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Quasi-plurisubharmonic envelopes 2: Bounds on Monge–Ampère volumes

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Abstract

In part 1 of the present series of papers, the authors developed a new approach to L^∞-a priori estimates for degenerate complex Monge–Ampère equations when the reference form is closed. This simplifying assumption was used to ensure the constancy of the volumes of Monge–Ampère measures. Here, we study the way these volumes stay away from zero and infinity when the reference form is no longer closed. We establish a transcendental version of the Grauert–Riemenschneider conjecture, partially answering conjectures of Demailly–Păun [Ann. of Math. **159** (2004), 1247–1274] and Boucksom–Demailly–Păun–Peternell [J. Algebraic Geom. **22** (2013), 201–248]. Our approach relies on a fine use of quasi-plurisubharmonic envelopes. The results obtained here are used in the third part of the series for solving degenerate complex Monge–Ampère equations on compact Hermitian varieties.

Introduction

The study of complex Monge–Ampère equations on compact Hermitian (non-Kähler) manifolds has gained considerable interest in the last decade after Tosatti and Weinkove established an appropriate version of Yau's theorem in [TW10]. The smooth Gauduchon–Calabi–Yau conjecture has been further solved by Székelyhidi–Tosatti–Weinkove [STW17], while the pluripotential theory has been partially extended by Dinew, Kołodziej, and Nguyen [DK12, KN15, Din16, KN19].

As in Yau's original proof [Yau78], the method of [TW10] consists in establishing a priori estimates along a continuity path, and the most delicate estimate again turns out to be the a priori L^∞-estimate. The fact that the reference form is not closed introduces several new difficulties: there are many extra terms to handle when using Stokes' theorem, and it becomes non-trivial to get uniform bounds on the total Monge–Ampère volumes involved in the estimates.

In [GL21a], we developed a new approach for establishing uniform a priori estimates, restricting to the context of Kähler manifolds for simplicity. While the pluripotential approach consists in measuring the Monge–Ampère capacity of sublevel sets $(\varphi < -t)$, we directly measure the volume of the latter, avoiding delicate integration by parts. Our approach applies in the Hermitian

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setting, once certain Monge–Ampère volumes are under control. Understanding the behavior of these volumes is the main focus of this article, while [GL21b] is concerned with the resolution of degenerate complex Monge–Ampère equations.

We let X denote a compact complex manifold of complex dimension n, equipped with a Hermitian metric ω_X. The first difficulty we face is to decide whether

$$v_+(\omega_X) := \sup\left\{\int_X (\omega_X + dd^c\varphi)^n \colon \varphi \in \mathrm{PSH}(X,\omega_X) \cap L^\infty(X)\right\}$$

is finite. Here $d = \partial + \overline{\partial}$, $d^c = (1/2i)(\partial - \overline{\partial})$, and $\mathrm{PSH}(X,\omega_X)$ is the set of ω_X-plurisubharmonic functions: these are functions $u\colon X \to \mathbb{R} \cup \{-\infty\}$ that are locally given as the sum of a smooth and a plurisubharmonic function, and such that $\omega_X + dd^c u \geqslant 0$ is a positive current. The complex Monge–Ampère measure $(\omega_X + dd^c u)^n$ is well defined by [BT82].

Building on works of Chiose [Chi16b] and Guan–Li [GL10], we provide several results that ensure that the condition $v_+(\omega_X) < +\infty$ is satisfied

- for any compact complex manifold X of dimension $n \leqslant 2$,
- for any threefold that admits a pluriclosed metric $dd^c\tilde{\omega}_X = 0$,
- as soon as there exists a metric $\tilde{\omega}_X$ such that $dd^c\tilde{\omega}_X = 0$ and $dd^c\tilde{\omega}_X^2 = 0$,
- as soon as X belongs to the Fujiki class $\mathcal{C}$.

The Fujiki class is the class of compact complex manifolds that are bimeromorphically equivalent to Kähler manifolds.

We also need to bound the Monge–Ampère volumes from below. Given a semi-positive form ω, we introduce several positivity properties:

- We say that ω is *non-collapsing* if there is no bounded ω-plurisubharmonic (ω-psh) function u such that $(\omega + dd^c u)^n \equiv 0$.
- The form ω satisfies condition (B) if there exists a constant $B > 0$ such that
$$-B\omega^2 \leqslant dd^c\omega \leqslant B\omega^2 \quad \text{and} \quad -B\omega^3 \leqslant d\omega \wedge d^c\omega \leqslant B\omega^3\,.$$
- We say that ω is *uniformly non-collapsing* if
$$v_-(\omega) := \inf\left\{\int_X (\omega + dd^c u)^n \colon u \in \mathrm{PSH}(X,\omega) \cap L^\infty(X)\right\} > 0\,.$$

The non-collapsing condition is the minimal positivity condition one should require. We show in Proposition 2.8 that it implies the *domination principle*, a useful extension of the classical maximum principle. We provide a simple example showing that having positive volume $\int_X \omega^n > 0$ does not prevent from being collapsing (see Example 3.5).

After providing a simplified proof of the Kołodziej–Nguyen modified comparison principle (see [KN15, Theorem 0.5] and Theorem 1.5), we show that condition (B) implies non-collapsing. Condition (B) is, for example, satisfied by any form ω that is the pull-back of a Hermitian form on a singular Hermitian variety.

When ω is closed, simple integration by parts reveals that $v_-(\omega) = \int_X \omega^n$ is positive as soon as ω is positive at some point. Bounding $v_-(\omega)$ from below is a much more delicate issue in general. We show in Proposition 3.4 that ω is uniformly non-collapsing if one restricts to ω-psh functions that are uniformly bounded by a fixed constant M:

$$v_M^-(\omega) := \inf\left\{\int_X (\omega + dd^c u)^n \colon u \in \mathrm{PSH}(X,\omega) \text{ with } -M \leqslant u \leqslant 0\right\} > 0\,.$$

For non-uniformly bounded functions, we show the following.

Theorem A. *The condition $v_+(\omega_X) < +\infty$ is independent of the choice of ω_X; it is moreover invariant under bimeromorphic change of coordinates. The condition $v_-(\omega_X) > 0$ is also independent of the choice of ω_X and invariant under bimeromorphic change of coordinates. In particular, these conditions both hold true if X belongs to the Fujiki class.*

We are not aware of a single example of a compact complex manifold such that $v_+(\omega_X) = +\infty$ or $v_-(\omega_X) = 0$. This is an important open problem.

The proof of Theorem A relies on a fine use of quasi-plurisubharmonic envelopes. These envelopes have been systematically studied in [GLZ19] in the Kähler framework. Adapting and generalizing [GLZ19] to this Hermitian setting, we prove the following in Section 2.

Theorem B. *Let ω be a semi-positive $(1,1)$-form. Given a Lebesgue measurable function $h\colon X \to \mathbb{R}$, we define the ω-plurisubharmonic envelope of h by*

$$P_\omega(h) := (\sup\{u \in \mathrm{PSH}(X,\omega)\colon u \leqslant h\})^*,$$

where the star means that we take the upper semi-continuous regularization. If h is bounded below, quasi-lower-semi-continuous, and $P_\omega(h) < +\infty$, then

- *$P_\omega(h)$ is a bounded ω-plurisubharmonic function;*
- *$P_\omega(h) \leqslant h$ in $X \setminus P$, where P is pluripolar;*
- *$(\omega + dd^c P_\omega(h))^n$ is concentrated on the contact set $\{P_\omega(h) = h\}$.*

An influential conjecture of Grauert–Riemenschneider [GR70] asked whether the existence of a semi-positive holomorphic line bundle $L \to X$ with $c_1(L)^n > 0$ implies that X is Moishezon (that is, bimeromorphically equivalent to a projective manifold). This conjecture has been solved positively by Siu in [Siu84] (with complements by [Siu85] and Demailly [Dem87]).

Demailly and Păun have proposed a transcendental version of this conjecture (see [DP04, Conjecture 0.8]): given a nef class $\alpha \in H^{1,1}_{\mathrm{BC}}(X,\mathbb{R})$ with $\alpha^n > 0$, they conjectured that α should contain a Kähler current, that is, a positive closed $(1,1)$-current that dominates a Hermitian form. Recall that the Bott–Chern cohomology group $H^{1,1}_{\mathrm{BC}}(X,\mathbb{R})$ is the quotient of closed real smooth $(1,1)$-forms by the image of $\mathcal{C}^\infty(X,\mathbb{R})$ under the dd^c-operator.

This influential conjecture has been further reinforced by Boucksom–Demailly–Păun–Peternell, who proposed a weak transcendental form of Demailly's holomorphic Morse inequalities [BDPP13, Conjecture 10.1]. This stronger conjecture has been solved recently by Witt Nyström when X is projective [Wit19].

Building on works of Chiose [Chi16a], Xiao [Xia15], and Popovici [Pop16], we obtain the following answer to the qualitative part of these conjectures.

Theorem C. *Let $\alpha, \beta \in H^{1,1}_{\mathrm{BC}}(X,\mathbb{C})$ be nef classes such that $\alpha^n > n\alpha^{n-1} \cdot \beta$. The following properties are equivalent:*

(i) *The class $\alpha - \beta$ contains a Kähler current.*

(ii) $v_+(\omega_X) < +\infty$.

(iii) *The manifold X belongs to the Fujiki class.*

A consequence of our analysis is that the conjectures of Demailly–Păun and Boucksom–Demailly–Păun–Peternell can be extended to non-closed forms, making sense outside the Fujiki class. Progress in the theory of complex Monge–Ampère equations on compact Hermitian manifolds has indeed shown that it is useful to consider dd^c-perturbations of non-closed nef forms. It is therefore natural to try and consider an extension of Theorem C. This is the content of Theorem 4.6 (when $\beta = 0$) and Theorem 4.15 (when $\beta \neq 0$).

1. Non-collapsing forms

In the whole article we let X denote a compact complex manifold of complex dimension $n \geqslant 1$, and we fix a smooth semi-positive $(1,1)$-form ω on X.

1.1 Positivity properties

1.1.1 *Monge–Ampère operators.* A function is quasi-plurisubharmonic (quasi-psh) if it is locally given as the sum of a smooth and a psh function.

Given an open set $U \subseteq X$, quasi-psh functions $\varphi\colon U \to \mathbb{R} \cup \{-\infty\}$ satisfying $\omega_\varphi \geqslant 0$ for $\omega_\varphi := \omega + dd^c\varphi$ in the weak sense of currents are called ω-psh functions on U. Constant functions are ω-psh functions since ω is semi-positive. A $\mathcal{C}^2$-smooth function $u \in \mathcal{C}^2(X)$ has bounded Hessian; hence εu is ω-psh on X if $0 < \varepsilon$ is small enough and ω is positive (that is, Hermitian).

DEFINITION 1.1. We let $\mathrm{PSH}(X,\omega)$ denote the set of all ω-plurisubharmonic functions that are not identically $-\infty$.

The set $\mathrm{PSH}(X,\omega)$ is a closed subset of $L^1(X)$ for the L^1-topology. We refer the reader to [Dem12, GZ17, Din16] for basic properties of ω-psh functions.

We briefly explain how to adapt [BT82] to the Hermitian context; we refer the reader to [DK12] for a more systematic treatment. Let u be a bounded quasi-psh function on X, and fix a Hermitian form ω_X such that $\omega_X + dd^c u \geqslant 0$. Let U be an open subset of X such that $\omega_X \leqslant \beta = dd^c\rho$ for some smooth function ρ in U. Since u is β-psh in U, the current $(\beta + dd^c u)^p$ is a well-defined closed positive current for $1 \leqslant p \leqslant k$, as follows from Bedford–Taylor's theory. We can thus set

$$(dd^c u)^k := \sum_{p=0}^{k} (-1)^p \binom{k}{p} (\beta + dd^c u)^{k-p} \wedge \beta^p\,.$$

This is a closed current of bidegree (k,k). We claim that $(dd^c u)^k$ does not depend on the choice of β. To see this, assume that γ is another Kähler form in U that dominates ω_X in U. By Demailly's regularization result [Dem92], we can find a sequence (u_j) of smooth ω_X-psh functions decreasing to u. Then

$$\sum_{p=0}^{k} (-1)^p \binom{k}{p} (\beta + dd^c u_j)^{k-p} \wedge \beta^p = (dd^c u_j)^k = \sum_{p=0}^{k} (-1)^p \binom{k}{p} (\gamma + dd^c u_j)^{k-p} \wedge \gamma^p\,.$$

When we let $j \to +\infty$, Bedford–Taylor's convergence theorem [BT82] ensures that

$$\sum_{p=0}^{k} (-1)^p \binom{k}{p} (\beta + dd^c u)^{k-p} \wedge \beta^p = \sum_{p=0}^{k} (-1)^p \binom{k}{p} (\gamma + dd^c u)^{k-p} \wedge \gamma^p\,,$$

proving the claim. Covering X by such small open sets U_j, we thus obtain a closed (k,k)-current $(dd^c u)^k$ globally defined on X. In particular, $(dd^c u)^n$ is a signed measure on X with total mass 0.

