Algebraic Geometry 9 (5) (2022) 513–573
doi:10.14231/AG-2022-016

Birational geometry for d-critical loci and wall-crossing in Calabi–Yau 3-folds

Yukinobu Toda

Abstract

The notion of d-critical loci was introduced by Joyce in order to give classical shadows of (-1)-shifted symplectic derived schemes. In this paper, we discuss birational geometry for d-critical loci, by introducing notions such as "d-critical flips" and "d-critical flops". They are not birational maps of the underlying spaces, but rather should be understood as virtual birational maps. We show that several wall-crossing phenomena of moduli spaces of stable objects on Calabi–Yau 3-folds are described in terms of d-critical birational geometry. Among them, we show that wall-crossing diagrams of Pandharipande–Thomas (PT) stable pair moduli spaces, which are relevant in showing the rationality of PT generating series, form a d-critical minimal model program.

1. Introduction

1.1 Background and motivation

Let X be a smooth projective variety over $\mathbb{C}$. For a given numerical class $v \in H^{2*}(X, \mathbb{Q})$ and a stability condition σ on the derived category of coherent sheaves on X (for example, Bridgeland stability condition [Bri07], weak stability condition [Tod10a]), we denote by $M_\sigma(v)$ the coarse moduli space of S-equivalence classes of σ-semistable objects on X with Chern character v. (The existence of $M_\sigma(v)$ is not obvious in general, and we discuss assuming that it exists. It is proved in [AHH21] that $M_\sigma(v)$ exists if σ-semistable objects with Chern character v are bounded.) The moduli space $M_\sigma(v)$ depends on a choice of a stability condition σ. In general, we have *wall-crossing phenomena*; that is, there is a wall-chamber structure on the space of stability conditions such that $M_\sigma(v)$ is constant if σ lies on a chamber but may change if σ crosses a wall. It is an interesting question how the moduli spaces $M_\sigma(v)$ vary under wall-crossing of σ. More precisely, suppose that σ lies on a wall and $\sigma^\pm$ lie on its adjacent chambers. Then we have the following diagram (called *wall-crossing diagram*):

$$\begin{tikzcd} M_{\sigma^+}(v) \arrow[dr] & & M_{\sigma^-}(v) \arrow[dl] \\ & M_\sigma(v)\,. & \end{tikzcd} \tag{1.1}$$

Received 15 November 2018, accepted in final form 26 December 2021.
2020 Mathematics Subject Classification 14E30, 14D23, 14N35.
Keywords: D-critical locus, Bridgeland stability condition, Birational geometry.

The author is supported by World Premier International Research Center Initiative (WPI initiative), MEXT, Japan, and Grant-in Aid for Scientific Research grant (No. 26287002) from MEXT, Japan.

If $M_{\sigma^{\pm}}(v)$ are smooth (or singular with mild singularities) and birational, then it makes sense to ask whether the diagram (1.1) is a flip or flop in birational geometry [KM98]. Note that if this happens, we have the inequality of canonical line bundles of $M_{\sigma^{\pm}}(v)$ (see Definition 2.8)

$$M_{\sigma^{+}}(v) \geqslant_{K} M_{\sigma^{-}}(v)\,. \tag{1.2}$$

Indeed, this is true in some cases, and birational geometry of the diagram (1.1) has been especially studied when X is an algebraic surface [EG95, FQ95, MW97, NY11, BM14a, BM14a, ABCH13, Tod14]. In these articles, birational geometry of the diagram (1.1) has been important in understanding birational geometry of classical moduli spaces (for example, Hilbert schemes of points) or applications to enumerative geometry (for example, Donaldson invariants).

However, there is a limitation of this research direction. In general, the moduli spaces $M_{\sigma^{\pm}}(v)$ can have worse singularities than those that appear in birational geometry [KM98], for example, terminal singularities, canonical singularities. In fact, by Vakil's Murphy's law [Vak06], any singularity can appear on such moduli spaces, so they may not be irreducible, may not be reduced, may not be equidimensional, and so on. In such bad cases, it is not even clear what are birational maps between them, nor what are their canonical line bundles. Moreover, even if $M_{\sigma^{\pm}}(v)$ are smooth, they are not birational in general. So it does not make sense to ask whether the wall-crossing diagram (1.1) is a flip or a flop or satisfies the inequality (1.2).

If we focus on the case that X is a Calabi–Yau (CY for short) 3-fold, we still have the same issue as above. However, in this case, we have additional structures on the moduli spaces $M_{\sigma^{\pm}}(v)$ (more precisely, on their stable parts) called *d-critical structures*. This notion was introduced by Joyce [Joy15] in order to give classical shadows of (-1)-shifted symplectic structures on derived moduli spaces of stable objects on CY 3-folds [PTVV13]. In particular, we have the notion of *virtual canonical line bundles* on such moduli spaces.

The purpose of this paper is to introduce the notions of *d-critical flips, d-critical flops*, etc. for morphisms from d-critical loci to schemes or analytic spaces. They are not birational maps of the underlying spaces, but rather should be understood as "virtual" birational maps. We then show that, despite the possible bad singularities of $M_{\sigma^{\pm}}(v)$, several wall-crossing diagrams (1.1) for a CY 3-fold X fit into these notions of d-critical birational transformations. In particular, they satisfy an analog of the inequality (1.2) for virtual canonical line bundles.

1.2 D-critical birational geometry

By definition, a *d-critical locus* introduced by Joyce [Joy15] consists of data

$$(M, s)\,, \quad s \in \Gamma\big(M, \mathcal{S}_M^0\big)\,,$$

where M is a $\mathbb{C}$-scheme or an analytic space and $\mathcal{S}_M^0$ is a certain sheaf of $\mathbb{C}$-vector spaces on M (see Definition 3.1). The section s is called a *d-critical structure* of M. Roughly speaking, if M admits a d-critical structure s, this means that M is locally written as a critical locus of some function on a smooth space, and the section s remembers how M is locally written as a critical locus.

Let $\big(M^{\pm}, s^{\pm}\big)$ be two d-critical loci, and consider a diagram of morphisms of $\mathbb{C}$-schemes or analytic spaces

$$\begin{tikzcd} M^{+} \arrow[dr] & & M^{-} \arrow[dl] \\ & A\,. & \end{tikzcd} \tag{1.3}$$

We introduce the notion of a *d-critical flip* (respectively, *d-critical flop*); this is a diagram (1.3) satisfying the following: for any $p \in A$, there is a commutative diagram

$$\begin{tikzcd} Y^+ \arrow[rr, dotted, "\phi"] \arrow[dr] \arrow[ddr, bend right, "w^+"'] & & Y^- \arrow[dl] \arrow[ddl, bend left, "w^-"] \\ & Z \arrow[d, "g"] & \\ & \mathbb{C}, & \end{tikzcd}$$

where $\phi\colon Y^+ \dashrightarrow Y^-$ is a flip (respectively, flop) of smooth varieties (or complex manifolds), such that locally near $p \in A$, there exist isomorphisms between $M^\pm$ and $\{dw^\pm = 0\}$ as d-critical loci (see Definition 3.7 for details). Other notions such as a d-critical divisorial contraction, a d-critical Mori fiber space (MFS for short), and their generalized versions will be also defined in a similar way. We also introduce the inequality of virtual canonical line bundles on d-critical loci, as an analog of (1.2) (see Definition 3.23):

$$\left(M^+, s^+\right) \geqslant_K \left(M^-, s^-\right). \tag{1.4}$$

For example, d-critical flips and d-critical flops satisfy (1.4).

We remark that a diagram (1.3) being a d-critical flip or a d-critical flop does not imply anything on birational geometry of $M^\pm$ themselves, even when $M^\pm$ are smooth. Indeed, there is an example of a d-critical flip where $M^\pm$ are smooth but their dimensions are different (see Example 3.8), so in particular they are not birational. Therefore, we should interpret these notions as "virtual" birational maps rather than birational maps of the underlying spaces $M^\pm$.

1.3 Wall-crossing in Calabi–Yau 3-folds

Suppose that X is a smooth projective CY 3-fold, and consider a wall-crossing diagram (1.1). If v is primitive, then $M_{\sigma^\pm}(v)$ consist of $\sigma^\pm$-stable objects, and $M_{\sigma^\pm}(v)$ admit canonical d-critical structures by [BBBJ15]. Indeed, $M_{\sigma^\pm}(v)$ are classical truncations of derived schemes with (-1)-shifted symplectic structures [PTVV13], and the derived Darboux theorem [BBBJ15] for (-1)-shifted symplectic derived schemes yield d-critical structures on them.

Therefore, we can ask whether the diagram (1.1) is a d-critical flip, a d-critical flop, or so on. We will answer this question via an "analytic neighborhood theorem" given in Theorem 6.1. This theorem describes the diagram (1.1) analytic-locally on $M_\sigma(v)$ in terms of a wall-crossing diagram of moduli spaces of representations of a certain quiver with a (formal but convergent) super-potential. A similar result was already proved in [Tod18] for moduli spaces of semistable sheaves, and we will see that the argument can be generalized to our setting, assuming the existence of good moduli spaces of Bridgeland semistable objects. The latter existence problem is settled by Alper–Halpern–Leistner–Heinloth [AHH21]. The analytic neighborhood theorem reduces the above question on the diagram (1.1) to studying birational maps of moduli spaces of representations of some quivers without relations.

Using the analytic neighborhood theorem, we study d-critical birational geometry of wall-crossing diagrams (1.1) that appeared in the context of enumerative geometry on CY 3-folds, for example, Donaldson–Thomas (DT) invariants [Tho00], Pandharipande–Thomas (PT) invariants [PT09] and also Gopakumar–Vafa (GV) invariants [MT18]. The results are summarized below.

Theorem 1.1 (Theorems 8.3 and 9.13). (i) *In the case of wall-crossing of* 1*-dimensional stable sheaves, the diagram* (1.1) *is a d-critical generalized flop.*

(ii) *In the case of wall-crossing of PT stable pair moduli spaces, the diagram* (1.1) *is a d-critical generalized flip at any point in* $\operatorname{Im}\pi^-$ *and a d-critical MFS at any point in* $M_\sigma(v)\setminus\operatorname{Im}\pi^-$.

The wall-crossing diagrams in the above cases have applications to enumerative geometry. The wall-crossing diagrams of 1-dimensional stable sheaves in item (i) are used in [Tod22b] to show that GV invariants defined in [MT18] are independent of Bridgeland stability conditions, and also invariant under flops. The wall-crossing diagrams of stable pair moduli spaces in item (ii) are used in [Bri11, Tod09a, Tod10b, Tod12a] (also in [Dia12] for local curve case) to show the rationality conjecture of the generating series of PT invariants. In this case, we have wall-crossing diagrams that relate PT invariants and L invariants in [Dia12], and Theorem 1.1(ii) shows that they form a d-critical MMP (see Corollary 9.17).

