l$ depends on H if $x \in l$, although H does not appear in the notation.

For $e \geqslant 2$, we denote by $\mathcal{R}_e$ and $\mathcal{V}_e$ the vector bundles on $\mathbb{L}_6$ whose fibers over the point $(x, l, H) \in \mathbb{L}_6$ are $\mathrm{H}^0(\mathbb{P}^n, \mathcal{I}_{x\cup l}(e))$ and $\mathrm{H}^0(x \cup l, \mathcal{O}_{x\cup l}(e))$, respectively. It follows from the exact sequence

$$0 \to \mathcal{R}_e \to S^eV \otimes \mathcal{O}_{\mathbb{L}_6} \to \mathcal{V}_e \to 0$$

on $\mathbb{L}_6$ that the vector bundle $(\wedge^i\mathcal{R}_e) \otimes \det \mathcal{V}_e$ is globally generated for $i \geqslant 0$. From this, we infer that the vector bundle $\wedge^j(\mathcal{R}_e^{\oplus\alpha}) \otimes (\det \mathcal{V}_e)^\alpha$ is globally generated for $j, \alpha \geqslant 0$ because it is a direct sum of vector bundles that are tensor products of vector bundles of the form $(\wedge^i\mathcal{R}_e) \otimes \det \mathcal{V}_e$.

2.4 Zero set of a section

The result of this subsection is used in the proof of Proposition 4.1.

Lemma 2.8. (1) *Let C be a line or a smooth conic on $\mathbb{P}^2$ and x a point of C. Let $G \subset \mathrm{GL}_3$ be the subgroup consisting of elements that stabilize the curve C and the point x, and let $s \in \mathrm{H}^0(C, \mathcal{O}(d))$ be an element vanishing at x. Assume that the dimension of the orbit $G \cdot s \subset \mathrm{H}^0(C, \mathcal{O}(d))$ is at most 1. Then either s is zero, or the zero set of the section s is $\{x\}$.*

(2) *Consider $\mathbb{P}^2$ with homogeneous coordinates $[u_0 : u_1 : u_2]$, lines $l_1 = (u_2 = 0)$, $l_2 = (u_1 = 0)$ and the point $x = [1 : 0 : 0]$. Let $G \subset \mathrm{GL}_3$ be the subgroup consisting of elements that stabilize l_1 and l_2, and let $s \in \mathrm{H}^0(l_1 \cup l_2, \mathcal{O}(d))$ be an element vanishing at x. If the dimension of the orbit $G \cdot s \subset \mathrm{H}^0(l_1 \cup l_2, \mathcal{O}(d))$ is at most 3, then either $s|_{l_i} \in \mathrm{H}^0(l_i, \mathcal{O}(d))$ is identically zero for $i = 1$ or 2, or we have $\sharp\,\mathrm{Supp}\, Z(s|_{l_1}) + \sharp\,\mathrm{Supp}\, Z(s|_{l_2}) \leqslant 3$, where $Z(s|_{l_i})$ is the zero divisor of the section $s|_{l_i}$.*

(3) *Consider $\mathbb{P}^2$ with homogeneous coordinates $[u_0 : u_1 : u_2]$, a line $l = (u_2 = 0)$ and the point $x = [1 : 0 : 0]$. Let $C \subset \mathbb{P}^2$ be the double line with support l, that is, the non-reduced conic defined by $u_2^2 = 0$. Let $G \subset \mathrm{GL}_3$ be the subgroup consisting of elements that stabilize the line l and the point x, and let $s \in \mathrm{H}^0(C, \mathcal{O}(d))$ be an element vanishing at x. If the dimension of the*

orbit $G \cdot s \subset \mathrm{H}^0(C, \mathcal{O}(d))$ *is at most* 4*, then either* $s|_l$ *is identically zero, or the zero set of* $s|_l$ *consists of at most two points.*

(4) *Consider* $\mathbb{P}^2$ *with homogeneous coordinates* $[u_0 : u_1 : u_2]$*, lines* $l_1 = (u_2 = 0)$*,* $l_2 = (u_1 = 0)$ *and points* $x = [0 : 1 : 0]$*,* $z = [1 : 0 : 0]$*. Let* $G \subset \mathrm{GL}_3$ *be the subgroup consisting of elements that stabilize* l_1 *and* l_2 *and the point* x*, and let* $s \in \mathrm{H}^0(l_1 \cup l_2, \mathcal{O}(d))$ *be an element vanishing at* x*. Assume that the dimension of the orbit* $G \cdot s \subset \mathrm{H}^0(l_1 \cup l_2, \mathcal{O}(d))$ *is at most* 2*. Then one of the following holds:*

(a) *The restriction* $s|_{l_1} \in \mathrm{H}^0(l_1, \mathcal{O}(d))$ *is zero.*
(b) *We have* $s|_{l_1} \neq 0$*, and the zero set of* $s|_{l_1}$ *is either* $\{x\}$ *or* $\{x, z\}$*.*
(c) *We have* $s|_{l_1} \neq 0$*, the zero set of* $s|_{l_1}$ *contains a point other than* x *and* z*, and* $s|_{l_2}$ *is zero.*

Moreover, case (c) *occurs only if* $\dim G \cdot s = 2$*.*

Proof. (1) From the assumption, we infer that $G \cdot s = \mathbb{C} \cdot s$. Thus for any $g \in G$, the sections $g \cdot s$ and s have the same zero set. Since G acts transitively on $C \setminus \{x\}$, the zero set of s is either C or $\{x\}$.

(2) Assume $s|_{l_i} \not\equiv 0$ for $i = 1, 2$. By the hypothesis that $\dim G \cdot s \leqslant 3$, the zero divisor $Z(g \cdot s)$ varies in an at most 2-dimensional family as $g \in G$ varies.

We first see that for $i = 1, 2$, the support of $Z(s|_{l_i})$ consists of at most two points. Indeed, if the support of, say, $Z(s|_{l_1})$ consists of more than two points, then $Z(g \cdot s|_{l_1})$ varies in a 2-dimensional family. Then $Z(g \cdot s|_{l_2})$ must stay constant as $g \in G$ varies. Thus we have $s|_{l_2} = u_2^d$. The automorphisms $(u_0, u_1, u_2) \mapsto (au_0, au_1, bu_2)$, $a, b \in \mathbb{C}^\times$, multiply $s|_{l_1}$ and $s|_{l_2}$ by a^d and b^d, respectively, without changing their divisors of zeros. Then we have $\dim G \cdot s \geqslant 4$, which gives a contradiction.

If $\sharp \operatorname{Supp} Z(s|_{l_i}) = 2$ for $i = 1, 2$, then, after a change of homogeneous coordinates if necessary,

$$s|_{l_1} = u_0^{d-j_1} u_1^{j_1} \quad \text{and} \quad s|_{l_2} = u_0^{d-j_2} u_2^{j_2} \quad \text{for some} \quad j_1, j_2 \geqslant 1\,.$$

The automorphisms $(u_0, u_1, u_2) \mapsto (u_0, au_1, bu_2)$, $a, b \in \mathbb{C}^\times$, multiply $s|_{l_1}$ and $s|_{l_2}$ by a^{j_1} and b^{j_2}, respectively, without changing their divisors of zeros. Then we have $\dim G \cdot s \geqslant 4$, which gives a contradiction.

(3) We identify $\mathrm{H}^0(C, \mathcal{O}(d))$ with the degree d part of the graded ring $\mathbb{C}[u_0, u_1, u_2]/\big(u_2^2\big)$ and express the section $s \in \mathrm{H}^0(C, \mathcal{O}(d))$ as $A(u_0, u_1) + B(u_0, u_1)u_2$, where A and B are homogeneous polynomials of degree d and $d-1$, respectively, and $A(1, 0) = 0$. To prove the lemma, it suffices to show that the dimension of the orbit $G \cdot (s|_l) \subset \mathrm{H}^0(l, \mathcal{I}_x(d))$ is at most 2. Suppose that it is at least 3. Then the fiber dimension of the restriction morphism $G \cdot s \to G \cdot (s|_l)$ is at most 1. For $c, e \in \mathbb{C}$, define $\varphi_{c,e} \colon \mathbb{C}^3 \to \mathbb{C}^3$ by $(u_0, u_1, u_2) \mapsto (u_0 + cu_2, u_1 + eu_2, u_2)$. This is an element of G, and $\varphi_{c,e} \cdot s$ is expressed as

$$A(u_0, u_1) + c\frac{\partial A}{\partial u_0}(u_0, u_1)u_2 + e\frac{\partial A}{\partial u_1}(u_0, u_1)u_2 + B(u_0, u_1)u_2\,.$$

From this, we see that $(\varphi_{c,e} \cdot s)|_l = s|_l$. Therefore, $\partial A/\partial u_0$ and $\partial A/\partial u_1$ are linearly dependent, so $A(u_0, u_1) = \alpha u_1^d$ for some $\alpha \in \mathbb{C}$. But this contradicts the hypothesis $\dim G \cdot (s|_l) \geqslant 3$.

(4) By assumption, the zero divisors of sections in the orbit $G \cdot s$ form a family of dimension at most 1. Supposing that the zero set of $s|_{l_1}$ contains a point other than x and z and that $s|_{l_2} \neq 0$, we will derive a contradiction. Since the zero divisor $Z(s|_{l_1})$ of $s|_{l_1}$ moves in a 1-dimensional family under the action of G, the zero set of $s|_{l_2}$ must be $\{z\}$. The automorphisms

$(u_0, u_1, u_2) \mapsto (\alpha u_0, \alpha u_1, \beta u_2)$, $\alpha, \beta \in \mathbb{C}^\times$, act on s without changing its zero divisor. Then we have $\dim G \cdot s \geqslant 3$, which gives a contradiction. The last assertion of the lemma is clear. □

3. Base locus of linear system

In §§ 3 and 4, we will work under the setup we now describe in § 3.1.

3.1 Setup

Fix integers $d \geqslant 3$, $n \geqslant 3$ and $1 \leqslant k < n-1$, and let $V = \mathbb{C}^{n+1}$. Throughout this paper, we consider hypersurfaces of $\mathbb{P}^n = \mathbb{P}(V)$. Put $S = \mathrm{H}^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d)) \setminus \{0\}$, and let $\mathcal{X} \subset \mathbb{P}^n \times S$ be the universal family of hypersurfaces of degree d. Denote by p the projection $\mathcal{X} \to S$. Note that the natural action of GL_{n+1} on $\mathbb{P}^n$ induces natural action on S and $\mathcal{X}$. Let S' be a smooth variety with a GL_{n+1}-action and $b\colon S' \to S$ a GL_{n+1}-equivariant étale morphism. Then $\mathcal{X}' := \mathcal{X} \times_S S'$ has the induced GL_{n+1}-action, and the projection $p'\colon \mathcal{X}' \to S'$ is GL_{n+1}-equivariant. We assume that the projection p' is a smooth morphism. Let $\mathcal{Y}$ be a smooth variety with a GL_{n+1}-action, $p_1\colon \mathcal{Y} \to S'$ a GL_{n+1}-equivariant smooth projective morphism with connected fibers of dimension k and $f\colon \mathcal{Y} \to \mathcal{X}'$ a GL_{n+1}-equivariant generically finite morphism with $p' \circ f = p_1$.

$$\begin{tikzcd}
\mathcal{Y} \arrow[r, "f"] \arrow[dr, "p_1"'] & \mathcal{X}' \arrow[r] \arrow[d, "p'"] \arrow[rr, bend left, "h"] & \mathcal{X} \arrow[r] \arrow[d, "p"] & \mathbb{P}^n \\
& S' \arrow[r, "b"] & S &
\end{tikzcd}$$

The normal sheaf $\mathcal{N}_f$ of f is defined by the exact sequence

$$0 \to \mathcal{T}_{\mathcal{Y}} \to f^*\mathcal{T}_{\mathcal{X}'} \to \mathcal{N}_f \to 0\,.$$

The injectivity of the map $\mathcal{T}_{\mathcal{Y}} \to f^*\mathcal{T}_{\mathcal{X}'}$ is a consequence of our assumption that f is generically finite. Since $\mathcal{Y}$ and $\mathcal{X}'$ are smooth over S', the sheaf $\mathcal{N}_f$ fits into an exact sequence

$$0 \to \mathcal{T}_{\mathcal{Y}/S'} \to f^*\mathcal{T}_{\mathcal{X}'/S'} \to \mathcal{N}_f \to 0\,. \tag{3.1}$$

The projections of $\mathcal{Y}$ and $\mathcal{X}'$ to $\mathbb{P}^n$ are GL_{n+1}-equivariant, and the action of GL_{n+1} on $\mathbb{P}^n$ is transitive; thus $\mathcal{Y}$ and $\mathcal{X}'$ are smooth over $\mathbb{P}^n$. So $\mathcal{N}_f$ also fits into an exact sequence

$$0 \to \mathcal{T}_{\mathcal{Y}/\mathbb{P}^n} \to f^*\mathcal{T}_{\mathcal{X}'/\mathbb{P}^n} \to \mathcal{N}_f \to 0\,. \tag{3.2}$$

Since the relative tangent bundle $\mathcal{T}_{\mathcal{X}/\mathbb{P}^n}$ of $\mathcal{X}$ over $\mathbb{P}^n$ is isomorphic to the pull-back to $\mathcal{X}$ of the vector bundle $\mathcal{M}^d_{\mathbb{P}^n}$ on $\mathbb{P}^n$ (see § 2.3.1 for the notation $\mathcal{M}^d_{\mathbb{P}^n}$) and $\mathcal{X}' \to \mathcal{X}$ is étale, there is a natural isomorphism

$$\mathcal{T}_{\mathcal{X}'/\mathbb{P}^n} \simeq h^*\mathcal{M}^d_{\mathbb{P}^n}\,, \tag{3.3}$$

where $h : \mathcal{X}' \to \mathbb{P}^n$ is the projection.

3.2 Base locus

In this subsection, we associate with a general point $y \in \mathcal{Y}$ a linear subsystem of the linear system of quadratic hypersurfaces passing through the point $x = (h \circ f)(y) \in \mathbb{P}^n$ and study its base locus.

Let y be a general point of $\mathcal{Y}$, and let $T := \mathcal{T}_{\mathcal{Y}/\mathbb{P}^n}|_y$ and $x := (h \circ f)(y)$. We have an injection $f_*\colon T \to M^d_x$, that is, the composite

$$T = \mathcal{T}_{\mathcal{Y}/\mathbb{P}^n}|_y \hookrightarrow f^*\mathcal{T}_{\mathcal{X}'/\mathbb{P}^n}|_y = (h \circ f)^*\mathcal{M}^d_{\mathbb{P}^n}|_y = M^d_x\,.$$

From now on until the end of § 4, we assume that

$$\text{for general } s' \in S', \text{ we have } \mathrm{H}^0(\mathcal{Y}_{s'}, K_{\mathcal{Y}_{s'}}) = 0\,, \tag{3.4}$$

$$\text{the inequality } d > \frac{7n}{5} + \frac{2}{5}(9-k) \text{ holds}\,. \tag{3.5}$$

We choose general $Q_1, \dots, Q_l \in M^{d-2}$, and we denote by $\gamma(l)$ the dimension of the subspace $\sum_{i=1}^{l} M_x^2 \cdot Q_i + T/T$ of M_x^d/T. The sequence $(\gamma(i+1) - \gamma(i))_{i \geqslant 0}$ is non-increasing. Put

$$s^{(j)} = \min\{i \mid \gamma(i+1) - \gamma(i) \leqslant j\}\,.$$

Lemma 3.1. *The inequality $d - n \leqslant 2s^{(0)}$ holds.*

Proof. Recall that $p_1 \colon \mathcal{Y} \to S'$ is the projection, and put $s' := p_1(y)$ and $f' := f|_{\mathcal{Y}_{s'}} \colon \mathcal{Y}_{s'} \to \mathcal{X}'$. Since $\sum_{i=1}^{s^{(0)}} M_x^2 \cdot Q_i + T/T = M_x^d/T = \mathcal{N}_f|_y$, the map induced by (3.2)

$$\bigoplus_{i=1}^{s^{(0)}} (h \circ f')^* \mathcal{M}_{\mathbb{P}^n}^2 \cdot Q_i \to \mathcal{N}_f|_{\mathcal{Y}_{s'}}$$

is generically surjective. Then $\det((\mathcal{N}_f|_{\mathcal{Y}_{s'}})/(\text{torsion})) \otimes (h \circ f')^* \mathcal{O}_{\mathbb{P}^n}\big(2s^{(0)}\big)$ is generically globally generated since $\big(\wedge^i \mathcal{M}_{\mathbb{P}^n}^2\big)(2)$ is globally generated by Proposition 2.7. It follows from this and (3.1) that

$$\mathrm{H}^0\big(\mathcal{Y}_{s'}, K_{\mathcal{Y}_{s'}} \otimes f_{s'}^* K_{\mathcal{X}'_{s'}}^{-1} \otimes (h \circ f')^* \mathcal{O}_{\mathbb{P}^n}\big(2s^{(0)}\big)\big) \neq 0\,,$$

equivalently,

$$\mathrm{H}^0\big(\mathcal{Y}_{s'}, K_{\mathcal{Y}_{s'}} \otimes (h \circ f')^* \mathcal{O}_{\mathbb{P}^n}\big(2s^{(0)} + n + 1 - d\big)\big) \neq 0\,.$$

By (3.4), the number $2s^{(0)} + n + 1 - d$ must be positive. □

Let

$$\lambda_i \colon M_x^2 \xrightarrow{\times Q_i} M_x^d \Big/ \Big(T + \sum_{j=1}^{i-1} M_x^2 \cdot Q_j\Big) \tag{3.6}$$

be the map defined by $\lambda_i(A) = \overline{A \cdot Q_i}$, and put $K_i = \mathrm{Ker}(\lambda_i) \subset M_x^2$.

Lemma 3.2. *Assume $s^{(i-1)} - s^{(i)} \geqslant 2$.*

1. *The vector subspace $K_{s^{(i)}+1} \subset M_x^2$ does not depend on the choice of general elements $Q_1, \dots, Q_{s^{(i)}+1} \in M^{d-2}$.*
2. *We have an inclusion*

$$M^{d-2} \cdot K_{s^{(i)}+1} \subset T + \sum_{j=1}^{s^{(i)}} M_x^2 \cdot Q_j\,.$$

Proof. Although the proof is the same as that of [CR04, Lemma 3.2], we give a proof for readers' convenience. We write $K(Q_1, \dots, Q_{i-1}; Q_i)$ for the kernel K_i of λ_i in (3.6). It follows from the definition that

$$K(Q_1, \dots, Q_{i-1}; Q_i) \subset K(Q_1, \dots, Q_{i-1}, R; Q_i)$$

for $Q_1, \dots, Q_i, R \in M^{d-2}$. As we vary the general $Q_i \in M^{d-2}$, the rank of the map (3.6) is constant. It follows from this that

$$K(Q_1, \dots, Q_{i-1}; Q_i) \cdot M^{d-2} \subset T + \sum_{j=1}^{i} M_x^2 \cdot Q_j\,,$$

so we have

$$K(Q_1,\dots,Q_{i-1};Q_i)\subset K(Q_1,\dots,Q_{i-1},Q_i;R)$$

for general elements $Q_1,\dots,Q_i,R\in M^{d-2}$. Under the assumption $s^{(i-1)}-s^{(i)}\geqslant 2$, we have

$$\begin{aligned}K(Q_1,\dots,Q_{s^{(i)}};Q_{s^{(i)}+1})&=K(Q_1,\dots,Q_{s^{(i)}},R;Q_{s^{(i)}+1})\,,\\ K(Q_1,\dots,Q_{s^{(i)}};Q_{s^{(i)}+1})&=K(Q_1,\dots,Q_{s^{(i)}},Q_{s^{(i)}+1};R)\end{aligned}$$

for general elements $Q_1,\dots,Q_{s^{(i)}},R\in M^{d-2}$ because both sides have the same dimension. Then we have

$$\begin{aligned}K(Q_1,\dots,Q_{s^{(i)}};Q_{s^{(i)}+1})&=K(Q_1,\dots,Q_{s^{(i)}},Q_{s^{(i)}+1};R)\\ &=K(Q_1,\dots,Q_{s^{(i)}};R)\,.\end{aligned}$$

This means that $K(Q_1,\dots,Q_{s^{(i)}};Q_{s^{(i)}+1})$ is independent of $Q_{s^{(i)}+1}$. Similarly, we have

$$\begin{aligned}K(Q_1,\dots,Q_{s^{(i)}};Q_{s^{(i)}+1})&=K(Q_1,\dots,Q_{s^{(i)}},R;Q_{s^{(i)}+1})\\ &=K(Q_1,\dots,Q_{s^{(i-1)}},R;Q_{s^{(i)}+1})\,.\end{aligned}$$

This means that $K(Q_1,\dots,Q_{s^{(i)}};Q_{s^{(i)}+1})$ is independent of $Q_{s^{(i)}}$. By symmetry, it does not depend on the general choice of $Q_1,\dots,Q_{s^{(i)}}$.

(2) By definition, we have $Q_{s^{(i)}+1}\cdot K_{s^{(i)}+1}\subset T+\sum_{j=1}^{s^{(i)}}M_x^2\cdot Q_j$. Since $K_{s^{(i)}+1}$ is independent of the general choice of $Q_{s^{(i)}+1}$, this implies that $M^{d-2}\cdot K_{s^{(i)}+1}\subset T+\sum_{j=1}^{s^{(i)}}M_x^2\cdot Q_j$. □

Lemma 3.3. (1) *For some $1\leqslant j\leqslant 4$, we have $s^{(j-1)}-s^{(j)}\geqslant 2$.*
(2) *If $s^{(3)}-s^{(4)}\geqslant 2$, then $s^{(0)}-s^{(4)}\geqslant 5$.*

Proof. (1) Suppose $s^{(j-1)}-s^{(j)}\leqslant 1$ for every $1\leqslant j\leqslant 4$. Then

$$\begin{aligned}n-1-k=\dim M_x^d/T\geqslant\sum_{i=0}^{4}s^{(i)}&=5s^{(0)}-\sum_{j=1}^{4}(5-j)\big(s^{(j-1)}-s^{(j)}\big)\\ &\geqslant 5s^{(0)}-10\geqslant\frac{5}{2}(d-n)-10\,,\end{aligned}$$

where the last inequality follows from Lemma 3.1. But this contradicts the inequality (3.5).

(2) If $s^{(3)}-s^{(4)}\geqslant 2$ and $s^{(0)}-s^{(4)}\leqslant 4$, then above inequalities still hold, and we obtain a contradiction. □

Define m by $m=\max\big\{j\mid 1\leqslant j\leqslant 4\text{ and }s^{(j-1)}-s^{(j)}\geqslant 2\big\}$.

Now we study the base locus $\mathrm{Bs}|K_{s^{(m)}+1}|\subset\mathbb{P}^n$ of the linear system $|K_{s^{(m)}+1}|$ of quadratic hypersurfaces passing through x. (We do not consider the scheme structure of the base locus.)

Proposition 3.4. (1) *If $m=4$, then one of the following holds:*

- (A) *We have $\mathrm{Bs}|K_{s^{(4)}+1}|=l_1\cup l_2$, where l_1 is a line passing through x and l_2 is another line intersecting l_1 at a point different from x. In this case, $K_{s^{(4)}+1}=M^2_{l_1\cup l_2}$.*
- (B) *We have $\mathrm{Bs}|K_{s^{(4)}+1}|=\{x\}\cup l\cup$(possibly some finite points), where l is a line not passing through x. In this case, $K_{s^{(4)}+1}$ is a codimension 1 subspace of $M^2_{x\cup l}$.*
- (C) *We have $\mathrm{Bs}|K_{s^{(4)}+1}|=l\cup$(possibly some finite points), where l is a line passing through x. In this case, $K_{s^{(4)}+1}$ is a codimension 2 subspace of M^2_l.*
- (D) *We have $\mathrm{Bs}|K_{s^{(4)}+1}|=C$, where C is a smooth conic passing through x. In this case, $K_{s^{(4)}+1}=M^2_C$.*

(E) *We have* $\mathrm{Bs}|K_{s^{(4)}+1}| = l_1 \cup l_2$, *where* l_1 *and* l_2 *are distinct lines both passing through* x. *In this case,* $K_{s^{(4)}+1} = M^2_{l_1 \cup l_2}$.