For $u \in \mathrm{PSH}(X,\omega) \cap L^\infty(X)$ and $1 \leqslant k \leqslant n$, we define

$$(\omega + dd^c u)^k := \sum_{p=0}^{k} \binom{k}{p} (dd^c u)^p \wedge \omega^{k-p}\,.$$

This current is positive. Indeed, when $\omega > 0$ is Hermitian, we can approximate u by smooth ω-psh functions u_j as above and obtain that $(\omega + dd^c u)^k$ is the limit of the smooth positive forms $(\omega + dd^c u_j)^k$. The general case follows by approximating ω by $\omega + \varepsilon\omega_X$ and letting $\varepsilon \to 0$. In particular, $(\omega + dd^c u)^n$ is a positive Radon measure on X.

If u, v are bounded ω-psh functions, we define similarly the mixed products

$$(dd^c u)^k \wedge (dd^c v)^l := \sum_{0 \leqslant p \leqslant k} \sum_{0 \leqslant q \leqslant l} (-1)^{p+q} \binom{k}{p} \binom{l}{q} \beta_u^{k-p} \wedge \beta_v^{l-q} \wedge \beta^{k+l-p-q}\,,$$

where $\beta_u := \beta + dd^c u$ and β is a Kähler form that dominates ω in U. The mixed Monge–Ampère measures

$$\omega_u^j \wedge \omega_v^{n-j} := (\omega + dd^c u)^j \wedge (\omega + dd^c v)^{n-j}$$

are thus similarly well defined as positive Radon measures for any $0 \leqslant j \leqslant n$ and any bounded ω-psh functions u, v.

1.1.2 *Maximum principle.* We will need the following maximum principle.

LEMMA 1.2. *Let u, v, φ be bounded ω-psh functions in $U \subseteq X$. For all $1 \leqslant j \leqslant n$,*

$$\mathbf{1}_{\{u>v\}} \omega_{\max(u,v)}^j \wedge \omega_\varphi^{n-j} = \mathbf{1}_{\{u>v\}} \omega_u^j \wedge \omega_\varphi^{n-j}\,, \tag{1}$$

and

$$\omega_{\max(u,v)}^j \wedge \omega_\varphi^{n-j} \geqslant \mathbf{1}_{\{u>v\}} \omega_u^j \wedge \omega_\varphi^{n-j} + \mathbf{1}_{\{u \leqslant v\}} \omega_v^j \wedge \omega_\varphi^{n-j}\,. \tag{2}$$

If moreover $u \geqslant v$, then $\mathbf{1}_{\{u=v\}} \omega_u^j \wedge \omega_\varphi^{n-j} \geqslant \mathbf{1}_{\{u=v\}} \omega_v^j \wedge \omega_\varphi^{n-j}$.

Proof. We prove (1). Since it is a local property, we can assume that there is a Kähler form β in U such that $\beta \geqslant \omega$. We write β_u for $\beta + dd^c u$. By definition, the left-hand side is equal to

$$\mathbf{1}_{\{u>v\}} \sum_{p,q} \binom{j}{p} \binom{n-j}{q} (-1)^{n-p-q} \beta_{\max(u,v)}^{j-p} \wedge \beta_\varphi^{n-j-q} \wedge (\beta - \omega)^{p+q}\,.$$

The result thus follows from Bedford–Taylor's maximum principle: we can use [GZ17, Theorem 3.27] for $T = \beta_\varphi^{n-j-q}$ (in [GZ17, Theorem 3.27], the current T is of bidegree $(n-p, n-p)$, but the same proof works here without modification).

To prove (2), we apply (1) to $\max(u, v+\varepsilon)$:

$$\begin{aligned}\omega_{\max(u,v+\varepsilon)}^j \wedge \omega_\varphi^{n-j} &\geqslant \mathbf{1}_{\{u>v+\varepsilon\}} \omega_u^j \wedge \omega_\varphi^{n-j} + \mathbf{1}_{\{v+\varepsilon>u\}} \omega_v^j \wedge \omega_\varphi^{n-j} \\ &\geqslant \mathbf{1}_{\{u>v+\varepsilon\}} \omega_u^j \wedge \omega_\varphi^{n-j} + \mathbf{1}_{\{v\geqslant u\}} \omega_v^j \wedge \omega_\varphi^{n-j}\,.\end{aligned}$$

Since the sets $\{u > v + \varepsilon\}$ increase to $\{u > v\}$ as $\varepsilon \searrow 0$, letting $\varepsilon \to 0$, we obtain the second statement.

The last statement follows by multiplying (2) with $\mathbf{1}_{\{u=v\}}$. □

1.1.3 *Condition* (B) *and non-collapsing.* We always assume in this article that $\int_X \omega^n > 0$. On a few occasions, we will need to assume positivity properties that are possibly slightly stronger.

DEFINITION 1.3. We say that ω satisfies condition (B) if there exists a $B \geqslant 0$ such that

$$-B\omega^2 \leqslant dd^c\omega \leqslant B\omega^2 \quad \text{and} \quad -B\omega^3 \leqslant d\omega \wedge d^c\omega \leqslant B\omega^3 \, .$$

Here are three different contexts where this condition is satisfied:

- Any Hermitian metric $\omega > 0$ satisfies condition (B).
- If $\pi \colon X \to Y$ is a desingularization of a singular compact complex variety Y and ω_Y is a Hermitian metric, then $\omega = \pi^*\omega_Y$ satisfies condition (B).
- If ω is semi-positive and closed, then it satisfies condition (B).

Combining these, one obtains further settings where condition (B) is satisfied.

DEFINITION 1.4. We say that ω is *non-collapsing* if for any bounded ω-psh function, the complex Monge–Ampère measure $(\omega + dd^c u)^n$ has positive mass: $\int_X \omega_u^n > 0$.

We shall see in Corollary 1.6 below that condition (B) implies non-collapsing.

1.2 Comparison principle

The comparison principle plays a central role in Kähler pluripotential theory. Its proof breaks down in the Hermitian setting, as it heavily relies on the closedness of the reference form ω through the preservation of Monge–Ampère masses. In that context, the following "modified comparison principle" has been established by Kołodziej–Nguyen [KN15, Theorem 0.2].

THEOREM 1.5. *Assume that ω satisfies condition (B), and let u, v be bounded ω-psh functions. For $\lambda \in (0,1)$, we set $m_\lambda = \inf_X\{u - (1-\lambda)v\}$. Then*

$$\left(1 - \frac{4B(n-1)^2 s}{\lambda^3}\right)^n \int_{\{u<(1-\lambda)v+m_\lambda+s\}} \omega^n_{(1-\lambda)v} \leqslant \int_{\{u<(1-\lambda)v+m_\lambda+s\}} \omega_u^n$$

for all $0 < s < \lambda^3/(32B(n-1)^2)$.

The proof by Kołodziej–Nguyen relies on the main result of [DK12], together with extra fine estimates. We propose here a simplified proof.

Proof. Set $\phi := \max(u, (1-\lambda)v + m_\lambda + s)$ and $U_{\lambda,s} := \{u < (1-\lambda)v + m_\lambda + s\}$. Set $T_k := \omega_u^k \wedge \omega_\phi^{n-k}$ for $0 \leqslant k \leqslant n$ and $T_l = 0$ if $l < 0$. Set $a = Bs\lambda^{-3}(n-1)^2$. We are going to prove by induction on $k = 0, 1, \ldots, n-1$ that

$$(1 - 4a) \int_{U_{\lambda,s}} T_k \leqslant \int_{U_{\lambda,s}} T_{k+1} \, . \tag{3}$$

The conclusion follows since $(\omega_\phi)^n = (\omega_{(1-\lambda)v})^n$ in the plurifine open set $U_{\lambda,s}$.

We first prove (3) for $k = 0$. Since $u \leqslant \phi$, Lemma 1.2 ensures that

$$\mathbf{1}_{\{u=\phi\}}\omega_\phi^n \geqslant \mathbf{1}_{\{u=\phi\}}\omega_u \wedge \omega_\phi^{n-1} \, .$$

Observing that $U_{\lambda,s} = \{u < \phi\}$, we infer

$$\int_X dd^c(\phi - u) \wedge \omega_\phi^{n-1} = \int_X (\omega_\phi - \omega_u) \wedge \omega_\phi^{n-1} \geqslant \int_{U_{\lambda,s}} \omega_\phi^n - \int_{U_{\lambda,s}} \omega_u \wedge \omega_\phi^{n-1} \, .$$

A direct computation shows that

$$\begin{aligned} dd^c\omega_\phi^{n-1} &= (n-1)dd^c\omega \wedge \omega_\phi^{n-2} + (n-1)(n-2)d\omega \wedge d^c\omega \wedge \omega_\phi^{n-3} \\ &\leqslant (n-1)B\omega^2 \wedge \omega_\phi^{n-2} + (n-1)(n-2)B\omega^3 \wedge \omega_\phi^{n-3} \end{aligned}$$

since ω satisfies condition (B). As $\phi - u \geqslant 0$, it follows from Stokes' theorem that

$$\int_X dd^c(\phi - u) \wedge \omega_\phi^{n-1} \leqslant (n-1)B \left\{ \int_X (\phi - u)\omega^2 \wedge \omega_\phi^{n-2} + (n-2) \int_X (\phi - u)\omega^3 \wedge \omega_\phi^{n-3} \right\}.$$

Observe that

- $\lambda\omega \leqslant \omega_{(1-\lambda)v}$; hence $\omega^j \wedge \omega_\phi^{n-j} \leqslant \lambda^{-j}(\omega_{(1-\lambda)v})^j \wedge \omega_\phi^{n-j}$;
- $(\omega_{(1-\lambda)v})^j \wedge \omega_\phi^{n-j} = \omega_\phi^n$ in the plurifine open set $U_{\lambda,s}$;
- $0 \leqslant \phi - u \leqslant s$ and $\phi - u = 0$ on $X \setminus U_{\lambda,s}$.

We conclude that $\int_X (\phi - u)\omega^j \wedge \omega_\phi^{n-j} \leqslant s\lambda^{-j} \int_{U_{\lambda,s}} \omega_\phi^n$ for $j = 2, 3$; hence

$$\int_{U_{\lambda,s}} \omega_\phi^n - \int_{U_{\lambda,s}} \omega_u \wedge \omega_\phi^{n-1} \leqslant \int_X dd^c(\phi - u) \wedge \omega_\phi^{n-1} \leqslant \frac{Bs(n-1)^2}{\lambda^3} \int_{U_{\lambda,s}} \omega_\phi^n$$

since $\lambda^{-2} \leqslant \lambda^{-3}$. This yields (3) for $k = 0$.

