



Logarithmic Gromov–Witten theory with expansions

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ABSTRACT

We construct relative Gromov–Witten theory with expanded degenerations in the normal crossings setting and establish a degeneration formula for the resulting invariants. Given a simple normal crossings pair (X, D) , we show that there exist proper moduli spaces of curves in X with prescribed boundary conditions along D , equipped with virtual classes. Each point in such a moduli space parameterizes a map from a nodal curve to an expanded degeneration of X that is dimensionally transverse to the strata. In the context of maps to a simple normal crossings degeneration, the virtual fundamental class is known to decompose as a sum over tropical maps. We use the expanded formalism to prove the degeneration formula – we reconstruct the virtual class attached to a tropical map in terms of spaces of maps to expansions attached to the vertices.

1. Introduction

1.1. The problem. Let X be a smooth projective algebraic variety, and let $D \subset X$ be a simple normal crossings divisor with components $D_1, \dots, D_k$. Relative Gromov–Witten theory concerns maps of pairs

$$(C, p_1, \dots, p_n) \rightarrow (X, D),$$

where C is a smooth genus g curve in a homology class β , meeting the component D_i at p_j with contact order $c_{ij} \in \mathbb{N}$. For each j , the contact order c_{ij} is non-zero for at most one index i . These numerical data are packaged by the symbol Γ , and the divisor D is implicit in the notation X . There is a non-proper Deligne–Mumford stack $K_\Gamma^\circ(X)$ parameterizing such maps.

The problem considered in this paper is to build a proper moduli space $K_\Gamma(X)$ equipped with universal marked curve and target families

$$\begin{array}{ccccc} \mathcal{C} & \xrightarrow{\quad} & \mathcal{X} & \longrightarrow & X \\ & \searrow & \swarrow & & \\ & & K_\Gamma(X) & & \end{array}$$

and sections $s_i: K_\Gamma(X) \rightarrow \mathcal{C}$. Two requirements are sought:

(A) *Virtual toroidality.* The moduli space $K_\Gamma(X)$ is virtually smooth over an equidimensional

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logarithmically smooth stack. It is therefore equipped with a virtual fundamental class giving rise to a system of Gromov–Witten invariants of X *relative* to D .

- (B) *Transversality.* Fix a point p of $K_\Gamma(X)$, and denote by $(\mathcal{C}_p, s_i) \rightarrow \mathcal{X}_p$ the map at this point. Then $\mathcal{C}_p$ meets $\mathcal{X}_p$ at s_i along a divisor $\mathcal{D}_{j,p}$ with contact order c_{ij} . The divisor $\mathcal{D}_{j,p}$ maps to D_j on projecting to X .

Canonicity of the spaces is not demanded, and we permit discrete choices. However, the resulting Gromov–Witten theory must be independent of choices. The problem is solved in the first part of the paper. The solution leads to a gluing formula for the Gromov–Witten theory of simple normal crossings degenerations, which occupies the second part of the paper.

1.2. A brief history. In algebraic geometry, this problem was first considered by Li in the case where D is smooth, following work of Gathmann and Vakil [Gat02, Vak00]. In order to obtain a compactification, he studies maps to *expansions* of X along D , called “accordions”. These are obtained by iteratively degenerating X to the normal cone of D in X and form the fibres of $\mathcal{X}$ above. After replacing the target by an accordion, transverse stable limits of families of relative maps exist and are unique. The resulting moduli space $\mathrm{Li}_\Gamma(X)$ is a proper Deligne–Mumford stack with a virtual fundamental class [Li01]. Li’s theory is central to enumerative geometry. The original construction of the virtual class is technical, but the advent of new methods – orbifolds and logarithmic structures – offered simplifications [AF16, Kim10]. Important conceptual advances were made in [AMW14].

There have been several attempts to generalize Li’s approach to the simple normal crossings set-up, but it appears that the idea was abandoned in favour of the logarithmic approach. Remarkably, the transversality condition demanded above can be avoided entirely for the purpose of *defining* relative invariants. In 2001, Siebert suggested that one could study stable maps within the category of logarithmic schemes. This leads to a concise solution to the problem. In the logarithmic category, a map $C \rightarrow X$ can have components that are scheme-theoretically contained in D and nevertheless carry a well-defined contact order at marked points. Abramovich–Chen and Gross–Siebert constructed moduli spaces $\mathrm{ACGS}_\Gamma(X)$ of logarithmic maps to a simple normal crossings pair (X, D) , equipped with a virtual class [AC14, Che14b, GS13], building on striking work of Nishinou and Siebert [NS06]. Crucially, the work of Gross, Nishinou, and Siebert inserted tropical geometry into the theory of stable maps, and this is fundamental to the direction proposed here.

The logarithmic theory of maps is powerful, but the transversality condition is still desirable. A basic reason is to complete the conceptual parallel with Li’s theory, but there are more compelling ones. Many applications of relative Gromov–Witten theory come via the degeneration formula [AF16, Che14a, Li02]. The formula relates spaces of maps with matching boundary conditions with the space of maps to a general fibre. In the logarithmic theory, a parallel degeneration formula has not materialized, though there is encouraging recent work in this direction [ACGS20a, ACGS20b, KLR21]. The degeneration formula is the second contribution of our paper. In the logarithmic context, a basic difficulty arises when gluing logarithmic structures in families of stable maps. Gluing the curves compatibly with the underlying schematic map to X is necessary but rarely sufficient. The remaining “purely logarithmic” gluing presents several challenges. In our framework, the logarithmic data is recorded in a schematic fashion by a factorization of a map to X through an expansion. The role of the logarithmic structures is diminished, and traditional schematic methods are implemented.

1.3. Main results. Let (X, D) be a toroidal pair without self-intersections.

THEOREM A (Stable maps to expansions). *There exists a proper and virtually toroidal moduli space $K_\Gamma(X)$ equipped with flat source and target families*

$$\begin{array}{ccccc} \mathcal{C} & \xrightarrow{\quad} & \mathcal{X} & \longrightarrow & X \\ & \searrow \pi_s & \swarrow \pi_t & & \\ & & K_\Gamma(X) & & \end{array}$$

such that the universal curve is transverse to the universal target at every point on the base. The space is equipped with a virtual fundamental class and with (i) a forgetful morphism to the stack of curves $\mathfrak{M}_{g,n}$ and (ii) evaluation morphisms to the strata of X . Gromov–Witten invariants may be defined by pulling back Chow cohomology classes along these morphisms and integrating against the virtual fundamental class.

The reader may consult Example 4.4.9, which exhibits a stable limit of a family in the space of expanded maps and compares it to the limit as a logarithmic map. At present, we note the following features of the construction:

- (A) *Target expansions.* The fibres of π_t are toroidal degenerations of X . Each component of such a degeneration is birational to an equivariant partial compactification of a torus bundle over a stratum of (X, D) .
- (B) *Smooth intersections.* Given a fibre $\mathcal{X}_p$ of π_t , the intersection of any two irreducible components of $\mathcal{X}_p$ is a smooth, possibly non-proper divisor in each.
- (C) *Comparison morphism.* There is a canonical morphism

$$K_\Gamma(X) \rightarrow \text{ACGS}_\Gamma(X)$$

which is a logarithmic modification compatible with virtual structures and, in particular, identifies virtual classes; see Section 4.6.

In order to prove this result, we start with the space of logarithmic maps and then use toroidal modifications of the curve and target family to obtain the requisite transversality. We perform semistable reduction for the family by toroidal modifications of the base. These constructions are first made at the combinatorial level, then for maps to the Artin fan, and finally for curves in X .

The construction produces an infinite family of moduli spaces of transverse maps controlled by tropical moduli data. In the main text, these spaces are denoted by $K_\Gamma^\lambda(\cdot)$, with λ denoting an auxiliary polyhedral choice. The choices of λ form an inverse system, analogous to the inverse system of refinements of a given fan. The virtual classes form a compatible system in Chow homology; see Proposition 4.6.1. The properties in Theorem A are true *level-by-level* on a cofinal subsystem, and not only on the inverse limit.

The transversality condition is sufficiently robust to accommodate a cycle-theoretic gluing formula for the Gromov–Witten invariants of degenerations.

THEOREM B (Degeneration, decomposition, and gluing). *Let $\mathcal{Y} \rightarrow \mathbb{A}^1$ be a toroidal degeneration without self-intersections, with general fibre Y_η and special fibre Y_0 . There exist moduli spaces $K_\Gamma(\cdot)$ of maps to expansions with the following properties:*

- (A) *Virtual deformation invariance.* There is an equality of virtual classes

$$[K_\Gamma(Y_0)] = [K_\Gamma(Y_\eta)]$$

in the Chow group of the space of maps $K_\Gamma(\mathcal{Y})$.

- (B) *Decomposition.* The virtual class of maps to Y_0 decomposes as a sum over combinatorial splittings of the discrete data (that is, tropical stable maps)

$$[K_\Gamma(Y_0)] = \sum_{\rho} m_{\rho} [K_{\rho}(Y_0)],$$

where the $m_{\rho} \in \mathbb{Q}$ are explicit combinatorial multiplicities depending on the splitting ρ . Each space $K_{\rho}(Y_0)$ is a space of maps to expansions, marked by splitting type.

- (C) *Gluing.* For each splitting ρ with graph type G , there are moduli spaces $K_{\rho}(X_v)$ of maps to expansions of components X_v of Y_0 . There is a virtual birational model of their product

$$\times_v K_{\rho}(X_v) \rightarrow \prod_v K_{\rho}(X_v),$$

and there is an explicit formula relating the virtual class $[K_{\rho}(Y_0)]$ with the virtual class $[\times_v K_{\rho}(X_v)]$ and the class of the relative diagonal of the universal divisor expansion.

We refer the reader to the main text for the precise birational modification above. Its genesis is simple. The product of the moduli spaces of maps to X_v encodes disconnected maps to expansions of components of the normalization. If we are to glue two expansions, a bubbling of the target on one side may force a bubbling of the target on the other side.¹ Since the targets on the different factors of the product must interact, a “criss-cross” blowup is necessary.

1.4. Using the formula. The formula is more complicated than the corresponding double point formula, and it can be non-trivial to extract numerical consequences. However, the complexity can be understood concretely and worked with. It arises from the fact that the gluing condition is imposed by a diagonal condition coming not from the strata of (X, D) but rather from the strict transform of this class along a birational modification, as described above. The modification is obtained by subdividing the product of tropical moduli spaces. The modification is completely described by piecewise-linear geometry on the associated family of tropical curves. Describing the blowup is a finite problem that can be coded. Intersection theory computations with these modifications can be carried out by using the Fulton–Macpherson blowup formula for comparing strict and total transforms [Ful98, Section 6.7]. The blowups are logarithmic, so the blowup formula is captured by Chow operators on the Artin fans of these moduli spaces of maps. The rational Chow cohomology of Artin fans is the ring of piecewise polynomials [MPS21, MP21, Pay06, Ran20]. We expect these to become a natural calculus for statements in logarithmic intersection theory.

Even without a systematic calculus, we note that these formulas can often be “black boxed” to derive structural results. A prototypical example is provided by work of Pandharipande and Pixton [PP17, Sections 1.2 and 5]. They compare the bi-relative Gromov–Witten and stable pairs theories by working with an instance of the formula above and deduce the Gromov–Witten/pairs correspondence for a large class of varieties.

There are cases in the existing literature where the formula collapses; these are recorded in the final section of the paper. We also present a simplified formula for genus 0 invariants in the case of a triple point degeneration; see Section 7.5.

¹“The fates of the two sides are tied,” says Pandharipande.

1.5. Further discussion. Our results complete the simple normal crossings generalization of Li’s theory. The virtual class of Li’s theory also satisfies a remarkable torus localization formula [GV05, MR19]. The parallel formula in the expanded theory is the natural next step. If proved, this will yield reconstruction theorems for Gromov–Witten invariants via semistable reduction. A basic test for these developments would be to show that the logarithmic theory of (X, D) is determined by the absolute theory of the strata, generalizing the calculation schemes of Gathmann and Maulik–Pandharipande [Gat03, MP06]. See [BNR22, Corollary Y] for progress in genus 0.

A cousin of the theory in the present paper is *punctured* logarithmic Gromov–Witten theory, recently developed by Abramovich–Chen–Gross–Siebert [ACGS20b]. Superficially, maps to expansions are different beasts than punctured maps, but we expect that the gluing formulas will be similar. A version of the expanded theory for non-rigid targets can be constructed, which form the analogue of punctured maps; see [CN21].

The principle underlying our approach is that the category of all logarithmic stable maps, rather than the minimal or basic maps considered in the literature, can be worked with in its own right. The minimality condition amounts to the tropical moduli of a logarithmic family being as large as possible. We vary this condition by allowing the tropical moduli to be supported on faces of a refinement of the tropical moduli space. This leads to subcategories consisting of non-minimal maps that are represented over schemes by logarithmic modifications. The same ideas arise in earlier work of Santos-Parker, Wise, and the author, which is a source of inspiration [RSW19a, RSW19b]. Although non-archimedean geometry makes no overt appearance here, the ideas are parallel to Raynaud’s approach to analytic geometry via admissible formal schemes.

1.6. Inverse limits for Chow groups and moduli spaces. Our methods produce an infinite family of virtually birational moduli spaces for which a gluing formula holds. The results could be reformulated using an elegant idea of Aluffi [Alu05]. Given a logarithmic scheme or stack X , consider the category $\mathcal{C}_X$ whose objects are logarithmic modifications $X' \rightarrow X$, see [AW18, Kat94], and whose morphisms are the commuting triangles. Systems $\mathcal{C}_X$ and $\mathcal{C}_Y$ are equivalent if they contain objects with isomorphic source. The category $\mathcal{C}_X$ is equipped with a Chow group defined as the inverse limit of Chow groups of modifications and an operational Chow ring obtained from the direct limit. Virtual fundamental classes of the spaces constructed here give a class on the inverse limit, and Gromov–Witten invariants are integrals of operational Chow classes against the virtual class. The gluing formula is then an expression for the virtual class in the Chow groups of these categories. Recent work on logarithmic intersection theory by Barrott realizes these ideas [Bar18].

One could try to pass to the inverse limit of the spaces themselves. A natural category in which to do this is that of locally topologically ringed spaces. The limit resembles an object in Foster and Payne’s theory of adic tropicalizations [FP19]. Indeed, this approach was suggested in [Ran15, Remark 4.3.2]. One could also sheafify the category of logarithmic schemes in the topology of logarithmically étale modifications and formulate the gluing formula on the *valuativization*. The approaches would provide an elegant repackaging of our framework, but we prefer more widely known techniques.

1.7. Symplectic geometry and exploded manifolds. There is a rich parallel story in the symplectic category due to Li–Ruan, Ionel–Parker, Ionel, Tehrani–McLean–Zinger, and Tehrani developing Gromov–Witten theories in relative and logarithmic geometries [Ion15, IP03, IP04,

LR01, Teh17, TMZ18]. The symplectic side was motivated by considerations in Donaldson–Floer theory and predates the algebro-geometric versions.

In a series of papers, Parker has developed an approach to Gromov–Witten theory relative to normal crossings divisors, using exploded manifolds [ParB11, ParB12a]. He proves a gluing formula for Gromov–Witten invariants similar to the one we establish here [ParB17]. It appears that one can compare Parker’s category to logarithmic schemes by base changing to $\mathrm{Spec}(\mathbb{R}_{\geq 0} \rightarrow \mathbb{C})$, and then passing to the inverse limit as above. We work with the inverse system here, rather than the inverse limit. Our theory is formally consistent with exploded Gromov–Witten theory, with several parallels. The approach here is analogous to Parker’s *tropical completions*, while punctured Gromov–Witten theory is analogous to *cut curves*; see [ParB12b, Section 13]. We hope that this paper will lead to a wider understanding of exploded manifolds.

1.8. User’s guide. In the first part of the paper, we study logarithmic maps to expansions. The types of expansions we consider are obtained by polyhedral subdivisions of tropical targets, which are defined at the beginning of Section 3. The spaces of tropical maps to these expansions with the requisite transversality properties are constructed in Section 3.7 using toroidal weak semistable reduction [AK00]. These combinatorial constructions are lifted to statements about logarithmic stable maps using Artin fan techniques in Section 4.4, leading to Theorem A. We begin the second part of the paper by studying the degeneration formula in a purely combinatorial setting using extended tropicalizations. The appropriate moduli spaces of transverse maps are constructed in Section 6. Special attention is paid to the geometry of the product in Section 6.4. Finally, the combinatorial constructions are used to prove the virtual gluing formula in Section 7.3, leading to Theorem B.

Our results are stated for toroidal embeddings without self-intersection, rather than all logarithmically smooth targets. In light of advances in logarithmically étale descent, this is largely an expositional choice [ACMW17, AW18]. Logarithmic Gromov–Witten theory is insensitive to logarithmic modifications, so this presents no restrictions in applications.

1.9. Recent progress. The present paper was first circulated in 2018, and there has been significant progress in the intervening years. A parallel logarithmic theory of Donaldson–Thomas invariants has been developed, via the construction of a moduli space of higher-rank expansions [MP20]. The methods in [MP20] can be applied to construct the spaces in Theorem A in a manner that is logically independent from the foundational papers in logarithmic Gromov–Witten theory [AC14, Che14b, GS13], and this is explained by Carocci–Nabijou [CN21]. On the other hand, punctured Gromov–Witten theory gives rise to a different solution to gluing problems for logarithmic maps [ACGS20b]. The approaches appear to have complementary strengths. Additional progress has been made in logarithmic intersection theory; see [Bar18, Her19, MPS21, MP21].

Conventions. We work exclusively with fine and saturated logarithmic schemes over $\mathbb{C}$ that are locally of finite type. All cones will be rational and polyhedral, equipped with a toric monoid of positive linear functions. Given a moduli space K equipped with a virtual fundamental class, we denote this class by $[K]$. The logarithmic structure sheaf will be banished from the notation, and the word “underlying” will be used when ignoring the logarithmic structure. The symbols $\sqsubset$, $\triangleright$ and variants, being piecewise linear, will be used to denote tropical curves, in a continuing effort to popularize a convention of Abramovich.