(2) *If* $m = 3$, *then one of the following holds:*

(F) *We have* $\mathrm{Bs}|K_{s^{(3)}+1}| = \{x\} \cup l$, *where* l *is a line not passing through* x. *In this case,* $K_{s^{(3)}+1} = M^2_{x \cup l}$.

(G) *We have* $\mathrm{Bs}|K_{s^{(3)}+1}| = l \cup$ *(possibly some finite points), where* l *is a line passing through* x. *In this case,* $K_{s^{(3)}+1}$ *is a codimension* 1 *subspace of* M^2_l.

(3) *If* $m = 2$, *then the following holds:*

(H) *We have* $\mathrm{Bs}|K_{s^{(2)}+1}| = l$, *where* l *is a line passing through* x. *In this case,* $K_{s^{(2)}+1} = M^2_l$.

(4) *The case* $m = 1$ *does not occur.*

Remark. We will show later that cases (B) and (F) do not occur (cf. Proposition 4.1).

For the proof of the proposition, we prepare some notation and a lemma.

We choose general $R_1, \dots, R_r \in M^{d-1}$ and put

$$\eta(r) = \dim \frac{\sum_{i=1}^{s^{(m)}} M^2_x \cdot Q_i + \sum_{i=1}^{r} M^1_x \cdot R_i + T}{T} \tag{3.7}$$

and

$$\sigma^{(j)} = \min\{i \mid \eta(i+1) - \eta(i) \leqslant j\}\,. \tag{3.8}$$

For simplicity of notation, we write σ for $\sigma^{(0)}$. Define

$$L_j := \mathrm{Ker}\left(M^1_x \xrightarrow{\times R_j} \frac{M^d_x}{\sum_{i=1}^{s^{(m)}} M^2_x \cdot Q_i + \sum_{i=1}^{j-1} M^1_x \cdot R_i + T} \right). \tag{3.9}$$

A similar argument as in the proof of Lemma 3.2 shows that if $\sigma^{(i-1)} - \sigma^{(i)} \geqslant 2$, then we have

$$L_{\sigma^{(i)}+1} \cdot M^{d-1} \subset \sum_{i=1}^{s^{(m)}} M^2_x \cdot Q_i + \sum_{j=1}^{\sigma^{(i)}} M^1_x \cdot R_j + T\,. \tag{3.10}$$

Lemma 3.5. (1) *Either* $\sigma^{(1)} - \sigma^{(2)} \geqslant 2$ *or* $\sigma - \sigma^{(1)} \geqslant 2$ *holds.*

(2) *We have* $\dim \mathrm{Bs}|K_{s^{(m)}+1}| \geqslant 1$.

Proof. (1) If $\sigma^{(1)} - \sigma^{(2)} \leqslant 1$ and $\sigma - \sigma^{(1)} \leqslant 1$, then

$$\begin{aligned} n - 1 - k &\geqslant 5s^{(4)} + \sum_{i=m+1}^{4} i\big(s^{(i-1)} - s^{(i)}\big) + \sigma^{(2)} + \sigma^{(1)} + \sigma \\ &\geqslant 5s^{(m)} - \frac{(5-m)(4-m)}{2} + 3\sigma - \varepsilon\,, \end{aligned} \tag{3.11}$$

where $0 \leqslant \varepsilon \leqslant 3$. A similar argument as the proof of Lemma 3.1 shows that

$$d - n \leqslant 2s^{(m)} + \sigma\,. \tag{3.12}$$

From (3.11) and (3.12), we obtain

$$\frac{7n}{5} + \frac{2}{5}(8 - k) \geqslant d\,,$$

which contradicts (3.5).

(2) Suppose $\dim \mathrm{Bs}|K_{s^{(m)}+1}| = 0$. Consider the case $\sigma^{(1)} - \sigma^{(2)} \geqslant 2$. Since $\dim M_x^1/L_{\sigma^{(2)}+1} = 2$, we have $M_H^1 = L_{\sigma^{(2)}+1}$ for a projective plane $H \subset \mathbb{P}^n$ passing through x. By (3.10), we have

$$M_H^d \subset \sum_{i=1}^{s^{(m)}} M_x^2 \cdot Q_i + \sum_{i=1}^{\sigma^{(2)}} M_x^1 \cdot R_i + T\,.$$

Consider the following subspaces of $M_x^d/M_H^d \simeq \mathrm{H}^0(H, \mathcal{I}_x(d))$:

$$W := \frac{K_{s^{(m)}+1} \cdot M^{d-2} + M_H^d}{M_H^d} \subset \frac{\sum_{i=1}^{s^{(m)}} M_x^2 \cdot Q_i + \sum_{j=1}^{\sigma^{(2)}} M_x^1 \cdot R_j + T}{M_H^d}\,.$$

Since we are assuming $\dim \mathrm{Bs}|K_{s^{(m)}+1}| = 0$, there are two hypersurfaces $D_1, D_2 \in |K_{s^{(m)}+1}|$ such that $Z := D_1 \cap D_2 \cap H$ is a zero-dimensional subscheme of length 4 containing x. Then $\mathrm{H}^0(H, \mathcal{I}_Z(d))$ is a codimension 3 subspace of $\mathrm{H}^0(H, \mathcal{I}_x(d))$. But $\mathrm{H}^0(H, \mathcal{I}_Z(d))$ is contained in W, and since $\sigma^{(1)} - \sigma^{(2)} \geqslant 2$, the codimension of W in $\mathrm{H}^0(H, \mathcal{I}_x(d))$ is at least 4. This gives a contradiction.

In the case $\sigma - \sigma^{(1)} \geqslant 2$, we have $M_L^1 = L_{\sigma^{(1)}+1}$ for a line $L \subset \mathbb{P}^n$ passing through x. Since we are assuming $\dim \mathrm{Bs}|K_{s^{(m)}+1}| = 0$, there is a hypersurface $D \in |K_{s^{(m)}+1}|$ such that $Z := D \cap L$ is a zero-dimensional subscheme of length 2 containing x. Arguing as in the case $\sigma^{(1)} - \sigma^{(2)} \geqslant 2$, we reach a contradiction. $\square$

Before we prove Proposition 3.4, we prove two claims.

CLAIM 3.5.1. *There is at most one curve in* $\mathrm{Bs}|K_{s^{(4)}+1}|$ *which does not pass through* x. *If such a curve exists, then it is a line.*

Proof of Claim 3.5.1. Let $C \subset \mathrm{Bs}|K_{s^{(4)}+1}|$ be a reduced connected curve not passing through x. Let $\mathbb{P}^e \subset \mathbb{P}^n$ be the smallest projective subspace containing C. If we denote by N the image of the restriction map $M^1 \to H^0(C, \mathcal{O}_C(1))$, then $\dim N = e + 1$. General hyperplanes H_1 and H_2 passing through x determine a surjective morphism $\mathcal{O}_C^{\oplus 2} \to \mathcal{O}_C(1)$. Tensoring with $\mathcal{O}_C(1)$, we obtain an exact sequence

$$0 \to \mathcal{O}_C \to \mathcal{O}_C(1)^{\oplus 2} \to \mathcal{O}_C(2) \to 0\,.$$

Considering the associated long exact sequence

$$0 \to \mathrm{H}^0(C, \mathcal{O}) \to \mathrm{H}^0(C, \mathcal{O}_C(1))^{\oplus 2} \xrightarrow{\alpha} \mathrm{H}^0(C, \mathcal{O}_C(2))\,,$$

we find that $\dim \alpha\big(N^{\oplus 2}\big) \geqslant 2e+1$. It then follows that the restriction map $\rho\colon M_x^2 \to \mathrm{H}^0(C, \mathcal{O}(2))$ has rank at last $2e+1$ because $\operatorname{Im} \rho$ contains $\alpha\big(N^{\oplus 2}\big)$. Since $K_{s^{(4)}+1}$ lies in the kernel of the map ρ and has codimension 4 in M_x^2, we have $e = 1$. So C is a line. If there are two lines l_1 and l_2 not passing through x in $\mathrm{Bs}|K_{s^{(4)}+1}|$, then by the above argument, l_1 and l_2 are disjoint. Then the restriction map $M_x^2 \to \mathrm{H}^0(l_1 \cup l_2, \mathcal{O}(2))$ has rank at least 5, and this is impossible. $\square$

CLAIM 3.5.2. *If* $C \subset \mathrm{Bs}|K_{s^{(4)}+1}|$ *is a reduced connected curve passing through* x, *then* C *is a line, or a smooth conic, or a union of two intersecting lines.*

Proof of Claim 3.5.2. Let $\mathbb{P}^e \subset \mathbb{P}^n$ be the smallest projective subspace containing C. Arguing as in the last claim, this time we find that $e \leqslant 2$. If $e = 1$, then C is a line. If $e = 2$, then C is a (possibly degenerate) conic because $h^0\big(\mathbb{P}^2, \mathcal{I}_x(2)\big) = 5$ and $K_{s^{(4)}+1}$ has codimension 4 in M_x^2. $\square$

Proof of Proposition 3.4. If $\mathrm{Bs}|K_{s^{(4)}+1}|$ does not contain any curve not passing through x, then by Claim 3.5.2, case (C), (D) or (E) holds. If $\mathrm{Bs}|K_{s^{(4)}+1}|$ contains a curve not passing through x and does not contain any curve passing through x, then case (B) holds. If $\mathrm{Bs}|K_{s^{(4)}+1}|$ contains both a curve l not passing through x and a curve C passing through x, then C must be a line; otherwise, we would have $M_C^2 = K_{s^{(4)}+1}$, and $\mathrm{Bs}|K_{s^{(4)}+1}|$ does not contain a curve not passing through x. If the lines l and C are disjoint, then the restriction map $M_x^2 \to \mathrm{H}^0(l \cup C, \mathcal{I}_x(2))$ has rank at least 5, and this is impossible. So l and C intersect, and case (A) holds.

The proofs of statements (2), (3) and (4) are similar. □

3.3 Dividing into subcases

For later use, we need to study some of the cases in Proposition 3.4 more closely and divide some of the cases into subcases.

We divide case (A) of Proposition 3.4 into three subcases:

(A1) $\sigma - \sigma^{(1)} \geqslant 2$,

(A2) $\sigma - \sigma^{(1)} \leqslant 1$ and $M^d_{x\cup l_2} \subset T + \sum_{i=1}^{s^{(0)}-2} M_x^2 \cdot Q_i$,

(A3) $\sigma - \sigma^{(1)} \leqslant 1$ and $M^d_{x\cup l_2} \not\subset T + \sum_{i=1}^{s^{(0)}-2} M_x^2 \cdot Q_i$.

Proposition 3.6. (1) *In case (A1), we have*

$$M^d_{l_1} \subset T + \sum_{i=1}^{s^{(4)}} M_x^2 \cdot Q_i + \sum_{i=1}^{\sigma^{(1)}} M_x^1 \cdot R_i\,.$$

(2) *In case (A2), we have* $K_{s^{(0)}-1} = M^2_{x\cup l_2}$.

(3) *In case (A3), we have the inequalities*

$$\dim(T/T') \cap \mathrm{H}^0(C, \mathcal{I}_{l_1}(d)) \leqslant d - \bigl(2\, s^{(4)} + \sigma - \varepsilon\bigr)\,, \tag{3.13}$$

$$\dim(T/T') \cap \mathrm{H}^0(C, \mathcal{I}_{x\cup l_2}(d)) \leqslant d - s^{(0)} + 2\,, \tag{3.14}$$

where $C = l_1 \cup l_2$, $T' := T \cap M_C^d$ and $\varepsilon := \sigma - \sigma^{(1)}$. Note that T/T', $\mathrm{H}^0(C, \mathcal{I}_{l_1}(d)) = M^d_{l_1}/M^d_C$ and $\mathrm{H}^0(C, \mathcal{I}_{x\cup l_2}(d)) = M^d_{x\cup l_2}/M^d_C$ are subspaces of M^d_x/M^d_C, so their intersections make sense.

Proof. (1) We have $L_{\sigma^{(1)}+1} = M^1_L$ for a line $L \subset \mathbb{P}^n$ passing through x, and we have $M^d_L \subset T + \sum_{i=1}^{s^{(4)}} M_x^2 \cdot Q_i + \sum_{i=1}^{\sigma^{(1)}} M_x^1 \cdot R_i$ by (3.10). If $L \neq l_1$, then we can find a $D \in |K_{s^{(4)}+1}|$ such that $L \cap D$ is a zero-dimensional scheme of length 2 containing x. Arguing as in the proof of Lemma 3.5(2), we obtain a contradiction.

(2) Since $M^d_{x\cup l_2} \subset T + \sum_{i=1}^{s^{(0)}-2} M_x^2 \cdot Q_i$, we have $M^2_{x\cup l_2} \subset K_{s^{(0)}-1}$. By Lemma 2.1(1), we infer that $M^2_{x\cup l_2} = K_{s^{(0)}-1}$.

(3) The subspace

$$(T/T') \cap \mathrm{H}^0(C, \mathcal{I}_{l_1}(d)) + \sum_{i=1}^{s^{(4)}} \mathrm{H}^0(C, \mathcal{I}_{l_1}(2)) \cdot \overline{Q_i} + \sum_{i=1}^{\sigma-1-\varepsilon} \mathrm{H}^0(C, I_{l_1}(1)) \cdot \overline{R_i}$$

of $\mathrm{H}^0(C, \mathcal{I}_{l_1}(d))$ is contained in the subspace

$$\frac{T + \sum_{i=1}^{s^{(4)}} M_x^2 \cdot Q_i + \sum_{i=1}^{\sigma-1-\varepsilon} M_x^1 \cdot R_i}{M_C^d}\,.$$

Since $\sigma - \sigma^{(1)} \leqslant 1$, the latter subspace does not contain $\mathrm{H}^0(C, \mathcal{I}_{l_1}(d))$. So the dimension of the former subspace is smaller than that of $\mathrm{H}^0(C, \mathcal{I}_{l_1}(d))$:

$$\dim(T/T') \cap \mathrm{H}^0(C, \mathcal{I}_{l_1}(d)) + 2\, s^{(4)} + \sigma - 1 - \varepsilon < d\,,$$

and the equality (3.13) follows.

The subspace

$$(T/T') \cap \mathrm{H}^0(C, \mathcal{I}_{x \cup l_2}(d)) + \sum_{i=1}^{s^{(0)}-2} \mathrm{H}^0(C, \mathcal{I}_{x \cup l_2}(2)) \cdot \overline{Q_i}$$

of $\mathrm{H}^0(C, \mathcal{I}_{x \cup l_2}(d))$ is contained in

$$\frac{T + \sum_{i=1}^{s^{(0)}-2} M_x^2 \cdot Q_i}{M_C^d}\,.$$

The latter subspace does not contain $\mathrm{H}^0(C, \mathcal{I}_{x \cup l_2}(d))$. So the dimension of the former subspace is smaller than that of $\mathrm{H}^0(C, \mathcal{I}_{x \cup l_2}(d))$:

$$\dim(T/T') \cap \mathrm{H}^0(C, \mathcal{I}_{x \cup l_2}(d)) + s^{(0)} - 2 < d - 1\,,$$

and the equality (3.14) follows. □

PROPOSITION 3.7. *In case (B) of Proposition 3.4, we have $\sigma^{(1)} - \sigma^{(2)} \geqslant 2$ and*

$$M_{x \cup l}^d \subset \sum_{i=1}^{s^{(4)}} M_x^2 \cdot Q_i + \sum_{i=1}^{\sigma^{(2)}} M_x^1 \cdot R_i + T\,.$$

Proof. By Lemma 3.5(1), we have either $\sigma^{(1)} - \sigma^{(2)} \geqslant 2$ or $\sigma - \sigma^{(1)} \geqslant 2$. We suppose the inequality $\sigma - \sigma^{(1)} \geqslant 2$ holds and will derive a contradiction. We have $L_{\sigma^{(1)}+1} = M_L^1$ for a line L passing through x. We have

$$M_L^d \subset \sum_{i=1}^{s^{(4)}} M_x^2 \cdot Q_i + \sum_{i=1}^{\sigma^{(1)}} M_x^1 \cdot R_i + T$$

by (3.10). Since $L \not\subset \mathrm{Bs}|K_{s^{(4)}+1}|$, there exists a $D \in |K_{s^{(4)}+1}|$ such that $D \cap L$ is a zero-dimensional scheme of length 2 containing x. Arguing as in the proof of Lemma 3.5(2), we obtain a contradiction.

Now that we have $\sigma^{(1)} - \sigma^{(2)} \geqslant 2$, we have $M_H^1 = L_{\sigma^{(2)}+1}$ for a plane $H \subset \mathbb{P}^n$ passing through x, and

$$M_H^d \subset \sum_{i=1}^{s^{(4)}} M_x^2 \cdot Q_i + \sum_{i=1}^{\sigma^{(2)}} M_x^1 \cdot R_i + T$$

by (3.10). We claim that $l \subset H$. If $l \not\subset H$, then there exist $D_1, D_2 \in |K_{s^{(4)}+1}|$ such that $Z := D_1 \cap D_2 \cap H$ is a zero-dimensional scheme of length 4. Then as in the proof of Lemma 3.5(2), we get a contradiction.

Since $\mathrm{Bs}|K_{s^{(4)}+1}| = \{x\} \cup l \cup$ (possibly some finite points), there are $D_1, D_2 \in |K_{s^{(4)}+1}|$ such that $D_i \cap H = l_i \cup l$, where l_i is a line passing through x with $l_1 \neq l_2$. Thus we have

$$M_{x \cup l}^d \subset \sum_{i=1}^{s^{(4)}} M_x^2 \cdot Q_i + \sum_{i=1}^{\sigma^{(2)}} M_x^1 \cdot R_i + T\,. \qquad \square$$

Proposition 3.8. (1) *We divide case (C) of Proposition 3.4 into subcases:*

(C1) $\sigma - \sigma^{(1)} \geqslant 2$,

(C2) $\sigma - \sigma^{(1)} \leqslant 1$.

Then in case (C1), we have

$$M_l^d \subset \sum_{i=1}^{s^{(4)}} M_x^2 \cdot Q_i + \sum_{i=1}^{\sigma^{(1)}} M_x^1 \cdot R_i + T\,.$$

In case (C2), there exists a plane H containing l such that $K_{s^{(4)}+1} = M_C^2$, where $C \subset H$ is a double line with support l. And the inequality

$$\dim T/T' \cap \mathrm{H}^0(C, \mathcal{I}_l(d)) \leqslant d - \left(2\, s^{(4)} + \sigma - \varepsilon\right) \tag{3.15}$$

holds, where $T' = T \cap M_C^d$ and $\varepsilon = \sigma - \sigma^{(1)}$.

(2) *In case (G) of Proposition 3.4, we have $\sigma - \sigma^{(1)} \geqslant 2$ and*

$$M_l^d \subset \sum_{i=1}^{s^{(3)}} M_x^2 \cdot Q_i + \sum_{i=1}^{\sigma^{(1)}} M_x^1 \cdot R_i + T\,.$$

Proof. (1) In case (C1), we have $M_L^1 = L_{\sigma^{(1)}+1}$ for a line $L \subset \mathbb{P}^n$ passing through x. A similar argument as in Proposition 3.6(1) shows that $L = l$ and shows the required inclusion.

Next we consider case (C2). By Lemma 3.5(1), we have $\sigma^{(1)} - \sigma^{(2)} \geqslant 2$. We have $M_H^1 = L_{\sigma^{(2)}+1}$ for a plane $H \subset \mathbb{P}^n$ passing through x. We have

$$M_H^d \subset \sum_{i=1}^{s^{(4)}} M_x^2 \cdot Q_i + \sum_{i=1}^{\sigma^{(2)}} M_x^1 \cdot R_i + T$$

by (3.10), and a similar argument as in the proof of Proposition 3.7 shows that $l \subset H$.

Consider the subspace

$$\left(K_{s^{(4)}+1} + M_H^2\right)/M_H^2 \subset M_l^2/M_H^2 = \mathrm{H}^0(H, \mathcal{I}_l(2))\,.$$

There are three cases: $\dim\left(K_{s^{(4)}+1} + M_H^2\right)/M_H^2 = 1, 2$ or 3. If $\dim\left(K_{s^{(4)}+1} + M_H^2\right)/M_H^2 = 3$, then $K_{s^{(4)}+1} + M_H^2 = M_l^2$, so $M_l^d \subset \sum_{i=1}^{s^{(4)}} M_x^2 \cdot Q_i + \sum_{i=1}^{\sigma^{(2)}} M_x^1 \cdot R_i + T$. But this contradicts $\sigma^{(1)} - \sigma^{(2)} \geqslant 2$.

If $\dim\left(K_{s^{(4)}+1} + M_H^2\right)/M_H^2 = 2$, then we will also derive a contradiction. There are $D_1, D_2 \in |K_{s^{(4)}+1}|$ such that $D_i \cap H = l \cup l_i$, $i = 1, 2$, where l_1 and l_2 are distinct lines on H. Put $\{z\} = l_1 \cap l_2$. Let $\mathcal{J} \subset \mathcal{O}_H$ be the product of the defining ideal of z and that of l in H. Let $B \subset H$ be the subscheme of H defined by $\mathcal{J}$. Then we have

$$M_B^d \subset \sum_{i=1}^{s^{(4)}} M_x^2 \cdot Q_i + \sum_{i=1}^{\sigma^{(2)}} M_x^1 \cdot R_i + T\,. \tag{3.16}$$

The kernel of the map

$$M_x^1 \xrightarrow{\times R_{\sigma^{(2)}+2}} M_x^d \Big/ \left(\sum_{i=1}^{s^{(4)}} M_x^2 \cdot Q_i + \sum_{i=1}^{\sigma^{(2)}+1} M_x^1 \cdot R_i + T\right)$$

contains the kernel of the map

$$M_x^1 \xrightarrow{\times R_{\sigma^{(2)}+2}} M_x^d / \left(M_B^d + M_x^1 \cdot R_{\sigma^{(2)}+1} \right).$$

The latter properly contains M_B^d by Lemma 2.1(2); so does the former. This contradicts the inequality $\sigma^{(1)} - \sigma^{(2)} \geqslant 2$.

Finally, we consider the case $\dim \left(K_{s^{(4)}+1} + M_H^2 \right) / M_H^2 = 1$. In this case, all $D \in |K_{s^{(4)}+1}|$ not containing H give rise to the same subscheme $D \cap H \subset H$. Since we have $\mathrm{Bs}|K_{s^{(4)}+1}| = l \cup$ (possibly some finite points), it must be a double line with support l, which we denote by C. Then $K_{s^{(4)}+1} \subset M_C^2$. Since both $K_{s^{(4)}+1}$ and M_C^2 have codimension 4 in M_x^2, they are equal.

The proof of the inequality (3.15) is similar to that of (3.13).

(2) Suppose $\sigma - \sigma^{(1)} \leqslant 1$. Then arguing as in case (C2), we obtain $K_{s^{(3)}+1} \subset M_C^2$. This is impossible because $K_{s^{(3)}+1}$ is of codimension 3 in M_x^2. □

Proposition 3.9. *We divide case (E) of Proposition 3.4 into two subcases:*

(E1) $\sigma - \sigma^{(1)} \geqslant 2$,

(E2) $\sigma - \sigma^{(1)} \leqslant 1$.

