



# Basepoint-freeness thresholds and higher syzygies on abelian threefolds

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## ABSTRACT

For a polarized abelian variety, Z. Jiang and G. Pareschi introduced an invariant and showed that the polarization is basepoint-free or projectively normal if the invariant is small. Their result was generalized to higher syzygies by F. Caucci; that is, the polarization satisfies property  $(N_p)$  if the invariant is small.

In this paper, we study a relation between the invariant and degrees of abelian subvarieties with respect to the polarization. For abelian threefolds, we give an upper bound of the invariant using degrees of abelian subvarieties. In particular, we affirmatively answer some questions on abelian varieties asked by the author, V. Lozovanu and F. Caucci in the three-dimensional case.

## 1. Introduction

Throughout this paper, we work over the complex number field  $\mathbb{C}$ . In [JP20], Z. Jiang and G. Pareschi introduce *cohomological rank functions* of  $\mathbb{Q}$ -twisted (complexes of) coherent sheaves on polarized abelian varieties. Using cohomological rank functions, they define the *basepoint-freeness threshold*  $\beta(l)$  for a polarized abelian variety  $(X, l)$  and show the following:

- (1) We have  $0 < \beta(l) \leq 1$ , and  $\beta(l) < 1$  if and only if any line bundle  $L$  representing  $l$  is basepoint-free.
- (2) If  $\beta(l) < \frac{1}{2}$ , any line bundle  $L$  representing  $l$  is projectively normal.

F. Caucci generalized assertion (ii) to higher syzygies, proving that any line bundle  $L$  representing  $l$  satisfies property  $(N_p)$  if  $\beta(l) < 1/(p+2)$  in [Cau20a] (Caucci proves the result on an algebraically closed field of any characteristic.) We refer the readers to [Laz04a, Section 1.8.D], [Eis05] for the definition of  $(N_p)$ . We just note here that the  $(N_p)$  form an increasing sequence of positivity properties. For example,  $L$  satisfies  $(N_0)$  if and only if  $L$  defines a projectively normal embedding, and  $L$  satisfies  $(N_1)$  if and only if  $L$  satisfies  $(N_0)$  and the homogeneous ideal of the embedding is generated by quadrics.

In [Cau20a], Caucci asks the following question.

**Question 1.1** (Question 6.2, [Cau20a, Section 1, p. 950]). Let  $(X, l)$  be a polarized abelian variety. Set  $B_t = tl$  for a positive rational number  $t > 0$ . Assume that  $(B_t^{\dim Z}.Z) > (\dim Z)^{\dim Z}$  for any

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abelian subvariety  $Z \subset X$  with  $\dim Z \geq 1$ . Then does it hold that  $\beta(l) < t$ ?

For  $\dim X \leq 2$ , Question 1.1 is answered affirmatively; this can be checked easily for elliptic curves. For abelian surfaces, it is deduced from [Ito18, Proposition 3.1] and [Cau20a, Proposition 1.4] as explained in [Cau20a]. For  $\dim X = 3$ , the proof of [Loz18, Theorem 1.1] with [Cau20a, Proposition 1.4] give a partial affirmative answer, namely  $\beta(l) < t$  if  $(B_t^3) > 59$ ,  $(B_t^2.S) > 4$ ,  $(B_t.C) > 2$  for any abelian surface  $S \subset X$  and any elliptic curve  $C \subset X$ . In any dimension, it is not difficult to show  $\beta(l) < t$  under the stronger assumption  $(B_t^{\dim Z}.Z) > (\dim X \cdot \dim Z)^{\dim Z}$  by combining results in [LPP11] and [Nak96] (see Proposition 3.3).

In this paper, we give an affirmative answer to Question 1.1 in the three-dimensional case.

**THEOREM 1.2.** *Let  $(X, l)$  be a polarized abelian threefold. Set  $B_t = tl$  for a positive rational number  $t > 0$ . Assume that  $(B_t^3) > 27$ ,  $(B_t^2.S) > 4$ ,  $(B_t.C) > 1$  for any abelian surface  $S \subset X$  and any elliptic curve  $C \subset X$ . Then  $\beta(l) < t$ .*

We obtain the following corollary, which answers [Ito18, Question 4.2] and [Loz18, Conjecture 7.1] affirmatively in the three-dimensional case.

**COROLLARY 1.3.** *Let  $L$  be an ample line bundle on an abelian threefold  $X$  and  $p \geq 0$  be an integer. Assume that  $(L^3) > 27(p+2)^3$ ,  $(L^2.S) > 4(p+2)^2$ ,  $(L.C) > p+2$  for any abelian surface  $S \subset X$  and any elliptic curve  $C \subset X$ . Then  $L$  satisfies  $(N_p)$ , and the Koszul cohomology group  $K_{p,q}(X, L; dL)$  is zero for any  $q, d \geq 1$ .*

The basepoint-freeness on abelian threefolds is studied in, for example, [BLR93, Rub99, Fuk01]. To the best of the author's knowledge, the following criterion did not appear in the literature.

**COROLLARY 1.4.** *Let  $L$  be an ample line bundle on an abelian threefold  $X$  with  $(L^3) \geq 30$  or, equivalently,  $h^0(L) \geq 5$ . Then  $L$  is basepoint-free if and only if  $(L^2.S) \geq 6$  and  $(L.C) \geq 2$  for any abelian surface  $S \subset X$  and any elliptic curve  $C \subset X$ .*

This paper is organized as follows. In Section 2, we prepare some notation and lemmas. We also show two propositions on basepoint-freeness thresholds. In Section 3, we see a relation between basepoint-freeness thresholds and Seshadri constants. In particular, we give a partial answer to Question 1.1 in arbitrary dimensions in Proposition 3.3. In Section 4, we study basepoint-freeness thresholds on abelian surfaces. In Section 5, we give an easy but important observation. In Section 6, we show Theorem 6.4, which reduces Question 1.1 to another question. In Section 7, we show two lemmas on curves and hypersurfaces in abelian varieties. In Section 8, we give an affirmative answer to the question in Section 6 for  $\dim X \leq 3$  and prove Theorem 1.2.

In the proof of Theorem 1.2, we do not need results in Sections 3 through 5 other than Proposition 5.3. Hence the readers need read only Section 2, Proposition 5.3 and Sections 6 through 8 for the proof of Theorem 1.2.

## 2. Preliminaries

### 2.1 Basepoint-freeness threshold

Let  $X$  be an abelian variety of dimension  $g$ . An *abelian subvariety*  $Y \subset X$  is a subvariety that is closed under the group law of  $X$ . In particular,  $o \in Y$  and  $Y$  can be  $\{o\}$  or  $X$ . A *polarization*  $l$  of  $X$  is an ample class in  $\text{Pic}(X)/\text{Pic}^0(X)$ . For  $b \in \mathbb{Z}$ , the multiplication-by- $b$  isogeny is denoted by

$$\mu_b: X \rightarrow X, \quad p \mapsto bp.$$

It is known that  $\mu_b^* l = b^2 l$  and  $\deg \mu_b = b^{2g}$ . Frequently,  $L$  denotes an ample line bundle representing  $l$ .

In [JP20], Jiang and Pareschi introduce the *cohomological rank function* as follows.

DEFINITION 2.1 ([JP20, Definition 2.1]). Let  $(X, l)$  be a  $g$ -dimensional polarized abelian variety and  $\mathcal{F} \in D^b(X)$  be a bounded complex of coherent sheaves on  $X$ .

- (i) For  $i \in \mathbb{Z}$ , we denote by  $h_{\text{gen}}^i(X, \mathcal{F})$  the dimension of the hypercohomology  $H^i(X, \mathcal{F} \otimes P_\alpha)$  for general  $\alpha \in \hat{X} = \text{Pic}^0(X)$ , where  $P_\alpha$  is the numerically trivial line bundle on  $X$  corresponding to  $\alpha$ .
- (ii) The *cohomological rank function*  $h_{\mathcal{F}, l}^i: \mathbb{Q} \rightarrow \mathbb{Q}_{\geq 0}$  is defined as

$$h_{\mathcal{F}, l}^i(x) = h_{\mathcal{F}}^i(xl) := b^{-2g} h_{\text{gen}}^i(X, \mu_b^* \mathcal{F} \otimes L^{ab})$$

for  $x = a/b \in \mathbb{Q}$ ,  $b > 0$  and an ample line bundle  $L$  representing  $l$ .

DEFINITION 2.2 ([JP20, Section 8]). Let  $(X, l)$  be a polarized abelian variety. The *basepoint-freeness threshold*  $\beta(l)$ , or *Bpf threshold* for short, is defined by

$$\beta(X, l) = \beta(l) := \inf \{x \in \mathbb{Q} \mid h_{\mathcal{I}_p}^1(xl) = 0\}$$

for a closed point  $p \in X$ .

We note that  $h_{\mathcal{F}, l}^i(x)$  does not depend on the choices of  $L$  or the representation  $x = a/b$ , and  $\beta(l)$  does not depend on the choice of  $p \in X$ .

THEOREM 2.3 ([JP20, Theorem D, Corollary E], [Cau20a, Theorem 1.1]). Let  $(X, l)$  be a polarized abelian variety and  $p \geq 0$  be an integer.

- (i) We have  $0 < \beta(l) \leq 1$ , and  $\beta(l) < 1$  if and only if any line bundle  $L$  representing  $l$  is basepoint-free.
- (ii) If  $\beta(l) < 1/(p+2)$ , any line bundle  $L$  representing  $l$  satisfies property  $(N_p)$ .

For a coherent sheaf  $\mathcal{F}$  on  $X$  and  $x \in \mathbb{Q}$ , a  $\mathbb{Q}$ -twisted coherent sheaf  $\mathcal{F}\langle xl \rangle$  is the equivalence class of the pair  $(\mathcal{F}, xl)$ , where the equivalence is defined by

$$(\mathcal{F} \otimes L^m, xl) \sim (\mathcal{F}, (x+m)l)$$

for any line bundle  $L$  representing  $l$  and any  $m \in \mathbb{Z}$ . In [JP20], the usual notions of generic vanishing are extended to the  $\mathbb{Q}$ -twisted setting. A coherent sheaf  $\mathcal{F}$  on  $X$  is said to be  $IT(0)$  if  $h^i(X, \mathcal{F} \otimes P_\alpha) = 0$  for any  $i > 0$  and any  $\alpha \in \hat{X}$ . A  $\mathbb{Q}$ -twisted coherent sheaf  $\mathcal{F}\langle xl \rangle$  for  $x = a/b$  is said to be  $IT(0)$  if  $\mu_b^* \mathcal{F} \otimes L^{ab}$  is. We do not state the usual definition of the notion  $GV$  here, but we can define  $\mathcal{F}\langle xl \rangle$  to be  $GV$  if  $\mathcal{F}\langle (x+x')l \rangle$  is  $IT(0)$  for any rational number  $x' > 0$ , by [JP20, Theorem 5.2].

For a line bundle  $N$  on  $X$ ,  $N\langle xl \rangle$  is  $IT(0)$  if and only if  $N\langle xl \rangle$  is ample, that is,  $N + xl$  is ample. For the non-twisted case, this is shown by [PP08, Example 3.10(1)]. The  $\mathbb{Q}$ -twisted case follows from the non-twisted case. Hence  $N\langle xl \rangle$  is  $GV$  if and only if  $N\langle xl \rangle$  is nef, that is,  $N + xl$  is nef.

These notions are closely related to  $\beta(l)$  as follows.

LEMMA 2.4 ([JP20, Section 8], [Cau20a, Lemma 3.3]). Let  $(X, l)$  be a polarized abelian variety and  $x \in \mathbb{Q}$ . Then  $\beta(l) < x$  if and only if  $\mathcal{I}_p\langle xl \rangle$  is  $IT(0)$  for some (and hence for any) point  $p \in X$ .

*Remark 2.5.* If  $X$  is an elliptic curve, it is easy to check that  $h_{\mathcal{I}_p}^1(xl) = 0$  if and only if  $x \deg(l) - 1 \geq 0$ , by the Riemann–Roch theorem. Hence we have  $\beta(l) = \deg(l)^{-1}$ . This follows from Lemma 2.4 as well.

For a basepoint-free ample line bundle  $N$  and a line bundle  $N'$  on a projective variety  $X$ , the Koszul cohomology group  $K_{p,q}(X, N'; N)$  is defined to be the cohomology of the Koszul-type complex

$$\bigwedge^{p+1} H^0(N) \otimes H^0(N' \otimes N^{q-1}) \rightarrow \bigwedge^p H^0(N) \otimes H^0(N' \otimes N^q) \rightarrow \bigwedge^{p-1} H^0(N) \otimes H^0(N' \otimes N^{q+1})$$

for  $p, q \geq 0$ . For  $p \geq 0$ , it is known that  $N$  satisfies  $(N_p)$  if and only if  $K_{i,q}(X, \mathcal{O}_X; N) = 0$  for any  $0 \leq i \leq p$  and  $q \geq 2$ .

Let  $M_N$  be the kernel of the surjective map  $H^0(X, N) \otimes \mathcal{O}_X \rightarrow N$ . By [EL93b, Section 1], the vanishing  $K_{p,q}(X, N'; N) = 0$  follows from  $h^1(X, \bigwedge^{p+1} M_N \otimes N^{q-1} \otimes N') = 0$ . The following proposition is a variant of [Cau20a, Proposition 3.5].