Part I. The expanded theory

2. Curves and automorphisms

2.1. Logarithmic curves and tropicalizations. We begin by recalling the definition of a tropical curve.

DEFINITION 2.1.1. An n -marked tropical curve $\square$ or simply a tropical curve is a finite graph G with vertex and edge sets V and E , equipped with

- (1) a marking function $m: \{1, \dots, n\} \rightarrow V$,
- (2) a genus function $g: V \rightarrow \mathbb{N}$,
- (3) a length function $\ell: E \rightarrow \mathbb{R}_+$.

The genus of a tropical curve $\square$ is defined to be

$$g(\square) = h_1(G) + \sum_{v \in V} g(v),$$

where $h_1(G)$ is the first Betti number of the graph G .

The tropical curve $\square$ has a *metric realization*. First endow the graph G with its evident metric from the edge-length function. Then, for each marking i with $m(i) = v$, attach a copy of $\mathbb{R}_{\geq 0}$ to v at the point 0. The point at infinity of this unbounded edge will be understood as “the” marked point p_i .

If ℓ is allowed to take values in an arbitrary monoid, we obtain families of tropical curves.

DEFINITION 2.1.2. Let σ be a cone with dual cone S_σ . A family of n -marked tropical curves over σ is a graph G with marking and genus function as in Definition 2.1.1 and whose length function takes values in S_σ .

A point of σ is a monoid homomorphism $\varphi: S_\sigma \rightarrow \mathbb{R}_{\geq 0}$. If this homomorphism is applied to the edge length $\ell(e) \in S_\sigma$, we obtain a positive real length for each edge and thus a tropical curve.

2.2. Logarithmic curves. Let (S, M_S) be a logarithmic scheme. A family of logarithmically smooth curves over S is a logarithmically smooth, flat, and proper morphism

$$\pi: (C, M_C) \rightarrow (S, M_S)$$

with connected and reduced fibres of dimension 1; see [Kat00].

Associated with a logarithmic curve $C \rightarrow S$ is a family of tropical curves, called its *tropicalization*. Given a node of C , its deformation parameter is an element of the characteristic monoid of S .

DEFINITION 2.2.1 (The tropicalization of a log smooth curve). Let $C \rightarrow S$ be a family of logarithmically smooth curves, and assume that the underlying scheme of S is a geometric point. The *tropicalization* of C , denoted by $\square$, is obtained as follows:

- (1) The underlying graph is the marked dual graph of C equipped with the standard genus and marking functions.
- (2) Given an edge e , the generalized length $\ell(e) = \delta_e \in \overline{M}_S$ is the deformation parameter of the corresponding node of C .

2.3. Automorphisms. We use the following adjectives for curves *without a chosen logarithmic structure*. A *prestable* curve is a possibly marked, nodal curve. A prestable curve is *semistable* if each rational component of the normalization contains at least two distinguished points. A *stable curve* is one where each rational component of the normalization hosts at least three distinguished points.

We examine the infinitesimal automorphisms of logarithmically smooth curves. This is not original but only records the analysis of infinitesimal automorphisms of logarithmically smooth curves due to Santos-Parker in his thesis, advised by Wise [ParK17, Section 3]. The facts are implicit in the paper [RSW19a].

The subject of automorphisms of non-minimal logarithmic curves has not been expressly considered in the literature and is more subtle than one would initially expect. We begin with a definition [Che14b, Definition B.3.1].

DEFINITION 2.3.1. An *automorphism of a logarithmically smooth curve* $(C, M_C) \rightarrow (S, M_S)$ is a commutative diagram

$$\begin{array}{ccc} (C, M_C) & \longrightarrow & (C, M_C) \\ \downarrow & & \downarrow \\ (S, M_S) & \longrightarrow & (S, M_S) \end{array}$$

such that

- (1) the horizontal arrows are isomorphisms of logarithmic schemes,
- (2) the underlying map $S \rightarrow S$ is the identity,
- (3) the underlying map $C \rightarrow C$ is an isomorphism.

Note that the logarithmic structure on the base is allowed to change.

2.4. A first example. Let (C, p) be a smooth rational curve equipped with its divisorial logarithmic structure. The natural $\mathbb{G}_a$ -action that stabilizes p yields automorphisms of the logarithmic structure. As a result, logarithmically smooth curves containing a rational component with only one distinguished point will always have infinitely many logarithmic automorphisms.

On the other hand, if X is a semistable curve, the natural torus action coming from scaling the components does induce automorphisms of the logarithmic structure, leading to the next example.

2.5. A second example. Let C/S be a logarithmic curve over a geometric point, and let C_1 and C_2 be components of C meeting at a node. In multiplicative notation, the equation of the logarithmic structure at this node is given by $xy = t$, where t is a parameter on the base. Assume that C_1 is strictly semistable. We will analyze the circumstances in which the scaling action on C_1 may be extended to a logarithmic automorphism of C/S .

The $\mathbb{G}_m$ -action on C_1 induces a non-trivial action on the parameter x , by scaling it. In order to build a logarithmic automorphism φ , the new curve $\varphi(C)$ must also satisfy the equation $xy = t$. To achieve this, we must either scale C_2 inversely and fix t , or additionally scale t .

Assume that C_2 is stable. This rules out a possible scaling action of the parameter y to balance the action on x . The alternative is to scale t , which requires a non-trivial action on the base logarithmic structure. If the base logarithmic structure is minimal, then t is not the

smoothing parameter for any other node of C besides the one between C_1 and C_2 , and there is no obstacle to this scaling. We consider the alternative scenario: Assume that there exists a third *stable* component C_3 meeting the stable component C_2 at a node with local equation $yz = t$. In this case, scaling t does not result in an automorphism since one of either C_2 and C_3 would also have to be scaled in order to produce an automorphism of C/S . It follows that the standard $\mathbb{G}_m$ -action on C_1 does not extend to a logarithmic automorphism of C/S .

2.6. The general case. The example above illustrates a general situation. When the base logarithmic structure is not minimal, the finiteness of automorphisms is not required for the finiteness of logarithmic automorphisms.

Recall that given a prestable curve $\underline{C} \rightarrow \underline{S}$, there is a *minimal* logarithmic structure obtained by pulling back the divisorial structures on the moduli stack of prestable curves and its universal curve. By identifying the stacks of prestable curves and prestable minimal logarithmic curves, we obtain the following.

LEMMA 2.6.1. *Let C/S be a minimal family of prestable logarithmic curves. Then there is an identification*

$$\mathrm{Aut}^{\log}(C/S) \cong \mathrm{Aut}(\underline{C}/\underline{S}).$$

DEFINITION 2.6.2. A logarithmic prestable curve C/S is said to be *stable as a logarithmic curve* if it has finite logarithmic automorphism group.

To characterize such curves, we note the following.

DEFINITION 2.6.3. Let C/S be a logarithmically smooth curve, and let S be a geometric point. Given components C_1 and C_2 and a path P between them, the *length* of P is an element in the characteristic $\overline{M}_S$ given by the sum of the deformation parameters corresponding to the edges in P .

A length $\delta \in \overline{M}_S$ is *strongly stable* if it is the length of a path P between two stable components.

DEFINITION 2.6.4. Let $\delta \in \overline{M}_S$ be a section of the characteristic monoid. We say that δ is *stable* if it lies in the $\mathbb{Q}$ -span of the lengths of strongly stable paths of C . That is, there exist paths P_i with strongly stable lengths δ_i and rational numbers q_i such that

$$\delta = \sum_i q_i \delta_i \in \overline{M}_S \otimes \mathbb{Q}.$$

This notion of stability is stated for sections of $\overline{M}_S$ but certainly depends on C .

THEOREM 2.6.5 (Santos-Parker [ParK17, Proposition 6]). *Let C/S be a logarithmically smooth curve over a geometric point. Let σ_S be the dual of the monoid of the base, and let $\mathcal{M}^{\mathrm{trop}}$ be the cone of tropical curves of combinatorial type equal to that of C . The curve has trivial infinitesimal logarithmic automorphisms if and only if all of the following conditions are satisfied:*

- (A) *The underlying curve is semistable.*
- (B) *The tropical moduli map $\sigma_S \rightarrow \mathcal{M}^{\mathrm{trop}}$ is injective.*
- (C) *Every strict semistable rational component supports a node with smoothing parameter t that is stable in $\overline{M}_S$.*

Proof. We refer the reader to the proof given by Santos-Parker in [ParK17], noting that the essential strategy of the proof is to globalize the observation in the example above. $\square$

PROPOSITION 2.6.6. *Let $C \rightarrow S$ be a stable logarithmic curve over a geometric point. Let $\tilde{C} \rightarrow C$ be a logarithmic modification such that the composition $\tilde{C} \rightarrow S$ is also a family of logarithmic curves. Then $\tilde{C} \rightarrow S$ is logarithmically stable.*

Proof. We check the conditions of the characterization above. All such families arise by pulling back a subdivision on the tropicalization. It suffices to consider the subdivision at a single edge, producing $\tilde{\square}$ from $\square$. The modified curve is clearly semistable since it is obtained by subdividing the tropical curve. Let e be the subdivided edge, and first assume that e is bounded (that is, not a marked point). The edge e has endpoints v_1 and v_2 . Let v_0 be the newly formed vertex, and let δ_1 and δ_2 be the deformation parameters of the edges incident to v_0 . Since v_1 and v_2 are stable by hypothesis, the parameters δ_1 and δ_2 are stable parameters. Since v_0 supports these parameters, the curve remains stable. If e is an unbounded edge, a similar argument applies with a single new bounded edge and a single new stable parameter supported on the new vertex. $\square$

Let X be a logarithmic scheme. Given a logarithmic map from a curve, $f: C \rightarrow X$, the *logarithmic automorphisms of the map* are those automorphisms of C that commute with the map. A *stable logarithmic map* is a logarithmic map whose automorphism group is finite.

COROLLARY 2.6.7. *Let*

$$\begin{array}{ccc} C & \longrightarrow & X \\ \downarrow & & \\ S & & \end{array}$$

be a stable logarithmic map over a geometric point. Let $\tilde{C} \rightarrow C$ be a logarithmic modification such that the composition $\tilde{C} \rightarrow S$ is also a family of logarithmic curves. The resulting map $\tilde{C} \rightarrow X$ is logarithmically stable.

Remark 2.6.8 (Semistable curves). Logarithmic curves with strictly semistable components, but finite automorphism group, have appeared in the literature. The moduli spaces of radially aligned curves constructed in [RSW19a] are examples where every curve admits *canonical* rational bubbling depending on the tropical structure. A related phenomenon appears in Smyth’s genus 1 curves with Gorenstein singularities [Smy11, Corollary 2.4]. They are also a constant presence in Parker’s exploded manifolds.

3. Combinatorial semistable reduction

We construct tropical moduli stacks of maps that mimic the requirements of Theorem A. In this section, we will use Σ for a cone complex. It will play the role of the target in the moduli problem for maps, just as X plays such a role on the geometric side. We occasionally use Σ_X to denote the cone complex associated with X , in order to stress the connection to geometry.

3.1. Tropical maps: Static target. Let Σ be a rational cone complex in the sense of [KKMS73, Section 2.1]. The reader may wish to assume that Σ is a fan. Given a tropical curve $\square$, a *tropical map* is a morphism of rational cone complexes

$$F: \square \rightarrow \Sigma.$$

In other words, F is continuous and maps each polyhedron (that is, vertex or edge) of $\sqsubset$ to exactly one polyhedron in Σ . Moreover, upon restriction to any edge e of $\sqsubset$, the map F is piecewise linear. In particular, an edge e of $\sqsubset$ maps into a cone σ_e . If it is non-contracted, then its image is a segment, which lies on a line L_e in the vector space σ_e^{gp} . The induced map $e \rightarrow L_e$ has a well-defined slope up to a sign depending on the orientation of the edge. We refer to its absolute value as the *expansion factor* of F along e .

If $\sqsubset \rightarrow \Delta$ is a family of tropical curves over a cone complex Δ , a family of tropical maps is a diagram of polyhedral complexes

$$\begin{array}{ccc} \sqsubset & \xrightarrow{\quad} & \Sigma \times \Delta \\ & \searrow & \swarrow \\ & \Delta & \end{array}$$

Let e_p be a marked end of $\sqsubset$. Let $\sqsubset \rightarrow \Sigma$ be a tropical map with cone σ_p containing e_p .

DEFINITION 3.1.1. The *contact order* of p , denoted by c_p , is the scalar multiple of the primitive integral vector on the ray in σ_p parallel to the image of e_p by the slope along e_p . If e_p is contracted, declare its contact order to be zero. The contact order takes values in $\Sigma(\mathbb{N})$.

3.2. Tropical maps: Moving target. We introduce the combinatorial target expansions.

DEFINITION 3.2.1. A *tropical expansion* of Σ over Δ is a cone complex $\tilde{\Sigma}$ together with morphisms of cone complexes

$$\begin{array}{ccc} \tilde{\Sigma} & \xrightarrow{\pi} & \Sigma \\ \downarrow & & \\ \Delta & & \end{array}$$

satisfying the following conditions:

- (1) *Equidimensionality.* Every cone of $\tilde{\Sigma}$ surjects onto a cone of Δ .
- (2) *Reducedness.* The image of the lattice of any cone $\sigma \in \tilde{\Sigma}$ is equal to the lattice of the image cone $\pi(\sigma)$ in Δ .
- (3) *Modification.* For each point p of Δ , the map $\tilde{\Sigma}_p \rightarrow \Sigma$ is an embedding of a subcomplex of a polyhedral subdivision of Σ .
- (4) *Generic isomorphism.* The fibre over 0 of $\tilde{\Sigma}$, that is, $\pi^{-1}(0) \subset \tilde{\Sigma}$, maps isomorphically onto a union of faces in Σ .

The terms *equidimensionality* and *reducedness* reflect the following geometric situation. If $f: X \rightarrow B$ is a toroidal morphism of toroidal embeddings, the induced morphism F of cone complexes satisfies the conditions above if and only if the map f satisfies the corresponding algebro-geometric conditions; see [AK00, Lemmas 4.1 and 5.2]. If both are satisfied, the morphism is weakly semistable, and therefore flat [Mol21, Theorem 2.1.5].

Remark 3.2.2. If the cone complexes $\tilde{\Sigma}$, Σ , and Δ above are fans embedded in a vector space, not necessarily complete, with linear maps between them, toric geometry outputs a family $\mathcal{Y}(\tilde{\Sigma}) \rightarrow S(\Delta)$. It is a flat family of broken toric varieties over the toric base $S(\Delta)$ associated with the fan Δ . The generic fibre is the toric variety associated with the preimage of $0 \in \Delta$ of the map $\tilde{\Sigma} \rightarrow \Delta$. By hypothesis, this is an open invariant subvariety of the toric variety $Y(\Sigma)$ associated

with Σ . In general, since we do not demand that the fibres of a tropical expansion be complete decompositions, the fibres of the flat family need not be proper.

Remark 3.2.3. Given a point of Δ , the fibre in $\tilde{\Sigma}$ is a polyhedral subcomplex of a subdivision of Σ . The third condition allows us to take subdivisions and then discard higher-dimensional cones if desired. The edge lengths of the bounded cells of this subdivision vary in a piecewise-linear fashion based on the parameters in the base. An example of a family of polyhedral subdivisions over a 1-dimensional cone is in Figure 1.

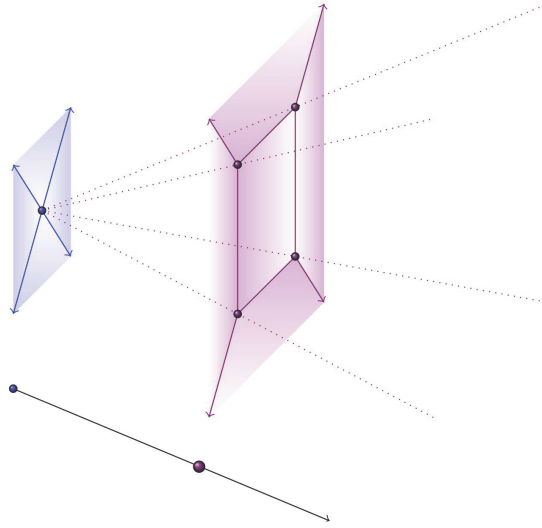


FIGURE 1. A family of tropical expanded targets over $\mathbb{R}_{\geq 0}$. The fibre over 0 is dual to a family of copies of $\mathbb{P}^1 \times \mathbb{P}^1$, while the non-trivial fibre that is shown is dual to a union of four copies of $\mathbb{P}^2$ glued along boundary curves, and all four copies of $\mathbb{P}^2$ meet at a point.

3.3. Conventions on combinatorial boundedness. A basic result in the logarithmic theory of stable maps is combinatorial finiteness. After fixing numerical invariants, there exist finitely many combinatorial types of tropical stable maps which can arise as combinatorial types of logarithmic stable maps to a fixed target [GS13, Section 3.1]. When we speak of tropical moduli stacks, we fix a finite set of combinatorial types, closed under specialization. This guarantees that the tropical stacks are of finite type. The combinatorial type here is the data comprising of the stable dual graph of the source curve, the strata to which each component maps, the curve classes of the components, and the contact orders of edges and half-edges corresponding to nodes and marked points. See [ACGS20a].

3.4. Tropical moduli stacks. Let Σ be a cone complex. Fix the discrete data $\Gamma = (g, n, \mathbf{c}, L)$, where g is the genus, n is the number of unbounded edges, $\mathbf{c}$ is the contact order of the markings, and L is a finite partially ordered set of combinatorial types, closed under specialization of types. An *automorphism* of a map $\square \rightarrow \Sigma$ is an automorphism of $\square$ commuting with the map to Σ .

We study moduli spaces of tropical maps to Σ having combinatorial type described by Γ . The moduli problem determines a combinatorial stack in the formalism of Cavalieri–Chan–Ulirsch–Wise [CCUW20, Definition 2.3.5].