In summary, Theorem 1.1 gives an interpretation of wall-crossing diagrams in CY 3-folds relevant in enumerative geometry in terms of d-critical birational geometry. In Appendix B, we also discuss some other examples of wall-crossing diagrams in CY 3-folds in terms of d-critical birational geometry, DT/PT correspondence, local K3 surfaces (see Theorems B.1 and B.3). They also have applications to enumerative geometry [Bri11, Tod10a, Tod12b].

1.4 Speculation toward a d-critical D/K equivalence conjecture

Let $Y^+ \dashrightarrow Y^-$ be a birational map between smooth projective varieties satisfying the relation $Y^+ \geqslant_K Y^-$. Then by Bondal–Orlov [BO95] and Kawamata [Kaw02], it is conjectured that there exists a fully faithful functor of derived categories of coherent sheaves $D^b(Y^-) \hookrightarrow D^b(Y^+)$, which is an equivalence if $Y^+ =_K Y^-$. We call this conjecture the *D/K equivalence conjecture.*

We expect that a similar conjecture may hold for d-critical loci or (-1)-shifted symplectic derived schemes. Namely, for a d-critical locus (M,s) (probably induced by a (-1)-shifted symplectic derived scheme equipped with additional data), there may exist a certain triangulated category $\mathcal{D}(M,s)$ such that if the relation (1.4) holds, we have a fully faithful functor

$$\mathcal{D}(M^-,s^-) \hookrightarrow \mathcal{D}(M^+,s^+)\,, \tag{1.5}$$

which is an equivalence if (1.4) is an equality. The category $\mathcal{D}(M^-,s^-)$ may be constructed as a gluing of $\mathbb{Z}/2\mathbb{Z}$-periodic triangulated categories of matrix factorizations defined locally on each d-critical chart, though its construction seems to be a hard problem at this moment (see [Joy13, Problem (J)], [Toë14, Section 6.1] and also constructions for local surfaces [Tod19, Tod22a]). If it exists, the category $\mathcal{D}(M,s)$ may be interpreted as a kind of "Fukaya category" of d-critical loci, or (-1)-shifted symplectic derived schemes (see [JS19, Conjecture 1.2]). Moreover, we expect that the numerical realization of semi-orthogonal complement of the embedding (1.5) recovers the wall-crossing formula of DT invariants on CY 3-folds established in [JS12, KS08]. Thus our d-critical birational geometry gives a link between two research subjects developed independently, the wall-crossing formula of DT invariants and the D/K equivalence conjecture.

If $M^\pm$ are smooth, so in particular $s^\pm = 0$, we can use usual derived categories of coherent sheaves $D^b(M^\pm)$ to ask an analog of the above question. In [Tod21], we address this question in the case of simple wall-crossing diagrams of stable pair moduli spaces.

$$\begin{tikzcd}
 & \boxed{\text{Wall-crossing in DT theory}} \arrow[d, "\text{categorification}"] \\
\boxed{\text{D-critical birational geometry}} \arrow[ur, "\text{geometric interpretation}"] \arrow[dr, "\text{analogy}"'] & \\
 & \boxed{\text{D/K equivalence conjecture}}
\end{tikzcd}$$

1.5 Outline of the paper

In Section 2, we review basic terminology of birational geometry. In Section 3, we recall Joyce's d-critical loci and introduce notions of d-critical birational transformations. In Section 4, we introduce moduli spaces of semistable objects on CY 3-folds and formulate the question on their wall-crossing diagrams. In Section 5, we set notation of moduli spaces of representations of quivers with convergent super-potentials. In Section 6, we state the analytic neighborhood theorem for wall-crossing diagrams in CY 3-folds and give an outline of the proof. In Section 7, we investigate wall-crossing phenomena in symmetric and extended quivers. In Section 8, we describe wall-crossing diagrams of 1-dimensional stable sheaves on CY 3-folds in terms of d-critical flops. In Section 9, we describe wall-crossing diagrams of stable pair moduli spaces in terms of d-critical flips. In Appendix A, we review basics on Bridgeland stability conditions. In Appendix B, we give some more examples of wall-crossing in CY 3-folds and describe them in terms of d-critical birational geometry. In Appendix C, we recall how wall-crossing diagrams in this paper have been relevant in the study of Donaldson–Thomas invariants.

1.6 Notation and convention

In this paper, all varieties and schemes are defined over $\mathbb{C}$. For a smooth variety or a complex manifold M, we denote by K_M its canonical divisor and by $\omega_M = \mathcal{O}_M(K_M)$ its canonical line bundle. For a smooth projective variety X and $\beta, \beta' \in H_2(X, \mathbb{Z})$, we write $\beta > \beta'$ if $\beta - \beta'$ is the class of an effective 1-cycle on X. For a projective morphism $f \colon Y \to Z$ of varieties, we denote by $\rho(Y/Z)$ its relative Picard number. When f is birational, its exceptional locus is denoted by $\mathrm{Ex}(f)$. For a scheme M, we denote by $D^b(M) := D^b(\mathrm{Coh}(M))$ the bounded derived category of coherent sheaves on M.

2. Review of birational geometry

In this section, we review some basics on birational geometry and recall several terminologies. A standard reference is [KM98].

2.1 Terminology from birational geometry

Let Y be a projective variety with at worst terminal singularities (for example, Y is smooth). A *minimal model program* (*MMP* for short) of Y is a sequence of birational maps

$$Y = Y_1 \dashrightarrow Y_2 \dashrightarrow \cdots \dashrightarrow Y_{N-1} \dashrightarrow Y_N \tag{2.1}$$

satisfying the following:

(i) Each Y_i is a projective variety with at worst terminal singularities.

(ii) Each birational map $Y_i \dashrightarrow Y_{i+1}$ is either a divisorial contraction or a flip.

(iii) The projective variety Y_N either is a minimal model, that is, K_{Y_N} is nef, or has a Mori fiber space structure $Y_N \to Z$. The former occurs if and only if the Kodaira dimension of Y is non-negative.

Here a line bundle L on a variety Y is called *nef* if for any projective curve $C \subset Y$, we have $\deg(L|_C) \geqslant 0$. A Cartier divisor D on Y is also called *nef* if the associated line bundle $\mathcal{O}_Y(D)$ is nef. The minimal model Y_N is not necessary unique, but two birational minimal models are known to be connected by a sequence of flops [Kaw08].

The above notions in birational geometry are summarized in the following definitions. In this paper, we only treat the case of smooth varieties, but the definitions are the same for varieties with terminal singularities.

Definition 2.1. Let Y be a smooth variety (respectively, complex manifold) and $f\colon Y \to Z$ a projective morphism of varieties (respectively, analytic spaces) with $\rho(Y/Z) = 1$ and $f_*\mathcal{O}_Y = \mathcal{O}_Z$. Then f is called

(i) a *divisorial contraction* if $\dim Y = \dim Z$ (that is, f is birational or bimeromorphic), $-K_Y$ is f-ample, and $\mathrm{Ex}(f)$ is a divisor;

(ii) an (*anti*) *flipping contraction* if $\dim Y = \dim Z$, $-K_Y$ (respectively, K_Y) is f-ample, and f is isomorphic in codimension 1;

(iii) a *flopping contraction* if $\dim Y = \dim Z$, f is crepant (that is, $K_Y \cdot C = 0$ for any curve $C \subset Y$ such that $f(C)$ is a point), and f is isomorphic in codimension 1;

(iv) a *MFS* if $\dim Z < \dim Y$ and $-K_Y$ is f-ample.

We can formulate the relevant birational transformations using the diagram (2.2) below in a unified way.

Definition 2.2. Let Y^+, Y^- be smooth varieties (or complex manifolds), and consider a diagram

$$\begin{tikzcd} Y^+ \arrow[dr, "f^+"'] & & Y^- \arrow[dl, "f^-"] \\ & Z, & \end{tikzcd} \tag{2.2}$$

where f^+, f^- are projective morphisms of varieties (respectively, analytic spaces) satisfying $f^{\pm}_*\mathcal{O}_{Y^{\pm}} = \mathcal{O}_Z$ if $Y^{\pm} \neq \emptyset$. Then the diagram (2.2) is called

(i) a *divisorial contraction* if f^+ is a divisorial contraction and f^- is an isomorphism;

(ii) a *flip* if f^+ is a flipping contraction and f^- is an anti-flipping contraction;

(iii) a *flop* if f^+, f^- are flopping contraction and the birational map $Y^+ \dashrightarrow Y^-$ is not an isomorphism;

(iv) a *MFS* if f^+ is a MFS and $Y^- = \emptyset$.

2.2 Generalized flips, flops and MFS

We also use the following generalized terminology, without assuming the condition of relative Picard numbers, etc.

Definition 2.3. In the situation of Definition 2.2, we call a diagram (2.2)

(i) a *generalized flip* if $f^{\pm}$ are birational (or bimeromorphic) morphisms, $-K_{Y^+}$ is f^+-ample, and K_{Y^-} is f^--ample;

(ii) a *generalized flop* if $f^{\pm}$ are crepant birational (or bimeromorphic) morphisms, isomorphisms in codimension 1, and there exists a f^+-ample divisor on Y^+ whose strict transform to Y^- is f^--anti-ample;

(iii) a *generalized MFS* if $-K_{Y^+}$ is f^+-ample and $Y^- = \emptyset$.

Remark 2.4. In the definition of a generalized flip, we do not assume that f^+ is isomorphic in codimension 1. For example, a divisorial contraction is a generalized flip. On the other hand, the morphism f^- for a generalized flip is always an isomorphism in codimension 1 by [KM98, Lemma 3.38].[1]

Remark 2.5. The conditions (i) and (ii) in Definition 2.3 are not complementary. Indeed, a diagram (2.2) is both of generalized flip and generalized flop if and only if f^+ and f^- are isomorphisms.

Remark 2.6. In the definition of a generalized MFS, we do not assume $\dim Z < \dim Y^+$, so f^+ can be birational. Indeed, such a case may happen in wall-crossing of stable pair moduli spaces in Section 9 (see Remark 9.16).

Remark 2.7. If the diagram (2.2) is a generalized flip, by applying a MMP relative to Z proved in [BCHM10], the birational map $Y^+ \dashrightarrow Y^-$ decomposes into divisorial contractions and flips over Z (though the intermediate varieties may have terminal singularities). Similarly, a generalized flop decomposes into flops over Z by [Kaw08].

2.3 Inequalities of canonical divisors

Following Kawamata (for example, see [Kaw18]), we introduce the inequalities of canonical divisors (with a slight modification allowing empty sets).

