Algebraic Geometry 9 (6) (2022) 651–687
doi:10.14231/AG-2022-020

The Hrushovski–Lang–Weil estimates

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Abstract

In this work, we give an algebro-geometric proof of Hrushovski's generalization of the Lang–Weil estimates on the number of points in the intersection of a correspondence with the graph of the Frobenius map. This result has numerous applications to various areas of mathematics, including model theory, algebraic dynamics, group theory and arithmetic algebraic geometry.

1. Introduction

André Weil in his seminal work on bounding exponential sums introduced the now-famous eponymous conjectures, which motivated much of the modern arithmetic and algebraic geometry that came out of Grothendieck's school. Weil himself proved the Riemann hypothesis for curves, and later in joint work with Lang established the Lang–Weil estimates [LW54] using "non-cohomological" methods. Till now these constitute the best possible diophantine estimate that one can give for the number of rational points on an arbitrary projective variety over a finite field. From a motivic point of view, one can view these estimates as giving uniform bounds to the number of points in the intersection of the diagonal correspondence with the graph of the Frobenius map.

In 2004 Hrushovski [Hru04] proved a vast generalization of the Lang–Weil estimates to the case when the diagonal correspondence is replaced by a "generic" correspondence. Hrushovski's proof combines Deligne's purity results with techniques from model theory and intersection theory.

It turned out that the theorem of Hrushovski [Hru04, Theorem 1.1] has numerous applications in various areas of mathematics, including model theory, algebraic dynamics, group theory and arithmetic algebraic geometry. For example, they were used to prove that every mapping torus of any free group endomorphism is residually finite [BS05] and to prove Gieseker's conjecture on stratified vector bundles [EM10]. Hrushovski's estimates also imply the Zariski density of periodic points under a surjective self map of a variety over a finite field [Fak03] and have consequences for dynamics over number fields (see, for example, [Ame11]).

Given the significance of Hrushovski's result, a natural question was to provide a purely algebro-geometric proof of it. Over the years, there were many attempts to do it and a few partial successes (see Section 1.5 below). The goal of this work is to give a purely algebro-geometric proof

Received 22 August 2021, accepted in final form 16 April 2022.
2020 Mathematics Subject Classification 14G17 (primary), 14C17 (secondary).
Keywords: intersection theory, finite fields, cohomological methods.

This work was supported by the Israel Science Foundation (Grant No. 1017/13).

of Hrushovski's theorem [Hru04, Theorem 1.1] in full generality.

To formulate the result, we first introduce some notation.

1.1 *Geometric degree.* Let k be an algebraically closed field. Recall that with every closed subscheme $X \subseteq \mathbb{P}^n_k$ of the projective space $\mathbb{P}^n_k$, one can associate a natural number, its *degree* $\deg(X)$ (see [Ful98, Example 2.5.2]).

(a) For a closed reduced subscheme $X \subseteq \mathbb{P}^n_k$, we denote by $\operatorname{gdeg}(X)$ the sum of the degrees of *all* irreducible components of X and call it the *geometric degree* of X (compare [BM93]).

(b) More generally, for a (locally closed) reduced subscheme $X \subseteq \mathbb{P}^n_k$, we denote by $\operatorname{gdeg}(X)$ the minimum $\min\{\operatorname{gdeg}(X_1) + \operatorname{gdeg}(X_2)\}$, taken over all pairs X_1, X_2 of closed reduced subschemes of $\mathbb{P}^n_k$ with $X = X_1 \smallsetminus X_2$. Note that if $X \subseteq \mathbb{P}^n_k$ is closed, then the geometric degree $\operatorname{gdeg}(X)$ defined in item (b) coincides with that of item (a).

(c) For a reduced subscheme C of $\mathbb{P}^m_X \subseteq \mathbb{P}^m_k \times \mathbb{P}^n_k$, we denote by $\operatorname{gdeg}(C)$ the geometric degree of C, viewed as a subscheme of $\mathbb{P}^{mn+m+n}_k$ via the Segre embedding.

1.2 *Setup.* (a) Let k be an algebraically closed field of characteristic $p > 0$, let q be a power of p, and let $\sigma_q : \operatorname{Spec} k \to \operatorname{Spec} k$ be the absolute qth Frobenius automorphism.

(b) Let $X^0 \subseteq \mathbb{P}^n_k$ be a locally closed reduced subscheme, let $\left(X^0\right)^{(q)}$ be the base change of X^0 along σ_q, and let $C^0 \subseteq X^0 \times \left(X^0\right)^{(q)}$ be a closed reduced subscheme. Then C^0 is a reduced subscheme of $\mathbb{P}^n_k \times \mathbb{P}^n_k$; thus we can talk about the geometric degree $\operatorname{gdeg}\left(C^0\right)$.

(c) We denote by $c^0 = \left(c^0_1, c^0_2\right) \colon C^0 \hookrightarrow X^0 \times \left(X^0\right)^{(q)}$ the inclusion map.

(d) Let $\mathrm{F}_{X^0/k,q}$ be the induced relative Frobenius map $X^0 \to \left(X^0\right)^{(q)}$ over k, and denote by $\Gamma^0 := \Gamma_{X^0,q} \subseteq X^0 \times \left(X^0\right)^{(q)}$ the graph of $\mathrm{F}_{X^0/k,q}$.

Now we are ready to formulate the main result of this work.

THEOREM 1.3 ([Hru04, Theorem 1.1]). *For every $n, \delta \in \mathbb{N}$, there exists a constant $M = M(n, \delta)$ with the following properties:*

(a) *Let $\left(k, q, X^0, C^0\right)$ be a quadruple as in Section 1.2 such that c^0_2 is quasi-finite,* $\operatorname{gdeg}\left(X^0\right) \leqslant \delta$, $\operatorname{gdeg} C^0 \leqslant \delta$ *and $q \geqslant M$. Then the intersection $C^0 \cap \Gamma^0$ is finite, and its cardinality satisfies*

$$\#\left(C^0 \cap \Gamma^0\right)(k) \leqslant M q^{\dim(C^0)} . \tag{1}$$

(b) *Let $\left(k, q, X^0, C^0\right)$ be a quadruple as in part* (a) *such that C^0 is irreducible and c^0_1 is dominant. Then the finite number $\#\left(C^0 \cap \Gamma^0\right)(k)$ satisfies*

$$\left| \#\left(C^0 \cap \Gamma^0\right)(k) - \frac{\deg\left(c^0_1\right)}{\deg\left(c^0_2\right)_{\mathrm{insep}}} q^{\dim(C^0)} \right| \leqslant M q^{\dim(C^0) - 1/2} . \tag{2}$$

Remark 1.4. (a) The assumptions of Theorem 1.3 imply that X^0 is irreducible, $\dim\left(X^0\right) = \dim\left(C^0\right)$, and both c^0_1 and c^0_2 are generically finite. In particular, we can talk about the degree $\deg\left(c^0_1\right)$ of c^0_1 and the inseparable degree $\deg\left(c^0_2\right)_{\mathrm{insep}}$ of c^0_2.

(b) The result can be slightly generalized. Namely, it suffices to assume that either c^0_1 or c^0_2 is quasi-finite in Theorem 1.3(a) and that either c^0_1 is quasi-finite and c^0_2 is dominant, or c^0_2 is quasi-finite and c^0_1 is dominant in Theorem 1.3(b).

1.5 *A particular case.* Now assume that k is an algebraic closure of a finite field $\mathbb{F}_q$. For any scheme X^0 over $\mathbb{F}_q$, we have a natural isomorphism $\left(X^0\right)^{(q)} \cong X^0$. Then one obtains the following particular case of Theorem 1.3.

Corollary 1.6 ([Shu19b, Theorem 1.3]). *Let $c^0 : C^0 \to X^0 \times X^0$ be a morphism of schemes of finite type over k such that X^0 is defined over $\mathbb{F}_q$ and either c_1^0 or c_2^0 is quasi-finite. Then there exists an integer M such that for every $n \geqslant M$,*

(a) *the set $\left(c^0\right)^{-1}\left(\Gamma_{X^0,q^n}\right)(k)$ is finite and $\#\left(c^0\right)^{-1}\left(\Gamma_{X^0,q^n}\right)(k) \leqslant Mq^{n\dim(C^0)}$;*

(b) *if in addition C^0 is irreducible with c_1^0 and c_2^0 dominant, then we have an inequality*

$$\left|\#\left(c^0\right)^{-1}\left(\Gamma_{X^0,q^n}\right)(k) - \frac{\deg\left(c_1^0\right)}{\deg\left(c_2^0\right)_{\mathrm{insep}}} q^{n\dim(C^0)}\right| \leqslant Mq^{n(\dim(C^0)-1/2)}.$$

We also would like to mention the following particular case of Corollary 1.6, which has applications to various fields of mathematics including algebraic dynamics [Fak03, Ame11], group theory [BS05] and algebraic geometry [EM10, ESB16].

Corollary 1.7 ([Hru04, Corollary 1.2], [Var18, Theorem 0.1]). *Let $c^0 : C^0 \to X^0 \times X^0$ be a morphism of schemes of finite type over k such that C^0 is irreducible, c_1^0 and c_2^0 are dominant, c_1^0 or c_2^0 is quasi-finite, and X^0 is defined over $\mathbb{F}_q$. Then for every sufficiently large n, the preimage $\left(c^0\right)^{-1}\left(\Gamma_{X^0,q^n}\right)(k)$ is non-empty.*

1.8 *Outline of the proof.* To prove Theorem 1.3, we follow closely the strategy of [Var18].[1] The only (but essential) difference is that unlike in [Var18], the refined intersection theory of Fulton is used.

Our argument goes as follows. Using Noetherian induction, we reduce Theorem 1.3 to Theorem 1.3(b). Next, replacing X^0 with an open subscheme, we reduce to a particular case of Theorem 1.3(b), where X^0 is smooth and c_2^0 decomposes as a composition of a flat universal homeomorphism and an étale morphism. In this case, the intersection $C^0 \cap \Gamma^0$ is finite when $q > \deg(c_2)_{\mathrm{insep}}$, which we assume from now on.

We denote by $X \subseteq \mathbb{P}^n$ and $C \subseteq X \times X^{(q)}$ the closures of X^0 and C^0, respectively, and let $c = (c_1, c_2) : C \to X \times X^{(q)}$ be the inclusion map. We say that $\partial X := X \smallsetminus X^0$ is *locally c-invariant* if every point $x \in X$ has an open neighborhood $U \subseteq X$ such that we have an inclusion $c_2^{-1}\left(\partial X^{(q)} \cap U^{(q)}\right) \cap c_1^{-1}(U) \subseteq c_1^{-1}(\partial X \cap U)$. By a *twisted* version of an argument of [Var18], we reduce to the situation where ∂X is locally c-invariant. Namely, this happens after we replace X^0 with an open subscheme and X with a blow-up.

Our main observation is that in this case, it is possible to write a formula for the cardinality $\#\left(C^0 \cap \Gamma^0\right)(k)$ in cohomological terms.

By a theorem of De Jong [dJo96], there exists an alteration $f : X' \to X$ where X' is smooth and the boundary $\partial X' := f^{-1}(\partial X)_{\mathrm{red}}$ is a union of smooth divisors $\{X_i'\}_{i\in I}$ with normal crossings. Mimicking [Pin92, Var18], we consider the blow-up $\widetilde{Y} := \mathrm{Bl}_{\cup_{i\in I}(X_i' \times (X_i')^{(q)})}\left(X' \times X'^{(q)}\right)$ and

[1] An interesting question is whether it is possible to prove our result by extending the method of the proof of Corollary 1.6 in [Shu19b]. Unlike those of [Hru04] and [Var18], the method of [Shu19b] does not make use of alterations and uses techniques from ℓ-adic cohomology, properties of the intersection complex and the theory of weights instead. One of the obstacles we do not know how to resolve at the moment is how to define a suitable "norm" on cohomological correspondences, compatible with the norm of algebraic cycles given in Appendix B.

denote by $\pi\colon \widetilde{Y} \to X' \times X'^{(q)}$ the projection map. Let $\Gamma \subseteq X' \times X'^{(q)}$ be the graph of $\mathrm{F}_{X'/k,q}$, and denote by $\widetilde{C} \subseteq \widetilde{Y}$ and $\widetilde{\Gamma} \subseteq \widetilde{Y}$ the strict preimages of C and Γ, respectively.

Set $X'^0 := f^{-1}(X^0) \subseteq X'$, and let $g^0\colon X'^0 \times (X'^0)^{(q)} \to X^0 \times (X^0)^{(q)}$ be the restriction of $f \times f^{(q)}$. Then $\pi\colon \widetilde{Y} \to Y$ in an isomorphism over $X'^0 \times (X'^0)^{(q)}$; therefore there exists a canonical open embedding $X'^0 \times (X'^0)^{(q)} \hookrightarrow \widetilde{Y}$, which induces an open embedding $(g'^0)^{-1}(C^0) \hookrightarrow \widetilde{C} \subseteq \widetilde{Y}$.

Let d be the dimension of X^0. Denote by $[C^0] \in A_d(C^0)$ the class of C^0, where $A_d(-)$ denotes the group of dimension d cycle classes, let $\alpha^0 := (g^0)^*([C^0]) \in A_d((g'^0)^{-1}(C^0))$ be the refined Gysin pullback of $[C^0]$, and let $\widetilde{\alpha} \in A_d(\widetilde{Y})$ be the "closure" of α^0. (More formally, $\widetilde{\alpha}$ is the class of the closure of a cycle representing α^0.) Since $\widetilde{Y}$ is smooth of dimension $2d$, we can consider the intersection number $\widetilde{\alpha} \cdot [\widetilde{\Gamma}] \in \mathbb{Z}$.

Our strategy is to calculate the intersection number $\widetilde{\alpha} \cdot [\widetilde{\Gamma}]$ in two ways: local and global. Arguing as in [Var18], we show that if q is greater than the ramification of c'_2 along $\partial X'^{(q)}$, then we have $\widetilde{C} \cap \widetilde{\Gamma} \subseteq \pi^{-1}(X'^0 \times (X'^0)^{(q)})$. Combining this with the refined projection formula and a local calculation, we get an equality

$$\widetilde{\alpha} \cdot [\widetilde{\Gamma}] = \deg\,(c_2^0)_{\text{insep}} \deg(f) \#(C^0 \cap \Gamma^0)(k)\,. \tag{3}$$

To get a second expression, we proceed as follows: for every subset $J \subseteq I$, we denote by X'_J the intersection $\cap_{i \in J} X'_i$. In particular, $X'_\emptyset = X'$. For every $J \subseteq I$, the cycle class $\widetilde{\alpha} \in A_d(\widetilde{Y})$ gives rise to a cycle class $\widetilde{\alpha}_J \in A_{d-|J|}(X'_J \times (X'_J)^{(q)})$.

For every prime number $\ell \neq \mathrm{char}(k)$ and $i \in \mathbb{Z}$, the class $\widetilde{\alpha}_J$ gives rise to a homomorphism $H^i(\widetilde{\alpha}_J)\colon H^i(X'_J, \mathbb{Q}_\ell) \to H^i((X'_J)^{(q)}, \mathbb{Q}_\ell)$ between ℓ-adic cohomology groups, hence to an endomorphism $\mathrm{F}^*_{X'_J,q} \circ H^i(\widetilde{\alpha}_J)$ of $H^i(X'_J, \mathbb{Q}_\ell)$. We then show the equality

$$\widetilde{\alpha} \cdot [\widetilde{\Gamma}] = \sum_{J \subseteq I} (-1)^{|J|} \sum_{i=0}^{2(d-|J|)} (-1)^i \operatorname{Tr}\left(\mathrm{F}^*_{X'_J,q} \circ H^i(\widetilde{\alpha}_J), H^i(X'_J, \mathbb{Q}_\ell)\right). \tag{4}$$

It is well known that each $\operatorname{Tr}\left(\mathrm{F}^*_{X'_J,q} \circ H^i(\widetilde{\alpha}_J)\right)$ is in $\mathbb{Z}$ and is independent of ℓ. Combining equalities (3), (4) and the identity $\operatorname{Tr}\left(\mathrm{F}^*_{X',q} \circ H^{2d}(\widetilde{\alpha}_\emptyset)\right) = \deg\,(c_1^0) \deg(f)$, to show Theorem 1.3 it suffices to show that there exists a constant $N = N(n, \delta)$ such that for every $J \subseteq I$ and $i \in \mathbb{N}$, we have the inequality $\left|\operatorname{Tr}\left(\mathrm{F}^*_{X'_J,q} \circ H^i(\widetilde{\alpha}_J)\right)\right| \leqslant N q^{i/2}$.

To prove this inequality, we use a combination of Deligne's purity theorem and a variant of the argument of [Hru04]. Namely, we introduce a modification of "Hrushovski's norm" on cycle classes and show that it is "submultiplicative". Actually, we deduce the submultiplicativity from a general claim asserting that the refined Gysin pullback is "uniformly bounded". The boundedness of refined Gysin pullbacks is also used to "bound" the cycle classes $\widetilde{\alpha}_J$.

Remark 1.9. (a) Equality (3) was the only missing ingredient in [Var18], which was needed in order to prove Corollary 1.6 rather than Corollary 1.7.

(b) Though the proof of equality (3) is rather straightforward, the refined intersection theory of Fulton is needed both for its formulation (construction of $\widetilde{\alpha}$) and the proof.

1.10 *Plan of the paper.* The paper is organized as follows.

In Section 2, we introduce twisted versions of constructions and results of [Var18]. Namely, we study correspondences with locally invariant boundaries in Section 2.1 and a version of Pink's construction in Section 2.2.

In Section 3, we give a cohomological expression for the number of points in the intersection. Namely, we review definitions, constructions and results from intersection theory and ℓ-adic cohomology in Section 3.1,[2] introduce our main construction and formulate our main formula in Section 3.2 and finally prove the formula in Section 3.3.

In Section 4, we carry out the proof of Theorem 1.3. Namely, in Section 4.1, we review "boundedness results" shown in the appendices; in Section 4.2, we reduce the result to a particular case of Theorem 1.3(b); while in Section 4.3, we deduce the final estimate from the formula for the number of points in the intersection, obtained in Section 3.

We finish the paper with two appendixes in which we show the "boundedness" results used in Section 4. In particular, the reader interested only in the non-uniform estimates (Corollary 1.6) may entirely skip the appendices.

In Appendix A, we introduce the notion of a bounded collection and show that "boundedness" is preserved by various standard algebro-geometric operations. In the case of quasi-projective varieties, we also relate this notion to a much more naive notion of boundedness of the geometric degree. (In particular, we give an elementary proof of a theorem of Kleiman, following a suggestion of Hrushovski.)

In Appendix B, we introduce a norm on the Chow groups of quasi-projective schemes and study its properties. Namely, we show that various standard operations, that is, proper push-forwards, smooth pullbacks and refined Gysin pullbacks, are "uniformly bounded" and deduce bounds on the spectral radius. We would like to stress that though our notion of a norm is much more "naive" than the one used by Hrushovski, our results are strongly motivated by his results. In particular, the last subsection is almost identical to the corresponding part of [Hru04].

Note that though results in Appendix A seem to be well known to experts, some of the results of Appendix B could be of independent interest.

1.11 *Conventions.* In this work, k will always denote an algebraically closed field, and all schemes over k will be assumed to be of finite type. By a variety, we mean an integral scheme (of finite type) over k. For two schemes X and Y over k, we will write $X \times Y$ instead of $X \times_k Y$. For a scheme X over k and an automorphism σ of k, we denote by ${}^{\sigma}X$ the base change of X along $\sigma\colon \operatorname{Spec} k \to \operatorname{Spec} k$. We also use a similar convention for morphisms of schemes.

2. Twisted versions of results of [Var18]

Let k be an algebraically closed field, and let $\sigma \in \operatorname{Aut}(k)$ be a field automorphism.

2.1 Correspondences with locally invariant boundary

In this subsection, we recall twisted versions of the constructions and results of [Var18], where the case $\sigma = \mathrm{Id}$ is treated.

Notation 2.1.1. Let k be an algebraically closed field.

(a) By a *correspondence*, we mean a morphism of schemes $c = (c_1, c_2)\colon C \to X \times {}^{\sigma}X$, where X and C are schemes of finite type over k.

(b) For a correspondence $c\colon C \to X \times {}^{\sigma}X$ and an open subscheme $U \subseteq X$, we denote by $c|_U\colon c_1^{-1}(U) \cap c_2^{-1}({}^{\sigma}U) \to U \times {}^{\sigma}U$ the restrictions of c.

[2] Note that though we follow Fulton's book [Ful98] rather closely, our notation is slightly different.

(c) Let $c\colon C\to X\times X$ and $\widetilde{c}\colon \widetilde{C}\to \widetilde{X}\times {}^{\sigma}\widetilde{X}$ be two correspondences. By a *morphism* from $\widetilde{c}$ to c, we mean a pair of morphisms $[f]=(f,f_C)$ making the following diagram commutative:

$$\begin{tikzcd} \widetilde{X} \arrow[d, "f"'] & \widetilde{C} \arrow[l, "\widetilde{c}_1"'] \arrow[r, "\widetilde{c}_2"] \arrow[d, "f_C"'] & {}^{\sigma}\widetilde{X} \arrow[d, "{}^{\sigma}f"] \\ X & C \arrow[l, "c_1"'] \arrow[r, "c_2"] & {}^{\sigma}X\,. \end{tikzcd} \tag{5}$$

(d) Suppose that we are given correspondences $\widetilde{c}$ and c as in item (c) and a morphism $f\colon \widetilde{X}\to X$. We say that $\widetilde{c}$ *lifts* c if there exists a morphism $f_C\colon \widetilde{C}\to C$ such that $[f]=(f,f_C)$ is a morphism from $\widetilde{c}$ to c.

DEFINITION 2.1.2. Let $c\colon Y\to X\times {}^{\sigma}X$ be a correspondence, and let $Z\subseteq X$ be a closed subset, that is, a closed reduced subscheme.

(a) We say that Z is *c-invariant* if $c_1\big(c_2^{-1}({}^{\sigma}Z)\big)$ is set-theoretically contained in Z.

(b) We say that Z is *locally c-invariant* if for every point $x\in Z$, there exists an open neighborhood $U\subseteq X$ of x such that $Z\cap U\subseteq U$ is $c|_U$-invariant.