Let **RPC** be the category of rational polyhedral cones. The definitions give rise to a tropical moduli functor

$$\mathrm{ACGS}_\Gamma(\Sigma) : \mathbf{RPC}^{\mathrm{op}} \rightarrow \mathbf{Groupoids}$$

by associating with a cone τ the groupoid of tropical prestable maps $\square \rightarrow \Sigma$ over τ with discrete data Γ . The following is a modest adaptation of the results in [CCUW20].

LEMMA 3.4.1 ([CCUW20, Adaptation of Theorem 1]). *There is a stack $\mathrm{ACGS}_\Gamma(\Sigma)$ over the category of cone complexes representing the fibred category over cones of tropical stable maps to Σ with discrete data Γ .*

3.5. Combinatorial transversality. A basic transversality statement motivates what follows. Let X be a toroidal embedding and $C \rightarrow X$ a logarithmic stable map. Consider its tropicalization

$$\square \rightarrow \Sigma_X.$$

We seek a tropical condition for the underlying map $C \rightarrow X$ to be transverse in the following sense: (i) the complement of the marked points $C^\circ \subset C$ maps into the open stratum of X , and (ii) the marked points map to the complement of the closed codimension 2 strata in X . This holds precisely when the image of every vertex of $\square$ is the unique vertex 0 of Σ_X and the image of every unbounded edge is a ray of Σ_X . For a more general map, vertices and edges of $\square$ may map into high-dimensional cones of Σ_X . Geometrically, these are situations where components and nodes of C map into high-codimension boundary strata of X .

Toroidal blowups of X , which are *conical* subdivisions of Σ_X , do not improve the situation. However, given a general map $\square \rightarrow \Sigma_X$, if we are allowed to make *polyhedral* subdivisions that are not necessarily conical, the image of every vertex can be arranged to be a vertex, and the image of every edge an edge. We refer to this as *combinatorial transversality*. Such polyhedral subdivisions determine expansions of the target. A reader who is unfamiliar with such expansions may wish to consult [NS06, Section 3] or [Gub13, Sections 6 and 7]. A discussion of subdivisions may be found in [AW18, Kat94, KKMS73].

3.6. The construction on standard points. The main construction is a more delicate version of the following simple “pointwise” construction. Given a single tropical stable map $\square \rightarrow \Sigma$, viewing $\square$ as a metric graph, one can choose a polyhedral – not necessarily conical – subdivision $\tilde{\Sigma}$ such that the image of $\square$ is the support of a polyhedral subcomplex; see Figure 2. Once this subdivision is made, the map $\square \rightarrow \tilde{\Sigma}$ may not be polyhedral. To remedy this, a subdivision to $\square$ can be made by adding vertices to $\square$ at the preimages of the vertices of $\tilde{\Sigma}$.

Remark 3.6.1. There always exists a subdivision of the target to make a map transverse. Fix a tropical map $\square \rightarrow \Sigma$. Replace Σ by a conical subdivision so that it becomes simplicial. Since the question is local, it suffices to deal with the case where Σ is $\mathbb{R}_{\geq 0}^n$. For each of the n projections to $\mathbb{R}_{\geq 0}$, choose a polyhedral subdivision T_i such that the image of every vertex is a vertex and the image of every non-contracted bounded edge is a union of bounded edges. The product of the T_i yields a cubical subdivision of $\mathbb{R}_{\geq 0}^n$ where the image of every vertex is a vertex. The image of an edge is the diagonal in a facet connecting two edges. By further triangulating, we guarantee that the image of each edge is an edge in the subdivision, as required. By passing to the 1-skeleton of this subdivision, and if necessary discarding unnecessary vertices and edges, we obtain a non-complete polyhedral subdivision of Σ .

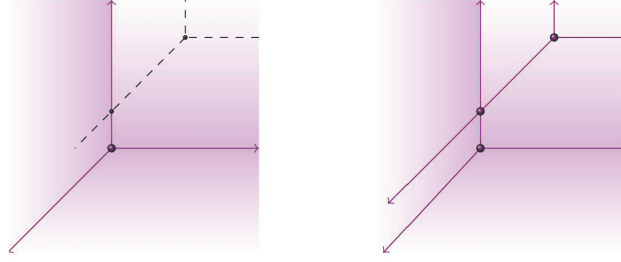


FIGURE 2. The dashed figure on the left is a tropical curve in the fan $\Delta_{\mathbb{P}^2}$. It is clearly not combinatorially transverse because the vertices and edges of the tropical curve do not map onto polyhedral subcomplexes of the target. There is a natural subdivision of this on the right, obtained by introducing new vertices and edges. The tropical curve on the right is not drawn on the right, but it is the one obtained by composition with the subdivision.

Geometrically, this procedure expands the target to accommodate a given one-parameter family of curves with smooth general fibre. The same is true if the base is a finite-rank valuation ring or, more generally, a valuativized logarithmic scheme. In these cases, the new families are guaranteed to be flat. Over a general base, this ceases to be true, and the base must be modified to ensure flatness.

3.7. The general case. We establish the tropical version of Theorem A by globalizing the construction above. The tropical moduli stack comes with a curve and map diagram:

$$\begin{array}{ccc} \square & \xrightarrow{F} & \text{ACGS}(\Sigma) \times \Sigma \\ & \searrow & \swarrow \\ & \text{ACGS}(\Sigma) & \end{array} \quad (1)$$

The construction will subdivide the base, curve, and target to obtain a universal family of stable maps to a family of expansions of Σ such that the universal map is transverse to the universal target.

THEOREM 3.7.1. *There exists a diagram of cone stacks*

$$\begin{array}{ccc} \square^\lambda & \xrightarrow{F^\lambda} & \tilde{\Sigma}^\lambda \\ & \searrow & \swarrow \\ & K_F^\lambda(\Sigma) & \end{array}$$

where each position in the diagram is a subdivision of the corresponding position in diagram (1). Furthermore, we have the following properties:

- (A) *Equidimensionality.* For both vertical arrows, every cone of the source surjects onto a cone of $K_F^\lambda(\Sigma)$.
- (B) *Reducedness.* For both vertical arrows, the image of the lattice of any cone σ is equal to the lattice in the image cone in $K_F^\lambda(\Sigma)$.
- (C) *Transversality.* The image of F^λ is a union of faces of $\tilde{\Sigma}^\lambda$.

Proof. The basic idea of the proof is to perform weak semistable reduction for the universal map

at a combinatorial level, following [AK00, Sections 4 and 5]. We guidepost the proof based on the intended geometric property that each step reflects.

Step I: Target expansion. The morphism

$$F: \square \rightarrow \text{ACGS}(\Sigma) \times \Sigma$$

is linear on each cone of the source. Given a cone τ of $\square$, the image $F(\tau)$ is a conical subset of $K_\Gamma(\Sigma) \times \Sigma$. Choose a subdivision of $\tilde{\Sigma}^{\text{aux}}$ of $\text{ACGS}(\Sigma) \times \Sigma$ such that the image of every cone of $\square$ is a union of cones. This can be done using a procedure developed by Abramovich and Karu; see [AK00, Lemma 4.3]. First assume that $\text{ACGS}(\Sigma) \times \Sigma$ consists of a single maximal cone σ . Then the image $F(\tau)$ is a strictly convex conical subset of σ . This subset is equal to the intersection of half-spaces given by linear functions $\ell_{\tau,j}$. The function $\psi_\sigma = -\sum_{\tau,j} |\ell_{\tau,j}|$ is piecewise linear, and the domains of linearity define a subdivision of σ such that τ maps to a cone.

Once a subdivision of σ is made such that the image of a cone is a union of cones, a further refinement does not affect the property. Repeat this procedure for all cones mapping to σ , thus defining a subdivision where the image of every cone is a *union* of cones. Now choose an arbitrary piecewise-linear function $\bar{\psi}_\sigma$ on the full target $\text{ACGS}(\Sigma) \times \Sigma$ extending the function ψ_σ constructed above. Consider the piecewise-linear function

$$\psi = \sum_{\sigma} \bar{\psi}_\sigma.$$

The domains of linearity of ψ give the requisite subdivision such that the image of every cone of the curve is a union of cones of the target. Call this subdivision of the target $\tilde{\Sigma}^{\text{aux}}$.

Step II: Transversality. Pull back the piecewise-linear function ψ to $\square$. Its bending locus yields a subdivision of the source curve, and the resulting map sends cones onto cones. Pass to the subcomplex $\tilde{\Sigma}_\circ^{\text{aux}}$ of $\tilde{\Sigma}^{\text{aux}}$ that contains the images of the cones of $\square^{\text{aux}}$. The newly constructed morphism of cone complexes

$$F^{\text{aux}}: \square^{\text{aux}} \rightarrow \tilde{\Sigma}_\circ^{\text{aux}}$$

has the property that the image of every cone of $\square^{\text{aux}}$ is a cone of $\tilde{\Sigma}_\circ^{\text{aux}}$, and the map is surjective.

Step III: Fixing the base. We have thus far ignored the base. The modification $\tilde{\Sigma}_\circ^{\text{aux}}$ maps surjectively onto $K_\Gamma(\Sigma)$. Indeed, the map factors through the product $\text{ACGS}_\Gamma(\Sigma) \times \Sigma$. Another application of [AK00, Lemma 4.3 and Proposition 4.4] yields subdivisions of the target and base

$$\tilde{\Sigma}_\circ^\lambda \rightarrow K_\Gamma^\lambda(\Sigma)$$

such that the image of every cone of the source is a cone of the target. Pull back this subdivision of $\tilde{\Sigma}_\circ^\lambda \rightarrow \tilde{\Sigma}_\circ^{\text{aux}}$ to the curve to obtain $\square^\lambda$. The image of any cone of $\square^\lambda$ is a cone in $K_\Gamma^\lambda(\Sigma)$ by the commutativity of the diagram. Finally, by refining the lattice on the base, we guarantee the reducedness condition in the statement. $\square$

In the transversality step, we passed from $\tilde{\Sigma}_\circ$ to the subcomplex $\tilde{\Sigma}_\circ^{\text{aux}}$. The step can be skipped without significant consequences, but its effect here is that we produce a non-proper target expansion that nonetheless contains the image of the curve.

Remark 3.7.2. The proof above makes no reference to the economics of the chosen subdivisions. For computational purposes, particular choices may be advantageous. A systematic understanding is likely to prove important in later work.

3.8. Smooth divisor: Li from ACGS. We specialize the construction to the case considered traditionally, where D is a smooth divisor. Combinatorially, the fan Σ is $\mathbb{R}_{\geq 0}$. At each point of $\text{ACGS}(\mathbb{R}_{\geq 0})$, there is a piecewise-linear map

$$F: \square \rightarrow \mathbb{R}_{\geq 0}.$$

For each vertex v in $\square$, its image $F(v)$ is generally not the vertex of $\mathbb{R}_{\geq 0}$. The images of all vertices of $\square$ form the vertices in a unique polyhedral subdivision T of $\mathbb{R}_{\geq 0}$. Geometrically, this canonically determines an expansion of the target in the traditional sense, by pulling back the subdivision along the tropicalization. However, the map $\square \rightarrow \mathsf{T}$ no longer maps cones to cones; see Figure 3. To remedy this, the subdivision may be pulled back to $\square$ to obtain a subdivision of the source curve.

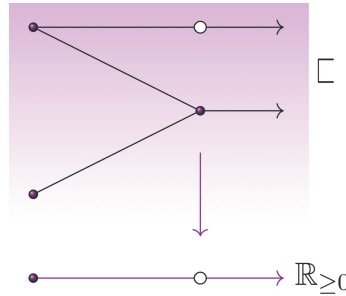


FIGURE 3. Ignoring the white vertices, the map in the figure above is a tropical map arising from points in the tropicalization ACGS space. Subdivision of the target is the expansion arising in Jun Li’s space. The subdivision of the source is made by marking the preimages of the newly introduced white vertex.

The curve and target families are no longer equidimensional over the base because the newly introduced vertices in the curve (or the target) form a cone in the total space of the curve (or the target) over $\text{ACGS}(\mathbb{R}_{\geq 0})$. The image of this cone is not necessarily a cone in the base. The subdivision of the base produces the subdivision $\text{Li} \rightarrow \text{ACGS}$. This combinatorics is analyzed in some detail in [CMR17, Proposition 16, Example 17]. See also [GS13, Section 6].

Remark 3.8.1. With hindsight, the paper of Chen on logarithmic stable maps may be viewed in different light, as reversing the construction above [Che14b]. Assume the existence of the moduli space Kim of logarithmic maps to expansions of the universal target $[\mathbb{A}^1/\mathbb{G}_m]$, as in [Kim10]. Then we may blow down the universal expanded target family to the constant family $[\mathbb{A}^1/\mathbb{G}_m]$ and partially stabilize the source curve, over Kim . Observe that the universal map continues to be logarithmic, with the pushforward logarithmic structure [AMW14, Appendix B]. This gives a new logarithmic source and target family over Kim . We observe that combinatorial types in Kim come with a partial ordering, where two otherwise identical combinatorial types are distinct if the images of two vertices in $\mathbb{R}_{\geq 0}$ have different orders. By forgetting this partial ordering, we obtain a collapsing of some faces in Kim , and consequently a blowdown map $\text{Kim} \rightarrow \text{Chen}$, where the universal curve continues to be flat. This outlines why the space of logarithmic maps to the universal target, without expansions, is representable by a stack. Of course, it is unlikely that the spaces would have been discovered in this way to begin with.

3.9. Auxiliary combinatorial choices. The procedure in the proof of Theorem 3.7.1 produces infinitely many diagrams satisfying the requirements of the theorem. Indeed, one can perform an

arbitrary further refinement of the base moduli space and pull back the source and target families to obtain a new moduli space satisfying the same conditions. Given any two such solutions to this moduli problem, there is a common subdivision of all parts of the diagram.

LEMMA 3.9.1. *Let $K_\Gamma^1(\Sigma)$ and $K_\Gamma^2(\Sigma)$ be two diagrams, equipped with universal families, arising in the result above. Then there exists a third diagram $(K_\Gamma^\nu(\Sigma), \square^\nu, \tilde{\Sigma}^\nu)$ together with subdivisions*

$$\text{sub}_i: K_\Gamma^\nu(\Sigma) \rightarrow K_\Gamma^i(\Sigma)$$

that are natural subdivisions of the pullbacks

$$\square^\nu \rightarrow \text{sub}_i^{-1} \square^i$$

and

$$\tilde{\Sigma}^\nu \rightarrow \text{sub}_i^{-1} \tilde{\Sigma}^i.$$

Proof. The existence of $\text{sub}_i: K_\Gamma^\nu(\Sigma) \rightarrow K_\Gamma^i(\Sigma)$ is straightforward. After pulling back the universal tropical curve and target from the two spaces, further subdivisions can be made. The equidimensionality procedure described in the proof of the previous theorem constructs the requisite diagram. $\square$

4. Virtual semistable reduction

Recall that (X, D) is a toroidal pair without self-intersection. In this section, we establish Theorem A. The moduli spaces $K_\Gamma^\lambda(\cdot)$ are first constructed when the target is the Artin fan of X , and then for X itself in Section 4.4. The transversality is established in Lemma 4.4.8. The virtual class is discussed in Section 4.5.

The procedure in this section has been used before, for instance in [AW18, CMR17, Ran17]. When the moduli space of maps is logarithmically unobstructed, for instance when considering genus 0 logarithmic maps to toric varieties, subdivisions of the tropical moduli spaces produce toroidal modifications of the geometric moduli spaces. If (X, D) is replaced by its Artin fan $\mathcal{A}_X$, the moduli space of maps becomes a toroidal embedding in the smooth topology. This toroidal embedding maps to an associated cone stack. Subdivisions from the previous section are applied to this cone stack, and then pulled back to maps X itself. We now formalize the idea.

4.1. Artin fans. A toroidal embedding without self-intersection (X, D) determines an associated cone complex [KKMS73]. At every point of X , there is a Zariski open set $U \subset X$ together with an étale map $U \rightarrow V_{\sigma(p)}$, where $V_{\sigma(p)}$ is an affine toric variety with cone $\sigma(p)$. The association $p \mapsto \sigma(p)$ is constant on the locally closed toroidal strata of X . By considering the specialization relations among generic points of strata, these can be glued to form a cone complex Σ_X . When the boundary of X is simple normal crossings, this is the cone over the dual complex of the boundary divisor D .

The cone complex Σ_X determines a 0-dimensional algebraic stack possessing the same combinatorial properties as X , referred to as the *Artin fan of X* . For a general construction, we refer the reader to [ACMW17, AW18, CCUW20, Uli19] and their precursor [Ols03]. In our case, the following will suffice.

The cone complex Σ_X above is a colimit of a diagram $\mathcal{D}_X$ of cones [ACP15, Remark 2.2.1]. For each cone σ in the diagram, there is a canonically associated toric stack $[V_\sigma/T]$, where T is the dense torus of V_σ . The diagram $\mathcal{D}_X$ immediately yields a diagram of algebraic stacks, whose

colimit exists as an algebraic stack is called the *Artin fan*. There is a canonical morphism

$$X \rightarrow \mathcal{A}_X.$$

This morphism is strict: the logarithmic structure on X is the pullback of the divisorial logarithmic structure on $\mathcal{A}_X$. The category of cone complexes is equivalent to the category of Artin fans without monodromy [Uli19, Proposition 3.3].

4.2. Toroidal modifications. Given a logarithmic scheme or stack, a *logarithmic modification*, sometimes called a toroidal modification, is a proper, birational, logarithmically étale morphism $X' \rightarrow X$. If X is a toroidal embedding, such modifications $X' \rightarrow X$ correspond in a functorial manner to polyhedral refinements $\Sigma' \rightarrow \Sigma$ of the corresponding cone complexes, and thus to logarithmic modifications of the associated Artin fans. See [ACMW17, Section 3.1].

4.3. Maps to the Artin fan. The primary purpose of working with Artin fans is that it is natural to map algebraic varieties to them.