Definition 2.8. In the situation of Definition 2.2, we write

(i) $Y^+ >_K Y^-$ if either $Y^- = \emptyset$ (while $Y^+ \neq \emptyset$), or there is a commutative diagram

$$\begin{tikzcd} & W \arrow[dl, "g^+"'] \arrow[dr, "g^-"] & \\ Y^+ \arrow[dr, "f^+"'] & & Y^- \arrow[dl, "f^-"] \\ & Z & \end{tikzcd} \tag{2.3}$$

such that $g^{\pm}$ are birational and $(g^+)^*K_{Y^+} - (g^-)^*K_{Y^-}$ is linearly equivalent to an effective divisor on W;

(ii) $Y^+ =_K Y^-$ if either $Y^{\pm} = \emptyset$, or there is a commutative diagram (2.3) for birational maps $g^{\pm}$ such that $(g^+)^*K_{Y^+}$ and $(g^-)^*K_{Y^-}$ are linearly equivalent.

[1]The reference for this fact was pointed out to the author by Chen Jiang.

The canonical divisors decrease by divisorial contractions and flips, while flops keep them (see [KM98, Lemma 3.38]). Therefore, a MMP is interpreted as a process decreasing the canonical divisors; that is, for a MMP (2.1), we have the inequalities of canonical divisors

$$Y = Y_1 >_K Y_2 >_K \cdots >_K Y_{N-1} >_K Y_N \,.$$

Moreover, we have $Y_N =_K Y'_{N'}$ if $Y'_{N'}$ is another birational minimal model of Y. By Remark 2.7 (or by using [KM98, Lemma 3.38]), if the diagram (2.2) is a generalized flip (respectively, generalized flop), we have $Y^+ >_K Y^-$ (respectively, $Y^+ =_K Y^-$).

3. Birational transformations for d-critical loci

The notion of d-critical loci was introduced by Joyce [Joy15] as a classical shadow of (-1)-shifted symplectic derived schemes [PTVV13]. In this section, we recall its definition and introduce an analog of the birational transformations in Section 2 for d-critical loci.

3.1 D-critical locus

Let M be a complex scheme (respectively, complex analytic space). In [Joy15], it is proved that there exists a canonical sheaf of $\mathbb{C}$-vector spaces $\mathcal{S}_M$ on M satisfying the following property: for any Zariski (respextively, analytic) open subset $U \subset M$ and a closed embedding $i\colon U \hookrightarrow Y$ into a smooth scheme (respectively, complex manifold) Y, there is an exact sequence

$$0 \longrightarrow \mathcal{S}_M|_U \longrightarrow \mathcal{O}_Y/I^2 \xrightarrow{d_{\mathrm{DR}}} \Omega_Y/I\cdot\Omega_Y \,.$$

Here $I \subset \mathcal{O}_Y$ is the ideal sheaf that defines U, and d_{DR} is the de Rham differential. Moreover, there is a natural decomposition $\mathcal{S}_M = \mathcal{S}_M^0 \oplus \mathbb{C}_M$, where $\mathbb{C}_M$ is the constant sheaf on M. The sheaf $\mathcal{S}_M^0$ restricted to U is the kernel of the composition

$$\mathcal{S}_M|_U \hookrightarrow \mathcal{O}_Y/I^2 \twoheadrightarrow \mathcal{O}_{U^{\mathrm{red}}} \,.$$

For example, suppose that $w\colon Y \to \mathbb{C}$ is an algebraic (respectively, holomorphic) function such that

$$U = \{dw = 0\}\,, \quad w|_{U^{\mathrm{red}}} = 0\,. \tag{3.1}$$

Then $I = (dw)$ and $w + (dw)^2$ is an element of $\Gamma\big(U, \mathcal{S}_M^0|_U\big)$.

Definition 3.1 ([Joy15]). A pair (M, s) for a complex scheme (respectively, analytic space) M and $s \in \Gamma\big(M, \mathcal{S}_M^0\big)$ is called an *algebraic* (respectively, *analytic*) *d-critical locus* if for any $x \in M$, there exist a Zariski (respectively, analytic) open neighborhood $x \in U \subset M$, a closed embedding $i\colon U \hookrightarrow Y$ into a smooth scheme (respectively, complex manifold) Y and an element $w \in \Gamma(\mathcal{O}_Y)$ satisfying (3.1) such that $s|_U = w + (dw)^2$ holds. In this case, the data

$$(U, Y, w, i) \tag{3.2}$$

is called a *d-critical chart*. The section s is called a *d-critical structure* of M.

Remark 3.2. If M is smooth, then $\mathcal{S}_M^0 = 0$, so there is a unique (trivial) choice of its d-critical structure, $s = 0$.

Given a d-critical locus (M, s), there exists a line bundle $\omega_{M,s}$ on M^{red} called the *virtual canonical line bundle*[2] (see [Joy15, Section 2.4]). It satisfies that, for any d-critical chart (3.2),

[2] In [Joy15, Section 2.4], it was just called the canonical line bundle. We put "virtual" in order to distinguish it from the usual canonical line bundle.

there is a natural isomorphism

$$\omega_{M,s}|_{U^{\mathrm{red}}} \xrightarrow{\cong} \omega_Y^{\otimes 2}|_{U^{\mathrm{red}}} . \tag{3.3}$$

Remark 3.3. For a derived scheme M^{der} with a (-1)-shifted symplectic structure [PTVV13], its classical truncation M carries a canonical d-critical structure by [BBBJ15]. In this case, the virtual canonical line bundle of M is the determinant of the cotangent complex of M^{der} restricted to M.

We introduce the following relative version of a d-critical chart.

Definition 3.4. Let (M, s) be an algebraic (respectively, analytic) d-critical locus and $\pi \colon M \to A$ a morphism of schemes (respectively, analytic spaces). For a Zariski (respectively, analytic) open subset $U \subset A$, suppose that there is a commutative diagram

$$\begin{tikzcd} \pi^{-1}(U) \arrow[r, hook, "i"] \arrow[d, "\pi"] & Y \arrow[d, "f"] \arrow[dr, "w"] & \\ U \arrow[r, hook, "j"'] & Z \arrow[r, "g"'] & \mathbb{C}\, , \end{tikzcd} \tag{3.4}$$

where $f \colon Y \to Z$ is a morphism of schemes (respectively, analytic spaces), Y is smooth, i and j are closed immersions, and $g \in \Gamma(\mathcal{O}_Z)$ is such that the data $\left(\pi^{-1}(U), Y, w, i\right)$ is a d-critical chart. In this case, we call the diagram (3.4) a *π-relative d-critical chart.*

3.2 D-critical birational transformations

We formulate the terminology of birational contractions for d-critical loci, using relative d-critical charts from Definition 3.4.

Definition 3.5. Let (M, s) be an algebraic (respectively, analytic) d-critical locus and

$$\pi \colon M \longrightarrow A \tag{3.5}$$

a morphism of schemes (respectively, analytic spaces). We call the morphism (3.5) an *algebraic* (respectively, *analytic*) *d-critical divisorial contraction, d-critical* (*anti*) *flipping contraction, d-critical flopping contraction at a point* $p \in A$ if there exist a Zariski (respectively, analytic) open neighborhood $U \subset A$ of p and a π-relative d-critical chart (3.4) such that $f \colon Y \to Z$ is a divisorial contraction, (anti) flipping contraction, flopping contraction, MFS, respectively, as in Definition 2.1.

We call the morphism (3.5) an *algebraic* (respectively, *analytic*) *d-critical divisorial contraction, d-critical* (*anti*) *flipping contraction, d-critical flopping contraction, d-critical MFS*, if the above corresponding condition holds for any $p \in A$.

A d-critical birational contraction need not to be birational between the underlying spaces. Indeed, we have the following example.

Example 3.6. Let $U^{\pm}$ be the following affine schemes with d-critical structures $s^{\pm}$

$$U^{\pm} := \operatorname{Spec} \mathbb{C}\left[x, y^{\pm}\right] / \left(xy^{\pm}, y^{\pm 2}\right), \quad s^{\pm} = xy^{\pm 2} + \left(d\left(xy^{\pm 2}\right)\right)^2 .$$

By gluing U^+ and U^- at the smooth open subset $\operatorname{Spec} \mathbb{C}\left[x, x^{-1}\right]$, we obtain an algebraic d-critical locus (M, s) with $M = U^+ \cup U^-$ and $s|_{U^{\pm}} = s^{\pm}$. Note that $M^{\mathrm{red}} = \mathbb{P}^1$, and M is non-reduced at the points $\{0\}$ and $\{\infty\}$. The structure morphism $\pi \colon M \to \operatorname{Spec} \mathbb{C}$ is an algebraic d-critical divisorial contraction, though M and $\operatorname{Spec}\mathbb{C}$ are not birational in the usual sense. Indeed, we have the π-relative d-critical chart

$$\begin{tikzcd} M \arrow[r, hook, "i"] \arrow[d, "\pi"] & \widehat{\mathbb{C}}^2 \arrow[d, "f"] \arrow[dr, "w"] & \\ \operatorname{Spec}\mathbb{C} \arrow[r, hook, "j"'] & \mathbb{C}^2 \arrow[r, "g"'] & \mathbb{C}\,, \end{tikzcd}$$

where f is the blow-up at $0 \in \mathbb{C}^2$ and g is the function $g(u,v) = uv$.

We also formulate a d-critical version of birational transformations below.

Definition 3.7. Let (M^+, s^+) and (M^-, s^-) be algebraic (respectively, analytic) d-critical loci, and consider a diagram

$$\begin{tikzcd} M^+ \arrow[dr, "\pi^+"'] & & M^- \arrow[dl, "\pi^-"] \\ & A\,, & \end{tikzcd} \tag{3.6}$$

where $\pi^\pm$ are morphisms of schemes (respectively, analytic spaces). Then we call the diagram (3.6) an *algebraic* (respectively, *analytic*) *d-critical divisorial contraction, d-critical* (*generalized*) *flip, d-critical* (*generalized*) *flop, d-critical* (*generalized*) *MFS at* $p \in A$ if there exist a Zariski (respectively, analytic) open neighborhood $U \subset A$ of p and $\pi^\pm$-relative d-critical charts

$$\begin{tikzcd} (\pi^\pm)^{-1}(U) \arrow[r, hook, "i^\pm"] \arrow[d, "\pi^\pm"] & Y^\pm \arrow[d, "f^\pm"] \arrow[dr, "w^\pm"] & \\ U \arrow[r, hook, "j"'] & Z \arrow[r, "g"'] & \mathbb{C}\,, \end{tikzcd} \tag{3.7}$$

where $g \in \Gamma(\mathcal{O}_Z)$ and j are independent of $\pm$, such that the diagram

$$Y^+ \xrightarrow{f^+} Z \xleftarrow{f^-} Y^-$$

is a divisorial contraction, (generalized) flip, (generalized) flop, (generalized) MFS, as in Definitions 2.2 and 2.3, respectively.

We call the diagram (3.6) an *algebraic* (respectively, *analytic*) *d-critical divisorial contraction, d-critical* (*generalized*) *flip, d-critical* (*generalized*) *flop, d-critical MFS*, respectively, if the corresponding condition holds for any $p \in A$.