**PROPOSITION 2.6.** *Let  $X$  be an abelian variety and  $N, N'$  be line bundles on  $X$  such that  $N$  is ample and basepoint-free. Let  $n \in \text{Pic } X / \text{Pic}^0(X)$  be the class of  $N$  and  $p, q$  be non-negative integers. Assume that there exists a rational number  $t > 0$  such that  $(q-1-t)N + N'$  is nef and  $\beta(n) < t/(p+1+t)$ . Then  $h^1(X, \bigwedge^{p+1} M_N \otimes N^{q-1} \otimes N') = 0$  and hence  $K_{p,q}(X, N'; N) = 0$ .*

*In particular, if  $l$  is a polarization of  $X$  and  $\beta(l) < 1/(p+2)$  for  $p \geq 0$ , then  $L$  is projectively normal and  $K_{p,q}(X, L; dL) = 0$  for any  $q, d \geq 1$  and any  $L$  representing  $l$ .*

*Proof.* Set  $s(n) = \inf \{x \in \mathbb{Q} \mid h_{M_N}^1(xl) = 0\}$  as in [JP20]. By [JP20, Theorem D] and the inequality  $\beta(n) < t/(p+1+t)$ , we have

$$s(n) = \frac{\beta(n)}{1 - \beta(n)} < \frac{t}{p+1},$$

and hence  $M_N \langle (t/(p+1))n \rangle$  is IT(0) by [JP20, Section 8], [Cau20a, Lemma 3.3]. As a  $\mathbb{Q}$ -twisted sheaf,  $M_N^{\otimes p+1} \otimes N^{q-1} \otimes N'$  is written as

$$\left( M_N \left\langle \frac{t}{p+1} n \right\rangle \right)^{\otimes p+1} \otimes (N^{q-1} \otimes N') \langle -tn \rangle.$$

Since  $(q-1-t)N + N'$  is nef by assumption,  $(N^{q-1} \otimes N') \langle -tn \rangle$  is GV. Thus  $M_N^{\otimes p+1} \otimes N^{q-1} \otimes N'$  is IT(0) by [Cau20a, Proposition 3.4], and hence  $h^1(X, M_N^{\otimes p+1} \otimes N^{q-1} \otimes N') = 0$ . Since  $\bigwedge^{p+1} M_N \otimes N^{q-1} \otimes N'$  is a direct summand of  $M_N^{\otimes p+1} \otimes N^{q-1} \otimes N'$ , we therefore have  $h^1(X, \bigwedge^{p+1} M_N \otimes N^{q-1} \otimes N') = 0$  and hence  $K_{p,q}(X, N'; N) = 0$ .

For the last statement, the projective normality of  $L$  follows from  $\beta(l) < 1/(p+2) \leq 1/2$  and Theorem 2.3(ii). Let  $N = dL$ ,  $N' = L$  and  $t = d^{-1}$  for  $d \geq 1$ . Then

$$\beta(n) = \beta(dl) = d^{-1}\beta(l) < d^{-1} \frac{1}{p+2} \leq \frac{d^{-1}}{p+1+d^{-1}}$$

for any  $d \geq 1$ . Since  $(q-1-d^{-1})N + N' = (q-1)N$  is nef for  $q \geq 1$  and  $\beta(n) < d^{-1}/(p+1+d^{-1})$ ,  $K_{p,q}(X, L; dL) = 0$  follows.  $\square$

*Remark 2.7.* By Proposition 2.6, we can generalize [AL16, Theorems 2.4 and 2.9] as follows:

Let  $N, N'$  be line bundles on  $X$  such that  $N$  is ample and basepoint-free as in Proposition 2.6. Assume that  $g = \dim X \geq 2$  and that  $N - N'$  is ample, and set  $b = \min\{s \in \mathbb{R} \mid sN - N' \text{ is nef}\}$ ; we have  $b < 1$ . Then  $K_{p,1}(X, N'; N) = 0$  for

$$r - b - \frac{g - 1 - b}{\beta(n)} < p \leq r - g, \quad (1)$$

where  $r = h^0(N) - 1$ .

In fact, by Serre duality and the Kodaira vanishing theorem, we have

$$\begin{aligned} h^1(\wedge^{p+1} M_N \otimes N') &= h^{g-1}(\wedge^{r-p-1} M_N \otimes N \otimes N'^{-1}) \\ &= h^1(\wedge^{r-p-g+1} M_N \otimes N^{g-1} \otimes N'^{-1}) \end{aligned} \quad (2)$$

as in the proof of [AL16, Lemma 2.1]. Here we use  $p \leq r - g$  and the ampleness of  $N - N'$ .

Set  $p' = r - p - g \geq 0$ . By (2), the vanishing  $K_{p,1}(X, N'; N) = 0$  follows from the vanishing  $h^1(\wedge^{p'+1} M_N \otimes N^{g-1} \otimes N'^{-1}) = 0$ . Hence it suffices to find a  $t > 0$  such that  $(g - 1 - t)N - N'$  is nef and  $\beta(n) < t/(p' + 1 + t)$ , by Proposition 2.6.

Note that  $g - 1 - b$  is positive because  $g \geq 2$  and  $b < 1$ . We show that a rational  $t > 0$  with  $0 < (g - 1 - b) - t \ll 1$  satisfies these conditions.

By the definition of  $b$  and the inequality  $g - 1 - t > b$ , the  $\mathbb{Q}$ -line bundle  $(g - 1 - t)N - N'$  is ample. On the other hand, the first inequality of (1) is equivalent to

$$\beta(n) < \frac{g - 1 - b}{r - b - p} = \frac{g - 1 - b}{p' + 1 + (g - 1 - b)}.$$

Since  $t$  is sufficiently close to  $g - 1 - b$ , we have  $\beta(n) < t/(p' + 1 + t)$ . Hence we obtain the vanishing  $K_{p,1}(X, N'; N) = 0$ .

The following lemma is well known to experts, but we give a proof for completeness.

LEMMA 2.8. *Let  $(X, l)$  be a polarized abelian variety and  $\mathcal{F}$  be a coherent sheaf on  $X$ . Let  $x \in \mathbb{Q}$ .*

- (i) *Let  $0 \rightarrow \mathcal{F}_1 \rightarrow \mathcal{F} \rightarrow \mathcal{F}_2 \rightarrow 0$  be an exact sequence of coherent sheaves on  $X$ . If  $\mathcal{F}_1\langle xl \rangle$  and  $\mathcal{F}_2\langle xl \rangle$  are IT(0), so is  $\mathcal{F}\langle xl \rangle$ .*
- (ii) *Let  $(X', l')$  be a polarized abelian variety and  $\mathcal{F}'$  be a coherent sheaf on  $X'$ . If  $\mathcal{F}\langle xl \rangle$  and  $\mathcal{F}'\langle x l' \rangle$  are IT(0), so is  $\mathcal{F} \boxtimes \mathcal{F}'\langle x(l \boxtimes l') \rangle$ , where  $\mathcal{F} \boxtimes \mathcal{F}'$  (respectively,  $l \boxtimes l'$ ) is the tensor product of pullbacks of  $\mathcal{F}$  and  $\mathcal{F}'$  (respectively,  $l$  and  $l'$ ) on  $X \times X'$ .*
- (iii) *Let  $f: X' \rightarrow X$  be an isogeny. Then  $\mathcal{F}\langle xl \rangle$  is IT(0) if and only if  $f^*\mathcal{F}\langle x f^* l \rangle$  is.*
- (iv) *Let  $X' \subset X$  be an abelian subvariety and  $\mathcal{F}'$  be a coherent sheaf on  $X'$ . Then  $\mathcal{F}'\langle x l|_{X'} \rangle$  is IT(0) on  $X'$  if and only if  $\iota_*\mathcal{F}'\langle xl \rangle$  is on  $X$ , where  $\iota: X' \rightarrow X$  is the inclusion morphism.*

*Proof.* Let  $x = a/b$  and  $\mu'_b$  be the multiplication-by- $b$  isogeny on  $X'$ .

(i) For  $\alpha \in \widehat{X}$ , we have  $h^i(\mu_b^* \mathcal{F}_j \otimes L^{ab} \otimes P_\alpha) = 0$  for  $i > 0$  if  $\mathcal{F}_j\langle xl \rangle$  is IT(0) for  $j = 1, 2$ . Tensoring  $L^{ab} \otimes P_\alpha$  with the pullback of the exact sequence in item (i) by  $\mu_b$ , we see that  $h^i(\mu_b^* \mathcal{F} \otimes L^{ab} \otimes P_\alpha) = 0$  for  $i > 0$ . Hence  $\mathcal{F}\langle xl \rangle$  is IT(0).

(ii) Let  $(\alpha, \alpha') \in \widehat{X} \times \widehat{X'} = \widehat{X \times X'}$ . By the Künneth formula,

$$H^i(X \times X', (\mu_b \times \mu'_b)^*(\mathcal{F} \boxtimes \mathcal{F}') \otimes (L \boxtimes L')^{ab} \otimes (P_\alpha \boxtimes P_{\alpha'}))$$

is the direct sum of

$$H^k(X, \mu_b^* \mathcal{F} \otimes L^{ab} \otimes P_\alpha) \otimes H^{i-k}(X', \mu'^*_b \mathcal{F}' \otimes L'^{ab} \otimes P_{\alpha'}) \quad (3)$$

for  $0 \leq k \leq i$ . If  $i > 0$ , then (3) is zero for any  $0 \leq k \leq i$  since  $\mathcal{F}\langle xl \rangle$  and  $\mathcal{F}'\langle xl' \rangle$  are IT(0). Hence  $\mathcal{F} \boxtimes \mathcal{F}'\langle x(l \boxtimes l') \rangle$  is IT(0) as well.

(iii) For  $\alpha \in \widehat{X}$ , it holds that

$$\begin{aligned} H^i(X', \mu_b'^* f^* \mathcal{F}' \otimes (f^* L)^{ab} \otimes f^* P_\alpha) &= H^i(X', f^* \mu_b^* \mathcal{F} \otimes f^* L^{ab} \otimes f^* P_\alpha) \\ &= \bigoplus_{\gamma \in \hat{f}^{-1}(\delta')} H^i(X, \mu_b^* \mathcal{F} \otimes L^{ab} \otimes P_\alpha \otimes P_\gamma), \end{aligned}$$

where  $\delta' \in \widehat{X}'$  is the origin and  $\hat{f}: \widehat{X} \rightarrow \widehat{X}'$  is the dual isogeny. Hence  $h^i(X, \mu_b^* \mathcal{F} \otimes L^{ab} \otimes P_\alpha) = 0$  for any  $\alpha \in \widehat{X}$  if and only if  $h^i(X', \mu_b'^* f^* \mathcal{F}' \otimes (f^* L)^{ab} \otimes f^* P_\alpha) = 0$  for any  $\alpha \in \widehat{X}$ . Since  $f^* P_\alpha = P_{\hat{f}(\alpha)}$  and  $\hat{f}$  is surjective, this is equivalent to the condition that  $h^i(X', \mu_b'^* f^* \mathcal{F}' \otimes (f^* L)^{ab} \otimes P_{\alpha'})$  is zero for any  $\alpha' \in \widehat{X}'$ . Hence  $\mathcal{F}\langle xl \rangle$  is IT(0) if and only if  $f^* \mathcal{F}\langle x f^* l \rangle$  is.

(iv) Let  $\hat{l}: \widehat{X} \rightarrow \widehat{X}'$  be the dual surjection. For the non-twisted case,  $\mathcal{F}'$  on  $X'$  is IT(0) if and only if  $\iota_* \mathcal{F}'$  is on  $X$  since  $\hat{l}$  is surjective and

$$h^i(X, \iota_* \mathcal{F}' \otimes P_\alpha) = h^i(X', \mathcal{F}' \otimes P_\alpha|_{X'}) = h^i(X', \mathcal{F}' \otimes P_{i(\alpha)})$$

for  $\alpha \in \widehat{X}$ . Hence we do not need to distinguish  $\iota_* \mathcal{F}'$  from  $\mathcal{F}'$  when we consider the property IT(0) for non-twisted sheaves. Furthermore,  $\mathcal{F}'$  is IT(0) if and only if  $t_p^* \mathcal{F}'$  is for some  $p \in X$ , where  $t_p: X \rightarrow X$  is the translation by  $p$ .

Let  $g = \dim X$  and  $g' = \dim X'$ . Then  $\mu_b^{-1}(X')$  is the disjoint union of  $b^{2(g-g')}$  copies  $X'_1, \dots, X'_{b^{2(g-g'')}}$  of  $X'$ . For each  $j$ , there exists a  $p_j \in \mu_b^{-1}(o)$  such that the translation by  $p_j$  gives an isomorphism  $t_j: X' \xrightarrow{\sim} X'_j \subset X$ . Then  $\mu_b^* \iota_* \mathcal{F}' \otimes L^{ab}$  is the direct sum of  $(\mu_b^* \iota_* \mathcal{F}' \otimes L^{ab})|_{X'_j}$  for  $1 \leq j \leq b^{2(g-g')}$ . Since  $\mu_b|_{X'_j} \circ t_j: X' \xrightarrow{\sim} X'_j \rightarrow X \subset X$  coincides with  $\mu_b'$  because  $p_j \in \mu_b^{-1}(o)$ , it holds that

$$t_j^*((\mu_b^* \iota_* \mathcal{F}' \otimes L^{ab})|_{X'_j}) = \mu_b'^* \mathcal{F}' \otimes t_j^*(L^{ab}|_{X'_j}).$$

Hence  $(\mu_b^* \iota_* \mathcal{F}' \otimes L^{ab})|_{X'_j}$  is IT(0) if and only if  $\mu_b'^* \mathcal{F}' \otimes t_j^*(L^{ab}|_{X'_j})$  is, which holds if and only if  $\mu_b'^* \mathcal{F}' \otimes L|_{X'}^{ab}$  is. Since  $\mu_b^* \iota_* \mathcal{F}' \otimes L^{ab}$  is the direct sum of  $(\mu_b^* \iota_* \mathcal{F}' \otimes L^{ab})|_{X'_j}$ , the sheaf  $\mu_b^* \iota_* \mathcal{F}' \otimes L^{ab}$  is IT(0) if and only if  $\mu_b'^* \mathcal{F}' \otimes L|_{X'}^{ab}$  is. Hence  $\iota_* \mathcal{F}'\langle xl \rangle$  is IT(0) if and only if  $\mathcal{F}'\langle xl|_{X'} \rangle$  is.  $\square$

For any integer  $m > 0$  and any polarization  $l$ , it holds that  $\beta(ml) = m^{-1}\beta(l)$ . Hence for a rational ample class  $\xi \in N^1(X)_{\mathbb{Q}} := (\text{Pic}(X)/\text{Pic}^0(X)) \otimes_{\mathbb{Z}} \mathbb{Q}$ , we can define  $\beta(\xi) := m^{-1}\beta(m\xi)$  for sufficiently divisible  $m > 0$ . Thus we have a function  $\beta: \text{Amp}(X) \cap N^1(X)_{\mathbb{Q}} \rightarrow \mathbb{R}$ , where  $\text{Amp}(X) \subset N^1(X) := (\text{Pic}(X)/\text{Pic}^0(X)) \otimes_{\mathbb{Z}} \mathbb{R}$  is the ample cone.

