



Finite torsors on projective schemes defined over a discrete valuation ring

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ABSTRACT

Given a Henselian and Japanese discrete valuation ring A and a flat and projective A -scheme X , we follow the approach of Biswas and dos Santos [J. Inst. Math. Jussieu **10** (2011), no. 2, 225–234] to introduce a *full subcategory* of coherent modules on X which is then shown to be Tannakian. We then prove that, under normality of the generic fibre, the associated affine and flat group is pro-finite in a strong sense (so that its ring of functions is a Mittag-Leffler A -module) and that it classifies finite torsors $Q \rightarrow X$. This establishes an analogy to Nori’s theory of the essentially finite fundamental group. In addition, we compare our theory with the ones recently developed by Mehta–Subramanian and Antei–Emsalem–Gasbarri. Using the comparison with the former, we show that any quasi-finite torsor $Q \rightarrow X$ has a reduction of the structure group to a finite one.

1. Introduction

Let A be a discrete valuation ring and X a projective flat A -scheme carrying an A -point x_0 . There has been some interest in constructing an affine and flat group scheme $\Pi(X, x_0)$ over A with the property that morphisms to finite and flat group schemes $\Pi(X, x_0) \rightarrow G$ should canonically correspond to pointed G -torsors over X ; see [Gas03, AEG20, MS13]. This is an analogue of Nori’s theory [Nor76] (Nori deals with a base *field*) and hence can be developed through

- (**SS**) a Tannakian category of semi-stable vector bundles [Nor76, Definition after Lemma 3.6],
- (**F**) the construction of fibre products of torsors [Nor82, Definition II.2] or
- (**T**) a Tannakian category of vector bundles trivialized by proper and surjective morphisms [BdS11].

The papers [Gas03] and [AEG20] follow (**F**), and [MS13] follows a variant of (**SS**). That is to say, [Gas03] and [AEG20] construct $\Pi(X, x_0)$ by producing a universal $\Pi(X, x_0)$ -torsor $\tilde{X} \rightarrow X$,

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and this is achieved by filtering the category of torsors under finite group schemes [AEG20, Theorem 4.2]. On the other hand, [MS13] looks at locally free \mathcal{O}_X -modules on X which lie in the categories considered by Nori when pulled back to the fibres of X ; see [MS13, beginning of Section 3, top of p. 346]. In the present work we focus on (\mathbf{T}) to construct the affine and flat group scheme $\Pi(X, x_0)$. It is therefore useful to recall briefly the overall picture behind (\mathbf{T}) in its original context. Given a smooth and projective variety V over a field, we look at those coherent sheaves which become trivial after being pulled back along a projective and surjective morphism $W \rightarrow V$. Then, this category ends up *coinciding* with that of essentially finite vector bundles (used in (\mathbf{SS})) if V is at least normal; see [BdS11, AM11, TZ20].

Once the approach (\mathbf{T}) is successfully implemented for X (details are to be reviewed further below), the category $\text{Rep}_A(\Pi(X, x_0))$ is realised as a *full* abelian subcategory of $\mathbf{coh}(X)$ so that not only can we construct the affine group scheme parametrising torsors, but we can naturally regard its category of representations in geometric terms. This facet is missing in [AEG20] (since it is not understood what coherent modules are obtained by twisting representations via the fundamental torsor) as well as in [MS13] (since the authors there focus on representations on free A -modules).

Let us review the remaining sections of the paper in order to give the reader an idea of our most relevant results. Section 2 serves to gather some facts from algebraic geometry (Lemmas 2.1, 2.2 and 2.3), to fix notation for a class of morphisms which is used all over in the paper (see Definition 2.4) and to put forward criteria allowing us to decide when a morphism of group schemes is faithfully flat (Lemmas 2.8 and 2.9).

It is in Section 3 where our theory starts to take form. In it, we introduce the central notion of modules which are *relatively trivial* (Definition 3.1) and the category \mathfrak{T}_φ of modules which are *relatively trivialized* by a proper and surjective morphism φ ; see Definition 3.4. The first useful properties of the category of such modules are also developed in Section 3: we show how to find more convenient trivializations (see Lemma 3.6, Definition 3.7 and Lemma 3.9) and how to make the first steps towards controlling kernels and cokernels of morphisms between relatively trivial modules (Proposition 3.10).

In Section 4, the work initiated in Section 3 is developed further, and we show that – under mild assumptions on the base scheme X – the category of modules which become relatively trivial after being pulled back along a suitable morphism $\varphi: Y \rightarrow X$, call it \mathfrak{T}_φ , is in fact abelian; see Theorem 4.3. In addition, concentrating on the objects in \mathfrak{T}_φ , which can be “dominated” by locally free ones, and using the fibre at an A -point x_0 of X , we construct a neutral Tannakian category (the terminology is that of [DH18, Definition 1.2.5]) over A . This category is called $\mathfrak{T}_\varphi^{\text{tan}}$ in Definition 4.6(3). We consequently obtain an affine and flat group scheme $\Pi(X, \varphi, x_0)$ (see [DH18, Theorem 1.2.6] or [Saa72, théorème II.4.1.1]), whose category of representations is naturally a full subcategory of $\mathbf{coh}(X)$. This is the *first main result*.

If K stands for the field of fractions of A and $X \otimes_A K$ is normal, Section 5 explains how to obtain information about $\Pi(X, \varphi, x_0) \otimes_A K$ from the category of essentially finite vector bundles (as defined in [Nor76, Definition after Lemma 3.6]) on $X \otimes_A K$; see Theorem 5.3 and Corollary 5.4.

Section 6 studies \mathfrak{T}_φ (Definition 3.4) by means of an idea introduced in [HdS21]. In this work we found a property of representations of a flat and affine group scheme G over a complete A , called *prudence*, which allows us to verify when the A -module of functions $A[G]$ is free. This is useful here since *prudence* is roughly Grothendieck’s algebraization in geometry. This fact leads to our *second main result*, which is that the full Galois groups constructed from \mathfrak{T}_φ are in many

cases *finite* group schemes; see Corollary 6.2. The findings here also serve to substantiate a claim made in [MS13, Lemma 3.1 and Theorem 4.1]. (See also our summary of Section 11 below.)

Section 7 recalls the reduced fibre theorem and adapts it for future application; see Theorem 7.1. This theorem comes with the (mild) hypothesis that a certain generic fibre should be geometrically reduced; since we wish to profit from this result even in the absence of such an assumption, we benefit from this accessory section to explain how to get rid of schemes which fail to be geometrically reduced; see Lemma 7.2. It is perhaps worth pointing out that it is at this point that additional properties of A – that it be Henselian and Japanese – start being required.

Recall that in Sections 3 and 4, we associated with a morphism $\varphi: Y \rightarrow X$ (satisfying certain technical assumptions) a tensor category \mathfrak{T}_φ , which, once polished, becomes neutral Tannakian. The task of putting the \mathfrak{T}_φ together for varying φ comes to the front. In Section 8, we address this issue. Our *third main result*, Theorem 8.1, provides the technical conditions for putting the \mathfrak{T}_φ together, while Corollary 8.2 expresses the consequence that the “system” of the \mathfrak{T}_φ is “filtered”. We are then able to construct a neutral Tannakian category \mathfrak{T}_X which is a full subcategory of $\mathbf{coh}(X)$; see Definition 8.3. The group scheme associated with \mathfrak{T}_X via the fibre functor $\bullet|_{x_0}: \mathfrak{T}_X \rightarrow A\text{-mod}$, call it $\Pi(X, x_0)$, has the distinctive property that its category of representations $\text{Rep}_A(\Pi(X, x_0))$ is equivalent, by construction, to the full subcategory \mathfrak{T}_X of $\mathbf{coh}(X)$. Under the assumption of normality of X , we are then able to show that $\Pi(X, x_0)$ is *pro-quasi-finite* (in the sense of Definition 8.7); see Theorem 8.8. If A is in addition complete, the results of Section 6 apply directly, and we obtain that $\Pi(X, x_0)$ is *strictly* pro-finite; see Definition 8.7 and Theorem 8.8. Finally, an indirect argument allows us to deduce from this our *fourth main result*: $\Pi(X, x_0)$ is *strictly* pro-finite even if A is simply Henselian and Japanese; see Theorem 8.10. As a consequence, the ring of functions of $\Pi(X, x_0)$ is a Mittag-Leffler A -module (Corollary 8.11).

In Section 9 we make a brief digression to give examples and applications of the previous theory to invertible sheaves; see Proposition 9.1. These examples served as inspiration and may be useful to the reader, although they are not used anywhere else in the paper.

Section 10 gives conditions for a finite group scheme – appearing as the structure group of a torsor over X – to be a quotient of $\Pi(X, x_0)$; see Proposition 10.3 and Corollary 10.4.

In Section 11 we review parts of [MS13]. Let us explain the reason for this review before summarising the contents of Section 11. If X_0 stands for the special fibre of X , one of the main points of [MS13] is to use a technique in [DW05], see pp. 574–576, especially the proof of Theorem 17 and Lemma 19, to show, speaking colloquially, how to trivialize a vector bundle $E \in \mathbf{VB}(X_0)$ on the special fibre of a *flat* A -scheme Y endowed with a finite morphism to X ; see the *proof* of [MS13, Lemma 3.1]. In our opinion, [MS13] only offers a sketch of how to implement this beautiful idea: in the proof of [MS13, Lemma 3.1], the necessary “base change” to an extension of A is not mentioned; neither is [DW05, Theorem 17]. In addition, as far as we can see, the proof of [MS13, Theorem 4.1] – which should extend Lemma 3.1 of op. cit. – offers solely a construction of a smooth curve inside a projective and smooth A -scheme. Given all these considerations, we set out to put the method of Deninger and Werner in a more robust form. In doing so, we are able to avoid Liu’s theorem on the existence of semi-stable models of relative curves (a crucial point in [DW05] and hence in [MS13, Lemma 3.1]) by means of the reduced fibre theorem (Theorem 7.1). The advantage is that the latter result holds for more general schemes than just curves. This allows us to show Theorem 11.1 saying that an F -trivial vector bundle E (see the introduction of Section 11 for the definition) on X_0 becomes relatively trivial (Definition 3.1) after pull-back by a finite morphism $Y \rightarrow X$ from a flat A -scheme. In the

same vein, we offer Theorem 11.2 explaining how to trivialize any vector bundle $\mathcal{E} \in \mathbf{VB}(X)$ with essentially finite fibres by a finite morphism $Y \rightarrow X$ from a *flat* A -scheme. (It should be noted that the proof of this result is technically intricate, but the main idea comes straight from Theorem 11.1.) The section then ends with a direct comparison between the category \mathfrak{T}_X and the one studied by [MS13]; see Corollary 11.4, where it is shown that vector bundles on \mathfrak{T}_X agree with those introduced in op. cit. This is our *fifth main finding*.

Section 12 gives further applications to the theory of torsors and shows our *sixth main result*, Theorem 12.1; it proves that torsors under quasi-finite group schemes actually come from finite ones. Another noteworthy point developed in this section is the comparison to the theory of [AEG20] and to the “classification” problem mentioned at the start of this introduction. By demonstrating that $\Pi(X, x_0)$ “classifies” pointed quasi-finite torsors over X , which is our *seventh most relevant result* and Theorem 12.3, we show that $\Pi(X, x_0)$ agrees with the pro-quasi-finite fundamental group, denoted by $\pi_1(X, x_0)^{\text{qf}}$ in op. cit.; see Theorem 12.4.

Notation and conventions

On the base ring

- (1) We let A be a discrete valuation ring with uniformizer π , field of fractions K and residual field k . The quotient ring $A/(\pi^{n+1})$ is denoted by A_n .
- (2) Given an object W over A (a scheme, a module, etc.) and an A -algebra B , we find useful to write W_B instead of $W \otimes_A B$. If context prevents any misunderstanding, we also employ W_n instead of W_{A_n} .
- (3) The characteristic of a discrete valuation ring is the ordered pair (r, p) , where r is the characteristic of the field of fractions and p that of the residue field.

On general algebraic geometry

- (1) A *vector bundle* over a scheme is a locally free sheaf of finite rank. A vector bundle is said to be trivial if it is isomorphic to a direct sum of copies of the structure sheaf (in particular, the rank is constant).
- (2) Let R be a noetherian ring and Y a proper R -scheme. We shall say that Y is H^0 -flat over R if it is flat and cohomologically flat [BLR90, p. 206, paragraph preceding Theorem 8.1/7] in degree zero over $\text{Spec } R$. In this work we shall employ constantly that if Y is R -flat, then (a) H^0 -flatness amounts to exactness of $M \mapsto H^0(\mathcal{O}_Y \otimes_R M)$, see [Har77, Proposition III.12.5 and Corollary III.12.6], and (b) if the fibres of Y are geometrically reduced [EGA, IV₂, définition 4.6.2], then Y is H^0 -flat over R ; see [EGA, III₂, proposition 7.8.6].
- (3) If R , respectively T , is an \mathbb{F}_p -algebra, respectively \mathbb{F}_p -scheme, we write $F_R: R \rightarrow R$, respectively $F_T: T \rightarrow T$, to denote the *Frobenius morphism*. If T is in addition a scheme over a perfect field, we adopt the notation of [Jan87, Section I.9.1] with the exception that we write $\text{Fr}^s: T^{(t)} \rightarrow T^{(s+t)}$ while Jantzen uses $F_{T^{(t)}}^s: T^{(t)} \rightarrow T^{(t+s)}$.
- (4) If R is a discrete valuation ring and $y: \text{Spec } R \rightarrow Y$ is an R -point of a scheme Y , we shall write y_{gen} for the image of the generic point of $\text{Spec } R$ in Y .
- (5) Let $X \rightarrow S$ be a morphism of schemes possessing in addition a section $x: S \rightarrow X$. Given morphisms $\varphi: X' \rightarrow X$ and $f: S' \rightarrow S$, any morphism of S -schemes $x': S' \rightarrow X'$ such that $\varphi x' = x f$ shall be called an S' -point of X' above x . (That is, for the sake of economy we choose not to make base changes to S' .)

On group schemes

- (1) To avoid repetitions, “*group scheme*” is a synonym for “*affine group scheme*”. If G is a group scheme over a ring R and R' is an R -algebra, we write $R'[G]$ instead of $\mathcal{O}(R' \otimes_R G)$.
- (2) The category of flat group schemes over a ring R is denoted by (\mathbf{FGSch}/R) .
- (3) Let G be a flat group scheme over the noetherian ring R . When dealing with representations of G , we follow the conventions of [Jan87, Section I.2] with the exception that the word “representation” is reserved for G -modules which are of finite type over R . The category of representations is denoted by $\mathrm{Rep}_R(G)$. The full subcategory of representations which underlie locally free R -modules is denoted by $\mathrm{Rep}_R^\circ(G)$.
- (4) The *right-regular*, respectively *left-regular*, G -module [Jan87, Section I.2.7] shall be denoted by $R[G]_{\mathrm{right}}$, respectively $R[G]_{\mathrm{left}}$.
- (5) If $f: G \rightarrow H$ is an arrow of (\mathbf{FGSch}/R) , we let $f^\#: \mathrm{Rep}_R(H) \rightarrow \mathrm{Rep}_R(G)$ be the restriction functor.

On torsors

- (1) Let R be a ring, X an R -scheme, G and H group schemes over R , $P \rightarrow X$ a G -torsor and $Q \rightarrow X$ an H -torsor. A *generalized morphism* from P to Q is an ordered pair (f, ρ) consisting of an arrow of X -schemes $f: P \rightarrow Q$ and a morphism of group schemes $\rho: G \rightarrow H$ such that, for points with value on arbitrary R -algebras, we have $f(yg) = f(y)\rho(g)$. In this case, we say that the generalized morphism *covers* the morphism ρ . If $G = H$, then a morphism of torsors from P to Q is simply a G -equivariant morphism of X -schemes or a generalized morphism covering the identity.
- (2) If R is a ring, $\rho: G \rightarrow H$ is an arrow of (\mathbf{FGSch}/R) and $P \rightarrow X$ is a G -torsor, we let $P \times^G H$ or $P \times^\rho H$ be the associated H -torsor; see [DG70, section III.4.3.2, p. 368] or [Jan87, Section I.5.14].
- (3) If G is a flat group scheme over a ring R and $P \rightarrow X$ is a G -torsor and M a representation of G , then we let $\theta_P(M)$ stand for the coherent sheaf constructed by twisting P by M ; see [Jan87, Sections I.5.8 and I.5.9], where it is denoted by $\mathcal{L}_{P/G}(M)$.

Miscellaneous

- (1) All tensor categories and functors are to be taken in the sense of [DM82, Section 1]. Let (\mathfrak{C}, \otimes) be a rigid tensor category [DM82, Definition 1.7]. If \mathfrak{C} is additive, then, for $a = (a_1, \dots, a_m)$ and $b = (b_1, \dots, b_m)$ in \mathbb{N}^m and $E \in \mathfrak{C}$, we write $\mathbf{T}^{a,b}E$ to denote the object $\bigoplus_i E^{\otimes a_i} \otimes \check{E}^{\otimes b_i}$. If \mathfrak{C} is in addition abelian, we let $\langle E; \mathfrak{C} \rangle_\otimes$ stand for the full subcategory of \mathfrak{C} whose objects are subquotients of some $\mathbf{T}^{a,b}E$.
- (2) If X is a proper and reduced scheme over a field, we let $\mathbf{EF}(X)$ denote the category of essentially finite vector bundles on X ; see [Nor76, Definition after Lemma 3.6].

2. Subsidiary material

We collect here some simple facts which are useful in developing our arguments.

LEMMA 2.1. *Let Y be a proper, reduced and connected scheme over the field K . If Y has a K -point, then $K = H^0(\mathcal{O}_Y)$.*

Proof. We know that $H^0(\mathcal{O}_Y) = R_1 \times \dots \times R_m$, where each R_i is a local Artin algebra. Since Y

is connected, we must have $m = 1$. So $H^0(\mathcal{O}_Y)$ is a local Artin algebra. Since Y is reduced, $H^0(\mathcal{O}_Y)$ is reduced, so $H^0(\mathcal{O}_Y)$ is a finite field extension of K . The existence of a K -rational point determines a morphism of K -algebras $H^0(\mathcal{O}_Y) \rightarrow K$, and this forces $H^0(\mathcal{O}_Y)$ to be K . \square

LEMMA 2.2. *Let Y be a flat and proper A -scheme. If $H^0(\mathcal{O}_{Y \otimes_A K}) = K$, then $H^0(\mathcal{O}_Y) = A$. In particular, if $Y \otimes_A K$ is reduced and connected and has a K -point, then $H^0(\mathcal{O}_Y) = A$.*

Proof. We know that $H^0(\mathcal{O}_Y)$ is a finite A -algebra and that $H^0(\mathcal{O}_Y) \otimes K = K$ by flat base change [Har77, Proposition III.9.3] together with the assumption. Then, $A \subset H^0(\mathcal{O}_Y) \subset K$. Since A is normal, we must have $A = H^0(\mathcal{O}_Y)$. The last claim follows immediately from Lemma 2.1. \square

LEMMA 2.3. *Let Y be a proper and H^0 -flat A -scheme. Let \mathcal{F} be a vector bundle on Y whose base changes \mathcal{F}_K and \mathcal{F}_k are trivial. Then \mathcal{F} is trivial.*

Proof. It is not difficult to see that the function $y \mapsto \text{rank } \mathcal{F}_y$ has to be constant and equal to n , say. From the equalities

$$\begin{aligned} \dim_k H^0(Y_k, \mathcal{F}_k) &= \dim_k H^0(Y_k, \mathcal{O}_{Y_k})^{\oplus n} \\ &= \dim_K H^0(Y_K, \mathcal{O}_{Y_K})^{\oplus n} \\ &= \dim_K H^0(Y_K, \mathcal{F}_K) \end{aligned}$$

and [Mum70, Corollary 5.2], we conclude that the canonical morphism

$$H^0(Y, \mathcal{F}) \longrightarrow H^0(Y_k, \mathcal{F}_k)$$

is surjective. Pick sections $s_1, \dots, s_n \in H^0(Y, \mathcal{F})$ such that $\{s_i|_{Y_k}\}_{i=1}^n$ is a basis of \mathcal{F}_k . Then, for each point y of the closed fibre, Nakayama's lemma shows that $\{s_{i,y}\}_{i=1}^n$ is a basis of the free \mathcal{O}_y -module \mathcal{F}_y , so that $\text{Supp}(\mathcal{F} / \sum \mathcal{O}_Y s_i)$ is a closed subset of Y disjoint from the special fibre. Such a property is only possible if $\mathcal{F} = \sum \mathcal{O}_Y s_i$. The proof is then complete once we note that, for a $y \in Y$, the n generators $\{s_{i,y}\}_{i=1}^n$ of $\mathcal{F}_y \simeq \mathcal{O}_y^{\oplus n}$ do not admit any non-trivial relation. \square

In organising our findings, we shall make repeated use of a certain class of morphisms of proper schemes. In order to avoid repetitions and to serve as a reference for the reader, we put forward the following.