In case (E1), for $i = 1$ or 2, we have $L_{\sigma^{(1)}+1} = M_{l_i}^1$ and

$$M_{l_i}^d \subset \sum_{i=1}^{s^{(4)}} M_x^2 \cdot Q_i + \sum_{i=1}^{\sigma^{(1)}} M_x^1 \cdot R_i + T\,. \tag{3.17}$$

In case (E2), for $i = 1, 2$, we have

$$\dim T/T' \cap \mathrm{H}^0(C, \mathcal{I}_{l_i}(d)) \leqslant d - \left(2\, s^{(4)} + \sigma - \varepsilon\right), \tag{3.18}$$

where $\varepsilon = \sigma - \sigma^{(1)}$.

Proof. In case (E1), we have $M_L^1 = L_{\sigma^{(1)}+1}$ for a line $L \subset \mathbb{P}^n$ passing through x. A similar argument as in Proposition 3.6(1) shows that $L = l_i$ for $i = 1$ or 2 and shows the required inclusion.

The proof of the inequality (3.18) is similar to that of (3.13). □

4. Intersection of a line or conic with the hypersurface $\mathcal{X}'_{s'}$

We retain the notation used in the preceding section; we work under the setup described in § 3.1 and under the assumptions (3.4) and (3.5). Recall that $y \in \mathcal{Y}$ is a general point, $x = (h \circ f)(y)$ and $s' = p_1(y)$. We classified the base locus of the linear system $|K_{s^{(m)}+1}|$ in Proposition 3.4 and introduced more subcases in § 3.3. In this section, we study the intersection of the line or conic appearing in the base locus $\mathrm{Bs}|K_{s^{(m)}+1}|$ with the hypersurface $\mathcal{X}'_{s'}$.

Before stating the goal of this section, we would like here to summarize what will be proved for each case in this and next section.

Case (A1) In Proposition 5.1, we prove that $l_1 \subset \mathcal{X}'_{s'}$.

Case (A2) In Proposition 4.1, we prove that this case does not occur.

Case (A3) In Proposition 5.1, we prove that $l_1 \subset \mathcal{X}'_{s'}$.

Case (B) In Proposition 4.1, we prove that this case does not occur.

Case (C1) In Proposition 5.1, we prove that $l \subset \mathcal{X}'_{s'}$.

Case (C2) In Proposition 5.1, we prove that $l \subset \mathcal{X}'_{s'}$.

Case (D) In Proposition 5.1, we prove that $C \subset \mathcal{X}'_{s'}$.

Case (E1) In Proposition 5.1, we prove that $l_i \subset \mathcal{X}'_{s'}$.

Case (E2) In Proposition 5.1, we prove that exactly one of l_1 and l_2 is contained in $\mathcal{X}'_{s'}$.

Case (F) In Proposition 4.1, we prove that this case does not occur.

Case (G) In Proposition 5.1, we prove that $l \subset \mathcal{X}'_{s'}$.

Case (H) In Proposition 5.1, we prove that $l \subset \mathcal{X}'_{s'}$.

Now the goal of this section is to prove the following proposition.

Proposition 4.1. (1) *In case (A3), the line l_1 is contained in $\mathcal{X}'_{s'}$, or the intersection $l_1 \cap \mathcal{X}'_{s'}$ consists of at most two points.*

(2) *Cases (A2), (B) and (F) do not occur.*

(3) *In case (C2), the line l is contained in $\mathcal{X}'_{s'}$, or the intersection $l \cap \mathcal{X}'_{s'}$ consists of at most two points.*

(4) *In case (D), the smooth conic C is contained in $\mathcal{X}'_{s'}$, or the intersection $C \cap \mathcal{X}'_{s'}$ consists of one point x.*

(5) *In case (E2), either at least one of l_1 and l_2 is contained in $\mathcal{X}'_{s'}$, or we have the inequality $\sharp(l_1 \cap \mathcal{X}'_{s'}) + \sharp(l_2 \cap \mathcal{X}'_{s'}) \leqslant 3$.*

(6) *In case (A1), the line l_1 is contained in $\mathcal{X}'_{s'}$, or the intersection $l_1 \cap \mathcal{X}'_{s'}$ consists of one point x. In case (E1), the line l_i is contained in $\mathcal{X}'_{s'}$, or the intersection $l_i \cap \mathcal{X}'_{s'}$ consists of one point x. In cases (C1), (G) and (H), the line l is contained in $\mathcal{X}'_{s'}$, or the intersection $l \cap \mathcal{X}'_{s'}$ consists of one point x.*

Remark. As one can see from the summary before Proposition 4.1, the possible situations after "or" in the proposition are actually proved, in § 5, not to occur.

First in § 4.1, we prove item (6). This is a line case, and we closely follow the proof in [CR04]. But in proving the integrability of the vector bundle $\mathcal{T}'$, we give a slightly refined argument. In § 4.2, we prove item (2), which is also a line case. In § 4.3, we prove items (1), (3), (4) and (5). These are conic cases. The proof of item (1) is the most involved.

4.1 Proof of Proposition 4.1(6)

We give a proof for case (H); the other cases are similar. By associating with a general point y of $\mathcal{Y}$ the pair (x, l), where l is the line passing through x determined in case (H) of Proposition 3.4, we obtain a rational map $g\colon \mathcal{Y} \to \mathbb{L}_1$ (see § 2.2.1 for the notation $\mathbb{L}_1$). By blowing up $\mathcal{Y}$ $\mathrm{GL}(V)$-equivariantly and shrinking S' if necessary, we may assume that g is a morphism.

$$\begin{tikzcd}
 & & & \mathbb{L}_1 \arrow[d, "q_1"] \\
\mathcal{Y} \arrow[urrr, bend left, "g"] \arrow[r, "f"] \arrow[dr, "p_1"'] & \mathcal{X}' \arrow[r] \arrow[rr, bend left, "h"] \arrow[d, "p'"] & \mathcal{X} \arrow[r] \arrow[d, "p"] & \mathbb{P}^n \\
 & S' \arrow[r, "b"] & S &
\end{tikzcd}$$

We have the commutativity $q_1 \circ g = h \circ f$. We have the induced map $g_*\colon T = \mathcal{T}_{\mathcal{Y}/\mathbb{P}^n}|_y \to \mathcal{T}_{\mathbb{L}_1/\mathbb{P}^n}|_{(x,l)} = T_l \mathbb{G}_1^x$ between tangent spaces for a general point $y \in \mathcal{Y}$.

Recall that $T = \mathcal{T}_{\mathcal{Y}/\mathbb{P}^n}|_y$ is considered as a subspace of $\mathcal{T}_{\mathcal{X}'/\mathbb{P}^n}|_{f(y)} \simeq M_x^d$ by the injection f_*. Put $T' := T \cap M_l^d$.

Recall that $R_1, \ldots, R_r \in M^{d-1}$ are general elements, and $\eta(r)$, $\sigma^{(j)}$ and σ are defined in (3.7) and (3.8). We have $d - n \leqslant 2s^{(2)} + \sigma$ (3.12). By Lemma 3.2(2), we have

$$M_l^d = M^{d-2} \cdot M_l^2 \subset T + \sum_{j=1}^{s^{(2)}} M_x^2 \cdot Q_j \,, \tag{4.1}$$

and we find that $\dim T/T' = d - (2s^{(2)} + \sigma)$ using Lemma 2.1(1).

By varying the general point $y \in \mathcal{Y}$, we obtain, over a Zariski open subset of $\mathcal{Y}$, a subbundle $\mathcal{T}'$ of the relative tangent bundle $\mathcal{T}_{\mathcal{Y}/\mathbb{P}^n}$ such that $\mathcal{T}'|_y = T'$.

Proposition 4.2. (1) *The subspace T' is in the kernel of the map $g_* \colon T \to T_l\mathbb{G}_1^x$.*
(2) *The subbundle $\mathcal{T}'$ is integrable.*

We first show that Proposition 4.1(6) follows from this proposition.

Proof of "Proposition 4.2 ⇒ Proposition 4.1(6)". Pick a general point y_0 in $\mathcal{Y}$, and put $l_0 = g(y_0)$, $x_0 = (h \circ f)(y_0)$ and $\mathcal{Y}'_{y_0} := g^{-1}(x_0, l_0)$. We have $\dim \mathcal{Y}'_{y_0} = \dim S + k - 2n + 1$.

Consider the map $\rho \colon \mathcal{Y}'_{y_0} \to \mathrm{H}^0(l_0, \mathcal{O}(d))$ defined by $\rho(y) = F_y|_{l_0}$, where F_y is a degree d polynomial corresponding to the point $b(p_1(y)) \in S = \mathrm{H}^0(\mathbb{P}^n, \mathcal{O}(d)) \setminus \{0\}$, which is the defining equation of the hypersurface $\mathcal{X}'_{p_1(y)}$. By Proposition 4.2(2), for general $y \in \mathcal{Y}'_{y_0}$, there is a submanifold $L \subset (h \circ f)^{-1}(x_0)$ passing through y such that the tangent bundle $\mathcal{T}_L$ of L is equal to $\mathcal{T}'|_L$. We have $\dim L = \dim T' = \dim S - n + k - d + 2s^{(2)} + \sigma$. By Proposition 4.2(1), the map g is constant on L, thus $L \subset \mathcal{Y}'_{y_0}$. By the definition of $\mathcal{T}'$, the map ρ is constant on L. Therefore, we have

$$\dim \operatorname{Im} \rho \leqslant \dim \mathcal{Y}'_{y_0} - \dim L = d - n + 1 - 2s^{(2)} - \sigma \leqslant 1 \,.$$

If $G \subset \mathrm{GL}(V)$ denotes the subgroup consisting automorphisms of V whose induced automorphisms of $\mathbb{P}^n = \mathbb{P}(V)$ fix x and the line l_0, then G acts on $\mathcal{Y}'_{y_0}$ and $\mathrm{H}^0(l_0, \mathcal{O}(d))$, and the map ρ is G-equivariant; thus the orbit $G \cdot \rho(y_0)$ is in $\operatorname{Im} \rho$. Since $\dim \operatorname{Im} \rho \leqslant 1$, either we have $\rho(y_0) = 0$, or $\rho(y_0)$ is a section defining the divisor $d \cdot x$ by Lemma 2.8(1). □

It remains to prove Proposition 4.2. We denote by ϵ the composition $T \to M_x^d \to M_x^d/M_l^d = \mathrm{H}^0(l, \mathcal{I}_x(d))$. If τ_1 and τ_2 are sections of $\mathcal{T}'$ defined over a neighborhood of $y \in \mathcal{Y}$, then $[\tau_1, \tau_2]$ is a section of $\mathcal{T}_{\mathcal{Y}/\mathbb{P}^n}$, so it makes sense to consider $\epsilon([\tau_1, \tau_2]|_y)$.

Lemma 4.3. *For sections τ_1 and τ_2 of $\mathcal{T}'$ defined over a neighborhood of $y \in \mathcal{Y}$, we have*

$$\epsilon([\tau_1, \tau_2]|_y) = g_*(\tau_1|_y) \bullet (\overline{\tau_2|_y}) - g_*(\tau_2|_y) \bullet (\overline{\tau_1|_y}) \,,$$

where $\overline{\tau_i|_y}$ is the image of $\tau_i|_y \in T' \subset M_l^d$ in M_l^d/M_{2l}^d, and the pairing $\bullet$ is the one in (2.4).

Proof. The proof is the same as that of [Cle03, Lemma 3.1]. □

The lemma shows that assertion (2) of Proposition 4.2 follows from assertion (1).

Lemma 4.4. *Put $T'' = T \cap M_{2l}^d$; we have $T'' \subset \operatorname{Ker}(g_* \colon T \to T_l\mathbb{G}_1^x)$.*

Proof. We suppose $g_*(v) \neq 0$ for some $v \in T''$ and will derive a contradiction.

Denote by K the kernel of the map

$$g_*(v)\bullet\colon M_l^d/M_{2l}^d \to M_x^d/M_l^d = \mathrm{H}^0(l,\mathcal{I}_x(d))\,, \tag{4.2}$$

and put $\gamma = \dim\big(T'/T''+K\big)/\big(T'/T''\big)$. We have

$$\begin{aligned}
\dim g_*(v)\bullet(T'/T'') &= \dim T'/T'' - \dim\big((T'/T'')\cap K\big)\\
&= \dim T'/T'' - \dim K + \gamma\\
&= \dim T'/T'' - \big(\dim M_l^d/M_{2l}^d - \operatorname{rank}(g_*(v)\bullet)\big) + \gamma\\
&= \operatorname{rank}(g_*(v)\bullet) - \dim \frac{M_l^d/M_{2l}^d}{T'/T''+K}.
\end{aligned} \tag{4.3}$$

Since $v \in T''$, we have $g_*(v)\bullet(T'/T'') \subset T/T'$ by Lemma 4.3. So we have

$$d - \big(2s^{(2)}+\sigma\big) = \dim T/T' \geqslant \dim g_*(v)\bullet(T'/T'')\,.$$

The map $g_*(v)\bullet$ is surjective by Lemma 2.2(2), so we have $\operatorname{rank}(g_*(v)\bullet) = d$. From these, we deduce

$$\dim \frac{M_l^d/M_{2l}^d}{T'/T''+K} \geqslant 2s^{(2)}+\sigma\,. \tag{4.4}$$

On the other hand, we claim the following.

Claim 4.4.1. *We have*

$$M_l^d = T' + \sum_{i=1}^{s^{(2)}} M_l^2\cdot Q_i\,, \tag{4.5}$$

and the inequality

$$\dim \frac{M_l^d/M_{2l}^d}{T'/T''+K} \leqslant 2s^{(2)} \tag{4.6}$$

holds.

Proof of Claim 4.4.1. First we observe that the natural sequence

$$0 \longrightarrow \frac{M_l^d}{T'+\sum_{i=1}^k M_l^2\cdot Q_i} \xrightarrow{\alpha_k} \frac{M_x^d}{T+\sum_{i=1}^k M_l^2\cdot Q_i} \xrightarrow{\beta_k} \frac{M_x^d}{T+M_l^d+\sum_{i=1}^k M_l^2\cdot Q_i} \longrightarrow 0 \tag{4.7}$$

is exact for $0 \leqslant k \leqslant s^{(2)}$. Indeed, we only have to show the injectivity of α_k. By the definition of T', the map α_0 is injective. Consider the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & M_l^2 \arrow[r] \arrow[d, "\cdot Q_k"] & M_x^2 \arrow[r] \arrow[d, "\cdot Q_k"] & M_x^2/M_l^2 \arrow[r] \arrow[d, "\cdot Q_k"] & 0\\
0 \arrow[r] & \frac{M_l^d}{T'+\sum_{i=1}^{k-1} M_l^2\cdot Q_i} \arrow[r, "\alpha_{k-1}"] & \frac{M_x^d}{T+\sum_{i=1}^{k-1} M_l^2\cdot Q_i} \arrow[r, "\beta_{k-1}"] & \frac{M_x^d}{T+M_l^d+\sum_{i=1}^{k-1} M_l^2\cdot Q_i} \arrow[r] & 0
\end{tikzcd}$$

for $0 < k \leqslant s^{(2)}$, and assume that the lower sequence is exact. The right vertical arrow is injective because it is injective for $k = s^{(2)}+1$. Thus α_k is injective by the snake lemma.

Consider the sequence (4.7) for $k = s^{(2)}$. It follows from (4.1) that $\alpha_{s^{(2)}}$ is the zero map. Therefore, the left term in (4.7) is zero, which implies (4.5).

By (4.5), the composition

$$\oplus_{i=1}^{s^{(2)}} M_l^2\cdot Q_i \longrightarrow M_l^d \longrightarrow M_l^d/M_{2l}^d \longrightarrow \frac{M_l^d/M_{2l}^d}{K} \longrightarrow \frac{M_l^d/M_{2l}^d}{T'/T''+K} \tag{4.8}$$

of morphisms is surjective. The image of $\oplus_{i=1}^{s^{(2)}} M_l^2 \cdot Q_i$ in the penultimate term $(M_l^d/M_{2l}^d)/K$, which we identify with M_x^d/M_l^d via the map $g_*^x(v)\bullet$, is contained in the subspace $\sum_{i=1}^{s^{(2)}} \big(M_x^2/M_l^2\big)\cdot \overline{Q_i} \subset M_x^d/M_l^d$, whose dimension is at most $2s^{(2)}$, where $\overline{Q_i}$ is the image of Q_i in M^{d-2}/M_l^{d-2}. Therefore, the rank of the composed morphism (4.8) is at most $2s^{(2)}$. So we obtain (4.6). This is the end of the proof of Claim 4.4.1. $\square$

Proof of Lemma 4.4, continued. By (4.4) and (4.6), we get $2s^{(2)} + \sigma \leqslant 2s^{(2)}$. But we have $\sigma \geqslant 4$ since $s^{(1)} - s^{(2)} \geqslant 2$. This gives a contradiction. $\square$

Finally, we prove Proposition 4.2(1).

Proof of Proposition 4.2(1). We want to show that $g_*|_{T'}$ is the zero map. By Lemma 4.4, the map $g_*|_{T'}$ induces a map $\overline{g_*}\colon T'/T'' \to T_l\mathbb{G}_1^x$. Consider the map

$$[\ ,\ ]\colon T'/T'' \otimes T'/T'' \to M_x^d/M_l^d$$

defined by

$$[v, v'] = \overline{g_*}(v)\bullet v' - \overline{g_*}(v')\bullet v\,, \tag{4.9}$$

where the pairing $\bullet$ is the one in (2.4). Since the image of the map $[\ ,\ ]$ is contained in $T/T' \subset M^d/M_l^d$ by Lemma 4.3, we have

$$\operatorname{rank}[\ ,\ ] \leqslant d - \big(2s^{(2)} + \sigma\big)\,. \tag{4.10}$$

Next we give a lower bound of $\operatorname{rank}[\ ,\ ]$. Denote by c the codimension of T'/T'' in M_l^d/M_{2l}^d. By (4.5), there are $D_i \in M_l^d$, $1 \leqslant i \leqslant c$, such that

$$M_l^d/M_{2l}^d = T'/T'' \oplus \bigoplus_{i=1}^{c} \mathbb{C}\overline{D_i}$$

and that each D_i is an element of $M_l^2 \cdot Q_j$ for some $1 \leqslant j \leqslant s^{(2)}$. We extend $\overline{g_*}$ to the map $M_l^d/M_{2l}^d \to T_l\mathbb{G}_1^x$ by defining $\overline{g_*}(\overline{D_i}) = 0$ and define the map

$$[\ ,\ ]^\sharp\colon \big(M_l^d/M_{2l}^d\big) \otimes \big(M_l^d/M_{2l}^d\big) \to M_x^d/M_l^d$$

by the same equation as (4.9). Then we have

$$\operatorname{Im}[\ ,\ ]^\sharp = \operatorname{Im}[\ ,\ ] + \sum_{i=1}^{c} \overline{g_*}(T'/T'')\bullet \overline{D_i}\,.$$

Since $\overline{g_*}(T'/T'')\bullet \overline{D_i} \subset M_x^2/M_l^2 \cdot \overline{Q_j}$ for some $1 \leqslant j \leqslant s^{(2)}$, we have

$$\operatorname{rank}[\ ,\ ]^\sharp \leqslant \operatorname{rank}[\ ,\ ] + 2s^{(2)}\,. \tag{4.11}$$

Now suppose that $\overline{g_*}$ is not the zero map. There exists a $u\otimes v \in M_l^1 \otimes M^{d-1}/M_l^{d-1} \simeq M_l^d/M_{2l}^d$ such that $\overline{g_*}(u\otimes v) \neq 0$. Then we have

$$\overline{g_*}(u\otimes v)\bullet M_l^d/M_{2l}^d = M_x^d/M_l^d \quad\text{and}\quad \overline{g_*}\big(M_l^d/M_{2l}^d\big)\bullet (u\otimes v) \subset \big(M_x^1/M_l^1\big)\cdot v\,,$$

where the former follows from Lemma 2.2(2), and the latter from the definition (2.4) of $\bullet$. From this, we have $\dim\big[u\otimes y, M_l^d/M_{2l}^d\big]^\sharp \geqslant d-1$; thus $\operatorname{rank}[\ ,\ ]^\sharp \geqslant d-1$. From this and (4.11), we get

$$\operatorname{rank}[\ ,\ ] \geqslant d - 1 - 2\,s^{(2)}\ . \tag{4.12}$$

From (4.10) and (4.12), we have $\sigma \leqslant 1$, which contradicts the equality $\sigma \geqslant 4$. $\square$

4.2 Proof of Proposition 4.1(2)

We give a proof only for case (F); the proofs for cases (A2) and (B) are similar.

By associating with a general point y of $\mathcal{Y}$ the pair (x, l), where l is the line determined in case (F), we obtain a rational map $g\colon \mathcal{Y} \to \mathbb{P}^n \times \mathbb{G}_1$ (see § 2.2.1 for the notation $\mathbb{G}_1$). By blowing up $\mathcal{Y}$ $\mathrm{GL}(V)$-equivariantly and shrinking S' if necessary, we may assume that g is a morphism. The morphism g induces a map $g_*\colon T = \mathcal{T}_{\mathcal{Y}/\mathbb{P}^n}|_y \to T_l\mathbb{G}_1$ between tangent spaces.

Put $T' = T \cap M^d_{x\cup l}$. By Lemma 2.1(1), we have $\dim T/T' = d + 1 - \big(3\,s^{(3)} + 2\sigma - \varepsilon\big)$, where $\varepsilon := \sigma - \sigma^{(1)} \leqslant 1$.

By varying the general point $y \in \mathcal{Y}$, we obtain, over a Zariski open subset of $\mathcal{Y}$, a subbundle $\mathcal{T}'$ of the relative tangent bundle $\mathcal{T}_{\mathcal{Y}/\mathbb{P}^n}$ such that $\mathcal{T}'|_y = T'$.

Proposition 4.5. (1) *The subspace T' is in the kernel of the map $g_*\colon T \to T_l\mathbb{G}_1$.*
(2) *The subbundle $\mathcal{T}'$ is integrable.*

Let us show that Proposition 4.5 implies that case (F) does not occur.

Proof of "Proposition 4.2 ⇒ Proposition 4.1(2) (case (F))". Arguing as in the proof of "Proposition 4.2 ⇒ Proposition 4.1(6)," we can find a submanifold $L \subset (h \circ f)^{-1}(x_0)$ passing through a general point $y_0 \in \mathcal{Y}$ with $\mathcal{T}_L = \mathcal{T}'|_L$ and show that $L \subset \mathcal{Y}'_{y_0} := g^{-1}(x_0, l_0)$, where $x_0 = (h \circ h)(y_0)$ and $l_0 = g(y_0)$. Rewriting $\dim L \leqslant \dim \mathcal{Y}'_{y_0}$ using

$$\mathcal{Y}'_{y_0} = \dim S + k - \dim \mathbb{P}^n - \dim \mathbb{G}_1 = \dim S + k - 3n + 2$$

and

$$\begin{aligned}\dim L &= \dim S - n + k - (d+1) + 3\,s^{(3)} + 2\sigma - \varepsilon \\ &\geqslant \dim S - n + k - (d+1) + \frac{3}{2}(d-n) + \frac{\sigma}{2} - \varepsilon\,,\end{aligned}$$

we obtain

$$n + \frac{11}{5} < \frac{d+n}{2} \leqslant 3 + \varepsilon - \frac{\sigma}{2} \leqslant 4\,,$$

where the first inequality is a consequence of (3.5). This is absurd. (Recall that $n \geqslant 3$.) □

Denote by ϵ the composition $T \to M^d_x \to M^d_x / M^d_{x\cup l}$.