Here we give some examples of d-critical flips and d-critical flops.

Example 3.8. Let V^+, V^- be finite-dimensional $\mathbb{C}$-vector spaces with dimension a, b on which $\mathbb{C}^*$ acts by weight $1, -1$, respectively. We denote by $\vec{x} = (x_1, \ldots, x_a)$, $\vec{y} = (y_1, \ldots, y_b)$ coordinates of V^+, V^-, respectively. For $c \in \mathbb{Z}_{\geqslant 0}$, let $U = \mathbb{C}^c$ (respectively, an analytic open neighborhood $U \subset \mathbb{C}^c$ of 0) with a trivial $\mathbb{C}^*$-action. By taking geometric invariant theory (GIT) quotients of $V^+ \times V^- \times U$ by the $\mathbb{C}^*$-action with respect to the character $\pm\mathrm{id}\colon \mathbb{C}^* \to \mathbb{C}^*$, we obtain

$$\begin{aligned} Y^+ &:= \mathrm{Tot}_{\mathbb{P}(V^+)}\big(\mathcal{O}_{\mathbb{P}(V^+)}(-1) \otimes V^-\big) \times U\,, \\ Y^- &:= \mathrm{Tot}_{\mathbb{P}(V^-)}\big(\mathcal{O}_{\mathbb{P}(V^-)}(-1) \otimes V^+\big) \times U\,. \end{aligned}$$

Then by setting

$$Z := \operatorname{Spec} \mathbb{C}[x_i y_j \colon 1 \leqslant i \leqslant a, 1 \leqslant j \leqslant b] \times U\,,$$

we obtain the diagram

$$Y^+ \xrightarrow{f^+} Z \xleftarrow{f^-} Y^-\,, \tag{3.8}$$

which is a standard toric flip if $a > b \geqslant 2$ and a standard toric flop if $a = b \geqslant 2$ (see [Rei92]). Let us consider $g \in \Gamma(\mathcal{O}_Z)$ of the form

$$g = \sum_{i=1}^{a} \sum_{j=1}^{b} w_{ij}(\vec{u}) x_i y_j\,, \quad w_{ij}(\vec{u}) \in \Gamma(\mathcal{O}_U)\,.$$

We define $w^\pm$ by the commutative diagram

$$\begin{tikzcd} Y^+ \arrow[r, "f^+"] \arrow[dr, "w^+"'] & Z \arrow[d, "g"] & Y^- \arrow[l, "f^-"'] \arrow[dl, "w^-"] \\ & \mathbb{C}\,. & \end{tikzcd}$$

If we have $c \gg a, b$ and the $w_{ij}(\vec{u})$ are sufficiently general, then the critical loci

$$M^\pm := \big\{ dw^\pm = 0 \big\} \subset Y^\pm \tag{3.9}$$

are smooth of dimension $\pm(a-b)+c-1$. Moreover, $f^\pm\big(M^\pm\big)$ are contained in $\{0\} \times U$. Therefore, the diagram

$$M^+ \xrightarrow{\pi^+} U \xleftarrow{\pi^-} M^- \tag{3.10}$$

is an algebraic (respectively, analytic) d-critical flip if $a > b \geqslant 2$ and an algebraic (respectively, analytic) d-critical flop if $a = b \geqslant 2$. Here as $M^\pm$ are smooth, the d-critical structures $s^\pm$ on $M^\pm$ must be zero. Note that in the former case, the dimensions of $M^\pm$ are different. The fibers of $\pi^\pm$ at $u \in U$ are linear subspaces in $\mathbb{P}(V^\pm)$, whose dimensions depend on u.

Here is an example of analytic d-critical flips and flops for a diagram of smooth projective varieties.

Example 3.9. Let C be a smooth projective curve with genus g, and let $S^k(C)$ be the kth symmetric product of C:

$$S^k(C) := (\overbrace{C \times \cdots \times C}^{k})/\mathfrak{S}_k\,.$$

Note that $S^k(C)$ is a smooth projective variety with dimension k. Let $\operatorname{Pic}^k(C)$ be the moduli space of line bundles on C with degree k, which is a g-dimensional complex torus. For each $n > 0$, we consider the classical diagram of Abel–Jacobi maps

$$\begin{tikzcd} M_+ = S^{n+g-1}(C) \arrow[dr, "\pi^+"'] & & M_- = S^{-n+g-1}(C) \arrow[dl, "\pi^-"] \\ & \operatorname{Pic}^{n+g-1}(C)\,. & \end{tikzcd} \tag{3.11}$$

Here the morphisms $\pi^\pm$ are given by $\pi^+(Z \subset C) = \mathcal{O}_C(Z)$ and $\pi^-(Z' \subset C) = \omega_C(-Z')$.

The diagram (3.11) appears as a special case of wall-crossing of stable pair moduli spaces discussed in Theorem 9.22 (see Example 9.24). By loc. cit., at a point $[L] \in \mathrm{Pic}^{n+g-1}(C)$, we see that the diagram (3.11) is an analytic d-critical flip if $h^1(L) > 1$, an analytic d-critical divisorial contraction if $h^1(L) = 1$ and an analytic d-critical MFS if $h^1(L) = 0$. Moreover, relative d-critical charts are analytic-locally on $\mathrm{Pic}^{n+g-1}(C)$ given as in Example 3.8. Note that $S^{\pm n+g-1}(C)$ are smooth projective varieties, which are not birational for $n > 0$ (as the dimensions are different).

Here is an example of a d-critical flop between non-reduced d-critical loci.

Example 3.10. Let us consider the case $a = b = 2$ and $c = 0$ in Example 3.8. In this case, the diagram (3.8) is the simplest example of a flop, called an *Atiyah flop*. Let us take $g \in \Gamma(\mathcal{O}_Z)$ to be $g = x_1x_2y_1^2$ and define $M^\pm$ as in (3.9). We have d-critical structures on $M^\pm$ defined by $s^\pm = w^\pm + \left(dw^\pm\right)^2$ and an algebraic d-critical flop

$$M^+ \xrightarrow{\pi^+} Z \xleftarrow{\pi^-} M^- .$$

The schemes $M^\pm$ are described as follows. Let $\mathbb{P}^1 = C^\pm \subset Y^\pm$ be the exceptional loci of $f^\pm$. Then the reduced part of M^+ is a smooth divisor on Y^+ that contains C^+. However, M^+ is non-reduced along two disjoint curves on M^+. On the other hand, the reduced part of M^- is a union of C^- and a smooth divisor on Y^- that intersects C^- at a point y. The scheme M^- is non-reduced along two curves on the above divisor that intersect at y. The singularities of schemes $M^\pm$ are not treated in birational geometry.

Remark 3.11. As in Remark 2.5, the d-critical generalized flips and flops at $p \in A$ in Definition 3.7 include the case that both of f^+ and f^- in the diagram (3.6) are isomorphisms. In this case, the left vertical arrows in (3.7) are closed immersions. This also includes the case that $\left(\pi^\pm\right)^{-1}(U) = \emptyset$. Indeed, one can take a closed embedding $U \subset Z$ for a smooth Z that admits a smooth morphism $g\colon Z \to \mathbb{C}$. Then by taking $Y^\pm = Z$, $f^\pm = \mathrm{id}$ and $w^\pm = g$, we have $\left\{dw^\pm\right\} = \emptyset$.

Remark 3.12. In the notation of Definition 3.7, suppose that the function g satisfies $g \in m_0^2$, where $m_0 \subset \mathcal{O}_Z$ is the ideal sheaf of $0 := j(p) \in Z$. Then as $g_*\colon T_{Z,0} \to T_{\mathbb{C},g(0)}$ is a zero map, we see that (set-theoretically) $\left(f^\pm\right)^{-1}(0) \subset \left\{dw^\pm = 0\right\}$. In particular, if $f^\pm$ contracts a curve in $Y^\pm$ to a point $0 \in Z$, then it lies on $M^\pm$ and $\pi^\pm$ also contracts it to p.

Remark 3.13. If the condition of Remark 3.12 is not satisfied, it is possible that $\pi^\pm\colon M^\pm \to A$ do not contract any curve while $f^\pm$ do. For example, let us consider the diagram

$$\begin{tikzcd} \widehat{\mathbb{C}}^2 \arrow[r, "f^+"] \arrow[dr, "w^+"'] & \mathbb{C}^2 \arrow[d, "g"] & \mathbb{C}^2 \arrow[l, "\mathrm{id}"'] \arrow[dl, "w^-"] \\ & \mathbb{C}, & \end{tikzcd}$$

where f^+ is the blow-up at the origin and g is the projection onto one of the factors of $\mathbb{C}^2$. By taking the critical locus of $w^\pm$, we obtain the diagram

$$\left\{dw^+ = 0\right\} = \mathrm{Spec}\,\mathbb{C} \xrightarrow{\mathrm{id}} \left(0 \in \mathbb{C}^2\right) \longleftarrow \left\{dw^- = 0\right\} = \emptyset\,. \tag{3.12}$$

Although f^+ contracts a $\mathbb{P}^1$ to a point, this diagram does not contract curves.

In order to avoid situations as in Remarks 3.11 and 3.13, we introduce the following strict notion of birational transformations.

Definition 3.14. In the situation of Definition 3.7, we call a diagram (3.6) *strict at* $p \in A$ if $\dim \left(\pi^+\right)^{-1}(p) \geqslant 1$, that is, π^+ is not a finite morphism at p. We call a diagram (3.6) *strict* if it is strict at some $p \in A$, that is, π^+ is not a finite morphism.

Remark 3.15. The diagram (3.10) is strict if $a \geqslant 2$ by Remark 3.12. On the other hand, the diagram (3.12) is not strict.

3.3 D-critical MMP

We define the following d-critical version of a MMP as follows.

Definition 3.16. Let (M, s) be an algebraic (respectively, analytic) d-critical locus. A *d-critical MMP* of (M, s) is a sequence

$$\begin{tikzcd} M_1 \arrow[dr, "\pi_1^+"'] & & M_2 \arrow[dl, "\pi_1^-"'] \arrow[dr, "\pi_2^+"'] & & \cdots \arrow[dl] \arrow[dr] & & M_N \arrow[dl, "\pi_{N-1}^-"] \\ & A_1 & & A_2 & & A_{N-1}\,, & \end{tikzcd} \tag{3.13}$$

where each (M_i, s_i) is an algebraic (respectively, analytic) d-critical locus, $(M_1, s_1) = (M, s)$ are d-critical loci, and for each i, the diagram

$$M_i \xrightarrow{\pi_i^+} A_i \xleftarrow{\pi_i^-} M_{i+1} \tag{3.14}$$

is an algebraic (respectively, analytic) d-critical generalized flip at any point in $\operatorname{Im}\pi_i^-$ and d-critical generalized MFS at any point in $A_i \setminus \operatorname{Im}\pi_i^-$. A d-critical MMP is called *strict* if each diagram (3.14) is strict in the sense of Definition 3.14.