We now assume that (3) holds for all $j \leqslant k-1$, and we check that it still holds for k. Observe that

$$\begin{aligned} dd^c\big(\omega_u^k \wedge \omega_\phi^{n-[k+1]}\big) &= k dd^c\omega \wedge \omega_u^{k-1} \wedge \omega_\phi^{n-[k+1]} + (n-[k+1]) dd^c\omega \wedge \omega_u^k \wedge \omega_\phi^{n-[k+2]} \\ &\quad + 2k(n-[k+1]) d\omega \wedge d^c\omega \wedge \omega_u^{k-1} \wedge \omega_\phi^{n-[k+2]} \\ &\quad + k(k-1) d\omega \wedge d^c\omega \wedge \omega_u^{k-2} \wedge \omega_\phi^{n-[k+1]} \\ &\quad + (n-[k+1])[n-(k+2)] d\omega \wedge d^c\omega \wedge \omega_u^k \wedge \omega_\phi^{n-[k+3]}. \end{aligned}$$

The same arguments as above therefore show that

$$\begin{aligned} \int_{U_{\lambda,s}} (T_k - T_{k+1}) &\leqslant \int_X (T_k - T_{k+1}) = \int_X (\phi - u) dd^c\big(\omega_u^k \wedge \omega_\phi^{n-[k+1]}\big) \\ &\leqslant \frac{Bs}{\lambda^3} \int_{U_{\lambda,s}} \big(k(k-1)T_{k-2} + 2k[n-k]T_{k-1} + (n-[k+1])^2 T_k\big) \\ &\leqslant a\left(\frac{1}{(1-4a)^2} + \frac{1}{1-4a} + 1\right) \int_{U_{\lambda,s}} T_k \leqslant 4a \int_{U_{\lambda,s}} T_k\,, \end{aligned}$$

where in the third inequality above, we have used the induction hypothesis, while the fourth inequality follows from the upper bound $4a < 1/8$. From this we obtain (3) for k, finishing the proof. ☐

Corollary 1.6. *If ω satisfies condition (B), then ω is non-collapsing.*

Proof. It follows from Theorem 1.5 that the domination principle holds (see [LPT21, Proposition 2.2]). The latter implies, in particular, that if u, v are ω-psh and bounded, then $e^{-v}(\omega + dd^c v)^n \geqslant e^{-u}(\omega + dd^c u)^n$ implies $v \leqslant u$ (see [LPT21, Proposition 2.3]). There can thus be no bounded ω-psh function u such that $(\omega + dd^c u)^n = 0$. Otherwise, the previous inequality applied with a constant function $v = A$ yields $u \geqslant A$ for any A, giving a contradiction. ☐

2. Envelopes

Here we consider envelopes of ω-psh functions, extending some results of [GLZ19] that have been established for Kähler manifolds.

2.1 Basic properties

Definition 2.1. A Borel set $E \subset X$ is (*locally*) *pluripolar* if it is locally contained in the $-\infty$ locus of some psh function: for each $x \in X$, there exist an open neighborhood U of x and a $u \in \mathrm{PSH}(U)$ such that $E \cap U \subset \{u = -\infty\}$.

Definition 2.2. Given a Lebesgue measurable function $h\colon X \to \mathbb{R}$, we define the *ω-psh envelope* of h by

$$P_\omega(h) := (\sup\{u \in \mathrm{PSH}(X,\omega)\colon u \leqslant h \text{ quasi-everywhere in } X\})^*,$$

where the star means that we take the upper semi-continuous regularization, while quasi-everywhere means outside a locally pluripolar set.

When ω is Hermitian and h is $\mathcal{C}^{1,1}$-smooth, so is $P_\omega(h)$ (see [Ber19, CZ19, CM21]), and one can show that

$$(\omega + dd^c P_\omega(h))^n = \mathbf{1}_{\{P_\omega(h)=h\}}(\omega + dd^c h)^n .$$

For merely semi-positive ω and a less regular obstacle h, we have the following.

Theorem 2.3. *If h is bounded from below, quasi-lower-semicontinuous, and $P_\omega(h) < +\infty$, then*

- *$P_\omega(h)$ is a bounded ω-plurisubharmonic function;*
- *$P_\omega(h) \leqslant h$ in $X \setminus P$, where P is pluripolar;*
- *$(\omega + dd^c P_\omega(h))^n$ is concentrated on the contact set $\{P_\omega(h) = h\}$.*

Recall that a function h is quasi-lower-semicontinuous (quasi-l.s.c.) if for any $\varepsilon > 0$, there exists an open set G of capacity smaller than ε such that h is continuous in $X \setminus G$. Quasi-psh functions and differences thereof are quasi-continuous (see [BT82]).

Proof. The proof is an adaptation of [GLZ19, Proposition 2.2, Lemma 2.3, Proposition 2.5], which deal with the case when ω is Kähler.

Since $P_\omega(h)$ is bounded from above, up to replacing h with $\min(h, C)$ with $C > \sup_X P_\omega(h)$, we can assume that h is bounded.

Step 1: h is smooth, ω is Hermitian. Building on Berman's work [Ber19], it was shown by Chu–Zhou in [CZ19] that the smooth solutions φ_β to

$$(\omega + dd^c\varphi_\beta)^n = e^{\beta(\varphi_\beta - h)}\omega^n$$

converge uniformly to $P_\omega(h)$ along with uniform $\mathcal{C}^{1,1}$-estimates. As a consequence, the measures $(\omega + dd^c\varphi_\beta)^n$ converge weakly to $(\omega + dd^c P_\omega(h))^n$. For each fixed $\varepsilon > 0$, we have the inclusions of open sets $\{P_\omega(h) < h - 2\varepsilon\} \subset \{\varphi_\beta < h - \varepsilon\}$ for β large enough, yielding

$$\begin{aligned}\int_{\{P_\omega(h)<h-2\varepsilon\}} (\omega + dd^c P_\omega(h))^n &\leqslant \liminf_{\beta\to+\infty} \int_{\{P_\omega(h)<h-2\varepsilon\}} (\omega + dd^c\varphi_\beta)^n \\ &\leqslant \liminf_{\beta\to+\infty} \int_{\{P_\omega(h)<h-2\varepsilon\}} e^{-\beta\varepsilon}\omega^n = 0 .\end{aligned}$$

Step 2: h is lower semi-continuous (l.s.c.), ω is Hermitian. If h is continuous, we can approximate it uniformly by smooth functions h_j. Set $u_j := P(h_j)$; the previous step ensures that

$$\int_X (h_j - u_j)(\omega + dd^c u_j)^n = 0 .$$

As $h_j \to h$ uniformly, we also have that $u_j \to u := P(h)$ uniformly, and the desired property follows from Bedford–Taylor's convergence theorem.

When h is merely l.s.c., we let (h_j) denote a sequence of continuous functions that increase pointwise to h and set $u_j = P(h_j)$. Then $u_j \nearrow u$ almost everywhere on X for some bounded function $u \in \mathrm{PSH}(X,\omega)$. Since $u_j \leqslant h_j \leqslant h$ quasi-everywhere on X, we infer $u \leqslant h$ quasi-everywhere on X; hence $u \leqslant P(h)$. For each $k < j$, the second step ensures that

$$\int_{\{u<h_k\}} (\omega + dd^c u_j)^n \leqslant \int_{\{u_j<h_j\}} (\omega + dd^c u_j)^n = 0\,.$$

Since $\{u < h_k\}$ is open, letting $j \to +\infty$ and then $k \to +\infty$, we arrive at

$$\int_{\{u<h\}} (\omega + dd^c u)^n = 0\,.$$

We also have that $P(h) \leqslant h$ quasi-everywhere on X; hence

$$\int_{\{u<P(h)\}} (\omega + dd^c u)^n = 0\,,$$

and [LPT21, Proposition 2.2] then ensures that $u = P(h)$.

Step 3: h is quasi-l.s.c., ω is Hermitian. By [GLZ19, Lemma 2.4], we can find a decreasing sequence (h_j) of l.s.c. functions such that $h_j \searrow h$ quasi-everywhere on X and $h_j \to h$ in capacity. Then $u_j := P(h_j) \searrow u := P(h)$. By Step 2, we know that for all $j > k$,

$$\int_{\{u_k<h\}} (\omega + dd^c u_j)^n \leqslant \int_{\{u_j<h_j\}} (\omega + dd^c u_j)^n = 0\,.$$

Since $\{u_k < h\}$ is quasi-open and the functions u_j are uniformly bounded, letting $j \to +\infty$, we obtain

$$\int_{\{u_k<h\}} (\omega + dd^c u)^n = 0\,.$$

Letting $k \to +\infty$ yields the desired result.

Step 4: *the general case.* We approximate $\omega \geqslant 0$ by the Hermitian forms $\omega_j = \omega + j^{-1}\omega_X$, which are positive. Observe that $j \mapsto u_j = P_{\omega_j}(h)$ decreases to $u = P_\omega(h)$ as j increases to $+\infty$. For $0 < k < j$, the previous step ensures that

$$\int_{\{u_k<h\}} (\omega + j^{-j}\omega_X + dd^c u_j)^n = 0\,.$$

As the set $\{u_k < h\}$ is quasi-open and u_j is uniformly bounded, we can let $j \to +\infty$ and use Bedford–Taylor's convergence theorem to get

$$\int_{\{u_k<h\}} (\omega + dd^c u)^n = 0\,.$$

We finally let $k \to +\infty$ to conclude. □

For later use, we extend the last item of Theorem 2.3 to a setting where $P_\omega(h)$ is not necessarily globally bounded.

Corollary 2.4. *If h is quasi-l.s.c. and $P_\omega(h)$ is locally bounded in a non-empty open set $U \subset X$, then $(\omega + dd^c P_\omega(h))^n$ is a well-defined positive Borel measure in U that vanishes in $U \cap \{P_\omega(h) < h\}$.*

Proof. Let (h_j) be a sequence of l.s.c. functions decreasing to h quasi-everywhere. Then $u_j := P_\omega(h_j)$ is a bounded ω-psh function such that $(\omega + dd^c u_j)^n = 0$ on $\{u_j < h_j\}$. Since u_j decreases to $u := P_\omega(h)$, Bedford–Taylor's convergence theorem ensures that $\omega_{u_j}^n \to \omega_u^n$ in U.

Fix a relatively compact open set $U' \Subset U$. For each fixed k, the set $\{u_k < h\}$ is quasi-open, and the functions u_j and u are uniformly bounded in U'; hence

$$\liminf_{j \to +\infty} \int_{\{u_k < h\} \cap U'} \omega_{u_j}^n \geqslant \int_{\{u_k < h\} \cap U'} \omega_u^n ,$$

which implies, after letting $k \to +\infty$, that ω_u^n vanishes in $U' \cap \{u < h\}$. We finally let U' increase to U to conclude. □

We will use the following later on.

LEMMA 2.5. *Let u, v be bounded ω-psh functions, and let $\varphi = P_\omega(\min(u, v))$ denote the ω-psh envelope of $\min(u, v)$. Then*

(i) $(\omega + dd^c \varphi)^n \leqslant \mathbf{1}_{\{\varphi = u < v\}} \omega_u^n + \mathbf{1}_{\{\varphi = v\}} \omega_v^n$;
(ii) *if $(\omega + dd^c u)^n = f dV_X$ and $(\omega + dd^c v)^n = g dV_X$, then*

$$(\omega + dd^c \varphi)^n \leqslant \max(f, g) dV_X ,$$

while

$$(\omega + dd^c \varphi)^n \geqslant \min(f, g) dV_X .$$

Proof. Since $\min(u, v)$ is quasi-continuous, it follows from Theorem 2.3 that the Monge–Ampère measure ω_φ^n is supported on $\{\varphi = \min(u, v)\} \subset \{\varphi = u\} \cup \{\varphi = v\}$. Thus

$$\omega_\varphi^n \leqslant \mathbf{1}_{\{\varphi = u < v\}} \omega_\varphi^n + \mathbf{1}_{\{\varphi = v\}} \omega_\varphi^n . \tag{4}$$

Since $\varphi = P(\min(u, v)) \leqslant u$ and $\varphi = P(\min(u, v)) \leqslant v$, Lemma 1.2 yields

$$\mathbf{1}_{\{\varphi = u\}} \omega_\varphi^n \leqslant \mathbf{1}_{\{\varphi = u\}} \omega_u^n \quad \text{and} \quad \mathbf{1}_{\{\varphi = v\}} \omega_\varphi^n \leqslant \mathbf{1}_{\{\varphi = v\}} \omega_v^n .$$

Combining this with (4), we obtain item (i).

When $(\omega + dd^c u)^n = f dV_X$ and $(\omega + dd^c v)^n = g dV_X$, we obtain

$$\mathbf{1}_{\{\varphi = u < v\}} \omega_\varphi^n \leqslant \mathbf{1}_{\{\varphi = u < v\}} f dV_X \leqslant \mathbf{1}_{\{\varphi = u < v\}} \max(f, g) dV_X$$

and $\mathbf{1}_{\{\varphi = v\}} \omega_\varphi^n \leqslant \mathbf{1}_{\{\varphi = v\}} g dV_X \leqslant \mathbf{1}_{\{\varphi = v\}} \max(f, g) dV_X$, hence

$$\omega_\varphi^n \leqslant \left\{ \mathbf{1}_{\{\varphi = u < v\}} + \mathbf{1}_{\{\varphi = v\}} \right\} \max(f, g) dV_X \leqslant \max(f, g) dV_X .$$

The second item follows from Lemma 1.2. □

2.2 Locally versus globally pluripolar sets

A classical result of Josefson asserts that a locally pluripolat set E in $\mathbb{C}^n$ is *globally pluripolar*; that is, there exists a psh function $u \in \mathrm{PSH}(\mathbb{C}^n)$ such that $E \subset \{u = -\infty\}$. This result has been extended to compact Kähler manifolds in [GZ05] and to the Hermitian setting in [Vu19]: if $E \subset X$ is locally pluripolar and ω_X is a Hermitian form, one can find a $u \in \mathrm{PSH}(X, \omega_X)$ such that $E \subset \{u = -\infty\}$.