LEMMA 2.1.3. *Let $[f]=(f,f_C)$ be a morphism from a correspondence $\widetilde{c}\colon \widetilde{C}\to \widetilde{X}\times {}^{\sigma}\widetilde{X}$ to $c\colon C\to X\times {}^{\sigma}X$. If $Z\subseteq X$ is a locally c-invariant closed subset, then $f^{-1}(Z)_{\mathrm{red}}\subseteq \widetilde{X}$ is locally $\widetilde{c}$-invariant.*

Proof. Repeat the argument of [Var18, Lemma 1.3], where the case $\sigma=\mathrm{Id}$ is shown. □

Notation 2.1.4. Let $c\colon Y\to X\times {}^{\sigma}X$ be a correspondence and $Z\subseteq X$ a closed subset. Following [Var18], with this data we associate closed subsets

$$F(c,Z):=c_2^{-1}({}^{\sigma}Z)\cap c_1^{-1}(X\smallsetminus Z)\subseteq Y\,, \tag{6}$$

$$G(c,Z):=\cup_{S\in\mathrm{Irr}(F(c,Z))}\big[\overline{c_1(S)}\cap {}^{\sigma^{-1}}\overline{c_2(S)}\big]\subseteq X\,, \tag{7}$$

where $\mathrm{Irr}(F(c,Z))$ denotes the set of (reduced) irreducible components of $F(c,Z)$, and $\overline{c_1(S)}\subseteq X$ and $\overline{c_2(S)}\subseteq {}^{\sigma}X$ are the closures of $c_i(S)$.

Remark 2.1.5. Arguing as in [Var18, Remark 1.5(a) and Corollary 1.7(a)], one sees that a closed subset $Z\subseteq X$ is c-invariant (respectively, locally c-invariant) if and only if $F(c,Z)=\emptyset$ (respectively, $G(c,Z)=\emptyset$).

Notation 2.1.6. For a closed subscheme $Z\subseteq X$, we denote by $\mathcal{I}_{Z,X}\subseteq\mathcal{O}_X$ the sheaf of ideals of Z.

The following result is a twisted variant of [Var18, Proposition 2.3].

PROPOSITION 2.1.7. *Let $c\colon C\to X\times {}^{\sigma}X$ be a correspondence over k such that X and C are irreducible, while c_2 is dominant quasi-finite. Let $X^0\subseteq X$ be a non-empty open subset of X.*

Then there exist a non-empty open subset $V\subseteq X^0$ and a blow-up $\pi\colon \widetilde{X}\to X$ that is an isomorphism over V such that for every correspondence $\widetilde{c}\colon \widetilde{C}\to \widetilde{X}\times\widetilde{X}$ lifting c, the closed subset $\widetilde{X}\smallsetminus\pi^{-1}(V)\subseteq\widetilde{X}$ is locally $\widetilde{c}$-invariant.

Proof. The proof is a twisted analog of that of [Var18, Lemma 2.2 and Proposition 2.3].

We set $U_0:=X^0$, and by induction, we define for every $j\geqslant 0$ open subsets $V_j\subseteq U_j\subseteq X^0$ by the rules

$$V_j:=U_j\smallsetminus\overline{c_1\big(c_2^{-1}({}^{\sigma}X\smallsetminus{}^{\sigma}U_j)\big)}\,, \tag{8}$$

$$Z_j:=G(c|_{U_j},U_j\smallsetminus V_j),\ \text{ and } U_{j+1}:=U_j\smallsetminus Z_j\subseteq U_j\,. \tag{9}$$

Arguing as in [Var18, Lemma 2.2], we show that each V_j is non-empty and set $V := V_{\dim X}$ and $F := F(c, X \smallsetminus V) = c_2^{-1}({}^{\sigma}X \smallsetminus {}^{\sigma}V) \cap c_1^{-1}(V)$.

For every $S \in \mathrm{Irr}(F)$, we denote by $Z_S := \overline{c_1(S)} \cap {}^{\sigma^{-1}}\overline{c_2(S)} \subseteq X$ the schematic intersection, let $Z \subseteq X$ be a closed subscheme such that the sheaf of ideals $\mathcal{I}_{Z,X} \subseteq \mathcal{O}_X$ equals the product $\mathcal{I}_{Z,X} = \prod_{S \in \mathrm{Irr}(F)} \mathcal{I}_{Z_S,X} \subseteq \mathcal{O}_X$, let $\widetilde{X}$ be the blow-up $\mathrm{Bl}_Z(X)$ and denote by $\pi\colon \widetilde{X} \to X$ the canonical projection.

Then the argument of [Var18, Proposition 2.3 and Lemma 2.2] implies that V and π satisfy the required properties.[3] $\square$

2.2 A version of Pink's construction

2.2.1 *The construction.* (a) Let X' be a smooth scheme of dimension d over k, and let $\{X'_i\}_{i\in I}$ be a finite collection of smooth divisors with normal crossings. We set $\partial X' := \cup_{i\in I} X'_i \subseteq X'$ and $X'^0 := X' \smallsetminus \partial X'$.

(b) For every $J \subseteq I$, we set $X'_J := \cap_{i\in J} X'_i$. In particular, $X'_\emptyset = X'$. Then every X'_J is either empty or smooth over k of relative dimension $d - |J|$. We also set $\partial X'_J := \cup_{i \in I \smallsetminus J} X'_{J \cup \{i\}}$ and $(X'_J)^0 := X'_J \smallsetminus \partial X'_J$.

(c) For every $i \in I$, we set $Y_i := X'_i \times {}^{\sigma}(X'_i)$. We also set $Y := X' \times {}^{\sigma}(X')$, $\partial Y := \cup_{i\in I} Y_i$ and $Y^0 := Y \smallsetminus \partial Y$. In particular, we have $X'^0 \times {}^{\sigma}\big(X'^0\big) \subseteq Y^0$.

(d) We denote by $\mathcal{K} := \prod_{i\in I} \mathcal{I}_{Y_i,Y} \subseteq \mathcal{O}_Y$ the product of the sheaves of ideals of the Y_i, let $\widetilde{Y}$ be the blow-up $\mathrm{Bl}_{\mathcal{K}}(Y)$ and let $\pi\colon \widetilde{Y} \to Y$ be the projection map.

(e) By construction, π is an isomorphism over Y^0, hence over $X'^0 \times {}^{\sigma}\big(X'^0\big)$. Therefore, we have an open embedding $j\colon X'^0 \times {}^{\sigma}\big(X'^0\big) \simeq \pi^{-1}\big(X'^0 \times {}^{\sigma}\big(X'^0\big)\big) \hookrightarrow \widetilde{Y}$.

(f) For every $J \subseteq I$, we set $Y_J := X'_J \times {}^{\sigma}(X'_J) \subseteq Y$ and $E_J := \pi^{-1}(Y_J) \subseteq \widetilde{Y}$, denote by i_J the inclusion $E_J \hookrightarrow \widetilde{Y}$ and by π_J the projection $E_J \to Y_J$.

The following lemma is an analog of [Var18, Lemma 3.4], and the proof there goes over verbatim.

Lemma 2.2.2. *In the situation of §2.2.1, for every $J \subseteq I$, the closed subscheme $E_J \subseteq \widetilde{Y}$ is smooth of dimension $2d - |J|$.*

Notation 2.2.3. For every morphism $c' = (c'_1, c'_2)\colon C' \to Y = X \times X^{(q)}$, we denote by $\widetilde{c}\colon \widetilde{C} \to \widetilde{Y}$ the *strict preimage* of c'. Explicitly, $\widetilde{C}$ is the schematic closure of $c'^{-1}\big(Y^0\big) \times_Y \widetilde{Y}$ in $C' \times_Y \widetilde{Y}$. Notice that $\widetilde{c}$ is a closed embedding (respectively, finite) if c' is such.

2.2.4 *Setup.* In the situation of § 2.2.1, assume that k is an algebraically closed field of characteristic p, that q is a power of p and that $\sigma = \sigma_q$, the qth Frobenius automorphism; that is, $\sigma_q(x) = x^q$ for all $x \in k$.

As in the introduction, for a scheme X over k, we write $X^{(q)}$ instead of ${}^{\sigma_q}X$, and we denote by $\mathrm{F}_{X,q} = \mathrm{F}_{X/k,q}\colon X \to X^{(q)}$ the geometric Frobenius morphism and by $\Gamma_{X,q} \subseteq X \times X^{(q)}$ the graph of $\mathrm{F}_{X/k,q}$.

[3] Alternatively, one can observe that in the arguments of [Var18, Lemma 2.2 and Proposition 2.3], one does not use the assumption that the morphisms c_1 and c_2 are k-linear. Therefore, our assertion follows from the proof of [Var18, Lemma 2.2 and Proposition 2.3] in the case of the correspondence $X \xleftarrow{c_1} C \xrightarrow{c'_2} X$, where c'_2 is the composition $C \xrightarrow{c_2} {}^{\sigma}X \xrightarrow{\sigma} X$.

2.2.5 *The ramification degree.* (a) Recall that for a Noetherian scheme X, the sheaf of ideals $\mathcal{I}_{X_{\mathrm{red}},X} \subseteq \mathcal{O}_X$ is nilpotent. By the *thickness* of X, we mean the smallest $r \in \mathbb{N}$ such that $(\mathcal{I}_{X_{\mathrm{red}},X})^r = 0$.

(b) Let $f\colon Y \to X$ be a morphism of Noetherian schemes, and let Z be a closed subset of X or, what is the same, a closed reduced subscheme. We denote by $\mathrm{ram}(f, Z)$ the thickness of the schematic preimage $f^{-1}(Z) \subseteq Y$.

(c) For a morphism $f\colon Y \to X$ of Noetherian schemes, we denote by $\mathrm{ram}(f) \in \mathbb{N} \cup \{\infty\}$ the supremum $\sup_x \mathrm{ram}(f, x)$, taken over all closed points $x \in X$. Actually, $\mathrm{ram}(f)$ is always finite (see Lemma A.1.12).

The following result is a slight generalization of [Var18, Lemma 3.7].

Lemma 2.2.6. *Let $c'\colon C' \to X' \times X'^{(q)}$ be a correspondence such that the closed subset $\partial X' \subseteq X'$ is locally c'-invariant and $q > \mathrm{ram}\left(c'_2, \partial X'^{(q)}\right)$, and let $\widetilde{\Gamma} \subseteq \widetilde{Y}$ be the strict preimage of $\Gamma := \Gamma_{X',q}$. Then $\widetilde{c}(\widetilde{C}) \cap \widetilde{\Gamma} \subseteq \pi^{-1}\left(X'^0 \times \left(X'^0\right)^{(q)}\right)$.*

Proof. Note that the assertion is local on X' in the étale topology. Moreover, as observed in the proof of [Var18, Lemma 3.7], we may assume $X' = \mathbb{A}^n$ and $X'_i = Z(x_i)$. In this case, X' and X'_i are defined over $\mathbb{F}_q$; thus the assertion is proven in [Var18, Lemma 3.7]. □

The following lemma is a slight generalization of [Var18, Lemma 3.9].

Lemma 2.2.7. *In the situation of § 2.2.4, for every $J \subseteq I$, we set $\Gamma_J := \Gamma_{X'_J,q} \subseteq X'_J \times (X'_J)^{(q)}$ and $\Gamma^0_J := \Gamma_{X'_J,q} \subseteq X^0_J \times \left(X^0_J\right)^{(q)}$. Then the schematic closure $\overline{\pi_J^{-1}\left(\Gamma^0_J\right)} \subseteq E_J$ is smooth of dimension d, and the schematic preimage $\pi_J^{-1}(\Gamma_J) \subseteq E_J$ is a schematic union of $\overline{\pi_{J'}^{-1}\left(\Gamma^0_{J'}\right)}$ with $J' \supseteq J$.*

Proof. Again the assertion is local on X' in the étale topology. Therefore, arguing as in Lemma 2.2.6, the assertion follows from [Var18, Lemma 3.9]. □

3. The formula for the number of points in the intersection

In this section, we will show a formula for the number of points in the intersection $\#\left(C^0 \cap \Gamma^0\right)(k)$ appearing in Theorem 1.3.

3.1 Preliminaries from intersection theory and ℓ-adic cohomology

All schemes are assumed to be of finite type over a fixed field k. An integral scheme will be called a variety.

Notation 3.1.1. Let X be a scheme and $i \in \mathbb{N}$.

(a) We denote by $Z_i(X)$ the group of i-cycles on X, that is, the free abelian group generated by the symbols $[Z]$, where Z runs over all closed subvarieties of X of dimension i.

(b) We denote by $A_i(X)$ the group of i-cycles on X modulo rational equivalence (see [Ful98, Section 1.3]). We denote by $A_*(X)$ (respectively, $Z_*(X)$) the disjoint union of the $A_i(X)$ (respectively, $Z_i(X)$).

(c) For a closed equidimensional subscheme $Z \subseteq X$ of dimension i, we denote by $[Z]_X$ or simply $[Z]$ its class in $Z_i(X)$ or $A_i(X)$ (see [Ful98, Section 1.5]).

(d) If X is equidimensional of dimension d with irreducible components $\{X_a\}_a$, then $A_d(X) = Z_d(X)$ is a free abelian group with basis $\{[X_a]\}_a$.

3.1.2 *Proper pushforward.* (a) For a proper morphism $f\colon X \to Y$ of schemes, we denote by $f_*\colon A_i(X) \to A_i(Y)$ the induced morphism (see [Ful98, Section 1.4]). It is characterized by the property that for every closed subvariety $Z \subseteq X$, the pushforward $p_*([Z])$ equals $\deg(p|_Z)[p(Z)]$ if the induced map $p|_Z\colon Z \to p(Z)$ is generically finite and is zero otherwise.

(b) If X is proper over k, then the projection $p_X\colon X \to \operatorname{Spec} k$ gives rise to the degree map $\deg := (p_X)_*\colon A_0(X) \to A_0(\operatorname{Spec} k) = \mathbb{Z}$.

(c) For every closed subscheme $X' \subseteq X$, we denote by $f|_{X'}\colon X' \to f(X')$ the induced morphism and denote the pushforward $(f|_{X'})_*\colon A_i(X') \to A_i(f(X'))$ simply by f_*.

3.1.3 *Flat pullback.* (a) For a flat morphism $f\colon X \to Y$ of schemes of relative dimension n, we denote by $f^*\colon A_i(Y) \to A_{i+n}(X)$ the pullback map (see [Ful98, Theorem 1.7]). It is characterized by the condition that for every closed subvariety $Z \subseteq Y$, we have the equality $f^*([Z]) = \left[f^{-1}(Z)\right]$, where $f^{-1}(Z)$ denotes the schematic pullback.

(b) Flat pullbacks are compatible with proper pushforwards. In other words, for every Cartesian diagram

$$\begin{tikzcd} X' \arrow[r, "g_X"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ Y' \arrow[r, "g_Y"] & Y, \end{tikzcd}$$

where f and f' are proper and g_X and g_Y are flat of relative dimension n, we have the equality $(g_Y)^* \circ f_* = f'_* \circ (g_X)^*$ of morphisms $A_i(X) \to A_{i+n}(Y')$.

3.1.4 (*Refined*) *Gysin map*. (a) Let $f\colon X \to Y$ be a morphism between smooth connected schemes of dimensions d_X and d_Y, respectively, let $Y' \subseteq Y$ be a closed subscheme, and set $X' := f^{-1}(Y') \subseteq X$. Then for every i, we have the refined Gysin map $f^*\colon A_i(Y') \to A_{i+d_X-d_Y}(X')$ (see [Ful98, Definition 8.1.2]; Fulton sometimes denotes it by $f^!$).[4]

(b) The refined Gysin maps are compatible with compositions (by [Ful98, Propostion 8.1.1(a) and Corollary 8.1.3]).

(c) Assume in addition that f is flat. Then the induced map $f'\colon X' \to Y'$ is flat as well, and the refined Gysin map f^* coincides with the pullback f'^* from § 3.1.3(a) (by [Ful98, Proposition 8.1.3(a)]).

3.1.5 (*Refined*) *intersection pairing*. Let X be a smooth connected scheme of dimension d.

(a) For a pair of closed subschemes $Y_1, Y_2 \subseteq X$, we have an intersection pairing

$$\cap\colon A_i(Y_1) \times A_j(Y_2) \to A_{i+j-d}(Y_1 \cap Y_2)$$

(compare with [Ful98, Definition 8.1.1], but note that Fulton's notation is different). Let $\Delta\colon X \to X \times X$ be the diagonal map. Using § 3.1.4, we can form the Gysin map $\Delta^*\colon A_{i+j}(Y_1 \times Y_2) \to A_{i+j-d}(Y_1 \cap Y_2)$, and the intersection pairing is characterized by the condition that for closed subvarieties $Z_1 \subseteq Y_1$ and $Z_2 \subseteq Y_2$, we have $[Z_1]_{Y_1} \cap [Z_1]_{Y_2} = \Delta^*([Z_1 \times Z_2])$.

(b) In the situation of item (a), assume that $Y_1 \cap Y_2$ is proper over k. Then we have an intersection pairing $\cdot := \deg \circ \cap\colon A_i(Y_1) \times A_{d-i}(Y_2) \to \mathbb{Z}$.

[4] More generally, [Ful98, Section 6.6] defines the refined Gysin pullback when f is a local complete intersection morphism, that is, $f = g \circ i$, where i is a closed regular embedding and g is a smooth morphism.

(c) In the situation of item (b), let $Y_1' \subseteq Y_1$ and $Y_2' \subseteq Y_2$ be closed subschemes, and let $i_1 \colon Y_1' \hookrightarrow Y_1$ and $i_2 \colon Y_2' \hookrightarrow Y_2$ be the inclusion maps. Then for every $\alpha_1 \in A_i(Y_1')$ and $\alpha_2 \in A_{d-i}(Y_2')$, we have the equality $(i_1)_*(\alpha_1) \cdot (i_2)_*(\alpha_2) = \alpha_1 \cdot \alpha_2$ (by [Ful98, Theorem 6.2(a)]).

(d) An open embedding $j \colon X^0 \hookrightarrow X$ induces open embeddings $j_1 \colon Y_1^0 := X^0 \cap Y_1 \hookrightarrow Y_1$, $j_2 \colon Y_2^0 := X^0 \cap Y_2 \hookrightarrow Y_2$ and $j_{12} \colon Y_1^0 \cap Y_2^0 \hookrightarrow Y_1 \cap Y_2$. Then it follows from § 3.1.4(c) that for every $\alpha_1 \in A_i(Y_1)$ and $\alpha_2 \in A_j(Y_2)$, we have the equality $j_1^*(\alpha_1) \cap j_2^*(\alpha_2) = j_{12}^*(\alpha_1 \cap \alpha_2)$.

(e) In the situation of item (d), assume that $Y_1 \cap Y_2$ is proper over k and $Y_1 \cap Y_2 \subseteq j(X^0)$. Then, by item (d), we have the equality $j_1^*(\alpha_1) \cdot j_2^*(\alpha_2) = \alpha_1 \cdot \alpha_2$.

(f) Let $f \colon X' \to X$ be a proper map, where X' is a smooth connected scheme of dimension d', and let $Y' \subseteq X'$ and $Y \subseteq X$ be closed subschemes such the intersection $f(Y') \cap Y$ is proper over k. Then the intersection $Y' \cap f^{-1}(Y)$ is proper over k as well, and for every $\alpha \in A_i(X')$ and $\beta \in A_{d-i}(Y')$, we have the equality $\alpha \cdot f^*(\beta) = f_*(\alpha) \cdot \beta$, called the *projection formula* (see [Ful98, Example 8.1.7]).

Example 3.1.6. Let X be a smooth connected scheme of dimension d, and let $Y, Z \subseteq X$ be closed subvarieties such that $Y \cap Z$ is proper over k and $\dim(X) + \dim(Y) = d$. Then we can form the intersection $[Y]_Y \cdot [Z]_Z$ (see § 3.1.5(b)).

(a) If X is proper, then we have the equality $[Y]_Y \cdot [Z]_Z = [Y]_X \cdot [Z]_X$ (by § 3.1.5(c)).

(b) If $Y \cap Z$ is finite, then we have the equality $[Y]_Y \cap [Z]_Z = \sum_{x \in Y \cap Z} l(\mathcal{O}_{Y \cap Z, x})[x]$, hence $[Y]_Y \cdot [Z]_Z = \sum_{x \in Y \cap Z} l(\mathcal{O}_{Y \cap Z, x})$. Indeed, since $X \subseteq X \times X$ is locally given by d equations, the assumption that the intersection $Y \cap Z = X \cap (Y \times Z) \subseteq X \times X$ is finite implies that $Y \times Z$ is Cohen–Macaulay at every point $x \in Y \cap Z \subseteq Y \times Z$. Therefore, the assertion follows from [Ful98, Proposition 8.2(b)].

From now on, we assume that k is an algebraically closed field and ℓ is a prime different from the characteristic of k.

3.1.7 *Morphism of the cohomology.* Let X_1 and X_2 be smooth connected proper schemes over k of dimension d.

(a) Let $c = (c_1, c_2) \colon C \to X_1 \times X_2$ be a correspondence, where C is a scheme of dimension d as well. Then c gives rise to a morphism

$$H^i(c) \colon H^i(X_1, \mathbb{Q}_\ell) \xrightarrow{(c_1)^*} H^i(C, \mathbb{Q}_\ell) \xrightarrow{(c_2)_*} H^i(X_2, \mathbb{Q}_\ell)\,,$$

where $(c_2)_* \colon H^i(C, \mathbb{Q}_\ell) \to H^i(X_2, \mathbb{Q}_\ell)$ is the pushforward map corresponding by Poincaré duality to the pairing $H^i(C, \mathbb{Q}_\ell) \times H^{2d-i}(X_2, \mathbb{Q}_\ell(d)) \to \mathbb{Q}_\ell$ defined by $(x, y) \mapsto \mathrm{Tr}_{C/k}(x \cup c_2^*(y))$.

(b) As explained, for example, in [Var18, Section 4.5], every class $\alpha \in A_d(X_1 \times X_2)$ induces a morphism $H^i(\alpha) \colon H^i(X_1, \mathbb{Q}_\ell) \to H^i(X_2, \mathbb{Q}_\ell)$. Namely, it is characterized by the condition that the map $\alpha \mapsto H^i(\alpha)$ is a group homomorphism and that if $C \subseteq X_1 \times X_2$ is a closed subvariety of dimension d and $c \colon C \hookrightarrow X_1 \times X_2$ is the inclusion map, then the morphism $H^i([C])$ coincides with the map $H^i(c)$ from item (a).