DEFINITION 2.4. Let X be a connected, proper and flat A -scheme carrying an A -point x_0 . Let $\mathfrak{S}(X, x_0)$ (or simply \mathfrak{S} if context prevents any misunderstanding) be the set of all X -schemes $\varphi: Y \rightarrow X$ such that:

- Ⓔ1. The morphism φ is proper and surjective.
- Ⓔ2. The scheme Y is H^0 -flat over A .
- Ⓔ3. The canonical arrow $Y \rightarrow \text{Spec } H^0(\mathcal{O}_Y)$ admits a section y_0 such that φy_0 extends x_0 . (There exists an $H^0(\mathcal{O}_Y)$ -point in Y above x_0 .)

We shall denote by $\mathfrak{S}^+(X, x_0)$ (or simply \mathfrak{S}^+) the subset of morphisms $\varphi: Y \rightarrow X$ in \mathfrak{S} which, in addition, satisfy:

- Ⓔ4. The canonical morphism $Y \rightarrow \text{Spec } H^0(\mathcal{O}_Y)$ is flat.

Remark 2.5. Let $\varphi: Y \rightarrow X$ belong to $\mathfrak{S}(X, x_0)$ as in Definition 2.4; it turns out that in this case, the connectedness of X is automatic because of property Ⓔ3. Indeed, let y_0 be the alluded $H^0(\mathcal{O}_Y)$ -point of Y above x_0 . Suppose that $e \in H^0(\mathcal{O}_X)$ is idempotent and that $x_0^\#(e) = 0$. Then, $\varphi^\#(e) \in H^0(\mathcal{O}_Y)$ is such that $y_0^\#(\varphi^\#(e)) = 0$, and this implies that $\varphi^\#(e) = 0$. Since φ

is surjective, we conclude that $e = 0$ (since e has to vanish on each local ring of X). Hence, the only connected component of X must be the one containing the image of x_0 .

We should now gather material on affine group schemes.

DEFINITION 2.6. Let $G \in (\mathbf{FGSch}/A)$. We say that G is pseudo-finite if both its fibres are finite group schemes over the respective residue fields.

Remark 2.7. We have no examples of pseudo-finite group schemes which are not quasi-finite to offer.

As Lemma 2.8 below recalls, one advantage of finite group schemes over a field is that the standard criterion for verifying when a morphism is faithfully flat in terms of representation categories [DM82, Proposition 2.21] admits a considerable simplification. This is then transmitted to pseudo-finite group schemes as argued by Lemma 2.9.

LEMMA 2.8 [Nor82, Proposition II.3, pp. 87–88]. *Let $u: H \rightarrow G$ be a morphism of group schemes over a field, and let*

$$H \xrightarrow{q} I \xrightarrow{i} G$$

be its factorisation into a faithfully flat morphism q and a closed immersion i . Assuming that I is finite (which is the case if either G or H is finite), a necessary and sufficient condition for u to be faithfully flat is that $\dim \mathcal{O}(G)^H = 1$.

Proof. We leave the proof of necessity in the statement to the reader and from now on assume $\dim \mathcal{O}(G)^H = 1$. Clearly, the equality $I = G$ is equivalent to faithful flatness of u . Since $\mathcal{O}(G)^H = \mathcal{O}(G)^I$, the assumptions translate into $\dim \mathcal{O}(G)^I = 1$. Now, we know that $\mathcal{O}(G)$ is a projective $\mathcal{O}(G)^I$ -module whose rank equals $\dim \mathcal{O}(I)$ (see [Mum70, Theorem 12.1(B)] or [DG70, section III.2.4]). Then, since $\dim \mathcal{O}(G)^I = 1$, we conclude that $\dim \mathcal{O}(I) = \dim \mathcal{O}(G)$, which means that $I = G$. \square

LEMMA 2.9. *Let $u: H \rightarrow G$ be an arrow in (\mathbf{FGSch}/A) , and assume that G is pseudo-finite. Then the following conditions are equivalent:*

- (1) *Both equalities*

$$A \cdot 1 = (A[G]_{\text{right}})^H \quad \text{and} \quad k \cdot 1 = (k[G]_{\text{right}})^H$$

are true.

- (2) *The morphism u is faithfully flat.*

- (3) *Let*

$$\mathfrak{s}(G) = \left\{ \begin{array}{l} V \subset A[G]_{\text{right}} \text{ is } G\text{-invariant, is finitely generated} \\ \text{as an } A\text{-module and contains the constants} \end{array} \right\}$$

and

$$\mathfrak{s}_0(G) = \left\{ \begin{array}{l} M \subset k[G]_{\text{right}} \text{ is } G\text{-invariant, is finitely generated} \\ \text{as a } k\text{-space and contains the constants} \end{array} \right\}.$$

Then, for each $V \in \mathfrak{s}(G)$ and $M \in \mathfrak{s}_0(G)$, we have

$$A \cdot 1 = V^H \quad \text{and} \quad k \cdot 1 = M^H.$$

Proof. (1) \Rightarrow (2). The assumptions show that

$$(K[G]_{\text{right}})^{H_K} = K \quad \text{and} \quad (k[G]_{\text{right}})^{H_k} = k.$$

By Lemma 2.8, we conclude that $u_K: H_K \rightarrow G_K$ and $u_k: H_k \rightarrow G_k$ are faithfully flat. Because of [DH18, Theorem 4.1.1], the morphism u is faithfully flat.

(2) \Rightarrow (3). This is trivial.

(3) \Rightarrow (1). Any $a \in A[G]_{\text{right}}$, respectively $b \in k[G]_{\text{right}}$, belongs to a certain $V \in \mathfrak{s}(G)$, respectively $M \in \mathfrak{s}_0(G)$, because of “local finiteness”; see [Ser68, section 1.5, corollaire]. The conclusion then follows. \square

3. Modules trivialized by a proper and H^0 -flat scheme

In this section, we introduce the category of coherent sheaves on which all further developments hinge: the category of sheaves which became “trivial” after a pull-back by a proper morphism. Since we wish to work with schemes over a discrete valuation ring (d.v.r.), the notion of triviality of a coherent module becomes more subtle than the one over a field, and we need to account for modules coming from the base ring. This is a source of difficulty specially because the “base” of the scheme effectuating the trivialization might grow.

DEFINITION 3.1. Let Y be any A -scheme. We say that $\mathcal{F} \in \mathbf{coh}(Y)$ is trivial relatively to A if there exists a coherent sheaf (that is, a finite A -module) F such that $\mathcal{F} \cong \mathcal{O}_Y \otimes_A F$.

Remark 3.2. Note that if A is a field, then a coherent sheaf on Y is trivial relatively to A if and only if it is trivial according to our conventions in Section 1.

In Definition 3.1, it is to be expected that several different choices concerning the “descended” module are possible. The next lemma, which we state here for future use, explains how to be more canonical.

LEMMA 3.3. Let Y be proper and H^0 -flat over A . Write $B = H^0(\mathcal{O}_Y)$, and let F be a finite A -module. Then the canonical morphism

$$\mathcal{O}_Y \otimes_B H^0(\mathcal{O}_Y \otimes_A F) \longrightarrow \mathcal{O}_Y \otimes_A F$$

is an isomorphism.

Proof. By the definition of H^0 -flatness, the canonical arrow

$$\sigma: B \otimes_A F \longrightarrow H^0(\mathcal{O}_Y \otimes_A F)$$

is bijective. We then consider the commutative diagram

$$\begin{array}{ccc} \mathcal{O}_Y \otimes_B H^0(\mathcal{O}_Y \otimes_A F) & \longrightarrow & \mathcal{O}_Y \otimes_A F, \\ \uparrow \sim \text{id} \otimes \sigma & \nearrow \sim & \\ \mathcal{O}_Y \otimes_B (B \otimes_A F), & & \end{array}$$

where the horizontal arrow is the one of the statement. \square

Here is the central definition of this section.

DEFINITION 3.4. If $\varphi: Y \rightarrow X$ is a morphism of A -schemes, we let \mathfrak{T}_φ stand for the full subcategory of $\mathbf{coh}(X)$ consisting of those coherent sheaves \mathcal{E} such that the coherent \mathcal{O}_Y -module $\varphi^*(\mathcal{E})$ is trivial relatively to A .

Let X be a connected, proper and flat A -scheme, and let x_0 be an A -point of X . Let $\varphi: Y \rightarrow X$ be an object of $\mathfrak{S}(X, x_0)$ (see Definition 2.4). If $B = H^0(\mathcal{O}_Y)$, we let y_0 stand for the B -point of Y above x_0 . We note that the A -module B is finite and torsion-free, hence free. The following will be useful further ahead.

LEMMA 3.5. *In the above notation, a coherent module $\mathcal{F} \in \mathbf{coh}(X)$ belongs to \mathfrak{T}_φ if and only if $\varphi^*(\mathcal{F})$ is trivial relatively to B .*

Proof. We only show the “if” clause. In this case, $\varphi^*(\mathcal{F}) \simeq \mathcal{O}_Y \otimes_B F$, and using the point y_0 , we conclude that $F \simeq \varphi^*(\mathcal{F})|_{y_0}$. Now, since y_0 is taken to the A -point x_0 , we have that $\varphi^*(\mathcal{F})|_{y_0} \simeq B \otimes_A (\mathcal{F}|_{x_0})$, so that $\varphi^*(\mathcal{F}) \simeq \mathcal{O}_Y \otimes_A (\mathcal{F}|_{x_0})$. \square

We now give ourselves an object \mathcal{E} of \mathfrak{T}_φ : this means that $\varphi^*(\mathcal{E})$ is isomorphic to the pull-back of a certain A -module to Y . We shall notice that $\mathcal{E}|_{x_0}$ is a possible choice.

LEMMA 3.6. *Let $E := \mathcal{E}|_{x_0}$. Then there exists an isomorphism $\alpha: \mathcal{O}_Y \otimes_A E \xrightarrow{\sim} \varphi^*(\mathcal{E})$. In addition, if \bar{E} is an A -module such that $\varphi^*(\mathcal{E}) \simeq \mathcal{O}_Y \otimes_A \bar{E}$, then \bar{E} must in fact be isomorphic to E .*

Proof. Write $\varphi^*(\mathcal{E}) \simeq \mathcal{O}_Y \otimes_A \bar{E}$ for some other A -module \bar{E} . Using y_0 , we conclude that $B \otimes_A E \simeq B \otimes_A \bar{E}$. Now this shows that $\mathcal{O}_Y \otimes_A E \simeq \mathcal{O}_Y \otimes_A \bar{E}$ because $\mathcal{O}_Y \otimes_B (B \otimes_A (-)) = \mathcal{O}_Y \otimes_A (-)$. To handle the final statement, the isomorphism $B \otimes_A E \simeq B \otimes_A \bar{E}$ is sufficient. Here are the details. Agreeing to write $A(\ell) = A/(\pi^\ell)$, we have

$$E \simeq A^r \oplus A(\delta_1) \oplus \cdots \oplus A(\delta_m) \quad \text{and} \quad \bar{E} \simeq A^{\bar{r}} \oplus A(\bar{\delta}_1) \oplus \cdots \oplus A(\bar{\delta}_{\bar{m}});$$

here either m is a positive integer, in which case $\delta_1 \geq \cdots \geq \delta_m$ are also positive integers, or $m = 0$ and the factors $A(\delta_i)$ are to be dropped, and analogous considerations are in force for \bar{E} . As B is free over A , of rank $s \geq 1$ say, we have, as A -modules,

$$B \otimes_A E \simeq A^{rs} \oplus \underbrace{A(\delta_1) \oplus \cdots \oplus A(\delta_1)}_s \oplus \cdots \oplus \underbrace{A(\delta_m) \oplus \cdots \oplus A(\delta_m)}_s$$

and

$$B \otimes_A \bar{E} \simeq A^{\bar{r}s} \oplus \underbrace{A(\bar{\delta}_1) \oplus \cdots \oplus A(\bar{\delta}_1)}_s \oplus \cdots \oplus \underbrace{A(\bar{\delta}_{\bar{m}}) \oplus \cdots \oplus A(\bar{\delta}_{\bar{m}})}_s,$$

so that the isomorphism $B \otimes_A E \simeq B \otimes_A \bar{E}$ implies that $r = \bar{r}$, $m = \bar{m}$ and $\delta_i = \bar{\delta}_i$. \square

We observe that the latter part of Lemma 3.6 shall not be employed in the rest of this text. Now, we note that while an A -module such that $\varphi^*(\mathcal{E}) \simeq \mathcal{O}_Y \otimes_A E$ is uniquely determined, one can find many isomorphisms $\mathcal{O}_Y \otimes_A E \simeq \varphi^*(\mathcal{E})$, and we now single out a *special class* of such in order to achieve a proof of Proposition 3.10 below. This proposition is a key step in endowing the category \mathfrak{T}_φ with kernels and cokernels (see Section 4).

From now on we write

$$\boxed{E := \mathcal{E}|_{x_0} \quad \text{and} \quad F = \varphi^*(\mathcal{E})|_{y_0} .}$$

Using y_0 and the fact that it “extends” x_0 , we arrive at canonical isomorphisms

$$\iota: F \xrightarrow{\sim} B \otimes_A E \quad \text{and} \quad I: \mathcal{O}_Y \otimes_B F \xrightarrow{\sim} \mathcal{O}_Y \otimes_A E. \quad (3.1)$$

DEFINITION 3.7. An isomorphism $\beta: \mathcal{O}_Y \otimes_B F \rightarrow \varphi^*(\mathcal{E})$ is called *adapted to y_0* if the composition

$$F \xrightarrow{\text{canonical}} (\mathcal{O}_Y \otimes_B F)|_{y_0} \xrightarrow{\beta} \varphi^*(\mathcal{E})|_{y_0}$$

is the identity.

We shall show that adapted isomorphisms exist. For that and future applications, we need the following.

LEMMA 3.8. *Let $\overline{\mathcal{E}}$ be another object of \mathfrak{T}_φ than \mathcal{E} , and write \overline{F} for the B -module $\varphi^*(\overline{\mathcal{E}})|_{y_0}$. Then the pull-back morphism*

$$\text{id} \otimes (-): \text{Hom}_B(F, \overline{F}) \longrightarrow \text{Hom}_{\mathcal{O}_Y}(\mathcal{O}_Y \otimes_B F, \mathcal{O}_Y \otimes_B \overline{F})$$

is bijective with inverse

$$(-)|_{y_0}: \text{Hom}_{\mathcal{O}_Y}(\mathcal{O}_Y \otimes_B F, \mathcal{O}_Y \otimes_B \overline{F}) \longrightarrow \text{Hom}_B(F, \overline{F}).$$

If $\beta: \mathcal{O}_Y \otimes_B F \xrightarrow{\sim} \varphi^*(\mathcal{E})$ and $\overline{\beta}: \mathcal{O}_Y \otimes_B \overline{F} \xrightarrow{\sim} \varphi^*(\overline{\mathcal{E}})$ are adapted, then the composition

$$\text{Hom}_{\mathcal{O}_Y}(\varphi^*(\mathcal{E}), \varphi^*(\overline{\mathcal{E}})) \xrightarrow{\overline{\beta}^{-1}(-)\beta} \text{Hom}_{\mathcal{O}_Y}(\mathcal{O}_Y \otimes_B F, \mathcal{O}_Y \otimes_B \overline{F}) \xrightarrow{(-)|_{y_0}} \text{Hom}_B(F, \overline{F})$$

is $(-)|_{y_0}$.

Proof. Let $\overline{\iota}: \overline{F} \xrightarrow{\sim} B \otimes_A \overline{E}$ and $\overline{I}: \mathcal{O}_Y \otimes_B \overline{F} \xrightarrow{\sim} \mathcal{O}_Y \otimes_A \overline{E}$ be the isomorphisms analogous to those in (3.1). It follows easily that

$$\begin{array}{ccc} \text{Hom}_{\mathcal{O}_Y}(\mathcal{O}_Y \otimes_B F, \mathcal{O}_Y \otimes_B \overline{F}) & \xrightarrow{\sim} & \text{Hom}_B(F, H^0(\mathcal{O}_Y \otimes_B \overline{F})) & \xrightarrow[\text{via } \overline{I}]{\sim} & \text{Hom}_B(F, H^0(\mathcal{O}_Y \otimes_A \overline{E})) \\ & & & & \uparrow \sim \\ & & & & \text{Hom}_B(F, B \otimes_A \overline{E}) \\ & \swarrow \text{id} \otimes (-) & & & \uparrow \sim \\ & & & & \text{Hom}(F, \overline{F}) \end{array}$$

commutes; this is enough to justify all claims of the first statement. Those of the second are very simple. \square

LEMMA 3.9. *Isomorphisms adapted to y_0 always exist.*

Proof. We choose any isomorphism $\beta: \mathcal{O}_Y \otimes_B F \xrightarrow{\sim} \varphi^*(\mathcal{E})$ and let b be the composition

$$F \xrightarrow{\text{canonical}} (\mathcal{O}_Y \otimes_B F)|_{y_0} \xrightarrow{\beta|_{y_0}} \varphi^*(\mathcal{E})|_{y_0}.$$

Let $\gamma: \mathcal{O}_Y \otimes_B F \xrightarrow{\sim} \mathcal{O}_Y \otimes_B F$ induce b^{-1} via $(-)|_{y_0}$ (which exists due to Lemma 3.8). Then $\beta\gamma$ is adapted to y_0 . \square

Granted these preparations, we can now have a better control over kernels and cokernels in \mathfrak{T}_φ .

PROPOSITION 3.10. *Let $\overline{\mathcal{E}}$ be another object of \mathfrak{T}_φ than \mathcal{E} and $u: \mathcal{E} \rightarrow \overline{\mathcal{E}}$ a morphism of \mathcal{O}_X -modules. Write \overline{F} for $\varphi^*(\overline{\mathcal{E}})|_{y_0}$. Let $\beta: \mathcal{O}_Y \otimes_B F \xrightarrow{\sim} \varphi^*(\mathcal{E})$ and $\overline{\beta}: \mathcal{O}_Y \otimes_B \overline{F} \xrightarrow{\sim} \varphi^*(\overline{\mathcal{E}})$ be adapted to y_0 . The following claims hold true:*

- (1) *If $v_0 := \varphi^*(u)|_{y_0}: F \rightarrow \overline{F}$, then $\varphi^*(u) = \overline{\beta} \circ (\text{id} \otimes_B v_0) \circ \beta^{-1}$. Put differently, $\varphi^*(u)$ “descends” to B .*
- (2) *Write \overline{E} for the A -module $\overline{\mathcal{E}}|_{x_0}$. Let $\overline{\iota}: \overline{F} \xrightarrow{\sim} B \otimes_A \overline{E}$ and $\overline{I}: \mathcal{O}_Y \otimes_B \overline{F} \xrightarrow{\sim} \mathcal{O}_Y \otimes_A \overline{E}$ be the isomorphisms analogous to those in (3.1). Define $\alpha = \beta I^{-1}$ and $\overline{\alpha} = \overline{\beta} \overline{I}^{-1}$. Then, the*

morphism of A -modules $u_0 := u|_{y_0} : E \rightarrow \overline{E}$ renders

$$\begin{array}{ccc} \varphi^* \mathcal{E} & \xrightarrow{\varphi^* u} & \varphi^* \overline{\mathcal{E}} \\ \alpha \uparrow & & \uparrow \bar{\alpha} \\ \mathcal{O}_Y \otimes_A E & \xrightarrow{\text{id} \otimes_A u_0} & \mathcal{O}_Y \otimes_A \overline{E} \end{array}$$

commutative. Put differently, $\varphi^*(u)$ “descends” to A .