We can show that  $\beta$  extends to a continuous function on  $\text{Amp}(X)$ , though we do not use this fact in the rest of the paper.

**PROPOSITION 2.9.** *Let  $X$  be an abelian variety.*

- (i) *Let  $l \in \text{Pic}(X)/\text{Pic}^0(X)$  be a polarization and  $n \in \text{Pic}(X)/\text{Pic}^0(X)$  be a nef (respectively, ample) class. Then  $\beta(l+n) \leq \beta(l)$  (respectively,  $\beta(l+n) < \beta(l)$ ).*
- (ii) *The function  $\beta: \text{Amp}(X) \cap N^1(X)_{\mathbb{Q}} \rightarrow \mathbb{R}$  extends to a continuous function on  $\text{Amp}(X)$ .*

*Proof.* (i) Let  $L$  and  $N$  be line bundles representing  $l$  and  $n$ , respectively.

For the case when  $n$  is nef, it suffices to show that  $x > \beta(l+n)$  for any  $x = a/b > \beta(l)$ . Take an integer  $m \gg 1$  such that  $x = a/b > (ma-1)/mb > \beta(l)$ . By Lemma 2.4, the sheaf



$\mu_{mb}^* \mathcal{I}_p \otimes L^{(ma-1)mb}$  is IT(0). On the other hand,  $L^{mb} \otimes N^{m^2 ab}$  is IT(0) since it is ample. Thus

$$\mu_{mb}^* \mathcal{I}_p \otimes (L \otimes N)^{m^2 ab} = (\mu_{mb}^* \mathcal{I}_p \otimes L^{(ma-1)mb}) \otimes (L^{mb} \otimes N^{m^2 ab})$$

is IT(0) as well by [PP11, Proposition 3.1]. Hence  $\mathcal{I}_p \langle x(l+n) \rangle$  is IT(0) because  $x = ma/mb$ , which is equivalent to  $\beta(l+n) < x$ .

If  $n$  is ample,  $cn - l$  is nef for some integer  $c \gg 1$ . Then it holds that

$$\beta(l+n) = c \cdot \beta(c(l+n)) = c \cdot \beta((c+1)l + (cn-l)) \leq c \cdot \beta((c+1)l) = \frac{c}{c+1} \beta(l) < \beta(l),$$

where the first inequality follows from the nef case.

(ii) It suffices to show that if a sequence  $\{\xi_i\}_{i \in \mathbb{N}}$  in  $\text{Amp}(X) \cap N^1(X)_{\mathbb{Q}}$  converges to some  $\xi \in \text{Amp}(X)$ , then  $\{\beta(\xi_i)\}_{i \in \mathbb{N}}$  also converges to a real number.

For any rational number  $0 < \delta < 1$ , there exists an  $N \geq 0$  such that both  $(1+\delta)\xi_n - \xi$  and  $\xi - (1-\delta)\xi_n$  are ample for any  $n \geq N$ . Hence  $(1+\delta)\xi_n - (1-\delta)\xi_m$  is ample for any  $m, n \geq N$ . By item (i), we have

$$(1+\delta)^{-1} \beta(\xi_n) = \beta((1+\delta)\xi_n) < \beta((1-\delta)\xi_m) = (1-\delta)^{-1} \beta(\xi_m).$$

Taking  $m, n \rightarrow \infty$  gives

$$(1+\delta)^{-1} \overline{\lim}_{n \rightarrow \infty} \beta(\xi_n) \leq (1-\delta)^{-1} \underline{\lim}_{m \rightarrow \infty} \beta(\xi_m).$$

Taking  $\delta \rightarrow 0$ , we obtain

$$\overline{\lim}_{n \rightarrow \infty} \beta(\xi_n) \leq \underline{\lim}_{m \rightarrow \infty} \beta(\xi_m).$$

Hence  $\{\beta(\xi_n)\}$  converges to a real number. □

## 2.2 Singularities and the non-klt locus

We recall some notation in birational geometry. Let  $X$  be a smooth variety and  $D = \sum_i d_i D_i$  be an effective  $\mathbb{Q}$ -divisor on  $X$ . Let  $f: Y \rightarrow X$  be a log resolution of  $(X, D)$ , and write

$$K_Y = f^*(K_X + D) + F$$

with  $F = \sum_j b_j F_j$ . Here we assume that if  $F_j$  is the strict transform of  $D_i$ , we take  $b_j = -d_i$ , and all the other  $F_j$  are exceptional divisors of  $f$ .

The pair  $(X, D)$  is called *log canonical*, or *lc* for short, (respectively, *klt*) at  $x \in X$  if  $b_j \geq -1$  (respectively,  $b_j > -1$ ) for any log resolution  $f$  and any  $j$  with  $x \in f(F_j)$ . The pair  $(X, D)$  is called log canonical (respectively, klt) if  $(X, D)$  is log canonical (respectively, klt) at any  $x \in X$ . A prime divisor  $F_j$  with  $b_j \leq -1$  is called a *non-klt place* of  $(X, D)$ . A subvariety  $Z \subset X$  is called a *non-klt center* of  $(X, D)$  if it is the image of a non-klt place. When  $(X, D)$  is log canonical at the generic point of a non-klt center  $Z$ , the subvariety  $Z$  is also called a *log canonical center*, or *lc center* for short.

The *log canonical threshold* of  $(X, D)$  at  $x \in X$  is

$$\text{lct}_x(X, D) = \text{lct}_x(D) := \max\{s \geq 0 \mid (X, sD) \text{ is log canonical at } x\}.$$

The *multiplier ideal*  $\mathcal{J}(X, D)$  of  $(X, D)$  is defined as

$$\mathcal{J}(X, D) := f_* \mathcal{O}_Y(\lceil F \rceil);$$

it does not depend on the choice of the log resolution  $f$ . Note that  $\text{Supp } \mathcal{O}_X / \mathcal{J}(X, D)$  coincides

with the *non-klt locus*

$$\mathrm{Nklt}(X, D) := \{x \in X \mid (X, D) \text{ is not klt at } x\}.$$

The following result about the existence of minimal lc centers is known (see [EL93a, Hel97, Kaw97]).

**THEOREM 2.10.** *Let  $X$  be a smooth variety and  $D$  be an effective  $\mathbb{Q}$ -divisor on  $X$  such that  $(X, D)$  is log canonical. Then every irreducible component of the intersection of two lc centers of  $(X, D)$  is also a lc center of  $(X, D)$ .*

*In particular, if  $(X, D)$  is log canonical but not klt at  $x \in X$ , there exists a unique minimal lc center  $Z$  of  $(X, D)$  containing  $x$ . Furthermore,  $Z$  is normal at  $x$ .*

The following lemma follows from the standard tie-breaking trick.

**LEMMA 2.11.** *Let  $X$  be a smooth projective variety and  $D$  be an effective  $\mathbb{Q}$ -divisor on  $X$  such that  $(X, D)$  is lc but not klt at a point  $x \in X$ . Let  $Z$  be the minimal lc center of  $(X, D)$  at  $x$ . Assume that  $B$  is an ample  $\mathbb{Q}$ -divisor on  $X$  such that  $B - D$  is ample. Then there exists an effective  $\mathbb{Q}$ -divisor  $F$  on  $X$  such that  $B - F$  is ample and  $\mathrm{Nklt}(X, F) = Z$  in a neighborhood of  $x$ .*

*Proof.* Let  $H \subset X$  be a general effective divisor containing  $Z$ . We may take  $F = (1 - \varepsilon)D + \varepsilon'H$  for  $0 < \varepsilon \ll \varepsilon' \ll 1$ .  $\square$

The following lemma is also well known.

**LEMMA 2.12.** *Let  $f: X \rightarrow T$  be a smooth morphism between smooth varieties and  $D$  be an effective  $\mathbb{Q}$ -divisor on  $X$ . Set  $X_t = f^{-1}(t)$  for  $t \in T$ . If  $t$  is general, it holds that*

$$\mathrm{Nklt}(X_t, D|_{X_t}) = \mathrm{Nklt}(X, D) \cap X_t.$$

*Proof.* By [Laz04b, Theorem 9.5.35], we have  $\mathcal{J}(X_t, D|_{X_t}) = \mathcal{J}(X, D)|_{X_t}$  if  $t$  is general. This lemma follows by considering the cosupports of the ideals.  $\square$

The following lemma is a special case of [Hel97, Proposition 3.2] and the first step of cutting down lc centers in [Hel97, Kaw97].

**LEMMA 2.13.** *Let  $X$  be a smooth projective variety of dimension  $n$ , let  $B$  be an ample  $\mathbb{Q}$ -divisor on  $X$ , and let  $D \equiv B$  be an effective  $\mathbb{Q}$ -divisor on  $X$ . For a point  $x \in X$ , assume that  $m := \mathrm{mult}_x(D) > n$ . Let  $c := \mathrm{lct}_x(D) \leq n/m < 1$ , and let  $Z \subset X$  be the minimal lc center of  $(X, cD)$  at  $x$ . If  $(B^{\dim Z} \cdot Z) \geq n^{\dim Z} \cdot \mathrm{mult}_x(Z)$ , there exists an effective  $\mathbb{Q}$ -divisor  $D_1 \equiv c_1 B$  for some  $c < c_1 < 1$  such that  $(X, D_1)$  is lc but not klt at  $x$  and the minimal lc center  $Z_1$  of  $(X, D_1)$  at  $x$  is properly contained in  $Z$ .*

*Proof.* Under the assumption of this lemma,

$$b_x(X, cD) := \sup\{\mathrm{mult}_x(D') \mid D' \text{ is an effective } \mathbb{Q}\text{-divisor such that } (X, cD + D') \text{ is lc at } x\}$$

is at most  $n - \mathrm{mult}_x(cD) = n - mc$ . Then the statement of this lemma follows from [Hel97, Proposition 3.2] since the condition

$$(B^{\dim Z} \cdot Z) > \left( \frac{b_x(X, cD)}{1 - c} \right)^{\dim Z} \cdot \mathrm{mult}_x(Z)$$

in [Hel97, Proposition 3.2] is satisfied because  $(B^{\dim Z} \cdot Z) \geq n^{\dim Z} \cdot \mathrm{mult}_x(Z)$ ,  $n - mc \geq b_x(X, cD)$  and  $m > n$ . We note that  $B$  is assumed to be a line bundle in [Hel97, Proposition 3.2], but the proof works for  $\mathbb{Q}$ -divisors.  $\square$



*Remark 2.14.* In the proof of Theorem 1.2, we use Lemma 2.13 when  $X$  is an abelian threefold and  $Z$  is a curve. In this case, the condition  $(B^{\dim Z}.Z) \geq n^{\dim Z} \cdot \text{mult}_x(Z)$  is equivalent to  $(B.Z) \geq 3$  since  $Z$  is smooth at  $x$  by Theorem 2.10.

### 3. Bpf thresholds and Seshadri constants

For a polarized abelian variety  $(X, l)$ , the *Seshadri constant* of  $(X, l)$  is defined by

$$\varepsilon(X, l) = \sup \{ t \geq 0 \mid \pi^*l - tE \text{ is nef} \},$$

where  $\pi: \text{Bl}_o X \rightarrow X$  is the blow-up of  $X$  at the origin  $o$  and  $E$  is the exceptional divisor. Equivalently, we may define

$$\varepsilon(X, l) = \inf_V^{\dim V} \sqrt{\frac{(l^{\dim V}.V)}{\text{mult}_o(V)}}, \quad (4)$$

where  $\text{mult}_o(V)$  is the multiplicity of  $V$  at  $o$  and we take the infimum over all subvarieties  $V \subset X$  containing  $o$  with  $\dim V \geq 1$ . We refer the readers to [Laz04a, Section 5] for details.

By definition,  $\varepsilon(X, l)^{-1} < x$  if and only if  $x\pi^*l - E$  is ample. On the other hand,  $\beta(l) < x$  if and only if  $\mathcal{I}_o\langle xl \rangle$  is IT(0). We note that  $x\pi^*l - E$  (respectively,  $\mathcal{I}_o\langle xl \rangle$ ) can be considered as a formal tensor product of  $\mathcal{O}(-E) = \pi^{-1}\mathcal{I}_o$  with  $x\pi^*l$  (respectively,  $\mathcal{I}_o$  with  $xl$ ). Furthermore, recall that a line bundle  $N$  on an abelian variety is IT(0) if and only if  $N$  is ample. Thus we might think that the definition of  $\beta(l)$  is similar to that of  $\varepsilon(X, l)^{-1}$ . In fact, the following direct relation between  $\beta(l)$  and  $\varepsilon(X, l)^{-1}$  is known.

**PROPOSITION 3.1** ([LPP11, Cau20a, Cau20b]). *Let  $(X, l)$  be a polarized abelian variety of dimension  $g$ . Then it holds that  $\varepsilon(X, l)^{-1} \leq \beta(l) \leq g \cdot \varepsilon(X, l)^{-1}$ .*

*Proof.* The first inequality is shown in [Cau20b, Proposition E]. For the second inequality, we use a threshold

$$r(l) = \inf \{ t \in \mathbb{Q} \mid \text{there exists an effective } \mathbb{Q}\text{-divisor } F \text{ on } X \text{ such that} \quad (5)$$

$$tl - F \text{ is ample and } \mathcal{J}(X, F) = \mathcal{I}_o \}$$

introduced in [Cau20a]. By the Nadel vanishing theorem, Caucci shows that  $\beta(l) \leq r(l)$  [Cau20a, Proposition 1.4]. On the other hand, [LPP11, Lemma 1.2] essentially says that  $r(l) \leq g \cdot \varepsilon(X, l)^{-1}$ . Hence we have the second inequality.  $\square$

For  $\varepsilon(X, l)^{-1}$ , the following upper and lower bounds are obtained from the degrees of abelian subvarieties.