DEFINITION 4.3.1. A *logarithmic prestable map to $\mathcal{A}_X$ over S* is a diagram of logarithmic stacks

$$\begin{array}{ccc} \mathcal{C} & \longrightarrow & \mathcal{A}_X \\ \downarrow & & \\ S, & & \end{array}$$

where $\mathcal{C}$ is a family of logarithmically smooth curves.

Fix discrete data Γ comprising the genus g , number of marked points n , and contact orders $\mathbf{c}$. The logarithmic tangent bundle of $\mathcal{A}_X$ is the zero sheaf, so logarithmic deformations of maps from curves are unobstructed over the moduli space of curves. This leads to the following result of Abramovich and Wise [AW18, Proposition 1.6.1].

PROPOSITION 4.3.2. *There exists a logarithmic algebraic stack $\mathfrak{M}_\Gamma(\mathcal{A}_X)$ which is toroidal in the smooth topology, of dimension $3g - 3 + n$, and represents the fibred category over logarithmic schemes of logarithmic maps to $\mathcal{A}_X$. The forgetful morphism to $\mathfrak{M}_{g,n}$ is logarithmically étale and birational.*

As constructed, the stack is only locally of finite type. Indeed, a logarithmic map $C \rightarrow \mathcal{A}_X$ over a logarithmic base S is equivalent to the data of C and a map $\square \rightarrow \Sigma_X$. Without a balancing condition on the maps, the problem is unbounded. However, the space of *stable* logarithmic maps $\text{ACGS}_\Gamma(X)$ is of finite type. We simply replace $\mathfrak{M}_\Gamma(\mathcal{A}_X)$ with any open substack that is of finite type and receives a morphism from $\text{ACGS}_\Gamma(X)$.

Remark 4.3.3. If Σ is a fan embedded in a vector space, the map $\square \rightarrow \Sigma$ can be required to be balanced. That is, we demand that the sum of the directional derivatives at every point of $\square$ be 0. It has been shown that there are only finitely many combinatorial types; this can be seen via a Newton polygon argument [NS06, Proposition 2.1]. In the case considered here, Σ can be embedded in a vector space, but tropicalizations of maps $C \rightarrow X$ need not be balanced. However, a modified balancing condition does hold, and this is used to establish boundedness in [GS13, Theorem 3.8]. Boundedness holds in greater generality [ACMW17, Section 1.1].

Let $\text{ACGS}_\Gamma(\mathcal{A}_X)$ be the resulting finite-type stack of logarithmic maps to $\mathcal{A}_X$. By enumerating the combinatorial types that are relevant to X , which we know are finite by boundedness,

we obtain the analogous tropical moduli stack $\text{ACGS}_\Gamma(\Sigma)$, where Σ is the cone complex of X . The moduli problem $\text{ACGS}_\Gamma(\Sigma)$ on cones from Section 3 can be turned into a moduli problem on logarithmic schemes, as follows. Given a logarithmic scheme (S, M_S) that is locally of finite type, we consider the moduli problem of tropical stable maps with discrete data Γ over S^{trop} . This extends the tropical moduli functor to logarithmic schemes; see [CCUW20, Part II].

By an insight of Gross and Siebert [GS13, Section 1], a family of logarithmic maps to $\mathcal{A}_X$ over S gives rise to a family of tropical stable maps to Σ over S^{trop} . Since $\text{K}_\Gamma(\Sigma)$ is universal for this moduli problem, we obtain a morphism

$$\text{trop}: \text{ACGS}_\Gamma(\mathcal{A}_X) \rightarrow \text{ACGS}_\Gamma(\Sigma).$$

The notion of a morphism from the logarithmic algebraic stack on the left to the cone stack on the right is explained in detail in [CCUW20]. Briefly, the authors prove that cone stacks form an equivalent 2-category to Artin fans, and the tropicalization map is constructed by using this categorical equivalence. In that context, the following proposition is essentially identical to [CCUW20, Theorem 4].

PROPOSITION 4.3.4. *The morphism $\text{trop}: \text{ACGS}_\Gamma(\mathcal{A}_X) \rightarrow \text{ACGS}_\Gamma(\Sigma)$ is surjective, smooth, and strict.*

Proof. The morphism is surjective by construction. By appealing to [GS13, Section 1], the minimal monoids of $\text{ACGS}_\Gamma(\mathcal{A}_X)$ are identified with the monoids dual to the tropical moduli space, so strictness is a tautology. Smoothness follows from the logarithmic smoothness of $\text{ACGS}_\Gamma(\mathcal{A}_X)$. $\square$

4.4. Spaces of maps from the subdivisions. With the tropicalization map handy, we pull back the subdivisions constructed in Section 3.7. These spaces will satisfy the transversality hypothesis in Theorem A.

Let X be a toroidal embedding without self-intersections, and let Σ be its cone complex. Let S be a logarithmic scheme with tropicalization Δ .

DEFINITION 4.4.1. Given a tropical expansion $\tilde{\Sigma}$ of Σ over Δ in the sense of Definition 3.2.1, the associated *logarithmic expansion* is obtained as the logarithmic modification

$$\tilde{\mathcal{X}} := (X \times S) \times_{\Sigma \times \Delta} \tilde{\Sigma} \rightarrow S.$$

The reader might be perturbed that the definition above has a logarithmic scheme on one side and a cone complex on the other – this is consistent with the formalism introduced in [CCUW20]. In practice, we pass to the associated Artin fan on the cone complex side and take the fibre product in fine and saturated logarithmic schemes.

As discussed previously, the family $\tilde{\mathcal{X}} \rightarrow S$ has reduced and equidimensional fibres by construction. In fact, it is also flat [Mol21, Theorem 2.1.5]. We construct moduli spaces of curves mapping to expansions of X .

Fix a tropical moduli diagram

$$\begin{array}{ccc} \square^\lambda & \xrightarrow{F^\lambda} & \tilde{\Sigma}^\lambda \\ & \searrow & \swarrow \\ & \text{K}_\Gamma^\lambda(\Sigma) & \end{array}$$

as constructed in Theorem 3.7.1. The subdivision $\text{K}_\Gamma^\lambda(\Sigma)$ is a subcategory of the fibred category $\text{ACGS}_\Gamma(\Sigma)$. By pulling back $\text{K}_\Gamma(\mathcal{A}_X)$ via the tropicalization map, we obtain a subcategory

$K_\Gamma^\lambda(\mathcal{A}_X)$. We similarly obtain a universal curve family $\mathcal{C}^\lambda$ and a target family $\mathcal{A}_X^\lambda$.

LEMMA 4.4.2 (Representability). *The categories $K_\Gamma^\lambda(\mathcal{A}_X)$, $\mathcal{C}^\lambda$, and $\mathcal{A}_X^\lambda$ are representable by algebraic stacks with logarithmic structure.*

Proof. Each of the categories is obtained locally by pulling back toric modifications, and possibly root constructions, of toric varieties, so representability is immediate. Note that the root constructions have the effect of adding finite isotropy groups to the strata and correspond to the passage to a finite-index sublattice at the level of cone stacks. The root constructions are required to ensure that the curve $\mathcal{C}^\lambda$ is reduced over the moduli space. See for instance [Mol21, Section 3] for a discussion. $\square$

PROPOSITION 4.4.3. *The morphism $\mathcal{C}^\lambda \rightarrow K_\Gamma^\lambda(\mathcal{A}_X)$ is a flat family of logarithmically smooth curves. The Artin fan target family $\mathcal{A}_X^\lambda \rightarrow K_\Gamma^\lambda(\mathcal{A}_X)$ is also flat.*

Proof. Given the modification $K_\Gamma^\lambda(\mathcal{A}_X) \rightarrow \text{ACGS}(\mathcal{A}_X)$, we may pull back the universal curve to obtain $\mathcal{C} \rightarrow K_\Gamma^\lambda(\mathcal{A}_X)$. This family is certainly flat since it is obtained by pulling back a flat family. The new universal curve is a modification $\mathcal{C}^\lambda \rightarrow \mathcal{C}$, and we wish to show that the composition $\mathcal{C}^\lambda \rightarrow K_\Gamma^\lambda(\mathcal{A}_X)$ remains flat. This can be checked using the polyhedral criteria [AK00, Sections 4 and 5], which are satisfied by construction. The same holds for the target family. $\square$

For properties relating to characteristic sheaves, X and $\mathcal{A}_X$ are indistinguishable [AW18, Lemma 4.1].

LEMMA 4.4.4. *The morphism $\text{ACGS}_\Gamma(X) \rightarrow \text{ACGS}_\Gamma(\mathcal{A}_X)$ is strict.*

We obtain the analogous variants of actual stable maps to expansions of X , denoted by $K_\Gamma^\lambda(X)$.

COROLLARY 4.4.5. *The curve and target families $\mathcal{C}^\lambda \rightarrow K_\Gamma^\lambda(X)$ and $\mathcal{X}^\lambda \rightarrow K_\Gamma^\lambda(X)$ are flat.*

Proof. The curve is pulled back from the space of maps to the Artin fan, so its flatness is clear. Similarly, over $K_\Gamma^\lambda(\mathcal{A}_X)$, there is a modification

$$\mathcal{A}_X^\lambda \rightarrow \mathcal{A}_X \times K_\Gamma^\lambda(\mathcal{A}_X).$$

The family $\mathcal{A}_X^\lambda$ is flat over $K_\Gamma^\lambda(\mathcal{A}_X)$ by construction. We also have the constant target family $X \times K_\Gamma^\lambda(\mathcal{A}_X) \rightarrow K_\Gamma^\lambda(\mathcal{A}_X)$. The modification yields $\mathcal{X}^\lambda \rightarrow K_\Gamma^\lambda(\mathcal{A}_X)$, which is the composition of a strict logarithmically smooth morphism $\mathcal{X}^\lambda \rightarrow \mathcal{A}_X^\lambda$ and a flat morphism, and therefore remains flat. The target family on $K_\Gamma^\lambda(X)$ is pulled back from the space of maps to the Artin fan, so it is flat. $\square$

LEMMA 4.4.6 (Properness). *The fibred category $K_\Gamma^\lambda(X)$ is representable by a proper Deligne–Mumford stack with logarithmic structure.*

Proof. The analogous stack $K_\Gamma^\lambda(\mathcal{A}_X)$ is algebraic by Lemma 4.4.2. The morphism

$$K_\Gamma^\lambda(\mathcal{A}_X) \rightarrow \text{ACGS}_\Gamma(\mathcal{A}_X)$$

is proper since it is pulled back from a subdivision. Therefore, $K_\Gamma^\lambda(X)$ is a proper modification of the proper Deligne–Mumford stack $\text{ACGS}_\Gamma(X)$ and therefore is representable and proper. $\square$

The subdivision may add arbitrary numbers of bubbles to the curve that map to an expansion of the target. Logarithmic stability is unaffected.

LEMMA 4.4.7 (Logarithmic stability). *Given a point of the logarithmic algebraic stack $\mathcal{K}_F^\lambda(X)$, the corresponding map*

$$C^\lambda \rightarrow \mathcal{X}^\lambda \rightarrow X$$

has finite logarithmic automorphism group.

Proof. This follows immediately from Corollary 2.6.7. $\square$

The stacks inherit the requisite transversality. Let $C^\lambda \rightarrow \mathcal{A}_X^\lambda \rightarrow \mathcal{A}_X$ be a point of $\mathcal{K}_F^\lambda(\mathcal{A}_X)$.

LEMMA 4.4.8 (Transversality). *Let p_i be a marked point of C^λ with contact order c_{ij} along the divisor $D_j \subset \mathcal{A}_X$ and zero contact order with the remaining divisors. Let D_j^λ be the stratum of the target $\mathcal{X}^\lambda$ containing the image of p_i . Then C^λ meets D_j at finitely many points, each marked, and the order of tangency at p_i is equal to c_{ij} .*

Proof. Consider a geometric point of the algebraic stack $\mathcal{K}_F^\lambda(\mathcal{A}_X)$. If the stalk of the characteristic sheaf at this point is a monoid P , then by pullback, we obtain a logarithmic map to $\mathcal{A}_X$ over $\text{Spec}(P \rightarrow \mathbb{C})$. The dual cone σ_P is a cone of $\mathcal{K}_F^\lambda(\Sigma)$. Let m denote the dimension of the cone σ_P . We examine the associated family of tropical maps $F: \square^\lambda \rightarrow \Sigma^\lambda \rightarrow \Sigma$. Let G be the underlying graph of $\square$. Every vertex v of G determines an m -dimensional cone of the cone complex $\square$. Note that this cone has relative dimension 0 over $\mathcal{K}_F^\lambda(\Sigma)$. Its image in Σ is also an m -dimensional cone, and therefore the generic point of the component C_v dual to v maps to the generic point of the component of $\mathcal{A}_X^\lambda$ dual to $F(v)$. Similarly, the edges of $\square$ which are dual to nodes and marked points map to either m - or $(m+1)$ -dimensional cones of Σ^λ , which are dual to either components or double intersections of components $\mathcal{A}_X^\lambda$. As a consequence, the components of C^λ map to components of $\mathcal{A}_X^\lambda$, and the nodes of C^λ map to double intersections of these components. Since all points of non-trivial contact order are marked, therefore distinct, the contact orders are as prescribed. $\square$

We detail the stable limit of a family of transverse curves in the expanded theory in the simplest non-trivial geometry. A discussion in the non-expanded theory may be found in [Ran17, Section 4].

Example 4.4.9. We work with the target $X = \mathbb{P}^2$ and fix homogeneous coordinates x, y, z . The logarithmic structure is given by

$$D = D_1 \cup D_2 = \{x = 0\} \cup \{y = 0\}.$$

Take the curve class to be that of a line in $\mathbb{P}^2$. Curves will carry two marked points p_1 and p_2 , and the contact order of p_1 with D_1 is 1 and with D_2 is 0. Similarly, the contact order of p_2 with D_2 is 1 and with D_1 is 0.

Work over the discretely valued field $\mathbb{C}((t))$, and examine the family of logarithmic stable maps given by

$$\mathbb{V}(x + y + tz) \subset \mathbb{P}^2.$$

The marked points are determined by the intersections with D_1 and D_2 , and the maps are inclusions. The flat limit of this embedded family of curves over $\text{Spec } \mathbb{C}[[t]]$ in the dual projective space is the line $\mathbb{V}(x + y)$. Observe that this limit would violate the contact order condition since this curve passes through $D_1 \cap D_2$. In the moduli space $\text{ACGS}(\mathbb{P}^2)$, the limiting curve has two source components, each isomorphic to $\mathbb{P}^1$, joined at a node. The limiting map is

$$C_0 \cup C_1 \rightarrow \mathbb{P}^2,$$

where the curve C_0 maps isomorphically onto $\mathbb{V}(x + y)$ and the component C_1 carries both markings and is collapsed to $[0, 0, 1]$.

In the expanded theory, the target expansion is constructed by blowing up

$$\mathbb{P}^2 \times \operatorname{Spec} \mathbb{C}[[t]] \rightarrow \operatorname{Spec} \mathbb{C}[[t]]$$

at the point $([0, 0, 1], 0)$. The special fibre is an expansion

$$p: \mathbb{F}_1 \cup \mathbb{P}^2 \rightarrow \mathbb{P}^2,$$

obtained as a union of $\mathbb{P}^2$ blown up at $[0, 0, 1]$ and a copy of $\mathbb{P}^2$. The polyhedral decomposition corresponding to this target degeneration is given in Figure 4 below. The exceptional divisor E attaches to a line in $\mathbb{P}^2$. The map p restricts to a proper surjective map on $\mathbb{F}_1$ and a constant map on $\mathbb{P}^2$. The limiting stable map is

$$C_0 \cup C_1 \rightarrow \mathbb{F}_1 \cup \mathbb{P}^2 \rightarrow \mathbb{P}^2,$$

with the same source curve as before. The composite map to $\mathbb{P}^2$ is also the one described earlier. We describe the map to the expansion. The curve C_0 maps isomorphically onto the strict transform in $\mathbb{F}_1$ of the curve $\mathbb{V}(x + y)$ in $\mathbb{P}^2$. The curve C_1 maps to a line in $\mathbb{P}^2$, with p_1 and p_2 mapping to distinct toric boundary curves of this $\mathbb{P}^2$. The node of the curve maps to the double divisor in the reducible target.

The unexpanded logarithmic map, the minimal logarithmic structure on the base, has characteristic $\mathbb{N}$. Equivalently, the tropical moduli space for curves of this type is $\mathbb{R}_{\geq 0}$. Since this fan admits no non-trivial conical subdivisions, the expanded space and the logarithmic mapping space, which a priori differ by toroidal modification, are isomorphic in a neighbourhood of this point. It is only the universal families that differ.

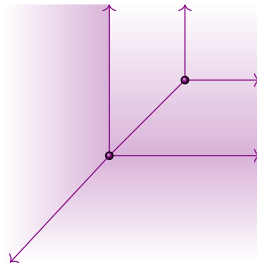


FIGURE 4. The cone over the fan pictured above gives rise to a toric degeneration of $\mathbb{P}^2$ over $\mathbb{A}^1$. The fibre over 0 is the degeneration described above. In the discussion above, the component C_0 maps to the target component dual to the central vertex, while C_1 maps to the component dual to the vertex on the upper right.

4.5. The virtual class. We have constructed a Deligne–Mumford stack $K_{\Gamma}^{\lambda}(X)$. There are three ways of constructing a virtual fundamental class.