We give an example of a d-critical MMP from a usual MMP.

Example 3.17. Let $\mathcal{X}$ be a complex manifold with a projective morphism $f\colon \mathcal{X} \to \Delta$, where $0 \in \Delta \subset \mathbb{C}$ is a small disc. Suppose that $f^{-1}(t)$ is a smooth minimal model for any $t \in \Delta \setminus \{0\}$. Also suppose that we have a f-relative MMP of $\mathcal{X}$ over Δ

$$\mathcal{X} = \mathcal{X}_1 \dashrightarrow \mathcal{X}_2 \dashrightarrow \cdots \dashrightarrow \mathcal{X}_{N-1} \dashrightarrow \mathcal{X}_N\,,$$

where $\mathcal{X}_N \to \Delta$ is a minimal model over Δ such that each $\mathcal{X}_i$ is smooth. (For example, such a MMP always exists when $\dim \mathcal{X} = 2$.) Then each birational map $\mathcal{X}_i \dashrightarrow \mathcal{X}_{i+1}$ fits into the diagram

$$\begin{tikzcd} \mathcal{X}_i \arrow[r, "\pi_i^+"] \arrow[dr, "f_i"'] & \mathcal{Y}_i \arrow[d, "g_i"] & \mathcal{X}_{i+1} \arrow[l, "\pi_i^-"'] \arrow[dl, "f_{i+1}"] \\ & \Delta\,, & \end{tikzcd}$$

where the top sequence is either a divisorial contraction or a flip. Let $h\colon \Delta \to \mathbb{C}$ be defined by $t \mapsto t^2$, and set

$$w_i := h \circ f_i \colon \mathcal{X}_i \to \mathbb{C}\,, \quad M_i := \{dw_i = 0\}\,, \quad A_i := \mathcal{Y}_i \times_\Delta \operatorname{Spec}\mathbb{C}[t]/t^m$$

for $m \gg 0$. Note that M_i and A_i are projective schemes with $f_i(M_i) \subset A_i$ for $m \gg 0$, and M_i admits a d-critical structure $s_i = w_i + (dw_i)^2$. Then we obtain a d-critical MMP (3.13), which is strict by Remark 3.12.

Remark 3.18. In Example 3.17, note that $(f_i)^{-1}(0) = M_i$ as a set, but their scheme structures may be different. For example, if $(f_i)^{-1}(0)$ is a curve with a nodal singularity at $x \in (f_i)^{-1}(0)$, then the scheme structure of M_i at x is given by $\widehat{\mathcal{O}}_{M_i,x} = \mathbb{C}[[x,y]]/\big(x^2y, xy^2\big)$, which is a critical locus of the function x^2y^2 and not isomorphic to the nodal singularity.

As an analogy of a minimal model in birational geometry, we introduce the following notion of minimal d-critical loci.

DEFINITION 3.19. A d-critical locus (M,s) is called *minimal* if the virtual canonical line bundle $\omega_{M,s}$ is nef.

EXAMPLE 3.20. (i) For a d-critical locus (M,s), if M is smooth, then $\omega_{M,s} = \omega_M^{\otimes 2}$. Therefore, (M,s) is minimal if and only if K_M is nef, that is, M is minimal in the usual sense.

(ii) In the situation of Example 3.17, the d-critical locus (M_N, s_N) is minimal, as $\omega_{M_N,s_N} = \omega_{\mathcal{X}_N}^{\otimes 2}|_{M_N}$ is nef. Note that M_N is a projective singular scheme if $(f_N)^{-1}(0)$ is singular.

(iii) There is also an example of a singular projective minimal d-critical locus (M,s) such that M^{red} is a smooth non-minimal model. Let Z be the A_1 surface singularity $Z = \big\{xy + z^2 = 0\big\} \subset \mathbb{C}^3$, and take the blow-up $f\colon Y \to Z$ at the origin, which is a crepant resolution of the singularity $0 \in Z$. We consider the commutative diagram

$$\begin{tikzcd} Y \arrow[r, "f"] \arrow[dr, "w"'] & Z \arrow[d, "(x,y,z)\mapsto x^2+y^2+z^2"] \\ & \mathbb{C}\,. \end{tikzcd}$$

We have the following d-critical locus:

$$(M,s)\,, \quad M = \{dw = 0\}\,, \quad s = w + (dw)^2\,.$$

Then $M^{\mathrm{red}} = \mathbb{P}^1$, and there are four points in $\mathbb{P}^1$ at which M is non-reduced. At these points, the scheme structure of M is given by the critical locus of the function $(x,y) \mapsto xy^2$ on $\mathbb{C}^2$. Note that $\omega_{M,s} = \omega_Y^{\otimes 2}|_{M^{\mathrm{red}}} \cong \mathcal{O}_{\mathbb{P}^1}$ as ω_Y is trivial. Therefore, the d-critical locus (M,s) is minimal, while M^{red} is not minimal.

As for the usual minimal model in birational geometry, we have the following lemma.

LEMMA 3.21. *Let $\big(M^+, s^+\big)$ be a minimal d-critical locus. Then there is no strict diagram*

$$M^+ \xrightarrow{\pi^+} A \xleftarrow{\pi^-} M^- \tag{3.15}$$

for a d-critical locus (M^-, s^-) that is a d-critical generalized flip at any point in $\operatorname{Im}\pi^-$ and a d-critical generalized MFS at any point in $A \setminus \operatorname{Im}\pi^-$.

Proof. Suppose that a diagram (3.15) exists. As π^+ is not a finite morphism, there is a projective curve $C \subset M^+$ such that $\pi(C) = p$ for some point $p \in A$. Let us take $\pi^{\pm}$-relative d-critical charts (3.4) for an open neighborhood $p \in U \subset A$. Then as $-K_{Y^+}$ is f-ample, we have

$$\deg(\omega_{M^+,s^+}|_C) = 2K_{Y^+} \cdot i^+(C) < 0\,,$$

which contradicts that $\big(M^+, s^+\big)$ is minimal. □

Remark 3.22. By Lemma 3.21, a strict *d*-critical MMP (3.13) terminates at M_N if M_N is either minimal or an empty set. In the latter case, the morphism $M_{N-1} \to A_{N-1}$ is a *d*-critical generalized MFS at any point in A_{N-1}.

We generalize the inequality of canonical divisors in Definition 2.8 to virtual canonical line bundles on d-critical loci.

Definition 3.23. In the situation of Definition 3.7, we write

$$\left(M^{+}, s^{+}\right) \geqslant_K \left(M^{-}, s^{-}\right) \tag{3.16}$$

if for any $p \in A$, there is a $\pi^{\pm}$-relative d-critical chart (3.7) such that $Y^{+} \geqslant_K Y^{-}$ as in Definition 2.8. The inequality (3.16) is strict if $Y^{+} >_K Y^{-}$ for some $p \in A$.

Remark 3.24. By (3.3), the inequality (3.16) is regarded as an inequality for virtual canonical bundles of d-critical loci.

If the diagram (3.6) is a d-critical divisorial contraction, (generalized) flip or (generalized) MFS, we have the inequality (3.16). In particular, for a d-critical MMP (3.13), we have the inequalities

$$(M_1, s_1) \geqslant_K (M_2, s_2) \geqslant_K \cdots \geqslant_K (M_N, s_N)\,.$$

All inequalities are strict if (3.13) is a strict d-critical MMP. Moreover, we have the equality of (3.16) if the diagram (3.6) is a d-critical (generalized) flop.

4. Moduli spaces of semistable objects on CY 3-folds

In this section, we discuss moduli spaces of Bridgeland semistable objects on CY 3-folds and introduce their wall-crossing diagrams as in the introduction. We address the general question of whether wall-crossing diagrams in CY 3-folds are described in terms of d-critical birational transformations introduced in Section 3, which is a main topic in this paper.

4.1 Moduli spaces of objects on CY 3-folds

Let X be a smooth projective CY 3-fold; that is, $\dim X = 3$, $K_X = 0$ and $H^1(\mathcal{O}_X) = 0$. Below we fix a trivialization

$$\mathcal{O}_X \xrightarrow{\cong} \omega_X\,. \tag{4.1}$$

We denote by $\mathcal{M}$ the 2-functor

$$\mathcal{M} \colon \mathcal{S}\mathrm{ch}/\mathbb{C} \longrightarrow \mathcal{G}\mathrm{roupoid}$$

sending a $\mathbb{C}$-scheme S to the groupoid of relatively perfect objects

$$\mathcal{E} \in D^b(X \times S) \tag{4.2}$$

such that for each $s \in S$, the derived restriction $\mathcal{E}_s$ to $X \times \{s\}$ satisfies $\mathrm{Ext}^{<0}(\mathcal{E}_s, \mathcal{E}_s) = 0$. By the result of Lieblich [Lie06], the 2-functor $\mathcal{M}$ is an Artin stack locally of finite type. We have the open substack $\mathcal{M}^{\mathrm{si}} \subset \mathcal{M}$ consisting of simple objects, that is, substacks of objects (4.2) that furthermore satisfy $\mathrm{Hom}(\mathcal{E}_s, \mathcal{E}_s) = \mathbb{C}$ for any $s \in S$. Then by [Lie06, Corollary 4.3.3] (also see [Ina02]), there is an algebraic space M^{si} locally of finite type with a morphism

$$\mathcal{M}^{\mathrm{si}} \longrightarrow M^{\mathrm{si}}\,, \tag{4.3}$$

which is an étale-locally trivial $B\mathbb{C}^*$-bundle (that is, $\mathbb{C}^*$-gerbe). By the result of [BBBJ15], we have the following.

Theorem 4.1 ([BBBJ15]). *There is a canonical d-critical structure on the stack $\mathcal{M}$ whose virtual canonical line bundle is given by*

$$\omega_{\mathcal{M}}^{\mathrm{vir}} := \det \mathbf{R}\mathcal{H}om_{\mathrm{pr}_{\mathcal{M}}}(\mathcal{E}, \mathcal{E})\,.$$

Here $\mathcal{E}$ is a universal sheaf on $X \times \mathcal{M}$, and $\mathrm{pr}_{\mathcal{M}} \colon X \times \mathcal{M} \to \mathcal{M}$ is the projection. The restriction of the above d-critical structure to $\mathcal{M}^{\mathrm{si}} \subset \mathcal{M}$ descends to a d-critical structure on M^{si}.

Remark 4.2. More precisely, the d-critical structure on $\mathcal{M}$ in Theorem 4.1 is canonically determined once we choose a trivialization (4.1).

Remark 4.3. The last statement that the d-critical structure on $\mathcal{M}^{\mathrm{si}}$ descends to M^{si} is not mentioned in [BBBJ15]. Here is a brief argument suggested by a referee.