(3) The coherent sheaves $\text{Ker}(\varphi^* u)$ and $\text{Coker}(\varphi^* u)$ are trivial relatively to A .

Proof. Claim (1) is a direct consequence of Lemma 3.8. Claim (2) is a simple consequence of claim (1). Claim (3) follows from claim (2) together with the fact that Y is flat over A . \square

4. Further properties of the category of coherent modules trivialized by a proper morphism

In this section, we let X be a proper and flat A -scheme with *reduced fibres* and x_0 an A -point of X . Let $\varphi : Y \rightarrow X$ be an object of $\mathfrak{S}(X, x_0)$. (Recall that in such a situation, X is connected; see Remark 2.5.) If $B = H^0(\mathcal{O}_Y)$, we let y_0 stand for the B -point of Y above x_0 . Let us gather some simple properties concerning the category \mathfrak{T}_φ of Definition 3.4.

LEMMA 4.1. *Let $u : \mathcal{E} \rightarrow \mathcal{F}$ be an arrow of \mathfrak{T}_φ . Then $\mathcal{C} = \text{Coker}(u)$ belongs to \mathfrak{T}_φ .*

Proof. As $\text{Coker}(\varphi^* u) \simeq \varphi^*(\text{Coker}(u))$, this is a straightforward consequence of Proposition 3.10(3). \square

LEMMA 4.2. *If $\mathcal{E} \in \mathfrak{T}_\varphi$ is A -flat, then \mathcal{E} is a vector bundle.*

Proof. By assumption, X_k and X_K are reduced schemes. It follows from [BdS11, Section 1, Remarks (a)] that \mathcal{E}_k and \mathcal{E}_K are *vector bundles*. (We note that the context in op. cit. presupposes the ground field to be algebraically closed, but this is not necessary for the proof to work.) Using the “fibre-by-fibre” flatness criterion [EGA, IV₃, proposition II.3.10], we are done. \square

We are now ready to state the main structure theorem concerning \mathfrak{T}_φ .

THEOREM 4.3. *The category \mathfrak{T}_φ is abelian, and the inclusion functor $\mathfrak{T}_\varphi \rightarrow \mathbf{Coh}(X)$ is exact.*

The main point of the proof of Theorem 4.3, given Lemma 4.1, is to show that the kernel of an arrow of \mathfrak{T}_φ is also in \mathfrak{T}_φ and, by Proposition 3.10, all we need to do is to relate $\varphi^*(\text{Ker}(u))$ and $\text{Ker}(\varphi^*(u))$ for an arrow u . The argument takes up the ensuing lines.

First, recall that an \mathcal{O}_X -module \mathcal{M} is said to be φ^* -acyclic if the left derived functors $L_i \varphi^*(\mathcal{M})$ vanish for $i > 0$. (These functors are obtained by means of resolutions by flat \mathcal{O} -modules; see [Har66, Section II.4, pp. 99]. In addition, note that from [Har66, Proposition II.4.4], if \mathcal{M} is coherent, then $L_i \varphi^*(\mathcal{M})$ is also coherent.)

PROPOSITION 4.4. *Any \mathcal{E} in \mathfrak{T}_φ is φ^* -acyclic.*

Proof. Let us first assume $\pi^* \mathcal{E} = 0$, so that \mathcal{E} is by assumption an \mathcal{O}_{X_k} -module which becomes trivial after being pulled back to Y_k . Then, we know that \mathcal{E} is locally isomorphic to $\mathcal{O}_{X_k}^r$; see [BdS11, Section 1, Remarks (a)]. Hence, there exists a faithfully flat morphism $\alpha : \overline{X} \rightarrow X$ (given by a disjoint union of open subsets of X) such that $\alpha^*(\mathcal{E}) \simeq \mathcal{O}_{\overline{X}_k}^r$.

Employing the notation introduced in the cartesian diagram

$$\begin{array}{ccc} \bar{Y} & \xrightarrow{\beta} & Y \\ \bar{\varphi} \downarrow & \square & \downarrow \varphi \\ \bar{X} & \xrightarrow{\alpha} & X, \end{array}$$

we now prove that $\mathcal{O}_{\bar{X}_k}^r$ is $\bar{\varphi}^*$ -acyclic. The exact sequence

$$0 \longrightarrow \mathcal{O}_{\bar{X}}^r \xrightarrow{\pi} \mathcal{O}_{\bar{X}}^r \longrightarrow \mathcal{O}_{\bar{X}_k}^r \longrightarrow 0 \quad (4.1)$$

gives rise to an exact sequence

$$0 \longrightarrow L_1 \bar{\varphi}^*(\mathcal{O}_{\bar{X}_k}^r) \longrightarrow \bar{\varphi}^*(\mathcal{O}_{\bar{X}}^r) \xrightarrow{\pi} \bar{\varphi}^*(\mathcal{O}_{\bar{X}}^r) \longrightarrow \bar{\varphi}^*(\mathcal{O}_{\bar{X}_k}^r) \longrightarrow 0,$$

and this proves that $L_1 \bar{\varphi}^*(\mathcal{O}_{\bar{X}_k}^r) = 0$ as $\bar{\varphi}^*(\mathcal{O}_{\bar{X}}^r) \simeq \mathcal{O}_{\bar{Y}}^r$ is A -flat. From sequence (4.1), we have, for any $i \geq 1$, another exact sequence

$$0 = L_{i+1} \bar{\varphi}^*(\mathcal{O}_{\bar{X}}^r) \longrightarrow L_{i+1} \bar{\varphi}^*(\mathcal{O}_{\bar{X}_k}^r) \longrightarrow L_i \bar{\varphi}^*(\mathcal{O}_{\bar{X}}^r) = 0;$$

this allows us to conclude that $\mathcal{O}_{\bar{X}_k}^r$ is $\bar{\varphi}^*$ -acyclic. As α^* and β^* are exact, we know that $\beta^* \circ L_i \varphi^* \simeq L_i \bar{\varphi}^* \circ \alpha^*$, and therefore we can say that, for each $i > 0$,

$$\begin{aligned} \beta^* \circ L_i \varphi^*(\mathcal{E}) &= L_i \bar{\varphi}^* \circ \alpha^*(\mathcal{E}) \\ &\simeq L_i \bar{\varphi}^*(\mathcal{O}_{\bar{X}_k}^r) = 0. \end{aligned}$$

But $L_i \varphi^*(\mathcal{E})$ is coherent and β is faithfully flat, so $\beta^*(L_i \varphi^*(\mathcal{E})) = 0$ implies $L_i \varphi^*(\mathcal{E}) = 0$, and we have finished the verification that \mathcal{E} is φ^* -acyclic.

For any $\mathcal{M} \in \mathbf{Coh}(X)$, let us agree to write

$$\mathrm{Tors}(\mathcal{M}) = \bigcup_{m \geq 1} \mathrm{Ker} \pi^m : \mathcal{M} \rightarrow \mathcal{M}$$

for the sheaf of sections annihilated by some power of π and define

$$t(\mathcal{M}) := \min\{m \in \mathbb{N} : \pi^m \cdot \mathrm{Tors}(\mathcal{M}) = 0\}.$$

We shall show by induction on $t(\mathcal{E})$ that \mathcal{E} is φ^* -acyclic. If $t(\mathcal{M}) = 0$, then $\mathrm{Tors}(\mathcal{M}) = 0$, so that, due to Lemma 4.2, the module \mathcal{E} is a vector bundle and a fortiori φ^* -acyclic.

Now, suppose $t(\mathcal{E}) \geq 1$ and that for all $\mathcal{F} \in \mathfrak{T}_\varphi$ with $t(\mathcal{F}) < t(\mathcal{E})$, the \mathcal{O}_X -module \mathcal{F} is φ^* -acyclic.

Write $\mathcal{E}' := \pi \mathcal{E}$. Then, if $e' \in \mathrm{Tors}(\mathcal{E}')(U)$ over some affine U , it follows that $e' = \pi e$, where $e \in \mathrm{Tors}(\mathcal{E})(U)$. This being so, we conclude that $\pi^{t(\mathcal{E})-1} e' = 0$ and hence that $t(\mathcal{E}') < t(\mathcal{E})$. Next, we consider the exact sequence

$$0 \longrightarrow \mathcal{E}' \xrightarrow{\rho} \mathcal{E} \xrightarrow{\sigma} \mathcal{E}'' \longrightarrow 0, \quad (4.2)$$

where $\mathcal{E}'' = \mathcal{E}/\pi \mathcal{E}$. Since $\mathcal{E}'' = \mathrm{Coker} \mathcal{E} \xrightarrow{\pi} \mathcal{E}$, it follows that $\mathcal{E}'' \in \mathfrak{T}_\varphi$ (there is no need to apply Lemma 4.1 here). Note that $\pi \mathcal{E}'' = 0$ and hence \mathcal{E}'' is φ^* -acyclic by the first step in the proof. Consequently, we have the exact sequence

$$0 \longrightarrow \varphi^*(\mathcal{E}') \longrightarrow \varphi^*(\mathcal{E}) \xrightarrow{\varphi^*(\sigma)} \varphi^*(\mathcal{E}'') \longrightarrow 0,$$

which says that $\varphi^*(\mathcal{E}') = \mathrm{Ker}(\varphi^*(\sigma))$. But both \mathcal{E} and \mathcal{E}'' are in \mathfrak{T}_φ so that Proposition 3.10(3) guarantees that \mathcal{E}' lies in \mathfrak{T}_φ . As $t(\mathcal{E}') < t(\mathcal{E})$, we can say that \mathcal{E}' is φ^* -acyclic and hence, using the exact sequence (4.2) above, it follows that \mathcal{E} is equally φ^* -acyclic. \square

Proof of Theorem 4.3. Let $u: \mathcal{E} \rightarrow \mathcal{F}$ be an arrow of \mathfrak{T}_φ . If $\mathcal{K} = \text{Ker}(u)$, $\mathcal{C} = \text{Coker}(u)$ and $\mathcal{I} = \text{Im}(u)$, then we have exact sequences

$$0 \longrightarrow \mathcal{K} \longrightarrow \mathcal{E} \longrightarrow \mathcal{I} \longrightarrow 0 \quad \text{and} \quad 0 \longrightarrow \mathcal{I} \longrightarrow \mathcal{F} \longrightarrow \mathcal{C} \longrightarrow 0.$$

Note that $\mathcal{C} \in \mathfrak{T}_\varphi$ by Lemma 4.1, so that Proposition 4.4 tells us that \mathcal{C} is φ^* -acyclic. This implies that the sequence

$$0 \longrightarrow \varphi^*(\mathcal{I}) \longrightarrow \varphi^*(\mathcal{F}) \longrightarrow \varphi^*(\mathcal{C}) \longrightarrow 0$$

is exact, and Proposition 3.10 implies that $\varphi^*(\mathcal{I})$ is trivial relatively to A . Otherwise said, \mathcal{I} belongs to \mathfrak{T}_φ . Proposition 4.4 then implies that \mathcal{I} is φ^* -acyclic. Applying the same reasoning, we conclude that $\mathcal{K} \in \mathfrak{T}_\varphi$. \square

COROLLARY 4.5. *The functor $\bullet|_{x_0}: \mathfrak{T}_\varphi \rightarrow A\text{-mod}$ is exact and faithful.*

Proof. We know that $\bullet|_{x_0}$ is *right* exact, so that we only need to show that $\bullet|_{x_0}$ preserves monomorphisms. Let $u: \mathcal{E} \rightarrow \overline{\mathcal{E}}$ be such a monomorphism in \mathfrak{T}_φ , and denote $\mathcal{E}|_{x_0}$, respectively $\overline{\mathcal{E}}|_{x_0}$, by E , respectively \overline{E} . Let $\alpha: \mathcal{O}_Y \otimes_A E \xrightarrow{\sim} \varphi^*(\mathcal{E})$ and $\overline{\alpha}: \mathcal{O}_Y \otimes_A \overline{E} \xrightarrow{\sim} \varphi^*(\overline{\mathcal{E}})$ be adapted to y_0 (see Definition 3.7 and Lemma 3.9). Using Proposition 3.10, we see that $\overline{\alpha}^{-1}\varphi^*(u)\alpha = \text{id}_{\mathcal{O}_Y \otimes_A E} u_0$ for some $u_0: E \rightarrow \overline{E}$. Because φ^*u is a monomorphism (due to Lemma 4.1 and Proposition 4.4), we conclude that u_0 has to be a monomorphism: otherwise, the pull-back functor $A\text{-mod} \rightarrow \mathbf{Coh}(Y)$ would fail to be exact and faithful. This implies that $\varphi^*(u)|_{y_0}: \varphi^*(\mathcal{E})|_{y_0} \rightarrow \varphi^*(\overline{\mathcal{E}})|_{y_0}$ is a monomorphism, which shows that $B \otimes_A u|_{x_0}$ is a monomorphism. Since B is a faithfully flat A -algebra, we conclude that $u|_{x_0}$ is also a monomorphism.

We now verify that $\bullet|_{x_0}$ is faithful by showing that if $\mathcal{E} \in \mathfrak{T}_\varphi$ is such that $\mathcal{E}|_{x_0} = 0$, then $\mathcal{E} = 0$. Under this assumption, we see that $\varphi^*(\mathcal{E})|_{y_0} = 0$, which, together with an isomorphism $\mathcal{O}_Y \otimes_A E \simeq \varphi^*(\mathcal{E})$, proves the equality $\varphi^*(\mathcal{E}) = 0$. Now, given any point $\xi \in X$, one easily sees using the surjectivity of φ that $\mathcal{E}|_\xi = 0$. Because of [Har77, Exercise II.5.8(c)] and the fact that X is reduced [Mat89, Corollary to Theorem 23.9], we conclude that $\mathcal{E} = 0$. \square

In possession of Theorem 4.3 and Corollary 4.5, we can now apply the theory of Tannakian categories over A to define certain fundamental group schemes.

DEFINITION 4.6 (The fundamental and Galois groups). (1) We shall let $\mathfrak{T}_\varphi^\circ$ denote the full subcategory of \mathfrak{T}_φ whose objects are vector bundles.

(2) Given $\mathcal{E} \in \mathfrak{T}_\varphi^\circ$, we denote by $\langle \mathcal{E}; \mathfrak{T}_\varphi \rangle_\otimes$ the full subcategory of \mathfrak{T}_φ having as objects subquotients of $\mathbf{T}^{a,b}\mathcal{E}$ for varying multi-indexes a, b .

(3) Following [HdS21], by $\mathfrak{T}_\varphi^{\text{tan}}$ we mean the full subcategory of \mathfrak{T}_φ whose objects are

$$\{\mathcal{V} \in \mathfrak{T}_\varphi: \mathcal{V} \text{ is a quotient of some } \mathcal{E} \in \mathfrak{T}_\varphi^\circ\} = \bigcup_{\mathcal{E} \in \mathfrak{T}_\varphi^\circ} \text{Ob}\langle \mathcal{E}, \mathfrak{T}_\varphi \rangle_\otimes.$$

(4) Given $\mathcal{E} \in \mathfrak{T}_\varphi^\circ$, we let $\text{Gal}'(\mathcal{E}, \mathfrak{T}_\varphi, x_0)$, or simply $\text{Gal}'(\mathcal{E})$ if context prevents any misunderstanding, stand for the group scheme obtained from $\langle \mathcal{E}; \mathfrak{T}_\varphi \rangle_\otimes$ via Tannakian duality (see [DH18, Theorem 1.2.6] or [Saa72, théorème II.4.1.1]) by the functor $\bullet|_{x_0}: \langle \mathcal{E}; \mathfrak{T}_\varphi \rangle_\otimes \rightarrow A\text{-mod}$.

(5) We let $\Pi(X, \varphi, x_0)$ be the flat group scheme obtained from $\mathfrak{T}_\varphi^{\text{tan}}$ via Tannakian duality by the functor $\bullet|_{x_0}: \mathfrak{T}_\varphi^{\text{tan}} \rightarrow A\text{-mod}$.

To end, we raise a point which was naturally suggested in the elaboration of the arguments in this section. (It should be compared to [Har77, Exercise II.5.8].)

Question 4.7. Let Z be a flat, reduced and noetherian A -scheme. Call a d.v.r.-point of Z any morphism $z: \text{Spec } R \rightarrow Z$ of A -schemes where R is a d.v.r. Now, for $\mathcal{F} \in \mathbf{coh}(Z)$ and $z: \text{Spec } R \rightarrow Z$ a d.v.r.-point of Z , define its “type” as the invariants of $\mathcal{F}|_z$. Now assume that the type of \mathcal{F} is the same for all z . Is it true that \mathcal{F} is trivial relatively to A ?

5. The generic fibre of the group scheme II

In this section, we let X be a proper and flat A -scheme with *reduced fibres* and x_0 an A -point of X . Let $\varphi: Y \rightarrow X$ be an object of $\mathfrak{S}(X, x_0)$.

Let \mathfrak{T}_{φ_K} stand for the full subcategory of $\mathbf{coh}(X_K)$ whose set of objects is

$$\{E \in \mathbf{coh}(X_K): \varphi_K^* E \text{ is trivial}\}.$$

We note that \mathfrak{T}_{φ_K} is a Tannakian category over K which is neutralized by the functor $\bullet|_{x_0K}: \mathfrak{T}_{\varphi_K} \rightarrow K\text{-mod}$. (The reader is right to object that we have not proved this last claim since in Section 4 we work with a discrete valuation ring, but the translation is evident and in fact much simpler.)

THEOREM 5.1. *Suppose that X_K is normal, and let $E \in \mathfrak{T}_{\varphi_K}$. Then, E is essentially finite and $\langle E; \mathfrak{T}_{\varphi_K} \rangle_{\otimes} = \langle E; \mathbf{EF}(X_K) \rangle_{\otimes}$. In particular, the K -group scheme associated with $\langle E; \mathfrak{T}_{\varphi_K} \rangle_{\otimes}$ via the fibre functor $\bullet|_{x_0K}$ is finite.*

Proof. Let \mathfrak{C} be the full subcategory of all vector bundles over X_K which become trivial after being pulled back via some proper and surjective morphism. Then [AM11, Theorem 1] or [TZ20, Theorem I] guarantees that $\mathfrak{C} = \mathbf{EF}(X_K)$ and, in particular, \mathfrak{T}_{φ_K} is a full subcategory of $\mathbf{EF}(X_K)$. Following [BdS11, Corollary 2.3], let us show that $\langle E; \mathfrak{C} \rangle_{\otimes} = \langle E; \mathfrak{T}_{\varphi_K} \rangle_{\otimes}$. Clearly, each object in $\langle E; \mathfrak{T}_{\varphi_K} \rangle_{\otimes}$ is also an object of $\langle E; \mathfrak{C} \rangle_{\otimes}$. Let then T be some tensor power of E and $q: T \rightarrow Q$ be a quotient morphism with $Q \in \mathfrak{C}$. Then, $\varphi_K^*(q)$ is a quotient morphism from $\varphi_K^*(T) \simeq \mathcal{O}_{Y_K}^{\oplus r}$ to φ_K^*Q . From Lemma 5.2, we see that φ_K^*Q is trivial, which means that $Q \in \mathfrak{T}_{\varphi_K}$. Using duality of vector bundles, we then conclude that $\langle E; \mathfrak{C} \rangle_{\otimes} = \langle E; \mathfrak{T}_{\varphi_K} \rangle_{\otimes}$. \square

The following result was employed in the proof of Theorem 5.1.