**LEMMA 3.2.** *Let  $(X, l)$  be a polarized abelian variety. Then*

$$\max_Z \frac{1}{\dim Z \sqrt{(l^{\dim Z}.Z)}} \leq \frac{1}{\varepsilon(X, l)} \leq \max_Z \frac{\dim Z}{\dim Z \sqrt{(l^{\dim Z}.Z)}},$$

where we take the maxima over all abelian subvarieties  $Z \subset X$  with  $\dim Z \geq 1$ .

*Proof.* The lower bound follows from (4) since  $\text{mult}_o(Z) = 1$  for any abelian subvariety  $Z$ . The upper bound follows from [Nak96, Lemma 3.3] or [Oht22, Theorem 1.2]. We note that  $\{ \dim Z \sqrt{(l^{\dim Z}.Z)^{-1}} \}_Z$  and  $\{ \dim Z \cdot \dim Z \sqrt{(l^{\dim Z}.Z)^{-1}} \}_Z$  have maxima since the  $(l^{\dim Z}.Z)$  are integers.  $\square$

We note that the affirmative answer to Question 1.1 is equivalent to

$$\beta(l) \leq \max_Z \frac{\dim Z}{\dim Z \sqrt[\dim Z]{(l^{\dim Z} \cdot Z)}},$$

where we take the maximum over all abelian subvarieties  $Z \subset X$  with  $\dim Z \geq 1$ . This upper bound is the same as that in Lemma 3.2. By Proposition 3.1 and Lemma 3.2, we have a partial answer to Question 1.1.

PROPOSITION 3.3. *Let  $(X, l)$  be a polarized abelian variety of dimension  $g$ . Then*

$$\beta(l) \leq \max_Z \frac{g \cdot \dim Z}{\dim Z \sqrt[\dim Z]{(l^{\dim Z} \cdot Z)}},$$

where we take the maximum over all abelian subvarieties  $Z \subset X$  with  $\dim Z \geq 1$ .

Equivalently,  $\beta(l) < t$  holds for  $t > 0$  if  $(B_t^{\dim Z} \cdot Z) > (g \cdot \dim Z)^{\dim Z}$  for any abelian subvariety  $Z \subset X$  with  $\dim Z \geq 1$ , where  $B_t := tl$ .

*Proof.* This is a direct consequence of Proposition 3.1 and Lemma 3.2.  $\square$

For lower bounds of  $\beta(l)$ , Caucci [Cau20b, Proposition 1.6.10] obtains the inequality  $\beta(l) \geq \sqrt[\dim Z]{(lg)^{-1}}$  from  $\beta(l) \geq \varepsilon(X, l)^{-1}$ . Similarly, we have  $\beta(l) \geq \max_Z \sqrt[\dim Z]{(l^{\dim Z} \cdot Z)^{-1}}$  by the inequality  $\beta(l) \geq \varepsilon(X, l)^{-1}$  and Lemma 3.2. The following lemma improves these lower bounds by the factor  $\sqrt[\dim Z]{(\dim Z)!}$  for each  $Z$ . Since  $\beta(l) < x$  if and only if  $\beta(xl) < 1$ , this lemma could be considered as the statement that if a  $\mathbb{Q}$ -line bundle  $xL$  is “basepoint-free,” then “ $h^0(xL) = x^g \chi(l)$ ” is greater than 1, and  $xL|_Z$  is “basepoint-free” for any abelian subvariety  $Z \subset X$ .

LEMMA 3.4. *Let  $(X, l)$  be a polarized abelian variety of dimension  $g$ . Then it holds that*

- (i)  $\beta(l) \geq \sqrt[\dim Z]{\chi(l)^{-1}} = \sqrt[\dim Z]{g!} \sqrt[\dim Z]{(lg)^{-1}},$
- (ii)  $\beta(l) \geq \beta(l|_Z)$  for any abelian subvariety  $Z \subset X$  with  $\dim Z \geq 1$ .

As a consequence, it holds that

$$\beta(l) \geq \max_Z \frac{1}{\dim Z \sqrt[\dim Z]{\chi(l|_Z)}} = \max_Z \sqrt[\dim Z]{\frac{(\dim Z)!}{(l^{\dim Z} \cdot Z)}}, \quad (6)$$

where we take the maxima over all abelian subvarieties  $Z \subset X$  with  $\dim Z \geq 1$ .

*Proof.* Equation (6) follows from assertions (i) and (ii) since we have  $\beta(l|_Z) \geq \sqrt[\dim Z]{\chi(l|_Z)^{-1}}$  by applying assertion (i) to  $(Z, l|_Z)$ . Hence it suffices to show assertions (i) and (ii).

- (i) Take a rational number  $x = a/b > \beta(l)$ . It is enough to show that  $x \geq \sqrt[\dim Z]{\chi(l)^{-1}}$ .

By the choice of  $x$ , the sheaf  $\mu_b^* \mathcal{I}_o \otimes L^{ab}$  is IT(0), and hence

$$H^0(X, L^{ab}) \rightarrow H^0(X, L^{ab} \otimes \mathcal{O}_X / \mu_b^* \mathcal{I}_o)$$

is surjective. Thus we have

$$\begin{aligned} (ab)^g \chi(l) &= \chi(L^{ab}) = h^0(X, L^{ab}) \\ &\geq h^0(X, L^{ab} \otimes \mathcal{O}_X / \mu_b^* \mathcal{I}_o) = \deg \mu_b = b^{2g} \end{aligned}$$

and hence  $x^g = (a/b)^g \geq \chi(l)^{-1}$ .

- (ii) Take a rational number  $x = a/b > \beta(l)$ . It suffices to show that  $\mathcal{I}_{o/Z} \langle xl|_Z \rangle$  is IT(0), where  $\mathcal{I}_{o/Z}$  is the ideal sheaf on  $Z$  corresponding to  $o \in Z \subset X$ .

Consider the diagram

$$\begin{array}{ccc} H^0(X, L^{ab} \otimes P_\alpha) & \xrightarrow{r} & H^0(X, L^{ab} \otimes P_\alpha \otimes \mathcal{O}_X / \mu_b^* \mathcal{I}_o) \\ R \downarrow & & \downarrow R' \\ H^0(Z, L|_Z^{ab} \otimes P_\alpha|_Z) & \xrightarrow{r'} & H^0(Z, L|_Z^{ab} \otimes P_\alpha|_Z \otimes \mathcal{O}_Z / \mu_{b,Z}^* \mathcal{I}_{o/Z}), \end{array}$$

where  $\alpha \in \widehat{X}$  and  $\mu_{b,Z}: Z \rightarrow Z$  is the multiplication-by- $b$  isogeny on  $Z$ . Since  $\mu_b^* \mathcal{I}_o \otimes L^{ab}$  is IT(0), the map  $r$  is surjective for any  $\alpha$ . Since  $R'$  is induced from the surjection  $\mathcal{O}_X / \mu_b^* \mathcal{I}_o \rightarrow \mathcal{O}_Z / \mu_{b,Z}^* \mathcal{I}_{o/Z}$  between skyscraper sheaves,  $R'$  is surjective as well. Hence  $r'$  is also surjective, and hence we have  $h^1(Z, \mu_{b,Z}^* \mathcal{I}_{o/Z} \otimes L|_Z^{ab} \otimes P_\alpha|_Z) = 0$ . Since the natural homomorphism  $\widehat{X} \rightarrow \widehat{Z}$  is surjective,  $\mu_{b,Z}^* \mathcal{I}_{o/Z} \otimes L|_Z^{ab}$  is IT(0). Hence so is  $\mathcal{I}_{o/Z} \langle xl|_Z \rangle$ .  $\square$

If Question 1.1 has an affirmative answer, we have

$$\max_Z \frac{\dim Z \sqrt{(\dim Z)!}}{\dim Z \sqrt{(l \dim Z \cdot Z)}} \leq \beta(l) \leq \max_Z \frac{\dim Z}{\dim Z \sqrt{(l \dim Z \cdot Z)}} \quad (7)$$

by Lemma 3.4. These inequalities are similar to those for  $\varepsilon(X, l)^{-1}$  in Lemma 3.2.

By the above similarity, we might expect that  $\beta(l)^{-1}$  and  $\varepsilon(X, l)$  could have similar properties. For example, both  $\beta(l)^{-1}$  and  $\varepsilon(X, l)$  are homogeneous; that is,  $\beta(ml)^{-1} = m\beta(l)^{-1}$  and  $\varepsilon(X, ml) = m\varepsilon(X, l)$  for any positive integer  $m$ . It is known that the Seshadri constant of a direct product  $(X \times X', l \boxtimes l')$  is the minimum of  $\varepsilon(X, l)$  and  $\varepsilon(X', l')$  (see [MR15, Proposition 3.4], for example). Bpf thresholds also have this property.

LEMMA 3.5. *Let  $(X, l)$  and  $(X', l')$  be polarized abelian varieties. Then*

$$\beta(X \times X', l \boxtimes l') = \max\{\beta(X, l), \beta(X', l')\}.$$

*Proof.* Let  $o \in X$  and  $o' \in X'$  be the origins. Since  $(X, l)$ ,  $(X', l')$  are polarized abelian subvarieties of  $(X \times X', l \boxtimes l')$ , we have  $\beta(X \times X', l \boxtimes l') \geq \max\{\beta(X, l), \beta(X', l')\}$  by Lemma 3.4.

Take a rational number  $x > \max\{\beta(X, l), \beta(X', l')\}$ . To prove the converse inequality  $\beta(X \times X', l \boxtimes l') \leq \max\{\beta(X, l), \beta(X', l')\}$ , it suffices to show that  $\beta(X \times X', l \boxtimes l') < x$ , in other words, that  $\mathcal{I}_{(o,o')} \langle x(l \boxtimes l') \rangle$  is IT(0).

Consider the exact sequence

$$0 \rightarrow \mathcal{I}_{\{o\} \times X'} \rightarrow \mathcal{I}_{(o,o')} \rightarrow \mathcal{I}_{(o,o')/\{o\} \times X'} \rightarrow 0$$

on  $X \times X'$ . Both  $\mathcal{I}_o \langle xl \rangle$  and  $\mathcal{O}_{X'} \langle xl' \rangle$  are IT(0) because  $\beta(X, l) < x$ . Hence  $\mathcal{I}_{\{o\} \times X'} \langle x(l \boxtimes l') \rangle$  is IT(0) by Lemma 2.8(ii). Since  $\beta(X', l') < x$ , the  $\mathbb{Q}$ -twisted sheaf  $\mathcal{I}_{o'} \langle xl' \rangle$  on  $X'$  is IT(0). Thus the  $\mathbb{Q}$ -twisted sheaf  $\mathcal{I}_{(o,o')/\{o\} \times X'} \langle x(l \boxtimes l') \rangle$  is IT(0) by Lemma 2.8(iv). Hence  $\mathcal{I}_{(o,o')} \langle x(l \boxtimes l') \rangle$  is IT(0) by Lemma 2.8(i).  $\square$

Another basic property of Seshadri constants is the inequality

$$\varepsilon(X, l + n) \geq \varepsilon(X, l) + \varepsilon(X, n)$$

for two polarizations  $l$  and  $n$  on  $X$ . Hence it would be interesting to ask if

$$\beta(l + n)^{-1} \geq \beta(l)^{-1} + \beta(n)^{-1}$$

holds or not in general.

#### 4. Bpf thresholds on abelian surfaces

In this section, we study Bpf thresholds on abelian surfaces. First, we show that the inequalities (7) hold for abelian surfaces. We note that the upper bounds of  $\beta(l)$  in the following proposition are stated in [Cau20a, Section 1, p. 950].

PROPOSITION 4.1. *Let  $(X, l)$  be a polarized abelian surface, and set*

$$\mu(X, l) = \sup\{t \geq 0 \mid \pi^*l - tE \text{ is big}\},$$

where  $\pi: \text{Bl}_o X \rightarrow X$  is the blow-up of  $X$  at the origin and  $E$  is the exceptional divisor. Let  $r(l) \in \mathbb{R}$  be the threshold defined in (5). Then it holds that

$$\begin{aligned} \max \left\{ \frac{\sqrt{2}}{\sqrt{(l^2)}}, \max_C \frac{1}{(l.C)} \right\} &\leq \beta(l) \leq r(l) \leq \max \left\{ \frac{2}{\mu(X, l)}, \max_C \frac{1}{(l.C)} \right\} \\ &\leq \max \left\{ \frac{2}{\sqrt{(l^2)}}, \max_C \frac{1}{(l.C)} \right\}, \end{aligned} \quad (8)$$

where we take the maxima over all elliptic curves  $C \subset X$ . In particular, (7) holds for abelian surfaces.

*Proof.* The first inequality follows from Lemma 3.4. The second inequality is shown by [Cau20a, Proposition 1.4]. The third inequality follows from [Ito18, Proposition 3.1]. Since  $\mu(X, l) \geq \sqrt{(l^2)}$  by the Riemann–Roch theorem, the last inequality follows.  $\square$

Remark 4.2. If  $\sqrt{(l^2)}$  is not an integer, we have a lower bound of  $\mu(X, l)$  that is slightly better than  $\sqrt{(l^2)}$  as follows.

We may assume that  $l$  is primitive; that is,  $l$  is not written as  $kn$  for some integer  $k \geq 2$  and some polarization  $n$ . Let  $(1, d)$  be the type of  $l$ . In particular,  $\sqrt{2d} = \sqrt{(l^2)}$  is not an integer. Let  $(k_0, m_0)$  be the primitive solution of Pell’s equation  $m^2 - 2dk^2 = 1$ . By the proof of [Bau98, Theorem A.1(a)], there exists an effective divisor  $D \equiv 2k_0l$  such that  $\text{mult}_o(D) = 2m_0$ . Hence we have

$$\mu(X, l) \geq \frac{m_0}{k_0} = \sqrt{2d + \frac{1}{k_0^2}} > \sqrt{2d} = \sqrt{(l^2)}.$$

If the Picard number of  $X$  is 1, we have  $\mu(X, l) = m_0/k_0$  by [Bau99, Theorem 6.1(b)].

If  $2/\sqrt{(l^2)} \leq \max_C 1/(l.C)$ , both the upper and lower bounds of (8) are  $\max_C 1/(l.C)$ , and hence we have  $\beta(l) = \max_C 1/(l.C)$ . We show that this equality holds under the weaker condition  $\sqrt{2}/\sqrt{(l^2)} \leq \max_C 1/(l.C)$ . We use the following lemma.