Let $\mathfrak{M}^{\mathrm{si}}$ be a derived enhancement of $\mathcal{M}^{\mathrm{si}}$ that is (-1)-shifted symplectic. There is a Hamiltonian $B\mathbb{C}^*$-action on $\mathfrak{M}^{\mathrm{si}}$ by scaling automorphisms, and its moment map is

$$\mathfrak{M}^{\mathrm{si}} \longrightarrow \mathfrak{g}^*[-1] = \mathrm{Spec}\,\mathbb{C}[\xi]\,,$$

where $\mathfrak{g}$ is the tangent complex of $B\mathbb{C}^*$ and $\deg(\xi) = -2$. The (-1)-shifted symplectic reduction is

$$\mathfrak{M}^{\mathrm{si}}_{\mathrm{red}} := \left[\mathfrak{M}^{\mathrm{si}}/B\mathbb{C}^*\right] \times_{[\mathfrak{g}^*[-1]/B\mathbb{C}^*]} \mathfrak{g}^*[-1]\,,$$

which is (-1)-shifted symplectic as it is a Lagrangian intersection over the 0-shifted symplectic derived stack $[\mathfrak{g}^*[-1]/B\mathbb{C}^*]$ (see [AC21, Section 2.2]). As its classical truncation is M^{si}, we have an induced d-critical structure on M^{si} that is a descendant of the d-critical structure on $\mathcal{M}^{\mathrm{si}}$.

4.2 Moduli spaces of Bridgeland semistable objects

We define $\Gamma_X \subset H^{2*}(X, \mathbb{Q})$ to be the image of the Chern character map

$$\mathrm{ch} \colon K(X) \longrightarrow H^{2*}(X, \mathbb{Q})\,.$$

Let $\mathrm{Stab}(X)$ be the space of Bridgeland stability conditions on the derived category $D^b(X)$ with respect to the Chern character map $\mathrm{ch} \colon K(X) \to \Gamma_X$ (see Appendix A.3). By definition, a point $\sigma \in \mathrm{Stab}(X)$ is written as

$$\sigma = (Z, \mathcal{A}) \in \mathrm{Stab}(X)\,, \quad \mathcal{A} \subset D^b(X)\,, \quad Z \colon \Gamma_X \longrightarrow \mathbb{C}\,, \tag{4.4}$$

where $\mathcal{A}$ is the heart of a bounded t-structure and Z is a group homomorphism, both satisfying some conditions.

Remark 4.4. So far it is not known whether $\mathrm{Stab}(X) \neq \emptyset$ for a projective CY 3-fold in general. At the present time, this is only known when X is an étale quotient of an abelian 3-fold [MP15, BMS16]. On the other hand, we can generalize the arguments below in a modified situation where it is easier to construct stability conditions, for example, $\mathrm{Stab}(X)$ for a non-compact CY 3-fold, $\mathrm{Stab}(\mathcal{D})$ for a triangulated subcategory $\mathcal{D} \subset D^b(X)$, etc.

For a stability condition σ as in (4.4) and an element $v \in \Gamma_X$, we have the substacks

$$\mathcal{M}^s_\sigma(v) \subset \mathcal{M}_\sigma(v) \subset \mathcal{M}\,, \tag{4.5}$$

where $\mathcal{M}_\sigma(v)$ consists of the σ-semistable objects $E \in \mathcal{A}$ with $\mathrm{ch}(E) = v$ and $\mathcal{M}^s_\sigma(v)$ is the σ-stable part of $\mathcal{M}_\sigma(v)$. Below we continue the discussion under the following assumption.

Assumption 4.5. The substacks (4.5) are open substacks of $\mathcal{M}$, and they are of finite type.

Remark 4.6. Assumption 4.5 holds in the cases where $\mathrm{Stab}(X)$ is known to be non-empty (see [PT19]). As proven in [PT19], the Bogomolov–Gieseker type inequality conjecture proposed in [BMT14, BMS16] implies both the constructions of stability conditions and Assumption 4.5.

Under Assumption 4.5, we can discuss good moduli spaces for Artin stacks (4.5) in the sense of [Alp13]. A general definition is as follows.

Definition 4.7 ([Alp13]). A morphism $p\colon \mathcal{M} \to M$, where $\mathcal{M}$ is an Artin stack and M an algebraic space, is called a *good moduli space* for $\mathcal{M}$ if the following conditions hold:

(i) The morphism p is quasi-compact, and $p_*\colon\ \mathrm{QCoh}(\mathcal{M}) \to \mathrm{QCoh}(M)$ is exact.

(ii) The natural map $\mathcal{O}_M \to p_*\mathcal{O}_{\mathcal{M}}$ is an isomorphism.

A good moduli space $p\colon \mathcal{M} \to M$ is universal for morphisms to algebraic spaces (see [Alp13, Theorem 6.6]). Namely, for a morphism $p'\colon \mathcal{M} \to M'$ for another algebraic space M', there is a unique factorization $p'\colon \mathcal{M} \stackrel{p}{\to} M \to M'$.

Let us return to moduli stacks of (semi)stable objects (4.5). We will use the following result which is proved in [AHH21, Theorem 7.25].

Theorem 4.8 ([AHH21]). *The stack $\mathcal{M}_\sigma(v)$ admits a good moduli space*

$$p_M\colon \mathcal{M}_\sigma(v) \longrightarrow M_\sigma(v)$$

for a separated algebraic space $M_\sigma(v)$ of finite type.

The good moduli space $M_\sigma(v)$ is the coarse moduli space of S-equivalence classes of σ-semistable objects with Chern character v. It follows that there is a one-to-one correspondence between closed points of $M_\sigma(v)$ and σ-polystable objects, that is, σ-semistable objects in $\mathcal{A}$ with Chern character v, isomorphic to direct sums of σ-stable objects.

Remark 4.9. The existence of the good moduli space is well known for moduli stacks given as GIT quotient stacks, say moduli stacks of Gieseker semistable sheaves (see [HL97]). In this case, we can proceed with the arguments without relying on [AHH21]. In general, it is not known whether $\mathcal{M}_\sigma(v)$ can be constructed as a GIT quotient stack, so we rely on [AHH21] for the existence of $M_\sigma(v)$.

Let $M^s_\sigma(v) \subset M_\sigma(v)$ be the open subspace consisting of the σ-stable objects. Then we have the morphism

$$p_M\colon \mathcal{M}^s_\sigma(v) \longrightarrow M^s_\sigma(v) \tag{4.6}$$

giving a good moduli space for $\mathcal{M}^s_\sigma(v)$. Note that $M^s_\sigma(v)$ is also an open subspace of the moduli space of simple objects M^{si} in Subsection 4.2, and the morphism (4.6) is a pull-back of the morphism (4.3) by the open immersion $M^s_\sigma(v) \subset M^{\mathrm{si}}$.

4.3 Wall-crossing diagram in CY 3-folds

By a general theory of Bridgeland stability conditions, there is a collection of locally finite codimension 1 submanifolds (called *walls*)

$$\{\mathcal{W}_\lambda\}_{\lambda\in\Lambda}, \quad \mathcal{W}_\lambda \subset \mathrm{Stab}(X)$$

such that the moduli stack $\mathcal{M}_\sigma(v)$ is constant if σ lies in a connected component of the complements of walls (called *chamber*) but may change if σ crosses a wall. Each wall $\mathcal{W}_\lambda$ is defined by the condition

$$Z(v_1) \in \mathbb{R}_{>0} Z(v_2)\,, \quad v_1 + v_2 = v \in \Gamma_X\,,$$

where v_1, v_2 are not proportional in $(\Gamma_X)_{\mathbb{Q}}$.

Below, we take $v \in \Gamma_X$ to be primitive; that is, v is not written as a multiple of some element in Γ_X. Let us take

$$\sigma = (Z, \mathcal{A}) \in \mathrm{Stab}(X)\,, \quad \sigma^{\pm} = \left(Z^{\pm}, \mathcal{A}^{\pm}\right) \in \mathrm{Stab}(X)\,,$$

where σ lies on a wall and $\sigma^{\pm}$ lie on its adjacent chambers. By applying $\mathbb{C}$-action on $\mathrm{Stab}(X)$ if necessary (see Remark A.5), we may assume $\Im Z(v) > 0$. Then any $\sigma^{\pm}$-semistable object $E \in \mathcal{A}^{\pm}$ with $\mathrm{ch}(E) = v$ is a σ-semistable object in $\mathcal{A}$. So we have open immersions $\mathcal{M}_{\sigma^{\pm}}(v) \subset \mathcal{M}_{\sigma}(v)$ and the commutative diagram

$$\begin{tikzcd} \mathcal{M}_{\sigma^+}(v) \arrow[r, hook] \arrow[d, "p_M^+"'] & \mathcal{M}_{\sigma}(v) \arrow[d, "p_M"] & \mathcal{M}_{\sigma^-}(v) \arrow[l, hook'] \arrow[d, "p_M^-"] \\ M_{\sigma^+}(v) \arrow[r, "q_M^+"'] & M_{\sigma}(v) & M_{\sigma^-}(v)\,. \arrow[l, "q_M^-"] \end{tikzcd} \tag{4.7}$$

Here the top arrows are open immersions, the vertical arrows are morphisms to the good moduli spaces, and the bottom arrows are induced by the universality of good moduli spaces. Moreover, we have $M^s_{\sigma^{\pm}}(v) = M_{\sigma^{\pm}}(v)$ as v is primitive and $\sigma^{\pm}$ lie on chambers. In particular, $M_{\sigma^{\pm}}(v)$ admit d-critical structures by Theorem 4.1, and the following question makes sense.

Question 4.10. For a primitive $v \in \Gamma_X$, is the bottom sequence in (4.7) a (generalized) d-critical flip or flop?

In what follows, we study Question 4.10 via moduli spaces of representations of Ext-quivers.

5. Representations of quivers with super-potentials

In this section, we construct d-critical birational transformations introduced in Section 4 via representations of quivers with convergent super-potentials. The descriptions in this section will give analytic-local models of d-critical birational transformations for wall-crossing diagrams in CY 3-folds considered in Section 4.