Lemma 4.6. *For sections τ_1 and τ_2 of $\mathcal{T}'$ defined over a neighborhood of $y \in \mathcal{Y}$, we have*

$$\epsilon([\tau_1, \tau_2]|_y) = g_*(\tau_1|_y) \bullet \overline{(\tau_2|_y)} - g_*(\tau_2|_y) \bullet \overline{(\tau_1|_y)}\,,$$

where $\overline{\tau_i|_y}$ is the image of $\tau_i|_y \in T' \subset M^d_{x\cup l}$ in $M^d_{x\cup l}/M^d_{x\cup 2l} \simeq M^d_l/M^d_{2l}$, and the pairing $\bullet$ is the one in (2.3).

Proof. The proof is similar to that of [Cle03, Lemma 3.1]. □

Lemma 4.7. *Put $T'' = T \cap M^d_{x\cup 2l}$; we have $T'' \subset \mathrm{Ker}(g_*\colon T \to T_l\mathbb{G}_1)$.*

Proof. Suppose $g_*(v) \neq 0$ for some $v \in T''$. The proof of the lemma is almost the same as that of Lemma 4.4. The main difference is that, in this case, the rank of the map

$$g_*(v)\bullet\colon M^d_l/M^d_{2l} \to M^d/M^d_l = \mathrm{H}^0(l, \mathcal{O}(d))$$

is d or $d+1$, by Lemma 2.2(1). Noting this difference and arguing as in the proof of Lemma 4.4, we obtain the equality $2\sigma - \varepsilon \leqslant 1$, which is absurd because $2\sigma - \varepsilon \geqslant 6$. □

Proof of Proposition 4.5(1). The proof is almost the same as that of Proposition 4.2(1). Noting the difference mentioned in the proof of Lemma 4.7 and arguing as in the proof of Proposition 4.2(1), we obtain the equality $2\sigma - \varepsilon \leqslant 3$, which is absurd again. □

4.3 Proof of Proposition 4.1(1), (3), (4) and (5)

In case (D), for a general point $y \in \mathcal{Y}$, we obtained a smooth conic C passing through x. In case (A3), as in Proposition 3.6, we put $C = l_1 \cup l_2$, where recall that l_1 passes through x. In case (C2), we denote by C the double line in Proposition 3.8. In case (E2), we put $C = l_1 \cup l_2$, where recall that $l_1 \cap l_2 = \{x\}$. In all cases, C is a (possibly singular) conic passing through x.

For cases (D), (E2) and (C2), by associating with a general point $y \in \mathcal{Y}$ the pair (x, C) (and the plane H on which C lies), we obtain rational maps $g\colon \mathcal{Y} \to \mathbb{L}_2$, $g\colon \mathcal{Y} \to \mathbb{L}_3$ and $g\colon \mathcal{Y} \to \mathbb{L}_4$, respectively. For case (A3), by associating with a general point $y \in \mathcal{Y}$ the triple (x, l_1, l_2) (and the plane H on which C lies), we obtain a rational map $g\colon \mathcal{Y} \to \mathbb{L}_5$.

By blowing up $\mathcal{Y}$ $\mathrm{GL}(V)$-equivariantly and shrinking S' if necessary, we may assume that g is a morphism. It induces a map

$$g_*\colon T = \mathcal{T}_{\mathcal{Y}/\mathbb{P}^n}|_y \to \mathcal{T}_{\mathbb{L}_\Box/\mathbb{P}^n}|_{(x,C)} = T_C \mathbb{G}^x_\Box$$

between tangent spaces for a general point $y \in \mathcal{Y}$, where $\Box$ is 2 in case (D), 3 in case (E2), 4 in case (C2) and 5 in case (A3).

Put $T' = T \cap M^d_C$. We have $\dim T/T' = 2d - \big(4\, s^{(4)} + 2\sigma - \varepsilon\big)$, where $\varepsilon = \sigma - \sigma^{(1)} \leqslant 1$. By varying the general point $y \in \mathcal{Y}$, we obtain, over a Zariski open subset of $\mathcal{Y}$, a subbundle $\mathcal{T}'$ of the relative tangent bundle $\mathcal{T}_{\mathcal{Y}/\mathbb{P}^n}$ such that $\mathcal{T}'|_y = T'$.

Denote by ϵ the composition

$$T \to M^d_x \to \mathrm{M}^d_x / M^d_C = \mathrm{H}^0(C, \mathcal{I}_x(d))\,.$$

If τ_1 and τ_2 are sections of $\mathcal{T}'$ defined over a neighborhood of $y \in \mathcal{Y}$, then $[\tau_1, \tau_2]$ is a section of $\mathcal{T}_{\mathcal{Y}/\mathbb{P}^n}$, so it makes sense to consider $\epsilon([\tau_1, \tau_2]|_y)$.

Lemma 4.8. *For sections τ_1 and τ_2 of $\mathcal{T}'$ defined over a neighborhood of $y \in \mathcal{Y}$, we have*

$$\epsilon([\tau_1, \tau_2]|_y) = \delta_{g_*(\tau_1|_y)}\big(\overline{\tau_2|_y}\big) - \delta_{g_*(\tau_2|_y)}\big(\overline{\tau_1|_y}\big)\,,$$

where $\overline{\tau_i|_y}$ is the image of $\tau_i|_y \in T' \subset M^d_C$ in M^d_C/M^d_{2C}, and the map $\delta_?$ is the one in (2.8).

Proof. Since $\mathbb{L}_3$, $\mathbb{L}_4$ and $\mathbb{L}_5^\circ$ are subvarieties of $\mathbb{L}_2$, we may only consider case (D). Imitating the proof of [Cle03, Lemma 3.1], we prove the lemma using local coordinates.

Let $\mathbf{y} = (y_1, y_2, \dots)$ be a local coordinate system of $(h \circ f)^{-1}(x)$ centered at y. Choose homogeneous coordinates $[z_0 : \cdots : z_n]$ of $\mathbb{P}^n$ in such a way that $x = [1 : 0 : \cdots : 0]$ and $H = \{z_3 = \cdots = z_n = 0\}$. We express the morphism g using the local coordinates as $g(\mathbf{y}) = (x, C(\mathbf{y}), H(\mathbf{y}))$. Let

$$\begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & b_{13}(\mathbf{y}) & \cdots & b_{1n}(\mathbf{y}) \\ 0 & 0 & 1 & b_{23}(\mathbf{y}) & \cdots & b_{2n}(\mathbf{y}) \end{bmatrix}$$

be the matrix such that the plane $H(\mathbf{y})$ (or, equivalently, 3-dimensional subspace of $\mathbb{C}^{n+1}$) is spanned by the rows. We take $[z_0 : z_1 : z_2]$ as homogeneous coordinates of $H(\mathbf{y})$ and let $c(\mathbf{y}; z_0, z_1, z_2) = \sum_{0 \leqslant i \leqslant j \leqslant 2} c_{ij}(\mathbf{y}) z_i z_j$ be the defining equation of $C(\mathbf{y})$ in $H(\mathbf{y})$. Let

$$\tilde{F}(\mathbf{y}) = \sum_I F_I(\mathbf{y}; z_0, z_1, z_2)(z_3 \dots z_n)^I$$

be the degree d polynomial corresponding to $(b\circ p_1)(\mathbf{y}) \in S$, where $I = (i_3, \dots, i_n)$ is a multi-index and $F_I(\mathbf{y}; z_0, z_1, z_2)$ is a degree $d - |I|$ homogeneous polynomial of z_0, z_1, z_2 with parameter $\mathbf{y}$.

Let $\tau_1 = \sum_k a_k\, \partial/\partial y_k$ and $\tau_2 = \sum_{k'} a'_{k'}\, \partial/\partial y_{k'}$ be the local expressions of τ_1 and τ_2. The condition that τ_1 is a section of $\mathcal{T}'$ is equivalent to

$$\sum_{k,I} a_k \frac{\partial}{\partial y_k} F_I(\mathbf{y}; z_0, z_1, z_2)(z_3 \dots z_n)^I \Big|_{C(\mathbf{y})} \equiv 0\,.$$

This means that there exists a degree $d-2$ homogeneous polynomial $D(\mathbf{y}; z_0, z_1, z_2)$ of z_0, z_1, z_2 with parameter $\mathbf{y}$ such that

$$\sum_{k,I=(i_3,\dots,i_n)} a_k \frac{\partial}{\partial y_k} F_I(\mathbf{y}; z_0, z_1, z_2) \prod_{3\leqslant l\leqslant n} (b_{1l}z_1 + b_{2l}z_2)^{i_l} = c(\mathbf{y}; z_0, z_1, z_2) D(\mathbf{y}; z_0, z_1, z_2)\,. \quad (4.13)$$

Likewise, for τ_2, we have, for some $D'(\mathbf{y}; z_0, z_1, z_2)$,

$$\sum_{k',I=(i_3,\dots,i_n)} a'_{k'} \frac{\partial}{\partial y_{k'}} F_I(\mathbf{y}; z_0, z_1, z_2) \prod_{3\leqslant l\leqslant n} (b_{1l}z_1 + b_{2l}z_2)^{i_l} = c(\mathbf{y}; z_0, z_1, z_2) D'(\mathbf{y}; z_0, z_1, z_2)\,. \quad (4.14)$$

Setting $\mathbf{y} = \mathbf{0}$ in (4.14), we get

$$\sum_{k'} a'_{k'}(\mathbf{0}) \frac{\partial}{\partial y_{k'}} F_O(\mathbf{0}; z_0, z_1, z_2) = c(\mathbf{0}; z_0, z_1, z_2) D'(\mathbf{0}; z_0, z_1, z_2)\,, \quad (4.15)$$

where $O = (0, \dots, 0)$.

Now we compute $\epsilon([\tau_1, \tau_2]|_y)$, which is the restriction of $[\tau_1, \tau_2]\tilde{F}|_{\mathbf{y}=\mathbf{0}}$ to $C = C(\mathbf{0})$,

$$\begin{aligned} \tau_1\tau_2\tilde{F} &= \tau_1 \left(\sum_{k',I} a'_{k'} \frac{\partial}{\partial y_{k'}} F_I(\mathbf{y}; z_0, z_1, z_2) \prod_{3\leqslant l\leqslant n} z_l^{i_l} \right) \\ &= \sum_{k,k',I} \left(a_k \frac{\partial a'_{k'}}{\partial y_k} \frac{\partial}{\partial y_{k'}} + a_k a'_{k'} \frac{\partial}{\partial y_k} \frac{\partial}{\partial y_{k'}} \right) F_I(\mathbf{y}; z_0, z_1, z_2) \prod_{3\leqslant l\leqslant n} z_l^{i_l}\,. \end{aligned}$$

Switching τ_1 and τ_2 in the above equation, we obtain the expression of $\tau_2\tau_1\tilde{F}$. Taking the difference, we obtain

$$[\tau_1, \tau_2]\tilde{F} = \sum_{k,k',I} \left(a_k \frac{\partial a'_{k'}}{\partial y_k} \frac{\partial}{\partial y_{k'}} - a'_{k'} \frac{\partial a_k}{\partial y_{k'}} \frac{\partial}{\partial y_k} \right) F_I(\mathbf{y}; z_0, z_1, z_2) \prod_{3\leqslant l\leqslant n} z_l^{i_l}\,. \quad (4.16)$$

We next compute $\delta_{g_*(\tau_1|_y)}(\overline{\tau_2|_y})$. Let $\pi\colon \oplus_{i=0}^n \mathbb{C}z_i \to U = \oplus_{i=0}^2 \mathbb{C}z_i$ be the natural projection and $\iota\colon R = \mathbb{C}\cdot c(\mathbf{0}; z_0, z_1, z_2) \hookrightarrow S^2U$ be the inclusion map. Let $\psi\colon R \to S^2U$ be the map defined by $\psi(c(\mathbf{0}; z_0, z_1, z_2)) = \tau_1 c(\mathbf{y}; z_0, z_1, z_2)\big|_{\mathbf{y}=\mathbf{0}}$, and let $\varphi\colon \oplus_{i=0}^n \mathbb{C}z_i \to U$ be the map defined by the matrix

$$\left.\begin{bmatrix} 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \tau_1 b_{13}(\mathbf{y}) & \cdots & \tau_1 b_{1n}(\mathbf{y}) \\ 0 & 0 & 0 & \tau_1 b_{23}(\mathbf{y}) & \cdots & \tau_1 b_{2n}(\mathbf{y}) \end{bmatrix}\right|_{\mathbf{y}=\mathbf{0}}.$$

Then the tangent vector $g_*(\tau_1|_y) \in T_{g(y)}\mathbb{G}_2^x$ corresponds to the first-order deformation defined by (φ, ψ) (see § 2.2.2).

We have

$$
\begin{aligned}
\tau_2|_y &= \tau_2 \tilde{F}|_{\mathbf{y}=\mathbf{0}} \\
&= \sum_{k', I\neq O} a'_{k'}(\mathbf{0}) \frac{\partial}{\partial y_{k'}} F_I(\mathbf{0}; z_0, z_1, z_2) \prod_{3\leqslant l\leqslant n} z_l^{i_l} + \sum_{k'} a'_{k'}(\mathbf{0}) \frac{\partial}{\partial y_{k'}} F_O(\mathbf{0}; z_0, z_1, z_2) \\
&= \sum_{k', I\neq O} a'_{k'}(\mathbf{0}) \frac{\partial}{\partial y_{k'}} F_I(\mathbf{0}; z_0, z_1, z_2) \prod_{3\leqslant l\leqslant n} z_l^{i_l} - c(\mathbf{0}; z_0, z_1, z_2) D'(\mathbf{0}; z_0, z_1, z_2)\,,
\end{aligned}
$$

where we used (4.15) in the last equality. Using (2.6), we see that

$$
\delta_{g_*(\tau_1|_y)}\left(\overline{\tau_2|_y}\right) \in S^d U / c(\mathbf{0}; z_0, z_1, z_2) \cdot S^{d-2} U
$$

is represented by

$$
-\sum_{\substack{k' \\ 3\leqslant l\leqslant n \\ 1\leqslant i\leqslant 2}} a'_{k'}(\mathbf{0}) \frac{\partial}{\partial y_{k'}} F_l(\mathbf{0}; z_0, z_1, z_2)(\tau_1 b_{il})(\mathbf{0}) z_i + (\tau_1 c)(\mathbf{0}; z_0, z_1, z_2) D'(\mathbf{0}; z_0, z_1, z_2),
$$

where F_l stands for $F_{(0,\ldots,0,1,0,\ldots,0)}$, with the lth component equal to 1. Therefore, $\delta_{g_*(\tau_1|_y)}(\overline{\tau_2|_y}) - \delta_{g_*(\tau_2|_y)}(\overline{\tau_1|_y})$ is represented by

$$
\begin{aligned}
&-\sum_{\substack{k' \\ 3\leqslant l\leqslant n \\ 1\leqslant i\leqslant 2}} a'_{k'}(\mathbf{0}) \frac{\partial}{\partial y_{k'}} F_l(\mathbf{0}; z_0, z_1, z_2)(\tau_1 b_{il})(\mathbf{0}) z_i + (\tau_1 c)(\mathbf{0}; z_0, z_1, z_2) D'(\mathbf{0}; z_0, z_1, z_2) \\
&+\sum_{\substack{k \\ 3\leqslant l\leqslant n \\ 1\leqslant i\leqslant 2}} a_k(\mathbf{0}) \frac{\partial}{\partial y_k} F_l(\mathbf{0}; z_0, z_1, z_2)(\tau_2 b_{il})(\mathbf{0}) z_i + (\tau_2 c)(\mathbf{0}; z_0, z_1, z_2) D(\mathbf{0}; z_0, z_1, z_2)\,.
\end{aligned}
\tag{4.17}
$$

Applying τ_2 and τ_1 to (4.13) and (4.14), respectively, and setting $\mathbf{y} = \mathbf{0}$ and subtracting the resulting equations, we can see that (4.17) is equal to

$$
\begin{aligned}
\sum_{k,k'} \left(a_k(\mathbf{0}) \frac{\partial a'_{k'}}{\partial y_k}(\mathbf{0}) \frac{\partial}{\partial y_{k'}} - a'_{k'}(\mathbf{0}) \frac{\partial a_k}{\partial y_{k'}}(\mathbf{0}) \frac{\partial}{\partial y_k} \right) F_O(\mathbf{0}; z_0, z_1, z_2) \\
+ c(\mathbf{0}; z_0, z_1, z_2)(\tau_2 D - \tau_1 D')(\mathbf{0}; z_0, z_1, z_2)\,.
\end{aligned}
\tag{4.18}
$$

By setting $\mathbf{y} = \mathbf{0}$ in (4.16), we see that $[\tau_1, \tau_2]\tilde{F}|_{\mathbf{y}=\mathbf{0}}$ coincides with (4.18) modulo $\mathrm{H}^0(\mathbb{P}^n, \mathcal{I}_C(d))$. This proves the lemma. □

Proposition 4.9. (1) *The subspace T' is in the kernel of the map $g_*\colon T \to T_C \mathbb{G}_2^x$.*
(2) *The subbundle $\mathcal{T}'$ is integrable; that is, $[\mathcal{T}', \mathcal{T}'] \subset \mathcal{T}'$.*

To prove the proposition, we need a lemma.

Lemma 4.10. *Put $T'' = T \cap M_{2C}^d$; we have $T'' \subset \mathrm{Ker}(g_*\colon T \to T_C \mathbb{G}_2^x)$.*

Proof. Suppose $g_*(v) \neq 0$ for some $v \in T''$. We will derive a contradiction.

Let K be the kernel of the map

$$
\delta_{g_*(v)}\colon M_C^d / M_{2C}^d \to \mathrm{M}_x^d / M_C^d = \mathrm{H}^0(C, \mathcal{I}_x(d))\,,
$$

and put

$$
\gamma = \dim \frac{T'/T'' + K}{T'/T''}\,.
$$

By Lemma 4.8, we have $\delta_{g_*(v)}(T'/T'') \subset T/T'$. We have

$$\begin{aligned}\dim \delta_{g_*(v)}(T'/T'') &= \dim T'/T'' - \dim K + \gamma \\ &= \dim T'/T'' - \dim M_C^d/M_{2C}^d + \operatorname{rank} \delta_{g_*(v)} + \gamma \\ &= \operatorname{rank} \delta_{g_*(v)} - \dim \frac{M_C^d/M_{2C}^d}{T'/T''+K}\,.\end{aligned} \tag{4.19}$$

As in the case of equation (4.5), we can show that

$$M_C^d = T' + \sum_{i=1}^{s^{(4)}} M_C^2 \cdot Q_i\,. \tag{4.20}$$

In the rest of the proof of this lemma, we consider each case separately.

Case (D). Arguing as for Claim 4.4.1, we obtain the inequality

$$\dim \frac{M_C^d/M_{2C}^d}{T'/T''+K} \leqslant 4s^{(4)}\,. \tag{4.21}$$

By Lemma 2.3, we have $\operatorname{rank} \delta_{g_*(v)} \geqslant 2d-3$. From these and (4.19), we obtain

$$2d - (4\,s^{(4)} + 2\sigma - \varepsilon) = \dim T/T' \geqslant \dim \delta_{g_*(v)}(T'/T'') \geqslant 2d - 3 - 4\,s^{(4)}\,.$$

So we have $2\sigma - \varepsilon \leqslant 3$. But we have $2\sigma - \varepsilon \geqslant 8$ because $s^{(3)} - s^{(4)} \geqslant 2$. This gives a contradiction.

Case (E2). We consider two cases (see § 2.2.3 for the notation E_{3,l_i}):
(i) $g_*(v) \notin E_{3,l_1} \cup E_{3,l_2}$ and (ii) $g_*(v) \in E_{3,l_1} \cup E_{3,l_2}$.

In case (i), arguing as for Claim 4.4.1, we obtain $\dim \big(M_C^d/M_{2C}^d\big)/\big(T'/T''+K\big) \leqslant 4s^{(4)}$. By Lemma 2.4, we have $\operatorname{rank} \delta_{g_*(v)} \geqslant 2d-3$. Arguing as in case (D), we have $2\sigma - \varepsilon \leqslant 3$, which contradicts the inequality $2\sigma - 3 \geqslant 8$.

Next we consider case (ii): say, $g_*(v) \in E_{3,l_1}$. Arguing as for Claim 4.4.1, we obtain the inequality $\dim \big(M_C^d/M_{2C}^d\big)/\big(T'/T''+K\big) \leqslant 2s^{(4)}$. By Lemma 2.4, we have $\operatorname{rank} \delta_{g_*(v)} \geqslant d-1$ and $\delta_{g_*(v)}(T'/T'') \subset T/T' \cap \mathrm{H}^0(C, \mathcal{I}_{l_1}(d))$. From these and (4.19), using the inequality (3.18), we obtain $\sigma - \varepsilon \leqslant 1$, again giving a contradiction.

Case (C2). We consider two cases (see § 2.2.4 for the notation $E_{4,l}$ and $E_{4,H}$):
(i) $g_*(v) \notin E_{4,l} \cup E_{4,H}$ and (ii) $g_*(v) \in E_{4,l} \cup E_{4,H}$.

In case (i), arguing as for Claim 4.4.1, we obtain $\dim \big(M_C^d/M_{2C}^d\big)/\big(T'/T''+K\big) \leqslant 4s^{(4)}$. By Lemma 2.5, we have $\operatorname{rank} \delta_{g_*(v)} \geqslant 2d-1$. Arguing as in case (D), we have $2\sigma - \varepsilon \leqslant 1$, which contradicts the inequality $2\sigma - 3 \geqslant 8$.

In case (ii), we obtain $\dim \big(M_C^d/M_{2C}^d\big)/\big(T'/T''+K\big) \leqslant 2s^{(4)}$. By Lemma 2.5, we have $\operatorname{rank} \delta_{g_*(v)} \geqslant d-1$ and $\delta_{g_*(v)}(T'/T'') \subset T/T' \cap \mathrm{H}^0(C, \mathcal{I}_l(d))$. From these and (4.19), using the inequality (3.15), we obtain $\sigma - \varepsilon \leqslant 1$, again giving a contradiction.

Case (A3). There are three cases (see § 2.2.5 for the notation E_{5,l_i}):
(i) $g_*(v) \notin E_{5,l_1} \cup E_{5,l_2}$, (ii) $g_*(v) \in E_{5,l_1}$ and (iii) $g_*(v) \in E_{5,l_2}$.

In case (i), arguing as for Claim 4.4.1, we obtain $\dim \big(M_C^d/M_{2C}^d\big)/\big(T'/T''+K\big) \leqslant 4s^{(4)}$. By Lemma 2.4, we have $\operatorname{rank} \delta_{g_*(v)} \geqslant 2d-3$. Arguing as in case (D), we have $2\sigma - \varepsilon \leqslant 3$, which contradicts the inequality $2\sigma - 3 \geqslant 8$.

In case (ii), we obtain $\dim \big(M_C^d/M_{2C}^d\big)/\big(T'/T''+K\big) \leqslant 2s^{(4)}$. By Lemma 2.6, we have $\operatorname{rank} \delta_{g_*(v)} \geqslant d-1$ and $\delta_{g_*(v)}(T'/T'') \subset T/T' \cap \mathrm{H}^0(C, \mathcal{I}_{l_1}(d))$. From these and (4.19), using the inequality (3.13), we obtain $\sigma - \varepsilon \leqslant 1$, again giving a contradiction.

In case (iii), we obtain $\dim\big(M_C^d/M_{2C}^d\big)/\big(T'/T''+K\big) \leqslant s^{(4)}$. By Lemma 2.6, we have $\operatorname{rank}\delta_{g_*(v)} = d-1$ and $\delta_{g_*(v)}(T'/T'') \subset T/T' \cap \mathrm{H}^0(C, \mathcal{I}_{x\cup l_2}(d))$. From these and (4.19), using the inequality (3.14), we obtain $s^{(0)} - s^{(4)} \leqslant 3$, which contradicts Lemma 3.3(2). This is the end of the proof of Lemma 4.10. □

Proof of Proposition 4.9. By Lemma 4.8, it suffices to show Proposition 4.9(1). The proof is similar to that of Proposition 4.2(1).