LEMMA 4.3. *Let  $(X, l)$  be a polarized abelian  $g$ -fold and  $S \subset X$  be an abelian subvariety of codimension 1. Then it holds that*

$$\beta(l|_S) \leq \beta(l) \leq \max \left\{ \beta(l|_S), \frac{g(l^{g-1}.S)}{(l^g)} \right\} = \max \left\{ \beta(l|_S), \frac{\chi(l|_S)}{\chi(l)} \right\}.$$

*Proof.* The first inequality follows from Lemma 3.4(ii).

Take a rational number  $x > \max \{ \beta(l|_S), g(l^{g-1}.S)/(l^g) \}$ . To prove the second inequality, it suffices to show that  $\mathcal{I}_o\langle xl \rangle$  is IT(0).

Since  $x > g(l^{g-1}.S)/(l^g)$ , we have  $((xl)^g) > g \cdot ((xl)^{g-1}.S)$ . Hence  $xl - S$  is big by [Laz04b, Theorem 2.2.15]. Since  $X$  is an abelian variety,  $xl - S$  is ample.

Consider the exact sequence

$$0 \rightarrow \mathcal{I}_S \rightarrow \mathcal{I}_o \rightarrow \mathcal{I}_{o/S} \rightarrow 0$$

on  $X$ . Since  $xl - S$  is ample,  $\mathcal{I}_S\langle xl \rangle = \mathcal{O}_X(-S)\langle xl \rangle$  is  $\text{IT}(0)$ . On the other hand,  $\mathcal{I}_{o/S}\langle xl|_S \rangle$  is  $\text{IT}(0)$  as a  $\mathbb{Q}$ -twisted sheaf on  $S$  because  $x > \beta(l|_S)$ . Hence  $\mathcal{I}_{o/S}\langle xl \rangle$  is  $\text{IT}(0)$  as a  $\mathbb{Q}$ -twisted sheaf on  $X$  by Lemma 2.8(iv). Then  $\mathcal{I}_o\langle xl \rangle$  is  $\text{IT}(0)$  by Lemma 2.8(i).  $\square$

**PROPOSITION 4.4.** *Let  $(X, l)$  be a polarized abelian surface. Assume that there exists an elliptic curve  $C \subset X$  such that  $\sqrt{(l^2)}/\sqrt{2} \geq (l.C)$ . Then it holds that*

$$\beta(l) = \varepsilon(X, l)^{-1} = \frac{1}{(l.C)}.$$

*Proof.* By Remark 2.5, we have  $\beta(l|_C) = \deg(l|_C)^{-1} = (l.C)^{-1}$ . By Lemma 4.3,

$$\frac{1}{(l.C)} = \beta(l|_C) \leq \beta(l) \leq \max \left\{ \beta(l|_C), \frac{2(l.C)}{(l^2)} \right\} = \max \left\{ \frac{1}{(l.C)}, \frac{2(l.C)}{(l^2)} \right\} = \frac{1}{(l.C)},$$

where the last equality follows from the assumption  $\sqrt{(l^2)}/\sqrt{2} \geq (l.C)$ . Hence we have  $\beta(l) = (l.C)^{-1}$ .

For the Seshadri constant, it holds that

$$(l.C) = \frac{1}{\beta(l)} \leq \varepsilon(X, l) \leq (l.C),$$

where the first inequality follows from Proposition 3.1 and the second inequality follows from (4). Thus we have  $\varepsilon(X, l) = \beta(l)^{-1} = (l.C)$ .  $\square$

Proposition 4.4 states that we can compute  $\beta(l)$  when  $X$  contains an elliptic curve whose degree  $(l.C)$  is small compared to the volume  $(l^2)$ , as in the following example. As far as the author knows, this is the first example of computing  $\beta(l)$  other than (the multiples of) non-basepoint-free polarizations.

*Example 4.5.* Let  $X$  be an abelian surface that contains an elliptic curve  $C$ . Let  $l'$  be a polarization on  $X$ , and set  $l = l' + a[C]$  for  $a \geq (l'.C)$ , where  $[C] \in \text{Pic } X / \text{Pic}^0 X$  is the class of  $C$ . Then  $\sqrt{(l^2)} \geq \sqrt{2}(l.C)$ , and hence we have  $\beta(l) = (l.C)^{-1} = (l'.C)^{-1}$  by Proposition 4.4.

*Remark 4.6.* We note that the inequalities in (8) are sharp in the sense that  $\beta(l)$  could coincide with  $\sqrt{2}/\sqrt{(l^2)}$ ,  $2/\sqrt{(l^2)}$  and  $\max_C (l.C)^{-1}$ , respectively, as follows.

If  $l$  is a principal polarization, then  $(l^2) = 2$  and any  $L$  representing  $l$  is not basepoint-free. Hence  $\beta(l) = 1 = \sqrt{2}/\sqrt{(l^2)}$ . If  $(l^2) = 4$ , then  $L$  is not basepoint-free since  $h^0(L) = 2$ . Hence  $\beta(l) = 1 = 2/\sqrt{(l^2)}$ . A case with  $\beta(l) = \max_C (l.C)^{-1}$  is given in Example 4.5.

For the next proposition, we recall three notions on higher-order embeddings. A line bundle  $N$  on a smooth projective variety  $M$  is called *k-jet ample* for an integer  $k \geq 0$  if the restriction map

$$H^0(L) \rightarrow H^0(L \otimes \mathcal{O}_M / \mathcal{I}_{p_1} \mathcal{I}_{p_2} \cdots \mathcal{I}_{p_{k+1}})$$

is surjective for any (not necessarily distinct)  $k+1$  points  $p_1, \dots, p_{k+1} \in M$ . A line bundle  $N$  is called *k-very ample* (respectively, *k-spanned*) if the restriction map  $H^0(M, N) \rightarrow H^0(Z, N|_Z)$  is surjective for any 0-dimensional subscheme  $Z$  (respectively, for any curvilinear 0-dimensional subscheme  $Z$ ) of  $M$  with  $\text{length}(\mathcal{O}_Z) = k+1$ . It is known that *k-jet ampleness* implies *k-very ampleness* (cf. [BS93a, Proposition 2.2]), which of course implies *k-spannedness*.

On abelian varieties, these notions are studied in, for example, [BS97a, BS97b, Ter98, PP04]. In [Cau20b, Theorem D], Caucci shows that an ample line bundle  $L$  on an abelian variety is  $k$ -jet ample if  $\beta(l) < 1/(k+1)$ . On abelian surfaces, these notions on higher-order embeddings are completely determined by  $\beta(l)$  or  $\varepsilon(X, l)$  under a suitable assumption.

**PROPOSITION 4.7.** *Let  $t > 0$  be a rational number and  $L$  be an ample line bundle on an abelian surface  $X$  such that  $(L^2) > 4/t^2$ . Then for  $l = [L]$  the polarization represented by  $L$ , the following are equivalent:*

- (i)  $\beta(l) < t$ .
- (ii)  $\varepsilon(X, l) > t^{-1}$ .
- (iii) We have  $(l.C) > t^{-1}$  for any elliptic curve  $C \subset X$ .

*If  $t = 1/(p+2)$  for an integer  $p \geq -1$  (in particular, we assume that  $(L^2) > 4(p+2)^2$ ), then assertions (i)–(iii) and the following are equivalent:*

- (iv)  $L$  is  $(p+1)$ -spanned,
- (v)  $L$  is  $(p+1)$ -very ample,
- (vi)  $L$  is  $(p+1)$ -jet ample.

*Moreover, if  $p \geq 0$ , then assertions (i)–(vi) are equivalent to*  
(vii)  $L$  satisfies  $(N_p)$ .

*Proof.* The implication (i)  $\Rightarrow$  (ii) follows from  $\beta(l) \geq \varepsilon(X, l)^{-1}$ . The implication (ii)  $\Rightarrow$  (iii) follows from (4). The implication (iii)  $\Rightarrow$  (i) follows from (8) and the assumption  $(l^2) > 4/t^2$ .

Assume that  $t = 1/(p+2)$  for an integer  $p \geq -1$ . The implication (i)  $\Rightarrow$  (vi) follows from [Cau20b, Theorem D]. As we have already seen, (vi)  $\Rightarrow$  (v) and (v)  $\Rightarrow$  (iv) hold in general.

If there exists an elliptic curve  $C \subset X$  such that  $(L.C) \leq t^{-1} = p+2$ , it is easy to find a subscheme  $Z \subset C$  with  $\text{length}(\mathcal{O}_Z) = p+2$  such that  $H^0(C, L|_C) \rightarrow H^0(Z, L|_Z)$  is not surjective (see [BS93b, Proposition 1.2], for example). Then  $H^0(X, L) \rightarrow H^0(Z, L|_Z)$  is not surjective as well, and hence  $L$  is not  $(p+1)$ -spanned. Thus we have (iv)  $\Rightarrow$  (iii), and the equivalence of assertions (i)–(vi) follows.

If  $p \geq 0$ , then the equivalence (iii)  $\Leftrightarrow$  (vii) is nothing but the main result of [KL19, Ito18].  $\square$

We note that the most parts of the equivalence of (ii)–(vii) are already known. For example, (ii)  $\Leftrightarrow$  (iii) follows from [Bau98, Theorem A.1(b)]. The equivalences (iii)  $\Leftrightarrow$  (iv)  $\Leftrightarrow$  (v) follow from [Ter98, Theorem 1.1]. As stated in the proof, the equivalence (iii)  $\Leftrightarrow$  (vii) is proved in [KL19, Ito18].

**Remark 4.8.** We see that the assumption  $(L^2) > 4/t^2$  is sharp at least for  $t = 1/(p+2)$  with  $p = -1$  or  $0$ , as follows.

Let  $N$  be an ample symmetric line bundle on a simple abelian surface  $X$  such that  $(N^2) = 4$ . Since  $h^0(N) = 2$ , the line bundle  $N$  is not basepoint-free, and hence  $\beta(n) = 1$ .

For  $p = -1$  and  $L = N$ , it holds that  $(L^2) = 4 = 4/t^2$  since  $t = 1/(p+2) = 1$ . In this case, assertions (i) and (iv)–(vi) do not hold. On the other hand, assertion (iii) holds since  $X$  is simple.

For  $p = 0$  and  $L = 2N$ , it holds that  $(L^2) = 16 = 4/t^2$  since  $t = 1/(p+2) = 1/2$ . In this case,  $\beta(l) = \beta(n)/2 = 1/2$ . Furthermore,  $L$  is not projectively normal, by [Ohb88], [Iye03, Remark 3.6, 2)]. Thus assertions (i), (vii) do not hold, but assertion (iii) holds since  $X$  is simple.



## 5. An observation

Higher syzygies on abelian surfaces and threefolds are studied in [KL19, Ito18, Loz18]. In these papers, the starting point is the following theorem in [LPP11].

**THEOREM 5.1** ([LPP11]). *Let  $p$  be a non-negative integer,  $X$  be an abelian variety and  $L$  be an ample line bundle on  $X$ . Assume that there exists an effective  $\mathbb{Q}$ -divisor  $F$  on  $X$  such that*

- (i)  $(p+2)^{-1}L - F$  is ample,
- (ii) the multiplier ideal  $\mathcal{J}(X, F)$  coincides with the maximal ideal  $\mathcal{I}_o$  of the origin  $o \in X$ .

Then  $L$  satisfies property  $(N_p)$ .

Note that we may replace item (ii) with  $\text{Nklt}(X, F) = \{o\}$  as in [Ito18, Remark 2.1]. Hence Theorem 5.1 closely resembles the following well-known fact, which is used in the study of Fujita's basepoint-freeness conjecture in, for example, [EL93a, Kol93, Hel97, Kaw97, YZ20].

**PROPOSITION 5.2.** *Let  $L$  be an ample line bundle on a smooth projective variety  $X$  and  $x \in X$  be a point. If there exists an effective  $\mathbb{Q}$ -divisor  $F$  on  $X$  such that  $L - F$  is ample and  $x$  is an isolated point of  $\text{Nklt}(X, F)$ , then  $K_X + L$  is basepoint-free at  $x$ .*

A difference between Theorem 5.1 and Proposition 5.2 is that the former assumes the equality  $\text{Nklt}(X, F) = \{o\}$  on *whole*  $X$  but the latter assumes that  $\text{Nklt}(X, F) = \{x\}$  *locally*. As stated in [Loz18, Section 7], this difference makes the construction of  $F$  in Theorem 5.1 harder even in the case  $\dim X = 3$ . To construct a divisor  $F$  in Theorem 5.1 on abelian threefolds, Lozovanu [Loz18] gives a detailed analysis on the restricted volume  $\text{vol}_{X'|E}(\pi^*L - tE)$  using (infinitesimal) Newton–Okounkov bodies and Seshadri constants, where  $\pi: X' \rightarrow X$  is the blow-up at the origin and  $E$  is the exceptional divisor.

An easy but important observation is the following proposition, whose proof is essentially the same as that of Proposition 5.2 (see [Laz04b, Section 10.4.C], for example).

**PROPOSITION 5.3.** *Let  $(X, l)$  be a polarized abelian variety and  $t > 0$  be a positive rational number. Let  $S \subset X$  be a finite set, and assume that there exists an effective  $\mathbb{Q}$ -divisor  $F$  on  $X$  such that*

- (i)  $tl - F$  is ample,
- (ii) each point in  $S$  is an isolated point of the non-klt locus  $\text{Nklt}(X, F)$ .