(c) A morphism $f \colon X_1 \to X_2$ induces a pullback map $f^* \colon H^i(X_2, \mathbb{Q}_\ell) \to H^i(X_1, \mathbb{Q}_\ell)$. Then for every $\alpha \in A_d(X_1 \times X_2)$, one can form an endomorphism $f^* \circ H^i(\alpha)$ of $H^i\big(X_1, \overline{\mathbb{Q}_l}\big)$. Let $\Gamma_f \subseteq X_1 \times X_2$ be the graph of f. Then (see, for example, the discussion in [Var18, Section 4.6]) we have the equality

$$\alpha \cdot [\Gamma_f] = \sum_{i=0}^{2d} (-1)^i \,\mathrm{Tr}\left(f^* \circ H^i(\alpha), H^i(X_1, \mathbb{Q}_\ell)\right). \tag{10}$$

3.1.8 *A calculation.* In the situation of § 3.1.7, with $\alpha \in A_d(X_1 \times X_2)$ we associate $\deg(p_1|_\alpha) \in \mathbb{Z}$ characterized by the condition $(p_1)_*(\alpha) = \deg(p_1|_\alpha)[X_1]$ (see § 3.1.1(d)). We claim that for every $\alpha \in A_d(X_1 \times X_2)$ and a generically finite morphism $f : X_1 \to X_2$, we have the equality

$$\operatorname{Tr}\left(f^* \circ H^{2d}(\alpha), H^{2d}(X_1, \mathbb{Q}_\ell)\right) = \deg(f) \deg(p_1|_\alpha) .$$

Indeed, since $\operatorname{Tr}\left(f^* \circ H^{2d}(\alpha)\right) = \operatorname{Tr}\left(H^{2d}(\alpha) \circ f^*\right)$ and $H^{2d}(X_2, \mathbb{Q}_\ell)$ is 1-dimensional, we need to show that $H^{2d}(\alpha) \circ f^* = \deg(f) \deg(p_1|_\alpha) \operatorname{Id}$. By additivity, we can assume $\alpha = [C]$, where $C \subseteq X_1 \times X_2$ is a closed subvariety of dimension d.

Let $c\colon C \hookrightarrow X_1 \times X_2$ be the inclusion map. Then, by the definition of $(p_1)_*$, the degree $\deg(p_1|_\alpha)$ equals $\deg(c_1)$ if c_1 is generically finite and zero otherwise. Therefore, combining § 3.1.7(b) and basic properties of the trace map, we see that the endomorphism $H^{2d}(\alpha) \circ f^* = (c_2)_* \circ (c_1)^* \circ f^*$ of $H^{2d}\left(X_1, \overline{\mathbb{Q}_l}\right)$ equals $\deg(f \circ c_1) \operatorname{Id} = \deg(f) \deg(c_1) \operatorname{Id}$ if c_1 is generically finite and zero otherwise.

3.1.9 *Independence of ℓ.* Let X be a smooth proper variety of dimension d over k. It is known that for every $\ell \neq \operatorname{char}(k)$ and $i \in \mathbb{N}$, the cohomology $H^i(X, \mathbb{Q}_\ell)$ is a finite-dimensional $\mathbb{Q}_\ell$-vector space. It follows from results of Katz–Messing [KM74] that for every $\alpha \in A_d(X \times X)$, the characteristic polynomial of the endomorphism $H^i(\alpha)$ of $H^i(X, \mathbb{Q}_\ell)$ has coefficients in $\mathbb{Z}$ and is independent of $\ell \neq \operatorname{char}(k)$.

Namely, in [KM74] (see also [And04, § 3.3.3]), this is shown under the assumption that k is an algebraic closure of a finite field (using purity [Del74]), while the general case follows by a standard spreading-out argument (see Claim B.3.3 and the “general case” in the proof of Proposition B.3.2 below).

3.2 The formula

The finiteness of the intersection is guaranteed by the following simple lemma.

Lemma 3.2.1. *Let $c = (c_1, c_2)\colon C \to X \times X^{(q)}$ be a correspondence, let $\Gamma \subseteq X \times X^{(q)}$ be the graph of $\mathrm{F}_{X/k,q}$, and let $y \in c^{-1}(\Gamma)(k) \subseteq C(k)$ be a point such that $c_2^{-1}(c_2(y))$ is finite and $q > \operatorname{ram}(c_2, c_2(y))$. Then $y \in c^{-1}(\Gamma)$ is an isolated point, and there exists an open neighborhood $C^0 \subseteq C$ of y such that the closed subschemes $C^0 \cap c^{-1}(\Gamma)$ and $c_2^{-1}(c_2(y)) \cap C^0$ of C^0 are equal.*

Proof. Set $x := c_1(y) \in X(k)$. Since the assertion is local, we can assume that X and C are affine. Then $c_2(y) = x^{(q)}$, and we form ideals $I := \left(\mathcal{I}_{c_2^{-1}(x^{(q)}),C}\right)_y \subseteq \mathcal{O}_{C,y}$ and $J := \left(\mathcal{I}_{c^{-1}(\Gamma),C}\right)_y \subseteq \mathcal{O}_{C,y}$, where $\mathcal{O}_{C,y}$ is the local ring of C at y. Since y is an isolated point in $c_2^{-1}\left(x^{(q)}\right)$, the radical $\sqrt{I}$ is the maximal ideal $\mathfrak{m}_{C,y} \subseteq \mathcal{O}_{C,y}$, and it suffices to show the equality $I = J$.

Let $(f_1, \ldots, f_n)$ be generators of $\mathfrak{m}_{X,x} \subseteq \mathcal{O}_{X,x}$, let $\left(f_1^{(q)}, \ldots, f_n^{(q)}\right)$ be the corresponding generators of $\mathfrak{m}_{X^{(q)},x^{(q)}} \subseteq \mathcal{O}_{X^{(q)},x^{(q)}}$, and let $a_i := c_1^{\cdot}(f_i) \in \mathcal{O}_{C,y}$ and $b_i := c_2^{\cdot}\left(f_i^{(q)}\right) \in \mathcal{O}_{C,y}$ be their pullbacks. Then we have the equalities $I = (b_1, \ldots, b_n)$ and $J = \left(b_1 - a_1^q, \ldots, b_n - a_n^q\right)$.

Since $\operatorname{ram}(c_2, x^{(q)}) < q$ and $a_i \in \mathfrak{m}_{C,y} = \sqrt{I}$ for all i, we have $a_i^{q-1} \in I$, hence $a_i^q \in \mathfrak{m}_{C,y} I \subseteq I$. Therefore, we have $b_i - a_i^q \in I$ and $b_i = \left(b_i - a_i^q\right) + a_i^q \in J + \mathfrak{m}_{C,y} I$ for all i. This implies the inclusions $J \subseteq I \subseteq J + \mathfrak{m}_{C,y} I$, from which the equality $I = J$ follows by Nakayama's lemma. $\square$

The following notion is not standard, but it is convenient for our purposes.

3.2.2 *Uh-étale morphisms.* We call a morphism of schemes $f\colon X \to Y$ *uh-étale*, where *uh* stands for *universal homeomorphism*, if it decomposes as $X \xrightarrow{f_i} Z \xrightarrow{f_s} Y$, where f_i is a flat universal homeomorphism and f_s is étale. Notice that if f is uh-étale and $U \subseteq X$ is an open subscheme, then $f|_U \colon U \to X$ is uh-étale.

Lemma 3.2.3. *Let $f\colon X \to Y$ be a generically finite morphism of finite type between integral Noetherian schemes.*

(a) *There exists an open dense subscheme $U \subseteq X$ such that $f|_U \colon U \to Y$ is uh-étale.*

(b) *If f is uh-étale, then we have the equality $l(\mathcal{O}_{f^{-1}(f(x)),x}) = \deg(f)_{\mathrm{insep}}$ for every $x \in X$.*

Proof. (a) The induced map on rational functions $f^*\colon \operatorname{Rat}(Y) \hookrightarrow \operatorname{Rat}(X)$ decomposes as a composition $\operatorname{Rat}(Y) \hookrightarrow L \hookrightarrow \operatorname{Rat}(X)$ such that $L/\operatorname{Rat}(Y)$ is separable and $\operatorname{Rat}(X)/L$ is purely inseparable. Replacing Y with an open dense subscheme and X with its preimage, we can assume that X and Y are affine, while f is finite.

In this case, f decomposes as a composition $X \xrightarrow{f_i} Z \xrightarrow{f_s} Y$, where Z is the affine scheme $\operatorname{Spec}(L \cap \Gamma(X, \mathcal{O}_X))$. Then $\operatorname{Rat}(Z) \simeq L$, the morphisms f_i and f_s are finite, f_i is generically a universal homeomorphism, while f_s is generically étale. Therefore, there exists an open dense subscheme $V \subseteq Z$ such that $f_s|_V \colon V \to Y$ is étale and $f_i|_V := f_i^{-1}(V) \to V$ is a flat universal homeomorphism (use [Sta22, Tag 0559]). Then $U := f_i^{-1}(V)$ satisfies the required property.

(b) First notice that in the notation of § 3.2.2, the scheme Z is irreducible, because X is irreducible and f_i is a universal homeomorphism, and that Z is reduced because Y is reduced, while f_s is étale. Thus Z is integral, so we can talk about $\deg(f_i)$. Then for every $x \in X$, we have the equality

$$l(\mathcal{O}_{f^{-1}(f(x)),x}) = l(\mathcal{O}_{f_i^{-1}(f_i(x)),x}) = \deg(f_i) = \deg(f)_{\mathrm{insep}}\,.$$

Indeed, the first equality holds since f_s is unramified, the second one holds since f_i is a flat universal homeomorphism, while the third one is clear. □

Assumptions 3.2.4. Let (k, q, X^0, C^0) be a quadruple as in Theorem 1.3(b), let $X \subseteq \mathbb{P}^n_k$ be the closure of X^0, let $C \subseteq X \times X^{(q)}$ be the closure of C^0, and let $c\colon C \hookrightarrow X \times X^{(q)}$ be the inclusion map. Assume that

(a) X^0 is smooth over k,

(b) the morphism c_2^0 is uh-étale (see § 3.2.2),

(c) the boundary $\partial X := X \smallsetminus X^0$ is locally c-invariant (see Definition 2.1.2).

Construction 3.2.5. (a) By [dJo96, Theorem 4.1], there exists an alteration $f\colon X' \to X$ such that X' is connected and smooth over k and the reduced preimage $\partial X' := f^{-1}(\partial X)_{\mathrm{red}} \subseteq X'$ is a strict normal crossing divisor, that is, a union of smooth divisors $\{X'_i\}_{i\in I}$ with normal crossings. From now on, we fix such an f. In particular, the twisted Pink construction given in § 2.2.1 applies.

(b) We set $X'^0 := X' \smallsetminus \partial X'$ and denote by $f^0\colon X'^0 \to X^0$ the restriction of f. Also we will denote the projection $f \times f^{(q)}\colon Y = X' \times X'^{(q)} \to X \times X^{(q)}$ by g and its restriction $f^0 \times (f^0)^{(q)}\colon X'^0 \times (X'^0)^{(q)} \to X^0 \times (X^0)^{(q)}$ by g^0.

(c) We set $C'^0 := (g^0)^{-1}(C^0) \subseteq X'^0 \times (X'^0)^{(q)}$, let $C' \subseteq Y$ be the closure of C'^0, and let $c' = (c'_1, c'_2)\colon C' \hookrightarrow Y = X' \times X'^{(q)}$ be the inclusion map. As in § 2.2.3, let $\widetilde{C} \subseteq \widetilde{Y}$ be the strict preimage of C'. By construction, the open embedding $j\colon X'^0 \times (X'^0)^{(q)} \hookrightarrow \widetilde{Y}$ (see § 2.2.1(e)) restricts to an open embedding $j_C\colon C'^0 \hookrightarrow \widetilde{C}$.

(d) Since g^0 is a morphism between smooth varieties, we can form the refined Gysin pullback $\alpha^0 := (g^0)^*\big([C^0]_{C^0}\big) \in A_d\big(C'^0\big)$ of $[C^0]_{C^0} \in A_d\big(C^0\big)$ (see §§ 3.1.1(c) and 3.1.4(a)).

(e) Choose a cycle $\widetilde{\alpha}^0 := \sum_i n_i[V_i] \in Z_d\big(C'^0\big)$ representing α^0, where each $V_i \subseteq C'^0$ is a closed irreducible subvariety. We denote by $\widetilde{\alpha}_{\widetilde{C}} \in A_d\big(\widetilde{C}\big)$ the class of $\sum_i n_i\big[\overline{V}_i\big] \in Z_d\big(\widetilde{C}\big)$, where $\overline{V}_i \subseteq \widetilde{C}$ is the closure of $j_C(V_i)$.

(f) We denote by $\widetilde{\alpha} \in A_d\big(\widetilde{Y}\big)$ the image of $\widetilde{\alpha}_{\widetilde{C}} \in A_d\big(\widetilde{C}\big)$ with respect to the closed embedding $\widetilde{C} \hookrightarrow \widetilde{Y}$. Recall that for every $J \subseteq I$, the map $i_J\colon E_J \hookrightarrow \widetilde{Y}$ is a closed embedding of codimension $|J|$ between smooth schemes (see Lemma 2.2.2), and set $\widetilde{\alpha}_J := \pi_{J*}i_J^*(\widetilde{\alpha}) \in A_{d-|J|}(Y_J)$. In particular, we have $\widetilde{\alpha}_J = 0$ if $Y_J = \emptyset$.

(g) By § 3.1.7, for every $i = 0, \ldots, 2(d-|J|)$, every class $\widetilde{\alpha}_J \in A_{d-|J|}\big(X'_J \times (X'_J)^{(q)}\big)$ gives rise to an endomorphism $\mathrm{F}^*_{X'_J,q} \circ H^i(\widetilde{\alpha}_J)$ of $H^i(X'_J, \mathbb{Q}_\ell)$.

Remark 3.2.6. Notice that in Construction 3.2.5, we made two choices. Namely, we chose an alteration $f\colon X' \to X$ in item (a) and a cycle $\widetilde{\alpha}^0$ in item (e).

The goal of this section is to show the following formula.

PROPOSITION 3.2.7. *In the situation of Assumptions 3.2.4 and notation of Construction 3.2.5, assume $q > \operatorname{ram}\big(c'_2, \partial X'^0\big)$ and $q > \operatorname{ram}\big(c_2^0\big)$ (see § 2.2.5). Set $\Gamma^0 := \Gamma_{X^0,q}$. Then the intersection $\big(C^0 \cap \Gamma^0\big)(k)$ is finite, and the difference*

$$\#\big(C^0 \cap \Gamma^0\big)(k) - \frac{\deg\big(c_1^0\big)}{\deg\big(c_2^0\big)_{\mathrm{insep}}} q^d$$

equals the product of $1/(\deg(f)\deg(c_2^0)_{\mathrm{insep}})$ and

$$\sum_{i=0}^{2d-1}(-1)^i \operatorname{Tr}\big(F^*_{X',q} \circ H^i(\widetilde{\alpha}_\emptyset)\big) + \sum_{J \neq \emptyset}(-1)^{|J|}\left(\sum_{i=0}^{2d-|J|}(-1)^i \operatorname{Tr}\big(F^*_{X'_J,q} \circ H^i(\widetilde{\alpha}_J)\big)\right). \tag{11}$$

3.3 Calculation of the intersection numbers

Let $\big[\widetilde{\Gamma}\big] \in A_d\big(\widetilde{Y}\big)$ be the class of $\widetilde{\Gamma} \subseteq \widetilde{Y}$ (see Lemma 2.2.6). To show Proposition 3.2.7, we are going to calculate the intersection number $\widetilde{\alpha} \cdot \big[\widetilde{\Gamma}\big]$ in $\widetilde{Y}$ in two different ways.

The following result is an analog of [Var18, Lemma 4.9] in our setup (which in turn was motivated by [Laf02, proposition IV.6]).

LEMMA 3.3.1. *Let $[\Gamma_J] \in A_{d-|J|}(Y_J)$ be the class of $\Gamma_J \subseteq Y_J$ (see Lemma 2.2.7). Then we have the equality*

$$\widetilde{\alpha} \cdot \big[\widetilde{\Gamma}\big] = \sum_{J \subseteq I}(-1)^{|J|}\widetilde{\alpha}_J \cdot [\Gamma_J]\,. \tag{12}$$

Proof. By the projection formula (see § 3.1.5(f)), we have the equalities $\widetilde{\alpha}_J \cdot [\Gamma_J] = \widetilde{\alpha} \cdot (i_{J*}\pi_J^*[\Gamma_J])$. Thus it suffices to show the identity $\big[\widetilde{\Gamma}\big] = \sum_{J \subseteq I}(-1)^{|J|} i_{J*}\pi_J^*[\Gamma_J]$. To prove the result, we argue as in [Var18, Lemma 4.9], where an untwisted version of this result is deduced from an analog of Lemma 2.2.7. □

LEMMA 3.3.2. *In the situation of Assumptions 3.2.4 and notation of Construction 3.2.5, assume $q > \operatorname{ram}\big(c'_2, \partial X'^0\big)$ and $q > \operatorname{ram}\big(c_2^0\big)$. Then we have the equality*

$$\widetilde{\alpha} \cdot \big[\widetilde{\Gamma}\big] = \deg(f)\big([C^0]_{C^0} \cdot [\Gamma^0]_{\Gamma^0}\big) = \deg(f)\deg\big(c_2^0\big)_{\mathrm{insep}} \cdot \#\big(C^0 \cap \Gamma^0\big)(k)\,. \tag{13}$$

Remark 3.3.3. Since $q > \mathrm{ram}\,(c_2^0)$, the intersection $C^0 \cap \Gamma^0$ is finite (by Lemma 3.2.1). Therefore, the intersection number $[C^0]_{C^0} \cdot [\Gamma^0]_{\Gamma^0}$ is defined (see § 3.1.5(b)).

Proof. We set $\Gamma'^0 := \Gamma_{X'^0,q}$. We claim that we have the sequence of equalities

$$\widetilde{\alpha} \cdot [\widetilde{\Gamma}] \overset{\text{(i)}}{=\!=} \widetilde{\alpha}_{\widetilde{C}} \cdot [\widetilde{\Gamma}]_{\widetilde{\Gamma}} \overset{\text{(ii)}}{=\!=} \alpha^0 \cdot [\Gamma'^0]_{\Gamma'^0} \overset{\text{(iii)}}{=\!=} (g^0)^*([C^0]_{C^0}) \cdot [\Gamma'^0]_{\Gamma'^0}$$
$$\overset{\text{(iv)}}{=\!=} [C^0]_{C^0} \cdot (g^0)_*([\Gamma'^0]_{\Gamma'^0}) \overset{\text{(v)}}{=\!=} \deg(f)([C^0]_{C^0} \cdot [\Gamma^0]_{\Gamma^0}).$$

Indeed,

- equality (i) follows from Example 3.1.6(a);
- since $q > \mathrm{ram}\,(c_2', \partial X'^0)$, we have the inclusion $\widetilde{C} \cap \widetilde{\Gamma} \subseteq \pi^{-1}(X'^0 \times X'^{(q)})$ (by Lemma 2.2.6), hence $\widetilde{C} \cap \widetilde{\Gamma} \subseteq j(X'^0 \times X'^{(q)})$ (see § 2.2.1(e)), so equality (ii) follows from § 3.1.5(e) and equality $j_C^*(\widetilde{\alpha}_{\widetilde{C}}) = \alpha^0$ (use Construction 3.2.5(c), (e));
- equality (iii) follows from the equality $\alpha^0 = (g^0)^*([C^0]_{C^0})$ (see Construction 3.2.5(d));
- equality (iv) follows from the projection formula (see § 3.1.5(f));
- equality (v) would follow if we show that $(g^0)_*([\Gamma'^0]_{\Gamma'^0}) = \deg(f)[\Gamma^0]_{\Gamma^0}$; to see this, notice that the restriction $g^0|_{\Gamma'^0} \colon \Gamma'^0 \to \Gamma^0$ is isomorphic to the projection $f^0 \colon X'^0 \to X^0$; in particular, it is generically finite of degree $\deg(f)$.

This shows the left equality of (13). To see the right one, it suffices to show that

$$[C^0]_{C^0} \cdot [\Gamma^0]_{\Gamma^0} = \sum_{y \in (C^0 \cap \Gamma^0)(k)} l(\mathcal{O}_{C^0 \cap \Gamma^0, y}) = \deg\,(c_2^0)_{\mathrm{insep}} \cdot \#(C^0 \cap \Gamma^0)(k)\,. \tag{14}$$

The first equality of (14) follows from § 3.1.6(b), while for the second one it suffices to show that

$$l(\mathcal{O}_{C^0 \cap \Gamma^0, y}) = l(\mathcal{O}_{(c_2^0)^{-1}(c_2^0(y)), y}) = \deg\,(c_2^0)_{\mathrm{insep}}\,. \tag{15}$$

Finally, the first equality in (15) follows from Lemma 3.2.1, while the second one follows from Lemma 3.2.3(b). □

Lemma 3.3.4. *Let $\widetilde{\alpha}_\emptyset = \pi_*(\widetilde{\alpha}) \in A_d(Y)$ be the class defined in Construction* 3.2.5(f). *Then we have the equality $g_*(\widetilde{\alpha}_\emptyset) = \deg(f)^2[C]$ in $A_d(X \times X^{(q)})$.*

Proof. We have to show that $(g \circ \pi)_*(\widetilde{\alpha}) = \deg(f)^2[C]$. Denote by $\widetilde{g}_C \colon \widetilde{C} \to C$ the restriction of $g \circ \pi$. It suffices to show the equality $(\widetilde{g}_C)_*([\widetilde{\alpha}]_{\widetilde{C}}) = \deg(f)^2[C]_C$ in $A_d(C) = Z_d(C)$. Therefore, it suffices to show the equality after pulling back to any open non-empty subscheme V of C.

Thus, taking the pullback to C^0, using § 3.1.3(b) and the equality $j_C^*(\widetilde{\alpha}) = (g^0)^*([C^0]_{C^0})$, it suffices to show that $(g^0)_*(g^0)^*([C^0]_{C^0}) = \deg(f)^2[C^0]_{C^0}$.

Let $U \subseteq X^0$ be an open dense subscheme such that $f_U \colon f^{-1}(U) \to U$ is finite flat. Since c_1^0 and c_2^0 are dominant, the open subscheme $C_U := C^0 \cap (U \times U^{(q)})$ of C^0 is non-empty. Set $g_U := f_U \times (f_U)^{(q)}$. Taking the pullback to C_U, using §§ 3.1.3(b) and 3.1.4(b),(c), it thus suffices to show that $(g_U)_*(g_U)^*([C_U]_{C_U}) = \deg(f)^2[C_U]_{C_U}$. Since $(g_U)^*([C_U]_{C_U}) = [g_U^{-1}(C_U)]_{g^{-1}(C_U)}$ (use § 3.1.4(b)) while the map $g_U^{-1}(C_U) \to C_U$ is finite flat of degree $\deg(f)^2$, the assertion follows. □

Corollary 3.3.5. *In the situation of Assumptions* 3.2.4 *and notation of Construction* 3.2.5, *we have the equality*

$$\mathrm{Tr}\,(F_{X',q}^* \circ H^{2d}(\widetilde{\alpha}_\emptyset)) = \deg(f)\deg(c_1)q^d\,.$$

Proof. By § 3.1.8, we have to show that

$$\deg(F_{X',q})\deg(p_1|_{\widetilde{\alpha}_\emptyset}) = \deg(f)\deg(c_1)q^d\,.$$

Since $\deg(F_{X',q}) = q^{\dim X'} = q^d$, we thus have to show that $\deg(p_1|_{\widetilde{\alpha}_\emptyset}) = \deg(f)\deg(c_1)$, that is, to show the equality $(p_1)_*(\widetilde{\alpha}_\emptyset) = \deg(f)\deg(c_1)[X']$ in $A_d(X')$.