LEMMA 5.2. *Let V be a proper scheme over a field and $\mathcal{O}_V^{\oplus r} \rightarrow \mathcal{Q}$ a quotient morphism to a vector bundle \mathcal{Q} of rank d . Suppose that for a certain surjective and proper morphism $f: V' \rightarrow V$, the pull-back $f^*(\det \mathcal{Q})$ is trivial. Then \mathcal{Q} is trivial.*

Proof. We let $\text{Grass}(r, d)$ be the Grassmann scheme as described in [Nit05, Example 5.1.5(2)] and denote by \mathcal{U} the universal quotient of $\mathcal{O}_{\text{Grass}(r, d)}^{\oplus r}$. By the construction of $\text{Grass}(r, d)$, there exists a morphism $\gamma: V \rightarrow \text{Grass}(r, d)$ such that $\mathcal{Q} = \gamma^*(\mathcal{U})$. Consequently, $f^*\gamma^*(\det \mathcal{U})$ is trivial. Let $\delta: \text{Grass}(r, d) \rightarrow \mathbb{P}^n$ ($n = \binom{r}{d} - 1$) be the determinant (Plücker) immersion, so that $\det(\mathcal{U}) = \delta^*\mathcal{O}(1)$; see [Nit05, Section 5.1.6, p. 114]. We then conclude that $f^*\gamma^*\delta^*\mathcal{O}(1)$ is trivial. Using Chow’s lemma [EGA, II, théorème 5.6.1], there is no loss of generality in supposing V' to be projective. Now, [Liu02, Exercise 8.1.7(a)] says that the closed schematic image of $\delta\gamma f$ in \mathbb{P}^n is finite over the ground field. Hence, the image of γ is a closed subset with finitely many points, so that $\gamma^*(\mathcal{U})$, which is \mathcal{Q} , is trivial. \square

For $E \in \mathfrak{T}_{\varphi_K}$, let us denote the group scheme over K associated with $\langle E; \mathfrak{T}_{\varphi_K} \rangle_{\otimes}$ via the functor $\bullet|_{x_0K}$ by $\text{Gal}(E, \mathfrak{T}_{\varphi_K}, x_0, K)$.

THEOREM 5.3. *Suppose that X_K is normal, and let $\mathcal{E} \in \mathfrak{T}_{\varphi}^{\circ}$. If E stands for \mathcal{E}_K , then we have $\text{Gal}'(\mathcal{E}, \mathfrak{T}_{\varphi}, x_0) \otimes K \simeq \text{Gal}(E, \mathfrak{T}_{\varphi_K}, x_0, K)$. In particular, $\text{Gal}'(\mathcal{E}, \mathfrak{T}_{\varphi}, x_0) \otimes K$ is finite.*

Proof. Let $\sigma: \langle \mathcal{E}; \mathfrak{T}_\varphi \rangle_\otimes \rightarrow \langle E; \mathfrak{T}_{\varphi_K} \rangle_\otimes$ be the base-change functor, G be the group scheme $\text{Gal}'(\mathcal{E}, \mathfrak{T}_\varphi, x_0)$ and H be $\text{Gal}(E, \mathfrak{T}_{\varphi_K}, x_{0K})$. Let $\theta: \text{Rep}_A(G) \rightarrow \langle \mathcal{E}; \mathfrak{T}_\varphi \rangle_\otimes$ be an inverse to $\bullet|_{x_0}: \langle \mathcal{E}; \mathfrak{T}_\varphi \rangle_\otimes \rightarrow \text{Rep}_A(G)$, and denote by $\tau: \text{Rep}_A(G) \rightarrow \text{Rep}_K(H)$ the composition

$$\text{Rep}_A(G) \xrightarrow{\theta} \langle \mathcal{E}; \mathfrak{T}_\varphi \rangle_\otimes \xrightarrow{\sigma} \langle E; \mathfrak{T}_{\varphi_K} \rangle_\otimes \xrightarrow{\bullet|_{x_{0K}}} \text{Rep}_K(H).$$

If $V \subset A[G]_{\text{right}}$ contains the constants, the fact that θ is fully faithful implies that $H^0(X, \theta(V)) \simeq A$. Hence, $H^0(X_K, \sigma\theta(V)) \simeq K$, by flat base change. Consequently, $\tau(V)^H \simeq K$.

Let $i: \text{Rep}_A(G) \rightarrow \text{Rep}_K(G_K)$ be the extension of scalars. It is not difficult to see that there exists a morphism $\xi: H \rightarrow G_K$ of group schemes such that

$$\begin{array}{ccc} \text{Rep}_A(G) & \xrightarrow{\tau} & \text{Rep}_K(H) \\ \downarrow i & \nearrow \xi^\# & \\ \text{Rep}_K(G_K) & & \end{array}$$

is commutative (up to natural isomorphism of tensor functors).

If $V \subset K[G]_{\text{right}}$, there exists a $V^b \subset A[G]_{\text{right}}$ containing the constants and an injection $V \rightarrow i(V^b)$. Then,

$$\xi^\#(V)^H \subset \tau(V^b)^H \simeq K.$$

This implies that $K \simeq K[G]_{\text{right}}^H$, and we conclude, with the help of Lemma 2.8 and the fact that H is finite (Theorem 5.1), that ξ is faithfully flat. On the other hand, since $E|_{x_{0K}}$ is a faithful representation of H , the standard criterion [DM82, Proposition 2.21] immediately shows that ξ is a closed immersion; it follows that ξ is an isomorphism. \square

As a simple consequence we have the following.

COROLLARY 5.4. *Suppose that X_K is normal. Then $\Pi(X, \varphi, x_0) \otimes K$ is pro-finite.*

Remark 5.5. We are unable to show that Theorem 5.3 holds without the finiteness assumption on $\text{Gal}(E, \mathfrak{T}_{\varphi_K}, x_0)$. This is because, once this assumption is removed, Lemma 2.8 cannot be applied, so that, in its place, we need the standard criterion guaranteeing faithful flatness [DM82, Proposition 2.21].

6. Prudence in the category \mathfrak{T}_*

We shall assume that A is complete. In [HdS21, Section 6], we introduced the notion of *prudence* of an affine flat group scheme over A . As shown in op. cit., this concept turns out to be equivalent to the Mittag-Leffler property (or freeness) of the A -module of functions of the group. In this section, we show that the A -module of functions of a Galois group (see Definition 4.6) is free by using prudence.

Let us briefly recall the concept of prudent affine group schemes over a complete d.v.r. Our reference is [HdS21, Section 6.2]. Let G be a flat affine group scheme over A and V be an object of $\text{Rep}_A(G)$, with the coaction $\rho: V \rightarrow V \otimes A[G]$. An element $v \in V$ is said to be *semi-invariant* if $\rho(v) = v \otimes \theta$ for some $\theta \in A[G]$. Assume furthermore that V is A -free. We say that V is *prudent* if an element \mathbf{v} whose reductions modulo $\pi^{n+1}V$ are semi-invariant for all n must necessarily be semi-invariant. We say that G is prudent if each $V \in \text{Rep}_A^\circ(G)$ is prudent.

Let us now fix the hypothesis on the ambient space in this section. Let X be a proper and flat A -scheme having reduced fibres. We give ourselves an A -point x_0 and an arrow $\varphi: Y \rightarrow X$ in $\mathfrak{S}^+(X, x_0)$ (see Definition 2.4). If $B = H^0(\mathcal{O}_Y)$, we let y_0 be the B -point of Y above x_0 whose existence is ensured by the definition of \mathfrak{S}^+ .

THEOREM 6.1. *Let $\mathcal{E} \in \mathfrak{T}_\varphi^\circ$. Then $\text{Gal}'(\mathcal{E}, \mathfrak{T}_\varphi, x_0)$ is prudent. In particular, its A -module of functions is free.*

(Recall that $\text{Gal}'(\mathcal{E}, \mathfrak{T}_\varphi, x_0)$ is introduced in Definition 4.6.) Before starting the proof, let us obtain the following.

COROLLARY 6.2. *Suppose that X_K is normal. Then $\text{Gal}'(\mathcal{E}, \mathfrak{T}_\varphi, x_0)$ is finite.*

Proof. We abbreviate $G := \text{Gal}'(\mathcal{E}, \mathfrak{T}_\varphi, x_0)$. By Theorem 5.3, the group scheme $G \otimes K$ is finite over K . On the other hand, Theorem 6.1 implies that the A -module $A[G]$ is free. Consequently, it has to be finite. \square

Proof of Theorem 6.1. Recall that by assumption, Y is H^0 -flat over A and flat over $B = H^0(\mathcal{O}_Y)$. We wish to show that $\text{Gal}'(\mathcal{E})$ is prudent [HdS21, Section 6], and for that we introduce the following setting.

Let $\mathcal{V} \in \langle \mathcal{E}; \mathfrak{T}_\varphi \rangle_\otimes$ be a vector bundle, and let $v_n: \mathcal{L}_n \rightarrow \mathcal{V}_n$ be a compatible family of monomorphisms of \mathcal{O}_X -modules (as customary, $\mathcal{V}_n = A_n \otimes_A \mathcal{V}$, $B_n = A_n \otimes_A B$, etc.) such that for each $n \in \mathbb{N}$, the \mathcal{O}_X -module \mathcal{L}_n belongs to \mathfrak{T}_φ . By Grothendieck's existence theorem [III05, Theorem 8.4.2], there exists an arrow of coherent \mathcal{O}_X -modules $v: \mathcal{L} \rightarrow \mathcal{V}$ such that v_n is none other than $v \otimes_A A_n$.

Write $\mathcal{M} := \varphi^*(\mathcal{L})$; our assumption on \mathcal{L}_n tells us that $\mathcal{M}_n = \varphi^*(\mathcal{L}_n)$ is trivial relatively to A . Then, since we assume that Y is H^0 -flat over A , Lemma 3.3 tells us that the canonical arrow

$$\mathcal{O}_Y \otimes_B H^0(\mathcal{M}_n) \xrightarrow{\theta_n} \mathcal{M}_n$$

is an *isomorphism*. We now require the following result.

LEMMA 6.3. *Under the above notation and assumption, the canonical morphism of \mathcal{O}_Y -modules*

$$\tau: \mathcal{O}_Y \otimes_B H^0(\mathcal{M}) \longrightarrow \mathcal{M}$$

is an isomorphism. In particular, \mathcal{M} is trivial relatively to B .

Proof. Consider the commutative diagram

$$\begin{array}{ccccccc} \mathcal{M}_{n+1} & \xrightarrow{\pi^{n+1}} & \mathcal{M}_{n+1} & \xrightarrow{q_n} & \mathcal{M}_n & \longrightarrow & 0 \\ \theta_{n+1} \uparrow & & \theta_{n+1} \uparrow & & \theta_n \uparrow & & \\ \mathcal{O}_Y \otimes_B H^0(\mathcal{M}_{n+1}) & \xrightarrow{\pi^{n+1}} & \mathcal{O}_Y \otimes_B H^0(\mathcal{M}_{n+1}) & \xrightarrow{\text{id} \otimes H^0(q_n)} & \mathcal{O}_Y \otimes_B H^0(\mathcal{M}_n) & \longrightarrow & 0, \end{array}$$

where q_n is the canonical arrow. By assumption, the vertical arrows are isomorphisms while the upper row is tautologically an exact sequence of \mathcal{O}_Y -modules. We conclude that the bottom row is exact, and faithful flatness of Y over B (faithfulness is guaranteed by [EGA, III₁, théorème 4.3.1]) shows that the sequence

$$H^0(\mathcal{M}_{n+1}) \xrightarrow{\pi^{n+1}} H^0(\mathcal{M}_{n+1}) \xrightarrow{H^0(q_n)} H^0(\mathcal{M}_n) \longrightarrow 0$$

is equally exact. Hence, if

$$\mathfrak{M} := \varprojlim \left(\cdots \longrightarrow H^0(\mathcal{M}_{n+1}) \xrightarrow{H^0(q_n)} H^0(\mathcal{M}_n) \longrightarrow \cdots \right),$$

it follows that \mathfrak{M} is a finitely generated B -module and that the projection $\mathfrak{M} \rightarrow H^0(\mathcal{M}_n)$ induces an isomorphism

$$B_n \otimes_B \mathfrak{M} \xrightarrow{\sim} H^0(\mathcal{M}_n).$$

(This follows from the fact that B is π -adically complete and [EGA, 0_I, proposition 7.2.9].) More importantly, letting $u_n: \mathcal{M} \rightarrow \mathcal{M}_n$ stand for the natural epimorphism, a direct application of the theorem of formal functions (see [Ill05, Theorem 8.2.4] or [Har77, Theorem III.11.1]) guarantees that the obvious arrow

$$H^0(\mathcal{M}) \xrightarrow{\varprojlim_n H^0(u_n)} \mathfrak{M}$$

is a bijection. Consequently, for any given $n \in \mathbb{N}$, the arrow $H^0(u_n): H^0(\mathcal{M}) \rightarrow H^0(\mathcal{M}_n)$ induces an isomorphism

$$U_n: B_n \otimes_B H^0(\mathcal{M}) \xrightarrow{\sim} H^0(\mathcal{M}_n).$$

To show that $\tau: \mathcal{O}_Y \otimes_B H^0(\mathcal{M}) \rightarrow \mathcal{M}$ is an isomorphism, by Grothendieck's algebraization theorem [Ill05, Theorem 8.2.9], we only need to show that for each n ,

$$B_n \otimes_B (\mathcal{O}_Y \otimes_B H^0(\mathcal{M})) \xrightarrow{\text{id} \otimes \tau} B_n \otimes_B \mathcal{M}$$

is an isomorphism. Now, we have the commutative diagram

$$\begin{array}{ccccc} B_n \otimes_B (\mathcal{O}_Y \otimes_B H^0(\mathcal{M})) & \xrightarrow{\text{id} \otimes \tau} & B_n \otimes_B \mathcal{M} & \xrightarrow[\sim]{\bar{u}_n} & \mathcal{M}_n \\ \sim \uparrow \text{canonic} & & & & \uparrow \sim \\ \mathcal{O}_Y \otimes_B (B_n \otimes_B H^0(\mathcal{M})) & \xrightarrow{\text{id} \otimes U_n} & \mathcal{O}_Y \otimes_B H^0(\mathcal{M}_n), & & \end{array}$$

where \bar{u}_n is associated with u_n , and the conclusion follows. \square

By virtue of Lemma 6.3, the module \mathcal{M} is trivial relatively to B ; using Lemma 3.5, we conclude that \mathcal{M} is trivial relatively to A . This shows that $\mathcal{L} \in \mathfrak{T}_\varphi$. As it is not difficult to show that v is a monomorphism by looking at the arrow $\mathcal{L}|_{x_0} \rightarrow \mathcal{V}|_{x_0}$, we see that \mathcal{L} does indeed belong to $\langle \mathcal{E}, \mathfrak{T}_\varphi \rangle_\otimes$.

We have then showed that the group scheme $\text{Gal}'(\mathcal{E}, \mathfrak{T}_\varphi)$ is prudent, and hence that its ring of functions is free as an A -module [HdS21, Section 6]. \square

7. Digression on the reduced fibre theorem

For future applications, we shall state a version of the reduced fibre theorem (see [BLR95, Theorem 2.1'] and [Tem10]) suitable for us.

THEOREM 7.1 (The reduced fibre theorem). *Let A be Henselian and Y be a flat A -scheme of finite type whose fibre Y_K is geometrically reduced. Then, there exist a finite extension of discrete*

valuation rings $B \supset A$ and a commutative diagram

$$\begin{array}{ccc}
 & & Z \\
 & & \downarrow \varepsilon \\
 Y & \longleftarrow & Y_B \\
 \downarrow & \square & \downarrow \\
 \text{Spec } A & \longleftarrow & \text{Spec } B,
 \end{array}$$

where:

RFT1) The B -scheme Z is flat and has geometrically reduced fibres.

RFT2) The morphism ε is finite and surjective.

RFT3) If $L = \text{Frac}(B)$, then

$$\varepsilon \otimes_B L: Z \otimes_B L \longrightarrow Y_B \otimes_B L$$

is an isomorphism.

In particular, the composition $Z \rightarrow Y_B \rightarrow Y$ is finite and surjective.

Proof. We show how to arrive at the conclusion starting from [Tem10, Theorem 3.5.5]. By this theorem (see also the notation in Section 3, p. 619 of op. cit.), we obtain the existence of an integral scheme S , a separable alteration (proper, dominant and inducing a finite and separable extension of function fields) $S \rightarrow \text{Spec } A$ and a commutative diagram with cartesian square

$$\begin{array}{ccc}
 & & \tilde{Y} \\
 & & \downarrow \delta \\
 Y & \longleftarrow & Y_S \\
 \downarrow & \square & \downarrow \\
 \text{Spec } A & \longleftarrow & S
 \end{array}$$

such that:

- The S -scheme \tilde{Y} is flat and has geometrically reduced fibres.
- The morphism δ is finite.
- The base change

$$\tilde{Y} \times_S S_K \xrightarrow{\delta \times_S S_K} Y_S \times_S S_K$$

is an isomorphism.

As $Y_S \rightarrow S$ is an open morphism [EGA, IV₂, théorème 2.4.6] and S_K is dense in S , we can say that $Y_S \times_S S_K$ is dense in Y_S . Consequently:

- The morphism δ is *surjective*.

If L is the function field of S and B is the integral closure of A in L (necessarily a d.v.r. and a finite A -module [Mat80, Section 31.B]), the properness of S gives the existence of a morphism of A -schemes $\text{Spec } B \rightarrow S$ extending $\text{Spec } L \rightarrow S$. We then arrive at a commutative diagram

with cartesian squares

$$\begin{array}{ccccc}
 & & \tilde{Y} & \longleftarrow & Z \\
 & & \downarrow \delta & \square & \downarrow \varepsilon \\
 Y & \longleftarrow & Y_S & \longleftarrow & Y_B \\
 \downarrow & & \downarrow & \square & \downarrow \\
 \text{Spec } A & \longleftarrow & S & \longleftarrow & \text{Spec } B,
 \end{array}$$

where:

- The B -scheme Z is flat and has geometrically reduced fibres (for the last condition, see [EGA, IV₂, proposition 4.6.1]).
- The morphism ε is finite and surjective (for the surjectivity, see [EGA, I, proposition 3.5.2]).
- The base-change morphism

$$\varepsilon \otimes_B L: Z \otimes_B L \longrightarrow Y_B \otimes_B L$$

is an isomorphism.

The proof of the last claim is quite simple, and we omit it. □

The ensuing result will prove useful for the application of Theorem 7.1. In it, we employ the notion of Japanese discrete valuation ring [EGA, 0_{IV}, définition 23.1.1]. (The reader should recall that all complete d.v.r. are Japanese, as are those whose field of fractions has characteristic zero. See [EGA, 0_{IV}, théorème 23.1.5 and proposition 23.1.2].)

LEMMA 7.2. *Let A be Japanese and S a flat A -scheme of finite type. There exists a finite purely inseparable extension \tilde{K} of K such that the following property holds: If \tilde{A} is the integral closure of A in \tilde{K} , then the \tilde{A} -scheme $\tilde{S} := (S \otimes_A \tilde{A})_{\text{red}}$ has geometrically reduced generic fibre and is flat.*

Proof. Let \tilde{K}/K be a finite and purely inseparable extension such that $(S \otimes_A \tilde{K})_{\text{red}}$ is geometrically reduced over \tilde{K} ; see [EGA, IV₂, proposition 4.6.6]. Now, in the notation of the statement, $\tilde{S}[1/\pi] = (S \otimes_A \tilde{K})_{\text{red}}$, and we are done. To prove that \tilde{S} is flat over \tilde{A} , let $\tilde{\pi}$ be a uniformizer of \tilde{A} . Let \tilde{f} be a function on some unspecified affine open subset of $S \otimes_A \tilde{A}$ such that $\tilde{\pi}\tilde{f}$ is nilpotent, say $\tilde{\pi}^m \tilde{f}^m = 0$. Then, the fact that $S \otimes_A \tilde{A}$ is \tilde{A} -flat says that \tilde{f} is nilpotent, and this shows that $(S \otimes_A \tilde{A})_{\text{red}}$ is \tilde{A} -flat. □

8. The fundamental group scheme

Let X be an irreducible, proper and flat A -scheme with geometrically reduced fibres and x_0 an A -point of X . We now wish to assemble the categories \mathfrak{T}_φ (see Definition 3.4 and Theorem 4.3) for varying φ in a single one, and for that we need the following. (We remind the reader that the category $\mathfrak{S}^+(X, x_0)$ mentioned below is introduced in Definition 2.4.)

THEOREM 8.1. *We suppose that A is Henselian and Japanese. Let $\varphi: Y \rightarrow X$ be a proper and surjective morphism. Then, there exists a commutative diagram of schemes*

$$\begin{array}{ccc} Z & \longrightarrow & Y \\ & \searrow \psi & \downarrow \varphi \\ & & X \end{array}$$

enjoying the following properties:

- (1) *The morphism ψ is surjective and proper.*
- (2) *The ring $C := H^0(\mathcal{O}_Z)$ is a discrete valuation ring and is a finite extension of A .*
- (3) *The canonical morphism $Z \rightarrow \text{Spec } C$ is flat and has geometrically reduced fibres.*
- (4) *The C -scheme Z has a C -point above x_0 .*

In particular, $\psi: Z \rightarrow X$ belongs to the category $\mathfrak{S}^+(X, x_0)$.