We further extend this result to the case of non-collapsing forms.

LEMMA 2.6. *If E is (locally) pluripolar and $\omega \geqslant 0$ is non-collapsing, then $E \subset \{u = -\infty\}$ for some $u \in \mathrm{PSH}(X, \omega)$.*

The proof is a consequence of Theorem 2.3 and analogous results established on Kähler manifolds.

Proof. As in [GZ05, Theorem 5.2], it is enough to check that $V^*_{E,\omega} \equiv +\infty$, where

$$V_{E,\omega}(x) = \sup\{\varphi(x) \colon \varphi \in \mathrm{PSH}(X,\omega) \text{ and } \varphi \leqslant 0 \text{ quasi-everywhere on } E\}\,.$$

Here quasi-everywhere means outside a locally pluripolar set. If it is not the case, then $V^*_{E,\omega}$ is a bounded ω-psh function on X. We can assume that $E \subset U \Subset V \Subset V'$ is contained in a holomorphic chart V'. By Josefson's theorem (see [GZ17, Theorem 4.4]), we can find a psh function $u \in L^1_{\mathrm{loc}}(V')$ in V' such that $E \subset \{u = -\infty\}$. Let u_j be a sequence of smooth psh functions in a neighborhood of V such that $u_j \searrow u$. Fix $N \in \mathbb{N}$, and for j large enough, set

$$K_{j,N} := \{x \in V \colon u_j(x) \leqslant -N\}\,, \quad \varphi_{j,N} := V^*_{K_{j,N},\omega}\,,$$

and note that $K_{j,N}$ is increasing in j and that $\varphi_{j,N} \searrow \varphi_N \in \mathrm{PSH}(X,\omega) \cap L^\infty(X)$ as $j \to +\infty$. We also have that $E \subset \cup_{j\geqslant 1} K_{j,N}$; hence $0 \leqslant \varphi_N \leqslant V^*_{E,\omega}$. We can thus find a j_N so large that $\varphi_{j,N} \leqslant \sup_X V^*_{E,\omega} + 1$ for all $j \geqslant j_N$.

Let ρ be a smooth psh function in V such that $dd^c\rho \geqslant \omega$. The Chern–Levine–Nirenberg inequality (see [GZ17, Theorem 3.14]) ensures that for $j \geqslant j_N$,

$$\int_{K_{j,N}} (\omega + dd^c\varphi_{j,N})^n \leqslant \int_{K_{j,N}} (dd^c(\rho + \varphi_{j,N}))^n \leqslant \frac{1}{N}\int_V |u_j|(dd^c(\rho+\varphi_{j,N}))^n \leqslant \frac{C}{N}$$

for some uniform constant $C > 0$. The function that is zero on $K_{j,N}$ and $+\infty$ elsewhere is l.s.c. on X since $K_{j,N}$ is compact. It thus follows from Theorem 2.3 that

$$\int_X (\omega + dd^c\varphi_{j,N})^n = \int_{K_{j,N}} (\omega + dd^c\varphi_{j,N})^n \leqslant \frac{C'}{N}\,.$$

Letting $j \to +\infty$, we obtain $\int_X (\omega + dd^c\varphi_N)^n \leqslant C'/N$. Now $\varphi_N \nearrow \varphi$ as $N \to +\infty$, for some $\varphi \in \mathrm{PSH}(X,\omega)$ that is bounded since $0 \leqslant \varphi_N \leqslant V^*_{E,\omega}$. We thus obtain $\int_X (\omega + dd^c\varphi)^n = 0$, yielding a contradiction since ω is non-collapsing and φ is bounded. $\square$

Since locally pluripolar sets are $\mathrm{PSH}(X,\omega)$-pluripolar, arguing as in the proof of [GLZ19, Proposition 2.2], one finally obtains the following.

Corollary 2.7. *Let f be a Borel function such that $P_\omega(f) \in \mathrm{PSH}(X,\omega)$. Then*

$$P_\omega(f) = (\sup\{u \in \mathrm{PSH}(X,\omega) \colon u \leqslant f \text{ in } X\})^*\,.$$

2.3 Domination principle

We now establish the following generalization of the domination principle.

Proposition 2.8. *Assume that ω is non-collapsing, and fix $c \in [0,1)$. If u, v are bounded ω-psh functions such that $\omega_u^n \leqslant c\omega_v^n$ on $\{u < v\}$, then $u \geqslant v$.*

The usual domination principle corresponds to the case $c = 0$ (see [LPT21, Proposition 2.2]).

Proof. Fixing $a > 0$ arbitrarily small, we are going to prove that $u \geqslant v - a$ on X. Assume to the contrary that $E = \{u < v - a\}$ is not empty. Since u, v are quasi-psh, the set E has positive Lebesgue measure. For $b > 1$, we set $u_b := P_\omega(bu - (b-1)v)$. It follows from Theorem 2.3 that $(\omega + dd^c u_b)^n$ is concentrated on the set

$$D := \{u_b = bu - (b-1)v\}\,.$$

Also note that $b^{-1}u_b + \big(1-b^{-1}\big)v \leqslant u$ with equality on D. Therefore,

$$\mathbf{1}_D\big(\omega + dd^c\big(b^{-1}u_b + \big(1-b^{-1}\big)v\big)\big)^n \leqslant \mathbf{1}_D\omega_u^n\,,$$

as follows from Lemma 1.2; hence

$$\mathbf{1}_D b^{-n}(\omega + dd^c u_b)^n + \mathbf{1}_D\big(1-b^{-1}\big)^n(\omega + dd^c v)^n \leqslant \mathbf{1}_D\omega_u^n\,.$$

We choose b so large that $\big(1-b^{-1}\big)^n > c$. Multiplying the above inequality by $\mathbf{1}_{\{u<v\}}$ and noting that $\omega_u^n \leqslant c\omega_v^n$ on $\{u<v\}$, we obtain

$$\mathbf{1}_{D\cap\{u<v\}}(\omega + dd^c u_b)^n = 0\,.$$

Since u_b is bounded and ω is non-collapsing, we know that $\omega_{u_b}^n(D) = \omega_{u_b}^n(X) > 0$. We infer that the set $D\cap\{u \geqslant v\}$ is not empty, and on this set we have

$$u_b = bu - (b-1)v \geqslant u \geqslant -C$$

since u is bounded. It thus follows that $\sup_X u_b$ is uniformly bounded from below. As $b\to+\infty$, the functions $u_b - \sup_X u_b$ converge to a function u_∞ that is $-\infty$ on E but not identically $-\infty$; hence it belongs to $\mathrm{PSH}(X,\omega)$. This implies that the set E has Lebesgue measure 0, giving a contradiction. □

Here is a direct consequence of the domination principle.

COROLLARY 2.9. *Assume that ω is non-collapsing, and let u, v be bounded ω-psh functions. Then for all $\varepsilon > 0$,*

$$e^{-\varepsilon v}(\omega + dd^c v)^n \geqslant e^{-\varepsilon u}(\omega + dd^c u)^n \implies v \leqslant u\,.$$

Proof. Fix $a > 0$. On the set $\{u < v - a\}$, we have $\omega_u^n \leqslant e^{-\varepsilon a}\omega_v^n$. Proposition 2.8 thus gives $u \geqslant v - a$. This is true for all $a > 0$; hence $u \geqslant v$. □

3. Bounds on Monge–Ampère masses

From here on, we fix a Hermitian form ω_X on X.

3.1 Global bounds

Since the semi-positive $(1,1)$-form ω is not necessarily closed, the mass of the complex Monge–Ampère measures $(\omega + dd^c u)^n$ is (in general) not constantly equal to $V_\omega := \int_X \omega^n > 0$.

DEFINITION 3.1. For $1 \leqslant j \leqslant n$, we consider

$$v_{-,j}(\omega) := \inf\left\{\int_X (\omega + dd^c u)^j \wedge \omega^{n-j} : u \in \mathrm{PSH}(X,\omega)\cap L^\infty(X)\right\}$$
$$v_{+,j}(\omega) := \sup\left\{\int_X (\omega + dd^c u)^j \wedge \omega^{n-j} : u \in \mathrm{PSH}(X,\omega)\cap L^\infty(X)\right\}.$$

We set $v_-(\omega) := v_{-,n}(\omega)$ and $v_+(\omega) = v_{+,n}(\omega)$. When $\omega > 0$ is Hermitian, the supremum and infimum in the definition of $v_{+,j}(\omega)$ and $v_{-,j}(\omega)$ can be taken over $\mathrm{PSH}(X,\omega)\cap C^\infty(X)$, as follows from Demailly's approximation [Dem92] and Bedford–Taylor's convergence theorem [BT76, BT82].

It is an interesting open problem to determine when $v_-(\omega_X)$ is positive and/or $v_+(\omega_X)$ is finite. These conditions may depend on the complex structure, but they are independent of the choice of a Hermitian metric.

3.1.1 *Monotonicity and invariance properties.*

Proposition 3.2. *Let $0 \leqslant \omega_1 \leqslant \omega_2$ be semi-positive $(1,1)$-forms. Then*

$$v_-(\omega_1) \leqslant v_-(\omega_2) \quad \text{and} \quad v_{+,j}(\omega_1) \leqslant v_{+,j}(\omega_2) \tag{5}$$

for all $1 \leqslant j \leqslant n$. Moreover, one has

(i) $v_{+,j}(\omega_X) < +\infty \Longleftrightarrow v_{+,j}(\omega'_X) < +\infty$ *for any other Hermitian metric ω'_X,*

(ii) $0 < v_-(\omega_X) \Longleftrightarrow 0 < v_-(\omega'_X)$ *for any other Hermitian metric ω'_X.*

Proof. Since any ω_1-psh function u is also ω_2-psh, we obtain

$$\int_X (\omega_1 + dd^c u)^j \wedge \omega_1^{n-j} \leqslant \int_X (\omega_2 + dd^c u)^j \wedge \omega_2^{n-j} \leqslant v_{+,j}(\omega_2)\,,$$

which shows that $v_{+,j}(\omega_1) \leqslant v_{+,j}(\omega_2)$. We now bound $v_-(\omega_2)$ from below. Let v be a bounded ω_2-psh function, and let $u = P_{\omega_1}(v)$ denote its ω_1-psh envelope. Then u is a bounded ω_1-psh function and $u \leqslant v$ on X. Lemma 1.2 and Theorem 2.3 thus ensure that

$$(\omega_1 + dd^c u)^n \leqslant \mathbf{1}_{\{u=v\}}(\omega_2 + dd^c u)^n \leqslant \mathbf{1}_{\{u=v\}}(\omega_2 + dd^c v)^n\,.$$

We therefore obtain $v_-(\omega_1) \leqslant v_-(\omega_2)$. This proves (5).

Now let ω, ω' be two Hermitian metrics (we simplify the notation). Observe that $v_\pm(A\omega) = A^n v_\pm(\omega)$ for all $A > 0$. Since $A^{-1}\omega' \leqslant \omega \leqslant A\omega$ for an appropriate choice of the constant $A > 1$, items (i) and (ii) follow from (5). □

We now establish bounds on the mixed Monge–Ampère quantities.