*If an ideal sheaf  $\mathcal{I}$  on  $X$  satisfies  $\mathcal{J}(X, F) \subset \mathcal{I}$  and  $\text{Supp } \mathcal{O}_X/\mathcal{I} \subset S$ , then  $\mathcal{I}\langle tl \rangle$  is  $IT(0)$ . In particular,  $\mathcal{I}_S\langle tl \rangle$  is  $IT(0)$ , where  $\mathcal{I}_S$  is the ideal sheaf of  $S$  with the reduced scheme structure.*

*Proof.* Let  $t = a/b$ . Since  $\mu_b^*\mathcal{J}(X, F) = \mathcal{J}(X, \mu_b^*F)$  and  $abL - \mu_b^*F \equiv abL - b^2F \equiv b^2(tl - F)$  is ample,  $h^1(L^{ab} \otimes \mu_b^*\mathcal{J}(X, F) \otimes P_\alpha) = 0$  for any  $\alpha \in \widehat{X}$  by the Nadel vanishing theorem. Hence the natural map

$$H^0(L^{ab} \otimes P_\alpha) \rightarrow H^0(L^{ab} \otimes P_\alpha \otimes \mathcal{O}_X/\mu_b^*\mathcal{J}(X, F))$$

is surjective.

By condition (ii), there exist ideal sheaves  $\mathcal{J}, \mathcal{J}'$  on  $X$  such that  $\mathcal{J}(X, F) = \mathcal{J} \cap \mathcal{J}'$ ,  $\text{Supp } \mathcal{O}_X/\mathcal{J} = S$  and  $S \cap \text{Supp } \mathcal{O}_X/\mathcal{J}' = \emptyset$ . Then we have

$$\mathcal{O}_X/\mu_b^*\mathcal{J}(X, F) = \mathcal{O}_X/\mu_b^*(\mathcal{J} \cap \mathcal{J}') = \mathcal{O}_X/\mu_b^*\mathcal{J} \oplus \mathcal{O}_X/\mu_b^*\mathcal{J}'$$

since  $\text{Supp } \mathcal{O}_X/\mu_b^* \mathcal{J} \cap \text{Supp } \mathcal{O}_X/\mu_b^* \mathcal{J}' = \emptyset$ . Thus

$$H^0(L^{ab} \otimes P_\alpha \otimes \mathcal{O}_X/\mu_b^* \mathcal{J}(X, F)) = H^0(L^{ab} \otimes P_\alpha \otimes \mathcal{O}_X/\mu_b^* \mathcal{J}) \oplus H^0(L^{ab} \otimes P_\alpha \otimes \mathcal{O}_X/\mu_b^* \mathcal{J}'),$$

and hence the natural map

$$H^0(L^{ab} \otimes P_\alpha) \rightarrow H^0(L^{ab} \otimes P_\alpha \otimes \mathcal{O}_X/\mu_b^* \mathcal{J})$$

is surjective as well.

The assumption on  $\mathcal{I}$  is equivalent to  $\mathcal{J} \subset \mathcal{I}$ . Since  $\mathcal{O}_X/\mu_b^* \mathcal{J}$  is a skyscraper sheaf, the natural map

$$H^0(L^{ab} \otimes P_\alpha \otimes \mathcal{O}_X/\mu_b^* \mathcal{J}) \rightarrow H^0(L^{ab} \otimes P_\alpha \otimes \mathcal{O}_X/\mu_b^* \mathcal{I})$$

is also surjective. Thus  $H^0(L^{ab} \otimes P_\alpha) \rightarrow H^0(L^{ab} \otimes P_\alpha \otimes \mathcal{O}_X/\mu_b^* \mathcal{I})$  is surjective, and hence  $H^1(\mu_b^* \mathcal{I} \otimes L^{ab} \otimes P_\alpha) = 0$  because  $H^1(L^{ab} \otimes P_\alpha) = 0$ . Since  $\alpha$  can be any point in  $\hat{X}$ , the sheaf  $\mu_b^* \mathcal{I} \otimes L^{ab}$  is IT(0). Thus  $\mathcal{I}\langle tl \rangle$  is IT(0).

The last statement follows from  $\mathcal{J}(X, F) \subset \mathcal{I}_S$  and  $\text{Supp } \mathcal{O}_X/\mathcal{I}_S = S$ .  $\square$

As a special case, we obtain the following corollary.

**COROLLARY 5.4.** *Let  $(X, l)$  be a polarized abelian variety and  $t > 0$  be a positive rational number. Assume that there exists an effective  $\mathbb{Q}$ -divisor  $F$  on  $X$  such that*

- (i)  $tl - F$  is ample,
- (ii)  $o \in X$  is an isolated point of  $\text{Nklt}(X, F)$ .

*Then  $\beta(l) < t$ . In particular, any  $L$  representing  $l$  satisfies  $(N_p)$  if  $t \leq 1/(p+2)$ .*

*Proof.* Recall that  $\beta(l) < t$  if and only if  $\mathcal{I}_o\langle tl \rangle$  is IT(0). Hence this corollary is the special case  $S = \{o\}$  of Proposition 5.3.  $\square$

We might regard the condition  $\beta(l) < t$ , which is equivalent to  $\beta(tl) < 1$ , as “the basepoint-freeness of the  $\mathbb{Q}$ -line bundle  $tL$ ” by Theorem 2.3. Since  $tL = K_X + tL$  for an abelian variety, Corollary 5.4 can be regarded as a  $\mathbb{Q}$ -line bundle case of Proposition 5.2 on abelian varieties.

Some techniques to construct  $F$  as in Proposition 5.2 are developed in the study of Fujita’s conjecture. The basic strategy is to cut down minimal lc centers so that the centers become 0-dimensional. The strategy and techniques to cut down lc centers are used in [Ito18, Loz18] to construct  $F$  as in Theorem 5.1. However, we can use them more directly due to Corollary 5.4 since we only need to consider the singularities of  $F$  locally as in Proposition 5.2. Recall the threshold  $r(l)$  in (5):

$$r(l) = \inf\{t \in \mathbb{Q} \mid \text{there exists an effective } \mathbb{Q}\text{-divisor } F \text{ on } X \text{ such that } tl - F \text{ is ample and } \mathcal{J}(X, F) = \mathcal{I}_o\}.$$

We define a modified threshold.

**DEFINITION 5.5.** Let  $(X, l)$  be a polarized abelian variety. We define

$$r'(l) = \inf\{t \in \mathbb{Q} \mid \text{there exists an effective } \mathbb{Q}\text{-divisor } F \text{ on } X \text{ such that } tl - F \text{ is ample and } \text{Nklt}(X, F) \text{ has an isolated point}\}.$$

*Remark 5.6.* If there exists an effective  $\mathbb{Q}$ -divisor  $F$  on  $X$  such that  $tl - F$  is ample and  $\text{Nklt}(X, F)$  has an isolated point, then  $r'(l) \leq t$  by definition. In fact, we have  $r'(l) < t$  since  $t'l - F$  is still ample if  $0 < t - t' \ll 1$ .

The following corollary is a restatement of Corollary 5.4.

**COROLLARY 5.7.** *Let  $(X, l)$  be a polarized abelian variety. Then  $\beta(l) \leq r'(l) \leq r(l)$ .*

*Proof.* We obtain the first inequality by applying Proposition 5.3 to  $S = \{p\}$ , where  $p$  is an isolated point of  $\text{Nklt}(X, F)$ . The second one follows from the definitions of  $r'(l)$  and  $r(l)$ .  $\square$

By Corollaries 5.4 and 5.7 and results in the study of Fujita's conjecture, we obtain the following proposition, though we do not use this in the proof of Theorem 1.2.

**PROPOSITION 5.8.** *Let  $(X, l)$  be a polarized abelian variety, and set  $B_t = tl$  for  $t > 0$ .*

- (i)  $\dim X = 1$ : We have  $\beta(l) = r'(l) = r(l) = (\deg l)^{-1}$ .
- (ii)  $\dim X = 2$  ([Cau20a]): If  $(B_t^2) > 4$  and  $(B_t.C) > 1$  for any elliptic curve  $C \subset X$ , then  $\beta(l) \leq r'(l) \leq r(l) < t$ .
- (iii)  $\dim X = 3$ : Assume that
  - $(B_t^3) > 27$ ,
  - $(B_t^2.S) \geq 9$  for any surface  $S \subset X$ ,
  - $(B_t.C) \geq 3$  for any curve  $C \subset X$ .
 Then  $\beta(l) \leq r'(l) < t$ .
- (iv)  $\dim X = 4$ : Assume that  $(B_t^{\dim Z}.Z) \geq 5^{\dim Z}$  for any subvariety  $Z \subset X$ . Then  $\beta(l) \leq r'(l) < t$ .
- (v) In any dimension: Assume that

$$(B_t^{\dim Z}.Z) > \binom{\dim X + 1}{2}^{\dim Z}$$

for any subvariety  $Z \subset X$ . Then  $\beta(l) \leq r'(l) < t$ .

*Proof.* (i) We have  $\beta(l) = (\deg l)^{-1}$  by Remark 2.5. On the other hand,  $r(l) = (\deg l)^{-1}$  can be checked from the definition. Hence part (i) follows from Corollary 5.7.

(ii) The inequality  $r(t) < t$  follows from Proposition 4.1. In fact, this is proved in [Cau20a] and we use his argument in the proof of Proposition 4.1.

(iii) Example 4.6 of [Hel99] shows that the numerical conditions in part (iii) imply the existence of an effective  $\mathbb{Q}$ -divisor  $D \equiv cB_t$  on  $X$  for some  $0 < c < 1$  such that  $o$  is the minimal lc center of  $(X, D)$ . Perturbing  $D$  slightly by Lemma 2.11, we obtain  $F$  as in the definition of  $r'(l)$ . Hence we have  $r'(l) < t$ .

For parts (iv) and (v), the same proof works if we replace [Hel99, Example 4.6] with the proof of [Kaw97, Theorem 4.1] and an algebraic proof [Kol97] of Angehrn and Siu's theorem [AS95], respectively.  $\square$

Other than [Hel99, Kaw97, Kol97], we could apply results in, for example, [Hei02, YZ20]. We note that we assume lower bounds of  $(B_t^{\dim Z}.Z)$  for *all subvarieties*  $Z$  in Proposition 5.8, contrary to the assumption in Question 1.1.

## 6. Induction argument to abelian subvarieties

Inspired by the works [HT11, LPP11, KL19] on  $(N_p)$  of polarized abelian varieties and a generalization of Fujita's conjecture [Kol97], the author asked the following question.

*Question 6.1* ([Ito18, Question 4.2]). Let  $p$  be a non-negative integer,  $X$  be an abelian variety and  $L$  be an ample line bundle on  $X$ . Set  $B = (p + 2)^{-1}L$ . Assume that  $(B^{\dim Z}.Z) > (\dim Z)^{\dim Z}$  for any abelian subvariety  $Z \subset X$  with  $\dim Z \geq 1$ . Then does  $L$  satisfy property  $(N_p)$ ?

Based on Theorem 2.3 and Question 6.1, Caucci asks the following question.

*Question 6.2.* Let  $(X, l)$  be a polarized abelian variety. Set  $B_t = tl$  for a positive rational number  $t > 0$ . Assume that  $(B_t^{\dim Z}.Z) > (\dim Z)^{\dim Z}$  for any abelian subvariety  $Z \subset X$  with  $\dim Z \geq 1$ . Then does it hold that  $\beta(l) < t$ ?

As we see in the previous section, techniques and results to cut down lc centers are quite useful to study these questions. However, we still have some difficulty to answer Question 6.2 even in dimension 3. A generalization of Fujita's conjecture [Kol97, Conjecture 5.4] claims that an adjoint line bundle  $K_X + L$  on a smooth projective variety  $X$  is basepoint-free if  $(L^{\dim X}) > (\dim X)^{\dim X}$  and  $(L^{\dim Z}.Z) \geq (\dim X)^{\dim Z}$  for all subvarieties  $Z \subsetneq X$ . Between Question 6.2 and [Kol97, Conjecture 5.4], there are the following two differences which cause some problems when we use Corollary 5.4 to answer Question 6.2:

- (1) The lower bound  $(\dim Z)^{\dim Z}$  in Question 6.2 is smaller than the bound  $(\dim X)^{\dim Z}$  in [Kol97, Conjecture 5.4] for an abelian subvariety  $Z \subsetneq X$ .
- (2) In Question 6.2, we do not assume any lower bounds on  $(B_t^{\dim Z}.Z)$  if  $Z \subsetneq X$  is not an abelian subvariety.

For difference (1), we consider the following question.

*Question 6.3.* For an ample  $\mathbb{Q}$ -divisor  $B$  on a  $g$ -dimensional abelian variety  $X$  with  $(B^g) > g^g$ , does there exist an effective  $\mathbb{Q}$ -divisor  $F$  on  $X$  such that  $B - F$  is nef and  $\text{Nklt}(X, F)$  contains an abelian subvariety of  $X$  as an irreducible component?

Question 6.3 is a weak version of a conjecture [Loz18, Conjecture 7.2] by Lozovanu, which requires  $\text{Nklt}(X, F)$  to be an abelian subvariety. He proves his conjecture for threefolds under the assumption  $(B^3) > 40$ .

Although Lozovanu does not give a direct relation between his conjecture and Question 6.1, he suggests that this kind of statement would be useful in induction-type arguments. Following his suggestion, we reduce Question 6.2 to Question 6.3.

**THEOREM 6.4.** *Let  $g$  be a positive integer. An affirmative answer to Question 6.3 for  $\dim X \leq g$  implies one to Question 6.2 for  $\dim X \leq g$ .*

Before the proof of Theorem 6.4, we summarize the relations among the three questions as follows:

$$\text{Question 6.3}_{\leq g} \rightsquigarrow \text{Question 6.2}_g \rightsquigarrow \text{Question 6.1}_g,$$

where  $Q \rightsquigarrow Q'$  means that an affirmative answer to  $Q$  implies one to  $Q'$ , and  $Q_g$  or  $Q_{\leq g}$  means that we only consider the case when  $\dim X = g$  or  $\dim X \leq g$ , respectively. The first implication follows from Theorem 6.4, and the second one from Theorem 2.3. An important point is that difference (1) does not cause any problem when we consider Question 6.3 since we do not need to cut a lc center anymore if the center is an abelian subvariety. For Question 6.3, difference (2) still causes a problem when we cut down lc centers that are not abelian subvarieties. We treat this problem in Sections 7 and 8.

To prove Theorem 6.4, we use Poincaré's reducibility theorem. See [BL04, Corollary 5.3.6] for the proof.