Proof. The construction requires several steps.

Step 1. Because X is *irreducible*, there exists an irreducible component Y' of Y such that $\varphi: Y' \rightarrow X$ is surjective.

Step 2. Let $j: Y'' \rightarrow Y$ be a reduced closed subscheme underlying Y' . Note that Y'' is integral and that $\varphi'' := \varphi \circ j: Y'' \rightarrow X$ is proper and surjective.

Step 3. Let $\nu: Y^\dagger \rightarrow Y''$ be the normalization [EGA, II, section 6.3, p. 119]. Since A is universally Japanese [EGA, IV₂, théorème 7.7.2], we conclude that ν is finite, and hence $\varphi^\dagger := \varphi'' \circ \nu$ is proper and surjective. In addition, $H^0(\mathcal{O}_{Y^\dagger})$ is a normal integral domain (see [Liu02, Propositions 2.4.17 and 4.1.5]). The morphism $Y^\dagger \rightarrow \text{Spec } A$ is surjective and hence flat [Liu02, Corollary 4.3.10]; it then follows that $H^0(\mathcal{O}_{Y^\dagger})$ is a finite and flat A -module. As such, it must in addition be a local ring since A is Henselian. Consequently, $H^0(\mathcal{O}_{Y^\dagger})$ is a discrete valuation ring. In hindsight, the canonical morphism $Y^\dagger \rightarrow \text{Spec } H^0(\mathcal{O}_{Y^\dagger})$ is surjective and flat [Liu02, Corollary 4.3.10].

Step 4. Let $B^\dagger := H^0(\mathcal{O}_{Y^\dagger})$, and write L^\dagger for its field of fractions. Let \tilde{L} be a purely inseparable extension of L^\dagger such that, denoting by \tilde{B} the integral closure of B^\dagger in \tilde{L} , the generic fibre of

$$(Y^\dagger \otimes_{B^\dagger} \tilde{B})_{\text{red}} \longrightarrow \text{Spec } \tilde{B}$$

is geometrically reduced (we employ Lemma 7.2). Writing $\tilde{Y} := (Y^\dagger \otimes_{B^\dagger} \tilde{B})_{\text{red}}$, we note that the composition

$$\tilde{Y} \longrightarrow Y^\dagger \otimes_{B^\dagger} \tilde{B} \xrightarrow{\text{pr}} Y^\dagger$$

induces a homeomorphism between the underlying topological spaces. From this, it follows that \tilde{Y} is integral and that the obvious morphism $\tilde{\varphi}: \tilde{Y} \rightarrow X$ is surjective. Since \tilde{B} is a finite B^\dagger -module – recall that A is Japanese – we conclude that $\tilde{\varphi}$ is proper. Since \tilde{B} is a Dedekind domain, the fact that $\tilde{Y} \rightarrow \text{Spec } \tilde{B}$ is surjective (note that $Y^\dagger \otimes_{B^\dagger} \tilde{B} \rightarrow \text{Spec } \tilde{B}$ is surjective by [EGA, I, proposition 3.5.2]) shows that it is in addition flat.

For future usage, we also remark that

$$\begin{aligned} H^0(\mathcal{O}_{\tilde{Y}}) &= H^0(\mathcal{O}_{Y^\dagger \otimes_{B^\dagger} \tilde{B}})_{\text{red}} && \text{(see [Liu02, Proposition 2.4.2(c)])} \\ &= (H^0(\mathcal{O}_{Y^\dagger}) \otimes_{B^\dagger} \tilde{B})_{\text{red}} && \text{(flat base change [Liu02, Proposition 3.1.24])} \\ &= (B^\dagger \otimes_{B^\dagger} \tilde{B})_{\text{red}} && \text{(definition of } B^\dagger) \\ &= \tilde{B}. \end{aligned}$$

Step 5. By the reduced fibre theorem (Theorem 7.1), there exist a finite extension $B^\natural \supset \tilde{B}$ of discrete valuation rings and a commutative diagram

$$\begin{array}{ccc}
 & & Y^\natural \\
 & & \downarrow \varepsilon \\
 \tilde{Y} & \longleftarrow & \tilde{Y} \otimes_{\tilde{B}} B^\natural \\
 \downarrow & & \downarrow \\
 \text{Spec } \tilde{B} & \longleftarrow & \text{Spec } B^\natural
 \end{array}$$

such that:

- The B^\natural -scheme Y^\natural is flat and proper and has geometrically reduced fibres.
- The morphism ε is finite and surjective.
- If $L^\natural = \text{Frac}(B^\natural)$, then

$$\varepsilon \otimes_{B^\natural} L^\natural: Y^\natural \otimes_{B^\natural} L^\natural \longrightarrow (\tilde{Y} \otimes_{\tilde{B}} B^\natural) \otimes_{B^\natural} L^\natural$$

is an isomorphism.

Furthermore,

$$\begin{aligned}
 H^0(\mathcal{O}_{Y^\natural \otimes_{B^\natural} L^\natural}) &\simeq H^0(\mathcal{O}_{\tilde{Y} \otimes_{\tilde{B}} L^\natural}) && \text{(via } \varepsilon \otimes_{B^\natural} L^\natural \text{)} \\
 &\simeq H^0(\mathcal{O}_{\tilde{Y}}) \otimes_{\tilde{B}} L^\natural && \text{(by flat base change)} \\
 &\simeq L^\natural && \text{(by Step 4).}
 \end{aligned}$$

Let $\varphi^\natural: Y^\natural \rightarrow X$ be the composition

$$Y^\natural \xrightarrow{\varepsilon} \tilde{Y} \otimes_{\tilde{B}} B^\natural \xrightarrow{\text{pr}} \tilde{Y} \xrightarrow{\tilde{\varphi}} X;$$

as $\tilde{\varphi}$ is surjective and proper, it is clear that φ^\natural is surjective and proper.

Step 6. Let $C \supset B^\natural$ be a finite extension of d.v.r.s such that Y^\natural has a C -point above the A -point x_0 (here we apply the valuative criterion of properness for the A -scheme $Y^\natural|_{x_0}$ to obtain a finite extension $A \subset C$). We then define $Z = Y^\natural \otimes_{B^\natural} C$ and note that the following claims hold true:

- The composition $Z \xrightarrow{\text{pr}} Y^\natural \xrightarrow{\varphi^\natural} X$ is surjective and proper.
- The C -scheme Z is flat and proper and has geometrically reduced fibres.
- Let $M = \text{Frac}(C)$. Then, $H^0(\mathcal{O}_{Z \otimes_C M}) = M$ (because of $H^0(\mathcal{O}_{Y^\natural \otimes_{B^\natural} L^\natural}) = L^\natural$ and by flat base change).
- The ring of global functions of Z is C (because of the previous claims and Lemma 2.2.)
- There is a C -point in Z above the A -point x_0 .

This proves all conclusions in the statement. □

COROLLARY 8.2. *We suppose that A is Henselian and Japanese. Let $\varphi: Y \rightarrow X$ and $\varphi': Y' \rightarrow X$ be surjective and proper morphisms. Then there exist a morphism $\psi: Z \rightarrow X$ in $\mathfrak{S}^+(X, x_0)$ and*

a commutative diagram

$$\begin{array}{ccc}
 & Z & \\
 \swarrow & & \searrow \\
 Y & & Y' \\
 \searrow & \psi & \swarrow \\
 & X & \\
 \nearrow & & \nwarrow \\
 \varphi & & \varphi'
 \end{array}$$

In addition, it is possible to find a Z such that the following extra conditions hold:

- (1) The ring $H^0(\mathcal{O}_Z)$ is a d.v.r. which is a finite extension of A .
- (2) The canonical morphism $Z \rightarrow \text{Spec } H^0(\mathcal{O}_Z)$ is flat with geometrically reduced fibres.

DEFINITION 8.3. We let \mathfrak{T}_X stand for the full subcategory of $\mathbf{coh}(X)$ whose objects are

$$\bigcup_{\varphi \in \mathfrak{S}(X, x_0)} \text{Ob } \mathfrak{T}_\varphi^{\text{tan}}.$$

In addition, \mathfrak{T}_X° is the full subcategory of \mathfrak{T}_X having objects which are in addition vector bundles.

In the terminology of the definition, we can say the following. (For the notion of neutral Tannakian category employed below, see [DH18, Definition 1.2.5].)

COROLLARY 8.4. *If A is Henselian and Japanese, then the category \mathfrak{T}_X is a full abelian subcategory of $\mathbf{coh}(X)$ which is in addition stable under tensor products. With this structure and with the functor $\bullet|_{x_0}: \mathfrak{T}_X \rightarrow A\text{-mod}$, the category \mathfrak{T}_X becomes a neutral Tannakian category over A .*

Proof. Let $\varphi: Y \rightarrow X$ and $\varphi': Y' \rightarrow X$ belong to $\mathfrak{S}(X, x_0)$ and $\psi: Z \rightarrow X$ be as in Corollary 8.2. Then, $\mathfrak{T}_\varphi^{\text{tan}}$ and $\mathfrak{T}_{\varphi'}^{\text{tan}}$ are full subcategories of $\mathfrak{T}_\psi^{\text{tan}}$, and all the claims made in the statement follow from Theorem 4.3 and Corollary 4.5. \square

DEFINITION 8.5. The fundamental group scheme of X at the point x_0 is the affine and flat group scheme obtained from \mathfrak{T}_X and the functor $\bullet|_{x_0}: \mathfrak{T}_X \rightarrow A\text{-mod}$ via Tannakian duality [Saa72, théorème II.4.1.1]. It shall be denoted by $\Pi(X, x_0)$.

If $\mathcal{E} \in \mathfrak{T}_X^\circ$, we let $\text{Gal}'(\mathcal{E}, \mathfrak{T}_X, x_0)$ be the flat group scheme over A defined by the category $\langle \mathcal{E} \rangle_\otimes$ and the functor $\bullet|_{x_0}$.

The next result clarifies the relation between \mathfrak{T}_X and its constituents \mathfrak{T}_φ .

PROPOSITION 8.6. *Let $\varphi: Y \rightarrow X$ be an arrow of $\mathfrak{S}(X, x_0)$ and $\mathcal{E} \in \mathfrak{T}_\varphi^\circ$. Then, the natural morphism $\nu: \Pi(X, x_0) \rightarrow \Pi(X, \varphi, x_0)$ is faithfully flat, while $\text{Gal}'(\mathcal{E}, \mathfrak{T}_X, x_0) \rightarrow \text{Gal}'(\mathcal{E}, \mathfrak{T}_\varphi, x_0)$ is an isomorphism.*

Proof. Let us write $G := \Pi(X, x_0)$ and $G^\varphi := \Pi(X, \varphi, x_0)$. In addition, given any A -linear category \mathfrak{C} , we denote by $\mathfrak{C}_{(k)}$ the full subcategory of \mathfrak{C} whose objects are “annihilated by π ”, meaning that multiplication by π coincides with 0. We shall prove that:

- (i) For any $V \in \text{Rep}_A^\circ(G^\varphi)$ and any quotient morphism $q: \nu^\#(V) \rightarrow Q$ with $Q \in \text{Rep}_A^\circ(G)$, there exists a $Q^\varphi \in \text{Rep}_A^\circ(G^\varphi)$ such that $\nu^\#(Q^\varphi) = Q$.
- (ii) For any $M \in \text{Rep}_A(G^\varphi)_{(k)}$ and any quotient morphism $q: \nu^\#(M) \rightarrow Q$ with $Q \in \text{Rep}_A(G)_{(k)}$, there exists a $Q^\varphi \in \text{Rep}_A(G^\varphi)$ such that $\nu^\#(Q^\varphi) = Q$.

If these two conditions are satisfied, then the evident morphisms $A[G^\varphi] \rightarrow A[G]$ and $k[G^\varphi] \rightarrow k[G]$ are injective (as follows from the “dual statements” in [DH18, Section 3.2.1] and [DM82, Proposition 2.21]), and [DH18, Theorem 4.1.1] proves that ν is faithfully flat.

To verify item (i), we give ourselves a morphism $\psi: Z \rightarrow X$ in $\mathfrak{S}(X, x_0)$, an object \mathcal{Q} from \mathfrak{T}_ψ° , an object \mathcal{V} of $\mathfrak{T}_\varphi^\circ$ and an epimorphism $g: \mathcal{V} \rightarrow \mathcal{Q}$, and aim at showing that \mathcal{Q} lies in $\mathfrak{T}_\varphi^\circ$. By Lemma 2.3, we only need to show that $\varphi^*(\mathcal{Q})|_{Y_k}$ and $\varphi^*(\mathcal{Q})|_{Y_K}$ are both trivial. But this is a direct consequence of Lemma 5.2.

To verify item (ii), we give ourselves an object \mathcal{M} of $(\mathfrak{T}_\varphi^{\text{tan}})_{(k)}$, a morphism $\psi: Z \rightarrow X$ in $\mathfrak{S}(X, x_0)$, an object \mathcal{Q} of $(\mathfrak{T}_\psi^{\text{tan}})_{(k)}$ and an epimorphism $g: \mathcal{M} \rightarrow \mathcal{Q}$. Since \mathcal{M} and \mathcal{Q} are locally free \mathcal{O}_{X_k} -modules [BdS11, Section 1, Remarks (a)], the exact same argument as before proves that \mathcal{Q} lies in $\mathfrak{T}_{\varphi, (k)}$. Being a quotient of \mathcal{M} , which belongs to $(\mathfrak{T}_\varphi^{\text{tan}})_{(k)}$, the \mathcal{O}_{X_k} -module \mathcal{Q} must be in $(\mathfrak{T}_\varphi^{\text{tan}})_{(k)}$.

The natural arrow $\text{Gal}'(\mathcal{E}, \mathfrak{T}, x_0) \rightarrow \text{Gal}'(\mathcal{E}, \mathfrak{T}_\varphi, x_0)$ is induced by the obvious fully faithful functor $\langle \mathcal{E}; \mathfrak{T}_\varphi \rangle_\otimes \rightarrow \langle \mathcal{E}; \mathfrak{T} \rangle_\otimes$. Now, any \mathcal{V} in $\langle \mathcal{E}; \mathfrak{T} \rangle_\otimes$ is of the form $\mathcal{V}'/\mathcal{V}''$, where $\mathcal{V}'' \subset \mathcal{V}' \subset \mathbf{T}^{a,b}\mathcal{E}$. Since ν is faithfully flat, [DH18, Theorem 4.1.2] shows that \mathcal{V}' and \mathcal{V}'' belong to $\mathfrak{T}_\varphi^\circ$; consequently, \mathcal{V} is an object of $\langle \mathcal{E}; \mathfrak{T}_\varphi \rangle_\otimes$, and the fully faithful functor $\langle \mathcal{E}; \mathfrak{T}_\varphi \rangle_\otimes \rightarrow \langle \mathcal{E}; \mathfrak{T} \rangle_\otimes$ is an equivalence. \square

In order to highlight some properties of $\Pi(X, x_0)$, let us make the following definitions.

DEFINITION 8.7. Let $G \in \mathbf{FGSch}/A$ be given. Let \mathbf{P} be one of the adjectives “finite”, “quasi-finite” or “pseudo-finite”. We say that G is pro- \mathbf{P} (respectively, strictly pro- \mathbf{P}) if there exist a directed set I and a diagram $\{\nu_{ij}: G_j \rightarrow G_i: i \leq j \in I\}$ in \mathbf{FGSch}/A where each G_i is \mathbf{P} (respectively, each G_i is \mathbf{P} and each ν_{ij} is faithfully flat) such that $G \simeq \varprojlim_i G_i$. (For the definition of quasi-finite, the reader is directed to [EGA, II, définition 6.2.3].)

THEOREM 8.8. *We suppose that A is Henselian and Japanese and, moreover, assume that X_K is normal.*

- (1) *The group scheme $\Pi(X, x_0)$ is pro-quasi-finite.*
- (1') *The group scheme $\Pi(X, x_0)$ is strictly pro-pseudo-finite.*
- (2) *If in addition A is complete, then, for each $\mathcal{E} \in \mathfrak{T}_X^\circ$, the group scheme $\text{Gal}'(\mathcal{E}, \mathfrak{T}_X, x_0)$ is finite and $\Pi(X, x_0)$ is strictly pro-finite.*

We note that this result shall be improved below – see Theorem 8.10 – by a careful application of assertion (2) and Lemma 2.9, but not to make the argument overly involved, we opt for less generality now.

Proof. We consider the set of isomorphism classes I of objects in \mathfrak{T}_X° and order it by decreeing that $\mathcal{E} \leq \mathcal{F}$ if and only if $\mathcal{E} \in \langle \mathcal{F} \rangle_\otimes$. Putting $G_\mathcal{E} := \text{Gal}'(\mathcal{E}, \mathfrak{T}_X, x_0)$, we obtain a system of group schemes $\{\nu_{\mathcal{E}, \mathcal{F}}: G_\mathcal{F} \rightarrow G_\mathcal{E}\}$ whose limit is $\Pi(X, x_0)$.

(1) and (1'). We know that for each $\mathcal{E} \in \mathfrak{T}_X^\circ$, the group $G_\mathcal{E} \otimes K$ is finite (by Proposition 8.6 and Theorem 5.3). Hence, the proof follows from the following lemma.

LEMMA 8.9. *Let $G \in \mathbf{FGSch}/A$ be such that $G \otimes K$ is a pro-finite group scheme. Then G is (a) strictly pro-pseudo-finite and (b) pro-quasi-finite.*

Proof. Following [DHdS18, Theorem 2.17], we write $G = \varprojlim_\alpha G_\alpha$, where each G_α lies in the category (\mathbf{FGSch}/A) , the transition morphisms are faithfully flat and each $G_\alpha \otimes K$ is of finite

type over K . Since $G \otimes K$ is pro-finite, $G_\alpha \otimes K$ is in fact finite. But for a flat A -module M , the inequality $\dim_K M \otimes K < \infty$ entails $\dim_k M \otimes k < \infty$, as we see by lifting a linearly independent set in $M \otimes k$ to M . We conclude that G_α is pseudo-finite. In addition, by the same theorem in [DHdS18], we know that G_α is a projective limit $\varprojlim_i G_{\alpha,i}$, where $G_{\alpha,i}$ is flat and of finite type over A and the transition morphisms $G_{\alpha,j} \rightarrow G_{\alpha,i}$ induce isomorphisms on the generic fibres. In particular, $G_{\alpha,i} \otimes K \simeq G_\alpha \otimes K$, and hence $G_{\alpha,i} \otimes K$ is again finite. By the same argument as before, G_α , being of finite type, is quasi-finite. \square

Proof of Theorem 8.8, continued. (2) In view of Corollary 6.2 and Proposition 8.6, only the final statement needs proof. Due to [DH18, Theorem 4.1.2], each arrow $\nu_{\mathcal{E}, \mathcal{F}}: G_{\mathcal{F}} \rightarrow G_{\mathcal{E}}$ is faithfully flat, which is enough argument. \square

We now present the amplification of Theorem 8.8 already mentioned before.

THEOREM 8.10. *We suppose that A is Henselian and Japanese and, moreover, assume that X_K is normal. Then $\Pi(X, x_0)$ is strictly pro-finite. In particular, if $\mathcal{E} \in \mathfrak{T}_X$, then $\text{Gal}'(\mathcal{E}, \mathfrak{T}, x_0)$ is finite.*

Proof. Theorem 8.8 tells us that $\Pi(X, x_0)$ is strictly pro-pseudo-finite. To show that it is strictly pro-finite, we let $\Pi(X, x_0) \rightarrow G$ be a faithfully flat morphism to a pseudo-finite flat group scheme over A and show that G is indeed finite over A .