Proposition 3.3. (i) *One always has $v_{+,1}(\omega) < +\infty$.*

(ii) *If ω is Hermitian, then $0 < v_{-,1}(\omega)$.*

(iii) *If $dd^c\omega^{n-2} = 0$, then $v_{+,2}(\omega) < +\infty$.*

(iv) *If $dd^c\omega = 0$ and $dd^c\omega^2 = 0$, then $v_{-,j}(\omega) = v_{+,j}(\omega) = V_\omega \in \mathbb{R}^*_+$.*

(v) *For all $0 \leqslant \ell \leqslant j \leqslant n$, one has $v_{+,\ell}(\omega) \leqslant 2^j v_{+,j}(\omega)$.*

(vi) *One has $v_{+,n-1}(\omega) < +\infty$ if and only if $v_{+,n}(\omega) < +\infty$.*

A Hermitian metric such that $dd^c\big(\omega^{n-2}\big) = 0$ is called Astheno–Kähler. These metrics play an important role in the study of harmonic maps (see [JY93]). A Hermitian metric satisfying $dd^c\omega = 0$ is called SKT or pluriclosed in the literature. When $n = 3$, the Astheno–Kähler and the pluriclosed condition coincide, and the third item is due to Chiose [Chi16b, Question 0.8]. Examples of compact complex manifolds admitting a pluriclosed metric can be found in [FPS04, Oti22].

Assertion (iv) was introduced by Guan–Li in [GL10]. It has been shown by Chiose [Chi16b] that it is equivalent to the invariance of Monge–Ampère masses: $\int_X (\omega + dd^c\varphi)^n = \int_X \omega^n$ for all smooth ω-psh functions if and only if $dd^c\omega^j = 0$ for all $j = 1, 2$. Note that any compact complex surface admits a Gauduchon metric $dd^c\omega = 0$, see [Gau77], which also satisfies $dd^c\omega^2 = 0$ for bidegree reasons.

Proof. One can assume without loss of generality that $\omega \leqslant \tilde{\omega}$, where $\tilde{\omega}$ is a Gauduchon metric. It follows that for any $\varphi \in \mathrm{PSH}(X,\omega) \cap L^\infty(X)$,

$$\int_X (\omega + dd^c\varphi) \wedge \omega^{n-1} \leqslant \int_X (\omega + dd^c\varphi) \wedge \tilde{\omega}^{n-1} = \int_X \omega \wedge \tilde{\omega}^{n-1}\,;$$

hence $v_{+,1}(\omega) \leqslant \int_X \omega \wedge \tilde{\omega}^{n-1} < +\infty$.

If ω is Hermitian, one can similarly bound ω from below by a Gauduchon form and conclude that $v_{-,1}(\omega) > 0$.

To prove assertion (iii), we fix $\varphi \in \mathrm{PSH}(X,\omega) \cap L^\infty(X)$ with $\sup_X \varphi = 0$. We want to show that $\int_X (\omega + dd^c\varphi)^2 \wedge \omega^{n-2} \leqslant M$ is uniformly bounded from above. It suffices to bound $\int_X (\omega_X + dd^c\varphi)^2 \wedge \omega^2$ from above, where ω_X is a Hermitian metric on X. By approximation, we can further assume that φ is smooth. A direct application of Stokes' theorem yields

$$\begin{aligned}\int_X (\omega_X + dd^c\varphi)^2 \wedge \omega^{n-2} &= \int_X \omega_X^2 \wedge \omega^{n-2} + 2\int_X \omega^{n-2} \wedge \omega_X \wedge dd^c\varphi + \int_X \omega^{n-2} \wedge (dd^c\varphi)^2 \\ &= \int_X \omega_X^2 \wedge \omega^{n-2} + 2\int_X \varphi dd^c\big(\omega^{n-2} \wedge \omega_X\big) - \int_X \varphi dd^c\omega^{n-2} \wedge dd^c\varphi\,.\end{aligned}$$

The last integral vanishes since $dd^c\omega^{n-2} = 0$. The second one is uniformly bounded since the functions φ belong to a compact subset of $L^1(X)$. Altogether, this shows that $v_{+,2}(\omega) < +\infty$ if $dd^c\big(\omega^{n-2}\big) = 0$.

Now assume that $dd^c\omega = 0$ and $dd^c\big(\omega^2\big) = 0$. Since $dd^c\big(\omega^2\big) = 2d\omega \wedge d^c\omega + 2\omega \wedge dd^c\omega$, this condition is equivalent to $dd^c\omega = 0$ and $d\omega \wedge d^c\omega = 0$, which is also equivalent to $dd^c\omega^k = 0$ for all $k = 1, \dots, n$. For $\varphi \in \mathcal{C}^\infty(X)$, the binomial expansion of the Monge–Ampère measure $(\omega + dd^c\varphi)^n$ yields

$$\int_X (\omega + dd^c\varphi)^n = \sum_{k=0}^n \binom{n}{k} (dd^c\varphi)^k \wedge \omega^{n-k}\,.$$

Setting $S = (dd^c\varphi)^p \wedge \omega^{n-p-1}$ and noting that $dd^cS = 0$, we have

$$(dd^c\varphi) \wedge S = (dd^c\varphi) \wedge S - \varphi dd^cS = d(d^c\varphi \wedge S - \varphi \wedge d^cS)\,.$$

Stokes' theorem thus yields $\int_X (\omega + dd^c\varphi)^n = \int_X \omega^n$. If $\varphi \in \mathrm{PSH}(X,\omega) \cap L^\infty(X)$, we approximate φ by a decreasing sequence of smooth ω_X-psh functions φ_j. For each j, we have $\int_X (\omega + dd^c\varphi_j)^n = \int_X \omega^n$. The signed measures $(\omega + dd^c\varphi_j)^n$ converge to $(\omega + dd^c\varphi)^n$; hence the mass equality holds for φ, proving assertion (iv).

Observe that for any $\varphi \in \mathrm{PSH}(X,\omega) \cap L^\infty(X)$ and $0 \leqslant \ell \leqslant j \leqslant n$, one has

$$\int_X (\omega + dd^c\varphi)^\ell \wedge \omega^{n-\ell} \leqslant \int_X (2\omega + dd^c\varphi)^j \wedge \omega^{n-j} \leqslant 2^j v_{+,j}(\omega)\,. \tag{6}$$

In particular, $v_{+,n-1}(\omega) \leqslant 2^n v_{+,n}(\omega)$; hence $v_{+,n}(\omega) < +\infty$ implies $v_{+,n-1}(\omega) < +\infty$. We finally show that, conversely, $v_{+,n-1}(\omega) < +\infty$ implies $v_{+,n}(\omega) < +\infty$ by proving

$$v_{+,n}(\omega) \leqslant 2^{2n-2} v_{+,n-1}(\omega)\,.$$

Indeed, observe that

$$\begin{aligned}0 &= \int_X (\omega + dd^c\varphi - \omega)^n \\ &= \int_X (\omega + dd^c\varphi)^n + \sum_{k=1}^n (-1)^k \binom{n}{k} (\omega + dd^c\varphi)^{n-k} \wedge \omega^k \\ &\geqslant \int_X (\omega + dd^c\varphi)^n - \sum_{1 \leqslant 2k+1 \leqslant n} \binom{n}{2k+1} (\omega + dd^c\varphi)^{n-2k-1} \wedge \omega^{2k+1}\,.\end{aligned}$$

Using (6), we thus get

$$v_{+,n}(\omega) \leqslant \sum_{1 \leqslant 2k+1 \leqslant n} \binom{n}{2k+1} 2^{n-1} v_{+,n-1}(\omega) = 2^{2n-2} v_{+,n-1}(\omega) . \qquad \square$$

3.1.2 *Uniformly bounded functions.* Restricting to uniformly bounded ω-psh functions, it is natural to consider

$$v_M^-(\omega) := \inf \left\{ \int_X (\omega + dd^c u)^n \colon u \in \mathrm{PSH}(X, \omega) \text{ with } -M \leqslant u \leqslant 0 \right\},$$

where $M \in \mathbb{R}^+$, and

$$v_M^+(\omega) := \sup \left\{ \int_X (\omega + dd^c u)^n \colon u \in \mathrm{PSH}(X, \omega) \text{ with } -M \leqslant u \leqslant 0 \right\}.$$

These quantities are always under control, as we now explain.

PROPOSITION 3.4. *Assume that ω is non-collapsing. For any $M \in \mathbb{R}^+$, one has*

$$0 < v_M^-(\omega) \leqslant v_M^+(\omega) < +\infty .$$

Proof. The finiteness of $v_M^+(\omega)$ follows easily from integration by parts; it is, for example, a simple consequence of [DK12, Theorem 3.5].

In order to show that $v_M^-(\omega)$ is positive, we argue by contradiction. Assume that there exist $u_j \in \mathrm{PSH}(X, \omega)$ such that $-M \leqslant u_j \leqslant 0$ and $\int_X (\omega + dd^c u_j)^n \leqslant 2^{-j}$. For $j \in \mathbb{N}$ fixed, the sequence

$$k \mapsto v_{j,k} := P_\omega(\min(u_j, u_{j+1}, \dots, u_{j+k}))$$

decreases toward a ω-psh function w_j such that $-M \leqslant w_j \leqslant 0$. It therefore follows from Lemma 2.5 that

$$\int_X (\omega + dd^c w_j)^n = \lim_{k \to +\infty} \int_X (\omega + dd^c v_{j,k})^n \leqslant \sum_{\ell=0}^{+\infty} \int_X (\omega + dd^c u_{j+\ell})^n \leqslant 2^{-j+1} .$$

Thus the sequence $j \mapsto w_j$ increases to a bounded ω-psh function w such that $(\omega + dd^c w)^n = 0$, which yields a contradiction. $\square$

Example 3.5. Here we provide an example of a semi-positive form ω such that $\int_X \omega^n > 0$ but ω is collapsing, in particular $v_-(\omega) = 0$. Let $X = Y \times Z$, where Y and Z are two compact complex manifolds of dimension $m \geqslant 1$ and $p \geqslant 1$, respectively, and $\dim X = n = p + m$. Fix a smooth function u on Y such that $\omega_Y + dd^c u < 0$ is negative in a small open set $U \subset Y$. Let $0 \leqslant \rho \leqslant 1$ be a cut-off function on Y supported in U. The smooth $(1,1)$-form ω defined by

$$\omega = \rho \circ \pi_1 (\pi_1^* \omega_Y + \pi_2^* \omega_Z)$$

is semi-positive on X and satisfies $\omega(y, z) = 0$ for $y \notin U$.

Now set $\phi := P_\omega(u \circ \pi_1)$, and let $\mathcal{C} := \{\phi = u \circ \pi_1\}$ denote the contact set. The Monge–Ampère measure $(\omega + dd^c \phi)^n$ is concentrated on $\mathcal{C}$. Arguing as in [Ber09, Proposition 3.1], one can show that $\mathcal{C} \subset \{x \in X,\ \omega + dd^c u \circ \pi_1(x) \geqslant 0\}$. Since $\omega + dd^c(u \circ \pi_1) < 0$ is negative in $U \times Z$, it follows that $\mathcal{C} \subset X \setminus (U \times Z)$. Now $\omega = 0$ outside $U \times Z$; hence

$$(\omega + dd^c \phi)^n \leqslant \mathbf{1}_{\mathcal{C}} (dd^c u \circ \pi_1)^n = 0$$

because $u \circ \pi_1$ depends only on y. It thus follows that $(\omega + dd^c \phi)^n = 0$ on X.

3.2 Bimeromophic invariance

Lemma 3.6. *Let $f\colon X \to Y$ be a proper holomorphic map between compact complex manifolds of dimension n, equipped with Hermitian forms ω_X, ω_Y. Then one has*

- $v_+(\omega_X) < +\infty \Longrightarrow v_+(\omega_Y) < +\infty$,
- $v_-(\omega_Y) > 0 \Longrightarrow v_-(\omega_X) > 0$ *if f has connected fibers.*

It follows from Zariski's main theorem that f has connected fibers if it is bimeromorphic.

Proof. Up to rescaling, we can assume that $f^*\omega_Y \leqslant \omega_X$. Fix $\varphi \in \mathrm{PSH}(Y, \omega_Y) \cap L^\infty(Y)$. Then $\varphi \circ f \in \mathrm{PSH}(X, \omega_X) \cap L^\infty(X)$ with

$$\int_Y (\omega_Y + dd^c\varphi)^n = \int_X (f^*\omega_Y + dd^c\varphi \circ f)^n \leqslant \int_X (\omega_X + dd^c\varphi \circ f)^n \leqslant v_+(\omega_X)\,;$$

thus $v_+(\omega_Y) \leqslant v_+(\omega_X)$, and the first assertion is proved.