**THEOREM 6.5** (Poincaré's reducibility theorem). *Let  $(X, l)$  be a polarized abelian variety and  $Y \subset X$  be an abelian subvariety. Then there exists an abelian subvariety  $Z \subset X$  such that the addition map  $f: Y \times Z \rightarrow X$  is an isogeny with  $f^*l = l|_Y \boxtimes l|_Z$ .*

The following is a key proposition in the proof of Theorem 6.4. This is a generalization of Corollary 5.4.

**PROPOSITION 6.6.** *Let  $(X, l)$  be a polarized abelian variety. Assume that there exist a rational number  $t > 0$ , an effective  $\mathbb{Q}$ -divisor  $F$  on  $X$  and an abelian subvariety  $Y \subset X$  such that*

- (i)  $tl - F$  is ample,
- (ii)  $Y$  is an irreducible component of  $\text{Nklt}(X, F)$ ,
- (iii)  $\beta(l|_Y) < t$ , where we set  $\beta(l|_Y) := -\infty$  if  $Y = \{o\}$ .

Then  $\beta(l) < t$ .

*Proof.* For a subset  $S \subset X$  and a point  $x \in X$ , the translations of  $S$  by  $x$  and  $-x$  are denoted by  $S + x, S - x \subset X$ , respectively. Let  $f: Y \times Z \rightarrow X$  be the isogeny in Theorem 6.5.

By condition (ii), there exists an open subset  $U \subset X$  such that  $\text{Nklt}(X, F) \cap U = Y \cap U \neq \emptyset$ . Take a general  $x \in X$ . In particular, the finite set  $Y \cap (Z - x)$  is contained in  $U$ . Applying Lemma 2.12 to  $(X, F)$  and the quotient homomorphism  $X \rightarrow X/Z$ , we have

$$\begin{aligned} \text{Nklt}(Z - x, F|_{Z-x}) \cap U &= \text{Nklt}(X, F) \cap (Z - x) \cap U \\ &= (Y \cap U) \cap (Z - x) = Y \cap (Z - x). \end{aligned}$$

Set  $W := (Y + x) \cap Z \subset U + x$  and  $D_Z := (F + x)|_Z \equiv F|_Z$ . Then

$$\text{Nklt}(Z, D_Z) \cap (U + x) = \text{Nklt}(Z, (F + x)|_Z) \cap (U + x) = (Y + x) \cap Z = W. \quad (9)$$

Recall that  $\beta(l) < t$  holds if and only if  $\mathcal{I}_x\langle tl \rangle$  is  $\text{IT}(0)$ . Since  $f$  is an isogeny,  $\mathcal{I}_x\langle tl \rangle$  is  $\text{IT}(0)$  if and only if  $\mathcal{I}_{f^{-1}(x)}\langle tf^*l \rangle$  is  $\text{IT}(0)$ , by Lemma 2.8(iii).

Since  $W = (Y + x) \cap Z$  and  $-Y = Y \subset X$ , it holds that  $f^{-1}(x) = \{(x - w, w) \in Y \times Z \mid w \in W\} \subset Y \times W$ . Hence we have an exact sequence

$$0 \rightarrow \mathcal{I}_{Y \times W} \rightarrow \mathcal{I}_{f^{-1}(x)} \rightarrow \mathcal{I}_{f^{-1}(x)}/\mathcal{I}_{Y \times W} \rightarrow 0$$

on  $Y \times Z$ . To prove that  $\mathcal{I}_{f^{-1}(x)}\langle tf^*l \rangle$  is  $\text{IT}(0)$ , it suffices to show that both  $\mathcal{I}_{Y \times W}\langle tf^*l \rangle$  and  $(\mathcal{I}_{f^{-1}(x)}/\mathcal{I}_{Y \times W})\langle tf^*l \rangle$  are  $\text{IT}(0)$ , by Lemma 2.8(i).

**CLAIM 6.7.** *The  $\mathbb{Q}$ -twisted sheaf  $\mathcal{I}_{Y \times W}\langle tf^*l \rangle$  is  $\text{IT}(0)$ .*

*Proof.* Since  $\mathcal{O}_Y\langle tl|_Y \rangle$  is  $\text{IT}(0)$  and  $\mathcal{I}_{Y \times W}\langle tf^*l \rangle = \mathcal{O}_Y \boxtimes \mathcal{I}_{W/Z}\langle t(l|_Y \boxtimes l|_Z) \rangle$ , it suffices to show that  $\mathcal{I}_{W/Z}\langle tl|_Z \rangle$  is  $\text{IT}(0)$ , by Lemma 2.8(ii). By condition (i),  $tl|_Z - D_Z \equiv tl|_Z - F|_Z$  is ample. By (9), each point in  $W$  is an isolated point of  $\text{Nklt}(Z, D_Z)$ . Hence  $\mathcal{I}_{W/Z}\langle tl|_Z \rangle$  is  $\text{IT}(0)$  by Proposition 5.3.  $\square$

**CLAIM 6.8.** *The  $\mathbb{Q}$ -twisted sheaf  $(\mathcal{I}_{f^{-1}(x)}/\mathcal{I}_{Y \times W})\langle tf^*l \rangle$  is  $\text{IT}(0)$ .*

*Proof.* Since  $f^{-1}(x) = \{(x - w, w) \in Y \times Z \mid w \in W\} \subset Y \times W$ , we have

$$\mathcal{I}_{f^{-1}(x)}/\mathcal{I}_{Y \times W} = \bigoplus_{w \in W} \mathcal{I}_{(x-w, w)/Y \times \{w\}}.$$

Since  $\beta(l|_Y) < t$  by condition (iii), the  $\mathbb{Q}$ -twisted sheaf  $\mathcal{I}_{x-w/Y}\langle tl|_Y \rangle$  is IT(0) for any  $w \in W$ . Hence  $\mathcal{I}_{(x-w,o)/Y \times \{o\}}\langle tf^*l \rangle$  is IT(0) by Lemma 2.8(iv). Then  $\mathcal{I}_{(x-w,w)/Y \times \{w\}}\langle tf^*l \rangle$  is IT(0), and hence so is the direct sum  $\mathcal{I}_{f^{-1}(x)}/\mathcal{I}_{Y \times W}\langle tf^*l \rangle = \bigoplus_{w \in W} \mathcal{I}_{(x-w,w)/Y \times \{w\}}\langle tf^*l \rangle$ .  $\square$

By Claims 6.7 and 6.8, the  $\mathbb{Q}$ -twisted sheaf  $\mathcal{I}_{f^{-1}(x)}\langle tf^*l \rangle$  is IT(0). Hence this proposition follows.  $\square$

Now we can prove Theorem 6.4.

*Proof of Theorem 6.4.* We prove this theorem by the induction on  $g$ . If  $g = 1$ , there is nothing to prove since we already see that Question 6.2 has an affirmative answer for  $\dim X = 1$  by Proposition 5.8(i).

Assume that this theorem holds for  $g - 1$ . Let  $(X, l)$ ,  $B_t$  be as in Question 6.2 with  $\dim X = g$ . What we need to show is that an affirmative answer to Question 6.3 $_{\leq g}$  implies  $\beta(l) < t$ .

Since  $(B_t^g) > g^g$ , we still have  $(B^g) > g^g$  for  $B = B_{t'}$  with  $0 < t - t' \ll 1$ . If we assume an affirmative answer to Question 6.3 $_g$ , there exists an effective  $\mathbb{Q}$ -divisor  $F$  on  $X$  such that  $B - F$  is nef and  $\text{Nklt}(X, F)$  contains an abelian subvariety  $Y \subset X$  as an irreducible component. Since  $tl - F = B_t - F = (B_t - B) + (B - F)$  is ample,  $F$  satisfies conditions (i) and (ii) in Proposition 6.6. If we assume an affirmative answer to Question 6.3 $_{\leq g-1}$ , we have an affirmative answer to Question 6.2 $_{\leq g-1}$  by the induction hypothesis. Hence  $\beta(l|_Y) < t$  follows. Thus condition (iii) in Proposition 6.6 is also satisfied, and  $\beta(l) < t$  follows from Proposition 6.6.  $\square$

## 7. Two lemmas on curves and hypersurfaces on abelian varieties

In this section, we show two lemmas on curves and hypersurfaces on abelian varieties. In the proof of Theorem 1.2, we will apply these lemmas to minimal lc centers of  $(X, D)$  for some  $D$  on an abelian threefold  $X$ .

Throughout this section, let  $X, B$  be as in Question 6.3; that is,  $X$  is an abelian  $g$ -fold and  $B$  is an ample  $\mathbb{Q}$ -divisor on  $X$  with  $(B^g) > g^g$ . For a subvariety  $V \subset X$ , denote by  $\langle V \rangle \subset X$  the smallest abelian subvariety that contains a translate of  $V$ . Let  $\pi: X' \rightarrow X$  be the blow-up at the origin  $o \in X$  and  $E \subset X'$  be the exceptional divisor.

The following lemma is a variant of [Loz18, Proposition 4.3].

LEMMA 7.1. *For a curve  $C \subset X$ , the following hold:*

- (i) *We have  $(B.C) > g^2/\sqrt[g]{g!}$  if  $\langle C \rangle = X$ .*
- (ii) *If  $S := \langle C \rangle$  is a hypersurface, then  $B - S$  is ample or  $(B.C) > g(g-1)/\sqrt[g-1]{(g-1)!}$ .*

*Proof.* (i) By [Deb94, Proposition 4.1], we have  $(B.C) \geq g\sqrt[g]{(B^g)/g!} > g^2/\sqrt[g]{g!}$ .

(ii) If  $(B^{g-1}.S) \leq g^{g-1}$ , it holds that  $(B^g) > g^g \geq g(B^{g-1}.S)$ . Hence  $B - S$  is big by [Laz04b, Theorem 2.2.15]. Since  $X$  is an abelian variety,  $B - S$  is ample.

If  $(B^{g-1}.S) > g^{g-1}$ , we have

$$(B.C) \geq (g-1)\sqrt[g-1]{(B^{g-1}.S)/(g-1)!} > g(g-1)/\sqrt[g-1]{(g-1)!}$$

by [Deb94, Proposition 4.1].  $\square$

In the following lemma, we combine the argument in [Hel97, Section 4] with [Nak05], [Loz18, Propositions 4.2 and 4.4]. For the asymptotic multiplicity  $\text{mult}_{S'}(\|\pi^*B - gE\|)$ , we refer the readers to [ELM<sup>+</sup>06, Loz18].

LEMMA 7.2. *Let  $S \subset X$  be a prime divisor such that  $\text{mult}_{S'}(\|\pi^*B - gE\|) \geq 1$ , where  $S' \subset X'$  is the strict transform of  $S$ . Then  $S$  is an abelian subvariety and  $B - S$  is ample.*

*Proof.* By  $\text{mult}_{S'}(\|\pi^*B - gE\|) \geq 1$  and [ELM<sup>+</sup>06, Lemma 3.3], we have  $\text{vol}(\pi^*B - gE) = \text{vol}(\pi^*B - gE - S')$ . Since  $\pi^*B - gE$  is big because  $(B^g) > g^g$ , so is  $\pi^*B - gE - S'$ . Hence  $\pi_*(\pi^*B - gE - S') = B - S$  is big as well. Since  $X$  is an abelian variety, the big divisor  $B - S$  is ample. Thus it remains to show that  $S$  is an abelian subvariety.

By the assumption  $\text{mult}_{S'}(\|\pi^*B - gE\|) \geq 1 > 0$ , the divisor  $S'$  is contained in the restricted base locus  $\mathbf{B}_-(\pi^*B - gE)$  by [ELM<sup>+</sup>06, Proposition 2.8]. In particular,  $S'$  is contained in the augmented base locus  $\mathbf{B}_+(\pi^*B - gE)$ . By [Loz18, Lemma 4.2], the origin  $o$  is contained in  $S$  and if  $S$  is smooth at  $o$ , then  $S$  is an abelian subvariety. Hence it suffices to show that  $\text{mult}_o(S) = 1$ .

Set  $\tau = \sup\{t \geq 0 \mid \pi^*B - tE \text{ is pseudo-effective}\}$  and  $m = \text{mult}_o(S)$ . We have  $\tau > g$  and  $m \geq 1$ . Define a function  $m_{S'}: [0, \tau] \rightarrow \mathbb{R}_{\geq 0}$  by  $m_{S'}(t) = \text{mult}_{S'}(\|\pi^*B - tE\|)$ .

By the definition of  $m_{S'}(t)$  and [ELM<sup>+</sup>06, Lemma 3.3], we have a bound for the restricted volume  $\text{vol}_{X'|E}(\pi^*B - tE)$  for  $t \in (0, \tau)$  as

$$\begin{aligned} 0 < \text{vol}_{X'|E}(\pi^*B - tE) &= \text{vol}_{X'|E}(\pi^*B - tE - m_{S'}(t)S') \\ &\leq \text{vol}_E((\pi^*B - tE - m_{S'}(t)S')|_E) \\ &= \text{vol}_{\mathbb{P}^{g-1}}\mathcal{O}_{\mathbb{P}^{g-1}}(t - mm_{S'}(t)) \end{aligned}$$

since  $S'|_E \sim \mathcal{O}_E(m)$ . Hence we have  $t - mm_{S'}(t) > 0$  for  $t \in (0, \tau)$  and

$$(B^g) = g \int_0^\tau \text{vol}_{X'|E}(\pi^*B - tE) dt \leq g \int_0^\tau (t - mm_{S'}(t))^{g-1} dt, \quad (10)$$

where the first equality follows from [LM09, Corollary C].

By [Nak96, Lemma 3.4] (see also [Loz18, Proposition 4.4]), we have  $m_{S'}(t) - m_{S'}(g) \geq t - g$  for  $t \in [g, \tau]$ . Thus it holds that

$$\begin{aligned} t - mm_{S'}(t) &\leq t - m(t - g + m_{S'}(g)) \\ &= -(m-1)(t - g) + g - mm_{S'}(g) \end{aligned} \quad (11)$$

for  $t \in [g, \tau]$ . For simplicity, set  $\alpha = g - mm_{S'}(g)$ . We have

$$0 < \alpha = g - mm_{S'}(g) \leq g - m \quad (12)$$

by the assumption  $m_{S'}(g) = \text{mult}_{S'}(\|\pi^*B - gE\|) \geq 1$ .