By Lemma 4.10, the map $g_*|_{T'}$ induces a map $\overline{g_*}\colon T'/T'' \to T_C\mathbb{G}_\square^x$. Consider the map

$$[\ ,\ ]\colon T'/T'' \otimes T'/T'' \to M_x^d/M_C^d$$

defined by

$$[v, v'] = \delta_{\overline{g_*}(v)}(v') - \delta_{\overline{g_*}(v')}(v)\,. \tag{4.22}$$

Denote by c the codimension of T'/T'' in M_C^d/M_{2C}^d. By (4.20), there are $D_i \in M_C^d$, $1 \leqslant i \leqslant c$, such that

$$M_C^d/M_{2C}^d = T'/T'' \oplus \bigoplus_{i=1}^{c} \mathbb{C}\overline{D_i}$$

and that each D_i is an element of $M_C^2 \cdot Q_j$ for some $1 \leqslant j \leqslant s^{(2)}$.

We extend $\overline{g_*}$ to the map $M_C^d/M_{2C}^d \to T_C\mathbb{G}_\square^x$ by defining $\overline{g_*}\big(\overline{D_i}\big) = 0$, and we define the map

$$[\ ,\ ]^\sharp\colon M_C^d/M_{2C}^d \otimes M_C^d/M_{2C}^d \to M_x^d/M_C^d$$

by the same equation as (4.22). Then we have

$$\operatorname{Im}[\ ,\ ]^\sharp = \operatorname{Im}[\ ,\ ] + \sum_{i=1}^{c} \delta_{\overline{g_*}(T'/T'')}\big(\overline{D_i}\big)\,.$$

Now suppose that $\overline{g_*}$ is not the zero map. We will derive a contradiction. We consider each case separately.

Case (D). Since $\delta_{\overline{g_*}(T'/T'')}\big(\overline{D_i}\big) \subset M_x^2/M_C^2 \cdot \overline{Q_j}$ for some $1 \leqslant j \leqslant s^{(2)}$, we have

$$\operatorname{rank}[\ ,\ ]^\sharp \leqslant \operatorname{rank}[\ ,\ ] + 4s^{(4)}\,. \tag{4.23}$$

Since $\operatorname{Im}[\ ,\ ] \subset T/T'$, we have

$$\operatorname{rank}[\ ,\ ] \leqslant 2d - \big(4\,s^{(4)} + 2\sigma - \varepsilon\big)\,. \tag{4.24}$$

Since $M_C^2 \cdot M^{d-2} = M_C^d$, we have $\overline{g_*}(\overline{u\cdot v}) \neq 0$ for some $u \in M_C^2$ and $v \in M^{d-2}$.

The dimension of $\delta_{\overline{g_*}(\overline{u\cdot v})}\big(M_C^d/M_{2C}^d\big)$ is at least $2d-3$ by Lemma 2.3, and we have the inclusion $\delta_{\overline{g_*}(M_C^d/M_{2C}^d)}(\overline{u\cdot v}) \subset M_x^2/M_C^2 \cdot \overline{v}$. From this, we infer that $\dim\big[\overline{u\cdot v}, M_C^d/M_{2C}^d\big]^\sharp \geqslant 2d-7$; thus

$$\operatorname{rank}[\ ,\ ]^\sharp \geqslant 2d - 7\,. \tag{4.25}$$

Combining (4.24), (4.23) and (4.25), we obtain $2\sigma - \varepsilon \leqslant 7$. This gives a contradiction because we have $2\sigma - \varepsilon \geqslant 8$, which follows from $s^{(3)} - s^{(4)} \geqslant 2$.

Case (E2). We consider two cases:
(i) $\operatorname{Im}\overline{g_*} \not\subset E_{3,l_1} \cup E_{3,l_2}$ and (ii) $\operatorname{Im}\overline{g_*} \subset E_{3,l_k}$ for $k = 1$ or 2.

We have $\delta_{\overline{g_*}(T'/T'')}\big(\overline{D_i}\big) \subset M_x^2/M_C^2 \cdot \overline{Q_j}$ in case (i) and $\delta_{\overline{g_*}(T'/T'')}\big(\overline{D_i}\big) \subset M_{l_k}^2/M_C^2 \cdot \overline{Q_j}$ in case (ii) for some $1 \leqslant j \leqslant s^{(2)}$. Thus we have

$$\operatorname{rank}[\ ,\ ]^\sharp \leqslant \begin{cases} \operatorname{rank}[\ ,\ ] + 4\, s^{(4)} & \text{in case (i)}\,, \\ \operatorname{rank}[\ ,\ ] + 2\, s^{(4)} & \text{in case (ii)}\,. \end{cases}$$

Since $\operatorname{Im}[\ ,\ ] \subset T/T'$ in case (i) and $\operatorname{Im}[\ ,\ ] \subset T/T' \cap \mathrm{H}^0(C, \mathcal{I}_{l_k}(d))$, we have by (3.18)

$$\operatorname{rank}[\ ,\ ] \leqslant \begin{cases} 2d - \left(4\, s^{(4)} + 2\sigma - \varepsilon\right) & \text{in case (i)}\,, \\ d - \left(2\, s^{(4)} + \sigma - \varepsilon\right) & \text{in case (ii)}\,. \end{cases}$$

Choose $u \in M_C^2$ and $v \in M^{d-2}$ with $\overline{g_*}(\overline{u \cdot v}) \neq 0$. In case (i), we choose such u and v with $\overline{g_*}(\overline{u \cdot v}) \notin E_{3,l_1} \cup E_{3,l_2}$. The dimension of $\delta_{\overline{g_*}(\overline{u \cdot v})}\left(M_C^d/M_{2C}^d\right)$ is at least $2d-3$ in case (i) and at least $d-1$ in case (ii), by Lemma 2.4. We have $\delta_{\overline{g_*}(M_C^d/M_{2C}^d)}(\overline{u \cdot v}) \subset M_x^2/M_C^2 \cdot \overline{v}$ in case (i) and $\delta_{\overline{g_*}(M_C^d/M_{2C}^d)}(\overline{u \cdot v}) \subset M_{l_k}^2/M_C^2 \cdot \overline{v}$ in case (ii). From this, we infer that $\dim \left[\overline{u \cdot v}, M_C^d/M_{2C}^d\right]^\sharp$ is at least $2d-7$ in case (i) and at least $d-3$ in case (ii); thus

$$\operatorname{rank}[\ ,\ ]^\sharp \geqslant \begin{cases} 2d - 7 & \text{in case (i)}\,, \\ d - 3 & \text{in case (ii)}\,. \end{cases}$$

Combining these inequalities, we obtain $2\sigma - \varepsilon \leqslant 7$ in case (i) and $\sigma - \varepsilon \leqslant 3$ in case (ii). In any case, we reach a contradiction because $2\sigma - \varepsilon \geqslant 8$.

Case (C2). We consider two cases:
(i) $\operatorname{Im} \overline{g_*} \not\subset E_{4,l} \cup E_{4,H}$ and (ii) $\operatorname{Im} \overline{g_*} \subset E_{4,l} \cup E_{4,H}$.

Arguing as in the previous case, we obtain $2\sigma - \varepsilon \leqslant 5$ in case (i) and $\sigma - \varepsilon \leqslant 3$ in case (ii). Again this gives a contradiction.

Case (A3). There are three cases:
(i) $\operatorname{Im} \overline{g_*} \not\subset E_{5,l_1} \cup E_{5,l_2}$, (ii) $\operatorname{Im} \overline{g_*} \subset E_{5,l_1}$ and (iii) $\operatorname{Im} \overline{g_*} \subset E_{5,l_2}$.

Arguing as in case (E2), we have

$$\operatorname{rank}[\ ,\ ]^\sharp \leqslant \begin{cases} \operatorname{rank}[\ ,\ ] + 4\, s^{(4)} & \text{in case (i)}\,, \\ \operatorname{rank}[\ ,\ ] + 2\, s^{(4)} & \text{in case (ii)}\,, \\ \operatorname{rank}[\ ,\ ] + s^{(4)} & \text{in case (iii)}\,, \end{cases}$$

$$\operatorname{rank}[\ ,\ ] \leqslant \begin{cases} 2d - \left(4\, s^{(4)} + 2\sigma - \varepsilon\right) & \text{in case (i)}\,, \\ d - \left(2\, s^{(4)} + \sigma - \varepsilon\right) & \text{in case (ii)}\,, \\ d - s^{(0)} + 2 & \text{in case (iii)}\,, \end{cases}$$

$$\operatorname{rank}[\ ,\ ]^\sharp \geqslant \begin{cases} 2d - 7 & \text{in case (i)}\,, \\ d - 3 & \text{in case (ii)}\,, \\ d - 2 & \text{in case (iii)}\,. \end{cases}$$

Combining these, we obtain $2\sigma - \varepsilon \leqslant 7$ in case (i), $\sigma - \varepsilon \leqslant 3$ in case (ii) and $s^{(0)} - s^{(4)} \leqslant 4$ in case (iii). These are incompatible with the inequality $2\sigma - \varepsilon \geqslant 8$ and with Lemma 3.3(2).

This is the end of proof of Proposition 4.9. □

Now let us consider the final step of the proof of Proposition 4.1(1), (3), (4) and (5). We argue as in the proof of "Proposition 4.2 ⇒ Proposition 4.1(6)."

Pick a general point y_0 in $\mathcal{Y}$, and let $\mathcal{Y}'_{y_0} := g^{-1}(g(y_0))$. Consider the map $\rho\colon \mathcal{Y}'_{y_0} \to \mathrm{H}^0(C_0, \mathcal{O}(d))$ defined by $\rho(y) = F_y|_{C_0}$, where F_y is the degree d polynomial corresponding to

$b(p_1(y)) \in S$, and $g(y_0) = (x_0, C_0)$.

We have

$$\dim \mathcal{Y}'_{y_0} = \begin{cases} \dim S + k - 3n & \text{in case (D)}\,, \\ \dim S + k - 3n + 2 & \text{in case (E2)}\,, \\ \dim S + k - 3n + 3 & \text{in case (C2)}\,, \\ \dim S + k - 3n + 1 & \text{in case (A3)}\,. \end{cases}$$

We put $a = 0$ in case (D), $a = 2$ in case (E2), $a = 3$ in case (C2) and $a = 1$ in case (A3).

We have $\dim T' = \dim S - n + k - 2d + \big(4\, s^{(4)} + 2\sigma - \varepsilon\big)$. Arguing as in the proof of "Proposition 4.2 $\Rightarrow$ Proposition 4.1(6)," we have

$$\dim \operatorname{Im} \rho \leqslant \dim \mathcal{Y}'_{y_0} - \dim T' = 2d - 2n - \big(4\, s^{(4)} + 2\sigma - \varepsilon\big) + a \leqslant \varepsilon + a\,, \tag{4.26}$$

where we used (3.12) in the last inequality.

If $G \subset \mathrm{GL}(V)$ denotes the subgroup consisting of automorphisms of V whose induced automorphisms of $\mathbb{P}^n = \mathbb{P}(V)$ fix x and the conic C_0, then the orbit $G \cdot \rho(y_0)$ is in $\operatorname{Im} \rho$. This fact together with the inequality (4.26) and Lemma 2.8 proves the proposition for cases (D), (E2) and (C2). To complete the proof for case (A3), we need to show that case (c) of Lemma 2.8(4) does not occur in our situation. We devote the rest of the proof to showing this.

Supposing that case (c) of Lemma 2.8(4) occurs in case (A3), we will derive a contradiction. By the last sentence of Lemma 2.8(4), this occurs only if $\varepsilon = 1$ and the inequalities in (4.26) are equalities; thus we have

$$\varepsilon = 1 \quad \text{and} \quad d - n = 2\, s^{(4)} + \sigma\,. \tag{4.27}$$

Recall from § 2.3.2 that $\mathbb{L}_6$ parametrizes triples (x, l, H), where H is a plane in $\mathbb{P}^n$, x is a point of H and l is a line on H, and that $q_6 \colon \mathbb{L}_6 \to \mathbb{P}^n$ is a projection. Let $u \colon \mathbb{L}_5 \to \mathbb{L}_6$ be the morphism given by $(x, l_1, l_2, H) \mapsto (x, l_2, H)$. Define the subvariety $\Delta \subset \mathbb{L}_6 \times S$ by

$$\Delta = \big\{((x, l, H), s) \mid x \cup l \subset p^{-1}(s)\big\}\,,$$

where recall that $S = \mathrm{H}^0(\mathbb{P}^n, \mathcal{O}(d)) \setminus \{0\}$ and $p \colon \mathcal{X} \to S$ is the projection of the universal family of hypersurfaces to S, so $p^{-1}(s)$ is the hypersurface of $\mathbb{P}^n$ defined by the homogeneous polynomial corresponding to s. Likewise, define

$$\mathbb{L}_6 \times S' \supset \Delta' := \big\{((x, l, H), s') \mid x \cup l \subset p'^{-1}(s)\big\}\,,$$

and let $r \colon \Delta' \to \mathbb{L}_6$ be the projection.

As we are now assuming that case (c) of Lemma 2.8(4) occurs, by associating $((x, l_2, H), p_1(y))$ with a general point $y \in \mathcal{Y}$, where $x = (h \circ f)(y)$, we obtain a rational map $g_1 \colon \mathcal{Y} \to \Delta'$. By blowing up $\mathcal{Y}$ $\mathrm{GL}(V)$-equivariantly and shrinking S' if necessary, we may assume that g_1 is a morphism.

$$\begin{tikzcd}
& & & \mathbb{L}_5 \arrow[d, "u"] \\
& \Delta' \arrow[r] \arrow[d] \arrow[rr, bend left, "r"] & \Delta \arrow[r] \arrow[d] & \mathbb{L}_6 \arrow[d, "q_6"] \\
\mathcal{Y} \arrow[r, "f"] \arrow[ur, "g_1"] \arrow[dr, "p_1"'] \arrow[uurrr, bend left, "g"] & \mathcal{X}' \arrow[r] \arrow[d, "p'"] \arrow[rr, bend left, "h"] & \mathcal{X} \arrow[r] \arrow[d, "p"] & \mathbb{P}^n \\
& S' \arrow[r, "b"'] & S &
\end{tikzcd}$$

Giving a point (x, l_1, l_2, H) of $\mathbb{L}_5$ is the same as giving a quadruple

$$\left(V \xrightarrow{\pi} U,\ U \xrightarrow{\pi'} A,\ U \xrightarrow{\tau_1} W_1,\ U \xrightarrow{\tau_2} W_2\right),$$

where U, A, W_i are vector spaces of dimension 3, 1, 2, respectively, and π, π', τ_i are surjective linear maps such that there is a linear map $\pi''\colon W_1 \to A$ with $\pi' = \pi'' \circ \tau_1$. The correspondence is given by $H = \mathbb{P}(U)$, $x = \mathbb{P}(A)$ and $l_i = \mathbb{P}(W_i)$. Denote by

$$\left(V \otimes \mathcal{O}_{\mathbb{L}_5} \xrightarrow{\pi} \mathcal{U},\ \mathcal{U} \xrightarrow{\pi'} \mathcal{A},\ \mathcal{U} \xrightarrow{\tau_1} \mathcal{W}_1,\ \mathcal{U} \xrightarrow{\tau_2} \mathcal{W}_2\right) \tag{4.28}$$

the universal family of such quadruples, where $\mathcal{U}$, $\mathcal{A}$, $\mathcal{W}_i$ are vector bundles on $\mathbb{L}_5$. Likewise, we have universal objects

$$\left(V \otimes \mathcal{O}_{\mathbb{L}_6} \xrightarrow{\pi} \mathcal{U},\ \mathcal{U} \xrightarrow{\pi'} \mathcal{A},\ \mathcal{U} \xrightarrow{\tau_2} \mathcal{W}_2\right) \tag{4.29}$$

on $\mathbb{L}_6$, where same objects appear in (4.28) and (4.29), but objects in (4.28) are pull-backs of the corresponding objects in (4.29) via the morphism $u\colon \mathbb{L}_5 \to \mathbb{L}_6$.

By the definition of Δ, we have $\Delta = \mathbf{V}\left(\mathcal{R}_d^\vee\right) \setminus (\text{zero section})$. One can show that the canonical line bundle K_Δ is isomorphic to the pull-back to Δ of the line bundle

$$(\det \mathcal{U})^{-n+3} \otimes \mathcal{A}^{d-4} \otimes (\det \mathcal{W}_2)^{\frac{1}{2}d(d+1)-4}$$

on $\mathbb{L}_6$. Since the morphism $\Delta' \to \Delta$ is étale, we have

$$K_{\Delta'} = r^*\left((\det \mathcal{U})^{-n+3} \otimes \mathcal{A}^{d-4} \otimes (\det \mathcal{W}_2)^{\frac{1}{2}d(d+1)-4}\right). \tag{4.30}$$

The normal sheaf $\mathcal{N}_{g_1}$ of the morphism g_1 is defined by the exact sequence

$$0 \to \mathcal{T}_{\mathcal{Y}} \to g_1^*\mathcal{T}_{\Delta'} \to \mathcal{N}_{g_1} \to 0.$$

For a general point $s' \in S'$, the sheaf $\mathcal{N}_{g_1}|_{\mathcal{Y}_{s'}}$ is isomorphic to the normal sheaf of $g_{1s'}\colon \mathcal{Y}_{s'} \to \Delta'_{s'}$, where the subscript s' means that we take the fibers over s'. The morphism $r\colon \Delta' \to \mathbb{L}_6$ is smooth, and the relative tangent bundle $\mathcal{T}_{\Delta'/\mathbb{L}_6}$ is isomorphic to $r^*\mathcal{R}_d$. Since $\mathcal{Y}$ is smooth over $\mathbb{L}_6^\circ$, we have, over the open subset $(r \circ g_1)^{-1}(\mathbb{L}_6^\circ)$, an exact sequence

$$0 \to \mathcal{T}_{\mathcal{Y}/\mathbb{L}_6} \to g_1^*\mathcal{T}_{\Delta'/\mathbb{L}_6} \to \mathcal{N}_{g_1} \to 0.$$

For a general point $y \in \mathcal{Y}$, we put $T^\natural := \mathcal{T}_{\mathcal{Y}/\mathbb{L}_6}|_y$, which we regard as a subspace of $M^d_{x\cup l_2}$ by the injection

$$T^\natural = \mathcal{T}_{\mathcal{Y}/\mathbb{L}_6}|_y \hookrightarrow g_1^*\mathcal{T}_{\Delta'/\mathbb{L}_6}|_y = (r \circ g_1)^*\mathcal{R}_d|_y = M^d_{x\cup l_2}.$$

By Proposition 4.9, we have $T' \subset T^\natural$. Thus, we have

$$\begin{array}{ccc} T^\natural & \subset & M^d_{x\cup l_2} \\ \cup & & \cup \\ T' & \subset & M^d_{l_1\cup l_2}. \end{array}$$

Using (4.27), one can compute dimensions:

$$\dim T^\natural/T' = 3 \quad \text{and} \quad \dim M^d_{l_1\cup l_2}/T' = 3n - k - 2d.$$

Put $\lambda = 3n - k - 2d$. Choose general members $P_1, \ldots, P_\lambda \in M^{d-2}$ and a general point $s' \in S'$. Let ψ: $(r \circ g_1)^*\mathcal{R}_d|_{\mathcal{Y}_{s'}} \to \mathcal{N}_{g_1}|_{\mathcal{Y}_{s'}}$ be the restriction to the fiber $\mathcal{Y}_{s'}$ of the composition

$$(r \circ g_1)^*\mathcal{R}_d \simeq g_1^*\mathcal{T}_{\Delta'/\mathbb{L}_6} \hookrightarrow g_1^*\mathcal{T}_{\Delta'} \to \mathcal{N}_{g_1},$$

and define a morphism

$$\theta\colon ((r\circ g_1)^*\mathcal{R}_2|_{\mathcal{Y}_{s'}})^{\oplus\lambda} \to \mathcal{N}_{g_1}|_{\mathcal{Y}_{s'}}$$

by $(\sigma_j)_{1\leqslant j\leqslant\lambda} \mapsto \sum_{1\leqslant j\leqslant\lambda}\psi(\sigma_j\cdot P_j)$.

Let $\mathcal{M}$ be the line bundle on $\mathbb{L}_5$ whose fiber at $(x,l_1,l_2,H)\in\mathbb{L}_5$ is $M^2_{x\cup l_2}/M^2_{l_1\cup l_2}$. There is a natural surjection $u^*\mathcal{R}_2\to\mathcal{M}$. Pulling this back to $\mathcal{Y}_{s'}$ via $g' := g|_{\mathcal{Y}_{s'}}\colon\mathcal{Y}_{s'}\to\mathbb{L}_5$, we obtain a surjection $\varphi\colon (r\circ g_1)^*\mathcal{R}_2|_{\mathcal{Y}_{s'}}\to (g')^*\mathcal{M}$.

Claim 4.10.1. *For general $P\in M^{d-2}$, the composition*

$$(r\circ g_1)^*\mathcal{R}_2|_{\mathcal{Y}_{s'}} \xrightarrow{\cdot P} (r\circ g_1)^*\mathcal{R}_d|_{\mathcal{Y}_{s'}} \xrightarrow{\psi} \mathcal{N}_{g_1}|_{\mathcal{Y}_{s'}} \to \operatorname{Coker}\theta/(\mathit{torsion}) \tag{4.31}$$

factors through the morphism φ; that is, there exists a morphism $\xi_P\colon (g')^*\mathcal{M}\to\operatorname{Coker}\theta/(\mathit{torsion})$ such that the composed morphism (4.31) is $\xi_P\circ\varphi$.

Proof of Claim 4.10.1. For a general point $y\in\mathcal{Y}_{s'}$, we have $M^d_{l_1\cup l_2}\subset T^\natural+\sum_{i=1}^{\lambda}M^2_{x\cup l_2}\cdot P_i$ because $\dim M^d_{l_1\cup l_2}/T'=\lambda$. It follows from this that the composed morphism (4.31) factors through φ over a non-empty open subset of $\mathcal{Y}_{s'}$. Since $\operatorname{Coker}\theta/(\text{torsion})$ is torsion-free, it factors through φ over $\mathcal{Y}_{s'}$. □

We have

$$\dim\left(T^\natural+\sum_{i=1}^{\lambda}M^2_{x\cup l_2}\cdot P_i\right)\Big/M^d_{l_1\cup l_2} = 3+\lambda < d-1 = \dim M^d_{x\cup l_2}/M^d_{l_1\cup l_2}\,,$$

where the inequality is a consequence of the numerical assumption (3.5), so we have $\operatorname{rank}\operatorname{Coker}\theta = d-\lambda-4$. Choose general $P_{\lambda+1},\dots,P_{d-4}\in M^{d-2}$, and consider the map

$$\xi=\sum_{i=\lambda+1}^{d-4}\xi_{P_i}\colon ((g')^*\mathcal{M})^{\oplus d-\lambda-4}\to\operatorname{Coker}\theta/(\text{torsion})$$

(see the statement of Claim 4.10.1 for the notation ξ_{P_i}). Then ξ is an injective morphism with torsion cokernel.