4.5.1 The first way. As a fibred category over logarithmic schemes, $K_{\Gamma}^{\lambda}(X)$ can be identified with a subcategory of $\operatorname{ACGS}_{\Gamma}(X)$, consisting of those logarithmic stable maps to (an unexpanded) X over S such that the tropical moduli map

$$S^{\operatorname{trop}} \rightarrow \operatorname{ACGS}(\Sigma)$$

factors through the subcomplex $K_F^\lambda(\Sigma)$. Restrict the universal family to this subcategory to obtain a curve family $\mathcal{C} \rightarrow K_F^\lambda(X)$ and a morphism $\mathcal{C} \rightarrow X$. The relative obstruction theory of the map

$$K_F^\lambda(X) \rightarrow K_F^\lambda(\mathcal{A}_X)$$

is perfect. Indeed, the argument for this is identical to the one explained in [AW18, Section 6.1]. Given an S -point of $K_F^\lambda(X)$, let S' be a strict infinitesimal extension, given by an ideal I . Consider the diagram of lifts below:

$$\begin{array}{ccccc} & & f & & \\ & & \curvearrowright & & \\ & & & & X \\ & & & \nearrow & \downarrow \\ C & \longrightarrow & C' & \longrightarrow & \mathcal{A}_X \\ \downarrow & & \downarrow & & \\ S & \longrightarrow & S' & & \end{array}$$

Since the extension is strict, the tropical data of C/S and C'/S' are identical. The map $C \rightarrow \mathcal{A}_X$ extends automatically to C' . Lifts of these data are a torsor on C of abelian groups $f^*T_X^{\log} \otimes I$. We thus obtain an obstruction theory for $K_F^\lambda(X)$ over $K_F^\lambda(\mathcal{A}_X)$, using the formalism of [Wis11, Section 5]. The virtual pullback of the fundamental class under the morphism $K_F^\lambda(X) \rightarrow K_F^\lambda(\mathcal{A}_X)$ yields a virtual fundamental class on $K_F^\lambda(X)$ by [Man12].

4.5.2 The midway. The first way ignores any subdivision that has been done to the universal family. That is, both curve and target are un bubbled. We may introduce bubbles in the curve while keeping the target and the obstruction theory unchanged.

The universal modified curve admits a contraction

$$\text{st}: \mathcal{C}^\lambda \rightarrow \mathcal{C}.$$

This morphism is a logarithmic modification, so it is a contraction of chains of rational bubbles. Consider the lifting problem for logarithmic maps $[g: C^\lambda \rightarrow X]$ from the modified curve C^λ/S to a strict infinitesimal extension S' given by an ideal I . Once again, the lifts are given by a torsor on C^λ of the abelian groups $g^*T_X^{\log} \otimes I$. By construction, the maps

$$C^\lambda \rightarrow X$$

factor through $C^\lambda \rightarrow C$. Since the bubbles are rational, pushing forward via $\text{Rst}_\star(\cdot)$ identifies the torsors controlling the obstruction theory coming from C^λ with those on C . Again, virtual pullback of the fundamental class from $K_F^\lambda(\mathcal{A}_X)$ gives rise to a virtual class, which by the discussion above is the same virtual class.

4.5.3 The second way. In the intermediate version above, we considered only the deformation theory of the collapsed map. Typically, one studies the obstructions to deformations of the map to the expansion. Again, the virtual class is unchanged.

This follows by considering the lifting problems as above. Specifically, we examine the composition

$$\mathcal{C}^\lambda \rightarrow \mathcal{X}^\lambda \rightarrow X \times K_F^\lambda(X) \rightarrow X.$$

The second arrow is logarithmically étale over $K_F^\lambda(X)$ since it is the pullback of a subdivision. It therefore identifies relative logarithmic tangent bundles. In turn, the relative logarithmic tangent bundle of the product is simply pulled back from the logarithmic tangent bundle of X .

Once again, the torsors controlling lifts over strict extensions are naturally identified, and the virtual pullback produces a virtual class. In all cases, Manolache’s results lead to the following [Man12].

THEOREM 4.5.1. *The Deligne–Mumford stack $K_\Gamma^\lambda(X)$ carries a virtual fundamental class in the expected dimension. This class is equal to the virtual pullback of the ordinary fundamental class along the morphism*

$$K_\Gamma^\lambda(X) \rightarrow K_\Gamma^\lambda(\mathcal{A}_X).$$

4.6 Virtual birationality. By construction, each of our moduli spaces come with a morphism

$$K_\Gamma^\lambda(X) \rightarrow \text{ACGS}_\Gamma(X).$$

This may be seen in a moduli-theoretic manner. Given a map to an expansion

$$C^\lambda \rightarrow \mathcal{X}^\lambda \rightarrow X$$

over S , stabilize the underlying morphism to X to obtain $[C \rightarrow X]$ over S , and equip it with the pushforward logarithmic structure [AMW14, Appendix B]. The logarithmic structure on S may not be minimal, but by the universal property of minimality, the map determines a minimal object.

The different approaches to the relative theory give the same invariants; cf. [AMW14].

PROPOSITION 4.6.1. *Pushforward under the morphism*

$$K_\Gamma^\lambda(X) \rightarrow \text{ACGS}_\Gamma(X)$$

identifies virtual classes. The logarithmic invariants of X are computed by the expanded theory.

Proof. The result follows from the discussion above, using the ideas in [AW18, Section 6]. Indeed, there is a birational morphism

$$K_\Gamma^\lambda(\mathcal{A}_X) \rightarrow \text{ACGS}_\Gamma(\mathcal{A}_X).$$

As discussed above, we may construct virtual classes for maps to $\mathcal{X}^\lambda$ and minimal maps to X relative to these two moduli spaces. However, the obstruction theory for $K_\Gamma(X) \rightarrow K_\Gamma(\mathcal{A}_X)$ clearly pulls back to the obstruction theory for $K_\Gamma^\lambda(X) \rightarrow K_\Gamma^\lambda(\mathcal{A}_X)$. The result follows from a theorem of Costello [Cos06, Theorem 5.0.1]. The statement about Gromov–Witten invariants follows from the projection formula. $\square$

Cosmetic changes to the above argument also yield the following.

PROPOSITION 4.6.2. *Given any two moduli spaces $K_\Gamma^\lambda(X)$ and $K_\Gamma^\mu(X)$ of transverse stable maps to expansions and a logarithmic modification*

$$K_\Gamma^\lambda(X) \rightarrow K_\Gamma^\mu(X),$$

pushforward identifies the virtual classes.

Remark 4.6.3. Since common refinements of subdivisions exist, the collection of logarithmic modifications of $K_\Gamma^\lambda(X)$ forms an inverse system. As a consequence of the proposition, there is a well-defined virtual class in the group $\varprojlim_\lambda A_\star(K_\Gamma^\lambda(X), \mathbb{Q})$, where the inverse limit is taken under proper pushforward maps; see [Alu05] for a discussion of such inverse limit Chow groups. Note that the inverse limit is not the logarithmic Chow *homology* groups [Bar18] but rather the much smaller direct limit Chow cohomology groups examined in [MPS21].

4.7 What is this a moduli space of? Each of the spaces $K_{\Gamma}^{\lambda}(X)$ constructed here is a solution to a moduli problem over *schemes*. Over logarithmic schemes, tautologically, they are defined as subcategories of $\text{ACGS}_{\Gamma}(X)$ of families of curves over a base whose tropicalizations lie in the subdivision indexed by λ . Indeed, logarithmic modifications are always subcategories when considered in this manner; see [Kat99, Definition 3.8] and [RSW19a, Section 2.5]. This category admits minimal objects – those families where at each closed point, the tropicalization map is an isomorphism onto a cone of the λ -subdivision. The category is therefore representable by a Deligne–Mumford stack [Wis16, Appendix B]. It ignores the modifications done to the curve and target family. The same category also carries a second modular interpretation, coming from the universal modified curve and target. Once again, over logarithmic schemes, one studies families of logarithmic maps to an expansion of X such that the universal curve and target are pulled back from the tropical families corresponding to the indexing element λ .

Part II. The gluing formula

A basic intuition guides this part of the paper. Let $\mathcal{Z} \rightarrow \mathbb{A}^1$ be a simple normal crossings degeneration, with general and special fibres Z_{η} and Z_0 , respectively. Given a curve C_{η} in Z_{η} , we can specialize it to obtain a curve in Z_0 . The reverse is a problem in deformation theory. Given a curve C_0 in Z_0 , one needs to smooth it out over $\mathbb{A}^1$ in order to describe a curve in Z_{η} . In general this problem is much too hard, and one must either work in the logarithmic category or place hypotheses on the curve in the special fibre. A natural hypothesis is that the components of C_0 interact nicely with the strata of Z_0 : each component meets only the codimension 1 strata of Z_0 at points and is disjoint from the higher-codimension strata. In this case, there is a *predeformability* condition: when two components meet at a point along a divisor of Z_0 , the orders of vanishing must be equal. This condition is necessary for a smoothing to exist.

In simple cases, such as rational curves in toric varieties, predeformability is sufficient for deformability [MR20, NS06, Ran17]. The general problem is hopeless, but for the purposes of working with the virtual geometry of the space of maps, one pretends that this condition suffices always, and Li’s relative Gromov–Witten theory is built from this point of view. When the special fibre of $\mathcal{Z}$ is not allowed to expand, the right component of the space of maps to the special fibre is the locus of logarithmic maps. However, the intuitive picture of gluing along components breaks down. One needs to study maps into the strata of the degeneration, which is where punctured logarithmic curves enter the picture [ACGS20b]. Using the previous part of the paper, we reintroduce the transversality requirement, obtaining a proper space of maps to expanded targets, and prove the gluing formula following this picture.

In order to work virtually, we work tropically. An inventive observation of Abramovich and Wise is that the space of maps to $\mathcal{Z}$ is virtually smooth over the space of tropical maps to the tropicalization [AW18]. We take this observation to its extreme by first proving the gluing formula entirely within the combinatorial framework. We degenerate the cone complex $\Sigma_{\mathcal{Z}}$ into an extended cone complex – a singular tropical object. Tropical nodal curves in the special fibre of the tropical degeneration can have nothing to do with maps to a general fibre, but we find that *predeformability is tropical deformability*. After modifying the tropical targets to ensure combinatorial transversality, the nodal tropical curves may always be smoothed out, leading to a tropical gluing formula for the cone complexes. Standard virtual pullback techniques imply the gluing formula.

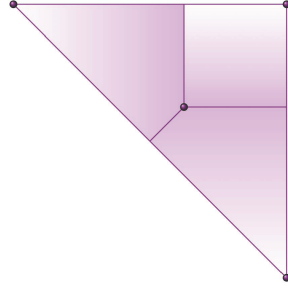


FIGURE 5. The extended tropicalization of $\mathbb{P}^2$ consists of three extended cones. The face at infinity perpendicular to each ray is the extended tropicalization of the dual stratum.

5. Extended tropicalizations

Extended tropicalizations are compactifications of fans and polyhedral complexes obtained by adding faces at infinity which are themselves such complexes. We will use them to describe degenerations of tropical varieties to singular ones. A careful treatment of extended cone complexes can be found in [ACP15, Section 2], and the relationship with Berkovich spaces and tropical geometry can be found in [Pay09, Thu07]. An interesting *pointification* viewpoint on the matter, considering piecewise-linear functions taking the value infinity, has been introduced by Huszar–Marcus–Ulirsch [HMU19]. This pointification is functorial and sets the discussion below on sound categorical footing.

Let Σ be a cone complex with integral structure. Each cone $\sigma \in \Sigma$ is obtained from its dual cone S_σ of linear functions as the space of monoid homomorphisms

$$\sigma = \operatorname{Hom}_{\mathbf{Mon}}(S_\sigma, \mathbb{R}_{\geq 0}).$$

By replacing $\mathbb{R}_{\geq 0}$ with the extended monoid $\mathbf{R}_\infty = \mathbb{R}_{\geq 0} \sqcup \{\infty\}$ equipped with its order topology, we obtain a compactification, the *extended cone*

$$\sigma \hookrightarrow \bar{\sigma} = \operatorname{Hom}_{\mathbf{Mon}}(S_\sigma, \mathbf{R}_\infty).$$

The cone complex Σ can be described as a colimit of a diagram of cones, and by replacing each cone with its extended cone, we obtain a canonical compactification

$$\Sigma \hookrightarrow \bar{\Sigma}.$$

WARNING 5.0.1. Pictures of the extended tropicalization can cause some consternation, as both order-reversing and order-preserving bijections are at play. The extended tropicalization of a toric variety X with fan Σ and dense torus $\mathbb{G}_m^r$ is equipped with an action of the vector group $\mathbb{R}^r$. The extended tropicalization is stratified by the orbits of this group action and is in order-preserving bijection with the orbits of the $\mathbb{G}_m^r$ -action on X .

There is an order-reversing bijection between the *closures* in the extended tropicalizations of the k -dimensional cones of Σ and the codimension k strata of X ; see Figure 5. Since each face at infinity can be interpreted as the extended tropicalization of a toric variety in the boundary of X , the order-reversing and order-preserving bijections interact in the boundary. While this may cause some initial confusion, it is an efficient tool in recording the necessary combinatorics for the degeneration problem.

Given a polyhedral complex $\mathcal{P}$ that is not necessarily a cone complex, one similarly obtains

a compactification $\overline{\mathcal{P}}$ of the asymptotic directions as follows. Let $\Sigma_{\mathcal{P}}$ be the cone over the polyhedral complex. The height 1 slice of the extended cone complex $\overline{\Sigma}_{\mathcal{P}}$ gives a compactification of $\mathcal{P}$.

Extended cones can be glued along extended faces by taking colimits to obtain a larger class of objects, as has already been suggested in the literature [ACP15, Remark 2.4.1]. We will be concerned with the following three non-exclusive examples:

- (A) *Nodal tropical curves*. Given a pointed tropical curve $\sqsubset$, its compactification is obtained by adding one point at infinity on each unbounded half-edge of $\sqsubset$. By gluing such extended tropical curves along their infinite points, one obtains objects that we call *nodal tropical curves*.
- (B) *Boundaries*. Given a cone complex Σ , the complement $\overline{\Sigma} \setminus \Sigma$ is a union of the extended faces. This has an irreducible component decomposition into compactified cone complexes corresponding to the rays of Σ .
- (C) *Degenerate fibres*. Given a cone complex Σ and a surjective map of cone complex $\Sigma \rightarrow \mathbb{R}_{\geq 0}$, pass to the associated map $\overline{\Sigma} \rightarrow \mathbf{R}_{\infty}$. The fibre over ∞ is a union of extended cone complexes glued along boundary divisors.

A picture that encapsulates all three examples is given Figure 6.

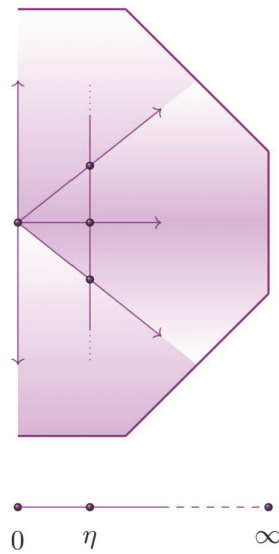


FIGURE 6. The figure describes a tropical degeneration over $\mathbf{R}_{\infty}$ which arises as the tropicalization of a toric degeneration of $\mathbb{P}^1$ over $\mathbb{A}^1$. The special fibre consists of three components, each equal to $\mathbb{R} \sqcup \{\pm\infty\}$. Over a general point η in $\mathbb{R}$, the fibre is a subdivided real line with three vertices dual to the three components in the special fibre.

6. The tropical gluing formula

In this section, we prove a gluing formula entirely within combinatorial geometry. A degeneration of a cone complex $\overline{\Sigma}$ over $\mathbf{R}_{\infty}$ gives rise to a degeneration of the space of tropical maps. We describe the components of the degenerate moduli space by gluing maps from curves to the components of the degenerate fibre.

6.1. The tropical moduli space of maps. Let $\Sigma \rightarrow \mathbb{R}_{\geq 0}$ be a surjective morphism of cone complexes, and let $\bar{\Sigma} \rightarrow \mathbf{R}_{\infty}$ be the associated morphism of extended complexes. Let Σ_r denote the fibre over the point $r \in \mathbf{R}_{\infty}$. We consider Σ_{∞} to be a degenerate limit of Σ_0 .

A family of tropical stable maps to a family $\Sigma \rightarrow \mathbb{R}_{\geq 0}$, parameterized by a cone τ , is a commutative square

$$\begin{array}{ccc} \square & \xrightarrow{F} & \Sigma \\ \downarrow & & \downarrow \\ \tau & \longrightarrow & \mathbb{R}_{\geq 0}, \end{array}$$

where $\square$ is a family of tropical curves and F is a family of tropical maps to Σ . We obtain a stack over the category of cones, whose fibre over τ is the groupoid of tropical stable maps over τ . As before, we restrict our attention to the subcategory of maps of a fixed finite set of combinatorial types. The result is a finite-type tropical moduli stack

$$\pi: \text{ACGS}_{\Gamma}(\Sigma) \rightarrow \mathbb{R}_{\geq 0}$$

parameterizing such maps. The morphism π presents the moduli space as a family over $\mathbb{R}_{\geq 0}$. Let ρ_i be a ray of the moduli space, and let v_i be its primitive integral generator. Then $\pi(v_i)$ generates a submonoid of some index in $\mathbb{N}$. Call this index m_{ρ_i} .

Passing to extended tropicalizations gives rise to a morphism

$$\pi: \overline{\text{ACGS}}_{\Gamma}(\bar{\Sigma}) \rightarrow \mathbf{R}_{\infty}.$$

The components lying above ∞ are in bijection with the rays ρ_i that dominate $\mathbb{R}_{\geq 0}$. As we will soon describe, each such extended component is a moduli stack of tropical maps. As π is a piecewise-linear function, we expect a “rational equivalence” of the form

$$[\overline{\text{ACGS}}_{\Gamma}(\bar{\Sigma}_0)] \approx \sum_i \frac{m_{\rho_i}}{\text{Aut}(\rho_i)} [\overline{\text{ACGS}}_{\rho_i}(\Sigma_{\infty})]. \quad (2)$$

Here, $\overline{\text{ACGS}}_{\rho_i}(\Sigma_{\infty})$ is a finite cover of the component at infinity dual to ρ_i , obtained by removing generic automorphisms. We will not need to make the rational equivalence here precise, but if we use the categorical equivalence of cone complexes with Artin fans, a precise translation can be made in the Chow group of an appropriate family of Artin stacks [CCUW20, Uli19], and this implies the statement in [ACGS20a, Section 4.1].