5.1 Representations of quivers

Recall that a *quiver* Q consists of data $Q = (V(Q), E(Q), s, t)$, where $V(Q)$, $E(Q)$ are finite sets and $s, t\colon E(Q) \to V(Q)$ are maps. The set $V(Q)$ is the set of vertices, and $E(Q)$ is the set of edges. For $e \in E(Q)$, the vertex $s(e)$ is the source of e, and $t(e)$ is the target of e. For $i, j \in V(Q)$, we use the following notation:

$$E_{i,j} := \{e \in E(Q)\colon s(e) = i, t(e) = j\}\,, \quad \mathbb{E}_{i,j} := \bigoplus_{e \in E_{i,j}} \mathbb{C} \cdot e\,; \tag{5.1}$$

that is, $E_{i,j}$ is the set of edges from i to j, and $\mathbb{E}_{i,j}$ is the $\mathbb{C}$-vector space spanned by $E_{i,j}$. The *dual quiver* $Q^{\vee}$ of Q is defined by

$$Q^{\vee} := \left(V(Q), E(Q), s^{\vee}, t^{\vee}\right)\,, \quad s^{\vee} := t\,, \quad t^{\vee} := s\,.$$

Recall that a *path* of a quiver Q is a composition of edges in Q

$$e_1 e_2 \ldots e_n\,, \quad e_i \in E(Q)\,, \quad t(e_i) = s(e_{i+1})\,.$$

The number n above is called the *length* of the path. The *path algebra* of a quiver Q is a $\mathbb{C}$-vector space spanned by paths in Q:

$$\mathbb{C}[Q] := \bigoplus_{n \geqslant 0} \bigoplus_{e_1,\ldots,e_n \in E(Q),\, t(e_i)=s(e_{i+1})} \mathbb{C} \cdot e_1 e_2 \ldots e_n\,.$$

Here a path of length zero is a trivial path at each vertex of Q, and the product on $\mathbb{C}[Q]$ is defined by the composition of paths.

A *Q-representation* consists of data

$$\mathbb{V} = \left\{(V_i, u_e)\colon i \in V(Q),\, e \in E(Q),\, u_e\colon V_{s(e)} \to V_{t(e)}\right\}, \tag{5.2}$$

where V_i is a finite-dimensional $\mathbb{C}$-vector space and u_e is a linear map. It is well known that a Q-representation is nothing but a left $\mathbb{C}[Q]$-module structure on $\bigoplus_{i \in V(Q)} V_i$. Also note that the dual of (5.2),

$$\mathbb{V}^\vee := \left\{\left(V_i^\vee, u_e^\vee\right)\colon i \in V(Q),\, e \in E(Q),\, u_e^\vee\colon V_{t(e)}^\vee \to V_{s(e)}^\vee\right\}, \tag{5.3}$$

is a $Q^\vee$-representation.

For a Q-representation $\mathbb{V}$ as in (5.2), the vector

$$\vec{m} = (m_i)_{i \in V(Q)}\,, \quad m_i = \dim V_i \tag{5.4}$$

is called the *dimension vector* of $\mathbb{V}$. For each $i \in V(Q)$, let S_i be the 1-dimensional Q-representation corresponding to the vertex i, whose dimension vector is denoted by $\vec{i}$. We set

$$\Gamma_Q := \bigoplus_{i \in V(Q)} \mathbb{Z} \cdot \vec{i}.$$

Note that the dimension vector (5.4) for a non-zero Q-representation (5.2) takes its value in the positive cone

$$\Gamma_{Q,>0} := \left\{\vec{m} = (m_i)_{i \in V(Q)} \in \Gamma_Q \colon m_i \geqslant 0\right\} \setminus \{0\} \subset \Gamma_Q\,.$$

For a given element $\vec{m} = (m_i)_{i \in V(Q)} \in \Gamma_{Q,>0}$, let V_i be $\mathbb{C}$-vector spaces with dimension m_i. Let us set

$$G := \prod_{i \in V(Q)} \mathrm{GL}(V_i)\,, \quad \mathrm{Rep}_Q(\vec{m}) := \prod_{e \in E(V)} \mathrm{Hom}(V_{s(e)}, V_{t(e)})\,.$$

The algebraic group G acts on $\mathrm{Rep}_Q(\vec{m})$ by

$$g \cdot u = \left\{g_{t(e)}^{-1} \circ u_e \circ g_{s(e)}\right\}_{e \in E(Q)}$$

for $g = (g_i)_{i \in V(Q)} \in G$ and $u = (u_e)_{e \in E(Q)}$. A Q-representation with dimension vector $\vec{m}$ is determined by a point in $\mathrm{Rep}_Q(\vec{m})$ up to G-action. The moduli stack of Q-representations with dimension vector $\vec{m}$ is given by the quotient stack $\mathcal{M}_Q(\vec{m}) := \left[\mathrm{Rep}_Q(\vec{m})/G\right]$. We have the natural morphism to the GIT quotient

$$p_Q\colon \mathcal{M}_Q(\vec{m}) \to M_Q(\vec{m}) := \mathrm{Rep}_Q(\vec{m}) /\!\!/ G\,. \tag{5.5}$$

Here, in general, if a reductive algebraic group G acts on an affine scheme $Y = \operatorname{Spec} R$, its GIT quotient is given by $Y /\!\!/ G := \operatorname{Spec}\left(R^G\right)$. A closed point of $M_Q(\vec{m})$ corresponds to a semi-simple Q-representation, that is, a direct sum of simple Q-representations, and p_Q sends a Q-representation to its semi-simplification. We have the commutative diagram

$$\begin{tikzcd} \mathrm{Rep}_Q(\vec{m}) \arrow[r] \arrow[dr, "\pi_Q"'] & \mathcal{M}_Q(\vec{m}) \arrow[d, "p_Q"] \\ & M_Q(\vec{m}) . \end{tikzcd} \tag{5.6}$$

The point $0 \in \mathrm{Rep}_Q(\vec{m})$ and its image $0 \in M_Q(\vec{m})$ by the map (5.5) correspond to the semi-simple Q-representation $\bigoplus_{i \in V(Q)} V_i \otimes S_i$. A Q-representation (5.2) is called *nilpotent* if any sufficiently large number of compositions of the linear maps u_e becomes zero. It is easy to see that a Q-representation is nilpotent if and only if it is an iterated extension of simple objects $\{S_i\}_{i \in V(Q)}$. In particular, the fiber $p_Q^{-1}(0) \subset \mathcal{M}_Q(\vec{m})$ for the morphism (5.5) consists of nilpotent Q-representations with dimension vector $\vec{m}$. The morphism (5.5) is a good moduli space of the stack $\mathcal{M}_Q(\vec{m})$ (see Definition 4.7).

5.2 Semistable quiver representations

For a quiver Q, let $K(Q)$ be the Grothendieck group of the abelian category of finite-dimensional Q-representations. Let $\mathcal{H} \subset \mathbb{C}$ be the upper half-plane, and take

$$\xi = (\xi_i)_{i \in V(Q)} \in \mathcal{H}^{\sharp V(Q)}, \quad \xi_i \in \mathcal{H} . \tag{5.7}$$

Then we have the group homomorphism

$$Z_\xi \colon K(Q) \xrightarrow{\mathbf{dim}} \Gamma_Q \longrightarrow \mathbb{C}, \quad [S_i] \longmapsto \xi_i . \tag{5.8}$$

Here $\mathbf{dim}$ is the map taking the dimension vectors of Q-representations. Then Z_ξ defines a Bridgeland stability condition on the abelian category of finite-dimensional Q-representations (see Appendix A). For simplicity, we refer to Z_ξ-(semi)stable objects as ξ-(semi)stable objects.

For a choice of ξ as in (5.7), let

$$\mathrm{Rep}_Q^\xi(\vec{m}) \subset \mathrm{Rep}_Q(\vec{m}) \tag{5.9}$$

be the Zariski open locus consisting of ξ-semistable Q-representations. We take the associated GIT quotients:

$$\mathcal{M}_Q^\xi(\vec{m}) := \left[\mathrm{Rep}_Q^\xi(\vec{m})/G\right], \quad M_Q^\xi(\vec{m}) := \mathrm{Rep}_Q^\xi(\vec{m}) /\!\!/ G .$$

We have the commutative diagram

$$\begin{tikzcd} \mathcal{M}_Q^\xi(\vec{m}) \arrow[r, hook, "j_Q^\xi"] \arrow[d, "p_Q^\xi"'] \arrow[dr, "r_Q^\xi"] & \mathcal{M}_Q(\vec{m}) \arrow[d, "p_Q"] \\ M_Q^\xi(\vec{m}) \arrow[r, "q_Q^\xi"'] & M_Q(\vec{m}) . \end{tikzcd} \tag{5.10}$$

Here j_Q^ξ is an open immersion, p_Q and p_Q^ξ are natural morphisms to the good moduli spaces, and q_Q^ξ is the morphism induced by the open immersion (5.9). By general GIT, the morphism q_Q^ξ is a projective morphism of irreducible varieties satisfying $q_{Q*}^\xi \mathcal{O}_{M_Q^\xi(\vec{m})} = \mathcal{O}_{M_Q(\vec{m})}$.

Let

$$M_Q^s(\vec{m}) \subset M_Q(\vec{m}), \quad M_Q^{\xi,s}(\vec{m}) \subset M_Q^\xi(\vec{m})$$

be the open subsets of the simple part and ξ-stable part, respectively. It is well known (for example, see [Rei08]) that both of $M^s_Q(\vec{m})$ and $M^{\xi,s}_Q(\vec{m})$ are smooth varieties. As any preimage $\left(q^\xi_Q\right)^{-1}(x)$ for $x \in M^s_Q(\vec{m})$ is one point, q^ξ_Q is a projective birational morphism if $M^s_Q(\vec{m}) \neq \emptyset$.

5.3 Quivers with convergent super-potentials

For a quiver Q, by taking the completion of the path algebra $\mathbb{C}[Q]$ with respect to the length of the path, we obtain the formal path algebra:

$$\mathbb{C}[\![Q]\!] := \prod_{n \geqslant 0} \bigoplus_{e_1, \ldots, e_n \in E(Q),\, t(e_i) = s(e_{i+1})} \mathbb{C} \cdot e_1 e_2 \ldots e_n \,.$$

Note that an element $f \in \mathbb{C}[\![Q]\!]$ is written as

$$f = \sum_{n \geqslant 0,\, \{1, \ldots, n+1\} \xrightarrow{\psi} V(Q)} \sum_{e_i \in E_{\psi(i), \psi(i+1)}} a_{\psi, e_\bullet} \cdot e_1 e_2 \ldots e_n \,. \tag{5.11}$$

Here $a_{\psi, e_\bullet} \in \mathbb{C}$ and $e_\bullet = (e_1, \ldots, e_n)$, and $E_{\psi(i), \psi(i+1)}$ is defined as in (5.1). The element f lies in $\mathbb{C}[Q]$ if and only if $a_{\psi, e_\bullet} = 0$ for $n \gg 0$.

Definition 5.1. The subalgebra $\mathbb{C}\{Q\} \subset \mathbb{C}[\![Q]\!]$ is defined to consist of the elements (5.11) such that $|a_{\psi, e_\bullet}| < C^n$ for some constant $C > 0$ that is independent of n. Here n is the length of the path $e_\bullet = (e_1, \ldots, e_n)$.