Since  $m_{S'}(t)$  is a convex function by [Loz18, Lemma 2.5],

$$\Lambda_1 = \{(t, y) \in \mathbb{R}^2 \mid 0 \leq t < \tau, 0 \leq y \leq t - mm_{S'}(t)\}$$

is a convex set. The set

$$\Lambda_2 = \{(t, y) \in \mathbb{R}^2 \mid t \geq g, y \geq -(m-1)(t - g) + \alpha\}$$

is also convex. Since  $(g, \alpha) \in \Lambda_1 \cap \Lambda_2$  and  $\Lambda_1$  and  $\Lambda_2$  do not intersect in the interiors by (11), there exists an  $\lambda \geq m-1$  such that  $t - mm_{S'}(t) \leq -\lambda(t - g) + \alpha$  for any  $t \in [0, \tau]$ .

Recall that we need to show  $m = 1$ . To prove this by contradiction, we assume  $m \geq 2$ . Then we have  $\lambda \geq m-1 \geq 1$ . Let  $(a, a)$  be the intersection point of two lines  $y = t$  and  $y = -\lambda(t - g) + \alpha$ , and let  $(b, 0)$  be the intersection point of two lines  $y = 0$  and  $y = -\lambda(t - g) + \alpha$ , that is,

$$a = \frac{g\lambda + \alpha}{1 + \lambda}, \quad b = \frac{g\lambda + \alpha}{\lambda}. \quad (13)$$



We note that  $\tau \leq b$  follows since  $0 < t - mm_{S'}(t) \leq -\lambda(t - g) + \alpha$  for any  $t \in [0, \tau)$ .

By (10) and the assumption  $(B^g) > g^g$ ,

$$\begin{aligned} g^g < (B^g) &\leq g \int_0^\tau (t - mm_{S'}(t))^{g-1} dt \\ &= g \int_0^a (t - mm_{S'}(t))^{g-1} dt + g \int_a^\tau (t - mm_{S'}(t))^{g-1} dt \\ &\leq g \int_0^a t^{g-1} dt + g \int_a^b (-\lambda(t - g) + \alpha)^{g-1} dt \\ &= a^g + \frac{1}{\lambda} a^g = a^{g-1} b \leq \left( \frac{(g-1)a + b}{g} \right)^g, \end{aligned}$$

where the last inequality follows from the fact that the geometric mean is bounded by the arithmetic mean. Thus we have

$$g < \frac{(g-1)a + b}{g},$$

which is equivalent to  $0 < (\alpha + 1 - g)g\lambda + \alpha$  by (13). Since  $\alpha \leq g - m$  by (12) and  $\lambda \geq m - 1 \geq 1$ , it holds that

$$\begin{aligned} 0 < ((g - m) + 1 - g)g\lambda + (g - m) &= -(m - 1)g\lambda + g - m \\ &\leq -(m - 1)^2 g + g - m. \end{aligned}$$

Hence  $(m - 1)^2 < 1 - m/g < 1$ , which contradicts  $m \geq 2$ . Thus we have  $\text{mult}_o(S) = m = 1$ .  $\square$

## 8. Proof of Theorem 1.2

We answer Question 6.3 affirmatively for  $\dim X \leq 3$ .

**THEOREM 8.1.** *Question 6.3 has an affirmative answer for  $\dim X \leq 3$ . That is, for an ample  $\mathbb{Q}$ -divisor  $B$  on a  $g$ -dimensional abelian variety  $X$  with  $(B^g) > g^g$  and  $g \leq 3$ , there exists an effective  $\mathbb{Q}$ -divisor  $F$  such that  $B - F$  is nef and  $\text{Nklt}(X, F)$  contains an abelian subvariety as an irreducible component.*

*Proof.* If  $\dim X = 1$ , we may take  $F = o$ . If  $\dim X = 2$ , under the assumption  $(B^2) > 4$ , the proof of [Ito18, Proposition 3.1] constructs an effective  $\mathbb{Q}$ -divisor  $D$  such that  $B - D$  is ample,  $(X, D)$  is lc, and a minimal lc center of  $(X, D)$  is a point or an elliptic curve. Perturbing  $D$  slightly by Lemma 2.11, we obtain  $F$  as in the statement of this theorem.

Assume that  $\dim X = 3$ , and let  $B$  be an ample  $\mathbb{Q}$ -divisor with  $(B^3) > 27$ . Let  $\pi: X' \rightarrow X$  be the blow-up at  $o \in X$  and  $E \subset X'$  be the exceptional divisor. Since  $(B^3) > 27$ , we can take a rational number  $\delta > 0$  such that  $(B^3) > (3 + \delta)^3$ . In particular,  $\pi^*B - (3 + \delta)E$  is big. Take a general effective  $\mathbb{Q}$ -divisor  $D' \equiv \pi^*B - (3 + \delta)E$ . Set  $D = \pi_*(D') \equiv B$  and

$$c := \text{lct}_o(X, D) \leq \frac{3}{3 + \delta} < 1. \quad (14)$$

If the minimal lc center of  $(X, cD)$  at  $o$  is  $\{o\}$ , a small perturbation  $F$  of  $cD$  satisfies the condition in the statement of this theorem. Thus we may assume that the minimal lc center of  $(X, cD)$  at  $o$  is a curve or a surface.

*Case 1:* The minimal lc center is a curve  $C$ .

*Case 1-1.*  $\dim\langle C \rangle = 3$ : In this case, we have  $(B.C) > 9/\sqrt[3]{6} > 3$  by Lemma 7.1(i). Thus we can apply Lemma 2.13 to the above  $X, B, D$  by Remark 2.14, and hence there exists an effective  $\mathbb{Q}$ -divisor  $D_1 \equiv c_1 B$  for some  $c_1 < 1$  such that  $(X, D_1)$  is lc but not klt at  $o$  and the minimal lc center is  $\{o\}$ . Perturbing  $D_1$  slightly, we have an effective  $\mathbb{Q}$ -divisor  $F$  on  $X$  such that  $B - F$  is ample and  $\{o\}$  is an isolated point of  $\text{Nklt}(X, F)$ .

*Case 1-2.*  $\dim\langle C \rangle = 2$ : In this case,  $S = \langle C \rangle$  is an abelian surface. By Lemma 7.1(ii),  $B - S$  is ample or  $(B.C) > 3\sqrt{2}$ .

If  $B - S$  is ample,  $F = S$  satisfies the condition in the statement of this theorem.

If  $(B.C) > 3\sqrt{2} > 3$ , we have an  $F$  such that  $B - F$  is ample and  $\{o\}$  is an isolated point of  $\text{Nklt}(X, F)$  as in Case 1-1.

*Case 1-3.*  $\dim\langle C \rangle = 1$ : In this case,  $C$  is an elliptic curve. A small perturbation  $F$  of  $cD$  satisfies the condition in the statement of this theorem.

*Case 2:* The minimal lc center is a surface  $S$ .

CLAIM 8.2. *Let  $S' \subset X'$  be the strict transform of  $S$ . Then  $\text{mult}_{S'}(\|\pi^*(3(3+\delta)^{-1}B) - 3E\|) \geq 1$ .*

*Proof of Claim 8.2.* Assume that  $\text{mult}_{S'}(\|\pi^*(3(3+\delta)^{-1}B) - 3E\|) < 1$ ; in other words, assume that  $\text{mult}_{S'}(\|\pi^*B - (3+\delta)E\|) < (3+\delta)/3$ . Then we have  $\text{mult}_{S'}(D') < (3+\delta)/3$  since we take a general  $D' \equiv \pi^*B - (3+\delta)E$ . On the other hand, since  $S$  is a lc center of  $(X, cD)$ , we have  $\text{mult}_{S'}(D') = \text{mult}_S(D) = c^{-1} \geq (3+\delta)/3$  by (14), which gives a contradiction.  $\square$

Since  $\text{mult}_{S'}(\|\pi^*(3(3+\delta)^{-1}B) - 3E\|) \geq 1$  and  $((3(3+\delta)^{-1}B)^3) = (3(3+\delta)^{-1})^3(B^3) > 27$ ,  $S$  is an abelian surface and  $3(3+\delta)^{-1}B - S$  is ample by Lemma 7.2. Hence  $F = S$  satisfies the condition in the statement of this theorem.  $\square$

*Proof of Theorem 1.2.* Theorem 1.2 is nothing but an affirmative answer to Question 6.2 for  $\dim X = 3$ . Hence it follows from Theorems 6.4 and 8.1.  $\square$

*Remark 8.3.* We note that Corollary 5.4 is not enough to prove Theorem 1.2 even if we use the lemmas in Section 7. The reason is as follows.

Let  $X, l, B_t = tl$  be as in Theorem 1.2, and take  $D \equiv B_t, c = \text{lct}_o(D)$  as in the proof of Theorem 8.1. Assume that the minimal lc center of  $(X, cD)$  at  $o$  is an elliptic curve  $C$  as in case 1-3. Then we cannot cut down  $C$  since we only assume  $(B_t.C) > 1$  and cannot apply Lemma 2.13. This is the case when we essentially need Theorem 6.4 or Proposition 6.6.

If we perturb  $cD$ , we have  $\mathcal{J}(X, cD) = \mathcal{I}_C$  in a neighborhood of the origin. It might be worth noting that we obtain  $\beta(l) < t$  if  $\mathcal{J}(X, cD) = \mathcal{I}_C$  holds on *whole*  $X$ , as follows: If  $\mathcal{J}(X, cD) = \mathcal{I}_C$ , then  $\mathcal{I}_C\langle tl \rangle = \mathcal{J}(X, cD)\langle tl \rangle$  is IT(0) by the Nadel vanishing theorem. Since  $\deg \mathcal{O}_C(-o)\langle tl \rangle = (B_t.C) - 1 > 0$  and  $C$  is an elliptic curve,  $\mathcal{I}_{o/C}\langle tl \rangle = \mathcal{O}_C(-o)\langle tl \rangle$  is IT(0) as well. Thus  $\mathcal{I}_o\langle tl \rangle$  is IT(0) and hence  $\beta(l) < t$  follows from the exact sequence

$$0 \rightarrow \mathcal{I}_C \rightarrow \mathcal{I}_o \rightarrow \mathcal{I}_{o/C} \rightarrow 0 \quad (15)$$

and Lemma 2.4(i). However, we do not know how to control the singularity of  $(X, cD)$  outside the origin and hence do not know whether  $\mathcal{J}(X, cD) = \mathcal{I}_C$  holds or not on whole  $X$ . Thus we cannot apply the above argument to prove Theorem 1.2, at least without further argument about the singularities of  $(X, cD)$ .

The readers might ask if we can use  $c' = \text{lct}(D) := \max\{s \geq 0 \mid (X, sD) \text{ is log canonical}\}$  instead of  $c = \text{lct}_o(D)$ . Then  $\mathcal{J}(X, c'D) = \mathcal{I}_{C'}$  holds on whole  $X$  for a minimal lc center  $C'$  of  $(X, c'D)$  after perturbing  $c'D$  and hence the above argument to use (15) works if  $C'$  is an elliptic

curve. However, if  $C'$  is a curve of general type, it is not clear whether  $\mathcal{I}_{o/C'}(tl)$  is IT(0) or not. Furthermore, we cannot apply Lemma 2.13 to cut down  $C'$  since  $C'$  might not contain the origin and hence there might not exist a point  $x \in C'$  with  $\text{mult}_x(D) > 3$ . Therefore, we are not sure how to treat the case when  $C'$  is a curve of general type if we consider  $\text{lct}(D)$  instead of  $\text{lct}_o(D)$ .

*Proof of Corollary 1.3.* By Theorem 1.2, we have  $\beta(l) < 1/(p+2)$ . Hence  $L$  satisfies property  $(N_p)$  by Theorem 2.3. The vanishing of  $K_{p,q}(X, L; dL)$  follows from Proposition 2.6.  $\square$

*Proof of Corollary 1.4.* If  $(L^2.S) \geq 6$  and  $(L.C) \geq 2$  for any abelian surface  $S \subset X$  and any elliptic curve  $C \subset X$ , then  $\beta(l) < 1$  follows from Theorem 1.2 and the assumption  $(L^3) \geq 30 > 27$ . Hence  $L$  is basepoint-free by Theorem 2.3(i).

On the other hand, if  $L$  is basepoint-free, so are  $L|_S$  and  $L|_C$  for any abelian surface  $S$  and any elliptic curve  $C$  in  $X$ . Since  $L|_S$  and  $L|_C$  are ample,  $h^0(S, L|_S) = (L|_S^2)/2 \geq 3$  and  $h^0(C, L|_C) = \deg(L|_C) \geq 2$ . Thus we have  $(L^2.S) \geq 6$  and  $(L.C) \geq 2$ .  $\square$

If  $h^0(L) = (L^3)/3! \leq 3$ , then  $L$  is not basepoint-free since  $L$  is ample. Hence if  $h^0(L) \neq 4$ , the basepoint-freeness of an ample line bundle  $L$  on an abelian threefold is completely determined by the numerical data  $(L^{\dim Z}.Z)$  for all abelian subvarieties  $Z \subset X$ , including the data for  $Z = X$ .

However, the basepoint-freeness is not determined by such data if  $h^0(L) = 4$ . For example, let  $(X_1, l_1)$  and  $(X_2, l_2)$  be very general polarized abelian threefolds of type  $(1, 1, 4)$  and  $(1, 2, 2)$ , respectively. Let  $L_i$  be a line bundle representing  $l_i$  for each  $i = 1, 2$ . Then  $(L_1^3) = (L_2^3) = 24$ . Furthermore,  $X_1$  and  $X_2$  are simple since  $(X_1, l_1)$  and  $(X_2, l_2)$  are very general. Hence the numerical data  $(L^{\dim Z}.Z)$  for all abelian subvarieties  $Z \subset X$  coincide for  $X_1$  and  $X_2$ . On the other hand,  $L_1$  is basepoint-free by [DHS94, Proposition 2], but  $L_2$  is not by [NR95, Corollary 2.6], [BL04, Remark 15.1.4].

In particular, the assumption  $h^0(L) \geq 5$  is sharp; that is, the statement of Corollary 1.4 does not hold under the weaker assumption  $h^0(L) \geq 4$ .

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