Define the functor $\theta: \text{Rep}_A(G) \rightarrow \mathfrak{T}_X$ as the composition of the natural functor $\text{Rep}_A(G) \rightarrow \text{Rep}_A(\Pi(X, x_0))$ with a tensor inverse to $\bullet|_{x_0}: \mathfrak{T}_X \rightarrow \text{Rep}_A(\Pi(X, x_0))$. (That such an inverse exists is proved in [Saa72, section I.4.4.2, p. 69].) Since $\Pi(X, x_0) \rightarrow G$ is faithfully flat, we conclude $\text{Rep}_A(G) \rightarrow \text{Rep}_A(\Pi(X, x_0))$ is fully faithful (see [DH18, Proposition 3.2.1(ii)], for example) and hence that θ is fully faithful.

Let \widehat{A} be the completion of A , and write \widehat{X} for the base change $X \otimes_A \widehat{A}$; note that \widehat{X} is a flat and proper \widehat{A} -scheme with geometrically reduced fibres and that x_0 induces an \widehat{A} -point \widehat{x}_0 on it. In addition, since A is assumed Japanese, we can say that \widehat{K} , the field of fractions of \widehat{A} , is a separable extension of K ; see [EGA, IV₂, corollaire 7.6.6]. Consequently, $\widehat{X} \otimes_{\widehat{A}} \widehat{K}$ is also a normal scheme [EGA, IV₂, proposition 6.7.4], and it is then a simple matter to deduce that \widehat{X} is also irreducible, so that all properties imposed on the morphism $X \rightarrow \text{Spec } A$ in the beginning of this section are valid for $\widehat{X} \rightarrow \text{Spec } \widehat{A}$.

Using the base-change functor $\sigma: \mathfrak{T}_X \rightarrow \mathfrak{T}_{\widehat{X}}$ and the equivalence

$$\bullet|_{\widehat{x}_0}: \mathfrak{T}_{\widehat{X}} \longrightarrow \text{Rep}_{\widehat{A}}(\Pi(\widehat{X}, \widehat{x}_0)),$$

we derive a tensor functor

$$\tau: \text{Rep}_A(G) \longrightarrow \text{Rep}_{\widehat{A}}(\Pi(\widehat{X}, \widehat{x}_0))$$

preserving forgetful functors (up to natural tensor isomorphisms). If $i: \text{Rep}_A(G) \rightarrow \text{Rep}_{\widehat{A}}(G_{\widehat{A}})$ stands for the base-extension functor, then τ can be extended to a tensor functor

$$\xi: \text{Rep}_{\widehat{A}}(G_{\widehat{A}}) \longrightarrow \text{Rep}_{\widehat{A}}(\Pi(\widehat{X}, \widehat{x}_0))$$

rendering

$$\begin{array}{ccc} \text{Rep}_A(G) & \xrightarrow{\tau} & \text{Rep}_{\widehat{A}}(\Pi(\widehat{X}, \widehat{x}_0)) \\ \downarrow i & \nearrow \xi & \\ \text{Rep}_{\widehat{A}}(G_{\widehat{A}}) & & \end{array}$$

commutative up to natural isomorphism of tensor functors. In addition, ξ preserves the forgetful functors. We contend that the morphism of group schemes induced by ξ is faithfully flat, and for that we rely on Lemma 2.9, whose notation is from now on in force.

Let $V \in \mathfrak{s}(G)$. Since θ is full, we conclude that $H^0(X, \theta(V)) \simeq A$. Hence, flat base change gives $H^0(\widehat{X}, \sigma\theta(V)) \simeq \widehat{A}$. This implies that

$$\tau(V)^{\Pi(\widehat{X}, \widehat{x}_0)} \simeq \widehat{A}.$$

Likewise, if $M \in \mathfrak{s}_0(G)$, we conclude that

$$\tau(M)^{\Pi(\widehat{X}, \widehat{x}_0)} \simeq k.$$

Now, as we learn from Section 1.4 and Proposition 2 of Section 1.5 in [Ser68], for any $V \in \mathfrak{s}(G_{\widehat{A}})$, respectively $M \in \mathfrak{s}_0(G_{\widehat{A}})$, there exist a $V^b \in \mathfrak{s}(G)$, respectively a $M^b \in \mathfrak{s}_0(G)$, and an injection $V \rightarrow i(V^b)$, respectively an injection $M \rightarrow i(M^b)$. Then,

$$\xi(V)^{\Pi(\widehat{X}, \widehat{x}_0)} \subset \tau(V^b)^{\Pi(\widehat{X}, \widehat{x}_0)} \simeq \widehat{A},$$

and

$$\xi(M)^{\Pi(\widehat{X}, \widehat{x}_0)} \subset \tau(M^b)^{\Pi(\widehat{X}, \widehat{x}_0)} \simeq k.$$

Consequently, Lemma 2.9 guarantees that the morphism $\Pi(\widehat{X}, \widehat{x}_0) \rightarrow G_{\widehat{A}}$ deduced from ξ is faithfully flat. Now, since $\Pi(\widehat{X}, \widehat{x}_0)$ is strictly pro-finite (Theorem 8.8), it is not hard to see that $G_{\widehat{A}}$ is actually finite, so that G must then be finite [Bou85, section I.3.6, proposition 11]. \square

Using [Sta22, Tag 0AS7], we have the following.

COROLLARY 8.11. *Let us adopt the hypotheses of Theorem 8.10. Then the ring of functions of $\Pi(X, x_0)$ is a Mittag-Leffler A -module.*

9. An application to the theory of torsion points on the Picard scheme

We assume that A is Henselian and Japanese. Let X be an irreducible, proper and flat A -scheme with geometrically reduced fibres and x_0 an A -point of X . Assume in addition that X_K is normal. The following result connects Theorem 6.1 with the theory of torsion points on abelian schemes.

PROPOSITION 9.1. *Suppose that A has characteristic $(0, p)$ and absolute ramification index e . We give ourselves a positive integer r and an invertible sheaf \mathcal{L} on X .*

- (1) *If $\mathcal{L} \in \text{Pic}(X)$ has order p^r and is taken upon reduction to the identity of $\text{Pic}(X_k)$, then $p^r - p^{r-1} \leq e$.*
- (2) *If Y is an abelian scheme over A and $y \in Y(A)$ is a point of order p^r which reduces to the identity in $Y(k)$, then $p^r - p^{r-1} \leq e$.*

The proof requires the following. (For the definition of the Néron blowup, see [WW80, Section 1].)

LEMMA 9.2. *Let A have characteristic $(0, p)$ and absolute ramification index e . Let r be a positive integer. Then, the Néron blowup $\tilde{\mu}_{p^r}$ of μ_{p^r} at the origin in the special fibre is finite if and only if*

$$p^r \leq er + \min_{0 \leq i < r} \{p^i - ei\}.$$

Proof. Let us write $\mu_{p^r} = \text{Spec } \mathcal{O}$, where $\mathcal{O} = A[t]/(t^{p^r} - 1)$. Putting $t = 1 + s$, we have $\mathcal{O} = A[s]/(\varphi)$, where $\varphi(s) = s^{p^r} + \sum_{n=1}^{p^r-1} \binom{p^r}{n} s^n$. According to [Liu02, Lemma 8.1.2(e)], the ring of the Néron blowup $\tilde{\mathcal{O}}$ is the quotient of the polynomial ring $\mathcal{O}[\tilde{s}]$ by the ideal

$$\{P \in \mathcal{O}[\tilde{s}]: \pi^d P \in (\pi\tilde{s} - s) \text{ for some } d \geq 0\}.$$

Therefore, if we agree to write $\tilde{\varphi}(\tilde{s}) = \varphi(\pi\tilde{s})$, then

$$\tilde{\mathcal{O}} = \frac{A[\tilde{s}]/(\tilde{\varphi})}{(\text{elements annihilated by some power of } \pi)} = \frac{A[\tilde{s}]}{\{P: \pi^d P \in (\tilde{\varphi}) \text{ for some } d \geq 0\}}.$$

Let $\nu: A[\tilde{s}] \rightarrow \mathbb{N} \cup \{\infty\}$ be the ‘‘Gauss valuation’’; it extends the normalized valuation of A and satisfies $\nu(\tilde{s}) = 0$. Write $\tilde{\varphi} = \pi^{\nu(\tilde{\varphi})} \tilde{\varphi}_0$, where $\nu(\tilde{\varphi}_0) = 0$. We note that the natural surjection $A[\tilde{s}]/(\tilde{\varphi}_0) \rightarrow \tilde{\mathcal{O}}$ is an isomorphism.

If $\text{ord}_p: \mathbb{Z} \setminus \{0\} \rightarrow \mathbb{N}$ denotes the p -adic valuation, then Legendre’s formula shows that

$$\text{ord}_p \binom{p^r}{n} = r - \text{ord}_p(n),$$

so that

$$\tilde{\varphi} = \pi^{p^r} \tilde{s}^{p^r} + \sum_{n=1}^{p^r-1} \pi^{n+er-e \cdot \text{ord}_p(n)} u_n \tilde{s}^n = \pi^{p^r} \tilde{s}^{p^r} + \sum_{n=1}^{p^r-1} \pi^{\alpha(n)} u_n \tilde{s}^n,$$

with $u_n \in A^\times$ and $\alpha(n) := n + er - e \cdot \text{ord}_p(n)$. Now, if $\text{ord}_p(n) = i$, then $\alpha(n) = n + er - ei$, so that $\alpha(p^i) \leq \alpha(n)$ in this case. It follows that

$$\min_{1 \leq n < p^r} \alpha(n) = \min_{0 \leq i < r} \alpha(p^i) = er + \min_{0 \leq i < r} \{p^i - ei\}.$$

Hence, $\nu(\tilde{\varphi})$ is the minimum between $er + \min_{0 \leq i < r} \{p^i - ei\}$ and p^r . The lemma is now proved once we note that $\tilde{\mathcal{O}}$ is finite if and only if $\tilde{\varphi}_0$ above has an invertible leading coefficient. \square

Proof of Proposition 9.1. (1) Let us first deal with the case $r = 1$. Using the fact that \mathcal{L} becomes trivial on the μ_p -torsor associated with it and Theorem 8.1, we see that $\mathcal{L} \in \mathfrak{T}_X$. Let $G = \text{Gal}'(\mathcal{L})$, and let $\lambda: G \rightarrow \mathbb{G}_m$ be the associated representation. Then, λ factors through $\mu_p \subset \mathbb{G}_m$, and $\lambda \otimes K$ induces an isomorphism $G \otimes K \rightarrow \mu_p \otimes K$. Since λ_k is trivial, the morphism λ factors through an arrow $\tilde{\lambda}: G \rightarrow \tilde{\mu}_p$, where $\tilde{\mu}_p$ is the Néron blowup of μ_p at the identity of the special fibre. Now, because $G \rightarrow \tilde{\mu}_p$ induces an isomorphism between generic fibres and a fortiori an injection among rings of functions, the finiteness of G (by Theorem 8.10) implies that of $\tilde{\mu}_p$, and Lemma 9.2 finishes the proof.

Let us now assume $r \geq 2$ and that the claim is true for all $i < r$. Since $\mathcal{L}^{\otimes p^{r-i}}$ has order p^i , we conclude that

$$p^i \leq e + p^{i-1} \tag{9.1}$$

for $i \in \{1, \dots, r-1\}$. Using the fact that \mathcal{L} becomes trivial on the μ_{p^r} -torsor associated with it and Theorem 8.1, we see that $\mathcal{L} \in \mathfrak{T}_X$. Let $G = \text{Gal}'(\mathcal{L})$, and let $\lambda: G \rightarrow \mathbb{G}_m$ be the associated representation. Just as for the particular case, we conclude that the blowup of μ_{p^r} at the identity of the special fibre is finite. Hence, by Lemma 9.2,

$$p^r \leq er + \min_{0 \leq i < r} \{p^i - ei\}.$$

But using (9.1), we see that

$$1 \geq p - e \geq p^2 - 2e \geq \dots \geq p^{r-1} - (r-1) \cdot e.$$

Consequently, $\min_{0 \leq i < r} \{p^i - ei\} = p^{r-1} - (r-1) \cdot e$, so that $p^r \leq er + p^{r-1} - (r-1) \cdot e$, and this is what we wanted.

(2) One takes X to be an abelian scheme such that Y is its dual and applies part (1). \square

Remark 9.3. Corollary 9.1(2) can be deduced from ‘‘Cassels’ theorem’’ on formal groups. The one-dimensional case is classical [Sil09, Theorem IV.6.1], while the higher-dimensional case can be found in [Gra13, proof of Theorem 1, p. 966].

10. Application to the theory of torsors

We suppose that A is Henselian and Japanese. Let X be an irreducible, proper and flat A -scheme with geometrically reduced fibres and x_0 an A -point of X .

PROPOSITION 10.1. *Let $G \in (\mathbf{FGSch}/A)$ be finite and $\varphi: Q \rightarrow X$ be a G -torsor. Then, there exists a $\psi: Z \rightarrow X$ in $\mathfrak{S}^+(X, x_0)$ such that $\theta_Q: \text{Rep}_A(G) \rightarrow \mathbf{coh}(X)$ takes values in $\mathfrak{T}_\psi^{\text{tan}}$.*

Proof. According to Theorem 8.1, there exist a $\psi: Z \rightarrow X$ in $\mathfrak{S}^+(X, x_0)$ and a commutative diagram

$$\begin{array}{ccc} Z & \longrightarrow & Q \\ & \searrow \psi & \downarrow \varphi \\ & & X \end{array}$$

Hence, for each $M \in \text{Rep}_A(G)$, we conclude that $\psi^*(\theta_Q(M))$ is trivial relatively to A since $\varphi^*\theta_Q(M) \simeq \mathcal{O}_Q \otimes_A M$ is trivial relatively to A . Finally, since each $M \in \text{Rep}_A(G)$ is a quotient of some object in $\text{Rep}_A^\circ(G)$, see [Ser68, proposition 3], it follows that θ_Q takes values in $\mathfrak{T}_\psi^{\text{tan}}$. \square

In [Nor82, Chapter II, pp. 87ff], Nori defines the notion of ‘‘reduced torsor’’ in order to understand which group schemes do come as quotients of his fundamental group. We follow the same idea here, but instead of starting off with Nori’s definition, we prefer to use an equivalent characterization [Nor82, Proposition II.3].

Before reading the definition to come, the reader might profit to recall that, for any $G \in (\mathbf{FGSch}/A)$ and any G -torsor $\varphi: Q \rightarrow X$ above X , the functor $\theta_Q: \text{Rep}_A(G) \rightarrow \mathbf{coh}(X)$ is exact and faithful since $\varphi^*\theta_Q$ is naturally isomorphic to $M \mapsto \mathcal{O}_Q \otimes_A M$ (see, for example, the proof of [Jan87, Proposition I.5.9(a)]).

DEFINITION 10.2. Let $G \in (\mathbf{FGSch}/A)$ be finite and $Q \rightarrow X$ be a G -torsor having an A -point q_0 above x_0 . We say that the data (Q, G, q_0) defines a Nori-reduced torsor if $\theta_Q: \text{Rep}_A(G) \rightarrow \mathbf{coh}(X)$ is a fully faithful functor.

PROPOSITION 10.3 (Compare to [MS13, Theorem 7.1]). *Let $G \in (\mathbf{FGSch}/A)$ be finite and $Q \rightarrow X$ be a G -torsor having an A -point q_0 above x_0 . Then, the following conditions are equivalent:*

- (1) *The triple (Q, G, q_0) is Nori-reduced.*
- (2) *The ring of global functions of Q_k is k .*
- (2') *The triple $(Q_k, G_k, q_{0,k})$ is reduced.*
- (3) *The A -scheme Q is H^0 -flat, and $A = H^0(Q, \mathcal{O}_Q)$.*

The notion of ‘‘reduced’’ used in item (2') is that introduced in [Nor82, Definition II.3].

Proof. The proof relies on the fact that $A[G]_{\text{left}}$ is an algebra in the category $\text{Rep}_A(G)$ and that the corresponding \mathcal{O}_X -algebra $\theta_Q(A[G]_{\text{left}})$ is simply \mathcal{O}_Q .

(1) \Rightarrow (2). Because the functor θ_Q is fully faithful and $\theta_Q(k[G]_{\text{left}}) \simeq \mathcal{O}_Q \otimes k$, we conclude that $k = H^0(X, \mathcal{O}_Q \otimes k)$. But $H^0(X, \mathcal{O}_Q \otimes k) = H^0(Q, \mathcal{O}_Q \otimes k)$, and hence the only regular functions on the k -scheme Q_k are the constants.

(2) \Leftrightarrow (2'). This is [Nor82, Proposition II.3].

(2) \Rightarrow (3). Recall that Q is flat over X and a fortiori over A . Now, as $k = H^0(Q, \mathcal{O}_Q \otimes_A k)$, we can employ [Har77, Proposition III.12.10] to conclude that Q is cohomologically flat of degree zero over A . Since $H^0(\mathcal{O}_Q)$ is a finite and flat A -module, the isomorphism $k \simeq k \otimes_A H^0(\mathcal{O}_Q)$ proves that $A \simeq H^0(\mathcal{O}_Q)$.

(3) \Rightarrow (1). Let φ denote the structural morphism $Q \rightarrow X$; by assumption, it belongs to $\mathfrak{S}^+(X, x_0)$. If we agree to write $H := \Pi(X, \varphi, x_0)$ (see Definition 4.6), the existence of the point q_0 gives an isomorphism between $(\bullet|_{x_0}) \circ \theta_Q$ and the forgetful functor $\text{Rep}_A(G) \rightarrow \mathbf{A}\text{-mod}$ and hence a morphism of group schemes $\rho: H \rightarrow G$ together with a commutative diagram

$$\begin{array}{ccc}
 \text{Rep}_A(G) & \xrightarrow{\theta_Q} & \mathfrak{T}_\varphi \\
 \rho^\# \downarrow & \swarrow \sim & \\
 \text{Rep}_A(H) & & (\bullet|_{x_0})
 \end{array} \tag{10.1}$$

Now,

$$k = H^0(X, \mathcal{O}_{Q_k}) \simeq H^0(X, \theta_Q(k[G]_{\text{left}})) \simeq (k[G]_{\text{left}})^H \simeq (k[G]_{\text{left}})^{H_k}.$$

According to Lemma 2.8, this is only possible when $\rho_k: H_k \rightarrow G_k$ is faithfully flat. Analogously, we have $(A[G]_{\text{left}})^H \simeq A$. This implies that $(K[G]_{\text{left}})^{H_K} \simeq K$, and hence ρ_K is faithfully flat according to Lemma 2.8. In conclusion, ρ_k and ρ_K are faithfully flat, and hence ρ must be faithfully flat [DH18, Theorem 4.1.1]. Combining this with [DH18, Proposition 3.2.1(ii)], we conclude that $\rho^\#$ is fully faithful, so that diagram (10.1) secures the full faithfulness of θ_Q . \square

We shall now keep the notation and assumptions of Proposition 10.3 and offer other properties equivalent to the ones in its statement. This will allow us, in passing, to render the connection with [MS13, Theorem 7.1] and [Nor82, Definition II.3] more transparent. First we develop some preliminaries.

Let us abbreviate $\Pi = \Pi(X, x_0)$. As in [Nor76, Section 2 and the proof of Proposition 3.8], there exists a Π -torsor $\tilde{X} \rightarrow X$ with an A -point \tilde{x}_0 above x_0 such that $\theta_{\tilde{X}} \circ (\bullet|_{x_0}) \simeq \text{id}$ and $(\bullet|_{x_0}) \circ \theta_{\tilde{X}} \simeq \text{id}$ as tensor functors. The quasi-coherent \mathcal{O}_X -algebra of \tilde{X} is a direct limit of coherent modules belonging to \mathfrak{T}_X and corresponds, in $\text{IndRep}_A(\Pi)$, to $A[\Pi]_{\text{left}}$; see [Nor76, Definition after Lemma 2.2]. The torsor \tilde{X} is called the *universal pointed torsor*.

Let $\rho: G' \rightarrow G$ be an arrow of (\mathbf{FGSch}/A) . We say that Q has a *reduction of structure group to ρ , or to G'* , if there exists a G' -torsor $Q' \rightarrow X$ together with an isomorphism $Q' \times^\rho G \rightarrow Q$. In addition, if Q' can be picked to come with an A -point q'_0 such that (q'_0, e) corresponds to q_0 under the aforementioned isomorphism, then the reduction is called *pointed*. Note that we do not assume ρ to be a closed embedding.