Since $f_*([X']) = \deg(f)[X]$, it thus suffices to show the equality

$$f_*(p_1)_*(\widetilde{\alpha}_\emptyset) = (p_1)_*g_*(\widetilde{\alpha}_\emptyset) = \deg(f)^2\deg(c_1)[X]$$

in $A_d(X)$. Since $f\circ p_1 = p_1\circ g : X'\times X'^{(q)}\to X$, the last equality follows from Lemma 3.3.4 and the identity $(p_1)_*[C] = \deg(c_1)[X]$. □

Now we are ready to show Proposition 3.2.7.

3.3.6. *Proof of Proposition* 3.2.7. Combining Lemmas 3.3.1 and 3.3.2, we have the equality

$$\deg(f)\deg\left(c_2^0\right)_{\mathrm{insep}}\cdot\#\left(C^0\cap\Gamma^0\right)(k) = \sum_{J\subseteq I}(-1)^{|J|}\widetilde{\alpha}_J\cdot[\Gamma_J]\,.$$

Therefore, the assertion follows from a combination of the equalities

$$\widetilde{\alpha}_J\cdot[\Gamma_J] = \sum_{i=0}^{2(d-|J|)}(-1)^i\operatorname{Tr}\left(\mathrm{F}^*_{X'_J,q}\circ H^i(\widetilde{\alpha}_J), H^i(X'_J,\mathbb{Q}_\ell)\right)$$

(see equality (10)) and Corollary 3.3.5. □

4. Proof of the main theorem

4.1 Review of results from the appendices

In this subsection, we will review the results from the appendices we need for the proof of Theorem 1.3.

Claims 4.1.1 (*Standard boundedness results*; see Lemma A.2.8 for assertions (a) and (b) and § A.2.6 for the remaining assertions). *Let $X\subseteq\mathbb{P}^n_k$, $X'\subseteq\mathbb{P}^m_X$ and $Z\subseteq X$ be reduced subschemes, and let $f\colon X'\to X$ be the composition $X'\hookrightarrow\mathbb{P}^m_X\twoheadrightarrow X$.*

(a) *We have the equality* $\operatorname{gdeg}\left(X^{(q)}\right) = \operatorname{gdeg}(X)$.

(b) *Let $\overline{X}\subseteq\mathbb{P}^n_k$ be the closure of X, and let $X_1,\dots,X_r$ be all irreducible components of X. Then we have the inequalities* $\operatorname{gdeg}\left(\overline{X}\right)\leqslant\operatorname{gdeg}(X)$, $r\leqslant\operatorname{gdeg}(X)$ *and* $\operatorname{gdeg}(X_i)\leqslant\operatorname{gdeg}(X)$ *for all i.*

(c) *Let $Y\subseteq\mathbb{P}^n_k$ be a reduced subscheme. Then $\operatorname{gdeg}((X\cap Y)_{\mathrm{red}})$ is bounded in terms of n and $\operatorname{gdeg}(X)$ and $\operatorname{gdeg}(Y)$.*

(d) *Assume that $Z\subseteq X$ is either open or closed. Then $\operatorname{gdeg}((X\smallsetminus Z)_{\mathrm{red}})$ is bounded in terms of n, $\operatorname{gdeg}(X)$ and $\operatorname{gdeg}(Z)$.*

(e) *The ramification degree $\operatorname{ram}(f,Z)$ and the geometric degree of $f^{-1}(Z)_{\mathrm{red}}\subseteq X'$ are bounded in terms of n, m, $\operatorname{gdeg}(X)$, $\operatorname{gdeg}(X')$ and $\operatorname{gdeg}(Z)$.*

(f) *The geometric degree of $\overline{f(X')}\subseteq X$ is bounded in terms of n, m, $\operatorname{gdeg}(X)$ and $\operatorname{gdeg}(X')$.*

(g) *Assume that X and X' are irreducible and f is generically finite. Then $\deg(f)$ and $\operatorname{ram}(f)$ are bounded in terms of n, m, $\operatorname{gdeg}(X)$ and $\operatorname{gdeg}(X')$.*

(h) *Let $U \subseteq X$ be the largest open subscheme such that U is smooth over k. Then* $\mathrm{gdeg}(U)$ *is bounded in terms of n and* $\mathrm{gdeg}(X)$.

(i) *In the situation of assertion* (g), *there exists an open dense subscheme $U \subseteq X'$ such that $f|_U$ is uh-étale (see §3.2.2) and* $\mathrm{gdeg}(U)$ *is bounded in terms of n, m,* $\mathrm{gdeg}(X)$ *and* $\mathrm{gdeg}(X')$.

(j) *Assume that $Z \subseteq X$ is closed. Then there exists a closed embedding $\mathrm{Bl}_Z(X) \hookrightarrow \mathbb{P}^{n'}_X$ over X such that n' and the geometric degree of $\mathrm{Bl}_Z(X) \subseteq \mathbb{P}^{n'}_X$ are bounded in terms of n,* $\mathrm{gdeg}(X)$ *and* $\mathrm{gdeg}(Z)$.

(k) *Assume that X is irreducible and $Z \subseteq X$ is closed. Then there exist an alteration $\widetilde{f}\colon \widetilde{X} \to X$ and a closed embedding $\widetilde{X} \hookrightarrow \mathbb{P}^{n'}_X$ over X, where $\widetilde{X}$ is smooth over k and $\widetilde{f}^{-1}(Z)_{\mathrm{red}} \subseteq \widetilde{X}$ is a union of smooth divisors with normal crossings, such that n' and* $\mathrm{gdeg}(\widetilde{X})$ *are bounded in terms of* $n, \mathrm{gdeg}(X)$ *and* $\mathrm{gdeg}(Z)$.

4.1.2 *Norm of a cycle class.* Suppose that we are in the situation of Claims 4.1.1.

(a) For every $\alpha \in A_*(X)$, we denote by $|\alpha|$ the minimum of $\sum_j |n_j| \,\mathrm{gdeg}(Z_j)$ taken over all representatives $\alpha' = \sum_j n_j[Z_j] \in Z_*(X)$ of α, where the $Z_j \subseteq X$ are irreducible subvarieties and $n_j \in \mathbb{Z}$.

(b) Assume that $X \subseteq \mathbb{P}^n_k$ and $X' \subseteq \mathbb{P}^m_X$ are closed. Then f is proper, and it follows from Lemma B.2.2(a) that for every $\alpha \in A_*(X')$, we have the inequality $|f_*(\alpha)| \leqslant |\alpha|$.

(c) Assume that X and X' are smooth and connected. Then we have a refined Gysin pullback map $f^*\colon A_*(Z) \to A_*\big(f^{-1}(Z)\big)$ (see §3.1.4). Moreover, it follows from Corollary B.2.7 that there exists a constant M depending on n, m, $\mathrm{gdeg}(X)$, $\mathrm{gdeg}(X')$ and $\mathrm{gdeg}(Z)$ such that for every class $\alpha \in A_*(Z)$, we have the inequality $|f^*(\alpha)| \leqslant M|\alpha|$.

4.1.3 *Main estimate.* Let k be an algebraically closed field of characteristic $p > 0$, let q be a power of p, and let $X \subseteq \mathbb{P}^n_k$ be a smooth closed subvariety of dimension d. By Corollary B.3.5, there exists a constant M depending on n and $\deg(X)$ such that for every $\alpha \in A_d\big(X \times X^{(q)}\big)$ and $i \in \mathbb{N}$, the trace $\mathrm{Tr}\left(F^*_{X,q} \circ H^i(\alpha)\right) \in \mathbb{Z}$ (see §3.1.9) satisfies

$$\left|\,\mathrm{Tr}\left(F^*_{X,q} \circ H^i(\alpha)\right)\right| \leqslant M|\alpha| q^{i/2}\,.$$

4.2 Main reduction

In this subsection, we are going to reduce Theorem 1.3 to a particular case of Theorem 1.3(b).

Proposition 4.2.1. *To prove Theorem 1.3, it suffices to show that "the conclusion of Theorem 1.3(b) holds for quadruples satisfying Assumptions 3.2.4".*

More precisely, in order to prove Theorem 1.3, it suffices to show that for every $n, \delta \in \mathbb{N}$, there exists a constant M such that the conclusion of Theorem 1.3(b) holds for all quadruples $\big(k, q, X^0, C^0\big)$ satisfying Assumptions 3.2.4.

Proof. For the convenience of the reader, we divide the proof into steps. First, we are going to show that Theorem 1.3(a) follows from Theorem 1.3(b).

4.2.2. Note that it suffices to show that for every fixed $d \in \mathbb{N}$, the conclusion of Theorem 1.3(a) holds for quadruples such that $\dim\big(C^0\big) = d$. Moreover, by induction, we can assume that the conclusion of Theorem 1.3(a) holds for quadruples such that $\dim\big(C^0\big) < d$.

4.2.3. We claim that it suffices to show Theorem 1.3(a) for quadruples such that C^0 is irreducible.

Indeed, let (k, q, X^0, C^0) be a quadruple as in Theorem 1.3(a), and let $C_1, \dots, C_r$ be all irreducible components of C^0. Since $r \leqslant \mathrm{gdeg}(C)$, $\mathrm{gdeg}(C_i) \leqslant \mathrm{gdeg}\left(C^0\right)$ (see Claims 4.1.1(b)), $\dim(C_i) \leqslant \dim\left(C^0\right)$ and $\left(C^0 \cap \Gamma^0\right) = \cup_i\left(C_i \cap \Gamma^0\right)$, we see that if each quadruple $\left(k, q, X^0, C_i\right)$ satisfies inequality (1) with constant M, then quadruple $\left(k, q, X^0, C^0\right)$ satisfies inequality (1) with constant δM.

4.2.4. We claim that it suffices to prove Theorem 1.3(a) for quadruples satisfying the assumptions of Theorem 1.3(b).

Indeed, by §§ 4.2.2 and 4.2.3, it suffices to show the conclusion of Theorem 1.3(a) for quadruples $\left(k, q, X^0, C^0\right)$ such that C^0 is irreducible and $\dim\left(C^0\right) = d$. Consider closed reduced subschemes $Z := \overline{c_1(C)} \subseteq X$ and $C_Z := c_2^{-1}\left(Z^{(q)}\right)_{\mathrm{red}} \subseteq C^0$. Then $C_Z \subseteq Z \times Z^{(q)}$ is a closed reduced subscheme, and we have the equality

$$C^0 \cap \Gamma_{X^0,q} = C_Z \cap \Gamma_{Z,q}\,. \tag{16}$$

Moreover, it follows from a combination of Claims 4.1.1(a), (e), (f) that the geometric degrees of Z and C_Z are bounded in terms of n and δ.

If $C_Z \subsetneq C^0$, then $\dim(C_Z) < d$; thus the assertion follows from equality (16) and the induction hypothesis (see § 4.2.2). Therefore, we can assume $C_Z = C^0$ and that the morphism $c_1\colon C^0 \to Z$ is dominant. Thus quadruple $\left(k, q, Z, C^0\right)$ satisfies all the assumptions of Theorem 1.3(b), and the assertion follows from equality (16).

4.2.5. Now we claim that Theorem 1.3(b) implies Theorem 1.3(a). Indeed, by § 4.2.4, it suffices to show the conclusion of Theorem 1.3(a) for quadruples $\left(k, q, X^0, C^0\right)$ from Theorem 1.3(b). Since $\deg\left(c_1^0\right)$ is bounded in terms of n and δ (see Claims 4.1.1(g)), the assertion follows.

Now we are going to show that it suffices to show the conclusion of Theorem 1.3(b) for quadruples satisfying Assumptions 3.2.4(a)–(c).

4.2.6. First, we claim that it suffices to show the conclusion of Theorem 1.3(b) for quadruples $\left(k, q, X^0, C^0\right)$ such that X^0 is smooth.

Indeed, let $\left(k, q, X^0, C^0\right)$ be a quadruple as in Theorem 1.3(b), let $U \subseteq X^0$ be the largest smooth open dense subscheme, and set

$$Z := \left(X^0 \smallsetminus U\right)_{\mathrm{red}}\,, \quad C_U := C \cap \left(U \times U^{(q)}\right) \quad \text{and} \quad C_Z := \left(C^0 \cap \left(Z \times Z^{(q)}\right)\right)_{\mathrm{red}}\,.$$

Then it follows from the combination of Claims 4.1.1(a), (c), (d), (e), (h) that the geometric degrees of U, Z, C_U and C_Z are bounded in terms of n and δ. Since $\dim(C_Z) < \dim\left(C^0\right)$, inequality (1) for quadruple (k, q, Z, C_Z) holds by the induction hypothesis (see § 4.2.2). Finally, since $C^0 \cap \Gamma_{X^0,q}$ decomposes as the union of $C_U \cap \Gamma_{U,q}$ and $C_Z \cap \Gamma_{Z,q}$, inequality (2) for $\left(k, q, X^0, C^0\right)$ follows from that for (k, q, U, C_U).

4.2.7. Next we claim that it suffices to show the conclusion of Theorem 1.3(b) for quadruples $\left(k, q, X^0, C^0\right)$ satisfying Assumptions 3.2.4(a), (b).

Indeed, let $\left(k, q, X^0, C^0\right)$ be a quadruple satisfying Assumption 3.2.4(a). Then by Claims 4.1.1(i), there exists an open dense subscheme $V \subseteq C^0$ subset such that $c_2^0|_V$ is uh-étale (see § 3.2.2) and $\mathrm{gdeg}(V)$ is bounded in terms of n and δ. We set $U := X^0 \smallsetminus \overline{c_1^0(C^0 \smallsetminus V)} \subseteq X^0$ and $C_U := C^0 \cap \left(U \times U^{(q)}\right)$. Then $\dim\left(C^0 \smallsetminus V\right) < \dim\left(C^0\right) = \dim\left(X^0\right)$; thus $U \subseteq X^0$ is open dense. By construction, we have the inclusions $C_U \subseteq \left(c_1^0\right)^{-1}(U) \subseteq V$; thus $c_2^0|_{C_U}$ is uh-étale.

Then arguing as in § 4.2.6, we can replace (k, q, X^0, C^0) with (k, q, U, C_U), thus assuming that c_2^0 is uh-étale.

4.2.8. Finally, we claim that it suffices to show the conclusion of Theorem 1.3(b) for quadruples (k, q, X^0, C^0) satisfying Assumptions 3.2.4(a)–(c).

Indeed, by § 4.2.7, it suffices to show the conclusion of Theorem 1.3(b) for quadruples satisfying Assumptions 3.2.4(a), (b). Let (k, q, X^0, C^0) be any such quadruple, and let $V \subseteq X^0$ and $\pi\colon \widetilde{X} \to X$ be the open subscheme and the blow-up constructed in Proposition 2.1.7. The proof of the following assertion will be given in § A.2.7.

Claim 4.2.9. (a) *The geometric degree of $V \subseteq X^0$ is bounded in terms of n and δ.*

(b) *There exist an $n' \in \mathbb{N}$ and a closed embedding $\widetilde{X} \hookrightarrow \mathbb{P}^{n'}_X \subseteq \mathbb{P}^{nn'+n+n'}_k$ over X such that n' and* gdeg $(\widetilde{X})$ *are bounded in terms of n and δ.*

Set $C_V := C \cap (V \times V^{(q)})$. Using Claims 4.1.1(a) and arguing as in § 4.2.6, one sees that the conclusion of Theorem 1.3(b) for (k, q, X^0, C^0) follows from that for (k, q, V, C_V). Next, since $\pi|_V\colon \pi^{-1}(V) \to V$ is an isomorphism, we have an open embedding $V \cong \pi^{-1}(V) \hookrightarrow \widetilde{X}$ with dense image, which induces the embedding $C_V \hookrightarrow V \times V^{(q)} \hookrightarrow \widetilde{X} \times \widetilde{X}^{(q)}$. Then combining Claim 4.2.9 and Claims 4.1.1(a), (c), (e), we see that the geometric degrees of $V \subseteq \widetilde{X}$ and $C_V \subseteq \widetilde{X} \times \widetilde{X}^{(q)}$ are bounded in terms of n and δ. In other words, the conclusion of Theorem 1.3(b) for $(k, q, V \subseteq \mathbb{P}^n_k, C_V)$ follows from that for $(k, q, V \subseteq \mathbb{P}^{nn'+n+n'}_k, C_V)$.

Now the reduction is immediate. Indeed, let $\widetilde{C}$ be the closure of the image of the embedding $C_V \hookrightarrow V \times V^{(q)} \hookrightarrow \widetilde{X} \times \widetilde{X}^{(q)}$, and let $\widetilde{c}\colon \widetilde{C} \hookrightarrow \widetilde{X} \times \widetilde{X}^{(q)}$ be the inclusion map. By Proposition 2.1.7, the closed subset $\widetilde{X} \smallsetminus V \subseteq \widetilde{X}$ is locally $\widetilde{c}$-invariant; hence the proof of Proposition 4.2.1 is complete. □

4.3 Completion of the proof

4.3.1 *Reformulation of the problem.* It follows from a combination of Propositions 4.2.1 and 3.2.7 that it suffices to show that for every $n, \delta \in \mathbb{N}$, there exists a constant M satisfying the following property:

For every quadruple satisfying Assumptions 3.2.4, there exist an alteration $f\colon X' \to X$ as in Construction 3.2.5(a) and a cycle $\widetilde{\alpha}^0 \in A_d(C'^0)$ as in Construction 3.2.5(e) such that

$$\operatorname{ram}(c_2', \partial X'^0) \leqslant M\,, \quad \operatorname{ram}(c_2^0) \leqslant M\,, \quad |I| \leqslant M\,, \tag{17}$$

and for all $J \subseteq I$ and $i \in \mathbb{N}$, the trace $\operatorname{Tr}(F^*_{X'_J,q} \circ H^i(\widetilde{\alpha}_J)) \in \mathbb{Z}$ satisfies

$$\left|\operatorname{Tr}(F^*_{X'_J,q} \circ H^i(\widetilde{\alpha}_J))\right| \leqslant M q^{i/2}\,. \tag{18}$$

4.3.2. By Claims 4.1.1(k), there exist $n', \delta' \in \mathbb{N}$ such that for every quadruple satisfying Assumptions 3.2.4, there exist an alteration $f\colon X' \to X$ as in Construction 3.2.5(c) and a closed embedding $X' \hookrightarrow \mathbb{P}^{n'}_X$ over X such that the (geometric) degree of X' is at most δ'.

From now on, we assume that we are given a quadruple satisfying Assumptions 3.2.4, a pair (n', δ') and a closed embedding $X' \hookrightarrow \mathbb{P}^{n'}_X$ as in § 4.3.2.

4.3.3. (a) By Claims 4.1.1(d), (e), the geometric degree of $\partial X' \subseteq X'$ is bounded in terms of n' and δ', hence in terms of n and δ. Then, by Claims 4.1.1(b), (c), the cardinality $|I|$ and geometric degrees of X'_i and X'_J are bounded in terms of n and δ as well.

(b) Using Claims 4.1.1(b), (e), we see that the geometric degree of $C' = \overline{C'^0} \subseteq X' \times X'^{(q)}$ is bounded in terms of n and δ. Hence combining part (a) with Claims 4.1.1(e), (g), we see that $\mathrm{ram}\left(c_2^0\right)$ and $\mathrm{ram}\left(c_2', \partial X'^0\right)$ are bounded in terms of n and δ. This proves the existence of a constant M satisfying (17).

(c) By a combination of part (a) and Claims 4.1.1(j), there exist a pair $n'', \delta'' \in \mathbb{N}$ and a closed embedding $\widetilde{Y} \hookrightarrow \mathbb{P}^{n''}_Y$ over Y such that $\mathrm{gdeg}\left(\widetilde{Y}\right) \leqslant \delta''$.

(d) Combining part (c) with Claims 4.1.1(e), we see that the geometric degrees of $E_J \subseteq \widetilde{Y}$ and $X'^0 \times \left(X'^0\right)^{(q)} \subseteq \widetilde{Y}$ are bounded in terms of n and δ.

From now on, we assume that we have chosen a pair (n'', δ'') and a closed embedding $\widetilde{Y} \hookrightarrow \mathbb{P}^{n''}_Y$ satisfying the condition of § 4.3.3(c).

4.3.4. (a) Combining §§ 4.3.3(d) and 4.1.2(c), we see that the norm $\left|\alpha^0\right|$ of the class $\alpha^0 \in A_d\left(C'^0\right)$ (see Construction 3.2.5(d)) is bounded in terms of n and δ.

(b) Choose a cycle $\widetilde{\alpha}^0 = \sum_i n_i [V_i]_{C'^0}$ representing α^0 such that $\left|\alpha^0\right| = \sum_i |n_i| \,\mathrm{gdeg}(V_i)$ (see § 4.1.2(a) and Construction 3.2.5(d)). We claim that this cycle satisfies the required properties.

(c) By Claims 4.1.1(b), the cycle class $\widetilde{\alpha} = \sum_i n_i \left[\overline{V_i}\right] \in A_d\left(\widetilde{Y}\right)$ (see Construction 3.2.5(f)) satisfies

$$|\widetilde{\alpha}| \leqslant \sum_i |n_i| \,\mathrm{gdeg}\left(\overline{V_i}\right) \leqslant \sum_i |n_i| \,\mathrm{gdeg}(V_i) = \left|\alpha^0\right|.$$

Hence, by part (a), the norm $|\widetilde{\alpha}|$ is bounded in terms of n and δ.

(d) Combining part (c) with §§ 4.3.3(d) and 4.1.2(b), (c), we see that the norm of the cycle class $\widetilde{\alpha}_J := (\pi_J)_*(i_J)^* \widetilde{\alpha} \in A_d(X'_J \times (X'_J)^{(q)})$ is bounded in terms of n and δ.

(e) Since $\mathrm{gdeg}(X'_J)$ is bounded in terms of n and δ (see § 4.3.3(a)), the existence of a constant M satisfying (18) follows from a combination of part (d) and § 4.1.3.

Appendix A. Bounded collections

A.1 Stability of bounded collections under standard operations

In this section, we introduce *bounded collections* and show that standard operations preserve *boundedness*.

Notation A.1.1. Let Λ be a set (or a class).