Our next task is to describe the infinite faces $\overline{\text{ACGS}}_{\rho_i}(\Sigma_{\infty})$ in terms of maps to the strata.

6.2. Extended faces. Fix a ray ρ of the tropical moduli space $\text{ACGS}_{\Gamma}(\Sigma)$. We now describe a moduli space $\overline{\text{ACGS}}_{\rho}(\Sigma_{\infty})$ which is a finite cover of the infinite face of $\overline{\text{ACGS}}_{\Gamma}(\Sigma)$ dual to ρ , as follows.

Choose a general point in the ray ρ of $\text{ACGS}_{\Gamma}(\Sigma)$. Since the multiplicity of vertical divisors will be equal to 0, we lose no generality in assuming that ρ surjects onto $\mathbb{R}_{\geq 0}$. Let the chosen general point map to $\eta \in \mathbb{R}_{\geq 0}$. The fibre Σ_{η} of the tropical degeneration is a polyhedral complex, and the fibre in $\text{ACGS}_{\Gamma}(\Sigma)$ over η intersected with ρ is a possibly stacky point of the moduli space. This determines a single tropical map

$$f_{\eta}: \square_{\eta} \rightarrow \Sigma_{\eta}$$

up to automorphisms. Since the point was chosen on a ray, the map is *rigid*: any deformation of the map $[f_{\eta}]$ changes the combinatorial type. We make the following additional assumption.

ASSUMPTION 6.2.1. The curve $\sqsubset_\eta$ is combinatorially transverse to Σ_η . That is, the image of every vertex of $\sqsubset$ is a vertex of Σ_η , and the image of every edge is an edge.

Justification. As described above, the ray ρ determines a family of tropical maps, and we may subdivide Σ , inducing a subdivision of Σ_η such that the transversality condition is met for the map $\sqsubset_\eta \rightarrow \Sigma_\eta$ for all η . It follows immediately from the arguments in [AW18] that this has the effect of replacing the moduli space with a modification. We are only interested in describing the component $\text{ACGS}_\rho(\Sigma_\infty)$ up to modifications, so this will have no effect.

We obtain a family of tropical maps to a family of targets, depending on a parameter η in $\mathbb{R}_{\geq 0}$. The curve $\sqsubset_\eta$ has bounded edges whose lengths are linear functions in η . The special fibre of the tropical family is obtained by taking $\eta \rightarrow \infty$. The point at infinity of ρ parameterizes a map

$$f_\infty: \sqsubset_\infty \rightarrow \Sigma_\infty.$$

LEMMA 6.2.1. *The edges of $\sqsubset_\infty$ are either (i) infinite rays associated with marked points or (ii) infinitely long edges, identified with the singular metric space $\mathbf{R}_\infty \sqcup_\infty \mathbf{R}_\infty$.*

Proof. If $\sqsubset_\infty$ contained bounded edges of finite length, then by uniformly scaling all bounded edges, we would produce a one-parameter deformation of the tropical map, which cannot exist by the rigidity hypothesis. $\square$

Moduli at the extended face. Fix a single tropical map $f_\infty: \sqsubset_\infty \rightarrow \Sigma_\infty$. Consider the moduli space $\text{ACGS}_\rho(\Sigma_\infty)$ parameterizing tropical stable maps

$$g_\infty: \triangleright \rightarrow \Sigma_\infty$$

from nodal tropical curves to Σ_∞ that (i) are deformations of a fixed map $f_\infty: \sqsubset_\infty \rightarrow \Sigma_\infty$ and (ii) come equipped with a uniform rescaling to $[f_\infty]$. Such maps are parameterized by a stack over the category of cones. Forgetting the rescaling information, we obtain a cover of the infinite face of $\text{ACGS}_\Gamma(\Sigma)$ that is dual to ρ , as in equation (2).

6.3 The cutting morphism. The special fibre Σ_∞ is glued from infinite faces. Each ray of the target Σ determines a single extended face that is the canonical compactification of a cone complex. The special fibre of $\bar{\Sigma}$ is glued from these extended faces:

$$\Sigma_\infty = \bigcup_v \bar{\Sigma}_v,$$

where v runs over the vertices. Given a map

$$\sqsubset \rightarrow \Sigma_\infty,$$

cut the target Σ_∞ into the pieces as above. For any map parameterized by a point of the moduli space $\text{ACGS}_\Gamma(\Sigma_\infty)$, we correspondingly split the curve into components, at its infinite bounded edges. After deleting the extended faces, we obtain maps $\{\sqsubset_v \rightarrow \Sigma_v\}_v$. Note that these tropical curves can in principle be disconnected. This produces a cutting morphism

$$\text{ACGS}_\rho(\Sigma_\infty) \rightarrow \prod_v \text{ACGS}_\rho(\Sigma_v),$$

where $\text{ACGS}_\rho(\Sigma_v)$ is the moduli stack of tropical maps to the cone complex Σ_v with the discrete data induced by ρ , by the star at v . The star at v can be identified in an elementary fashion with the fan of outwards directions at v in the polyhedral complex obtained by taking the fibre

of $\Sigma \rightarrow \mathbb{R}_{\geq 0}$ over the point 1. It corresponds to an irreducible component in the special fibre of an associated geometric degeneration.

6.4 Modifying the product. The stacks of maps $\text{ACGS}_\rho(\Sigma_v)$ can be used to describe the stack $\text{ACGS}_\rho(\Sigma_\infty)$, at least up to modifications. The product has the following universal diagram:

$$\begin{array}{ccccc} \bigcup_v \square_v & \xrightarrow{\quad\quad\quad} & K \times \bigcup_v \Sigma_v & \longrightarrow & \bigcup_v \Sigma_v \\ & \searrow & \swarrow & & \\ & K := \prod_v \text{ACGS}_\rho(\Sigma_v) & & & \end{array} \quad (3)$$

Let G_ρ be the graph given by the source of the map $[f_\infty]$. Each edge e of G_ρ has vertices v_1 and v_2 and determines a half-edge incident to each. In the component Σ_{v_1} containing v_1 , the point at infinity of this half-edge e_1 maps to a closed boundary stratum which we denote by Σ_e , noting that the other half-edge maps to the same stratum. This fits into an evaluation morphism

$$\text{ev}_e : \prod_v \text{ACGS}_\rho(\Sigma_v) \rightarrow \Sigma_e^2.$$

Note that in order to define this evaluation, we pass to the relative extended tropicalization in the universal curve, adding points at infinity, and inspect their images in Σ_e . This reflects the fact that these boundary-evaluation morphisms are not generally logarithmic when restricting the logarithmic structure on the target to a stratum, but become logarithmic when the generic logarithmic structure along the divisor is removed.

Remark 6.4.1. The naive strategy would be to now range over all the edges, pulling back the appropriate diagonal loci to obtain maps from collections of curves that glue to form a map to the target. However, the diagonal loci do not impose transverse conditions, and when this procedure is applied to moduli points in extended faces of the product, there is no natural way in which to smooth out the singular tropical curve. This reflects the geometric situation that the evaluations may map to small strata. We subdivide the target over the moduli space until the evaluations always map to vertices or geometrically to top-dimensional strata. This ensures that the gluing always imposes the expected number of geometric conditions, and smoothability follows.

The existence of the required moduli space modifying the product is a twist on Theorem 3.7.1, but we first discuss the necessity of the modification in greater detail. At first approximation, we must expand each tropical target for the maps $\square_v \rightarrow \Sigma_v$, which can be achieved by subdividing each factor as in the first part. Such a modification does not immediately suffice, however. Such expansions of two targets are independent of each other, and there is no guarantee that the resulting expansions can be glued together. This is exemplified in Figure 7.

Example 6.4.2. Let X be the mirror dual to $\mathbb{P}^1 \times \mathbb{P}^1$, whose fan is the fan over the faces of the square $(\pm 1, \pm 1)$. Consider the family of polyhedral complexes in $\mathbb{R}^2$ given at time $t \in \mathbb{R}_{\geq 0}$ by a bounded cell – the square $(\pm t, \pm t)$ – and rays beginning at the four points of this cell, with directions $(\pm 1, \pm 1)$ as depicted in Figure 7. This polyhedral complex forms a family $\Sigma \rightarrow \mathbb{R}_{\geq 0}$. The toric degeneration described by this family degenerates X to a union of four copies of $\mathbb{P}^2$.

The associated tropical degeneration consists of four copies of $\mathbb{P}_{\text{trop}}^2$ glued along faces, as shown in Figure 7. In understanding this picture, Warning 5.0.1 is best kept in mind. The picture depicts four partially compactified vector spaces, whose origins are the four points of the drawn square. The cross consists of four copies of $\mathbb{R} \sqcup \{\infty\}$, glued at the infinite points.

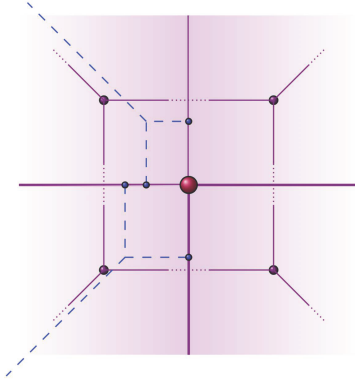


FIGURE 7. Depicted here is the extended tropicalization of four copies of $\mathbb{P}^2$ glued along their common boundaries. The axes surrounding the centre are boundary curves along which the surfaces meet, while the four smaller vertices in the interior are dual to the components themselves. Two dashed tropical curves are drawn, indicating how transversality in one component forces subdivisions of another.

The points where the dotted arrows meet the cross are the origins of compactified vector spaces $\mathbb{R} \sqcup \{\infty\}$.

In the figure, the dashed arrows in the northwest and southwest quadrants are non-transverse tropical curves, each corresponding to a point in the product. These two target components must be modified in order to maintain transversality. However, when an infinite edge is added to the interior, it forces a subdivision at infinity. In this case, the four smaller vertices on the central cross itself depict how subdivisions in the interior force subdivisions at infinity.

Example 6.4.2 indicates the general strategy. Subdividing the interior to accommodate an infinite edge of the curve forces a subdivision at infinity in that component. We choose a matching subdivision on the other side of the cross. Since marked points in our spaces always evaluate to the interiors of pairwise intersections of irreducible components, this can always be done. To make the appropriate construction, recall that we have inclusions of the infinite faces

$$\iota: \bigcup_v \bar{\Sigma}_v \rightarrow \Sigma_\infty \hookrightarrow \bar{\Sigma}.$$

Composing the universal map over $\prod_v \text{ACGS}_\rho(\Sigma_v)$ with the morphism ι , we think of the universal target² as being a constant family of $\bar{\Sigma}$.

The required variation of Theorem 3.7.1 in Part I is as follows.

THEOREM 6.4.3. *There exist a modification*

$$K^\lambda = \bigtimes_v^\lambda K_\rho(\Sigma_v) \rightarrow \prod_v \text{ACGS}_\rho(\Sigma_v)$$

²In fact, we only use the partial compactification of Σ determined by deleting the extended faces corresponding to cones of Σ of dimension 2 and higher.

of the product, modifications of its source and target family, and a moduli diagram

$$\begin{array}{ccccc} \bigcup_v \mathbb{C}_v^\lambda & \xrightarrow{\quad} & \overline{\Sigma}_{\text{exp}}^\lambda & \longrightarrow & \overline{\Sigma} \\ & \searrow & \swarrow & & \\ & \mathbb{K}^\lambda & & & \end{array}$$

with the following properties:

- (A) *Gluability.* The extended cone complex $\overline{\Sigma}_{\text{exp}}^\lambda$ is the canonical compactification of a cone complex $\Sigma_{\text{exp}}^\lambda$ that is combinatorially equidimensional over $\mathbb{K}^\lambda$.
- (B) *Curve family.* The universal curves $\mathbb{C}_v^\lambda$ are each a family of tropical curves over the base $\mathbb{K}^\lambda$.
- (C) *Degenerate transversality.* Over a point t of $\mathbb{K}^\lambda$ and for each v , the universal curve $\mathbb{C}_v[t]$ maps to the relative interior of an extended face $\Sigma_{\text{exp},v}[t]$ and is combinatorially transverse: the image of every polyhedron of the curve is a polyhedron of the target.

Proof. We first focus on the final requirement, demanding that each curve maps to the relative interior of a maximal infinite face of $\overline{\Sigma}$ but is combinatorially transverse within that stratum. Fix a vertex v , and consider the map $\mathbb{C}_v$ to the relative interior of an infinite face F_v of Σ_∞ . Introduce the notation

$$\text{PROD} = \prod_v \text{ACGS}_\rho(\Sigma_v)$$

to denote the product of moduli spaces attached to the vertices. To ensure the transversality of $\mathbb{C}_v$, we take $F_v \times \text{PROD}$ and make subdivisions of the total space, later modifying the curve and base to ensure the requisite conditions. For any fan, the collection of star subdivisions is cofinal in the inverse system of all subdivisions, so we may assume that transversality for $\mathbb{C}_v$ is achieved by making a sequence of star subdivisions of $F_v \times \text{PROD}$. However, as outlined in Example 6.4.2, the subdivisions of F_v may force subdivisions of other extended faces F_u . To see that a consistent choice can be made while maintaining gluability of the target, we instead make the subdivisions of the total space of the degeneration Σ that induce the given subdivision of F_v upon passing to canonical compactifications.

Given a cone τ of the fan $F_v \times \text{PROD}$ that we wish to perform a star subdivision of, observe that it determines a stratum of the canonical compactification $\overline{F}_v \times \overline{\text{PROD}}$. This is also a stratum of the larger compactification $\overline{\Sigma} \times \overline{\text{PROD}}$ and thus determines a cone τ^{lift} on the interior $\Sigma \times \text{PROD}$. The cone τ is a cone contained in a face at infinity of τ^{lift} . Further subdivisions do not affect the question at this stage, so assume that τ^{lift} is a simplicial cone inside a simplicial fan $\Sigma \times \text{PROD}$. Perform the star subdivision of τ^{lift} . Now passing to its canonical compactification, the induced subdivision on $\overline{F}_v \times \overline{\text{PROD}}$ is the star subdivision of $\overline{F}_v \times \overline{\text{PROD}}$. This is the extended face corresponding to the strict transform. By performing a sequence of these star subdivisions, we obtain a family of expansions of the faces at infinity, expanded compatibly, as well as a family of expansions of the tropical target Σ .

We now proceed as in Theorem 3.7.1. Pull back the resulting expansions of the faces $F_v \times \text{PROD}$ to the curves $\mathbb{C}_v$, flatten the source and target families, and obtain the modification

$$\bigtimes_v^\lambda \mathbb{K}_\rho(\Sigma_v) \rightarrow \prod_v \text{ACGS}_\rho(\Sigma_v)$$

with the requisite transversality properties. $\square$

Remark 6.4.4. When the boundary is a single smooth divisor, the divisor itself does not admit non-trivial modifications, as its tropicalization is a single point. As a result, the birational modification of the product is not necessary, and this accounts for the lack of modification of the product in traditionally considered cases [Che14a, KLR21, Li02].

As with the construction of the spaces in Part I of the paper, λ records a choice, and there is an inverse system of such moduli spaces obtained by common refinement.

6.5 Evaluations and tropical gluing. The next two steps are not strictly necessary for the geometric arguments but are combinatorial analogues of key steps in the next section. We first glue maps to form maps out of nodal tropical curves and then smooth them to maps from ordinary tropical curves.

Fix the rigid tropical type, and consider maps that are deformations of

$$f_\infty: \sqsubset_\infty \rightarrow \Sigma_\infty,$$

equipped with a fixed contraction to this map. Let G be the graph formed by considering the vertices and infinite bounded edges of $\sqsubset_\infty$. Previously, we constructed a cutting morphism

$$\text{ACGS}_\rho(\Sigma_\infty) \rightarrow \prod_v \text{ACGS}_\rho(\Sigma_v).$$

We also built a modification of the product with a transverse disconnected universal curve. The cutting morphism from $\text{ACGS}_\rho(\Sigma_\infty)$ does not immediately lift to the modified product. However, we may subdivide $\text{ACGS}_\rho(\Sigma_\infty)$ and its universal curve and target to obtain a space of transverse maps $K_\rho^\lambda(\Sigma_\infty)$ and a new cutting morphism

$$\kappa: K_\rho^\lambda(\Sigma_\infty) \rightarrow \bigtimes_v K_\rho(\Sigma_v).$$

We now wish to identify the image of this map. For a morphism to the product to come from a morphism to Σ_∞ , two conditions are apparent:

- (A) *Continuity.* For each edge e of G between v_1 and v_2 , the images of the two markings incident to v_1 and v_2 comprising e must coincide.
- (B) *Smoothability.* The construction of $K_\rho^\lambda(\Sigma_\infty)$ as a component of the special fibre of a tropical degeneration of maps demands that any map must arise as the limit of an $\mathbb{R}_{>\eta}$ -family of maps to Σ .

Over the modified product moduli space, we have an expanded target family $\Sigma_{\text{exp}}^\lambda$. An edge e in G determines a choice of codimension 2 stratum in the extended faces of this expanded target family. The infinite point of a marked end of the curve $\sqsubset_v$ containing one end of e has an evaluation morphism to the relative interior of the corresponding stratum of the target. We denote this relative divisor by $D_e(\Sigma_{\text{exp}}^\lambda)$. Since there are two vertices at the ends of each edge e , we have an evaluation morphism

$$\text{ev}_e: \bigtimes_v K_\rho(\Sigma_v) \rightarrow D_e^{\{2\}}(\Sigma_{\text{exp}}^\lambda).$$

The symbol on the right is the fibred second power of $D_e(\Sigma_{\text{exp}}^\lambda)$ over the moduli space $\bigtimes_v K_\rho(\Sigma_v)$. The subdivision has been chosen such that the image of a boundary marking is always a vertex of $D_e(\Sigma_{\text{exp}}^\lambda)$. We have a relative diagonal morphism

$$\Delta_e: D_e(\Sigma_{\text{exp}}^\lambda) \hookrightarrow D_e^{\{2\}}(\Sigma_{\text{exp}}^\lambda).$$

Note that in each fibre over the moduli space, the image of either evaluation morphism attached to e is a vertex of the divisor. The intersection of the image of Δ_e and the image of the map

$$\mathrm{ev}_e: \bigtimes_v^\lambda \mathcal{K}_\rho(\Sigma_v) \rightarrow D_e^{\{2\}}(\Sigma_{\mathrm{exp}}^\lambda)$$

is a subcomplex of $D_e(\Sigma_{\mathrm{exp}}^\lambda)$. By ranging over all edges, we obtain a tropical moduli space³

$$\mathcal{G}^\lambda(\Sigma_\infty) = \bigcap_e \mathrm{im}(\Delta_e) \cap \bigtimes_v^\lambda \mathcal{K}_\rho(\Sigma_v).$$

These are tropical maps to expansions of Σ_∞ that are transverse to the target and whose evaluation morphisms agree.