COROLLARY 10.4. *The equivalent properties appearing in Proposition 10.3 are also equivalent to each one of the following conditions:*

- (a) *If Q has a pointed reduction to $\rho: G' \rightarrow G$, then ρ is faithfully flat.*

- (b) *There exist a faithfully flat morphism $\rho: \Pi \rightarrow G$ and an isomorphism of pointed G -torsors $\tilde{X} \times^\rho G \xrightarrow{\sim} Q$. (That is, \tilde{X} defines a pointed reduction of Q .)*

Proof. Proposition 10.3(1) \Rightarrow (a). Let $\rho: G' \rightarrow G$ define a pointed reduction $Q' \rightarrow X$; it then follows that $\theta_{Q'} \circ \rho^\#$ is isomorphic to θ_Q . Since $\theta_{Q'}$ is faithful, we conclude that $\rho^\#: \text{Rep}_A(G) \rightarrow \text{Rep}_A(G')$ is full and faithful. By Lemma 2.9, the morphism ρ is faithfully flat.

(a) \Rightarrow (b). According to Proposition 10.1, the functor θ_Q takes values in \mathfrak{T}_X ; the existence of the point q_0 allows us to say that $\bullet|_{x_0} \circ \theta_Q$ is isomorphic to the forgetful functor $\text{Rep}_A(G) \rightarrow A\text{-mod}$, which gives us an arrow $\rho: \Pi \rightarrow G$ such that $\rho^\# \simeq \bullet|_{x_0} \circ \theta_Q$ (as tensor functors). Hence, $\theta_{\tilde{X}} \circ \rho^\# \simeq \theta_Q$, and we conclude, as in [Nor76, Proposition 2.9(c)], that Q has a pointed reduction to ρ . But condition (a) forces ρ to be faithfully flat.

(b) \Rightarrow Proposition 10.3(1). We know that $\theta_Q \simeq \theta_{\tilde{X}} \circ \rho^\#$ in this case; but, by construction, $\theta_{\tilde{X}}$ is fully faithful, as is $\rho^\#$ (by [DH18, Proposition 3.2.1(ii)], say). Therefore, θ_Q is fully faithful. \square

Remark 10.5. Let $Q \rightarrow X$ be as in the statement of Proposition 10.3. The condition that $\theta_Q: \text{Rep}_A^\circ(G) \rightarrow \mathfrak{T}_X^\circ$ be full is not enough to ensure that G is a faithfully flat quotient of Π (so that this is missing in [MS13, Theorem 7.1]).

11. Essentially finite vector bundles on the fibres: Reviewing a theory of Mehta and Subramanian

Let X be an irreducible, projective and flat A -scheme with geometrically reduced fibres and x_0 an A -point of X . (Recall that over a perfect field, an algebraic scheme is geometrically reduced if and only if it is reduced [EGA, IV₂, proposition 4.6.1]. Also note that X must be reduced.)

The following result, which is one of the main points in [MS13], is essentially a consequence of the method employed by Deninger and Werner in proving [DW05, Theorem 17] plus Section 8. Before putting forth its statement, let us recall the notion of an F -trivial vector bundle.

If M is a proper scheme over an unspecified *perfect* field of positive characteristic, a vector bundle E on M is called F -trivial [MS02, Section 2, p. 144] if for a certain $s \in \mathbb{N}$, the pull-back of E by a geometric Frobenius morphism $\text{Fr}^s: M^{(-s)} \rightarrow M$ is trivial.

THEOREM 11.1. *Suppose that A is Henselian and Japanese and that k is perfect of characteristic $p > 0$. Let E be an F -trivial vector bundle on X_k . Then, there exist a proper and surjective morphism $\psi: Z \rightarrow X$ such that:*

- (1) *The ring $B := H^0(\mathcal{O}_Z)$ is a discrete valuation ring and is a finite extension of A .*
- (2) *The canonical morphism $Z \rightarrow \text{Spec } B$ is flat and has geometrically reduced fibres.*
- (3) *The B -scheme Z has a B -point above x_0 .*
- (4) *Write ℓ for the residue field of B and denote by*

$$\psi_0: Z \otimes_B \ell \longrightarrow X_k$$

the morphism of k -schemes naturally induced by ψ . Then $\psi_0^(E)$ is trivial.*

Proof. *The case of characteristic $(0, p)$.* We assume that E is trivialized by $\text{Fr}^s: X_k^{(-s)} \rightarrow X_k$. Let $X \rightarrow \mathbb{P}_A^n$ be a closed immersion. Write $\Phi: \mathbb{P}_A^n \rightarrow \mathbb{P}_A^n$ for the evident A -morphism lifting the

k -linear Frobenius morphisms $\text{Fr}^s: \mathbb{P}_k^n \rightarrow \mathbb{P}_k^n$, and consider the cartesian diagram

$$\begin{array}{ccc} Y & \xrightarrow{\varphi} & X \\ \downarrow & \square & \downarrow \\ \mathbb{P}_A^n & \xrightarrow{\Phi} & \mathbb{P}_A^n. \end{array}$$

We note that Φ is a finite, flat and surjective morphism, so that $\varphi: Y \rightarrow X$ is likewise; in particular, this implies that Y is A -flat.

Base-changing by means of $A \rightarrow k$, we get the cartesian diagram

$$\begin{array}{ccc} Y_k & \xrightarrow{\varphi_k} & X_k \\ \downarrow & \square & \downarrow \\ \mathbb{P}_k^n & \xrightarrow{\text{Fr}^s} & \mathbb{P}_k^n, \end{array}$$

so that, since $X_k^{(-s)}$ is reduced, there exists a closed embedding $j: X_k^{(-s)} \rightarrow Y_k$, which *identifies* $X_k^{(-s)}$ with $Y_{k,\text{red}}$ and, in addition, produces a factorisation of $\text{Fr}^s: X_k^{(-s)} \rightarrow X_k$ like so:

$$\begin{array}{ccc} X_k^{(-s)} & \xrightarrow{\text{Fr}^s} & X_k \\ \downarrow j & \searrow & \downarrow \\ Y_k & \xrightarrow{\varphi_k} & X_k \\ \downarrow & \square & \downarrow \\ \mathbb{P}_k^n & \xrightarrow{\text{Fr}^s} & \mathbb{P}_k^n. \end{array}$$

See [DW05, Lemma 19]. Consequently, if V is any *reduced* scheme and $\alpha: V \rightarrow Y_k$ is a \mathbb{Z} -morphism, we conclude that $\alpha^* \varphi_k^* E$ is trivial since α factors as $V \rightarrow X_k^{(-s)} \xrightarrow{j} Y_k$.

A direct application of Theorem 8.1 now gives us a commutative diagram

$$\begin{array}{ccc} Z & \longrightarrow & Y \\ & \searrow \psi & \downarrow \varphi \\ & & X \end{array}$$

such that:

- (i) The morphism ψ is surjective and proper.
- (ii) The ring $B := H^0(\mathcal{O}_Z)$ is a discrete valuation ring and is a finite extension of A .
- (iii) The canonical morphism $Z \rightarrow \text{Spec } B$ is flat and has geometrically reduced fibres.
- (iv) The B -scheme Z has a B -point above x_0 .

In this situation, the proof is concluded by the observation preceding it. Indeed, if ℓ is the residue field of B and $\psi_0: Z \otimes_B \ell \rightarrow X_k$ is the arrow induced by ψ , we conclude that $\psi_0^*(E)$ is trivial because $Z \otimes_B \ell$ is reduced, so that ψ_0 factors through $Z \otimes_B \ell \rightarrow Y_k$.

The case of characteristic (p, p) . The idea behind the proof is much simpler, but notation and technicalities hinder its handling.

Suppose that $E \in \mathbf{VB}(X_k)$ becomes trivial after being pulled back by $F_{X_k}^s: X_k \rightarrow X_k$. Employing the commutative diagram of \mathbb{F}_p -schemes

$$\begin{array}{ccc} X & \xrightarrow{F_X^s} & X \\ \downarrow & & \downarrow \\ \text{Spec } A & \xrightarrow{F_A^s} & \text{Spec } A, \end{array}$$

we see that the morphism $(F_X^s)_0: X_k \rightarrow X_k$ induced on special fibres is none other than $F_{X_k}^s$. Hence, if F_A^s is a finite morphism, the choice $Z = X$ and $\Psi = F_X^s$ is sufficient to fulfil all but condition (3) of the statement. But finiteness of F_A^s is not always ensured, and we choose to argue as in [EGA, IV₃, section 8].

Let

$$\Lambda = \left\{ \begin{array}{l} B \text{ is a d.v.r. dominated by } A \text{ and dominating } A^{p^s} \\ \text{and such that } \text{Frac } B \text{ is a finite extension of } K^{p^s} \end{array} \right\},$$

and endow it with the partial order defined by domination of d.v.r. As A is Japanese, for any $B \in \Lambda$, the A^{p^s} -module B is finite, and any element in A belongs to some $B \in \Lambda$ (see [Mat89, Theorem 10.2 and Exercise 11.2]). Consequently, the limit $\varprojlim_{B \in \Lambda} \text{Spec } B$ in the category of A -schemes is simply $F_A^s: \text{Spec } A \rightarrow \text{Spec } A$. Employing [EGA, IV₃, théorème 8.8.2(ii)], we see that there exist an $\mathfrak{o} \in \Lambda$ and an \mathfrak{o} -scheme of *finite type* Y fitting into a cartesian commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{u} & Y \\ \downarrow & \square & \downarrow \\ \text{Spec } A & \longrightarrow & \text{Spec } \mathfrak{o}. \end{array} \tag{11.1}$$

In addition, if $u_B: X \rightarrow Y \otimes_{\mathfrak{o}} B$ stands for the canonical morphism, an application of [EGA, IV₃, proposition 8.2.5] shows that

$$(u_B): X \longrightarrow \varprojlim_{B \geq \mathfrak{o}} Y \otimes_{\mathfrak{o}} B$$

is in fact an isomorphism of \mathfrak{o} -schemes. Also by loc. cit., the canonical morphism

$$X \otimes_{A, F_A^s} A \longrightarrow \varprojlim_{B \geq \mathfrak{o}} X \otimes_{A, F_A^s} B$$

is also an isomorphism. The relative Frobenius morphism

$$\mathfrak{f}: X \longrightarrow X \otimes_{A, F_A^s} A$$

now gives rise, via [EGA, IV₃, théorème 8.8.2(i)], to a $B \geq \mathfrak{o}$ and a morphism of B -schemes

$$f: Y \otimes_{\mathfrak{o}} B \longrightarrow X \otimes_{A, F_A^s} B$$

such that $f \otimes_B A$ corresponds to \mathfrak{f} . Hence, if ψ stands for the composition of f with the projection $X \otimes_{A, F_A^s} B \rightarrow X$, we arrive at a commutative diagram

$$\begin{array}{ccc} Y \otimes_{\mathfrak{o}} B & \xrightarrow{\psi} & X \\ \downarrow & & \downarrow \\ \text{Spec } B & \longrightarrow & \text{Spec } A. \end{array} \tag{11.2}$$

In addition, $\psi \circ u_B = F_X^s$. Let us agree to write $Z = Y \otimes_o B$. Then, paralleling diagram (11.1), we have

$$\begin{array}{ccc} X & \xrightarrow{u_B} & Z \\ \downarrow & \square & \downarrow \\ \text{Spec } A & \longrightarrow & \text{Spec } B. \end{array} \tag{11.3}$$

CLAIM. *The following statements are true:*

- (1) *The morphism ψ is finite and surjective.*
- (2) *As a B -scheme, Z is flat and proper.*
- (3) *The geometric fibres of Z over B are reduced.*
- (4) *The ring of global functions of Z is B .*
- (5) *Write ψ_0 for the morphism induced from ψ between special fibres. Then $\psi_0^*(E)$ is trivial.*

Proof. (1) Surjectivity follows from $F_X^s = \psi \circ u_B$. Because \mathfrak{f} is finite and the inclusion $B \rightarrow A$ is faithfully flat, we conclude that f is finite [EGA, IV₂, proposition 2.7.1]. Hence, ψ is finite as $F_A^s: A \rightarrow B$ is finite.

(2) We note that the morphism $\text{Spec } A \rightarrow \text{Spec } B$ in diagram (11.3) is faithfully flat. Consequently, the claim is proved by employing [EGA, IV₂, propositions 2.5.1 and 2.7.1] and the fact that X is flat and proper over A .

(3) This is a direct consequence of diagram (11.3) and the fact that being geometrically reduced is independent of the field extension [EGA, IV₂, proposition 4.6.10].

(4) This is a direct consequence of flat base change applied to diagram (11.3) and $A = H^0(\mathcal{O}_X)$ (which follows from Lemma 2.2, say).

(5) Now let ℓ be the residue field of B ; it is clear that $B \rightarrow A$ in fact induces an isomorphism $\ell \xrightarrow{\sim} k$. Since $X = Z \otimes_B A$, it follows that $(u_B)_0: X \otimes_A k \rightarrow Z \otimes_B \ell$ is also an isomorphism. Hence, $\psi_0^*(E)$ is trivial because $\psi u_B = F_X^s$, so that $(u_B)_0^*(\psi_0^*(E))$ is trivial. The claim is proved. \square

Proof of Theorem 11.1, continued. To finish the proof, we note that $\psi: Z \rightarrow X$ satisfies all the conditions in the statement of Theorem 11.1 except for the existence of a B -point above x_0 . Now, the inverse image $\psi^{-1}(x_0)$ comes with a finite and surjective morphism to $\text{Spec } A$ (for the surjectivity, see [EGA, I, proposition 3.5.2]). Hence, it is possible to find a *finite* extension of d.v.r.s $B' \supset A$ and a point $\text{Spec } B' \rightarrow \psi^{-1}(x_0)$ which then gives a commutative diagram

$$\begin{array}{ccc} & & Z \\ & \nearrow^{z'_0} & \downarrow \\ \text{Spec } B' & \longrightarrow & \text{Spec } B \end{array}$$

such that z'_0 is a B' -point of X above x_0 and the induced arrow $B \rightarrow B'$ is a finite extension. Consequently, letting Z' be $Z \otimes_B B'$ and $\psi': Z' \rightarrow X$ the composition $Z' \xrightarrow{\text{pr}} Z \xrightarrow{\psi} X$, we see that $\psi': Z' \rightarrow X$ now satisfies all properties in the statement of the theorem. \square

As a consequence of Theorem 11.1, we can give a simple alternative description of the vector bundles in \mathfrak{T} and, in doing so, connect our theory to that of [MS13]. See Corollary 11.4.

THEOREM 11.2 (Compare to [MS13, Lemma 3.1]). *Suppose that A is Henselian and Japanese, that k is perfect and that X is in addition normal. Let $\mathcal{E} \in \mathbf{VB}(X)$ be such that \mathcal{E}_K and \mathcal{E}_k are essentially finite. Then, there exists a proper and surjective morphism*

$$\zeta: X' \longrightarrow X$$

such that:

- (1) *The ring $A' := H^0(\mathcal{O}_{X'})$ is a discrete valuation ring and a finite extension of A .*
- (2) *The canonical morphism $X' \rightarrow \text{Spec } A'$ is flat and has geometrically reduced fibres.*
- (3) *The A' -scheme X' has an A' -point above x_0 .*
- (4) *The vector bundle $\zeta^*(\mathcal{E})$ is trivial.*

Proof. As the vector bundle \mathcal{E}_k is essentially finite, it is possible to find a torsor under an étale group scheme $f: Y \rightarrow X_k$, and a fortiori an étale covering, such that $f^*(\mathcal{E}_k)$ is F -trivial; see [Nor76, Section 3]. In addition, Y can be chosen to possess a k -rational point y_0 above $x_{0,k}$ (cf. loc. cit.) and to satisfy $k = H^0(Y, \mathcal{O}_Y)$; see [Nor82, Proposition II.3]. Now, as A is Henselian, [Art69, Theorem 3.1] (the remarkable equivalence, Grothendieck's existence theorem and Artin approximation) allows us to find an étale covering $\tilde{f}: \tilde{Y} \rightarrow X$ lifting $Y \rightarrow X_k$. Looking at the finite and étale A -scheme $\tilde{f}^{-1}(x_0)$, applying one of the main properties of a Henselian local ring [Ray70, proposition VII.3.3] and making use of the k -point $y_0: \text{Spec } k \rightarrow \tilde{f}^{-1}(x_0)$, we can find an A -point \tilde{y}_0 in \tilde{Y} above x_0 . Note that \tilde{Y} inherits the following properties from X : it is flat and proper, has geometrically reduced fibres over A and is normal. In addition, \tilde{Y} is connected, and its normality then implies irreducibility. (Irreducibility might fail without normality.) Therefore, \tilde{Y} satisfies all hypothesis imposed on X at the beginning of the section and in the statement of the theorem. Note that, by construction, the restriction of $\tilde{f}^*\mathcal{E}$ to \tilde{Y}_k is F -trivial.

Because of the previous paragraph, we suppose, to lighten the notation, that \mathcal{E}_k is F -trivial already on X_k . Let us apply Theorem 11.1 to the vector bundle $E = \mathcal{E}_k$. Then, keeping with the notation of this theorem, we conclude that

$$\psi^*(\mathcal{E})|_{Z \otimes_B L}$$

is trivial. Let $L = \text{Frac}(B)$. Since $\psi^*(\mathcal{E})|_{Z \otimes_B L}$ is an essentially finite vector bundle, [Nor76, Section 3] ensures that we can find a torsor with finite structural group

$$\lambda^\circ: Q^\circ \longrightarrow Z \otimes_B L$$

such that

$$\lambda^{\circ*}(\psi^*(\mathcal{E})|_{Z \otimes_B L})$$

is trivial. In addition, Q° might be chosen to come with two extra properties, which are:

- Letting $z_0: \text{Spec } B \rightarrow Z$ be the point above x_0 mentioned in Theorem 11.1, the scheme Q° has an L -rational point q_0° above $z_{0,L}$. (This is not used in what follows.)
- The ring of global functions of Q° is L ; see [Nor82, Proposition II.3]. In particular, Q° is connected.

Let $Q^\square \rightarrow Q^\circ$ be the associated reduced scheme, and write $\lambda^\square: Q^\square \rightarrow Z \otimes_B L$ for the induced morphism. Clearly, Q^\square is connected, λ^\square is surjective and finite, and $\lambda^{\square*}(\psi^*(\mathcal{E})|_{Z \otimes_B L})$ is trivial.

Let $\mu: Z \otimes_B L \rightarrow Z$ be the natural immersion, and write

$$\lambda: Q \longrightarrow Z$$

for the integral closure of the quasi-coherent \mathcal{O}_Z -algebra $(\mu\lambda^\square)_*(\mathcal{O}_{Q^\square})$; see [EGA, II, section 6.3] or [Sta22, Tag 035H]. This means, according to [EGA, II, proposition 6.3.4], that for each affine open subset $V \subset Z$, the ring $\mathcal{O}_Q(\lambda^{-1}V)$ is the integral closure of $\mathcal{O}_Z(V)$ inside $\mathcal{O}_{Q^\square}((\mu\lambda^\square)^{-1}(V))$. In particular, Q is flat as a B -scheme, and

$$Q \otimes_B L = Q^\square.$$

Since A is universally Japanese [EGA, IV₂, théorème 7.7.2], for each affine and open subset V of Z , the ring $\mathcal{O}_Z(V)$ is universally Japanese and noetherian and hence is a Nagata ring [Sta22, Tag 032R]. As a consequence of [Sta22, Tag 03GH] and the fact that Q^\square is reduced, we see that λ is a *finite* morphism. Because $\lambda(Q)$ contains $Z \otimes_B L$ (recall that λ^\square is surjective), we conclude that λ is surjective. Finally, both $(\psi\lambda)^*(\mathcal{E})|_{Q \otimes_B L}$ and $(\psi\lambda)^*(\mathcal{E})|_{Q \otimes_B L}$ are trivial.

Theorem 8.1 can be applied to $\psi\lambda: Q \rightarrow X$, and this allows us to find a commutative diagram of schemes

$$\begin{array}{ccc} R & \longrightarrow & Q \\ & \searrow \theta & \downarrow \psi\lambda \\ & & X \end{array}$$

such that:

- The morphism θ is proper and surjective.
- The ring $C := H^0(\mathcal{O}_R)$ is a discrete valuation ring and a finite extension of A .
- The canonical morphism $R \rightarrow \text{Spec } C$ is flat and has geometrically reduced fibres.
- The C -scheme R has a C -point above x_0 .