Now we deduce the "positivity" of $\mathcal{N}_{g_1}|_{\mathcal{Y}_{s'}}$. Since ξ is injective with torsion cokernel, we have

$$\begin{aligned}\det\operatorname{Coker}\theta &\geqslant \det\operatorname{Coker}\theta/(\text{torsion})\\ &\geqslant (g')^*\mathcal{M}^{d-\lambda-4}\\ &= (g')^*\big(\det\mathcal{U}\otimes\mathcal{A}^{-1}\otimes\det\mathcal{W}_1\otimes(\det\mathcal{W}_2)^{-1}\big)^{d-\lambda-4}\,,\end{aligned}$$

where the last equality follows from the isomorphism

$$\mathcal{M}\simeq\det\mathcal{U}\otimes\mathcal{A}^{-1}\otimes\det\mathcal{W}_1\otimes(\det\mathcal{W}_2)^{-1}\,,$$

which is easy to see. It follows from the last sentence of § 2.3.2 that

$$\det\operatorname{Im}\theta\geqslant (g')^*(\det\mathcal{V}_2)^{-\lambda}\,.$$

One can see that $\det\mathcal{V}_2\simeq\det\mathcal{U}\otimes\mathcal{A}\otimes(\det\mathcal{W}_2)^2$, so we have

$$\det\operatorname{Im}\theta\geqslant (g')^*\big(\det\mathcal{U}\otimes\mathcal{A}\otimes(\det\mathcal{W}_2)^2\big)^{-\lambda}\,.$$

Combining these, we have

$$\det\mathcal{N}_{g_1}|_{\mathcal{Y}_{s'}}\geqslant (g')^*\big((\det\mathcal{U})^{d-2\lambda-4}\otimes\mathcal{A}^{-d+4}\otimes(\det\mathcal{W}_1)^{d-\lambda-4}\otimes(\det\mathcal{W}_2)^{-d-\lambda+4}\big)\,.$$

From this and (4.30), we obtain

$$K_{\mathcal{Y}_{s'}} \geqslant (g')^*\big((\det \mathcal{U})^{d-n-2\lambda-1} \otimes (\det \mathcal{W}_1)^{d-\lambda-4} \otimes (\det \mathcal{W}_2)^{\frac{1}{2}d(d+1)-d-\lambda}\big)\,.$$

Recall that $\lambda = 3n-k-2d$, and one can see that $d-\lambda-4 \geqslant 0$ and $\frac{1}{2}d(d+1)-d-\lambda \geqslant 0$ under the numerical assumption (3.5). The line bundles $\det \mathcal{U}$ and $\det \mathcal{W}_i$ are globally generated. Then by the assumption (3.4), we must have $d-n-2\lambda \leqslant 0$. This is equivalent to $d \leqslant \frac{1}{5}(7n-2k)$, which contradicts the assumption (3.5).

This is the end of the proof of Proposition 4.1.

5. Lines and conics contained in the hypersufaces

The purpose of this section is to show the following refinement of Proposition 4.1.

Proposition 5.1. *We can refine Proposition 4.1 as follows:*

1. *The line l_1 in cases (A1) and (A3), the line l in cases (C1), (C2), (G) and (H), and the line l_i in case (E1) are contained in $\mathcal{X}'_{s'}$.*
2. *In case (E2), exactly one of l_1 and l_2 is contained in $\mathcal{X}'_{s'}$.*
3. *In case (D), the smooth conic C is contained in $\mathcal{X}'_{s'}$.*

By Proposition 4.1, we know that if the line or conic associated with a general point $y \in \mathcal{Y}$ in Proposition 3.4 is not contained in the hypersurface $\mathcal{X}'_{s'}$, then it intersects $\mathcal{X}'_{s'}$ in one or two points. Propositions 5.2 and 5.3 rule out the possibility of lines intersecting $\mathcal{X}'_{s'}$ at one and two points, respectively. The degenerate and smooth conic cases are studied in §§ 5.3 and 5.4.

5.1 One-point line case

Consider the setup in § 3.1. Let Δ_d be the moduli space of triples (x, l, s'), where $s' \in S'$, l is a line in $\mathbb{P}^n$ and x is a point of l such that $l \cap \mathcal{X}'_{s'} \geqslant d \cdot x$, where $l \cap \mathcal{X}'_{s'}$ denotes a scheme-theoretic intersection. (Here if $l \subset \mathcal{X}'_{s'}$, then we understand that this inequality holds.) Let Δ_∞ be the subvariety of Δ_d parametrizing the $(x, l, s') \in \Delta_d$ such that $l \subset \mathcal{X}'_{s'}$. Denote by ρ the morphism $\Delta_d \to \mathcal{X}' \subset \mathbb{P}^n \times S'$ given by $(x, l, s') \mapsto (x, s')$.

Proposition 5.2. *Assume that there is a $\mathrm{GL}(V)$-equivariant rational map $g\colon \mathcal{Y} \to \Delta_d \setminus \Delta_\infty$ with $\rho \circ g = f$ and that the inequality*

$$\frac{d(d+1)}{2} - 3n + k + 1 \geqslant 0 \tag{5.1}$$

holds. Then for general $s' \in S'$, we have $\mathrm{H}^0(\mathcal{Y}_{s'}, K_{\mathcal{Y}_{s'}}) \neq 0$.

This proposition is just a restatement of [Cle03, Lemma 4.3].

5.2 Two-point line case

Consider the setup in § 3.1. Let r and s be positive integers with $r + s = d$. Let $\Delta_{r,s}$ be the moduli space of quadruples (x_1, x_2, l, s'), where $s' \in S'$ and x_1 and x_2 are points of l such that $l \cap \mathcal{X}'_{s'} \geqslant rx_1 + sx_2$. Let Δ_∞ be the subvariety of $\Delta_{r,s}$ parametrizing the $(x_1, x_2, l, s') \in \Delta_{r,s}$ such that $l \subset \mathcal{X}'_{s'}$. Denote by ρ the morphism $\Delta_{r,s} \to \mathcal{X}' \subset \mathbb{P}^n \times S'$ given by $(x_1, x_2, l, s') \mapsto (x_1, s')$.

Proposition 5.3. *Assume that there is a* $\mathrm{GL}(V)$*-equivariant rational map* $g\colon \mathcal{Y} \to \Delta_{r,s} \setminus \Delta_\infty$ *with* $\rho \circ g = f$ *and that the inequality*

$$\frac{d(d-1)}{2} - 3n + k + 1 \geqslant 0 \tag{5.2}$$

holds. Then for general $s' \in S'$*, we have* $\mathrm{H}^0(\mathcal{Y}_{s'}, K_{\mathcal{Y}_{s'}}) \neq 0$.

The "(-1)-twisted" version of this proposition is [Pac03, Proposition 1.4]. The proof there applies here as well.

5.3 Two-line conic case

Consider the setup in § 3.1. Let Γ° be the moduli space of quadruples (x, C, H, s) such that $(x, C, H) \in \mathbb{L}_3^\circ$, $s \in S$ and $\mathcal{X}_s \cap l_i \geqslant d \cdot x$ for $i = 1, 2$, where l_1 and l_2 are line components of C. Let Γ_∞° be the subvariety of Γ° consisting of the $(x, C, H, s) \in \Gamma^\circ$ such that $C \subset \mathcal{X}_s$. Put $\Gamma'^\circ := \Gamma^\circ \times_S S'$ and $\Gamma'^\circ_\infty := \Gamma^\circ_\infty \times_S S'$. Denote by ρ the morphism $\Gamma'^\circ \to \mathcal{X}' \subset \mathbb{P}^n \times S'$ given by $(x, C, H, s') \mapsto (x, s')$.

Proposition 5.4. *Assume that there is a* $\mathrm{GL}(V)$*-equivariant rational map* $g\colon \mathcal{Y} \to \Gamma'^\circ$ *with* $\rho \circ g = f$ *and that the inequalities*

$$\frac{d(d+3)}{2} - 4n + k - 2 \geqslant 0 \quad \textit{and} \quad \frac{d(d+1)}{2} - 3n + k + 2 \geqslant 0 \tag{5.3}$$

hold. Assume, moreover, that for general $y \in \mathcal{Y}$*, if we write* $g(y)$ *as* $(x, C = l_1 \cup l_2, H, s')$*, neither* l_1 *nor* l_2 *is contained in* $\mathcal{X}'_{s'}$*. Then for general* $s' \in S'$*, we have* $\mathrm{H}^0(\mathcal{Y}_{s'}, K_{\mathcal{Y}_{s'}}) \neq 0$.

Before beginning the proof of the proposition, we introduce some notation and construct a compactification (over S) of Γ'°.

Giving a point (x, C, H) of $\mathbb{L}_3$ is the same as giving surjective linear maps $\pi\colon V \to U$ and $\pi'\colon U \to A$ of vector spaces, where $\dim U = 3$ and $\dim A = 1$, together with a 1-dimensional subspace $R \subset S^2(\operatorname{Ker} \pi')$. The correspondence is that $x = \mathbb{P}(A)$ and $H = \mathbb{P}(U)$, and R defines C. Let

$$\pi\colon V \otimes \mathcal{O}_{\mathbb{L}_3} \to \mathcal{U}\,, \quad \pi'\colon \mathcal{U} \to \mathcal{A}\,, \quad \mathcal{R} \subset S^2(\operatorname{Ker} \pi')$$

be their universal family on $\mathbb{L}_3$. The fiber of the vector bundle $S^d\mathcal{U}/S^{d-2}\mathcal{U} \otimes \mathcal{R}$ at $(x, C, H) \in \mathbb{L}_3$ is isomorphic to $\mathrm{H}^0(C, \mathcal{O}(d))$. The fiber at $(x, C = l_1 \cup l_2, H) \in \mathbb{L}_3^\circ$ of the rank 2 vector subbundle $S^d(\operatorname{Ker} \pi')/S^{d-2}(\operatorname{Ker} \pi') \otimes \mathcal{R}$ of $S^d\mathcal{U}/S^{d-2}\mathcal{U} \otimes \mathcal{R}$ is isomorphic to the subspace of $\mathrm{H}^0(C, \mathcal{O}(d))$ consisting of the sections s of $\mathcal{O}_C(d)$ such that the zero divisor $Z(s|_{l_i})$ satisfies $Z(s|_{l_i}) \geqslant d \cdot x$ for $i = 1, 2$. Let $\mathcal{W}$ be the kernel of the composite of morphisms

$$S^dV \otimes \mathcal{O}_{\mathbb{L}_3} \to S^d\mathcal{U} \to S^d\mathcal{U}/S^{d-2}\mathcal{U} \otimes \mathcal{R} \to \frac{S^d\mathcal{U}/S^{d-2}\mathcal{U} \otimes \mathcal{R}}{S^d(\operatorname{Ker} \pi')/S^{d-2}(\operatorname{Ker} \pi') \otimes \mathcal{R}}\,.$$

Let $\mathcal{W}_\infty$ be the kernel of the composite of morphisms

$$S^dV \otimes \mathcal{O}_{\mathbb{L}_3} \to S^d\mathcal{U} \to S^d\mathcal{U}/S^{d-2}\mathcal{U} \otimes \mathcal{R}\,.$$

The fiber of $\mathcal{W}_\infty$ at $(x, C, H) \in \mathbb{L}_3$ is $\mathrm{H}^0(\mathbb{P}^n, \mathcal{I}_C(d))$. Put

$$\Gamma := \mathbf{V}\big(\mathcal{W}^\vee\big) \setminus (\text{zero section}) \quad \text{and} \quad \Gamma_\infty := \mathbf{V}\big(\mathcal{W}_\infty^\vee\big) \setminus (\text{zero section})\,.$$

There are natural projections $\Gamma \to \mathcal{X}$ and $\Gamma \to \mathbb{L}_3$. Put $\Gamma' := \Gamma \times_\mathcal{X} \mathcal{X}'$ and $\Gamma'_\infty := \Gamma_\infty \times_\mathcal{X} \mathcal{X}'$. Denote by r the composite $\Gamma' \to \Gamma \to \mathbb{L}_3$ of morphisms. Then $r^{-1}(\mathbb{L}_3^\circ) = \Gamma'^\circ$, so Γ' is a compactification of Γ'° over S'. The morphism $\rho\colon \Gamma'^\circ \to \mathcal{X}'$ extends to a morphism $\Gamma' \to \mathcal{X}'$, which is also denoted by ρ. We regard g as a rational map from $\mathcal{Y}$ to Γ'. Blowing up $\mathcal{Y}$ $\mathrm{GL}(V)$-equivariantly and shrinking S' if necessary, we may assume that g is a morphism from $\mathcal{Y}$ to Γ' and $\mathcal{Y}$ is still smooth over S'.

$$\begin{tikzcd}
& \Gamma' \arrow[r] \arrow[d, "\rho"] \arrow[rr, bend left, "r"] & \Gamma \arrow[r] \arrow[d, dashed] & \mathbb{L}_3 \arrow[d, "q_3"] \\
\mathcal{Y} \arrow[ur, "g"] \arrow[r, "f"] \arrow[dr, "p_1"'] & \mathcal{X}' \arrow[r] \arrow[d, "p'"] \arrow[rr, bend left, "h"] & \mathcal{X} \arrow[r] \arrow[d, "p"] & \mathbb{P}^n . \\
& S' \arrow[r, "b"'] & S &
\end{tikzcd}$$

One can compute the canonical bundle of Γ':

$$K_{\Gamma'} = r^*\Big((\det \mathcal{U})^{d^2-2d-n-2} \otimes \mathcal{A}^{2d-1} \otimes \mathcal{R}^{2-\frac{1}{2}(d^2-3d)}\Big). \tag{5.4}$$

The normal sheaf $\mathcal{N}_g$ of g is defined by the exact sequence

$$0 \to \mathcal{T}_{\mathcal{Y}} \to g^*\mathcal{T}_{\Gamma'} \to \mathcal{N}_g \to 0 .$$

Since $\mathrm{GL}(V)$ acts transitively on $\mathbb{L}_3^\circ$, the open subscheme $(r\circ g)^{-1}(\mathbb{L}_3^\circ)$ of $\mathcal{Y}$ is smooth over $\mathbb{L}_3^\circ$, so we have an exact sequence

$$0 \to \mathcal{T}_{\mathcal{Y}/\mathbb{L}_3} \to g^*\mathcal{T}_{\Gamma'/\mathbb{L}_3} \to \mathcal{N}_g \to 0 \tag{5.5}$$

on $(r\circ g)^{-1}(\mathbb{L}_3^\circ)$. We have isomorphisms

$$\mathcal{T}_{\Gamma'/\mathbb{L}_3} \simeq r^*\mathcal{W} \quad \text{and} \quad \mathcal{T}_{\Gamma'_\infty/\mathbb{L}_3} \simeq r^*\mathcal{W}_\infty .$$

Proof of Proposition 5.4. Pick a general point $s' \in S'$ and a general point y of $\mathcal{Y}_{s'}$, and let $(x, C = l_1 \cup l_2, H) := (r\circ g)(y)$. Looking at the fiber over y of (5.5), we have an inclusion $\mathcal{T}_{\mathcal{Y}/\mathbb{L}_3}|_y \hookrightarrow g^*\mathcal{T}_{\Gamma'/\mathbb{L}_3}|_y \simeq (r\circ g)^*\mathcal{W}|_y \subset M^d$. If F is the degree d homogeneous polynomial corresponding to $b(s') \in S$, then $F \in \mathcal{T}_{\mathcal{Y}/\mathbb{L}_3}|_y$. This is because for any $a \in \mathbb{C}^\times \subset \mathrm{GL}(V)$, $(r\circ g)(a\cdot y) = (r\circ g)(y)$, and a acts on S as multiplication by a^d. The fiber $g^*\mathcal{T}_{\Gamma'/\mathbb{L}_3}|_y = (r\circ g)^*\mathcal{W}|_y$ has $(r\circ g)^*\mathcal{W}_\infty|_y$ as a subspace, and if $F \in (r\circ g)^*\mathcal{W}_\infty|_y$, then this means that $C \subset \mathcal{X}'_{s'}$. This contradicts our assumption. So we have $F \notin (r\circ g)^*\mathcal{W}_\infty|_y$.

Claim 5.4.1. *We have $\mathcal{T}_{\mathcal{Y}/\mathbb{L}_3}|_y + (r\circ g)^*\mathcal{W}_\infty|_y = g^*\mathcal{T}_{\Gamma'/\mathbb{L}_3}|_y$.*

Proof of Claim 5.4.1. Choose homogeneous coordinates $[u_0 : \cdots : u_n]$ of $\mathbb{P}^n$ such that we have $x = [1 : 0 : \cdots : 0]$, $H = (u_3 = \cdots = u_n = 0)$, $l_1 = (u_1 = 0) \cap H$ and $l_2 = (u_2 = 0) \cap H$. Then F is written as

$$a_1 u_1^d + a_2 u_2^d + (\text{terms vanishing on } C)$$

with non-zero a_1 and a_2. Choose distinct integers k_1 and k_2. For $t \in \mathbb{C}^\times$, the diagonal matrix $\mathrm{diag}\big(1, t^{k_1}, t^{k_2}, 1, \ldots, 1\big) \in \mathrm{GL}_{n+1}$ acts on $\mathbb{P}^n$ as

$$[u_0 : u_1 : u_2 : \ldots, u_n] \mapsto \big[u_0 : t^{k_1}u_1 : t^{k_2}u_2 : \ldots, u_n\big].$$

This action transforms F as

$$F^t := a_1 t^{k_1 d} u_1^d + a_2 t^{k_2 d} u_2^d + (\text{terms vanishing on } C) .$$

The action preserves l_1 and l_2, so the orbit of y stays in the fiber of the map $(r\circ g)$, which implies that $\frac{d}{dt}F^t\big|_{t=1} \in \mathcal{T}_{\mathcal{Y}/\mathbb{L}_3}|_y$. Thus the image of the map $\mathcal{T}_{\mathcal{Y}/\mathbb{L}_3}|_y \to g^*\mathcal{T}_{\Gamma'/\mathbb{L}_3}|_y/(r\circ g)^*\mathcal{W}_\infty|_y$

contains $k_1a_1u_1^d+k_2a_2u_2^d$, where note that the 2-dimensional vector space $g^*\mathcal{T}_{\Gamma'/\mathbb{L}_3}|_y/(r\circ g)^*\mathcal{W}_\infty|_y$ is naturally isomorphic to $\mathbb{C}u_1^d\oplus\mathbb{C}u_2^d$. This proves the claim. $\square$

Proof of Proposition 5.4, continued. Write g_1 for the morphism $(r\circ g)|_{\mathcal{Y}_{s'}}\colon \mathcal{Y}_{s'}\to\mathbb{L}_3$. By the claim, the composite of morphisms

$$\theta\colon g_1^*\mathcal{W}_\infty\hookrightarrow g_1^*\mathcal{W}=g^*\mathcal{T}_{\Gamma'/\mathbb{L}_3}|_{\mathcal{Y}_{s'}}\to g^*\mathcal{T}_{\Gamma'}|_{\mathcal{Y}_{s'}}\to\mathcal{N}_g|_{\mathcal{Y}_{s'}}$$

is generically surjective. Let $\mathcal{V}$ be the vector bundle on $\mathbb{L}_3$ whose fiber over (x,C,H) is $\mathrm{H}^0(\mathbb{P}^n,\mathcal{I}_H(d))$. We have the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & g_1^*\mathcal{V} \arrow[r] \arrow[d, "\theta'"] & g_1^*\mathcal{W}_\infty \arrow[r] \arrow[d, "\theta"] & g_1^*(\mathcal{W}_\infty/\mathcal{V}) \arrow[r] \arrow[d, "\theta''"] & 0\\
0 \arrow[r] & \theta(g_1^*\mathcal{V}) \arrow[r] & \mathcal{N}_g|_{\mathcal{Y}_{s'}} \arrow[r] & \mathcal{N}_g|_{\mathcal{Y}_{s'}}/\theta(g_1^*\mathcal{V}) \arrow[r] & 0\,,
\end{tikzcd}$$

where θ' is surjective and θ'' is generically surjective. Put $\mathcal{N}':=\theta(g_1^*\mathcal{V})$ and $\mathcal{N}'':=\mathcal{N}_g|_{\mathcal{Y}_{s'}}/\theta(g_1^*\mathcal{V})$, and let $r_1=\operatorname{rank}\mathcal{N}'$ and $r_2=\operatorname{rank}\mathcal{N}''$.

Now we deduce the "positivity" of $\mathcal{N}_g|_{\mathcal{Y}_{s'}}$. Since $\mathcal{V}\otimes\det\mathcal{U}$ is globally generated by Proposition 2.7, we have $\det\mathcal{N}'\geqslant g_1^*(\det\mathcal{U})^{-r_1}$. The fiber of $\mathcal{W}_\infty/\mathcal{V}$ over $(x,C,H)\in\mathbb{L}_3$ is isomorphic to $\mathrm{H}^0(H,\mathcal{I}_C(d))$, so we have an isomorphism $\mathcal{W}_\infty/\mathcal{V}\simeq S^{d-2}\mathcal{U}\otimes\mathcal{R}$. Since $\mathcal{U}$ is globally generated, we can find an injective morphism $g_1^*\mathcal{R}^{\oplus r_2}\to\mathcal{N}''$ with torsion cokernel. Thus we obtain $\det\mathcal{N}''\geqslant g_1^*\mathcal{R}^{r_2}$. Combining these with (5.4), we obtain

$$K_{\mathcal{Y}_{s'}}\geqslant g_1^*\big((\det\mathcal{U})^{d^2-2d-n-r_1-2}\otimes\mathcal{A}^{2d-1}\otimes\mathcal{R}^{2-\frac{1}{2}(d^2-3d)+r_2}\big). \tag{5.6}$$

The variety $\mathbb{L}_3$ contains $\mathbb{L}_4$ as a divisor, and one can find that $\mathcal{O}_{\mathbb{L}_3}(\mathbb{L}_4)\simeq(\det\mathcal{U})^2\otimes\mathcal{A}^{-2}\otimes\mathcal{R}^{-2}$. Put

$$\eta=\left\lfloor\frac{1}{2}\left(\frac{d^2-3d}{2}-r_2-2\right)\right\rfloor\quad\text{and}\quad\varepsilon=\frac{d^2-3d}{2}-r_2-2-2\eta\,.$$

Then the right-hand side of (5.6) becomes

$$g_1^*\big((\det\mathcal{U})^{d^2-2d-n-r_1-2-2\eta-4\varepsilon}\otimes\mathcal{A}^{2d-1+2\eta}\otimes\big((\det\mathcal{U})^4\otimes\mathcal{R}^{-1}\big)^{\varepsilon}\otimes\mathcal{O}(\mathbb{L}_4)^{\eta}\big)\,. \tag{5.7}$$

The inequalities (5.3) imply that the exponents of $\det\mathcal{U}$ and $\mathcal{O}(\mathbb{L}_4)$ in (5.7) are non-negative. The line bundles $\det\mathcal{U}$ and $\mathcal{A}$ are globally generated, and $g_1^*\mathcal{O}(\mathbb{L}_4)$ has a non-zero section. Note that $(\det\mathcal{U})^4\otimes\mathcal{R}^{-1}$ is also globally generated because $S^2\mathcal{U}/\mathcal{R}$ is globally generated. Thus the line bundle (5.7) has a non-zero section, and we have $\mathrm{H}^0(\mathcal{Y}_{s'},K_{\mathcal{Y}_{s'}})\neq 0$ by (5.6). $\square$

Proposition 5.5. *Under the setup described in §3.1, consider the variety Γ'_∞ and morphisms $\rho|_{\Gamma'_\infty}\colon\Gamma'_\infty\to\mathcal{X}'$ and $r|_{\Gamma'_\infty}\colon\Gamma'_\infty\to\mathbb{L}_3$ introduced before the proof of Proposition 5.4. Assume that there exists a $\mathrm{GL}(V)$-equivariant rational map $g\colon\mathcal{Y}\to\Gamma'_\infty$ such that $f=(\rho|_{\Gamma'_\infty})\circ g$ and $((r|_{\Gamma'_\infty})\circ g)^{-1}(\mathbb{L}_3\setminus\mathbb{L}_4)\neq\emptyset$. Assume, moreover, that the inequalities*

$$\frac{d^2+3d}{2}\geqslant 3n-k\quad\text{and}\quad\frac{d^2+5d}{2}\geqslant 4n-k \tag{5.8}$$

hold. Then we have $\mathrm{H}^0(\mathcal{Y}_{s'},K_{\mathcal{Y}_{s'}})\neq 0$ for general $s'\in S'$.