6.6 Tropical smoothing. A final step remains to complete a tropical degeneration formula – the singular tropical curves constructed by gluing must be smoothed out. This step reflects the geometric phenomenon of having to build logarithmic enhancements of glued maps, which we discuss in the next section. In both steps, transversality of the curve is paramount.

Given a p point of $\mathcal{G}^\lambda(\Sigma_\infty)$, we obtain a map

$$g_\infty: \triangleright_\infty \rightarrow \Sigma_\infty^\lambda.$$

We wish to smooth it out to a family $\triangleright_\eta \rightarrow \Sigma_\eta$ for $\eta \in \mathbb{R}_{\geq 0}$, for η sufficiently large, whose limit is the map $[g_\infty]$. This involves “shrinking” the infinite edge lengths of $\triangleright_\eta$ to produce a continuous tropical map to Σ_η .

Consider a nodal edge e of $\triangleright_\infty$. Its image under g_∞ is an infinitely long edge of $\Sigma_\infty[p]$. However, over the point p , we have a canonical inclusion

$$\Sigma_\infty[p] \hookrightarrow \bar{\Sigma}_{\mathrm{exp}}^\lambda[p] \subset \bar{\Sigma}_{\mathrm{exp}}^\lambda.$$

Smoothing out g_∞ now amounts to resetting the lengths of the infinite edges of $\triangleright_\infty$ to finite values varying linearly in η , while maintaining continuity.

Given a nodal edge e of the curve, the image $g_\infty(e)$ is a nodal edge of the target. Over η , this nodal edge of the target determines a unique bounded edge $\Sigma_\eta[p]$. This edge has length ℓ_e . Let m_e denote the expansion factor on the edge e of $\triangleright_\infty$. Reset the edge length of $\triangleright_\eta$ to be ℓ_e/m_e . Consider the resulting map

$$\triangleright_\eta \rightarrow \Sigma_1[p].$$

We claim that it is continuous and piecewise linear. To see this, observe that each edge of $\triangleright_\eta$ maps onto an edge of $\Sigma_1[p]$ by a linear map with slope equal to the expansion factor m_e . When two edges meet at a vertex of $\triangleright_\eta$, that vertex is mapped to the corresponding vertex in $\Sigma_1[p]$ by construction. This implies continuity and piecewise linearity. In particular, we conclude that for any appropriate choice of λ , the morphism

$$\mathcal{K}_\rho^\lambda(\Sigma_\infty) \rightarrow \mathcal{G}^\lambda(\Sigma_\infty)$$

is a piecewise-linear homeomorphism.

³In full analogy with the geometric situation, this intersection of diagonals is a stand-in for the refined intersection product for regular embeddings, in the sense defined by Fulton.

7. The virtual gluing formula

In this final section, we prove the degeneration formula. We pull back the subdivisions constructed in the previous section to the moduli spaces of maps to the Artin fan and then to the degeneration. Two additional ingredients yield the gluing formula, both of which are reminiscent of the traditional case [Che14a, Section 7]. The first is the calculation of the number of logarithmic lifts of a glued, transverse map. The second is a comparison of obstruction theories.

7.1. The degeneration set-up. Let $\mathcal{Y} \rightarrow \mathbb{A}^1$ be a toroidal degeneration. There exists a stack of logarithmic stable maps to the fibres of the degeneration $\mathrm{ACGS}_\Gamma(\mathcal{Y})$. Let Y_η be the general fibre and Y_0 the special fibre, equipped with its logarithmic structure. The moduli space comes equipped with a structure map

$$\mathrm{ACGS}_\Gamma(\mathcal{Y}) \rightarrow \mathbb{A}^1.$$

The rays of the tropical moduli stack $\mathrm{ACGS}_\Gamma(\Sigma_{\mathcal{Y}})$ discussed in the previous section are indexed by finitely many *rigid tropical maps*. These determine divisors in the space of maps to the Artin fan, and hence virtual divisors of the stack above. We freely use the twofold interpretation of a ray ρ : as a ray in the tropical moduli space and as a combinatorial type of a rigid tropical map. Note that this combinatorial type includes a finite marked graph, genus splitting, degree splitting, and contact order data [ACGS20a, Definition 2.5.3].

In the previous section, we made the assumption that each rigid tropical map is supported on a subcomplex of $\Sigma_{\mathcal{Y}}$. The parallel assumption is made here.

ASSUMPTION 7.1.1. For each rigid tropical map ρ , we assume that the virtual codimension 0 strata of the component with type ρ correspond to maps $C \rightarrow Y_0$ that are transverse to the strata of Y_0 .

This can always be done after replacing $\mathcal{Y}$ by a toroidal modification such that the rigid tropical maps are transverse to the tropicalization of the degeneration. Indeed, the image of each rigid tropical map gives rise to a polyhedral complex in the dual complex obtained as the fibre over 1 in the tropical family of targets, denoted by $\Sigma_{\mathcal{Y}}[1]$. After a polyhedral subdivision of $\Sigma_{\mathcal{Y}}[1]$, this rigid tropical map can be made transverse – that is, such that the image of every vertex (respectively, edge) is a vertex (respectively, edge) of this subdivided complex. This induces a logarithmic modification of Y_0 , which achieves the requisite transversality claim. See the parallel Assumption 6.2.1 in the previous section.

Remark 7.1.1. Note that when $\mathcal{Y}$ is replaced by a logarithmic modification $\mathcal{Y}'$, the virtual modification of the moduli space $\mathrm{ACGS}_\Gamma(\mathcal{Y}')$ will acquire new rigid tropical curves. However, these new exceptional virtual divisors will not contribute to the Gromov–Witten theory numerically. Since any integral of interest will be pulled back from $\mathrm{ACGS}(\mathcal{Y})$, the projection formula will guarantee that these new virtual divisors will be zeroed out. Of course, if insertions are not pulled back, this ceases to be true. Nevertheless, once the numerical data is fixed, there are only finitely many rigid curves to sum over. We may always perform a logarithmic modification to $\mathcal{Y}$ to ensure that transversality holds for maps with each of these fixed rigid types. This is essentially the strategy employed in [NS06], though in that case the rigid tropical curves are 0-dimensional.

The *decomposition theorem* of Abramovich–Chen–Gross–Siebert equates the virtual class of maps to the special fibre with a sum of pieces,

$$[\mathrm{ACGS}(Y_\eta)] = \sum_i \frac{m_{\rho_i}}{\mathrm{Aut}(\rho_i)} [\mathrm{ACGS}_{\rho_i}(Y_0)] \in A_\star(\mathrm{ACGS}(\mathcal{Y})),$$

in parallel with equation (2). The space $\mathrm{ACGS}_{\rho_i}(Y_0)$ is a moduli space of logarithmic stable maps to Y_0 , equipped with a uniform rescaling to the fixed rigid type ρ_i . See [ACGS20a, Theorem 1.1.2].

7.2. The generic case: Transverse gluing. Fix a rigid tropical curve ρ with graph G . Write Y_0 as a union of irreducible components X_v for v a vertex of G . Our goal is now to describe the moduli space $\mathrm{ACGS}_{\rho}(Y_0)$, up to toroidal modifications, in terms of simpler pieces. Over the loci of generic curves in the Artin fan (that is, over the vertex in the tropical moduli space), this is straightforward. The (possibly empty) open stratum

$$\mathrm{ACGS}_{\rho}^{\circ}(Y_0) \subset \mathrm{ACGS}_{\rho}(Y_0)$$

of maps $C \rightarrow Y_0$ whose tropicalization is exactly ρ is exactly this locus of generic maps. Recall that we are assuming by modifications that such generic curves are supported away from higher-codimension strata in Y_0 . Consider the target normalization $\bigcup X_v \rightarrow Y_0$. We obtain a cutting morphism

$$\kappa^{\circ}: \mathrm{ACGS}_{\rho}^{\circ}(Y_0) \rightarrow \prod_v \mathrm{ACGS}_{\rho}^{\circ}(X_v).$$

For every edge e of G , there are a pair of scheme-theoretic evaluation morphisms at the corresponding marked points, giving us

$$\mathrm{ev}: \prod_v \mathrm{ACGS}_{\rho}^{\circ}(X_v) \rightarrow \prod_e (D_e^2)^{\circ},$$

where the genericity assumption tells us that the scheme-theoretic evaluations all map to the dense open stratum in each divisor. The diagonal copy of $\prod_e D_e^{\circ}$ in the square is a regular embedding since the interiors of these divisors are smooth. The scheme-theoretic fibre product gives rise to a new space

$$\begin{array}{ccc} \mathcal{G}^{\circ} & \longrightarrow & \prod_v \mathrm{ACGS}_{\rho}^{\circ}(X_v) \\ \downarrow & & \downarrow \mathrm{ev} \\ \prod_e D_e^{\circ} & \xrightarrow{\Delta^{\circ}} & \prod_e (D_e^2)^{\circ}. \end{array}$$

The constructions above yield

$$\kappa^{\circ} \times \mathrm{ev}: \mathrm{ACGS}_{\rho}^{\circ}(Y_0) \rightarrow \mathcal{G}^{\circ}.$$

This morphism is finite and étale. A moduli point of $\mathcal{G}^{\circ}$ is a logarithmic map from a disjoint union of curves to the components of Y_0 that scheme-theoretically glue. As the curve is transverse to the strata, the scheme-theoretic data has a logarithmic enhancement, and in fact finitely many such enhancements [Wis19, Theorem 1.1]. Since Δ° is a regular embedding, the stack $\mathcal{G}^{\circ}$ also inherits a virtual fundamental class from the product. The virtual class on the space $\mathrm{ACGS}_{\rho}^{\circ}(Y_0)$ can be pushed forward to $\mathcal{G}^{\circ}$, and the two classes agree on $\mathcal{G}^{\circ}$ up to a finite multiple.

Remark 7.2.1. The moduli spaces above are generally non-proper, and the maps parameterized by deeper strata in their natural compactifications $\mathrm{ACGS}_{\rho}(Y_0)$ are non-transverse. In certain special situations, invariants can be defined on the non-proper moduli spaces or be supported on the locus of transverse maps. In such situations, a degeneration formula can be proved. This is the situation in each of the papers [Bou19, CJMR21, GPS10, MR20, NS06]. Section 7.3 uses expansions to achieve such transversality in general.

7.3. The general case: Transversality via expansions. The antecedent paragraphs perform gluing in the locus that is dual to the tropically trivial part of the moduli space. We now extend the picture to the full moduli space. The modified spaces have been engineered in order to ensure that the transversality properties used in the case above are present at every point, so the arguments extend in a natural way.

We cease working with the Abramovich–Chen–Gross–Siebert space and replace $\text{ACGS}_\rho(Y_0)$ with a moduli stack $\mathcal{K}_\rho^\lambda(Y_0)$ over $\text{Spec}(\mathbb{N} \rightarrow \mathbb{C})$ defined as the stack parameterizing logarithmic stable maps to families of expansions of Y_0 with discrete data ρ , where λ is any of the elements in the inverse system of refinements constructed in Section 4.4. Consider a stable map over S

$$\begin{array}{ccccc} C & \xrightarrow{\quad} & \mathcal{Y}^\lambda & \longrightarrow & Y_0 \\ & \searrow & \swarrow & & \\ & & S & & \end{array}$$

that is transverse to the strata. Normalize Y_0 and observe that each component X_v comes equipped with a collection of divisors. These are either irreducible components of the normalization with divisors of Y_0 or preimages of the double divisors of Y_0 . Equip each irreducible component with the corresponding divisorial logarithmic structure. Similarly, we partially normalize $\mathcal{Y}^\lambda$ into pieces $\mathcal{X}_v^\lambda$, noting that each $\mathcal{X}_v^\lambda$ is a degeneration of the corresponding component of Y_0 . The curve also partially normalizes to produce, for each v , a morphism over S

$$f_i: C_v \rightarrow \mathcal{X}_v^\lambda \rightarrow X_v,$$

where the fibres over S of $\mathcal{X}_v^\lambda$ are expansions of X_v . The scheme-theoretic structure of this map is clear. In fact, it has a natural logarithmic structure. The curve C_v has a logarithmic structure obtained by marking the preimages of nodes as marked points. The scheme X_v has its divisorial logarithmic structure, while the expansion $\mathcal{X}_v^\lambda$ is a logarithmic modification of the trivial family of X_v .

LEMMA 7.3.1. *The morphism f_i is logarithmic, with the logarithmic structures given to the curve and target as above.*

Proof. The logarithmic structure at the curve C_v and of the map away from the preimages of the normalized nodes is clear. At these marked points, the curve is dimensionally transverse to the target, so pull back the divisorial logarithmic structure on X_v to C_v . The resulting logarithmic structure on C_v is generated in the neighbourhood of a smooth point by a power of the local parameter at the marked point, with exponent equal to the contact order. This local parameter determines an element of the logarithmic structure sheaf of the curve; the identification defines the desired logarithmic structure on the morphism. $\square$

This procedure gives rise to a cutting morphism

$$\kappa: \mathcal{K}_\rho^\lambda(Y_0) \rightarrow \prod_v \mathcal{K}_\rho^\lambda(X_v). \quad (4)$$

In order to glue, we apply Theorem 6.4.3 to first modify the product. As we have done in Part I of the paper, the subdivision

$$\bigtimes_v^\lambda \mathcal{K}_\rho(\Sigma_v) \rightarrow \prod_v \mathcal{K}_\rho^\lambda(\Sigma_v)$$

can be pulled back to produce a logarithmic modification of stacks

$$\times_v^\lambda \mathbf{K}_\rho(X_v) \rightarrow \prod_v \mathbf{K}_\rho^\lambda(X_v).$$

The stack $\times_v^\lambda \mathbf{K}_\rho(X_v)$ is equipped with a natural virtual fundamental class, coming from the analogous modified product of maps to the Artin fan. For a cofinal system of choices λ , we also have a modified cutting morphism

$$\kappa: \mathbf{K}_\rho^\lambda(Y_0) \rightarrow \times_v^\lambda \mathbf{K}_\rho(X_v).$$

We are guilty of a mild abuse of notation. The cutting morphism in equation (4) need not immediately lift to the modified product, but the source $\mathbf{K}_\rho^\lambda(Y_0)$ can be modified to lift the cutting morphism. The modification is formed by the fibre product of the corresponding tropical moduli spaces and therefore amounts to a different choice of λ in the inverse system. Since we have not specified a particular choice of λ , we avoid overburdening the notation.

7.4. Virtual classes. Given a point p of the modified product $\times_v^\lambda \mathbf{K}_\rho(X_v)$, the moduli space associates with it a disconnected universal curve, a glued target family, and a transverse map from each curve with a partial normalization of the target. Let $\mathcal{Y}_0^\lambda$ be the fibre of the target family over this chosen point of the product. Let

$$\mathcal{Y}_0^{\lambda,\circ}[p] \subset \mathcal{Y}_0^\lambda[p]$$

be the complement of the union of the closed codimension 2 strata in $\mathcal{Y}_0^\lambda[p]$. In other words, this interior consists of the irreducible components and their locally closed divisorial boundaries. The codimension 1 strata of $\mathcal{Y}_0^{\lambda,\circ}[p]$ are therefore smooth.

The same works in families. For the expanded target family, the fibrewise complement of the codimension 2 strata forms a flat family of open targets and a family of smooth divisors.

LEMMA 7.4.1. *Let $\mathcal{Y}_0^\lambda \rightarrow \times_v^\lambda \mathbf{K}_\rho(X_v)$ be the family of expanded, compatibly glued targets. The union of the $\mathcal{Y}_0^{\lambda,\circ}[p]$ ranging over p in the base form a flat family. Furthermore, the universal map*

$$\bigcup C_v \rightarrow \mathcal{Y}_0^{\lambda,\circ}$$

is proper.

Proof. Switching to the tropical picture, we have a map $\bigcup \square_v \rightarrow \Sigma_\infty$. By the transversality hypothesis, every cone of $\bigcup \square_v$ surjects onto a cone of the target Σ_∞ . This gives a union of cones, whose support determines an open subscheme $\mathcal{Y}_0^{\lambda,\circ}$ of the target family, which is flat. The properness of the universal map also follows by transversality: by design we have built target expansions such that the higher-codimension strata are never met by the curve. $\square$

The smoothness of the divisors is of particular use to us here, and we will refer to them as *open divisors*. Recall that the edges e of G produce boundary evaluations on the modified product stack:

$$\mathrm{ev}_e: \times_v^\lambda \mathbf{K}_\rho(X_v) \rightarrow D_e^{\{2\}}(\mathcal{X}^\lambda).$$

The symbol $D_e(\mathcal{X}^\lambda)$ denotes the open divisor family supporting the evaluations of marked points coming from e . The evaluation above is to its fibre square. Let $\mathcal{D}_G$ be the fibre product of the open divisor families $D_e(\mathcal{X}^\lambda)$, ranging over all nodal edges e of G . Let $\mathcal{D}_G^2$ be its fibre square. We

have an evaluation ev at points on each curve corresponding to the two flags of an edge e in G . Packaging these data leads to a fibre square in ordinary schemes

$$\begin{array}{ccc} \mathcal{G}^\lambda(Y_0) & \xrightarrow{\text{split}} & \times_v^\lambda \mathcal{K}_\rho(X_v) \\ \downarrow & & \downarrow \text{ev} \\ \mathcal{D}_G & \xrightarrow{\Delta} & \mathcal{D}_G^2. \end{array}$$

The fibre product parameterizes collections of logarithmic stable maps to expansions of X_v whose underlying maps glue to form a map to the scheme Y_0^λ . In particular, the maps have no natural logarithmic structure at the gluing nodes.