As the natural arrow $\text{Spec } C \rightarrow \text{Spec } B$ sends the generic, respectively special, point to the generic, respectively special, point, the triviality of $(\psi\lambda)^*(\mathcal{E})|_{Q \otimes_B L}$ and $(\psi\lambda)^*(\mathcal{E})|_{Q \otimes_B L}$ allows us to conclude that the restrictions of $\theta^*\mathcal{E}$ to the generic and special fibres of R over C are trivial. Because R is H^0 -flat over C (it has reduced fibres), we conclude by employing Lemma 2.3 that $\theta^*\mathcal{E}$ is trivial. \square

For the sake of discussion, let us introduce the following notion.

DEFINITION 11.3. The Mehta–Subramanian category of X , denoted by $\mathbf{MS}(X)$, is the full subcategory of $\mathbf{VB}(X)$ whose objects are

$$\{\mathcal{E} \in \mathbf{VB}(X): \mathcal{E}_k \text{ and } \mathcal{E}_K \text{ are essentially finite}\}.$$

An immediate consequence of [AM11, Theorem 1] (or [TZ20, Theorem I]) and Theorem 11.2 is then as follows.

COROLLARY 11.4. *Suppose that A is Henselian and Japanese, that k is perfect and that in addition to the hypothesis at the beginning of the section, X_k and X_K are normal. Then $\mathfrak{T}_X^\circ = \mathbf{MS}(X)$.*

Proof. Let $\mathcal{E} \in \mathbf{MS}(X)$. Since X is flat over A , normality of X_k and X_K implies normality of X , see [EGA, IV₂, corollaire 6.5.4], and Theorem 11.2 may be applied. Consequently, \mathcal{E} belongs to \mathfrak{T}_X° as ζ in Theorem 11.2 lies in $\mathfrak{S}^+(X, x_0)$. Conversely, let \mathcal{E} be a vector bundle in \mathfrak{T}_X° . Since X_k and X_K are normal, we know that \mathcal{E}_k and \mathcal{E}_K are essentially finite (due to [AM11, Theorem 1] or [TZ20, Theorem I]). \square

Remark 11.5. Since a point $x \in X$ above the generic fibre specializes to a point on the special fibre, normality of X is equivalent to normality of X on the points of X_k . Of course, X_k can easily fail to be normal even when X is regular.

12. Further applications to the theory of torsors

We assume that A is Henselian and Japanese and has a perfect residue field. Let X be an irreducible, projective and flat A -scheme with geometrically reduced fibres and x_0 an A -point of X .

THEOREM 12.1 (Compare to [MS13, Corollary 3.2]). *Let us add to the assumptions made at the start of this section that X is normal. Let $G \in (\mathbf{FGSch}/A)$ be quasi-finite over A , let $Q \rightarrow X$ be a G -torsor and q_0 an A -point of Q above x_0 .*

- (1) *There exist a $\zeta: X' \rightarrow X$ in $\mathfrak{S}^+(X, x_0)$ and an $\mathcal{E} \in \mathfrak{T}_\zeta^\circ$ such that $\theta_Q: \text{Rep}_A(G) \rightarrow \mathbf{coh}(X)$ takes values in $\langle \mathcal{E}; \mathfrak{T}_\zeta \rangle_\otimes$ (and a fortiori in \mathfrak{T}_X).*
- (2) *There exist a finite $H \in (\mathbf{FGSch}/A)$, a morphism $\rho: H \rightarrow G$, an H -torsor $R \rightarrow X$ and an A -point $r_0: \text{Spec } A \rightarrow R$ together with an isomorphism of torsors*

$$R \times^H G \xrightarrow{\sim} Q$$

sending the A -point (r_0, e) of $R \times^H G$ to q_0 . In addition, it is possible to choose ρ to be a closed immersion. Put differently, Q has a reduction of structure group to a finite group scheme.

- (3) *If $H^0(Q, \mathcal{O}_Q) \simeq A$, then G is in fact finite.*

Proof. As is well known (by adapting the proofs in [Wat79, Section 3.3]), the facts that G is of finite type and A is a d.v.r. allow us to find an $E \in \text{Rep}_A^\circ(G)$ such that the resulting morphism $G \rightarrow \mathbf{GL}(E)$ is a closed immersion or, in the terminology of [DHdS18, Section 3], that E is a faithful representation. We then write $\mathcal{E} = \theta_Q(E)$ and note that since $Q_k \rightarrow X_k$ and $Q_K \rightarrow X_K$ are finite principal bundles, the vector bundles $\mathcal{E}_k = \theta_{Q_k}(E_k)$ and $\mathcal{E}_K = \theta_{Q_K}(E_K)$ are in fact essentially finite [Nor76, Proposition 3.8]. Let $\zeta: X' \rightarrow X$, A' and x'_0 be as in Theorem 11.2 when applied to \mathcal{E} . Note that $\zeta: X' \rightarrow X$ is in $\mathfrak{S}^+(X, x_0)$ and that $\mathcal{E} \in \mathfrak{T}_\zeta$.

(1) We write Q' for the G -torsor $X' \times_X Q$. We know that for each $M \in \text{Rep}_A(G)$, the coherent $\mathcal{O}_{X'}$ -module $\zeta^* \theta_Q(M)$ is isomorphic to $\theta_{Q'}(M)$. Because each $\theta_Q(\mathbf{T}^{a,b}E)$ belongs to $\langle \mathcal{E}; \mathfrak{T}_\zeta \rangle_\otimes$, we conclude that each $\theta_{Q'}(\mathbf{T}^{a,b}E)$ is trivial.

Let $T \in \text{Rep}_A^\circ(G)$ be such that $\theta_Q(T)$ belongs to $\langle \mathcal{E}; \mathfrak{T}_\zeta \rangle_\otimes$. If $V \in \text{Rep}_A^\circ(G)$ is the target of an epimorphism $T \rightarrow V$, the exactness of $\theta_{Q'}$ produces an epimorphism

$$\mathcal{O}_{X'}^{\oplus r} \simeq \theta_{Q'}(T) \longrightarrow \theta_{Q'}(V).$$

Since $\theta_{Q'}(V) \otimes_A k$ and $\theta_{Q'}(V) \otimes_A K$ become trivial when pulled back via $Q' \otimes_A k \rightarrow X' \otimes_A k$ and $Q' \otimes_A K \rightarrow X' \otimes_A K$, we may apply Lemma 5.2 to conclude that $\theta_{Q'}(V) \otimes k$ and $\theta_{Q'}(V) \otimes K$ are trivial. Because X' is H^0 -flat over A , Lemma 2.3 says that $\theta_{Q'}(V)$ is equally trivial and hence that $\theta_Q(V) \in \langle \mathcal{E}; \mathfrak{T}_\zeta \rangle_\otimes$. By the same argument, now applied to \tilde{T} , we conclude that $\theta_Q(W)$ belongs to $\langle \mathcal{E}; \mathfrak{T}_\zeta \rangle_\otimes$ once $W \rightarrow T$ is a special subobject (we employ [dSa09, Definition 10]).

Now we know that for every $U \in \text{Rep}_A^\circ(G)$, there exist a special monomorphism $V \rightarrow \mathbf{T}^{a,b}E$ and an epimorphism $V \rightarrow U$; see [dSa09, Proposition 12]. From what was proved above, $\theta_Q(U)$ belongs to $\langle \mathcal{E}; \mathfrak{T}_\zeta \rangle_\otimes$.

To end the proof, let $M \in \text{Rep}_A(G)$ be arbitrary. Using [Ser68, section 2.2, corollaire], we find an equivariant presentation

$$0 \longrightarrow U_1 \longrightarrow U_0 \longrightarrow M \longrightarrow 0$$

with U_0 and U_1 in $\text{Rep}_A^\circ(G)$. An application of Lemma 4.1 (and the exactness of the functor θ_Q) shows that $\theta_Q(M)$ belongs to \mathfrak{T}_ζ and hence to $\langle \mathcal{E}; \mathfrak{T}_\zeta \rangle_\otimes$.

(2) We pick ζ and \mathcal{E} as in part (1) and define $H = \text{Gal}'(\mathcal{E}; \mathfrak{T}_X, x_0)$, so that $\bullet|_{x_0}: \langle \mathcal{E}; \mathfrak{T}_X \rangle_\otimes \rightarrow \text{Rep}_A(H)$ is an equivalence of tensor categories. Because of Theorem 8.10, the group scheme H is *finite* over A . Using the A -point q_0 of Q above x_0 , the functor $(\bullet|_{x_0}) \circ \theta_Q: \text{Rep}_A(G) \rightarrow \mathbf{A}\text{-mod}$ is naturally isomorphic to the forgetful functor, and hence we derive a morphism of group schemes $\rho: H \rightarrow G$ such that $(\bullet|_{x_0}) \circ \theta_Q \simeq \rho^\#$. Now, as in [Nor76, Section 2 and the proof of Proposition 3.8], there exists an H -torsor $R \rightarrow X$ with an A -point r_0 above x_0 such that $\theta_R \circ (\bullet|_{x_0}) \simeq \text{id}$ and $(\bullet|_{x_0}) \circ \theta_R \simeq \text{id}$ as tensor functors. (The quasi-coherent \mathcal{O}_X -algebra of the torsor R corresponds, in the category $\text{Rep}_A(H)$, to $A[H]_{\text{left}}$; see [Nor76, Definition after Lemma 2.2].) Now

$$\theta_Q \simeq \theta_R \circ (\bullet|_{x_0}) \circ \theta_Q \simeq \theta_R \circ \rho^\#,$$

which shows, as in [Nor76, Proposition 2.9(c)], that $Q \simeq R \times^H G$.

To verify the last statement, we note that ρ can be decomposed into $H \rightarrow H' \xrightarrow{\sigma} G$, where σ is a closed immersion and H' is finite. This being so, we have $Q \simeq (R \times^H H') \times^{H'} G$.

(3) Let $H \in (\mathbf{FGSch}/A)$ be finite, $Q \rightarrow X$ be an H -torsor, $H \rightarrow G$ be a closed immersion and $R \times^H G \simeq Q$ be an isomorphism as in part (2). Now, employing the arrow

$$R \times G \longrightarrow G, \quad (r, g) \longmapsto g^{-1},$$

we obtain an injection $\text{Mor}(G, \mathbb{A}^1) \rightarrow \text{Mor}(R \times G, \mathbb{A}^1)$ and hence an injection

$$\{H\text{-equivariant } G \rightarrow \mathbb{A}^1\} \longrightarrow \{H\text{-equivariant } R \times G \rightarrow \mathbb{A}^1\}.$$

Since the right-hand side is simply the ring of functions of $R \times^H G \simeq Q$, the hypothesis then forces $A = A[G]^H$. But $A[G]$ is a finite and locally free $A[G]^H$ -module whose rank equals that of $A[H]$ (see [DG70, section III.2.4]). It is then easy to see that the closed immersion $H \rightarrow G$ is an isomorphism. \square

We now gather some consequences of Theorem 12.1 and in doing so connect it to [Nor82, Chapter II] and [AEG20]. *The notation is as in the statement of Theorem 12.1.*

Let us abbreviate $\Pi = \Pi(X, x_0)$. As already explained in Section 10 (see the discussion preceding Corollary 10.4), there exists a Π -torsor $\tilde{X} \rightarrow X$ with an A -point \tilde{x}_0 above x_0 such that $\theta_{\tilde{X}} \circ (\bullet|_{x_0}) \simeq \text{id}$ and $(\bullet|_{x_0}) \circ \theta_{\tilde{X}} \simeq \text{id}$ as tensor functors. Recall that for each homomorphism $\rho: \Pi \rightarrow G$, the fpqc sheaf of the contracted product $\tilde{X} \times^\rho G$ (see [DG70, section III.4.3.2, p. 368] or [Jan87, Section I.5.14]) has the following description: it is the quotient of $\tilde{X} \times G$ by the right action of Π defined, on the level of points, by

$$(\tilde{x}, g) \cdot \gamma = (\tilde{x}\gamma, \rho(\gamma)^{-1}g). \tag{12.1}$$

Let us write

$$\chi_\rho: \tilde{X} \times G \longrightarrow \tilde{X} \times^\rho G$$

for the canonical quotient morphism. Then, the arrow

$$(\text{pr}_{\tilde{X}}, \chi_\rho): \tilde{X} \times G \longrightarrow \tilde{X} \times_X (\tilde{X} \times^\rho G)$$

is an isomorphism of G -torsors over \tilde{X} (see [DG70, section III.4.3.1] or [Jan87, Section I.5.14(3)]). In addition, if we let Π act on (the right of) $\tilde{X} \times G$ as implied by (12.1) and on $\tilde{X} \times_X (\tilde{X} \times^\rho G)$ by the action solely on \tilde{X} , then $(\text{pr}_{\tilde{X}}, \chi_\rho)$ is Π -equivariant, as a simple verification shows.

Now we note that $A = H^0(\mathcal{O}_{\tilde{X}})$ because $A[\Pi]_{\text{left}}$, which corresponds to the quasi-coherent \mathcal{O}_X -module $\mathcal{O}_{\tilde{X}}$, has only constant invariants and because of [Har77, Exercise II.1.11]. Since G is affine, any morphism of schemes $\tilde{X} \rightarrow G$ must factor through the structural morphism $\tilde{X} \rightarrow \text{Spec } A$; see [EGA, I, proposition 2.2.4]. We conclude that any arrow $\beta: \tilde{X} \times G \rightarrow \tilde{X} \times G$ between G -torsors must be of the form $(\tilde{x}, g) \mapsto (\tilde{x}, cg)$, where $c \in G(A)$. This being so, if β in addition fixes the A -point (\tilde{x}_0, e) , we see that $\beta = \text{id}$. This has the following pleasing consequence (implicit in [Nor76, Proposition 3.11]).

LEMMA 12.2. *Let $G \in \mathbf{FGSch}/A$, and let $\rho: \Pi \rightarrow G$ and $\sigma: \Pi \rightarrow G$ be arrows of group schemes over A . Let $\alpha: \tilde{X} \times^\rho G \rightarrow \tilde{X} \times^\sigma G$ be a morphism of G -torsors sending $\chi_\rho(\tilde{x}_0, e)$ to $\chi_\sigma(\tilde{x}_0, e)$. Then $\rho = \sigma$ and $\alpha = \text{id}$.*

Proof. We consider the commutative diagram

$$\begin{array}{ccc} \tilde{X} \times G & \xrightarrow{\tilde{\alpha}} & \tilde{X} \times G \\ (\text{pr}, \chi_\rho) \downarrow \sim & & \sim \downarrow (\text{pr}, \chi_\sigma) \\ \tilde{X} \times_X (\tilde{X} \times^\rho G) & \xrightarrow{\text{id} \times \alpha} & \tilde{X} \times_X (\tilde{X} \times^\sigma G) \end{array}$$

Since α takes $\chi_\rho(\tilde{x}_0, e)$ to $\chi_\sigma(\tilde{x}_0, e)$, we conclude that $\tilde{\alpha}(\tilde{x}_0, e) = (\tilde{x}_0, e)$ and hence, by the above discussion, $\tilde{\alpha}$ is the identity. Because (pr, χ_ρ) is Π -equivariant (for the actions explained above), we conclude that $\tilde{\alpha} = \text{id}$ is Π -equivariant, and this is only possible when $\rho = \sigma$. Since $\text{id} \times \alpha = \text{id}$, fpqc descent [EGA, IV₂, proposition 2.7.1] ensures that $\alpha = \text{id}$. \square

Let $G \in (\mathbf{FGSch}/A)$ be *quasi-finite*, and consider the category $\mathbf{Tors}_*(G)$ whose *objects* are ordered pairs (Q, q_0) consisting of a G -torsor over X and an A -point q_0 of Q above x_0 , and whose *arrows* are isomorphisms of G -torsors which preserve the A -rational point.

THEOREM 12.3. *We maintain the above notation.*

- (1) Write $|\mathbf{Tors}_*(G)|$ for the set of isomorphism classes in $\mathbf{Tors}_*(G)$ and $\tilde{X}[\rho]$ for the class of the ordered pair $(\tilde{X} \times^\rho G, \chi_\rho(\tilde{x}_0, e))$. Then the map

$$\text{Hom}(\Pi, G) \longrightarrow |\mathbf{Tors}_*(G)|, \quad \rho \longmapsto \tilde{X}[\rho]$$

is *bijjective*.

- (1') The category $\mathbf{Tors}_*(G)$ is *discrete*.

- (2) For each (Q, q_0) in $\mathbf{Tors}_*(G)$, there exists a unique *generalized morphism of torsors* $\chi: \tilde{X} \rightarrow Q$ taking \tilde{x}_0 to q_0 .

For the definition of discrete in part (1'), see [Mac78, Section I.2, p. 11].

Proof. (1) We first establish the surjectivity: the argument is identical to the one in [Nor76, Proposition 3.11]. From Theorem 12.1(1), the functor θ_Q takes values in $\mathfrak{T}_X^{\text{tan}}$. Using the A -point q_0 of Q above x_0 , we see that $(\bullet|_{x_0}) \circ \theta_Q: \text{Rep}_A(G) \rightarrow A\text{-mod}$ is naturally isomorphic to the forgetful functor and, since we have $\bullet|_{x_0}: \mathfrak{T}_X^{\text{tan}} \xrightarrow{\sim} \text{Rep}_A(\Pi)$, we derive a morphism of group schemes $\rho: \Pi \rightarrow G$ such that $(\bullet|_{x_0}) \circ \theta_Q \simeq \rho^\#$. Now

$$\theta_Q \simeq \theta_{\tilde{X}} \circ (\bullet|_{x_0}) \circ \theta_Q \simeq \theta_{\tilde{X}} \circ \rho^\#,$$

which shows, just as in [Nor76, Proposition 2.9(c)], that $Q \simeq \tilde{X} \times^\rho G$. The injectivity is a direct consequence of Lemma 12.2.

(1') This is a direct consequence of part (1) and Lemma 12.2.

(2) This is standard, but we run the argument for the convenience of the reader. Let $\tau: \tilde{X} \rightarrow Q$ and $\tau': \tilde{X} \rightarrow Q$ be morphisms as in the statement covering morphisms $\rho: \Pi \rightarrow G$ and $\rho': \Pi \rightarrow G$, respectively. We therefore deduce arrows $\bar{\tau}: \tilde{X} \times^\rho G \rightarrow Q$ and $\bar{\tau}': \tilde{X} \times^{\rho'} G \rightarrow Q$ in $\mathbf{Tors}_*(G)$ such that

$$\tilde{X} \xrightarrow{(\text{id}, e)} \tilde{X} \times G \xrightarrow{\chi_\rho} \tilde{X} \times^\rho G \xrightarrow{\bar{\tau}} Q$$

and

$$\tilde{X} \xrightarrow{(\text{id}, e)} \tilde{X} \times G \xrightarrow{\chi_{\rho'}} \tilde{X} \times^{\rho'} G \xrightarrow{\bar{\tau}'} Q$$

are, respectively, τ and τ' . This produces an arrow $\alpha: \tilde{X} \times^\rho G \rightarrow \tilde{X} \times^{\rho'} G$ in $\mathbf{Tors}_*(G)$. According to Lemma 12.2, we have $\rho = \rho'$ and $\alpha = \text{id}$, so that $\tau = \tau'$. \square

This allows us to compare Π to the fundamental group scheme introduced in [AEG20]. Recall that these authors show the existence of a pro-quasi-finite (see Definition 8.7) flat group scheme Π^* , a Π^* -torsor $X^* \rightarrow X$ and an A -point x_0^* above x_0 enjoying the following universal property: If G is flat and quasi-finite and $Q \rightarrow X$ is a G -torsor with a point q_0 above x_0 , then there exists a *unique* generalized morphism of torsors $X^* \rightarrow Q$ taking x_0^* to q_0 . (See the paragraph after the proof of [AEG20, Theorem 5.2].) From Theorem 12.3, we have the following.

COROLLARY 12.4. *Under the assumptions of Theorem 12.1, there exists a unique generalized isomorphism of torsors $\tilde{X} \rightarrow X^*$ taking \tilde{x}_0 to x_0^* . In particular, $\Pi \simeq \Pi^*$.*

In particular, Theorem 8.10 says that Π^* is in fact pro-finite.

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