The proof of this proposition is similar to but easier than that of Proposition 5.4, so we omit it.

5.4 Smooth conic case

Consider the setup in § 3.1. Let Ξ° be the moduli space of quadruples (x, C, H, s), where $s \in S$ and $(x, C, H) \in \mathbb{L}_2^\circ$ are such that $\mathcal{X}_s \cap C \geqslant 2d \cdot x$. Let Ξ°_∞ be the subvariety of Ξ° consisting of the (x, C, H, s) such that $C \subset \mathcal{X}_s$. Put $\Xi'^\circ = \Xi^\circ \times_S S'$ and $\Xi'^\circ_\infty = \Xi^\circ_\infty \times_S S'$. Denote by ρ the map $\rho\colon \Xi'^\circ \to \mathcal{X}'$ given by $(x, C, H, s') \mapsto (x, s')$.

Proposition 5.6. *Assume that there exists a* $\mathrm{GL}(V)$*-equivariant rational map* $g\colon \mathcal{Y} \to \Xi'^\circ \setminus \Xi'^\circ_\infty$ *such that* $\rho \circ g = f$*. Assume, moreover, that the inequalities*

$$\frac{d^2-d}{2} \geqslant 3n-k+2 \quad \text{and} \quad \frac{2d^2+5d}{3} \geqslant 4n-k+\frac{22}{3} \tag{5.9}$$

hold. Then for general $s' \in S'$*, we have* $\mathrm{H}^0(\mathcal{Y}_{s'}, K_{\mathcal{Y}_{s'}}) \neq 0$.

Before beginning the proof of the proposition, we introduce some notation and prepare some lemmas.

Giving a point $(x, C, H) \in \mathbb{L}_2$ is the same as giving surjective linear maps $\pi\colon V \to U$ and $\pi'\colon U \to A$ of vector spaces, where $\dim U = 3$ and $\dim A = 1$, and a 1-dimensional subspace $R \subset \mathrm{Ker}(S^2\pi')$, where $S^2\pi'\colon S^2U \to A^2$. Let

$$\pi\colon V \otimes \mathcal{O}_{\mathbb{L}_2} \to \mathcal{U}\,, \quad \pi'\colon \mathcal{U} \to \mathcal{A}\,, \quad \mathcal{R} \subset \mathrm{Ker}\left(S^2\pi'\right)$$

be their universal family on $\mathbb{L}_2$. Put $\mathbb{B} := \mathbb{L}_2 \setminus \mathbb{L}_2^\circ$. Then $\mathbb{B}$ is a divisor of $\mathbb{L}_2$ parametrizing triples (x, C, H) with C a degenerate conic. Note that we have a sequence of subschemes: $\mathbb{L}_2 \supset \mathbb{B} \supset \mathbb{L}_3 \supset \mathbb{L}_4$.

Define the map $\pi' \cdot \mathrm{id}\colon S^2\mathcal{U} \to \mathcal{U} \otimes \mathcal{A}$ by $(\pi' \cdot \mathrm{id})(uv) = u \otimes \pi'(v) + v \otimes \pi'(u)$ for sections u and v of $\mathcal{U}$. Tensoring this map with $\mathcal{A}^{-1}$, we obtain a map $\tau\colon S^2\mathcal{U} \otimes \mathcal{A}^{-1} \to \mathcal{U}$. Let ζ be the composite of morphisms

$$\left(\mathcal{R} \otimes \mathcal{A}^{-1}\right)^d \hookrightarrow S^d\left(S^2\mathcal{U} \otimes \mathcal{A}^{-1}\right) \xrightarrow{S^d\tau} S^d\mathcal{U} \to S^d\mathcal{U}/S^{d-2}\mathcal{U} \otimes \mathcal{R}\,.$$

The fiber of $S^d\mathcal{U}/S^{d-2}\mathcal{U} \otimes \mathcal{R}$ over a point $(x, C, H) \in \mathbb{L}_2$ is isomorphic to $\mathrm{H}^0(C, \mathcal{O}(d))$.

Lemma 5.7. *For a point* $(x, C, H) \in \mathbb{L}_2$*, the restriction* $\zeta|_{(x,C,H)}$ *of* ζ *to fibers over* (x, C, H) *is the zero map if and only if* $(x, C, H) \in \mathbb{L}_3$.

Proof. Let $\mathbb{H}$ be the moduli space parametrizing pairs (x, H), where H is a plane in $\mathbb{P}^n = \mathbb{P}(V)$ and x is a point of H. There is a natural projection $v\colon \mathbb{L}_2 \to \mathbb{H}$ given by $(x, C, H) \mapsto (x, H)$. Since $\mathrm{GL}(V)$ acts transitively on $\mathbb{H}$ and the morphism v is $\mathrm{GL}(V)$-equivariant, we may consider the lemma after fixing a pair $(x, H) \in \mathbb{H}$. By abuse of notation, we use the same letters when we fix $(x, H) \in \mathbb{H}$. For example, the fiber of the map $v\colon \mathbb{L}_2 \to \mathbb{H}$ over (x, H) is denoted by $\mathbb{L}_2$.

Choose homogeneous coordinates $[X_0 : \cdots : X_n]$ of $\mathbb{P}^n$ in such a way that $x = [1 : 0 : \cdots : 0]$ and $H = (X_3 = \cdots = X_n = 0)$. Then $\pi\colon V = \oplus_{i=0}^n \mathbb{C}X_i \to U = \oplus_{i=0}^2 \mathbb{C}X_i$ and $\pi'\colon U = \oplus_{i=0}^2 \mathbb{C}X_i \to A = \mathbb{C}X_0$ are natural projections, and

$$\mathbb{L}_2 = \left(\left\{aX_1^2 + bX_2^2 + 2cX_0X_1 + 2dX_0X_2 + 2eX_1X_2\right\} \setminus \{0\}\right)/\mathbb{C}^\times \simeq \mathbb{P}^4$$

is a 4-dimensional projective space with homogeneous coordinate $[a : b : c : d : e]$. We have $\mathbb{L}_2 \supset \mathbb{L}_3 = (c = d = 0)$.

By the definition of $\pi' \cdot \mathrm{id}\colon S^2U \to U \otimes A$, we have $(\pi' \cdot \mathrm{id})\left(X_0^2\right) = 2X_0 \otimes X_0$, $(\pi' \cdot \mathrm{id})(X_0X_i) = X_i \otimes X_0$ for $i \neq 0$ and $(\pi' \cdot \mathrm{id})(X_iX_j) = 0$ for $i, j \neq 0$.

Consider a point $[F] \in \mathbb{L}_2$ represented by a polynomial

$$F(X_0, X_1, X_2) = aX_1^2 + bX_2^2 + 2cX_0X_1 + 2dX_0X_2 + 2eX_1X_2\,.$$

The fiber of the line bundle $\mathcal{R} \subset S^2U \otimes \mathcal{O}_{\mathbb{L}_2}$ over $[F]$ is $\mathbb{C} \cdot F$, and $(\pi' \cdot \mathrm{id})(F) = 2(cX_1 + dX_2)$. Therefore, the image of $\zeta|_{[F]}$ is $\mathbb{C}\overline{(cX_1 + dX_2)^d} \subset S^dU/S^{d-2}U \cdot F$, which is zero if and only if $c = d = 0$. This proves the lemma. $\square$

It follows from the proof of Lemma 5.7 that if C is a smooth conic, then the image of $\zeta|_{(x,C,H)}$ is the 1-dimensional subspace of $\mathrm{H}^0(C, \mathcal{O}(d))$ spanned by a section s whose zero divisor $Z(s)$ is $2d \cdot x$. Let $u_1 \colon \mathbb{L}_2' \to \mathbb{L}_2$ be the blow-up of $\mathbb{L}_2$ along $\mathbb{L}_3$ and $E_1 = u_1^{-1}(\mathbb{L}_3)$ the exceptional divisor. Let $\mathbb{B}'$ be the strict transform of $\mathbb{B}$ in $\mathbb{L}_2'$. Let Z_1 be the reduced closed subscheme of $\mathbb{L}_2'$ with support $u_1^{-1}(\mathbb{L}_4) \cap \mathbb{B}'$.

Lemma 5.8. *The pull-back $u_1^*(\zeta)$ of the map ζ by u_1 factors as*

$$u_1^*\big(\mathcal{R} \otimes \mathcal{A}^{-1}\big)^d \hookrightarrow \big(u_1^*\big(\mathcal{R} \otimes \mathcal{A}^{-1}\big)(E_1)\big)^d \xrightarrow{\zeta'} u_1^*\big(S^d\mathcal{U}/S^{d-2}\mathcal{U} \otimes \mathcal{R}\big)\,. \tag{5.10}$$

Moreover, the locus where the map ζ' vanishes is Z_1.

Proof. As in the proof of Lemma 5.7, we fix a pair $(x, H) \in \mathbb{H}$ and use the notation introduced there such as $\mathbb{L}_2 = \mathbb{P}^4 = \{[a : b : c : d : e]\} \supset \mathbb{L}_3 = (c = d = 0)$.

The blow-up $\mathbb{L}_2'$ is

$$\mathbb{P}^4 \times \mathbb{P}^1 \supset \mathbb{L}_2' = \big\{([a : b : c : d : e], [c' : d'])|cd' = dc'\big\}\,.$$

It is covered by the two open subsets $\mathbb{L}_2' \cap (c' \neq 0)$ and $\mathbb{L}_2' \cap (d' \neq 0)$. By symmetry, we only have to consider the open subset $\mathbb{L}_2' \cap (c' \neq 0)$, which will be identified with $\mathbb{P}^3 \times \mathbb{A}^1$ by the isomorphism given by

$$\mathbb{P}^3 \times \mathbb{A}^1 \ni ([a : b : c : e], \bar{d}) \mapsto ([a : b : c : c\bar{d} : d], [1 : \bar{d}]) \in \mathbb{L}_2' \cap (c' \neq 0)\,.$$

The fiber of $u_1^*\mathcal{R}$ over a point $([a : b : c : e], \bar{d})$ is spanned by the polynomial $F = aX_1^2 + bX_2^2 + 2cX_0X_1 + 2c\bar{d}X_0X_2 + 2eX_1X_2$, and the morphism $u_1^*(\pi' \cdot \mathrm{id})$ sends it to $c(X_1 + \bar{d}X_2) \otimes X_0 \in U \otimes A$. Since E_1 is defined by $c = 0$, this shows that $u_1^*(\pi' {\cdot} \mathrm{id})$ factors as $u_1^*\mathcal{R} \hookrightarrow (u_1^*\mathcal{R})(E_1) \to U \otimes A \otimes \mathcal{O}_{\mathbb{L}_2'}$. This implies that $u_1^*(\zeta)$ factors as (5.10). By Lemma 5.7, the map $\zeta'|_{([a:b:c:e],\bar{d})}$ is non-zero if $c \neq 0$. If $c = 0$, then its image is spanned by $\overline{(X_1 + \bar{d}X_2)^d} \in S^dU/S^{d-2} \cdot \big(aX_1^2 + bX_2^2 + 2eX_1X_2\big)$, which is zero if and only if $aX_1^2 + bX_2^2 + 2eX_1X_2 = a(X_1 + \bar{d}X_2)^2$. So the locus where $\zeta'|_{([a:b:c:e],\bar{d})}$ is zero is given by $c = 0$, $a \neq 0$, $b = a\bar{d}^2$ and $e = a\bar{d}$. We can see that this coincides with Z_1. In fact, since $\mathbb{L}_2 \supset \mathbb{B} = \big(ad^2 + bc^2 - 2cde = 0\big)$, we have $\mathbb{B}' = \big(a\bar{d}^2 + b - 2\bar{d}e = 0\big)$ in the open subset $\mathbb{L}_2' \cap (c' \neq 0) = \mathbb{P}^3 \times \mathbb{A}^1$. The intersection of this with $u_1^{-1}(\mathbb{L}_4) = \big(c = ab - e^2 = 0\big)$ is $\big(c = 0, a \neq 0, b = a\bar{d}^2, e = a\bar{d}\big) \subset \mathbb{P}^3 \times \mathbb{A}^1$. $\square$

Define the ideal sheaf $\mathcal{I} \subset \mathcal{O}_{\mathbb{L}_2'}$ in such a way that the image of the map

$$(\zeta')^\vee \colon u_1^*\big(S^d\mathcal{U}/S^{d-2}\mathcal{U} \otimes \mathcal{R}\big)^\vee \to \big(u_1^*\big(\mathcal{R} \otimes \mathcal{A}^{-1}\big)(E_1)\big)^{-d}$$

is $\mathcal{I}\big(u_1^*\big(\mathcal{R} \otimes \mathcal{A}^{-1}\big)(E_1)\big)^{-d}$. By Lemma 5.8, we have $\mathrm{Supp}(\mathcal{O}_{\mathbb{L}_2'}/\mathcal{I}) = Z_1$.

Lemma 5.9. *We have $\mathcal{I} \supset (\mathcal{I}_{Z_1})^{2d-3}$.*

Proof. As in the proofs of Lemmas 5.7 and 5.8, we fix a pair $(x, H) \in \mathbb{H}$ and use the notation introduced there. By symmetry, it suffices to prove $\mathcal{I} \supset (\mathcal{I}_{Z_1})^{2d-3}$ only on $\mathbb{L}_2' \cap \{c' \neq 0\} = \mathbb{P}^3 \times \mathbb{A}^1$. We saw in the proof of Lemma 5.8 that $Z_1 \cap \big(\mathbb{P}^3 \times \mathbb{A}^1\big)$ is contained in $\big(\mathbb{P}^3 \times \mathbb{A}^1\big) \cap \{a \neq 0\}$. We identify $\big(\mathbb{P}^3 \times \mathbb{A}^1\big) \cap \{a \neq 0\}$ with $\mathbb{A}^3 \times \mathbb{A}^1 = \mathbb{A}^4$ by the isomorphism

$$\mathbb{A}^4 \ni \big(\bar{b}, \bar{c}, \bar{e}; \bar{d}\big) \mapsto \big([1 : \bar{b} : \bar{c} : \bar{e}], \bar{d}\big) \in \big(\mathbb{P}^3 \times \mathbb{A}^1\big) \cap \{a \neq 0\}\,.$$

The fiber of $u_1^*\mathcal{R}$ over $(\bar{b},\bar{c},\bar{e};\bar{d})$ is spanned by

$$F = X_1^2 + \bar{b}X_2^2 + 2\bar{c}X_0X_1 + 2\bar{c}\bar{d}X_0X_2 + 2\bar{e}X_1X_2\,.$$

Varying $(\bar{b},\bar{c},\bar{e};\bar{d}) \in \mathbb{A}^4$, we see that $\frac{1}{\bar{c}}F$ gives a nowhere vanishing section of $(u_1^*\mathcal{R})(E_1)$ over $\mathbb{A}^4$. The image of the nowhere vanishing section $\left(\frac{1}{\bar{c}}F\right)^d$ of $((u_1^*\mathcal{R}\otimes A)(E_1))^d$ over $\mathbb{A}^4$ by ζ' is $\overline{(X_1+\bar{d}X_2)^d} \in S^dU\otimes\mathcal{O}_{\mathbb{A}^4}/S^{d-2}U\cdot F$. Put $Y_1 = X_1+\bar{d}X_2$. Then F can be written as

$$F = Y_1^2 + \big(2\bar{c}X_0 + 2(\bar{e}-\bar{d})X_2\big)Y_1 + \big(\bar{b}+\bar{d}^2-2\bar{e}\bar{d}\big)X_2^2\,.$$

We make a coordinate change of $\mathbb{A}^4$. Put

$$s = 2\bar{d}\bar{e} - \bar{b} - \bar{d}^2\,,\quad t = -2\bar{c}\,,\quad v = 2\big(\bar{d}-\bar{e}\big)\,.$$

We use $(s,t,v;\bar{d})$ as a new coordinate of $\mathbb{A}^4$. Then $F = Y_1^2 - (tX_0+vX_2)Y_1 - sX_2^2$, and Z_1 is defined by $s=t=v=0$.

We define an isomorphism $S^dU\otimes\mathcal{O}_{\mathbb{A}^4}/S^{d-2}U\cdot F \simeq \mathcal{O}_{\mathbb{A}^4}^{\oplus 2d+1}$ as follows. For a degree d homogeneous polynomial $G(X_0,Y_1,X_2)$ of variables X_0, Y_1, X_2 with coefficients in $\mathbb{C}[s,t,v,\bar{d}]$, performing a division by F, we obtain the equation

$$G \equiv A(X_0,X_2)Y_1 + B(X_0,X_2) \pmod F\,,$$

where A and B are homogeneous polynomials in the variables X_0 and X_2 of degree $d-1$ and d, respectively, with coefficients in $\mathbb{C}[s,t,v,\bar{d}]$. Associating the $2d+1$ coefficients of A and B with G, we obtain an isomorphism $S^dU\otimes\mathcal{O}_{\mathbb{A}^4}/S^{d-2}U\cdot F \simeq \mathcal{O}_{\mathbb{A}^4}^{\oplus 2d+1}$. We want to consider the image of $\overline{(X_1+\bar{d}X_2)^d} = \overline{Y_1^d} \in S^dU\otimes\mathcal{O}_{\mathbb{A}^4}/S^{d-2}U\cdot F$ in $\mathcal{O}_{\mathbb{A}^4}^{\oplus 2d+1}$ via this isomorphism.

For $l \geqslant 2$, let $A_l(X_0,X_2)$ and $B_l(X_0,X_2)$ be the homogeneous polynomials in the variables X_0 and X_2 of degree $l-1$ and l, respectively, with coefficients in $\mathbb{C}[s,t,v,\bar{d}]$ such that

$$Y_1^l \equiv A_l(X_0,X_2)Y_1 + B(X_0,X_2) \pmod F\,.$$

Let I_l be the ideal of $\mathbb{C}[s,t,v,\bar{d}]$ generated by the coefficients of A_l and B_l. By the definition of $\mathcal{I}\subset\mathcal{O}_{\mathbb{L}_2'}$, we have $\mathcal{I}|_{\mathbb{A}^4} \simeq \widetilde{I_d}$. Therefore, the lemma follows from the next claim.

Claim 5.9.1. *For $l\geqslant 2$, the ideal I_l contains the elements t^is^j for $i+j=l-1$ and $t^is^jv^{2l-3-2i-2j}$ for $i+j\leqslant l-2$.*

Proof of Claim 5.9.1. Since $\bar{d}$ does not appear in the coefficients of F, neither does it in those of A_l and B_l. Therefore, we ignore $\bar{d}$ in the proof and consider I_l as an ideal of $\mathbb{C}[s,t,v]$.

We first make an observation. By the definitions of A_l and B_l, we have the relations

$$A_l = (tX_0+vX_2)A_{l-1} + B_{l-1} \quad\text{and}\quad B_l = sX_2^2A_{l-1}\,. \tag{5.11}$$

From this, we have

$$sX_2^2A_l - (tX_0+vX_2)B_l = sX_2^2B_{l-1}\,. \tag{5.12}$$

It follows from this and the second equation of (5.11) that

$$I_l \supset sI_{l-1}\,. \tag{5.13}$$

From the relation (5.11), we see easily, by induction on l, that if we set $\deg t = \deg v = 1$ and $\deg s = 2$, then the coefficients of A_l and B_l are homogeneous polynomials in t, v, s of degree $l-1$ and l, respectively. From this and the relation (5.11), we easily see, again by induction on l, that

(i) the coefficient of X_0^{l-1} in A_l is t^{l-1},

(ii) the coefficient of $X_0^{l-2}X_2$ in A_l is $(l-1)t^{l-2}v$,

(iii) the coefficient of $X_0^{l-1-i}X_i$ $(2 \leqslant i \leqslant l-1)$ in A_l is

$$\binom{l-1}{i} t^{l-1-i}v^i + t^{l-1-i} \cdot \big(\text{linear combination of } s^j v^{i-2j},\ 1 \leqslant j \leqslant \lfloor i/2 \rfloor\big)\,.$$

Now we prove the claim by induction on $l \geqslant 2$. We have $A_2 = tX_0 + vX_2$ and $B_2 = sX_2^2$; thus $I_2 = (t, v, s)$, so the claim holds for $l = 2$. Let $l \geqslant 3$, and assume that the claim holds up to degree $l-1$. From (5.13), assertion (i) and the induction hypothesis, we infer that I_l contains the elements $t^i s^j$ for $i + j = l-1$. Let us show next that I_l contains the elements $t^i s^j v^{2l-3-2i-2j}$ for $i + j \leqslant l-2$. If $j > 0$, then $t^i v^{2l-3-2i-2j} \in I_{l-j}$ by the induction hypothesis. By (5.13), we have $t^i s^j v^{2l-3-2i-2j} \in I_l$. It remains to show that $t^i v^{2l-3-2i} \in I_l$ for $0 \leqslant i \leqslant l-2$ or, equivalently, that $t^{l-1-k}v^{2k-1} \in I_l$ for $1 \leqslant k \leqslant l-1$. For $k = 1$, that we have $t^{l-1-k}v^{2k-1} \in I_l$ follows from assertion (ii). For $2 \leqslant k \leqslant l-1$, we see from assertion (iii) that

$$\binom{l-1}{k} t^{l-1-k}v^k + t^{l-1-k} \cdot \big(\text{linear combination of } s^j v^{k-2j},\ 1 \leqslant j \leqslant \lfloor k/2 \rfloor\big)$$

is contained in I_l. Multiplying this by v^{k-1}, we see that I_l contains the

$$\binom{l-1}{k} t^{l-1-k}v^{2k-1} + t^{l-1-k} \cdot \big(\text{linear combination of } s^j v^{2k-1-2j},\ 1 \leqslant j \leqslant \lfloor k/2 \rfloor\big)\,. \tag{5.14}$$

By the induction hypothesis, we have $t^{l-1-k}s^{j-1}v^{2k-1-2j} \in I_{l-1}$ for $1 \leqslant j \leqslant \lfloor k/2 \rfloor$. By (5.13), we see that the second term of (5.14) is in I_l, thus so is the first term. This is the end of the proof of Claim 5.9.1. □

This is the end of the proof of Lemma 5.9. □

Let $u_2 \colon \mathbb{L}_2'' \to \mathbb{L}_2'$ be the blow-up of $\mathbb{L}_2'$ along the smooth variety Z_1. Put $\mathcal{I}_2 := \operatorname{Im}(u_2^*\mathcal{I} \to \mathcal{O}_{\mathbb{L}_2''})$. By Lemma 5.9, we have

$$\mathcal{I}_2 \supset \mathcal{O}_{\mathbb{L}_2''}(-(2d-3)E_2)\,, \tag{5.15}$$

where $E_2 = u_2^{-1}(Z_1)$ is the u_2-exceptional divisor. Let $\widetilde{\mathbb{L}_2''} \to \mathbb{L}_2''$ be the blow-up of $\mathbb{L}_2''$ by the ideal $\mathcal{I}_2$, and let $\mathbb{L}_2''' \to \widetilde{\mathbb{L}_2''}$ be a $\mathrm{GL}(V)$-equivariant resolution. The composite $\mathbb{L}_2''' \to \widetilde{\mathbb{L}_2''} \to \mathbb{L}_2''$ is denoted by u_3. There exists an effective divisor E_3 on $\mathbb{L}_2'''$ such that $\operatorname{Im}\big((u_2 \circ u_3)^*\mathcal{I} \to \mathcal{O}_{\mathbb{L}_2'''}\big) = \mathcal{O}_{\mathbb{L}_2'''}(-E_3)$. The image of the map $(u_2 \circ u_3)^*(\zeta')^\vee$ given by

$$(u_2 \circ u_3)^* u_1^*\big(S^d\mathcal{U}/S^{d-2}\mathcal{U} \otimes \mathcal{R}\big)^\vee \to (u_2 \circ u_3)^*\big(u_1^*\big(\mathcal{R} \otimes \mathcal{A}^{-1}\big)(E_1)\big)^{-d}$$

is $(u_2 \circ u_3)^*\big(u_1^*\big(\mathcal{R} \otimes \mathcal{A}^{-1}\big)(E_1)\big)^{-d}(-E_3)$. Put $E := d(u_2 \circ u_3)^*(E_1) + E_3$ and $u := u_1 \circ u_2 \circ u_3$. Therefore, the map $u^*(\zeta)$ factors as

$$u^*\big(\mathcal{R} \otimes \mathcal{A}^{-1}\big)^d \hookrightarrow u^*\big(\mathcal{R} \otimes \mathcal{A}^{-1}\big)^d(E) \xrightarrow{\zeta'''} u^*\big(S^d\mathcal{U}/S^{d-2}\mathcal{U} \otimes \mathcal{R}\big)\,,$$

and the middle term is a subbundle of the last term via ζ'''.