Now consider $\psi \in \mathrm{PSH}(X, \omega_X) \cap L^\infty(X)$, and set $u = P_{f^*\omega_Y}(\psi)$. The function u is $f^*\omega_Y$, hence is plurisubharmonic on the fibers of f. If the latter are connected, we obtain that u is constant on them; that is, $u = \varphi \circ f$ for some function $\varphi \in \mathrm{PSH}(Y, \omega_Y) \cap L^\infty(Y)$. Since $(f^*\omega_Y + dd^c u)^n \leqslant \mathbf{1}_{\{u=\psi\}}(f^*\omega_Y + dd^c\psi)^n$, we infer

$$v_-(\omega_Y) \leqslant \int_Y (\omega_Y + dd^c\varphi)^n = \int_X (f^*\omega_Y + dd^c u)^n \leqslant \int_X (\omega_X + dd^c\psi)^n\,,$$

so that $v_-(\omega_Y) \leqslant v_-(\omega_X)$, proving the second assertion. □

We conversely show that the properties $v_+(\omega_X) < +\infty$ and $v_-(\omega_X) > 0$ are invariant under blow-ups and blow-downs with smooth centers.

Theorem 3.7. *Let X and Y be compact complex manifolds which are bimeromorphically equivalent. Then one has*

- $v_+(\omega_X) < +\infty \Longleftrightarrow v_+(\omega_Y) < +\infty$,
- $v_-(\omega_X) > 0 \Longleftrightarrow v_-(\omega_Y) > 0$.

Proof. A celebrated result of Hironaka (see, for example, [Mat04, Theorem 5.1]) ensures that any bimeromorphic map between compact complex manifolds is a finite composition of blow-ups and blow-downs with smooth centers. We can thus assume that $f\colon X \to Y$ is the blow-up of Y along a smooth center.

We fix a quasi-plurisubharmonic function ψ such that $\pi^*\omega_Y + dd^c\psi \geqslant \delta\omega_X$. The existence of ψ follows from a classical argument in complex geometry (see [LB70], [FT09, Proposition 3.2]). By Demailly's approximation theorem, we can further assume that ψ has analytic singularities. Up to scaling, we can assume without loss of generality that $\delta = 1$, and we set $\Omega = \{x \in X\colon \psi(x) > -\infty\}$.

We already know by Lemma 3.6 that $v_+(\omega_X) < +\infty \Longrightarrow v_+(\omega_Y) < +\infty$. Assume conversely that $v_+(\omega_Y) < +\infty$. For any $\varphi \in \mathrm{PSH}(X, \omega_X) \cap L^\infty(X)$,

$$\int_X (\omega_X + dd^c\varphi)^n \leqslant \int_\Omega (\pi^*\omega_Y + dd^c(\psi + \varphi))^n\,.$$

The function $u = \psi + \varphi$ is $\pi^*\omega_Y$-psh and bounded from above in Ω. It is constant on the fibers of π; hence $u = v \circ \pi$ with $v \in \mathrm{PSH}(\pi(\Omega), \omega_Y) \cap L^\infty(\pi(\Omega))$. As v is bounded from above, it extends trivially through the analytic set $\pi(\partial\Omega)$ as a ω_Y-psh function that is locally bounded in $\pi(\Omega)$. Thus

$$\int_\Omega (\pi^*\omega_Y + dd^c(\psi+\varphi))^n = \int_{\pi(\Omega)} (\omega_Y + dd^c v)^n$$
$$\leqslant \liminf_{j\to+\infty} \int_{\pi(\Omega)} (\omega_Y + dd^c \max(v,-j))^n \leqslant v_+(\omega_Y)$$

yields $v_+(\omega_X) \leqslant v_+(\omega_Y) < +\infty$.

We now assume that $v_-(\omega_X) > 0$. Pick $v \in \mathrm{PSH}(Y,\omega_Y)\cap L^\infty(Y)$, and set $u = P_{\omega_X}(v\circ\pi - \psi)$. Observe that $u \in \mathrm{PSH}(X,\omega_X)\cap L^\infty(X)$, and recall that $(\omega_X + dd^c u)^n$ is concentrated on the contact set $\mathcal{C} = \{u+\psi = v\circ\pi\}$ (see Theorem 2.3). Note that $\pi^*\omega_Y + dd^c(u+\psi) \geqslant \omega_X + dd^c u \geqslant 0$. Since $v\circ\pi$ is also $\pi^*\omega_Y$- psh, locally bounded in Ω, with $u+\psi \leqslant v\circ\pi$, it follows from Lemma 1.2 that

$$1_{\mathcal{C}}(\pi^*\omega_Y + dd^c(u+\psi))^n \leqslant 1_{\mathcal{C}}(\pi^*\omega_Y + dd^c v\circ\pi)^n \leqslant (\pi^*\omega_Y + dd^c v\circ\pi)^n\,.$$

Now $\pi^*\omega_Y + dd^c(u+\psi) \geqslant \omega_X + dd^c u$, and $(\omega_X + dd^c u)^n$ is concentrated on $\mathcal{C}$, so

$$1_{\mathcal{C}}(\pi^*\omega_Y + dd^c(u+\psi))^n \geqslant (\omega_X + dd^c u)^n\,.$$

We infer

$$v_-(\omega_X) \leqslant \int_X (\omega_X + dd^c u)^n \leqslant \int_{\mathcal{C}} (\pi^*\omega_Y + dd^c(u+\psi))^n$$
$$\leqslant \int_X (\pi^*\omega_Y + dd^c v\circ\pi)^n = \int_Y (\omega_Y + dd^c v)^n\,,$$

showing that $v_-(\omega_Y) \geqslant v_-(\omega_X) > 0$. The reverse implication $v_-(\omega_Y) > 0 \Longrightarrow v_-(\omega_X) > 0$ follows from Lemma 3.6. □

Recall that a compact complex manifold X belongs to the Fujiki class $\mathcal{C}$ if there exists a holomorphic bimeromophic map $\pi\colon Y \to X$, where Y is compact Kähler. Since $v_+(\omega_X) = v_-(\omega_X) = \int_X \omega_X^n \in \mathbb{R}_+^*$ when ω_X is a Kähler form, we obtain the following.

Corollary 3.8. *If X belongs to the Fujiki class $\mathcal{C}$, then*

$$0 < v_-(\omega_X) \leqslant v_+(\omega_X) < +\infty\,.$$

4. Weak transcendental Morse inequalities

4.1 Nef and big forms

Recall that the Bott-Chern cohomology group $H^{1,1}_{\mathrm{BC}}(X,\mathbb{R})$ is the quotient of closed real smooth $(1,1)$-forms by the image of $\mathcal{C}^\infty(X,\mathbb{R})$ under the dd^c-operator. This is a finite-dimensional vector space as X is compact.

Nefness and bigness are fundamental positivity properties of holomorphic line bundles in complex algebraic geometry (see [Laz04]). Their transcendental counterparts have been defined and studied by Demailly (see [Dem12]).

Definition 4.1. • A cohomology class $\alpha \in H^{1,1}_{\mathrm{BC}}(X,\mathbb{R})$ is *nef* if for any $\varepsilon > 0$, one can find a smooth closed real $(1,1)$-form $\theta_\varepsilon \in \alpha$ such that $\theta_\varepsilon \geqslant -\varepsilon\omega_X$.

• A *Hermitian current* on X is a positive current T of bidegree $(1,1)$ that dominates a Hermitian form; that is, there exists a $\delta > 0$ such that $T \geqslant \delta\omega_X$.

- A cohomology class $\alpha \in H^{1,1}_{\mathrm{BC}}(X,\mathbb{R})$ is *big* if it can be represented by a closed Hermitian current (a Kähler current).

It follows from an approximation result of Demailly [Dem92] that one can weakly approximate a Hermitian current by Hermitian currents with analytic singularities. In particular, a big cohomology class can be represented by a Kähler current with analytic singularities.

By analogy with the above setting, we propose the following definitions.

DEFINITION 4.2. Let ω be a smooth real $(1,1)$ form on X.

- We say that ω is *nef* if for any $\varepsilon > 0$, there exists a smooth quasi-plurisubharmonic function φ_ε such that $\omega + dd^c\varphi_\varepsilon \geqslant -\varepsilon\omega_X$.
- We say that ω is *big* if there exists a ω-psh function ρ with analytic singularities such that $\omega + dd^c\rho$ dominates a Hermitian form.

Note that $\mathrm{PSH}(X,\omega)$ is non-empty in both cases: indeed, $\rho \in \mathrm{PSH}(X,\omega)$ in the latter case, while one can extract $\varphi_{\varepsilon_j} \to \varphi \in \mathrm{PSH}(X,\omega)$ in the former, normalizing the potentials φ_{ε_j} by imposing $\sup_X \varphi_{\varepsilon_j} = 0$.

When X is a compact Kähler manifold and $\alpha \in H^{1,1}_{\mathrm{BC}}(X,\mathbb{R})$ is nef with $\alpha^n > 0$, a celebrated result of Demailly–Păun [DP04, Theorem 0.5] ensures the existence of a Kähler current representing α. This result is the key step in establishing a transcendental Nakai–Moishezon criterion (see [DP04, Main theorem]).

Further on, we study a possible extension of this result to the Hermitian setting. We thus need to extend the definition of v_- to nef forms.

DEFINITION 4.3. If ω is a nef $(1,1)$-form, we set

$$\hat{v}_-(\omega) := \inf_{\varepsilon>0} v_-(\omega + \varepsilon\omega_X)\,.$$

Although the form $\omega + \varepsilon\omega_X$ need not be semi-positive, one can find by definition a semi-positive form $\omega + \varepsilon\omega_X + dd^c\varphi_\varepsilon$ cohomologous to $\omega + \varepsilon\omega_X$, and it is understood here that

$$v_-(\omega + \varepsilon\omega_X) := v_-(\omega + \varepsilon\omega_X + dd^c\varphi_\varepsilon)\,.$$

By (5), the definition of $\hat{v}_-(\omega)$ is independent of the choice of the Hermitian form ω_X.

It is natural to expect that this definition is consistent with the previous one when ω is semi-positive and that $\hat{v}_-(\omega) = \alpha^n$ when ω is a closed form representing a nef class $\alpha \in H^{1,1}_{\mathrm{BC}}(X,\mathbb{R})$:

LEMMA 4.4. *If ω is semi-positive, then $\hat{v}_-(\omega) = v_-(\omega)$. If $v_+(\omega_X) < +\infty$ and ω is a closed form representing a nef class in $H^{1,1}_{\mathrm{BC}}(X,\mathbb{R})$, then $\hat{v}_-(\omega) = \alpha^n$.*

When X is Kähler, it is classical that any nef class $\alpha \in H^{1,1}_{\mathrm{BC}}(X,\mathbb{R})$ satisfies $\alpha^n \geqslant 0$. This inequality is no longer obvious on an arbitrary Hermitian manifold (we thank J.-P. Demailly for emphasizing this issue) but, as a consequence of Lemma 4.4, it remains true when $v_+(\omega_X) < +\infty$.

Proof. First assume that ω is semi-positive, and set $\omega_\varepsilon := \omega + \varepsilon\omega_X$ for $\varepsilon \in (0,1)$. Proposition 3.2 ensures that $v_-(\omega) \leqslant v_-(\omega_\varepsilon)$; hence $v_-(\omega) \leqslant \hat{v}_-(\omega)$. But for any $u \in \mathrm{PSH}(X,\omega) \cap L^\infty(X)$, we have

$$\begin{aligned}\int_X (\omega + dd^c u)^n &= \int_X (\omega_\varepsilon + dd^c u - \varepsilon\omega_X)^n \\ &\geqslant \int_X (\omega_\varepsilon + dd^c u)^n - C\varepsilon \geqslant \hat{v}_-(\omega) - C\varepsilon\,,\end{aligned}$$

where C is a constant depending on u, but it is harmless as we will let $\varepsilon \to 0$ while keeping u fixed. Doing so, we obtain $\int_X (\omega + dd^c u)^n \geqslant \hat{v}_-(\omega)$, and taking the infimum over such u, we obtain $v_-(\omega) \geqslant \hat{v}_-(\omega)$, proving the first statement.