The space $\mathcal{G}^\lambda(Y_0)$ has a natural virtual fundamental class described as follows. Since the open divisors do not meet, it follows that $\mathcal{D}_G$ is smooth over the base. The morphism Δ is therefore a regular embedding and defines a perfect obstruction theory given by its normal bundle [Ful93, Section 6.2]. We can pull back this obstruction theory to the top row. Define the class $[\mathcal{G}^\lambda(Y_0)]^{\text{vir}}$ as the virtual fundamental class obtained by virtual pull back of the virtual class on the top right:

$$[\mathcal{G}^\lambda(Y_0)]^{\text{vir}} := \Delta^! \left[\times_v^\lambda \mathcal{K}_\rho(X_v) \right]^{\text{vir}}.$$

This class has a standard description by unpacking the obstruction theory provided by the diagonal and Manolache’s virtual pullback [Man12]; a detailed description can be found, for instance, in [Bou19, Section 7.4] or [KLR21]. The upshot is the following. First let $\mathcal{A}_v$ be the Artin fan of X_v , and let $\times_v^\lambda \mathcal{K}_\rho(\mathcal{A}_v)$ be the Artin fan version of the modified product – pull back the tropical moduli space and its universal families to the product of the spaces of maps to the Artin fan. Consider the natural morphism from the cotangent complex of the diagonal:

$$\text{ev}^* \mathcal{L}_\Delta \rightarrow \text{split}^* \boxtimes_v (\mathcal{R}\pi_{v*} f_v^* T_v^{\log})^\vee,$$

where $T_v^{\log}$ is the logarithmic tangent bundle of the target X_v and π_v and f_v are the universal curve and map, respectively. The cone of this morphism defines an obstruction theory on the glued space and thus an obstruction theory relative to $\times_v^\lambda \mathcal{K}_\rho(\mathcal{A}_v)$. Geometrically, the deformations are controlled by those logarithmic tangent fields of the target that preserve the gluing conditions.

Remark 7.4.2. The relative diagonal used above is *not* the pullback of the diagonal from the projection to D^2 , although it is contained inside of it. The difference between the two accounts for the tropical transversality requirement in the previous section. In the usual degeneration formula, this diagonal will have a Künneth decomposition in cohomology, and Poincaré duality can then be used to distribute insertions across the gluing markings. In this case, the diagonal does not have a Künneth decomposition and must be worked with directly. A similar point arises in work of Pandharipande and Pixton on the Gromov–Witten/pairs correspondence for the quintic threefold, and the techniques developed there are likely to be useful here [PP17, Section 1.2]. In order to understand this conceptually, one can apply a similar and simpler version of the ideas in the present paper to the evaluation space – the space of maps from n points to a divisor D which are not allowed to touch the boundary of D . This will produce a modification of the evaluation space constructed in [ACGM10], and we will construct it in a separate paper.

By performing a base change for the degeneration $\mathcal{Y} \rightarrow \mathbb{A}^1$, we assume that its special fibre is reduced. By modifying λ , we ensure that the universal target family is also reduced.

LEMMA 7.4.3. *The morphism that forgets the logarithmic structure at the gluing nodes*

$$\mu: \mathcal{K}_\rho^\lambda(Y_0) \rightarrow \mathcal{G}^\lambda(Y_0)$$

is finite and étale of degree $\prod_e m_e$, where m_e is the expansion factor on the edge e of the rigid tropical curve.

Proof. This is a well-known fact that has appeared in various places in the literature. To see that the map is surjective, note that every point in $\mathcal{G}^\lambda(Y_0)$ gives rise to a map to the special fibre of an expansion of the degeneration $\mathcal{Y}$

$$C_0^\lambda \rightarrow \mathcal{Y}_0^\lambda \hookrightarrow \mathcal{Y}^\lambda$$

such that C_0^λ is transverse to the special fibre. In particular, the nodes of C_0^λ map to locally closed codimension 1 strata in Y_0^λ where two components meet. The special fibre Y_0^λ inherits a logarithmic structure from the degeneration $\mathcal{Y}^\lambda$, and since the curve is dimensionally transverse, the pullback of the logarithmic structure gives rise to a unique logarithmic lift in the category of fine but not necessarily saturated logarithmic schemes. The existence of lifts is immediate from transversality [AMW14, Lemma 4.2.2], while the uniqueness follows from [Wis19].

It remains to count the number of lifts and show that it is the claimed number. The calculation is well known and is recorded in complete generality and great detail in [ACGS20a, Theorem 5.3.3]. Note that [NS06, Proposition 7.1] and [Che14a, Lemma 7.9.1] both suffice here in the reduced case. $\square$

We have one final compatibility to check. Parallel to the tropical smoothing at the end of Section 6, we verify that once the map is transverse at the gluing nodes, there are no additional obstructions to promote a map to a logarithmic one.

LEMMA 7.4.4. *There is an equality*

$$\mu_*[\mathcal{K}_\rho^\lambda(Y_0)]^{\text{vir}} = \prod_e m_e \cdot [\mathcal{G}^\lambda(Y_0)]^{\text{vir}}.$$

This fact has appeared in the literature in nearly identical form. The reader can find the proof in detail in [Che14a, Section 7.10] and [AF16, Section 5.8.1]. We follow [AW18, Section 6] and [Wis11, Section 7.3].

Proof. Two approaches are available. The one used in [Che14a] is as follows. Since the morphism μ is étale, the obstruction theory on $\mathcal{G}^\lambda(Y_0)$ pulls back to an obstruction theory on $\mathcal{K}_\rho^\lambda(Y_0)$. This space already has an obstruction theory, so we must check that the two obstruction theories coincide. Since our target is expanded, simply delete the codimension 2 strata of Y_0^λ , so that over every point, the target is a union of smooth varieties meeting transversely along a divisor that is smooth in each. Now directly apply the calculation involving cotangent complexes in [Che14a, Proposition 7.10.1] to conclude that the two obstruction theories coincide, from which the result follows immediately.

We give the details on a second approach, seeing the compatibility of obstruction theories without reference to the cotangent complex by following [Wis11, Section 7.3]. The standard

obstruction theory for maps to $\mathcal{Y}$ is obtained by considering lifts over $K_\rho^\lambda(\mathcal{A}_{\mathcal{Y}})$:

$$\begin{array}{ccccc}
 & & f & \nearrow & Y_0 \\
 & & & \nearrow & \downarrow \\
 C & \longrightarrow & C' & \longrightarrow & \mathcal{A}_{\mathcal{Y}} \\
 \downarrow & & \downarrow & & \downarrow \\
 S & \longrightarrow & S' & \longrightarrow & \text{Spec}(\mathbb{N} \rightarrow \mathbb{C}),
 \end{array} \tag{5}$$

where $S \subset S'$ is a strict square zero thickening of S . Here, the map $Y_0 \rightarrow \mathcal{Y}$ is a strict logarithmic morphism. The lifts of this diagram form a torsor under the sheaf of abelian groups $f^*T_{Y_0}^{\log} \otimes I_{S \subset S'}$. The stack on S of such torsors on C defines an obstruction theory on the space of maps, relatively over $K_\rho^\lambda(\mathcal{A}_{\mathcal{Y}})$.

On the other hand, the obstruction theory pulled back from $\mathcal{G}^\lambda(Y_0)$ by definition considers the deformation problem for logarithmic maps from partial normalizations of a logarithmic curve to expansions of the components X_v with their divisorial logarithmic structure that are required to glue scheme-theoretically to form a map to the underlying scheme of Y_0 .

The lifting problem for this second obstruction theory is obtained from diagrams:

$$\begin{array}{ccc}
 & f & \nearrow \\
 C & \longrightarrow & C' \\
 \downarrow & & \downarrow \\
 S & \longrightarrow & S'
 \end{array} \quad \begin{array}{c} Y_0^\lambda \\ \downarrow \\ \mathcal{Y}^\lambda \end{array} \tag{6}$$

Over S' , there is a canonical closed embedding of schemes, including the special fibre as

$$\mathcal{Y}_0^\lambda \rightarrow \mathcal{Y}^\lambda,$$

where $\mathcal{Y}^\lambda \rightarrow \mathbb{A}^1$ is the relative degeneration over S' chosen in the construction of the modified product $\times_v^\lambda K_\rho(X_v)$. The map $C \rightarrow \mathcal{Y}_0^\lambda$ is transverse to the strata, and since the extension is strict, any lift from C' is automatically transverse. Endow $\mathcal{Y}_0^\lambda$ with the logarithmic structure coming from its inclusion as the special fibre of the degeneration. The transversality guarantees that this map lifts to a logarithmic map, compatible with the given logarithmic structures on C and C' .

Conversely, given a lift of diagram (5), simply forget the logarithmic structure at the marked nodes to obtain a lifting diagram (6). This canonically identifies the torsors defining the obstruction theories. The lifts of diagram (6) therefore also form a torsor under the sheaf of abelian groups $f^*T_{Y_0}^{\log} \otimes I_{S \subset S'}$. This identifies the obstruction theories, and virtual classes, as required. $\square$

7.5. Implementation and special cases. We summarize the previous sections and put the pieces together for easy access. The gluing formula is obtained as follows. For a rigid tropical stable map $[\rho]$ with underlying graph G , we construct a virtual birational modification of the product of moduli spaces of maps over the vertices. This space is equipped with boundary evaluations corresponding to flags of edges in G . The target of the boundary evaluation at e is generically the divisor D_e dual to e . Over deeper strata in the modified product, the target is

replaced by toric bundles over the strata of D_e . The gluing fibre diagram is the following one:

$$\begin{array}{ccc} \mathcal{G}^\lambda(Y_0) & \xrightarrow{\text{split}} & \times_v^\lambda \mathcal{K}_\rho(X_v) \\ \downarrow & & \downarrow \text{ev} \\ \mathcal{D}_G & \xrightarrow{\Delta} & \mathcal{D}_G^2. \end{array}$$

The two vertical arrows are sections of smooth fibrations. Up to the combinatorial factor $\prod_e m_e$, the virtual count attached to $[\rho]$ is computed by a virtual integral on $\mathcal{G}^\lambda(Y_0)$. Our degeneration formula for the virtual class involves a relative diagonal, that is, the class of the fibrewise diagonal in the fibre square of the divisor expansion. This complicates numerical applications of the degeneration formula. An identical phenomenon occurs in Parker’s theory. Parker’s refined cohomology, which supports his cycle-theoretic degeneration formula, does not satisfy a Künneth formula and hence has no diagonal splitting. Nonetheless, our formula collapses in special cases.

7.5.1 Examples in the existing literature. A class of examples are present in the literature already. As has already been mentioned, when the degeneration $\mathcal{Y}$ has no triple of components intersecting, the formula above collapses immediately to Li’s formula. Examples beyond this case can be found in surface geometries. In the papers [Bou19, CJMR21, GPS10], explicit arguments show that the relevant Gromov–Witten cycles can be supported away from loci where the universal divisor expands non-trivially. These arguments are recorded in [Bou19, Proposition 11], [CJMR21, Proposition 5.10], and [GPS10, Proposition 4.2] in the appropriate contexts. It follows that the relative and ordinary diagonals coincide for the purpose of calculating invariants, and a degeneration formula follows. On the other hand, when dealing with unobstructed geometries, such as genus 0 logarithmic descendant invariants of toric varieties, the moduli spaces are 0-dimensional, and again, no additional divisor bubbling arises. This provides a direct link to work Mandel and Ruddat [MR20] and earlier work of Nishinou and Siebert [NS06].

7.5.2 The triple point degeneration in genus 0. There is a simplified gluing formula that can be used for genus 0 for degenerations where the dual complex has dimension 2 and, in particular, the divisors are smooth pairs.

When the target is a toric surface degeneration, this simplification is implicit throughout the tropical literature; see for instance [GM07]. It is also explained by Parker in his framework in [ParB15, Section 4].

Let $\mathcal{Y} \rightarrow \mathbb{A}^1$ be a simple normal crossings degeneration with generic fibre Y_η and special fibre Y . Also assume that no four irreducible components of the special fibre Y_0 meet. The fibres of the associated tropical degeneration $\Sigma \rightarrow \mathbb{R}_{\geq 0}$ are therefore at most 2-dimensional polyhedral complexes.

We describe an algorithm for computing the Gromov–Witten theory of Y_η in genus 0 with stationary insertions using the degeneration above. Fix n points in Y_η , and let $p_1, \dots, p_n$ denote their limits in the special fibre. The tropicalizations of these points give rise to tropical point constraints $\{p_i^{\text{trop}}\}$ in Σ_η . Fix a rigid tropical stable map $\gamma = [f_\eta: \square_\eta \rightarrow \Sigma_\eta]$ meeting the point constraints $\{p_i^{\text{trop}}\}$. We compute the 0-cycle defined, for any indexing choice λ , as

$$\langle p_1, \dots, p_n \rangle_\gamma^{Y_0} := \int_{[\mathcal{K}_\gamma^\lambda(Y_0)]^{\text{vir}}} \prod_{i=1}^n \text{ev}_i^*(p_i) = \int_{[\text{ACGS}_\rho(Y_0)]^{\text{vir}}} \prod_{i=1}^n \text{ev}_i^*(p_i).$$

In this circumstance, observe that the irreducible component to which the evaluation morphism attached to an edge e maps is always D_e , rather than a bundle over a stratum of it. This follows from the fact that D_e has a 1-dimensional cone complex. The genus 0 condition precludes the possibility of parallel edges in G . These facts combine to simplify the gluing formula in this case to the following algorithm.

Algorithm 7.5.1. The invariant $\langle p_1, \dots, p_n \rangle_\gamma^{Y_0}$ is computed by the following procedure:

Step A: Orient the graph $\sqsubset_\eta$ such that (i) all ends of contact order 0 are incoming to their incident vertex, and (ii) every vertex has at most one outgoing edge.

Step B: For each vertex v of $\sqsubset_\eta$, let K_v be the moduli space of maps to expansions of the component Y_v with type given by the star of v in $[\sqsubset_\infty \rightarrow \Sigma_\infty]$.

Step C: For an edge e , let D_e be the stratum to which it is dual. Let v be a vertex with exiting edge e_v ; define the operator on cohomology

$$\text{GW}_v: \bigotimes_{e: \text{incoming}} H^*(D_e; \mathbb{Q}) \rightarrow H^*(D_{e_v}; \mathbb{Q})$$

as follows. Given a class in $\bigotimes_{e: \text{incoming}} H^*(D_e; \mathbb{Q})$, pull it back via the evaluation morphisms to the moduli space K_v , cap it with $[K_v]^{\text{vir}}$, and push it forward via the evaluation morphism ev_v .

Step D: Assign incidence cycles to edges e as follows. If e is an edge attached to a marked point with trivial contact, then define a class $\alpha(e)$ to be the class of a point. Otherwise, if e is the exiting edge of a vertex v , define

$$\alpha(e) = \text{GW}_v(\bigotimes_{e'} \alpha(e')),$$

where e' ranges over the incoming edges incident to v .

7.5.3 Validation of the algorithm. We provide a sketch that the simplified algorithm computes the invariants at hand. Fix an edge e of the curve $\sqsubset_\infty$. If we cut the graph $\sqsubset_\infty$ at e , we obtain two tropical curves. Let K_1 and K_2 denote the moduli spaces associated with these two tropical curves. According to the main theorem, we have the diagram

$$\widetilde{K_1 \times K_2} \rightarrow D_e^{\{2\}} \rightarrow D_e \times D_e, \quad (7)$$

where the first map is a section of a smooth fibration. Note that by construction, $D_e^{\{2\}}$ is an open subset of a logarithmic modification associated with the product

$$D_e^{\{2\}} \rightarrow K_1 \times K_2 \times D_e^2.$$

The second arrow in (7) is obtained by blowdown and projection onto the second factor in this product. By passing to a further subdivision, we can assume that there is a factorization

$$\widetilde{K_1 \times K_2} \xrightarrow{\vartheta} \widetilde{D_e \times D_e} \xrightarrow{\varpi} D_e \times D_e$$

such that ϑ is combinatorially flat and $\widetilde{D_e \times D_e}$ is smooth. The degeneration formula proved previously shows that the virtual class of the moduli space associated with γ can be computed in two steps: (i) take the diagonal D_e inside $D_e \times D_e$ and calculate its ϖ -strict transform, and (ii) pull back the resulting Chow cohomology class via ϑ , and let it operate on the virtual class.

We now use a simple observation. The morphism

$$K_1 \times K_2 \rightarrow D_e \times D_e$$

is *automatically* combinatorially flat. Indeed, the morphism is a product of two morphisms, and it suffices to check the combinatorial flatness on each factor. Since the tropical target has cones of dimension 0 or 1, the condition is immediate. Now consider the following intersection diagram:

$$\begin{array}{ccc}
 \widetilde{K_1 \times K_2} & \searrow & \\
 & (K_1 \times K_2)^\dagger \longrightarrow K_1 \times K_2 & \\
 & \downarrow & \downarrow \\
 \widetilde{D_e \times D_e} & \longrightarrow & D_e \times D_e.
 \end{array}$$

The square is defined by fine and saturated logarithmic base change. However, the right vertical arrow is combinatorially flat, so the square is a fibre square in the category of algebraic stacks as well. Since the diagonal in $\widetilde{D_e \times D_e}$ pushes forward to the diagonal in $D_e \times D_e$, it follows that the pushforward of the virtual class $[K_\gamma]$ to $K_1 \times K_2$ is precisely the pullback of the diagonal in $D_e \times D_e$ under the right vertical arrow above.

The algorithm is now justified for a single edge e by an elementary diagram chase; the general case follows by induction. $\square$

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