(a) We say that a collection of morphisms $\{f_\alpha \colon X_\alpha \to Y_\alpha\}_{\alpha\in\Lambda}$, where f_α is a morphism of schemes of finite type over an algebraically closed field k_α, is *bounded* if there exist a scheme S of finite type over $\mathbb{Z}$ and a morphism $f \colon X \to Y$ of schemes of finite type over S such that for every $\alpha \in \Lambda$, there exists a geometric point $s_\alpha \in S(k_\alpha)$ such that f_α is isomorphic to the geometric fiber f_{s_α} of f over s_α. In this case, we say that the family $f \colon X \to Y$ *parameterizes* the collection $\{f_\alpha\}_\alpha$ over S.

(b) Similarly, we say that a collection $\{X_\alpha\}_{\alpha\in\Lambda}$ of schemes of finite type over k_α is *bounded* if there exist a scheme S as in part (a) and a scheme X of finite type over S such that each X_α is isomorphic to a geometric fiber X_{s_α} over some $s_\alpha \in S(k_\alpha)$. Again we say that the family X *parameterizes* the collection $\{X_\alpha\}_\alpha$ over S.

(c) Let $\mathcal{P}$ be a class of morphisms of schemes that is stable under pullbacks. We say that a bounded collection $\{f_\alpha\}_\alpha$ of morphisms $f_\alpha \in \mathcal{P}$ is *$\mathcal{P}$-bounded* if there is a parameterizing family f that belongs to $\mathcal{P}$. In particular, we can talk about bounded collections of open, closed or locally closed embeddings.

(d) We say that a collection $\{Y_\alpha \subseteq X_\alpha\}_\alpha$, where $Y_\alpha \subseteq X_\alpha$ is an open (respectively, closed, locally closed) subscheme, is *bounded*, if the inclusions $\{Y_\alpha \hookrightarrow X_\alpha\}_\alpha$ form a bounded collection of open (respectively, closed, locally closed) embeddings.

(e) More generally, we define bounded collections of more complicated diagrams. For example,

(i) we say that a collection $\{X_\alpha \subseteq \mathbb{P}^n_{k_\alpha}\}_\alpha$, where $X_\alpha \subseteq \mathbb{P}^n_{k_\alpha}$ is a (locally) closed subscheme, is *bounded* if it is parameterized by a (locally) closed subscheme $X \subseteq \mathbb{P}^n_S$ for S as in part (a);

(ii) we say that a collection $\{X_\alpha \subseteq \mathbb{P}^n_{k_\alpha}, X'_\alpha \subseteq \mathbb{P}^m_{X_\alpha}\}_\alpha$, where $X_\alpha \subseteq \mathbb{P}^n_{k_\alpha}$ and $X'_\alpha \subseteq \mathbb{P}^m_{X_\alpha}$ are (locally) closed subschemes, is *bounded* if it is parameterized by a pair (locally) closed subschemes $X \subseteq \mathbb{P}^n_S$, $X' \subseteq \mathbb{P}^m_X$ for S as in part (a).

The following standard lemma is central for what follows.

Lemma A.1.2. *Let $f\colon X \to S$ be a morphism of schemes of finite type over $\mathbb{Z}$. Then there exists a surjective morphism $S' \to S$, where S' is a scheme of finite type over $\mathbb{Z}$, such that the pullback $X' := X \times_S S'$ satisfies the following properties:*

(a) *All geometric fibers of $X'_{\mathrm{red}} \to S'$ are reduced.*

(b) *Let $X'_1, \dots, X'_m$ be all the irreducible components of X'_{red}. Then all geometric fibers of $X'_i \to S'$ are integral.*

Proof. By Noetherian induction, we can assume that for every closed subscheme $Z \subseteq S$, the assertion holds for the morphism $f|_Z\colon f^{-1}(Z) \to Z$. Therefore, we can replace S with an open non-empty subscheme. In other words, we can assume that S is integral, and it suffices to show the existence of a dominant morphism $S' \to S$ satisfying the required properties. Then assertion (a) follows from [Sta22, Tag 0550 and Tag 0578], while assertion (b) follows from [Sta22, Tag 0551 and Tag 0559]. Alternatively, both results can be deduced from [GD66, théorème 9.7.7]. □

Corollary A.1.3. *Let $\{X_\alpha\}_\alpha$ be a bounded collection. Then the collections $\{(X_\alpha)_{\mathrm{red}}\}_\alpha$ and $\{X_{\alpha,i_\alpha}\}_{\alpha,i_\alpha}$, where X_{α,i_α} runs over the set of all irreducible components of $(X_\alpha)_{\mathrm{red}}$, are bounded.*

Proof. Let $X \to S$ parameterize the collection $\{X_\alpha\}_\alpha$, and let $X' \to S'$ be as in Lemma A.1.2. Then $X'_{\mathrm{red}} \to S'$ parameterizes the collection $\{(X_\alpha)_{\mathrm{red}}\}_\alpha$, while $\sqcup_{i=1}^m X'_i \to \sqcup_{i=1}^m S$ parameterizes $\{X_{\alpha,i_\alpha}\}_{\alpha,i_\alpha}$. □

Lemma A.1.4. (a) *Let $\{X_\alpha \subseteq \mathbb{P}^n_{k_\alpha}\}_{\alpha\in\Lambda}$ and $\{Y_\alpha \subseteq \mathbb{P}^n_{k_\alpha}\}_{\alpha\in\Lambda}$ be bounded collections (of locally closed subschemes). Then $\{X_\alpha, Y_\alpha \subseteq \mathbb{P}^n_{k_\alpha}\}_\alpha$ and $\{X_\alpha \cap Y_\alpha \subseteq \mathbb{P}^n_{k_\alpha}\}_\alpha$ are bounded collections.*

(b) *In the situation of part (a), let $\Lambda' \subseteq \Lambda$ consist of all α such that $Y_\alpha \subseteq X_\alpha$. Then the collection $\{Y_\alpha \subseteq X_\alpha\}_{\alpha\in\Lambda'}$ is bounded.*

(c) *Let $\{f_\alpha\}_\alpha$ be a bounded collection of morphisms such that each f_α is an open (respectively, closed, locally closed) embedding. Then $\{f_\alpha\}_\alpha$ is a bounded collection of open (respectively, closed, locally closed) embeddings in the sense of Notation A.1.1(c).*

(d) *Let $\{Y_\alpha \subseteq X_\alpha\}_\alpha$ be a bounded collection of closed (respectively, open) subschemes. Then the collection $\{X_\alpha \smallsetminus Y_\alpha\}_\alpha$ (respectively, $\{(X_\alpha \smallsetminus Y_\alpha)_{\mathrm{red}}\}_\alpha$) is bounded.*

(e) *Let $\{Z'_\alpha, Z''_\alpha \subseteq X_\alpha\}_\alpha$ be a bounded collection such that Z'_α and Z''_α are closed subschemes of X_α, and let $Z_\alpha \subseteq X_\alpha$ be the closed subscheme such that $\mathcal{I}_{Z_\alpha,X_\alpha} := \mathcal{I}_{Z'_\alpha,X_\alpha}\mathcal{I}_{Z''_\alpha,X_\alpha}$. Then the collection $\{Z_\alpha \subseteq X_\alpha\}_\alpha$ is bounded.*

(f) *Let $\{f_\alpha\colon X_\alpha \to Y_\alpha, Z_\alpha \subseteq Y_\alpha\}_\alpha$ be a bounded collection. Then the induced collection $\left\{f_\alpha^{-1}(Z_\alpha) \subseteq X_\alpha\right\}_\alpha$ is bounded.*

(g) *Let $\{f_\alpha\colon X_\alpha \to Y_\alpha\}_\alpha$ be a bounded collection. Then the collection $\left\{\overline{f_\alpha(X_\alpha)} \subseteq Y_\alpha\right\}_\alpha$ of schematic closures of images is bounded.*

Proof. (a) Assume that $X_0 \subseteq \mathbb{P}^n_{S_X}$ and $Y_0 \subseteq \mathbb{P}^n_{S_Y}$ parameterize the collections $\{X_\alpha \subseteq \mathbb{P}^n_{k_\alpha}\}_{\alpha\in\Lambda}$ and $\{Y_\alpha \subseteq \mathbb{P}^n_{k_\alpha}\}_{\alpha\in\Lambda}$, respectively, and set $S := S_X \times_{\mathbb{Z}} S_Y$. Then $X := X_0 \times_{S_X} S, Y := Y_0 \times_{S_Y} S \subseteq \mathbb{P}^n_S$ and $X \cap Y \subseteq \mathbb{P}^n_S$ parameterize $\{X_\alpha, Y_\alpha \subseteq \mathbb{P}^n_{k_\alpha}\}_\alpha$ and $\{X_\alpha \cap Y_\alpha \subseteq \mathbb{P}^n_{k_\alpha}\}_\alpha$, respectively.

(b) We use the notation of the proof of part (a). Consider the subset $S_0 \subseteq S$ consisting of all $s \in S$ such that the embedding $(Y \cap X)_s \hookrightarrow Y_s$ is an isomorphism. By [GD66, proposition 9.6.1(xi)], the subset S_0 is constructible. Therefore, there exists a morphism $S' \to S$ of schemes of finite type over $\mathbb{Z}$ whose image is S_0. Then $Y \times_S S' \subseteq X \times_S S' \subseteq \mathbb{P}^n_{S'}$ parameterizes $\{Y_\alpha \subseteq X_\alpha\}_{\alpha\in\Lambda'}$.

(c) Arguing as in part (b), it suffices to show that if $f\colon X \to Y$ is a morphism of schemes of finite type over S, then the set of $s \in S$ such that f_s is an open (respectively, closed, locally closed) embedding is constructible. But this follows from [GD66, proposition 9.6.1(viii), (ix), (x)].

(d) Let $Y \subseteq X$ be a closed (respectively, an open) subscheme representing $\{Y_\alpha \subseteq X_\alpha\}_\alpha$ over S (use part (c)). Then $X \smallsetminus Y$ (respectively, $(X \smallsetminus Y)_{\mathrm{red}}$) parameterizes the collection $\{X_\alpha \smallsetminus Y_\alpha\}_\alpha$ (respectively, $\{(X \smallsetminus Y)_{\mathrm{red},\alpha}\}_\alpha$). Since $((X \smallsetminus Y)_{\mathrm{red},\alpha})_{\mathrm{red}} = (X_\alpha \smallsetminus Y_\alpha)_{\mathrm{red}}$, the assertion follows from Corollary A.1.3.

(e) Let $Z', Z'' \subseteq X$ be closed subschemes parameterizing $\{Z'_\alpha, Z''_\alpha \subseteq X_\alpha\}_\alpha$ (use part (c)). Then the closed subscheme $Z \subseteq X$ such that $\mathcal{I}_{Z,X} := \mathcal{I}_{Z',X}\mathcal{I}_{Z'',X}$ parameterizes $\{Z_\alpha \subseteq X_\alpha\}_\alpha$.

(f) If $(f\colon X \to Y, Z \subseteq X)$ parameterizes $\{f_\alpha\colon X_\alpha \to Y_\alpha, Z_\alpha \subseteq Y_\alpha\}_\alpha$, then $f^{-1}(Z) \subseteq X'$ parameterizes $\left\{f_\alpha^{-1}(Z_\alpha) \subseteq X'_\alpha\right\}_\alpha$.

(g) Let $f\colon X \to Y$ parameterize $\{f_\alpha\colon X_\alpha \to Y_\alpha\}_\alpha$, and let $\overline{f(X)} \subseteq Y$ be the schematic closure of the image. By Noetherian induction, it remains to show that there exists an open non-empty subscheme $U \subseteq S$ such that $\overline{f_s(X_s)} = \overline{f(X)}_s$ for every $s \in U$. Replacing S with an open subscheme, we can assume that S is integral and the generic fiber X_η of f is dense in X. Now the assertion follows from [GD66, proposition 9.6.1(ii)]. □

Lemma A.1.5. (a) *Let $\{X_\alpha\}_\alpha$ be a bounded collection. Then the collection $\{U_\alpha \subseteq X_\alpha\}_\alpha$, where U_α is the largest open subscheme of X_α that is smooth over k_α, is bounded.*

(b) *Let $\{f_\alpha\colon X_\alpha \to Y_\alpha\}_\alpha$ be a bounded collection such that each f_α is a quasi-finite morphism between integral schemes over k_α. Then there exists a bounded collection $\{U_\alpha \subseteq X_\alpha\}_\alpha$, where U_α is an open dense subscheme of X_α, such that $f_\alpha|_{U_\alpha}$ is uh-étale (see § 3.2.2).*

Proof. (a) Let $f\colon X \to S$ be a morphism parameterizing $\{X_\alpha\}_\alpha$ such that S is reduced. By [Sta22, Tag 0529], there exists an open non-empty subset $V \subseteq S$ such that $f|_V\colon f^{-1}(V) \to V$ is flat. Let $U \subseteq f^{-1}(V)$ be the largest open subscheme where f is smooth. Then it follows from [Sta22, Tag 01V9] that $U_s \subseteq X_s$ is the largest subscheme of X_s that is smooth over s for every geometric point s of V. Now the assertion follows by Noetherian induction on S.

(b) Let $f\colon X \to Y$ be a family parameterizing collection $\{f_\alpha\}_\alpha$ over S. By [GD66, proposition 9.6.1(vii) and théorème 9.7.7], the set of points $t \in S$ such that $f_t\colon X_t \to Y_t$ is a quasi-finite morphism between geometrically integral schemes is a constructible subset. Thus, arguing as in Lemma A.1.4(b), we can assume that these properties are satisfied for all points $t \in S$. Using Noetherian induction on S and replacing S with an open subset, we can assume that X and Y

are integral. Then, by Lemma 3.2.3(a), there exists an open dense subscheme $U \subseteq X$ such that $f|_U$ is uh-étale. Replacing S with an open subscheme, we can assume that $U_t \subseteq X_t$ is dense for every $t \in S$. Then $U \subseteq X$ parameterizes the collection $\{U_\alpha \subseteq X_\alpha\}_\alpha$ we were looking for. □

The following standard fact seems to be missing from the literature.

Lemma A.1.6. *Let X be a scheme over S, and let $Z \subseteq X$ be a closed subscheme of X such that the normal cone $N_Z(X)$ is flat over S. Then $\mathrm{Bl}_Z(X) \to S$ commutes with all pullbacks; that is, for all morphisms $f\colon S' \to S$, the natural morphism*

$$\mathrm{Bl}_{Z\times_S S'}(X \times_S S') \xrightarrow{\sim} \mathrm{Bl}_Z(X) \times_S S'$$

is an isomorphism.

Proof. Since the normal cone $N_Z(X) = \mathcal{S}pec\big(\oplus_{n\geqslant 0} \big((\mathcal{I}_{Z,X})^n/(\mathcal{I}_{Z,X})^{n+1}\big)\big)$ is flat over S, we conclude that every $(\mathcal{I}_{Z,X})^n/(\mathcal{I}_{Z,X})^{n+1}$ is flat over S. By induction, we therefore conclude that every quotient $\mathcal{O}_X/(\mathcal{I}_{Z,X})^n$ is flat over S, which implies that $f^*(\mathcal{I}_{Z,X})^n \simeq (\mathcal{I}_{Z\times_S S', X\times_S S'})^n$ for every n. Since $\mathrm{Bl}_Z(X) = \mathcal{P}roj\big(\oplus_{n\geqslant 0} (\mathcal{I}_{Z,X})^n\big)$ and similarly for $\mathrm{Bl}_{Z\times_S S', X\times_S S'}$, the assertion follows. □

Corollary A.1.7. *Let $\{Z_\alpha \subseteq X_\alpha \subseteq \mathbb{P}^n_{k_\alpha}\}_\alpha$ be a bounded collection, where each $Z_\alpha \subseteq X_\alpha$ is a closed subscheme. Then there exist an $m \in \mathbb{N}$ and a closed embedding $i_\alpha\colon \mathrm{Bl}_{Z_\alpha}(X_\alpha) \hookrightarrow \mathbb{P}^m_{X_\alpha}$ over X_α for every α such that the collection of closed embeddings $\{i_\alpha\}_{\alpha\in\Lambda}$ is bounded.*

Proof. Let $Z \subseteq X \subseteq \mathbb{P}^n_S$ be a family parameterizing $\{Z_\alpha \subseteq X_\alpha \subseteq \mathbb{P}^n_{k_\alpha}\}_\alpha$ such that $Z \subseteq X$ is a closed subscheme and S is reduced. Then $\mathrm{Bl}_Z(X)$ is of finite type over S; thus there exists an open non-empty subset $V \subseteq S$ such that $N_Z(X)$ is flat over V (possibly empty). By Noetherian induction on S, we can restrict all schemes to V, thus assuming that S is affine and $N_Z(X)$ is flat over S. But S is affine; thus X has an ample line bundle. Hence there exists a closed embedding $i\colon \mathrm{Bl}_Z(X) \hookrightarrow \mathbb{P}^m_X$ over X. Then, by Lemma A.1.6, this closed embedding parameterizes a bounded collection of closed embeddings $\{i_\alpha\colon \mathrm{Bl}_{Z_\alpha}(X_\alpha) \hookrightarrow \mathbb{P}^m_{X_\alpha}\}_\alpha$ over $\{X_\alpha\}_\alpha$. □

Remark A.1.8. The argument of Corollary A.1.7 also shows that if $\{Z_\alpha \subseteq X_\alpha\}_\alpha$ is a bounded collection of closed subschemes, then the collection of projections $\{\mathrm{Bl}_{Z_\alpha}(X_\alpha) \to X_\alpha\}_\alpha$ is bounded.

Corollary A.1.9. *Let $\{Z_\alpha \subseteq X_\alpha\}_\alpha$ be a bounded family of closed subschemes. Then the collection of projections $\{N_{Z_\alpha}(X_\alpha) \to X_\alpha\}_\alpha$ is bounded.*

Proof. Let $\mathrm{Bl}_{Z_\alpha\times\{0\}}\left(X_\alpha \times \mathbb{A}^1\right)_0$ be the fiber of $\mathrm{Bl}_{Z_\alpha\times\{0\}}\left(X_\alpha \times \mathbb{A}^1\right) \to \mathbb{A}^1$ over $0 \in \mathbb{A}^1$. Then $N_{Z_\alpha}(X_\alpha)$ can be described as $N_{Z_\alpha}(X_\alpha) = \mathrm{Bl}_{Z_\alpha\times\{0\}}\left(X_\alpha\times\mathbb{A}^1\right)_0 \smallsetminus \mathrm{Bl}_{Z_\alpha}(X_\alpha)$ (compare with [Ful98, Section 5.1] or [Var07, Section 1.4]). Now the result follows by a combination of Remark A.1.8 and Lemma A.1.4(d), (f). □

Lemma A.1.10. *Let $\{Z_\alpha \subsetneq X_\alpha \subseteq \mathbb{P}^n_{k_\alpha}\}_\alpha$ be a bounded collection, where X_α is integral and $Z_\alpha \subsetneq X_\alpha$ is a closed subscheme. Then there exist an $m \in \mathbb{N}$, an alteration $f_\alpha\colon \widetilde{X}_\alpha \to X_\alpha$ and a closed embedding $i_\alpha\colon \widetilde{X}_\alpha \hookrightarrow \mathbb{P}^m_{X_\alpha}$ over X_α for every α such that $\widetilde{X}_\alpha$ is smooth, $f_\alpha^{-1}(Z_\alpha)_{\mathrm{red}}$ is a union of smooth divisors with normal crossings, and the collection of closed embeddings $\{i_\alpha\}_{\alpha\in\Lambda}$ is bounded.*

Proof. Let $Z \subsetneq X \subseteq \mathbb{P}^n_S$ be a family parameterizing $\{Z_\alpha \subsetneq X_\alpha \subseteq \mathbb{P}^n_{k_\alpha}\}_\alpha$. By [GD66, théorème 9.7.7 and corollaire 9.5.2], the set of points $s \in S$ such that X_s is geometrically integral and that $Z_s \neq X_s$ is a constructible subset. Thus, arguing as in Lemma A.1.4(b), we can assume that these properties are satisfied for all points $s \in S$. By Noetherian induction on S, we may replace S with an open non-empty subscheme, thus assuming that S is integral with generic fiber η and geometric generic fiber $\overline{\eta}$.

By a theorem of De Jong [dJo96, Theorem 4.1], there exists an alteration $f_{\overline{\eta}}\colon \widetilde{X}_{\overline{\eta}} \to X_{\overline{\eta}}$ such that $\widetilde{X}_{\overline{\eta}}$ is a smooth projective variety over $\overline{\eta}$ and $f_{\overline{\eta}}^{-1}(Z_{\overline{\eta}})_{\mathrm{red}}$ is a union of smooth divisors $Z_{j,\overline{\eta}}$ with normal crossings. In particular, there exists a closed embedding $i_{\overline{\eta}}\colon \widetilde{X}_{\overline{\eta}} \hookrightarrow \mathbb{P}^m_{X_{\overline{\eta}}}$ over $X_{\overline{\eta}}$.

For every morphism $S' \to S$, we set $X' := X \times_S S'$ and $Z' := Z \times_S S'$. By standard limit theorems, the morphism $\overline{\eta} \to S$ factors as $\overline{\eta} \to S' \to S$, where S' is an integral scheme of finite type over $\mathbb{Z}$ such that $\iota_{\overline{\eta}}$ is a pullback of a closed embedding $i\colon \widetilde{X}' \hookrightarrow \mathbb{P}^m_{X'}$ and each $Z_{j,\overline{\eta}}$ is a pullback of a closed subscheme $Z_j \subseteq \widetilde{X}'$. Consider the composition $f\colon \widetilde{X}' \hookrightarrow \mathbb{P}^m_{X'} \to X'$.

Replacing S' with an open subscheme, one can assume that $\widetilde{X}' \to S'$ is smooth, $\{Z_j\}_j$ are smooth divisors over S' with normal crossings, that is, $\cap_{j\in J} Z_j$ is smooth over S' of correct relative dimension, and $f^{-1}(Z')_{\mathrm{red}} = \cup_j Z_j$. Furthermore, we can assume that $X' \to S'$ and $\widetilde{X}' \to S'$ are flat and have geometrically integral fibers.