Now assume that ω is closed and $\{\omega\} \in H^{1,1}_{\mathrm{BC}}(X, \mathbb{R})$ is nef. We can also assume that $-\omega_X \leqslant \omega \leqslant \omega_X$. We pick $\varphi \in \mathrm{PSH}(X, \omega + \varepsilon\omega_X) \cap C^\infty(X)$ and observe that $\mathrm{PSH}(X, \omega + \varepsilon\omega_X) \subset \mathrm{PSH}(X, 2\omega_X)$ for $0 < \varepsilon \leqslant 1$; hence

$$\int_X (\omega + \varepsilon\omega_X + dd^c\varphi)^n = \int_X (\omega + dd^c\varphi)^n + \sum_{j=1}^{n} \binom{n}{j} \varepsilon^j \int_X \omega_X^j \wedge (\omega + dd^c\varphi)^{n-j}\,.$$

Writing $\omega + dd^c\varphi = (2\omega_X + dd^c\varphi) - (2\omega_X - \omega)$, expanding $(\omega + dd^c\varphi)^{n-j}$ accordingly, and using $0 \leqslant 2\omega_X - \omega \leqslant 3\omega_X$, we obtain that $\left|\int_X \omega_X^j \wedge (\omega + dd^c\varphi)^{n-j}\right|$ is bounded from above by a finite sum of terms $\int_X \omega_X^\ell \wedge (\omega_X + dd^c\varphi)^{n-\ell}$, each of which is bounded from above by $3^n v_+(\omega_X)$. Since $\int_X (\omega + dd^c\varphi)^n = \alpha^n$, we end up with

$$\alpha^n - C\varepsilon v_+(\omega_X) \leqslant \int_X (\omega + \varepsilon\omega_X + dd^c\varphi)^n \leqslant \alpha^n + C\varepsilon v_+(\omega_X)\,,$$

using that $\varepsilon^j \leqslant \varepsilon$ for all $1 \leqslant j \leqslant n$. We infer $\hat{v}_-(\omega) = \alpha^n$. □

4.2 Demailly–Păun conjecture

4.2.1 *Hermitian currents.* The following is a natural generalization of [DP04, Conjecture 0.8].

Question 4.5. Let X be a compact complex manifold. Let ω be a nef $(1,1)$-form such that $\hat{v}_-(\omega) > 0$. Does there exist a ω-psh function φ with analytic singularities such that the current $\omega + dd^c\varphi$ dominates a Hermitian form?

We provide a partial answer to Question 4.5 following some ideas of Chiose [Chi16a].

Theorem 4.6. *Let ω be a nef $(1,1)$-form.*

- *If $\hat{v}_-(\omega) > 0$ and $v_+(\omega_X) < +\infty$, then ω is big.*
- *Conversely, if ω is big and $v_-(\omega_X) > 0$, then $\hat{v}_-(\omega) > 0$.*

Proof. We assume without loss of generality that $\omega \leqslant \omega_X/2$.

We first assume that $\hat{v}_-(\omega) > 0$ and $v_+(\omega_X) < +\infty$, and we prove that ω is big. An application of the Hahn–Banach theorem as in [Lam99, Lemma 3.3] shows that the existence of a Hermitian current satisfying $\omega + dd^c\psi \geqslant \delta\omega_X$ is equivalent to the inequalities

$$\int_X \omega \wedge \theta^{n-1} \geqslant \delta \int_X \omega_X \wedge \theta^{n-1}$$

for all Gauduchon metrics θ. Assume to the contrary that there exists a sequence of Gauduchon metrics θ_j such that

$$\int_X \omega \wedge \theta_j^{n-1} \leqslant \frac{1}{j} \int_X \omega_X \wedge \theta_j^{n-1}\,.$$

We can normalize the latter so that $\int_X \omega_X \wedge \theta_j^{n-1} = 1$.

Set $\omega_j = \omega + (1/j)\omega_X$, and note that $\omega_j \leqslant \omega_X$ for $j \geqslant 2$. Since ω is nef, one can find $\psi_j \in \mathcal{C}^\infty(X, \mathbb{R})$ such that $\omega_j + dd^c\psi_j$ is a Hermitian form; hence the main result of [TW10] ensures that there exist constants $C_j > 0$ and $\varphi_j \in \mathrm{PSH}(X, \omega_j) \cap \mathcal{C}^\infty(X)$ such that $\sup_X \varphi_j = 0$ and $(\omega_j + dd^c\varphi_j)^n = C_j \omega_X \wedge \theta_j^{n-1}$. It follows from Proposition 3.2 that

$$C_j = \int_X (\omega_j + dd^c\varphi_j)^n \geqslant v_-(\omega_j) \geqslant \hat{v}_-(\omega) > 0\,,$$

while by assumption, $\int_X (\omega_j + dd^c\varphi_j)^{n-1} \wedge \omega_X \leqslant M := v_{+,n-1}(\omega_X)$ is bounded from above.

We set $\alpha_j := \omega_j + dd^c\varphi_j$ and consider

$$E := \left\{x \in X,\ \omega_X \wedge \alpha_j^{n-1} \geqslant 2M\omega_X \wedge \theta_j^{n-1}\right\}.$$

This set has small $\omega_X \wedge \theta_j^{n-1}$ measure since

$$\int_E \omega_X \wedge \theta_j^{n-1} \leqslant \frac{1}{2M} \int_E \omega_X \wedge \alpha_j^{n-1} \leqslant \frac{1}{2}\,;$$

thus $\int_{X\setminus E} \omega_X \wedge \theta_j^{n-1} \geqslant \frac{1}{2}$, thanks to the normalization $\int_X \omega_X \wedge \theta_j^{n-1} = 1$.

We can compare ω_X and α_j in $X \setminus E$ since

$$\omega_X \wedge \alpha_j^{n-1} \leqslant 2M\omega_X \wedge \theta_j^{n-1} = \frac{2M}{C_j}\alpha_j^n \leqslant \frac{2M}{\hat{v}_-(\omega)}\alpha_j^n\,.$$

Thus $\alpha_j \geqslant \hat{v}_-(\omega)/2nM\omega_X$ in $X \setminus E$, and we infer

$$\int_{X\setminus E} \alpha_j \wedge \theta_j^{n-1} \geqslant \frac{\hat{v}_-(\omega)}{2nM} \int_{X\setminus E} \omega_X \wedge \theta_j^{n-1} \geqslant \frac{\hat{v}_-(\omega)}{4nM} > 0\,,$$

which contradicts

$$\begin{aligned}\int_X \alpha_j \wedge \theta_j^{n-1} &= \int_X \omega \wedge \theta_j^{n-1} + \frac{1}{j}\int_X \omega_X \wedge \theta_j^{n-1} + \int_X dd^c\varphi_j \wedge \theta_j^{n-1} \\ &\leqslant \frac{2}{j}\int_X \omega_X \wedge \theta_j^{n-1} = \frac{2}{j} \to 0\,,\end{aligned}$$

where $\int_X dd^c\varphi_j \wedge \theta_j^{n-1} = 0$ follows from the Gauduchon property of θ_j.

We next assume that ω is big and $v_-(\omega_X) > 0$, and we prove that $\hat{v}_-(\omega) > 0$ by an argument similar to that of Theorem 3.7. Fix a ω-psh function ψ with analytic singularities such that $\omega + dd^c\psi \geqslant \delta\omega_X$ for some $\delta > 0$. We can assume that $\delta = 1$ and $\sup_X \psi = 0$. We prove that $v_-(\omega + \varepsilon\omega_X) \geqslant v_-(\omega_X)$ for all $\varepsilon > 0$. Fix an $\varepsilon > 0$ and a $u \in \mathrm{PSH}(X, \omega + \varepsilon\omega_X) \cap L^\infty(X)$, and set $v = P_{\omega_X}(u - \psi)$. The open set $G = \{\psi > -1\}$ is not empty; hence it is non-pluripolar. On G, we have $u \leqslant u - \psi \leqslant u + 1 \leqslant \sup_X u + 1$. We then get $v - \sup_X u - 1 \leqslant V^*_{G,\omega}$, where $V_{G,\omega}$ is the extremal function of E (see the proof of Lemma 2.6 for its definition). It follows that v is a bounded ω_X-psh function and $(\omega_X + dd^c v)^n$ is supported on the contact set $\mathcal{C} = \{v = u - \psi\} \subset \{\psi > -\infty\}$. Since $v + \psi \leqslant u$ with equality on $\{\psi > -\infty\} \cap \mathcal{C} = \mathcal{C}$, Lemma 1.2 ensures that

$$\mathbf{1}_{\{\psi>-\infty\}\cap\mathcal{C}}(\omega + \varepsilon\omega_X + dd^c(v + \psi))^n \leqslant \mathbf{1}_{\{\psi>-\infty\}\cap\mathcal{C}}(\omega + \varepsilon\omega_X + dd^c u)^n\,.$$

Using $\omega + dd^c\psi \geqslant \omega_X$ and the fact that $(\omega_X + dd^c v)^n(\psi = -\infty) = 0$ since v is bounded, we thus arrive at (noting that $(\omega_X + dd^c v)^n$ is supported on $\mathcal{C}$)

$$\int_X (\omega_X + dd^c v)^n = \int_{\mathcal{C}} (\omega_X + dd^c v)^n \leqslant \int_X (\omega + \varepsilon\omega_X + dd^c u)^n\,.$$

We thus get $v_-(\omega + \varepsilon\omega_X) \geqslant v_-(\omega_X) > 0$ for all $\varepsilon > 0$; hence $\hat{v}_-(\omega) > 0$. □

This result provides, in particular, the following answer to Question 4.5.

Corollary 4.7. *The answer to Question 4.5 is positive if*

- $n = 2$ *(X is any compact surface),*
- *or* $n = 3$ *and* X *admits a pluriclosed metric,*
- *or* n *is arbitrary and* X *belongs to the Fujiki class,*
- *or else* n *is arbitrary and* X *admits a Guan–Li metric.*

Let us stress that the 2-dimensional setting is due to Buchdahl [Buc99] and Lamari [Lam99]. The 3-dimensional case follows from Proposition 3.3.

4.2.2 *Transcendental Grauert–Riemenschneider conjecture.* Let $L \to X$ be a semi-positive holomorphic line bundle with $c_1(L)^n > 0$. An influential conjecture of Grauert–Riemenschneider [GR70] asked whether the existence of such a line bundle implies that X is Moishezon (that is, bimeromorphically equivalent to a projective manifold).

This conjecture has been solved positively by Siu in [Siu84] (see also [Dem87]). Demailly and Păun have proposed a transcendental version of this conjecture.

Conjecture 4.8 ([DP04, Conjecture 0.8]). Let X be a compact complex manifold of dimension n. Assume that X possesses a nef class $\alpha \in H^{1,1}_{\mathrm{BC}}(X, \mathbb{R})$ such that $\alpha^n > 0$. Then X belongs to the Fujiki class.

As a direct consequence of Theorem 4.6, Lemma 4.4, and Corollary 3.8, we obtain the following answer to the transcendental Grauert–Riemenschneider conjecture.

Theorem 4.9. *Let* X *be a compact* n*-dimensional complex manifold. Let* $\alpha \in H^{1,1}_{\mathrm{BC}}(X, \mathbb{R})$ *be a nef class such that* $\alpha^n > 0$. *The following are equivalent:*

- *The class* α *contains a Kähler current.*
- $v_+(\omega_X) < +\infty$.

Since a Kähler current with analytic singularities can be desingularized after finitely many blow-ups producing a Kähler form, we obtain the following.

Corollary 4.10. *Let* $\alpha \in H^{1,1}_{\mathrm{BC}}(X, \mathbb{R})$ *be a nef class such that* $\alpha^n > 0$. *Then* X *belongs to the Fujiki class if and only if* $v_+(\omega_X) < +\infty$.

4.3 Transcendental holomorphic Morse inequalities

The following conjecture has been proposed by Boucksom–Demailly–Păun–Peternell, as a transcendental counterpart to the holomorphic Morse inequalities for integral classes due to Demailly.

Conjecture 4.11 ([BDPP13, Conjecture 10.1.ii]). Let X be a compact n-dimensional complex manifold. Let $\alpha, \beta \in H^{1,1}_{\mathrm{BC}}(X, \mathbb{C})$ be nef classes such that $\alpha^n > n\alpha^{n-1} \cdot \beta$. Then $\alpha - \beta$ contains a Kähler current and $\mathrm{Vol}(\alpha - \beta) \geqslant \alpha^n - n\alpha^{n-1} \cdot \beta$.

Note that this contains [DP04, Conjecture 0.8] as a particular case ($\beta = 0$). This conjecture was recently established by Witt Nyström [Wit19] when X is projective. Building on works of Xiao [Xia15] and Popovici [Pop16], we propose the following characterization that answers the qualitative part.