Lemma 5.10. *The divisor $(d-1)u^*\mathbb{B} - E$ on $\mathbb{L}_2'''$ is effective.*

Proof. Since $-E_3 \geqslant u_3^*(-(2d-3)E_2)$ by (5.15), it suffices to show that

$$(d-1)u^*\mathbb{B} - d(u_2 \circ u_3)^*(E_1) - (2d-3)u_3^*E_2 \geqslant 0\,.$$

For this, we show that $(d-1)(u_1\circ u_2)^*\mathbb{B}-du_2^*(E_1)-(2d-3)E_2\geqslant 0$. Since $u_1^*\mathbb{B}=\mathbb{B}'+2E_1$, we need to show that

$$(d-1)u_2^*\mathbb{B}'+(d-2)u_2^*E_1-(2d-3)E_2\geqslant 0\,. \tag{5.16}$$

If $\mathbb{B}''$ and E_1'' are strict transforms of $\mathbb{B}'$ and E_1 in $\mathbb{L}_2''$, respectively, then $u_2^*\mathbb{B}'=\mathbb{B}''+E_2$ and $u_2^*E_1=E_1''+E_2$. Thus (5.16) follows. $\square$

Let $\mathcal{W}$ be the kernel of the map

$$S^dV\otimes\mathcal{O}_{\mathbb{L}_2'''}\xrightarrow{u^*S^d\pi}u^*S^d\mathcal{U}\to u^*\big(S^d\mathcal{U}/S^{d-2}\mathcal{U}\otimes\mathcal{R}\big)\to\frac{u^*\big(S^d\mathcal{U}/S^{d-2}\mathcal{U}\otimes\mathcal{R}\big)}{u^*\big(\mathcal{R}\otimes\mathcal{A}^{-1}\big)^d(E)}\,,$$

and let $\mathcal{W}_\infty$ be the kernel of the map

$$S^dV\otimes\mathcal{O}_{\mathbb{L}_2'''}\xrightarrow{u^*S^d\pi}u^*S^d\mathcal{U}\to u^*\big(S^d\mathcal{U}/S^{d-2}\mathcal{U}\otimes\mathcal{R}\big)\,.$$

Put

$$\Xi=\mathbf{V}\big(\mathcal{W}^\vee\big)\setminus(\text{zero section})\quad\text{and}\quad\Xi_\infty=\mathbf{V}\big(\mathcal{W}_\infty^\vee\big)\setminus(\text{zero section})\,.$$

There are natural morphisms $\Xi\to\mathcal{X}$ and $\Xi\to\mathbb{L}_2'''$. Put $\Xi'=\Xi\times_{\mathcal{X}}\mathcal{X}'$ and $\Xi_\infty'=\Xi_\infty\times_{\mathcal{X}}\mathcal{X}'$. Denote by r the composite of morphisms $\Xi'\to\Xi\to\mathbb{L}_2'''$. The inverse image of $\mathbb{L}_2^\circ$ via the morphism $u\circ r\colon\Xi'\to\mathbb{L}_2$ is Ξ'°. Thus Ξ' is a compactification of Ξ'° over S'. The morphism $\rho\colon\Xi'^\circ\to\mathcal{X}'$ extends to a morphism $\Xi'\to\mathcal{X}'$, which is also denoted by ρ. We regard g as a rational map from $\mathcal{Y}$ to Ξ'. By blowing up $\mathcal{Y}$ $\mathrm{GL}(V)$-equivariantly and shrinking S' if necessary, we may assume that g is a morphism from $\mathcal{Y}$ to Ξ' and $\mathcal{Y}$ is still smooth over S'

$$\begin{tikzcd}
 & \Xi' \arrow[r] \arrow[rr, bend left, "r"] \arrow[dd, "\rho"] & \Xi \arrow[r] \arrow[dd] & \mathbb{L}_2''' \arrow[d, "u"] \\
 & & & \mathbb{L}_2 \arrow[d, "q_2"] \\
\mathcal{Y} \arrow[uur, "g"] \arrow[r, "f"] & \mathcal{X}' \arrow[r] \arrow[rr, bend left, "h"] \arrow[d, "p'"] & \mathcal{X} \arrow[r] \arrow[d, "p"] & \mathbb{P}^n \\
 & S' \arrow[r, "b"'] & S &
\end{tikzcd}$$

There is an effective u-exceptional divisor D on $\mathbb{L}_2'''$ such that $K_{\mathbb{L}_2'''}=(u^*K_{\mathbb{L}_2})(D)$. One can compute the canonical bundle of Ξ':

$$K_{\Xi'}=(u\circ r)^*\Big((\det\mathcal{U})^{d^2-n-4}\otimes\mathcal{A}^{d-1}\otimes\mathcal{R}^{-\frac{1}{2}d(d+1)+5}\Big)\otimes r^*\mathcal{O}_{\mathbb{L}_2'''}(D-E)\,.$$

Define the normal sheaf $\mathcal{N}_g$ of g by the exact sequence

$$0\to\mathcal{T}_{\mathcal{Y}}\to g^*\mathcal{T}_{\Xi'}\to\mathcal{N}_g\to 0\,.$$

Since $(r\circ g)|_{(u\circ r\circ g)^{-1}(\mathbb{L}_2^\circ)}$ is smooth, we have an exact sequence

$$0\to\mathcal{T}_{\mathcal{Y}/\mathbb{L}_2'''}\to g^*\mathcal{T}_{\Xi'/\mathbb{L}_2'''}\to\mathcal{N}_g\to 0$$

on $(u\circ r\circ g)^{-1}(\mathbb{L}_2^\circ)$. We have isomorphisms $\mathcal{T}_{\Xi'/\mathbb{L}_2'''}\simeq r^*\mathcal{W}$ and $\mathcal{T}_{\Xi_\infty'/\mathbb{L}_2'''}\simeq r^*\mathcal{W}_\infty$.

Proof of Proposition 5.6. By assumption, we have $\operatorname{Im}g\not\subset\Xi_\infty'$. Pick a general point $s'\in S'$, and write g_1 for the morphism $(r\circ g)|_{\mathcal{Y}_{s'}}\colon\mathcal{Y}_{s'}\to\mathbb{L}_2'''$. Then the composite of maps

$$\theta\colon g_1^*\mathcal{W}_\infty\hookrightarrow g_1^*\mathcal{W}=g^*\mathcal{T}_{\Xi'/\mathbb{L}_2'''}|_{\mathcal{Y}_{s'}}\hookrightarrow g^*\mathcal{T}_{\Xi'}|_{\mathcal{Y}_{s'}}\to\mathcal{N}_g|_{\mathcal{Y}_{s'}}$$

is generically surjective (cf. the paragraphs before and after Claim 5.4.1). Let $\mathcal{V}$ be the vector bundle on $\mathbb{L}_2$ whose fiber over (x, C, H) is $\mathrm{H}^0(\mathbb{P}^n, \mathcal{I}_H(d))$. We have the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & (u\circ g_1)^*\mathcal{V} \arrow[r] \arrow[d, "\theta'"] & g_1^*\mathcal{W}_\infty \arrow[r] \arrow[d, "\theta"] & g_1^*(\mathcal{W}_\infty/u^*\mathcal{V}) \arrow[r] \arrow[d, "\theta''"] & 0 \\
0 \arrow[r] & \theta((u\circ g_1)^*\mathcal{V}) \arrow[r] & \mathcal{N}_g|_{\mathcal{Y}_{s'}} \arrow[r] & \mathcal{N}_g|_{\mathcal{Y}_{s'}}/\theta((u\circ g_1)^*\mathcal{V}) \arrow[r] & 0\,,
\end{tikzcd}$$

where θ' is surjective and θ'' is generically surjective. Put $\mathcal{N}' := \theta((u\circ g_1)^*\mathcal{V})$ and $\mathcal{N}'' := \mathcal{N}_g|_{\mathcal{Y}_{s'}}/\theta((u\circ g_1)^*\mathcal{V})$, and let $r_1 = \operatorname{rank}\mathcal{N}'$ and $r_2 = \operatorname{rank}\mathcal{N}''$. Arguing as in the proof of Proposition 5.4, we obtain

$$K_{\mathcal{Y}_{s'}} \geqslant (u\circ g_1)^*\big((\det\mathcal{U})^{d^2-n-4-r_1}\otimes\mathcal{A}^{d-1}\otimes\mathcal{R}^{-\frac{1}{2}d(d+1)+5+r_2}\big)\otimes g_1^*\mathcal{O}_{\mathbb{L}_2'''}(D-E)\,. \tag{5.17}$$

One can show that $\mathcal{O}_{\mathbb{L}_2}(\mathbb{B}) \simeq (\det\mathcal{U})^2\otimes\mathcal{R}^{-3}$. Put

$$\eta = \left\lfloor \frac{1}{3}\left(\frac{d(d+1)}{2} - r_2 - 5\right)\right\rfloor \quad \text{and} \quad \varepsilon = \frac{d(d+1)}{2} - r_2 - 5 - 3\eta\,.$$

The right-hand side of (5.17) becomes

$$\begin{aligned}(u\circ g_1)^*\big((\det\mathcal{U})^{d^2-n-4-r_1-2\eta-4\varepsilon}\otimes\mathcal{A}^{d-1}\otimes\mathcal{O}_{\mathbb{L}_2}(\mathbb{B})^{\eta-d+1}\otimes\big((\det\mathcal{U})^4\otimes\mathcal{R}^{-1}\big)^{\varepsilon}\big)\\ \otimes g_1^*\big(u^*\mathcal{O}_{\mathbb{L}_2}((d-1)\mathbb{B})\otimes\mathcal{O}_{\mathbb{L}_2'''}(-E)\big)\otimes g_1^*\mathcal{O}_{\mathbb{L}_2'''}(D)\,.\end{aligned} \tag{5.18}$$

The inequalities (5.9) imply that the exponents of $\det\mathcal{U}$ and $\mathcal{O}_{\mathbb{L}_2}(\mathbb{B})$ in (5.18) are non-negative. The line bundles $\det\mathcal{U}$, $\mathcal{A}$ and $(\det\mathcal{U})^4\otimes\mathcal{R}^{-1}$are globally generated, and $(u\circ g_1)^*\mathcal{O}_{\mathbb{L}_2}(\mathbb{B})$ and $g_1^*\mathcal{O}_{\mathbb{L}_2'''}(D)$ have a non-zero global section. Now, $g_1^*\big(u^*\mathcal{O}_{\mathbb{L}_2}((d-1)\mathbb{B})\otimes\mathcal{O}_{\mathbb{L}_2'''}(-E)\big)$ also has a non-zero global section, as follows from Lemma 5.10. Altogether, (5.18) has a non-zero section. Thus $\mathrm{H}^0(\mathcal{Y}_{s'}, K_{\mathcal{Y}_{s'}}) \neq 0$ by (5.17). □

5.5 Proof of Proposition 5.1

We first note that the inequalities (5.1), (5.2), (5.3), (5.9) and (5.8) hold under the assumption (3.5).

(1) If the line l_1 in case (A3) or the line l in case (C2) is not contained in $\mathcal{X}'_{s'}$, then we are in the situation of Proposition 5.3. So we have $\mathrm{H}^0(\mathcal{Y}_{s'}, K_{\mathcal{Y}_{s'}}) \neq 0$, contradicting our assumption (3.4). In the other cases, we obtain a contradiction applying Proposition 5.2.

(2) First suppose that neither l_1 nor l_2 is contained in $\mathcal{X}'_{s'}$. If $\sharp(l_1\cap\mathcal{X}'_{s'})+\sharp(l_2\cap\mathcal{X}'_{s'}) = 2$, then we obtain a contradiction applying Proposition 5.4. If $\sharp(l_1\cap\mathcal{X}'_{s'})+\sharp(l_2\cap\mathcal{X}'_{s'}) = 3$, then one of l_1 and l_2 intersects $\mathcal{X}'_{s'}$ in one point, and the other in two points. We can obtain a contradiction applying Proposition 5.2 or 5.3.

Now we have shown that at least one of l_1 and l_2 is contained in $\mathcal{X}'_{s'}$. If both l_1 are l_2 are contained in $\mathcal{X}'_{s'}$, then we can obtain a contradiction using Proposition 5.5.

(3) In case (D), if the conic C is not contained in $\mathcal{X}'_{s'}$, use Proposition 5.6 to obtain a contradiction. This is the end of the proof of Proposition 5.1.

6. Covered by lines and conics

To state the following two propositions, we introduce a setup. As in § 3.1, we consider the following diagram:

$$\begin{tikzcd} \mathcal{Y} \arrow[dr, "p_1"'] & \mathcal{X}' \subset \mathbb{P}^n \times S' \arrow[d, "p'"] & \\ & S' \arrow[r, "b"'] & S\,. \end{tikzcd}$$

Define the subvarieties $D \subset \mathbb{G}_1 \times S'$ and $D' \subset \mathbb{G}_2 \times S'$ by

$$\mathbb{G}_1 \times S' \supset D = \left\{(l, s') \middle| l \subset \mathcal{X}'_{s'}\right\},$$
$$\mathbb{G}_2 \times S' \supset D' = \left\{((C, H), s') \middle| C \subset \mathcal{X}'_{s'}\right\}.$$

Let $\pi_1 \colon D \to \mathbb{G}_1$, $\pi_2 \colon D \to S'$, $\pi'_1 \colon D' \to \mathbb{G}_2$ and $\pi'_2 \colon D' \to S'$ be the projections.

Proposition 6.1. *Assume that (3.4) holds and that there exists a* $\mathrm{GL}(V)$*-equivariant morphism* $j \colon \mathcal{Y} \to D$ *such that* $\pi_2 \circ j = p_1$*:*

$$\begin{tikzcd} \mathcal{Y} \arrow[r, "j"] \arrow[dr, "p_1"'] & D \subset \mathbb{G}_1 \times S' \arrow[r] \arrow[d, "\pi_2"] & \mathbb{G}_1 \\ & S'\,. & \end{tikzcd}$$

Assume, moreover, that the inequality $\frac{1}{2}(d^2 + 3d) \geqslant 3n - k - 2$ *holds. Then the morphism* j *is not generically finite.*

Proposition 6.2. *Assume that (3.4) holds and that there exists a* $\mathrm{GL}(V)$*-equivariant morphism* $j' \colon \mathcal{Y} \to D'$ *such that* $\pi'_2 \circ j' = p_1$ *and* $(j' \circ \pi'_1)^{-1}(\mathbb{G}_2^\circ) \neq \emptyset$*:*

$$\begin{tikzcd} \mathcal{Y} \arrow[r, "j'"] \arrow[dr, "p_1"'] & D' \subset \mathbb{G}_2 \times S' \arrow[r] \arrow[d, "\pi'_2"] \arrow[rr, bend left, "\pi'_1"] & \mathbb{G}_2 \\ & S'\,. & \end{tikzcd}$$

Assume, moreover, that the inequalities

$$\frac{2d^2 + 7d}{3} \geqslant 4n - k + \frac{7}{3} \quad \text{and} \quad \frac{d^2 + 3d}{2} \geqslant 3n - k + 4$$

hold. Then the morphism j' *is not generically finite.*

The proofs of these propositions are similar to (and easier than) those of Propositions 5.4 and 5.6, so we omit them.

Note that under the assumption (3.5), the inequalities in Propositions 6.1 and 6.2 hold.

Remark 6.3. In case $k = 1$, Proposition 6.1 already appeared in [RY20, Remark].

Theorem 6.4. *Let* $d, n \geqslant 3$ *and* $1 \leqslant k \leqslant n - 2$ *be integers satisfying (3.5). Let* $X \subset \mathbb{P}^n$ *be a very general hypersurface of degree* d*. If* $Z \subset X$ *is a* k*-dimensional subvariety of* X *such that its desingularization* $Y \to Z$ *has* $\mathrm{H}^0(Y, K_Y) = 0$*, then* Z *is a union of lines and conics.*

Proof. Put $S = \mathrm{H}^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d)) \setminus \{0\}$, and let $S^\circ \subset S$ be the open subset of S consisting of the degree d homogeneous polynomials F such that the hypersurface defined by F is smooth. Let

$\mathcal{X}^\circ \subset \mathbb{P}^n \times S^\circ$ be the universal family of smooth hypersurfaces of degree d. Let $H = \mathrm{Hilb}(\mathcal{X}^\circ/S^\circ)$ be the Hilbert scheme parametrizing the k-dimensional closed subvarieties of $\mathcal{X}^\circ/S^\circ$, and let $\pi\colon H \to S^\circ$ be the projection. Let $\mathcal{Z} \subset \mathbb{P}^n \times_{S^\circ} H$ be the universal family of subvarieties. We can stratify H as $H = \sqcup_{\lambda\in\Lambda} H_\lambda$ by countably many locally closed GL_n-invariant smooth subvarieties of H in such a way that for each $\lambda \in \Lambda$, there exist a smooth variety $\mathcal{Y}_\lambda$ with GL_n-action, a smooth projective GL_n-equivariant morphism $\mathcal{Y}_\lambda \to H_\lambda$ with connected fibers and a GL_n-equivariant morphism $f_\lambda\colon \mathcal{Y}_\lambda \to \mathcal{Z} \times_H H_\lambda$ over H_λ such that the restrictions of f_λ to the fibers over every point of H_λ are generically injective.

Let $\Lambda' \subset \Lambda$ be the subset of Λ consisting of the λ such that $\pi|_{H_\lambda}\colon H_\lambda \to S^\circ$ is not dominant. We show that if X is the hypersurface defined by a homogeneous polynomial $F \in S^\circ \backslash \cup_{\lambda\in\Lambda'} \overline{\pi(H_\lambda)}$ and if $Z \subset X$ is a k-dimensional subvariety such that its desingularization $Y \to Z$ satisfies $\mathrm{H}^0(Y, K_Y) = 0$, then Z is a union of lines and conics.

The point $[Z] \in H$ representing Z is in H_λ with $\lambda \in \Lambda\backslash\Lambda'$. There exist a smooth variety S' with a GL_n-action, a GL_n-equivariant étale morphism $b\colon S' \to S$ and a GL_n-equivariant morphism $j\colon S' \to H_\lambda$ over S such that the image of j passes through a general point of H_λ (cf. [CR04, § 1]). The base change by j of $f_\lambda\colon \mathcal{Y}_\lambda \to \mathcal{Z} \times_H H_\lambda \subset \mathcal{X} \times_{S^\circ} H_\lambda$ yields a morphism $f\colon \mathcal{Y} \to \mathcal{X}'$ as in § 3.1. By assumption, it satisfies (3.4). It follows from Proposition 5.1 that we can assign to a general point $y \in \mathcal{Y}$ a line or smooth conic passing through $(h \circ f)(y)$ which lies on $\mathcal{X}'_{s'}$. This assignment yields a morphism $j\colon \mathcal{Y} \to D$ or $j'\colon \mathcal{Y} \to D'$ as in Proposition 6.1 or 6.2 after blowing up $\mathcal{Y}$ GL_n-equivariantly and shrinking S' if necessary. Suppose that we are in the situation of Proposition 6.1. Since the restriction $j_{s'}\colon \mathcal{Y}_{s'} \to D_{s'}$ of j is not generically finite, the dimension of $j_{s'}(\mathcal{Y}_{s'}) \subset \mathbb{G}_1$ is at most $k-1$. By the definition of j, we have $f_{s'}(\mathcal{Y}_{s'}) \subset \cup_{l\in j_{s'}(\mathcal{Y}_{s'})} l$. Then we must have $f_{s'}(\mathcal{Y}_{s'}) = \cup_{l\in j_{s'}(\mathcal{Y}_{s'})} l$ for a dimensional reason. When we are in the situation of Proposition 6.2, we can argue similarly.

Now we have proved that for general $u \in H_\lambda$, the variety $\mathcal{Z} \times_H \{u\}$ is a union of lines and conics. Since the property of being a union of lines and conics is a closed condition, we see that Z is also a union of lines and conics. □

Corollary 6.5. *Let $d, n \geqslant 3$ be integers satisfying the inequality $d > \frac{1}{5}(7n+16)$. If C is a rational curve on a very general hypersurface $X \subset \mathbb{P}^n$ of degree d, then it is a line or a conic.*

We conclude this paper with questions. The "simplest" rational curve other than a line and a conic would be a twisted cubic. A dimension count suggests that if $4n > 3d$, then degree d hypersurfaces in $\mathbb{P}^n$ will contain twisted cubics. Then it would be natural to ask the following.

Question 6.6. If $\frac{1}{5}(7n+16) \geqslant d \geqslant \frac{4}{3}n$, then are there any rational curves other than lines and conics on a very general degree d hypersurface in $\mathbb{P}^n$?

In view of Theorem 6.4, one can ask the following.

Question 6.7. Assume $d \geqslant n+2$. Is every subvariety with geometric genus zero of a very general degree d hypersurface in $\mathbb{P}^n$ uniruled?

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References

Cle03 H. Clemens, *Lower bounds on genera of subvarieties of generic hypersurfaces*, Comm. Algebra **31** (2003), no. 8, 3673–3711; doi:10.1081/AGB-120022438.

CR04 H. Clemens and Z. Ran, *Twisted genus bounds for subvarieties of generic hypersurfaces*, Amer. J. Math. **126** (2004), no. 1, 89–120; doi:10.1353/ajm.2004.0003. *Erratum*, Amer. J. Math. **127** (2005), no. 1, 241–242; doi:10.1353/ajm.2005.0003.

CR20 I. Coskun and E. Riedl, *Clustered families and applications to Lang-type conjectures*, 2020, arXiv:2010.11301.

Ein88 L. Ein, *Subvarieties of generic complete intersections*, Invent. Math. **94** (1988), no. 1, 163–169; doi:10.1007/BF01394349.

Ein91 ———, *Subvarieties of generic complete intersections. II*, Math. Ann. **289** (1991), no. 3, 465–471; doi:10.1007/BF01446583.

Pac03 G. Pacienza, *Rational curves on general projective hypersurfaces*, J. Algebraic Geom. **12** (2003), no. 2, 245–267; doi:10.1090/S1056-3911-02-00328-4.

Pac04 ———, *Subvarieties of general type on a general projective hypersurface*, Trans. Amer. Math. Soc. **356** (2004), no. 7, 2649–2661; doi:10.1090/S0002-9947-03-03250-1.

RY20 E. Riedl and D. Yang, *Rational curves on general type hypersurfaces*, J. Differential Geom. **116** (2020), no. 2, 393–403; doi:10.4310/jdg/1603936816.

Voi96 C. Voisin, *On a conjecture of Clemens on rational curves on hypersurfaces*, J. Differential Geom. **44** (1996), no. 1, 200–213; doi:10.4310/jdg/1214458743.

Voi98 ———, *A correction: "On a conjecture of Clemens on rational curves on hypersurfaces"*, J. Differential Geom. **49** (1998), no. 3, 601–611; doi:10.4310/jdg/1214461112.

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