Then for every geometric point s' of S', the induced morphism $f_s\colon \widetilde{X}'_s \to X'_s$ is an alteration (being a surjective morphism between varieties of the same dimension), $\widetilde{X}'_s$ is smooth, and $f_s^{-1}(Z'_s)_{\mathrm{red}}$ is a union of smooth divisors with normal crossings. Now the assertion of the lemma follows by Noetherian induction. □

LEMMA A.1.11. *Let $\{f_\alpha\colon X_\alpha \to Y_\alpha\}_{\alpha\in\Lambda}$ be a bounded collection of morphisms, where X_α and Y_α are integral, while f_α is generically finite. Then the collection $\{\deg(f_\alpha)\}_\alpha$ of positive integers is bounded (in the usual sense).*

Proof. Let $f\colon X \to Y$ be a family parameterizing $\{f_\alpha\}_\alpha$. Note that if $\eta_\alpha \in Y_\alpha$ is a generic point, then we have the equality $\deg(f_\alpha) = \dim_{k(\eta_\alpha)} H^0\big(f_\alpha^{-1}(\eta_\alpha), \mathcal{O}_{f_\alpha^{-1}(\eta_\alpha)}\big)$. Therefore, it suffices to show that there exists a constant N such that for every $y \in Y$ such that $f^{-1}(y)$ is finite, we have $\dim_{k(y)} H^0\big(f^{-1}(y), \mathcal{O}_{f^{-1}(y)}\big) \leqslant N$. By Noetherian induction, we can replace Y with an open subscheme (and X with its preimage), thus assuming that Y is irreducible and f is finite flat. In this case, the function $y \mapsto \dim_{k(y)} H^0\big(f^{-1}(y), \mathcal{O}_{f^{-1}(y)}\big)$ is constant. □

LEMMA A.1.12. *Let $f\colon X \to S$ be a morphism between schemes of finite type over $\mathbb{Z}$. Then there exists a constant N such that for every geometric point s of S, the thickness of the geometric fiber X_s (see §2.2.5(a)) is at most N. In particular, the ramification $\mathrm{ram}(f)$ (see §2.2.5(c)) is finite.*

Proof. Let $f'\colon X' \to S'$ be as Lemma A.1.2, and let N be the thickness of X'. We claim that this N satisfies the required property. Indeed, by definition, we have $(\mathcal{I}_{X'_{\mathrm{red}},X'})^N = 0$. Hence for every geometric point s' of S', we have $(\mathcal{I}_{(X'_{\mathrm{red}})_{s'},X'_{s'}})^N = 0$. By our assumption, $(X'_{\mathrm{red}})_{s'}$ is reduced. Therefore, $(\mathcal{I}_{(X'_{s'})_{\mathrm{red}},X'_{s'}})^N = 0$; thus the thickness of $X'_{s'}$ is at most N. Since every geometric fiber X_s is isomorphic to some geometric fiber $X'_{s'}$, the assertion follows. □

COROLLARY A.1.13. (a) *For every bounded family $\{X_\alpha\}_\alpha$, the collection of thicknesses of X_α is bounded.*

(b) *Let $\{f_\alpha\colon X_\alpha \to Y_\alpha, Z_\alpha \subseteq Y_\alpha\}_{\alpha\in\Lambda}$, where $Z_\alpha \subseteq Y_\alpha$ is a closed subscheme, be a bounded collection. Then the collection $\{\mathrm{ram}(f_\alpha, Z_\alpha)\}_\alpha$ of positive integers is bounded.*

(c) *Let $\{f_\alpha\colon X_\alpha \to Y_\alpha\}_{\alpha\in\Lambda}$ be a bounded collection. Then the collection $\{\mathrm{ram}(f_\alpha)\}_\alpha$ of positive integers is bounded.*

Proof. (a) Let $f\colon X \to S$ be a family parameterizing the collection $\{X_\alpha\}_\alpha$. Then the assertion follows from Lemma A.1.12 applied to f.

(b) By Lemma A.1.4(f), the collection $\left\{f_\alpha^{-1}(Z_\alpha)\right\}_\alpha$ is bounded. Thus the result follows from part (a).

(c) This follows immediately from part (b). □

A.2 Boundedness of collections of quasi-projective varieties

The following result is central for what follows.

Theorem A.2.1. *For every $n, \delta \in \mathbb{N}$, the collection $\{X_\alpha \subseteq \mathbb{P}^n_{k_\alpha}\}_\alpha$, where $X_\alpha \subseteq \mathbb{P}^n_{k_\alpha}$ is a closed subvariety of $\mathbb{P}^n_{k_\alpha}$ of degree at most δ, is bounded.*

Theorem A.2.1 is a standard consequence of Kleiman's boundedness theorem [Kle71, corollaire 3.10(ii)], whose proof is rather involved. For the convenience of the reader, we include a much simpler argument suggested to us by Ehud Hrushovski.

Let k be an algebraically closed field. The following fact is standard.

Lemma A.2.2. *Let $X \subsetneq \mathbb{P}^n_k$ be a closed subvariety, let $p \in \mathbb{P}^n_k \smallsetminus X$ be a closed point, and let $\pi_p\colon \mathbb{P}^n_k \smallsetminus \{p\} \to \mathbb{P}^{n-1}_k$ be the linear projection from p. Then $\pi_p|_X$ is a finite map, and we have the inequality* $\deg(\pi_p(X)) \leqslant \deg(X)$.

Proposition A.2.3. *Let $X \subsetneq \mathbb{P}^n_k$ be a closed subvariety. Then X is an irreducible component of a complete intersection in $\mathbb{P}^n_k$ defined by equations of degree at most* $\deg(X)$.

Proof. We set $d := \dim(X)$ and proceed by induction on $n-d$. If $d = n-1$, then $X \subsetneq \mathbb{P}^n_k$ is a hypersurface, so the result is obvious. Now assume $d < n-1$, and let $p \in \mathbb{P}^n_k \smallsetminus X$ be a closed point. Then Lemma A.2.2 implies that $\pi_p(X) \subsetneq \mathbb{P}^{n-1}_k$ is a closed subvariety of dimension d and degree at most $\deg(X)$. Hence, by the induction hypothesis, $\pi_p(X)$ as an irreducible component of a complete intersection W' in $\mathbb{P}^{n-1}_k$ defined by equations of degree at most $\deg(\pi_p(X)) \leqslant \deg(X)$.

Denote by $W \subseteq \mathbb{P}^n_k$ the closure of $\pi_p^{-1}(W') \subseteq \mathbb{P}^n_k \smallsetminus \{p\}$. Then W is a complete intersection in $\mathbb{P}^n_k$ of dimension $d+1$, defined by equations of degree at most $\deg(\pi_p(X)) \leqslant \deg(X)$, and $X \subseteq W$.

Next, applying Lemma A.2.2 successively $(n-d-1)$ times, we get a linear projection $\pi_L\colon \mathbb{P}^n_k \smallsetminus L \to \mathbb{P}^{d+1}_k$ such that $\pi_L|_W\colon W \to \mathbb{P}^{d+1}_k$ is finite. Moreover, $\pi_L(X) \subsetneq \mathbb{P}^{d+1}_k$ is a hypersurface of degree $\deg(\pi_L(X)) \leqslant \deg(X)$.

Let $W'' \subseteq \mathbb{P}^n_k$ be the closure of $\pi_L^{-1}(\pi_L(X)) \subseteq \mathbb{P}^n_k \smallsetminus L$. Then W'' is a hypersurface of degree $\deg(\pi_L(X)) \leqslant \deg(X)$, and $X \subseteq W \cap W''$.

It remains to show that $W \cap W''$ is a complete intersection. But this follows from the fact that since W is a complete intersection, while $\pi_L|_W$ is finite, for each irreducible component $W_i \subseteq W$, we have $\pi_L(W_i) \nsubseteq \pi_L(X)$, thus $W_i \nsubseteq W''$. □

Proof of Theorem A.2.1. Since the collection of hypersurfaces in $\mathbb{P}^n_{k_\alpha}$ of degree at most δ is bounded, the assertion follows from a combination of Proposition A.2.3 and Corollary A.1.3. □

As a consequence, we obtain an easy criterion for when a collection of quasi-projective schemes is bounded.

Corollary A.2.4. *Let n be a fixed positive integer. Then the collection $\{X_\alpha \subseteq \mathbb{P}^n_{k_\alpha}\}_\alpha$ of reduced locally closed subschemes of $\mathbb{P}^n_{k_\alpha}$ is bounded if and only if the collection of geometric degrees $\{\mathrm{gdeg}(X_\alpha)\}_\alpha$ is bounded.*

Proof. Assume that the geometric degrees $\{\mathrm{gdeg}(X_\alpha)\}_\alpha$ are bounded. Then each X_α has a presentation $X_\alpha = X'_\alpha \smallsetminus X''_\alpha$, where $X'_\alpha, X''_\alpha \subseteq \mathbb{P}^n_{k_\alpha}$ are closed reduced subschemes, and the geometric degrees $\mathrm{gdeg}(X'_\alpha)$ and $\mathrm{gdeg}(X''_\alpha)$ are bounded. Thus, by Lemma A.1.4(a), (d), we reduce to the case when each $X_\alpha \subseteq \mathbb{P}^n_{k_\alpha}$ is closed. Since $\mathrm{gdeg}(X_\alpha)$ is the sum of degrees of irreducible components of X_α, the assertion in this case follows from a combination of Theorem A.2.1 with Lemma A.1.4(e) and Corollary A.1.3.

To show the converse, we have to show that if S is a scheme of finite type over $\mathbb{Z}$ and $X \subseteq \mathbb{P}^n_S$ is a locally closed subscheme, the geometric degrees of all reduced geometric fibers of $X \to S$ are bounded. Writing X as $X_1 \smallsetminus X_2$ such that $X_1, X_2 \subseteq \mathbb{P}^n_S$ are closed, we can assume that $X \subseteq \mathbb{P}^n_S$ is a closed subscheme. Let S', X' and $X'_1, \ldots, X'_m$ be as in Lemma A.1.2. Replacing $X \to S$ with $X'_i \to S'$, we can assume that all geometric fibers of $X \to S$ are integral. Replacing S with a disjoint union of a constructible stratification (and X with its pullback), we can assume that $X \to S$ is flat. In this case, the degrees of all geometric fibers of $X \to S$ are locally constant, hence bounded. □

Corollary A.2.5. *For every $n, m, \delta \in \mathbb{N}$, the following collection is bounded:*

$$\{X_\alpha \subseteq \mathbb{P}^n_{k_\alpha}, X'_\alpha \subseteq \mathbb{P}^m_{X_\alpha} \mid \mathrm{gdeg}(X_\alpha) \leqslant \delta \text{ and } \mathrm{gdeg}(X'_\alpha) \leqslant \delta\}_\alpha .$$

Proof. By Corollary A.2.4 and Lemma A.1.4(a), the collection

$$\left\{X_\alpha \subseteq \mathbb{P}^n_{k_\alpha}, X'_\alpha \subseteq \mathbb{P}^{nm+n+m}_{k_\alpha} \mid \mathrm{gdeg}(X_\alpha) \leqslant \delta \text{ and } \mathrm{gdeg}(X'_\alpha) \leqslant \delta\right\}_\alpha$$

is bounded. Therefore, the assertion follows from (the proof of) Lemma A.1.4(b). □

A.2.6. *Proof of Claims* 4.1.1(c)–(k). (c) By Corollary A.2.4 and Lemma A.1.4(a), the collection $\{X_\alpha, Y_\alpha \subseteq \mathbb{P}^n_{k_\alpha} \mid \mathrm{gdeg}(X_\alpha) \leqslant \delta \text{ and } \mathrm{gdeg}(Y_\alpha) \leqslant \delta\}_\alpha$ is bounded. Therefore, the collection $\{(X_\alpha \cap Y_\alpha)_{\mathrm{red}} \subseteq \mathbb{P}^n_{k_\alpha} \mid \mathrm{gdeg}(X_\alpha) \leqslant \delta \text{ and } \mathrm{gdeg}(Y_\alpha) \leqslant \delta\}_\alpha$ is bounded (by Lemma A.1.4(a) and Corollary A.1.3). Hence $\mathrm{gdeg}((X_\alpha \cap Y_\alpha)_{\mathrm{red}})$ is bounded in terms of n and δ (by Corollary A.2.4).

(d) Combining Corollary A.2.4 and Lemma A.1.4(a), (b), we get that the collection

$$\{Z_\alpha \subseteq X_\alpha \subseteq \mathbb{P}^n_{k_\alpha} \mid \mathrm{gdeg}(X_\alpha) \leqslant \delta \text{ and } \mathrm{gdeg}(Z_\alpha) \leqslant \delta\}_\alpha$$

is bounded. Now arguing as in part (c), the assertion follows from Lemma A.1.4(d).

(e) Arguing as above, the boundedness of $\mathrm{gdeg}\left(f^{-1}(Z)_{\mathrm{red}}\right)$ follows from Corollary A.2.5 and Lemma A.1.4(f), while the boundedness of $\mathrm{ram}(f, Z)$ follows from Corollary A.1.13(b).

The remaining assertions are proved similarly: assertion (f) follows from Lemma A.1.4(g); assertion (g) from Lemma A.1.11 and Corollary A.1.13(c); assertion (h) from Lemma A.1.5(a); assertion (i) from Lemma A.1.5(b); assertion (j) from Corollary A.1.7 and assertion (k) from Lemma A.1.10. □

A.2.7. *Proof of Claim* 4.2.9. Using previous results, both assertions follow from the explicit formulas in the proof of Proposition 2.1.7:

(a) Since $V = V_{\dim(X)}$, it suffices to show that the geometric degrees of the subschemes V_j, Z_j and U_j of $\mathbb{P}^n_k$ defined by the explicit formulas (6)–(9) are bounded in terms of n, δ and j. By induction on j, the assertion follows from a combination of Lemma A.2.8(a) and Claims 4.1.1(b)–(f).

(b) Consider the collection of quadruples $\left\{\left(k_\alpha, q_\alpha, X^0_\alpha, C^0_\alpha\right)\right\}_{\alpha\in\Lambda}$ from Theorem 1.3(b), satisfying Assumptions 3.2.4(a), (b). For $\alpha \in \Lambda$, we denote by X_α, F_α and Z_α the corresponding objects, which were denoted by X, F and Z, respectively, in the proof of Proposition 2.1.7.

By a combination of part (a), Lemma A.2.8(a) and Claims 4.1.1(b)–(f), we see that the geometric degrees of $\left\{\overline{c_1(S_\alpha)}, {}^{\sigma^{-1}}\overline{c_2(S_\alpha)} \subseteq \mathbb{P}^n_{k_\alpha}\right\}_{\alpha\in\Lambda, S_\alpha\in\mathrm{Irr}(F_\alpha)}$ and cardinalities $\{|\operatorname{Irr}(F_\alpha)|\}_{\alpha\in\Lambda}$ are bounded. Hence by a combination of Corollary A.2.4 and Lemma A.1.4(a), (e), we see that the collection $\{Z_\alpha \subseteq X_\alpha \subseteq \mathbb{P}^n_{k_\alpha}\}_{\alpha\in\Lambda}$ is bounded. Therefore, the assertion follows from Corollaries A.1.7 and A.2.4. $\square$

The following lemma lists simple properties of geometric degrees.

Lemma A.2.8. *Let $X \subseteq \mathbb{P}^n_k$ be a reduced subscheme and $\overline{X} \subseteq \mathbb{P}^n_k$ the closure of X.*

(a) *For every $\sigma \in \mathrm{Aut}(k)$, we have the equality $\mathrm{gdeg}({}^\sigma X) = \mathrm{gdeg}(X)$.*
(b) *There exists a closed reduced subscheme $X'' \subseteq \mathbb{P}^n_k$ such that $X = \overline{X} \smallsetminus X''$ and we have $\mathrm{gdeg}(X) = \mathrm{gdeg}\left(\overline{X}\right) + \mathrm{gdeg}(X'')$. In particular, we have $\mathrm{gdeg}\left(\overline{X}\right) \leqslant \mathrm{gdeg}(X)$.*
(c) *Let $X_1, \ldots, X_r$ be all irreducible components of X. Then we have $r \leqslant \mathrm{gdeg}(X)$ and $\mathrm{gdeg}(X_i) \leqslant \mathrm{gdeg}(X)$ for all i.*

Proof. (a) This part is clear.

(b) By definition, there exist closed reduced subschemes $X', X'' \subseteq \mathbb{P}^n$ such that $X = X' \smallsetminus X''$ and $\mathrm{gdeg}(X) = \mathrm{gdeg}(X') + \mathrm{gdeg}(X'')$. Then $X \subseteq X'$ is an open subscheme; thus $\overline{X}$ is a union of some irreducible components of X'. Then $X = \overline{X} \smallsetminus X''$ and $\mathrm{gdeg}\left(\overline{X}\right) \leqslant \mathrm{gdeg}(X')$, which implies the assertion.

(c) Note that $\overline{X}_1, \ldots, \overline{X}_r$ are irreducible components of $\overline{X}$. Then, by part (b), we have the inequalities $r \leqslant \mathrm{gdeg}\left(\overline{X}\right) \leqslant \mathrm{gdeg}(X)$. Choose $X'' \subseteq \mathbb{P}^n_k$ as in part (b). Then we have $X_i = \overline{X}_i \smallsetminus X''$, hence $\mathrm{gdeg}(X_i) \leqslant \deg\left(\overline{X_i}\right) + \mathrm{gdeg}(X'') \leqslant \mathrm{gdeg}\left(\overline{X}\right) + \mathrm{gdeg}(X'') = \mathrm{gdeg}(X)$. $\square$

Appendix B. Norm on cycles classes and uniform boundedness

Let k be an algebraically closed field. Using the notion of geometric degree, in this section, we define an integer-valued norm on the Chow group of any quasi-projective variety and study its properties.

B.1 Definition of the norm and basic properties

Motivated by Hrushovski's norm [Hru04, Notation 10.13] (see Remark B.1.2 below), we propose the following definition.

Definition B.1.1. For a (locally closed) subscheme $X \subseteq \mathbb{P}^n_k$ and $\alpha \in A_*(X)$, we define the *norm* $|\alpha|$ of α by the formula

$$|\alpha| := \min_{\sum_j m_j[Z_j]} \left\{ \sum_j |m_j| \, \mathrm{gdeg}(Z_j) \right\}, \tag{19}$$

where the minimum is taken over all representatives $\sum_j m_j[Z_j] \in Z_*(X)$ of α, where each Z_j is a closed subvariety of X.

Remark B.1.2. Note that though our Definition B.1.1 is motivated by [Hru04, Notation 10.13], the two notions are very different. Namely, while our norm is obviously finite and can be defined for an arbitrary quasi-projective variety, Hrushovski's norm $|\cdot|_{\mathrm{Hr}}$ is only defined when X is smooth and projective, and one has to use an "effective moving lemma" argument (see [Rob72] or [Hru04, Lemma 10.5]) in order to show that Hrushovski's norm is finite.[5]

[5] See also Remarks B.2.9 and B.2.13 below.

More precisely, one can show that there exists an effective constant C, which is polynomial in $\deg X$ and exponential in $\dim X$, such that for all $\alpha \in A_*(X)$, we have the inequalities

$$|\alpha| \leqslant |\alpha|_{\mathrm{Hr}} \leqslant C|\alpha| \,. \tag{20}$$

Remark B.1.3. By definition, the norm $|\alpha|$ depends on the embedding $\iota\colon X \hookrightarrow \mathbb{P}^n_k$. To indicate this dependence, we will sometimes write $|\alpha|_\iota$ instead of $|\alpha|$.

LEMMA B.1.4. (a) *For every $\alpha_1, \alpha_2 \in A_i(X)$ and $m_1, m_2 \in \mathbb{Z}$, we have the inequality*

$$|m_1\alpha_1 + m_2\alpha_2| \leqslant |m_1||\alpha_1| + |m_2||\alpha_2| \,.$$

(b) *Assume that $X \subseteq \mathbb{P}^n_k$ is closed. Then for every closed equidimensional subscheme $Y \subseteq X$ of dimension i, its class $[Y] \in A_i(X)$ satisfies $|[Y]| = \deg(Y)$.*

Proof. Assertion (a) is clear, so it remains to show assertion (b). By definition, the cycle $[Y] \in Z_i(X)$ equals $\sum_a n_a[Y_a]$, where the Y_a are the irreducible components of Y and we have $n_a \geqslant 0$ for every α. The assertion follows from the fact that for every representative $\sum_{j=1}^r m_j[Y_j'] \in Z_i(X)$ of $[Y] \in A_*(X)$, we have the inequality

$$\deg(Y) = \sum_j m_j \deg(Y_j') \leqslant \sum_j |m_j| \deg(Y_j') = \sum_j |m_j| \operatorname{gdeg}(Y_j') \,. \qquad \square$$

The following simple lemma is crucial for what follows.

PROPOSITION B.1.5. *For every reduced subschemes $X, Y \subseteq \mathbb{P}^n_k$, we have the inequality*

$$\operatorname{gdeg}\left((X \cap Y)_{\mathrm{red}}\right) \leqslant \operatorname{gdeg}(X) \operatorname{gdeg}(Y) \,.$$

Proof. When X and Y are closed subschemes, the bound was proved in [Ful98, Example 8.4.6] using the join construction. In general, assume $X = X' \smallsetminus X''$ and $Y = Y' \smallsetminus Y''$, where X', X'', Y', Y'' are closed reduced subschemes of $\mathbb{P}^n_k$ such that $\operatorname{gdeg}(X) = \operatorname{gdeg}(X') + \operatorname{gdeg}(X'')$ and $\operatorname{gdeg}(Y) = \operatorname{gdeg}(Y') + \operatorname{gdeg}(Y'')$. Using the identity $X \cap Y = (X' \cap Y') \smallsetminus (X'' \cup Y'')$, the result follows from [Ful98, Example 8.4.6]. $\square$

We will also be using the following standard lemma.

LEMMA B.1.6. *Let $X \subseteq \mathbb{P}^m_k$ and $Y \subseteq \mathbb{P}^n_k$ be closed subvarieties of degrees δ_X and δ_Y, respectively. Then the degree of $X \times Y \subseteq \mathbb{P}^m_k \times \mathbb{P}^n_k$ is $\binom{\dim(X)+\dim(Y)}{\dim(X)}\delta_X\delta_Y$.*

Proof. The Hilbert polynomial of $X \times Y$ under the Segre embedding equals the product of the Hilbert polynomials of X and Y. From this, the result follows. $\square$

LEMMA B.1.7. *Let $X \subseteq \mathbb{P}^n_k$ be a subvariety, $Z \subseteq \mathbb{P}^m_X$ a closed subvariety and $p\colon \mathbb{P}^m_X \to X$ the projection map.*

(a) *Assume that $X \subseteq \mathbb{P}^n_k$ is closed. Then $\deg(p_*([Z])) \leqslant \deg(Z)$.*

(b) *For a general X, we have the inequality $\operatorname{gdeg}(p_*([Z])) \leqslant \operatorname{gdeg}(Z) \operatorname{gdeg}(X)$.*

Proof. (a) The class of $[Z]$ in $\mathbb{P}^m_k \times \mathbb{P}^n_k$ is of the form $\sum_j n_j\left[\mathbb{P}^j_k \times \mathbb{P}^{\dim(Z)-j}_k\right]$, where all the n_j are non-negative [Ful98, Example 8.4.2]. On the other hand, the image of $p_*([Z])$ in $A_{\dim(Z)}(\mathbb{P}^n_k)$ is $n_0[\mathbb{P}^n_k]$, and we have $n_0 \leqslant \deg(Z)$.