Theorem 4.12. *Let $\alpha, \beta \in H^{1,1}_{\mathrm{BC}}(X, \mathbb{C})$ be nef classes such that $\alpha^n > n\alpha^{n-1} \cdot \beta$. The following are equivalent:*

- *The class $\alpha - \beta$ contains a Kähler current.*
- $v_+(\omega_X) < +\infty$.

Proof. If $\alpha - \beta$ contains a Kähler current, then X belongs to the Fujiki class, and we have already observed that $v_+(\omega_X) < +\infty$ (see Corollary 3.8).

We now assume that $v_+(\omega_X) < +\infty$. Let ω and ω' be smooth closed real $(1,1)$-forms representing α and β, respectively. We can assume without loss of generality that $\omega \leqslant \frac{1}{2}\omega_X$ and $\omega' \leqslant \frac{1}{2}\omega_X$. For each $\varepsilon > 0$, we fix smooth functions $\varphi_\varepsilon \in \mathrm{PSH}(X, \omega + \varepsilon\omega_X)$ and $\psi_\varepsilon \in \mathrm{PSH}(X, \omega' + \varepsilon\omega_X)$ such that $\omega_\varepsilon := \omega + \varepsilon\omega_X + dd^c\varphi_\varepsilon$ and $\omega'_\varepsilon = \omega' + \varepsilon\omega_X + dd^c\psi_\varepsilon$ are Hermitian forms.

We assume that $\alpha - \beta$ does not contain any Kähler current and deduce a contradiction. It follows from the Hahn–Banach theorem as in [Lam99, Lemma 3.3] that there exist Gauduchon metrics η_ε such that

$$\int_X (\omega_\varepsilon - \omega'_\varepsilon) \wedge \eta_\varepsilon^{n-1} \leqslant \varepsilon \int_X \omega'_\varepsilon \wedge \eta_\varepsilon^{n-1} . \tag{7}$$

We normalize η_ε so that $\int_X \omega'_\varepsilon \wedge \eta_\varepsilon^{n-1} = 1$.

Using [TW10], we can find normalized functions $u_\varepsilon \in \mathrm{PSH}(X, \omega_\varepsilon) \cap \mathcal{C}^\infty(X)$ and unique constants $c_\varepsilon > 0$ such that

$$(\omega_\varepsilon + dd^c u_\varepsilon)^n = c_\varepsilon \omega'_\varepsilon \wedge \eta_\varepsilon^{n-1}, \quad \sup_X u_\varepsilon = 0 .$$

Our normalization for η_ε yields $c_\varepsilon = \int_X (\omega_\varepsilon + dd^c u_\varepsilon)^n$. Applying Lemma 4.13 below with $\theta_1 = \omega_\varepsilon + dd^c u_\varepsilon$, $\theta_2 = c_\varepsilon \omega'_\varepsilon$, and $\theta_3 = \eta_\varepsilon$, and recalling that $\theta_1^n = \theta_2 \wedge \theta_3^{n-1}$ with $\int_X \theta_1^n = \int_X \theta_2 \wedge \theta_3^{n-1} = c_\varepsilon$, we obtain

$$\left(\int_X (\omega_\varepsilon + dd^c u_\varepsilon) \wedge \eta_\varepsilon^{n-1} \right) \left(\int_X (\omega_\varepsilon + dd^c u_\varepsilon)^{n-1} \wedge \omega'_\varepsilon \right) \geqslant \frac{c_\varepsilon}{n} .$$

Now $\int_X (\omega_\varepsilon + dd^c u_\varepsilon) \wedge \eta_\varepsilon^{n-1} = \int_X \omega_\varepsilon \wedge \eta_\varepsilon^{n-1}$ because η_ε is a Gauduchon metric, while (7) yields $\int_X \omega_\varepsilon \wedge \eta_\varepsilon^{n-1} \leqslant (1+\varepsilon) \int_X \omega'_\varepsilon \wedge \eta_\varepsilon^{n-1} = (1+\varepsilon)$; hence

$$(1+\varepsilon) \int_X (\omega_\varepsilon + dd^c u_\varepsilon)^{n-1} \wedge \omega'_\varepsilon \geqslant \frac{1}{n} \int_X (\omega_\varepsilon + dd^c u_\varepsilon)^n .$$

We finally claim that, as $\varepsilon \to 0$,

$$\int_X (\omega_\varepsilon + dd^c u_\varepsilon)^n \to \alpha^n \quad \text{and} \quad \int_X (\omega_\varepsilon + dd^c u_\varepsilon)^{n-1} \wedge \omega'_\varepsilon \to \alpha^{n-1} \cdot \beta ,$$

which yields $n\alpha^{n-1} \cdot \beta \geqslant \alpha^n$ and therefore a contradiction.

We first explain why $\int_X (\omega_\varepsilon + dd^c u_\varepsilon)^n \to \alpha^n$. Stokes' theorem yields

$$\begin{aligned} \alpha^n &= \int_X (\omega + dd^c(u_\varepsilon + \varphi_\varepsilon))^n = \int_X (\omega + \varepsilon\omega_X + dd^c(u_\varepsilon + \varphi_\varepsilon) - \varepsilon\omega_X)^n \\ &= \int_X (\omega_\varepsilon + dd^c u_\varepsilon)^n + \sum_{j=0}^{n-1} \binom{n}{j} \varepsilon^{n-j} (-1)^{n-j} \int_X (\omega_\varepsilon + dd^c u_\varepsilon)^j \wedge \omega_X^{n-j} . \end{aligned}$$

Since $\omega \leqslant \frac{1}{2}\omega_X$, the function $v_\varepsilon = u_\varepsilon + \varphi_\varepsilon$ is ω_X-psh for $0 < \varepsilon \leqslant \frac{1}{2}$; hence

$$0 \leqslant \int_X (\omega_\varepsilon + dd^c u_\varepsilon)^j \wedge \omega_X^{n-j} \leqslant \int_X (\omega_X + dd^c v_\varepsilon)^j \wedge \omega_X^{n-j} \leqslant 2^n v_+(\omega_X) ,$$

as follows from Proposition 3.3. We infer

$$\left|\alpha^n - \int_X (\omega_\varepsilon + dd^c u_\varepsilon)^n\right| \leqslant \sum_{j=0}^{n-1} \binom{n}{j} \varepsilon^{n-j} 2^n v_+(\omega_X) \leqslant 4^n \varepsilon v_+(\omega_X)\,.$$

The conclusion thus follows by letting $\varepsilon \to 0$.

We similarly can check that

$$\left|\alpha^{n-1}\cdot\beta - \int_X (\omega_\varepsilon + dd^c u_\varepsilon)^{n-1} \wedge \omega'_\varepsilon\right| \leqslant 2\cdot 6^n \varepsilon v_+(\omega_X)\,.$$

Using Stokes' theorem again, we indeed obtain that

$$\begin{aligned}\alpha^{n-1}\cdot\beta &= \int_X (\omega + dd^c\varphi_\varepsilon + dd^c u_\varepsilon)^{n-1} \wedge (\omega' + dd^c\psi_\varepsilon)\\ &= \int_X (\omega_\varepsilon + dd^c u_\varepsilon - \varepsilon\omega_X)^{n-1} \wedge (\omega'_\varepsilon - \varepsilon\omega_X)\\ &= \int_X (\omega_\varepsilon + dd^c u_\varepsilon)^{n-1} \wedge \omega'_\varepsilon + O(\varepsilon)\,.\end{aligned}$$

Each term $\int_X (\omega_X + dd^c v_\varepsilon)^\ell \wedge (\omega_X + dd^c\psi_\varepsilon)^p \wedge \omega_X^q$, with $\ell + p + q = n$, is bounded from above by $3^n v_+(\omega_X)$, as one can check by observing that the function $\frac{1}{3}(v_\varepsilon + \psi_\varepsilon)$ is ω_X-psh with

$$\int_X (\omega_X + dd^c v_\varepsilon)^\ell \wedge (\omega_X + dd^c\psi_\varepsilon)^p \wedge \omega_X^q \leqslant 3^n \int_X \left(\omega_X + dd^c \frac{v_\varepsilon + \psi_\varepsilon}{3}\right)^n. \qquad \square$$

We have used in the previous proof the following observation of Popovici.

LEMMA 4.13. *Let θ_1, θ_2, θ_3 be Hermitian forms on X. Then*

$$\left(\int_X \theta_1 \wedge \theta_3^{n-1}\right)\left(\int_X \theta_1^{n-1} \wedge \theta_2\right) \geqslant \frac{1}{n}\left(\int_X \sqrt{\frac{\theta_2 \wedge \theta_3^{n-1}}{\theta_1^n}}\,\theta_1^n\right)^2.$$

In particular, if $\theta_1^n = \theta_2 \wedge \theta_3^{n-1}$, then

$$\left(\int_X \theta_1 \wedge \theta_3^{n-1}\right)\left(\int_X \theta_1^{n-1} \wedge \theta_2\right) \geqslant \frac{1}{n}\left(\int_X \theta_1^n\right)^2.$$

We provide the proof as a courtesy to the reader.

Proof. It follows from the Cauchy–Schwarz inequality that

$$\left(\int_X \theta_1 \wedge \theta_3^{n-1}\right)\left(\int_X \theta_1^{n-1} \wedge \theta_2\right) \geqslant \left(\int_X \sqrt{\frac{\theta_1 \wedge \theta_3^{n-1}}{\theta_1^n}\,\frac{\theta_1^{n-1} \wedge \theta_2}{\theta_1^n}}\,\theta_1^n\right)^2.$$

The elementary pointwise estimate $Tr_{\theta_3}(\theta_1)Tr_{\theta_1}(\theta_2) \geqslant Tr_{\theta_3}(\theta_2)$ is [Pop16, Lemma 3.1]. Multiplying by θ_3^n/θ_1^n, we can reformulate this as

$$\frac{\theta_1 \wedge \theta_3^{n-1}}{\theta_1^n} \cdot \frac{\theta_2 \wedge \theta_1^{n-1}}{\theta_1^n} \geqslant \frac{1}{n}\frac{\theta_2 \wedge \theta_3^{n-1}}{\theta_1^n}\,.$$

The first inequality follows. Moreover, when $\theta_1^n = \theta_2 \wedge \theta_3^{n-1}$, we infer

$$\int_X \sqrt{\frac{\theta_2 \wedge \theta_3^{n-1}}{\theta_1^n}}\,\theta_1^n = \int_X \theta_1^n\,. \qquad \square$$

Motivated by possible extensions of the conjectures of Demailly–Păun and Boucksom–Demailly–Păun–Peternell, we introduce the following.

DEFINITION 4.14. Given Hermitian forms $\omega_1, \dots, \omega_n$, we consider

$$v_-(\omega_1, \dots, \omega_n) := \inf \left\{ \int_X (\omega_1 + dd^c\varphi_1) \wedge \cdots \wedge (\omega_n + dd^c\varphi_n),\ \varphi_j \in \mathcal{P}(\omega_j) \right\},$$
$$v_+(\omega_1, \dots, \omega_n) := \sup \left\{ \int_X (\omega_1 + dd^c\varphi_1) \wedge \cdots \wedge (\omega_n + dd^c\varphi_n),\ \varphi_j \in \mathcal{P}(\omega_j) \right\},$$

where $\mathcal{P}(\omega_j) := \mathrm{PSH}(X, \omega_j) \cap L^\infty(X)$. If the ω_j are merely nef, we set

$$\hat{v}_-(\omega_1, \dots, \omega_n) := \inf_{\varepsilon > 0} v_-(\omega_1 + \varepsilon\omega_X, \dots, \omega_n + \varepsilon\omega_X),$$
$$\hat{v}_+(\omega_1, \dots, \omega_n) := \inf_{\varepsilon > 0} v_+(\omega_1 + \varepsilon\omega_X, \dots, \omega_n + \varepsilon\omega_X).$$

A straightforward generalization of Theorem 4.12 along the lines of Theorem 4.6 is the following.

THEOREM 4.15. *Let X be a compact n-dimensional complex manifold such that $v_+(\omega_X) < +\infty$. Let ω, ω' be nef $(1,1)$-forms. If $\hat{v}_-(\omega) > n\hat{v}_+(\omega, \dots, \omega, \omega')$, then the form $\omega - \omega'$ is big.*

We leave the technical details to the reader.

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