(b) Note that $p(Z) \subseteq X$ is a closed subvariety. We may assume $\dim(p(Z)) = \dim(Z)$ (since otherwise $p_*([Z]) = 0$). In this case, it suffices to show the inequality

$$\deg(p|_Z) \operatorname{gdeg}(p(Z)) \leqslant \operatorname{gdeg}(Z) \operatorname{gdeg}(X) \,.$$

By Lemma A.2.8, there exists a closed reduced subscheme $X'' \subseteq \mathbb{P}^n_k$ such that $X = \overline{X} \smallsetminus X''$ and $\operatorname{gdeg}(X) = \operatorname{gdeg}\left(\overline{X}\right) + \operatorname{gdeg}(X'')$. Let $\overline{p}$ be the projection $\mathbb{P}^m_{\overline{X}} \to \overline{X}$, and let $\overline{Z} \subseteq \mathbb{P}^m_{\overline{X}}$ be the closure of Z. Then the image $\overline{p}\left(\overline{Z}\right) \subseteq \mathbb{P}^n_k$ is closed, and we have $p(Z) = \overline{p}\left(\overline{Z}\right) \smallsetminus X''$, thus

$$\operatorname{gdeg}(p(Z)) \leqslant \deg\left(\overline{p}\left(\overline{Z}\right)\right) + \operatorname{gdeg}(X'') \leqslant \deg\left(\overline{p}\left(\overline{Z}\right)\right)\left(1 + \operatorname{gdeg}(X'')\right).$$

Combining this with the equality $\deg(\overline{p}|_{\overline{Z}}) = \deg(p|_Z)$ and part (a), we conclude that

$$\deg(p|_Z)\operatorname{gdeg}(p(Z)) \leqslant \deg(\overline{p}|_{\overline{Z}})\deg\left(\overline{p}\left(\overline{Z}\right)\right)(1 + \operatorname{gdeg}(X'')) \leqslant \deg\left(\overline{Z}\right)\left(1 + \operatorname{gdeg}(X'')\right).$$

Since $1 + \operatorname{gdeg}(X'') \leqslant \operatorname{gdeg}(X)$ and $\deg\left(\overline{Z}\right) \leqslant \operatorname{gdeg}(Z)$ (by Lemma A.2.8), the assertion follows. □

B.2 Uniform boundedness of the standard operations

In this section, we study the behavior of the norm under standard operations in intersection theory. Our ultimate goal will be to study its behavior under Gysin pullback and hence under intersection products.

B.2.1 *Setup.* Let $X \subseteq \mathbb{P}^n_k$ and $Y \subseteq \mathbb{P}^m_X \subseteq \mathbb{P}^{m+n+mn}_k$ be subschemes, and let p be the composition $Y \hookrightarrow \mathbb{P}^m_X \twoheadrightarrow X$.

Lemma B.2.2. *Assume that we are in the situation of § B.2.1.*

(a) *There exists a constant M depending only on m, n and the geometric degrees of X_{red} and Y_{red} such that for every subvariety $Z \subseteq X$, we have the inequality*

$$\operatorname{gdeg}\left((p^{-1}(Z))_{\mathrm{red}}\right) \leqslant M \operatorname{gdeg}(Z).$$

(b) *Assume that p is smooth, and let M be as in part* (a). *Then for every $\alpha \in A_*(X)$, we have the inequality $|p^*(\alpha)| \leqslant M|\alpha|$.*

(c) *Assume that $Y \subseteq \mathbb{P}^m_X$ is closed. Then p is proper, and for every $\alpha \in A_*(Y)$, we have the inequality $|[p_*(\alpha)]| \leqslant |\alpha| \operatorname{gdeg}(X_{\mathrm{red}})$. Moreover, we have $|[p_*(\alpha)]| \leqslant |\alpha|$ if $X \subseteq \mathbb{P}^n_k$ is closed.*

Proof. (a) Note that $\left(p^{-1}(Z)\right)_{\mathrm{red}} = (Y \cap (\mathbb{P}^m \times Z))_{\mathrm{red}}$, where the intersection is taking place inside of $\mathbb{P}^{mn+m+n}_k$. Hence the lemma follows from Proposition B.1.5 and Lemma B.1.6.

(b) By Lemma B.1.4(a), it suffices to show that for every irreducible closed subvariety Z of X, we have the inequality $|p^*([Z])| \leqslant M \operatorname{gdeg}(Z)$. Since p is smooth, we have $p^*([Z]) = \left[p^{-1}(Z)\right]$, and $p^{-1}(Z)$ is reduced. Thus the assertion follows from part (a).

(c) Arguing as in part (b), the assertion follows from Lemma B.1.7 applied to X_{red}. □

As a consequence, we study the dependence of a norm on an embedding.

Corollary B.2.3. *Let $\iota_1 \colon X \hookrightarrow \mathbb{P}^{n_1}_k$ and $\iota_2 \colon X \hookrightarrow \mathbb{P}^{n_2}_k$ be two embeddings of X. Then there exists a constant M depending only on n_1, n_2 and the geometric degrees of $\iota_1(X)$ and $\iota_2(X)$ such that for every $\alpha \in A_*(X)$, we have $|\alpha|_{\iota_1} \leqslant M|\alpha|_{\iota_2}$.*

Proof. Using Lemma B.1.4(a), we may assume that α is the class $[Z]$ of an irreducible closed subvariety Z of X, and we want to show the inequality $|[\iota_1(Z)]| \leqslant M|[\iota_2(Z)]|$. Let

$$\iota \colon (\iota_1, \iota_2) \colon X \hookrightarrow \mathbb{P}^{n_1}_k \times \mathbb{P}^{n_2}_k \subseteq \mathbb{P}^{n_1 n_2 + n_1 + n_2}_k$$

be the diagonal embedding of X, and let $p_1 \colon \iota_1(X) \times \mathbb{P}^{n_2} \to \iota_1(X)$ be the projection. Using the identity

$$[\iota_1(Z)] = (p_1)_*\left(\left[(\mathbb{P}^{n_1}_k \times \iota_2(Z)) \cap \iota(X)\right]\right),$$

the result now follows from Proposition B.1.5 and Lemma B.2.2(a), (c). □

Next, we show the uniform boundedness of the Chern class operations.

We say that a collection $\{(X_\alpha \subseteq \mathbb{P}^n_{k_\alpha}, \mathcal{E}_\alpha)\}_\alpha$, where k_α is an algebraically closed field, X_α is a locally closed subscheme of $\mathbb{P}^n_{k_\alpha}$ and $\mathcal{E}_\alpha$ is a vector bundle on X_α, is *bounded* if there exist a locally closed embedding $X \hookrightarrow \mathbb{P}^n_S$ parameterizing $\{X_\alpha \subseteq \mathbb{P}^n_{k_\alpha}\}_\alpha$ (see Notation A.1.1(e), (i)) and a vector bundle on $\mathcal{E}$ on X whose pullback to each X_α is $\mathcal{L}_\alpha$.

Proposition B.2.4. (a) *Let $X \subseteq \mathbb{P}^n_k$ be a subscheme and $\mathcal{E}$ a vector bundle on X. Then there exists a constant M such that for every $\alpha \in A_*(X)$ and $i \in \mathbb{N}$, we have $|c_i(\mathcal{E}) \cap \alpha| \leqslant M|\alpha|$.*

(b) *Moreover, M only depends on the* numerical invariants of $(X \subseteq \mathbb{P}^n_k, \mathcal{E})$, *by which we mean that if the pair $(X \subseteq \mathbb{P}^n_k, \mathcal{E})$ belongs to a bounded collection $\{(X_\alpha \subseteq \mathbb{P}^n_{k_\alpha}, \mathcal{L}_\alpha)\}_\alpha$, then there exists a constant M that satisfies the property in part* (a) *for every $(X_\alpha \subseteq \mathbb{P}^n_{k_\alpha}, \mathcal{L}_\alpha)$.*

Proof. (a) Recall that the Chern class operations are homogeneous polynomials in the Segre class operations with integral coefficients [Ful98, Section 3.2]. Hence by Lemma B.1.4(a), it suffices to prove a similar assertion for the Segre class operations. Let $e+1$ be the rank of $\mathcal{E}$, let $p\colon P(\mathcal{E}) \to X$ be the projective bundle of lines in $\mathcal{E}$, and let $\mathcal{O}(1)$ be the canonical line bundle on $P(\mathcal{E})$. Recall (see [Ful98, Section 3.1]) that the Segre class operations are given by the formula

$$s_i(\mathcal{E}) \cap \alpha = p_*\big[c_1(\mathcal{O}(1))^{e+i} \cap p^*(\alpha)\big]\,.$$

Since p is smooth and projective, it follows from Lemma B.2.2 that it suffices to show the assertion in the case when $\mathcal{E}$ is a line bundle $\mathcal{L}$ and $i = 1$.

Since X is quasi-projective, $\mathcal{L}$ can be written as a difference of two very ample line bundles (use [Har77, Exercise II.5.7]). Using Lemma B.1.4(a), we may assume that $\mathcal{L}$ is very ample, hence there exists an embedding $\iota\colon X \hookrightarrow \mathbb{P}^{n'}_k$ such that $\mathcal{L}$ is the pullback of $\mathcal{O}(1)$. Using Lemma B.2.3, we can then assume that ι is the inclusion $X \hookrightarrow \mathbb{P}^n_k$.

In this case, we claim that $|c_1(\mathcal{L}) \cap \alpha| \leqslant |\alpha|$. Using Lemma B.1.4(a) again, it suffices to show that for every closed subvariety $Z \subseteq X$, we have the inequality $|c_1(\mathcal{L}) \cap [Z]| \leqslant \operatorname{gdeg}(Z)$. Let $H \subseteq \mathbb{P}^n_k$ be a hyperplane such that $Z \nsubseteq H$. By definition, $c_1(\mathcal{L}) \cap [Z]$ is the class $[H \cap Z]$ of the schematic intersection $H \cap Z \subseteq X$. Moreover, by Bertini's theorem [Jou83, théorème 6.3(4)], there exists an H such that $H \cap Z$ is an integral scheme if $\dim(Z) \geqslant 2$ and $H \cap Z$ is reduced if $\dim(Z) = 1$. In both cases, we have $|[H\cap Z]| \leqslant \operatorname{gdeg}((H\cap Z)_{\mathrm{red}}) \leqslant \operatorname{gdeg}(Z)$ by Proposition B.1.5.

(b) The assertion follows from the fact that the argument of part (a) "can be carried out in families". Namely, we can assume that S is affine. Then, as in part (a), for every locally closed subscheme $X \subseteq \mathbb{P}_S$, every line bundle is a difference of two very ample bundles. The rest of the argument follows by repeating the argument of part (a) word-for-word and using Corollary A.2.4 and the uniformity assertions in Lemmas B.2.3 and B.2.2. □

Finally, we show the uniform boundedness under Gysin pullbacks.

Theorem B.2.5. *Let $i\colon Y \hookrightarrow X$ be a closed regular embedding of codimension d of subschemes of $\mathbb{P}^n_k$, let $X' \subseteq \mathbb{P}^m_X$ be a subscheme, and let $i'\colon Y' := Y \times_X X' \hookrightarrow X'$ be the pullback of i.*

(a) *There exists a constant M such that for every cycle class $\alpha \in A_*(X')$, the refined pullback $i^*(\alpha) \in A_{*-d}(Y')$ satisfies $|i^*(\alpha)| \leqslant M|\alpha|$.*

(b) *Moreover, M only depends on* the numerical invariants of $\big(Y \overset{i}{\hookrightarrow} X \subseteq \mathbb{P}^n_k, X' \subseteq \mathbb{P}^m_X\big)$.

Remark B.2.6. As in Proposition B.2.4, part (b) means that if a diagram $\big(Y \overset{i}{\hookrightarrow} X \subseteq \mathbb{P}^n_k, X' \subseteq \mathbb{P}^m_X\big)$ belongs to a bounded collection $\big\{\big(Y_\alpha \overset{i_\alpha}{\hookrightarrow} X_\alpha \subseteq \mathbb{P}^n_{k_\alpha}, X'_\alpha \subseteq \mathbb{P}^m_{X_\alpha}\big)\big\}_\alpha$, then there exists a constant M that has the property in part (a) for every diagram $\big(Y_\alpha \overset{i_\alpha}{\hookrightarrow} X_\alpha \subseteq \mathbb{P}^n_{k_\alpha}, X'_\alpha \subseteq \mathbb{P}^m_{X_\alpha}\big)$.

Proof. We will carry out the proof in six steps.

Step 1. By Lemma B.1.4(a), it suffices to show that there exists a constant M, only depending on the numerical invariants of $\big(Y \overset{i}{\hookrightarrow} X \subseteq \mathbb{P}^n_k, X' \subseteq \mathbb{P}^m_X\big)$, such that for every closed subvariety $V' \subseteq X'$, we have the inequality $|i^*([V'])| \leqslant M\,\mathrm{gdeg}(V')$.

Step 2. Assume that the closed embedding $i'|_{V'}\colon i'^{-1}(V') \hookrightarrow V'$ is an isomorphism. In this case, we claim that we have the equality

$$i^*([V']_{X'}) = c_d(N_Y(X)|_{Y'}) \cap [i'^{-1}(V')]_{Y'}\,, \tag{21}$$

where $N_Y(X)|_{Y'}$ is the pullback to Y' of the normal bundle of Y in X. Indeed, consider the Cartesian diagram

$$\begin{tikzcd} i'^{-1}(V') \arrow[r, "i'|_{V'}", "\sim"'] \arrow[d, "q"'] & V' \arrow[d, "p"] \\ Y' \arrow[r, "i'"] \arrow[d] & X' \arrow[d] \\ Y \arrow[r, "i"] & X\,. \end{tikzcd}$$

Then using the self-intersection formula [Ful98, Corollary 6.3], the compatibility of Gysin pullbacks with proper pushforward [Ful98, Theorem 6.2(a)] and the projection formula for Chern classes [Ful98, Theorem 3.2(c)], we have the equalities

$$\begin{aligned} i^*([V']_{X'}) &= i^*(p_*([V']_{V'})) = q_*(i^*([V']_{V'})) = q_*\big(q^*(c_d(N_Y(X)|_{Y'})) \cap \big[i'^{-1}(V')\big]_{i'^{-1}(V')}\big) \\ &= c_d(N_Y(X)|_{Y'}) \cap q_*\big(\big[i'^{-1}(V')\big]_{i'^{-1}(V')}\big) = c_d(N_Y(X)|_{Y'}) \cap \big[i'^{-1}(V')\big]_{Y'}\,. \end{aligned}$$

We claim that the assertion of Step 1 holds. Indeed, if $(Y \overset{i}{\hookrightarrow} X \subseteq \mathbb{P}^n_k, X' \subseteq \mathbb{P}^m_X)$ belongs to a bounded collection, then the vector bundle $N_Y(X)|_{Y'}$ on Y' belongs a bounded collection (by Corollary A.1.9 and Lemma A.1.4(f)). Since $\big|\big[i'^{-1}(V')\big]\big| \leqslant \mathrm{gdeg}\,\big(i'^{-1}(V')\big) = \mathrm{gdeg}(V')$, the assertion of Step 1 now follows from a combination of equality (21) and Proposition B.2.4.

From now on, we assume that $i'|_{V'}\colon i'^{-1}(V') \hookrightarrow V'$ is not an isomorphism.

Step 3. Assume that $X \subseteq \mathbb{P}^n$ is a closed subscheme, $d = 1$ and $X' \to X$ is an open embedding with dense image.

For simplicity, we view Y and X' as subschemes of X. Then the schematic intersection $Y' \cap V' \subseteq V'$ is a Cartier divisor, and we have the equalities $i^*([V']) = i'^*([V']) = [Y' \cap V'] \in A_*(Y')$ (see [Ful98, Example 2.6.5 and Corollary 6.3]) and $i'_*i'^*([V']) = c_1(\mathcal{O}(Y')) \cap [V']$ (see [Ful98, Propositions 2.6(d) and 6.1(c)]).

(a) First assume $X' = X$ (hence also $Y' = Y$ and $i' = i$). Set $V := V'$. Then by Lemma B.1.4(b), we have the equality $|[Y \cap V]| = \deg(Y \cap V) = |i_*([Y \cap V])|$, which implies that $|i^*([V])| = |c_1(\mathcal{O}(Y)) \cap [V]|$.

Moreover, if $(Y \subseteq X \subseteq \mathbb{P}^n)$ belongs to a bounded collection, then the line bundle $\mathcal{O}(Y)$ on Y belongs to a bounded collection (by Lemma A.1.4(c)), so in this case the assertion of Step 1 follows from Proposition B.2.4.

(b) In general, let $V \subseteq X$ be the closure of V'. It suffices to show the inequality

$$|i^*([V'])| \leqslant |i^*([V])| \operatorname{gdeg}(X'_{\mathrm{red}})\,. \tag{22}$$

Indeed, the assertion of Step 1 would then follow from the inequality $\operatorname{gdeg}(V) \leqslant \operatorname{gdeg}(V')$ (see Lemma A.2.8(c)), Corollaries A.1.3 and A.2.4 and case (a), shown above.

Let $Z_1, \ldots, Z_r$ be all irreducible components of $Y \cap V$. Then each $Z'_j := Z_j \cap X'$ is either an irreducible components of $Y' \cap V'$ or empty. Choose a closed reduced subscheme $X_2 \subseteq \mathbb{P}^n_k$ such that $X' = X \smallsetminus X_2$ and $\operatorname{gdeg}(X'_{\mathrm{red}}) = \operatorname{gdeg}(X_{\mathrm{red}}) + \operatorname{gdeg}(X_2)$. Then we have $1 + \operatorname{gdeg}(X_2) \leqslant \operatorname{gdeg}(X'_{\mathrm{red}})$ and $Z'_j := Z_j \smallsetminus X_2$ for all j.

Note that the cycle $[Y \cap V] \in Z_*(Y)$ equals $\sum_{j=1}^r n_j [Z_j]$ for some $n_j \geqslant 0$. Hence we have the equality of cycles $[Y' \cap V'] = \sum_{j=1}^r n_j [Z'_j] \in Z_*(Y')$ and therefore the inequality

$$|i^*([V'])| \leqslant \sum_{j=1}^r n_j \operatorname{gdeg}(Z'_j) \leqslant \sum_{j=1}^r n_j \operatorname{gdeg}(Z_j) + \left(\sum_{j=1}^r n_j\right) \operatorname{gdeg}(X_2)\,.$$

Using the equality $\sum_{j=1}^r n_j \operatorname{gdeg}(Z_j) = \deg(Y \cap V) = |i^*([V])|$ (compare with Lemma B.1.4(b)), we conclude that $\left(\sum_{j=1}^r n_j\right) \leqslant |i^*([V])|$, implying inequality (22).

Step 4. Consider the Cartesian diagram

$$\begin{tikzcd} Y'' \arrow[r, "i''"] \arrow[d, "q"'] & X'' \arrow[d, "p"] \\ Y' \arrow[r, "i'"] \arrow[d] & X' \arrow[d] \\ Y \arrow[r, "i"] & X\,, \end{tikzcd}$$

where $X'' := \mathrm{Bl}_{Y'}(X')$ and $p\colon X'' \to X'$ is the projection map.

Let $V'' \subseteq X''$ be the strict transform of V'. Note that p is proper and that i (respectively, i'') is a regular immersion of codimension d (respectively, 1). Since $p_*([V'']) = [V']$, it follows by the compatibility of the Gysin pullback with proper pushforwards [Ful98, Theorem 6.2(a)] and the excess intersection formula [Ful98, Theorem 6.3] that we have the equality

$$i^*([V']) = i^*(p_*([V''])) = q_*(i^*([V''])) = q_*\big(c_{d-1}(\mathcal{E}) \cap (i'')^*([V''])\big)\,, \tag{23}$$

where $\mathcal{E} := q^*(N_Y(X)|_{Y'})/N_{Y''}(X'')$.

Step 5. Let $\overline{X} \subseteq \mathbb{P}^n_k$ be the closure of X, and let $\overline{X}', \overline{Y}' \subseteq \mathbb{P}^m_{\overline{X}}$ be the closures of X' and Y', respectively. We set $\overline{X}'' := \mathrm{Bl}_{\overline{Y}'}\big(\overline{X}'\big)$ and let $\overline{Y}'' \subseteq \overline{X}''$ be the exceptional divisor. Then we have the Cartesian diagram

$$\begin{tikzcd} Y'' \arrow[r, "i''"] \arrow[d] & X'' \arrow[d, "j"] \\ \overline{Y}'' \arrow[r, "\overline{i}''"] & \overline{X}''\,, \end{tikzcd}$$

where $\overline{i}''$ is a regular embedding of codimension 1 and j is an open embedding with dense image. In particular, we arrive at the situation of Step 3.

Step 6. We claim that there exist constants M_1, M_2, M_3, which only depend on the numerical invariants of $\big(Y \overset{i}{\hookrightarrow} X \subseteq \mathbb{P}^n_k, X' \subseteq \mathbb{P}^m_X\big)$, such that in the notation of Step 4, for each closed subvariety $V' \subseteq X'$ such that $i'|_{V'}\colon i'^{-1}(V') \hookrightarrow V'$ is not an isomorphism, we have the inequalities

$$|i^*([V'])| \leqslant M_1|(i'')^*([V''])| \leqslant M_1M_2 \operatorname{gdeg}(V'') \leqslant M_1M_2M_3 \operatorname{gdeg}(V')\,.$$

Indeed, assume that a diagram $\big(Y \overset{i}{\hookrightarrow} X \subseteq \mathbb{P}^n_k, X' \subseteq \mathbb{P}^m_X\big)$ belongs to a bounded collection. Then it follows from a combination of Lemma A.1.4(f), (g) and Corollary A.1.7 that the full Cartesian diagrams of Steps 4 and 5 belong to bounded collections. In particular, the existence of M_2 follows from Steps 3 and 5.

Next, it follows from Corollary A.1.9 that vector bundle $\mathcal{E}$ on Y', defined in Step 4, belongs to a bounded collection. Therefore, the existence of M_1 follows from equality (23) together with Proposition B.2.4, Lemma B.2.2(c) and Corollary A.2.4.

Finally, since V'' is an irreducible component of $p^{-1}(V')$, we have

$$\operatorname{gdeg}(V'') \leqslant \operatorname{gdeg}\big(p^{-1}(V')_{\mathrm{red}}\big) \quad \text{(by Lemma A.2.8(b))}\,,$$

so the existence of M_3 follows from Corollary A.2.4 and Lemma B.2.2(a). □

Corollary B.2.7. *Let $X \subseteq \mathbb{P}^n_k$ and $Y \subseteq \mathbb{P}^m_X$ be smooth subvarieties, let $Z \subseteq X$ be a closed reduced subscheme, and let f be a composition $Y \overset{i}{\hookrightarrow} \mathbb{P}^m_X \overset{p}{\twoheadrightarrow} X$. Then there exists a constant M depending on n, m and geometric degrees of X, Y and Z such that for every $\alpha \in A_*(Z)$, the refined pullback $f^*(\alpha) \in A_*\big(f^{-1}(Z)\big)$ satisfies $|f^*(\alpha)| \leqslant M|\alpha|$.*

Proof. Since $f^* = i^* \circ p^*$, the assertion follows from a combination of Lemma B.2.2(b), (the proof of) Corollary A.2.5 and Theorem B.2.5. □

Corollary B.2.8 (cf. [Hru04, Corollary 10.14]). *Let $X \subseteq \mathbb{P}^n_k$ be a smooth closed subscheme. Then there exists a constant M, depending on n and the geometric degree of X, such that for all $\alpha, \beta \in A_*(X)$, we have the inequality $|\alpha \cap \beta| \leqslant M|\alpha||\beta|$.*

Proof. Let $\Delta\colon X \hookrightarrow X \times X \subseteq \mathbb{P}^n_k \times \mathbb{P}^n_k$ be the diagonal embedding. By definition, we have $\alpha \cap \beta = \Delta^*(\alpha \times \beta)$. Thus using Theorem B.2.5 and Corollary B.2.3, it suffices to show the inequality $|\alpha \times \beta| \leqslant M|\alpha||\beta|$, where M depends only on n and the degree of X. As before, using Lemma B.1.4(a), we may assume $\alpha = [V]$ and $\beta = [W]$ for some closed subvarieties $V, W \subseteq X$. The result now follows from Lemma B.1.6. □

Remark B.2.9. (a) As is shown in [Hru04, Corollary 10.14], the constant M in Corollary B.2.8 can be replaced by 1 if one replaces our norm $|\cdot|$ with Hrushovski's norm $|\cdot|_{\mathrm{Hr}}$.

(b) Combining part (a) with inequalities (20), we see that the constant M in Corollary B.2.8 can be made effective (polynomial in $\operatorname{gdeg} X$ and exponential in $\dim X$).

We need the following estimate for the norm of a composition of correspondences.

B.2.10 *Composition of correspondences.* (a) Let X_1, X_2 and X_3 be a triple of d-dimensional smooth proper schemes over k. Then with every $u \in A_d(X_1 \times X_2)$ and $v \in A_d(X_2 \times X_3)$, one associates the composition $v \circ u := p_{13*}(p_{12}^*(u) \cap p_{23}^*(v)) \in A_d(X_1 \times X_3)$. Here, as usual, p_{ij} is the projection from $X_1 \times X_2 \times X_3$ to $X_i \times X_j$.

(b) In particular, when $X_1 = X_2 = X_3$, the above composition equips $A_d(X \times X)$ with the structure of a ring, whose unit is the class of the diagonal. Furthermore, the assignment $\alpha \mapsto H^i(\alpha)$ defines an action of $A_d(X \times X)$ on $H^i(X, \mathbb{Q}_\ell)$ for all $\ell \neq \operatorname{char}(k)$ and all $i \in \mathbb{N}$.

Corollary B.2.11 (cf. [Hru04, Lemma 10.17(3)]). *For every $n, \delta \in \mathbb{N}$, there exists a constant N depending on (n, δ) such that for all $d \leqslant n$, for*

- *all triples of d-dimensional smooth closed subvarieties $X_1, X_2, X_3 \subseteq \mathbb{P}^n_k$ (over an arbitrary algebraically closed field k) of degree δ and*
- *and all pairs $u \in A_d(X_1 \times X_2)$ and $v \in A_d(X_2 \times X_3)$,*

we have the inequality

$$|v \circ u| \leqslant N|v| \cdot |u| \,. \tag{24}$$

Here the norms are taken with respect to the Segre embedding of the product varieties.

Proof. Using the definition of $v \circ u$ (see § B.2.10(a)), the assertion follows from a combination of Corollary B.2.8 with Lemma B.2.2. □

DEFINITION B.2.12. (a) For $n, \delta \in \mathbb{N}$, we denote by $N(n, \delta)$ the smallest possible integer N satisfying the assertion of Corollary B.2.11.

(b) For $d \leqslant n$, a pair $X_1, X_2 \subseteq \mathbb{P}^n_k$ of d-dimensional smooth closed subvarieties of degree δ and a class $\alpha \in A_d(X_1 \times X_2)$, we define a *renormalized norm* of $||\alpha||$ of α by the formula $||\alpha|| := N(n, \delta)|\alpha|$.

Remark B.2.13. (a) As is shown in [Hru04, Corollary 10.17(3)], the constant N in Corollary B.2.11 can be replaced by $\binom{2d}{d}^2$ if one replaces our norm $|\cdot|$ with Hrushovski's norm $|\cdot|_{\mathrm{Hr}}$.

(b) Combining (a) with inequalities (20), we see that the constant $N(n, \delta)$ can be made effective (polynomial in δ and exponential in n).

Remark B.2.14. In the notation of Definition B.2.12, inequality (24) of Corollary B.2.11 can now be rewritten in the form

$$||v \circ u|| \leqslant ||v|| \cdot ||u|| \,. \tag{25}$$

B.3 A bound on the spectral radius

In this subsection, we apply the submultiplicativity of the renormalized norm (25) to obtain a bound on the spectral radius of the action of cycles on cohomology. First, we treat the case of varieties over finite fields, following [Hru04] very closely and using Katz–Messing [KM74] (see [Shu19a, Theorem 1.4] for a similar argument). Then we deduce the general case by a standard spreading-out argument.

B.3.1 *Spectral radius.* Let X be a smooth proper variety of dimension d over an algebraically closed field k, and let $\alpha \in A_d(X \times X)$. With every $\ell \neq \mathrm{char}(k)$ and a field embedding $\tau \colon \mathbb{Q}_\ell \hookrightarrow \mathbb{C}$, one associates an endomorphism $\tau\big(H^i(\alpha)\big)$ of the $\mathbb{C}$-vector space $H^i(X, \mathbb{Q}_\ell) \otimes_{\mathbb{Q}_\ell} \mathbb{C}$. Hence one can consider its spectral radius $\rho\big(\tau\big(H^i(\alpha)\big)\big) \in \mathbb{R}$, that is, the maximum of the norms of its eigenvalues.

Using the identity $\rho(A) = \limsup_n |\operatorname{Tr}(A^n)|^{1/n}$, it follows from § 3.1.9 that the spectral radius $\rho(H^i(\alpha)) := \rho(\tau(H^i(\alpha)))$ is independent of $\ell \neq \mathrm{char}(k)$ and τ.

PROPOSITION B.3.2 ([Hru04, Proposition 11.11]). *Let k be an algebraically closed field, and let $X \subseteq \mathbb{P}^n_k$ be a smooth closed subvariety over k of dimension d. Then for all $\alpha \in A_d(X \times X)$ and $i \in \mathbb{N}$, the spectral radius of $H^i(\alpha)$ (see § B.3.1) is at most $||\alpha||$ (see Definition B.2.12).*

Proof. Finite field case. First we are going to show the assertion in the case when k is an algebraic closure of a finite field $\mathbb{F}_q$. Replacing q with its power, we can assume that X is defined over $\mathbb{F}_q$. Let $\mathrm{F}_X \colon X \to X$ be the (geometric) Frobenius with respect to $\mathbb{F}_q$. By [KM74, Theorem 2], there exists a polynomial $P_i[T] \in \mathbb{Q}[T]$ such that the endomorphism $P_i(\mathrm{F}_X^*)$ of $H^j(X,\ \mathbb{Q}_\ell)$ is zero for $j \neq i$ and the identity for $j = i$. We choose $m \in \mathbb{N}$ such that $(mP_i)(T) \in \mathbb{Z}[T]$.

We denote by $\alpha^{\circ n} \in A_d(X \times X)$ the nth power of α in the ring $A_d(X \times X)$ (see § B.2.10), by $[\Gamma_X] \in A_d(X \times X)$ the class of the graph of F_X and by $(mP_i)([\Gamma_X]) \in A_d(X \times X)$ the element obtained from $[\Gamma_X]$ using the ring structure on $A_d(X \times X)$.

By a repeated application of equation (10), we get the equality

$$\operatorname{Tr}\left(H^i(\alpha)^n, H^i(X, \mathbb{Q}_\ell)\right) = \frac{(-1)^i}{m}\alpha^{\circ n} \cdot (mP_i)([\Gamma_X]) \in \mathbb{Q}. \tag{26}$$

Hence, by Corollaries B.2.8 and B.2.11 (in the form of inequality (25)), there exists a constant M such that for all $n \in \mathbb{N}$, we have the inequality

$$\left|\operatorname{Tr}\left(H^i(\alpha)^n, H^i(X, \mathbb{Q}_\ell)\right)\right| \leqslant M|(mP_i)([\Gamma_X])| \cdot ||\alpha||^n. \tag{27}$$

Since the spectral radius of $H^i(\alpha)$ equals $\limsup_n \left|\operatorname{Tr}\left(H^i(\alpha)^n, H^i(X, \mathbb{Q}_\ell)\right)\right|^{1/n}$, the proposition in this case follows from inequality (27).

General case. We claim that the general case follows from the finite field case, shown above. Indeed, fix a prime ℓ invertible in k. By Lemma B.1.4(a), we can assume that α is the class $[C]$ of a closed irreducible subvariety $C \subseteq X \times X$. We can assume that $\operatorname{Spec} k$ is a geometric generic point of a scheme S of finite type over $\mathbb{Z}$, that X is a geometric generic fiber of a closed subscheme $\widetilde{X} \subseteq \mathbb{P}^n_S$ and that C is the geometric generic fiber of a closed subscheme $\widetilde{C} \subseteq \widetilde{X} \times_S \widetilde{X}$. Moreover, shrinking S if necessary, we can assume that S is connected, ℓ is invertible in S, $\widetilde{X}$ is smooth over S and $\widetilde{C}$ is flat over S.

For every geometric point $\overline{s}$ of S, we denote by $\widetilde{X}_{\overline{s}}$ and $\widetilde{C}_{\overline{s}}$ the fibers of $\widetilde{X}$ and $\widetilde{C}$, respectively, over $\overline{s}$. Since every closed point of S is defined over a finite field, to show the reduction, it suffices to show that the modified norm $\left\|[\widetilde{C}_{\overline{s}}]\right\|$ and the spectral radius of the action of $H^i\left([\widetilde{C}_{\overline{s}}]\right)$ on $H^i\left(\widetilde{X}_{\overline{s}}, \mathbb{Q}_\ell\right)$ do not depend on $\overline{s}$. The assertion about $\left\|[\widetilde{C}_{\overline{s}}]\right\|$ follows from the equality $\left|[\widetilde{C}_{\overline{s}}]\right| = \deg\left(\widetilde{C}_{\overline{s}}\right)$ (see Lemma B.1.4(b)) and the observation that $\deg\left(\widetilde{C}_{\overline{s}}\right)$ and $\deg\left(\widetilde{X}_{\overline{s}}\right)$ do not depend on $\overline{s}$ because $\widetilde{X}$ and $\widetilde{C}$ are flat over S. The assertion about the spectral radius follows from Claim B.3.3 below. □

Claim B.3.3. *The characteristic polynomial of $H^i\left([\widetilde{C}_{\overline{s}}]\right)$ is independent of $\overline{s}$.*

Proof. It suffices to show that for every $n \in \mathbb{N}$, the trace $\operatorname{Tr}\left(H^i\left([\widetilde{C}_{\overline{s}}]\right)^n, H^i\left(\widetilde{X}_{\overline{s}}, \mathbb{Q}_\ell\right)\right)$ is independent of $\overline{s}$. To see this, it suffices to construct a global section $a(n) \in H^0(S, \mathbb{Q}_\ell)$ such that for every geometric point $\overline{s}$ of S, the pullback $a(n)_{\overline{s}} \in H^0(\overline{s}, \mathbb{Q}_\ell) = \mathbb{Q}_\ell$ of $a(n)$ equals $\operatorname{Tr}\left(H^i\left([\widetilde{C}_{\overline{s}}]\right)^n, H^i\left(\widetilde{X}_{\overline{s}}, \mathbb{Q}_\ell\right)\right)$. Indeed, the independence of $\overline{s}$ would then follow from the fact that S is connected, thus $a(n)$ is constant.

The construction of $a(n)$ is standard: Namely, since the projection $p\colon \widetilde{X} \to S$ is proper and smooth, the derived pushforward $R^ip_*(\mathbb{Q}_\ell)$ is a local system on S, and we have a natural isomorphism $R^ip_*(\mathbb{Q}_\ell)_{\overline{s}} \simeq H^i\left(\widetilde{X}_{\overline{s}}, \mathbb{Q}_\ell\right)$ for every $\overline{s}$. It therefore suffices to construct an endomorphism $\mathcal{H}^i\left(\widetilde{C}\right)\colon R^ip_*(\mathbb{Q}_\ell) \to R^ip_*(\mathbb{Q}_\ell)$ whose geometric fiber at $\overline{s}$ is $H^i\left(\widetilde{C}_{\overline{s}}\right)$.

Let $\pi\colon \widetilde{C} \to S$ be the projection and $(c_1, c_2)\colon \widetilde{C} \hookrightarrow \widetilde{X} \times \widetilde{X}$ the inclusion. Then we define $\mathcal{H}^i(\widetilde{C})$ to be the composition $(c_2)_* \circ (c_1)^*\colon R^ip_*(\mathbb{Q}_\ell) \to R^i\pi_*(\mathbb{Q}_\ell) \to R^ip_*(\mathbb{Q}_\ell)$, where $(c_1)^*$ is the canonical morphism $Rp_*(\mathbb{Q}_\ell) \to R(p \circ c_1)_*(\mathbb{Q}_\ell) = R\pi_*(\mathbb{Q}_\ell)$, while $(c_2)_*$ is defined by a manner similar to § 3.1.7, using the fact that p is smooth, while π is flat. □

COROLLARY B.3.4 ([Hru04, Corollary 11.12]). *Let k be an algebraically closed field of characteristic $p > 0$, let q be a power of p, and let $X \subseteq \mathbb{P}^n_k$ be a smooth closed subvariety over k of dimension d. Then for all $\alpha \in A_d\big(X \times X^{(q)}\big)$ and $i \in \mathbb{N}$, the spectral radius of the endomorphism $F^*_{X,q} \circ H^i(\alpha)$ of $H^i(X, \mathbb{Q}_\ell)$ (see §3.1.9(b)) is at most $q^{i/2}||\alpha||$.*

Proof. Finite field case. First, we are going to show the assertion in the case when k is an algebraic closure of a finite field $\mathbb{F}_q$. Using Lemma B.1.4(a), we may assume that α is the class $[C]$ of an irreducible subvariety $C \subseteq X \times X^{(q)}$ and that the closed subvarieties $X \subseteq \mathbb{P}^n_k$ and $C \subseteq X \times X^{(q)}$ are defined over $\mathbb{F}_{q^r}$. Then the Frobenius twist $X^{(q^r)}$ is naturally isomorphic to X, and the composition

$$F_X \colon X \xrightarrow{F_{X,q^r}} X^{(q^r)} \simeq X$$

is the geometric Frobenius endomorphism of X over $\mathbb{F}_{q^r}$.

The class $[C] \in A_d\big(X \times X^{(q)}\big)$ gives rise to classes $[C^{(q^i)}] \in A_d\big(X^{(q^i)} \times X^{(q^{i+1})}\big)$. Moreover, each $X^{(q^i)} \times X^{(q^{i+1})}$ is equipped with a natural embedding into $\mathbb{P}^n_k \times \mathbb{P}^n_k$, and $\big\|\big[C^{(q^i)}\big]\big\| = ||[C]||$ with respect to this embedding.

We claim that

(i) the endomorphism $\big(F^*_{X,q} \circ H^i([C])\big)^r$ of $H^i(X, \mathbb{Q}_\ell)$ equals

$$\mathrm{F}^*_X \circ \big(H^i\big(\big[C^{(q^{r-1})}\big]\big) \circ \cdots \circ H^i\big(\big[C^{(q)}\big]\big) \circ H^i([C])\big),$$

(ii) the endomorphisms F^*_X and $H^i\big(\big[C^{(q^{r-1})}\big]\big) \circ \cdots \circ H^i\big(\big[C^{(q)}\big]\big) \circ H^i([C])$ of $H^i(X, \mathbb{Q}_\ell)$ commute.

Indeed, assertion (i) follows from the equalities

$$\mathrm{F}^*_{X^{(j)},q} \circ H^i\big(\big[C^{(q^j)}\big]\big) = H^i\big(\big[C^{(q^{j-1})}\big]\big) \circ \mathrm{F}^*_{X^{(j-1)},q} \tag{28}$$

of endomorphisms of $H^i\big(X^{(q^i)}, \mathbb{Q}_\ell\big)$, the equality

$$\mathrm{F}_X = \mathrm{F}_{X,q^r} = \mathrm{F}_{X^{(q^{r-1})},q} \circ \cdots \circ \mathrm{F}_{X^{(q)},q} \circ \mathrm{F}_{X,q}$$

of endomorphisms of X and induction, while assertion (ii) follows from the same equalities together with the identifications $X^{(q^{j+r})} \simeq X^{(q^j)}$ and $C^{(q^{j+r})} \simeq C^{(q^j)}$.

Now the result easily follows: It suffices to show that the spectral radius of $\big(F^*_{X,q} \circ H^i([C])\big)^r$ is at most $q^{ir/2}||[C]||^r$. Using claims (i), (ii) and purity [Del74], it suffices to show that the spectral radius of

$$H^i\big(\big[C^{(q^{r-1})}\big]\big) \circ \cdots \circ H^i\big(\big[C^{(q)}\big]\big) \circ H^i([C]) = H^i\big(\big[C^{(q^{r-1})}\big] \circ \cdots \circ \big[C^{(q)}\big] \circ [C]\big)$$

is at most $||[C]||^r$. But this follows from a combination of Proposition B.3.2, inequality (25) and equalities $\big\|\big[C^{(q^i)}\big]\big\| = ||[C]||$.

General case. Again, we are going to deduce to the finite field case, shown above. We can assume that $\operatorname{Spec} k$ is a geometric generic point of a scheme S of finite type over $\mathbb{F}_p$ and X is a geometric generic fiber of a closed subscheme $\widetilde{X} \subseteq \mathbb{P}^n_S$. We let $\widetilde{X}^{(q)}$ be the pullback $\widetilde{X} \times_{S,F_{S,q}} S$ with respect to the absolute q-Frobenius morphism $F_{S,q} \colon S \to S$. Since $F_{S,q}$ is functorial in S, for every geometric point $\overline{s}$ of S, the geometric fiber $\big(\widetilde{X}^{(q)}\big)_{\overline{s}}$ of $\widetilde{X}^{(q)}$ is canonically isomorphic to $\big(\widetilde{X}_{\overline{s}}\big)^{(q)}$, and the geometric fiber over $\overline{s}$ of the relative Frobenius morphism $F_{\widetilde{X}/S,q} \colon \widetilde{X} \to \widetilde{X}^{(q)}$ is identified with $F_{X_{\overline{s}},q}$. In particular, the geometric generic fiber of $F_{\widetilde{X}/S,q}$ is identified with $F_{X,q} \colon X \to X^{(q)}$. Furthermore, we can assume that C is a geometric fiber of a closed subscheme

$\widetilde{C} \subseteq \widetilde{X} \times_S \widetilde{X}$, that $\widetilde{X}$ is smooth over S and that $\widetilde{C}$ is flat over S. To finish the argument, we repeat the last paragraph of the proof of Proposition B.3.2 and use Claim B.3.3. □

Corollary B.3.5. *In the situation of Corollary B.3.4, there exists a constant M depending on n and $\operatorname{gdeg}(X)$ such that the trace $\operatorname{Tr}\left(F^*_{X,q} \circ H^i(\alpha)\right) \in \mathbb{Z}$ (see §3.1.9) satisfies the inequality*

$$\left|\operatorname{Tr}\left(F^*_{X,q} \circ H^i(\alpha)\right)\right| \leqslant M|\alpha|q^{i/2}\,.$$

Proof. By §3.1.9, the Betti number $h^i(X) := \dim_{\mathbb{Q}_\ell} H^i(X, \mathbb{Q}_\ell)$ is independent of $\ell \neq \operatorname{char}(k)$. Using Corollary B.3.4, the identity $||\alpha|| = N(n, \deg(X))|\alpha|$ (see Definition B.2.12) and the inequality $\left|\operatorname{Tr}\left(F^*_{X,q} \circ H^i(\alpha)\right)\right| \leqslant \rho\big(F^*_{X,q} \circ H^i(\alpha)\big)h^i(X)$, the assertion follows from a combination of Corollary A.2.4 and Lemma B.3.6 below. □

Lemma B.3.6. *Let $\{X_\alpha \subseteq \mathbb{P}^n_{k_\alpha}\}$ be a bounded collection of closed smooth subvarieties. Then for every $i \in \mathbb{N}$, the collection of Betti numbers $\{h^i(X_\alpha)\}_\alpha$ is bounded.*

Proof. Let $X \subseteq \mathbb{P}^n_S$ be a family representing $\{X_\alpha \subseteq \mathbb{P}^n_{k_\alpha}\}$. Arguing as in Lemma A.1.4(b) and using the constructibility of the set of $s \in S$ such that X_s is smooth, we can assume that the projection $f\colon X \to S$ is smooth. We can fix a prime ℓ and assume that ℓ is invertible in S. By the proper base change theorem, for every geometric point $\overline{s}$ of S, we have an isomorphism $H^i(X_{\overline{s}}, \mathbb{Q}_\ell) \simeq R^i f_*(\mathbb{Q}_\ell)_{\overline{s}}$. Since $R^i f_*(\mathbb{Q}_\ell)$ is constructible, the assertion follows. □

Acknowledgements

This work would not be possible without the masterpiece [Hru04] of Ehud Hrushovski. We would also like to thank him for enlightening conversations over the years. In addition, the idea of the proof of Theorem A.2.1 belongs to him. We also thank Hélène Esnault, Mark Sapir and Luc Illusie for their interest and stimulating conversations. The first-named author would also like to thank Vasudevan Srinivas for his constant support and helpful discussions, especially around Kleiman's boundedness theorem. We also thank an anonymous referee for their comments and suggestions.

This work was has been conceived when the first-named author visited the Hebrew University of Jerusalem.

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