Note that $\mathbb{C}\{Q\}$ contains $\mathbb{C}[Q]$ as a subalgebra. A *convergent super-potential* of a quiver Q is an element $W \in \mathbb{C}\{Q\}/[\mathbb{C}\{Q\}, \mathbb{C}\{Q\}]$. A convergent super-potential W of Q is represented by a formal sum

$$W = \sum_{n \geqslant 1} \sum_{\substack{\{1, \ldots, n+1\} \xrightarrow{\psi} V(Q), \\ \psi(n+1) = \psi(1)}} \sum_{e_i \in E_{\psi(i), \psi(i+1)}} a_{\psi, e_\bullet} \cdot e_1 e_2 \ldots e_n \tag{5.12}$$

with $|a_{\psi, e_\bullet}| < C^n$ for a constant $C > 0$. This W is called *minimal* if $a_{\psi, e_\bullet} = 0$ for $e_\bullet = (e_1, \ldots, e_n)$ with $n \leqslant 2$.

For a dimension vector $\vec{m}$ of Q, let $\operatorname{tr} W$ be the formal function of $u = (u_e)_{e \in E(Q)} \in \operatorname{Rep}_Q(\vec{m})$ defined by

$$\operatorname{tr} W(u) := \sum_{n \geqslant 1} \sum_{\substack{\{1, \ldots, n+1\} \xrightarrow{\psi} V(Q) \\ \psi(n+1) = \psi(1)}} \sum_{e_i \in E_{\psi(i), \psi(i+1)}} a_{\psi, e_\bullet} \cdot \operatorname{tr}(u_n \circ u_{n-1} \circ \cdots \circ u_1) \,.$$

This formal function on $\operatorname{Rep}_Q(\vec{m})$ is G-invariant. By the arguments of [Tod18, Lemma 2.10] and [Tod22b, Lemma 4.9], there exist an analytic open neighborhood V and an analytic function $\overline{\operatorname{tr}}(W)$ with

$$0 \in V \subset M_Q(\vec{m}) \,, \quad \overline{\operatorname{tr}}(W) \colon V \longrightarrow \mathbb{C} \tag{5.13}$$

such that the formal function $\operatorname{tr} W$ absolutely converges on $\pi_Q^{-1}(V)$ (here π_Q is given in the diagram (5.6)) to give a G-invariant analytic function, which factors through $\pi_Q^{-1}(V) \to V$:

$$\operatorname{tr} W \colon \pi_Q^{-1}(V) \xrightarrow{\pi_Q} V \xrightarrow{\overline{\operatorname{tr}}(W)} \mathbb{C} \,. \tag{5.14}$$

Then we set

$$\begin{aligned}\mathrm{Rep}_{(Q,\partial W)}(\vec{m})|_V &:= \{d(\mathrm{tr}\, W)=0\} \subset \pi_Q^{-1}(V)\,,\\ \mathcal{M}_{(Q,\partial W)}(\vec{m})|_V &:= [\{d(\mathrm{tr}\, W)=0\}/G] \subset \left[\pi_Q^{-1}(V)/G\right],\\ M_{(Q,\partial W)}(\vec{m})|_V &:= \{d(\mathrm{tr}\, W)=0\}/\!\!/G \subset V\,.\end{aligned} \tag{5.15}$$

Here $(-)/\!\!/G$ is an analytic Hilbert quotient (see [HMP98, Gre15, Tod18]).

Remark 5.2. A Q-representation corresponding to a point in $\pi_Q^{-1}(V)$ satisfies the equation $\{d(\mathrm{tr}\, W)=0\}$ if and only if it satisfies the relation ∂W of the quiver Q given by derivations of W (see [Tod18, Subsection 2.6]).

Let ξ be data as in (5.7) that defines the ξ-stability on the category of Q-representations and

$$\mathrm{Rep}^{\xi}_{(Q,\partial W)}(\vec{m})|_V \subset \mathrm{Rep}_{(Q,\partial W)}(\vec{m})|_V \tag{5.16}$$

be the open locus consisting of the ξ-semistable Q-representations. Similarly to (5.15), we define

$$\begin{aligned}\mathcal{M}^{\xi}_{(Q,\partial W)}(\vec{m})|_V &:= \left[\mathrm{Rep}^{\xi}_{(Q,\partial W)}(\vec{m})|_V/G\right],\\ M^{\xi}_{(Q,\partial W)}(\vec{m})|_V &:= \mathrm{Rep}^{\xi}_{(Q,\partial W)}(\vec{m})|_V/\!\!/G\,.\end{aligned}$$

Then we have the commutative diagram

$$\begin{tikzcd} M^{\xi}_{(Q,\partial W)}(\vec{m})|_V \arrow[r, hook] \arrow[d, "q^{\xi}_{(Q,\partial W)}"'] & \left(q^{\xi}_Q\right)^{-1}(V) \arrow[d, "q^{\xi}_Q"'] \arrow[dr, "\mathrm{tr}^{\xi} W"] & \\ M_{(Q,\partial W)}(\vec{m})|_V \arrow[r, hook] & V \arrow[r, "\mathrm{tr} W"'] & \mathbb{C}\,. \end{tikzcd} \tag{5.17}$$

The $\hookrightarrow$ are closed embeddings (see [Tod18, Lemma 2.9]), q^{ξ}_Q is given in the diagram (5.10), $q^{\xi}_{(Q,\partial W)}$ is induced by the open immersion (5.16), and $\mathrm{tr}^{\xi} W$ is defined by the commutativity of (5.17).

Lemma 5.3. *Suppose that $M^{\xi,s}_Q(\vec{m}) = M^{\xi}_Q(\vec{m})$ holds. Then there is a d-critical structure on $M^{\xi}_{(Q,\partial W)}(\vec{m})|_V$ given by*

$$s = \mathrm{tr}^{\xi} W + \left(d\,\mathrm{tr}^{\xi} W\right)^2 \in \Gamma\left(\mathcal{S}^0_{M^{\xi}_{(Q,\partial W)}(\vec{m})|_V}\right)$$

such that the diagram (5.17) is its $q^{\xi}_{(Q,\partial W)}$-relative d-critical chart.

Proof. The assumption implies that $\mathcal{M}^{\xi}_{(Q,\partial W)}(\vec{m})|_V$ and $\left(r^{\xi}_Q\right)^{-1}(V)$ (see (5.10) for the definition of the map r^{ξ}_Q) are $\mathbb{C}^*$-gerbes over $M^{\xi}_{(Q,\partial W)}(\vec{m})|_V$ and $\left(q^{\xi}_Q\right)^{-1}(V)$, respectively. Since $\mathcal{M}^{\xi}_{(Q,\partial W)}(\vec{m})|_V$ is a critical locus of the function $\mathrm{tr}\, W|_{(r^{\xi}_Q)^{-1}(V)}$, it follows that $M^{\xi}_{(Q,\partial W)}(\vec{m})|_V$ is a critical locus of the function $\mathrm{tr}^{\xi} W$ on the smooth space $\left(q^{\xi}_Q\right)^{-1}(V)$. Therefore, the lemma holds. □

5.4 Wall-crossing of representations of quivers

Suppose that $\vec{m} \in \Gamma_{Q,>0}$ is primitive; that is, it is not divided by a positive integer greater than 1. For each decomposition $\vec{m} = \vec{m}_1 + \vec{m}_2$ for $\vec{m}_i \in \Gamma_{Q,>0}$, we set

$$\mathcal{W}_{(\vec{m}_1,\vec{m}_2)} := \left\{\xi \in \mathcal{H}^{\sharp V(Q)} : Z_{\xi}(\vec{m}_1) \in \mathbb{R}_{>0} Z_{\xi}(\vec{m}_2)\right\}. \tag{5.18}$$

As $\vec{m}$ is primitive, the vectors $\vec{m}_1$, $\vec{m}_2$ are not proportional in $(\Gamma_Q)_{\mathbb{Q}}$. Therefore, $\mathcal{W}_{(\vec{m}_1,\vec{m}_2)}$ is a real codimension 1 submanifold of $\mathcal{H}^{\sharp V(Q)}$. Since the possible decomposition $\vec{m} = \vec{m}_1 + \vec{m}_2$ is finite, these real codimension 1 submanifolds are finite.

The submanifolds (5.18) are the sets of walls of ξ-stability conditions on the category of Q-representations, and chambers are connected components of the complement of walls in $\mathcal{H}^{\sharp V(Q)}$. As is well known, we have the wall-crossing phenomena: the moduli spaces $M_Q^{\xi}(\vec{m})$ are constant if ξ lies on a chamber but may change if ξ crosses a wall. Note that as $\vec{m}$ is primitive, we have $M_Q^{\xi,s}(\vec{m}) = M_Q^{\xi}(\vec{m})$ if ξ lies on a chamber; hence it is a smooth variety.

Let W be a convergent super-potential of Q, and take $\xi^{\pm} \in \mathcal{H}^{\sharp V(Q)}$ that lie on chambers. Then there exist an analytic open subset $V \subset M_Q(\vec{m})$ containing 0 and a diagram (see the diagram (5.17))

$$\begin{tikzcd}
M^{\xi^+}_{(Q,\partial W)}(\vec{m})|_V \arrow[dr, "q^{\xi^+}_{(Q,\partial W)}"'] & & M^{\xi^-}_{(Q,\partial W)}(\vec{m})|_V \arrow[dl, "q^{\xi^-}_{(Q,\partial W)}"] \\
& M_{(Q,\partial W)}(\vec{m})|_V\,. &
\end{tikzcd} \tag{5.19}$$

By Lemma 5.3, the following corollary immediately follows.

Corollary 5.4. *The diagram* (5.19) *is an analytic* (*generalized*) *d-critical flip, flop, MFS if the diagram*

$$\begin{tikzcd}
M^{\xi^+}_{Q}(\vec{m}) \arrow[dr, "q^{\xi^+}_{Q}"'] & & M^{\xi^-}_{Q}(\vec{m}) \arrow[dl, "q^{\xi^-}_{Q}"] \\
& M_{Q}(\vec{m}) &
\end{tikzcd} \tag{5.20}$$

is a (*generalized*) *flip, flop, MFS, respectively.*

Remark 5.5. If $M_Q^s(\vec{m}) \neq \emptyset$, which follows if the condition in Theorem 7.1 is satisfied, then the diagram (5.20) is a birational map of smooth varieties $M_Q^{\xi^+}(\vec{m}) \dashrightarrow M_Q^{\xi^-}(\vec{m})$. Therefore, the diagram (5.20) is a (generalized) MFS only if $M_Q^s(\vec{m}) = \emptyset$ holds.

6. Analytic neighborhood theorem of wall-crossing in CY 3-folds

In this section, we give an analytic neighborhood theorem for wall-crossing diagrams in CY 3-folds. This theorem describes such diagrams in term of wall-crossing diagrams for quivers with convergent super-potentials. A similar result was already proved in [Tod18] in the case of moduli spaces of semistable sheaves. We will see that the same argument applies to the case of Bridgeland semistable objects if we assume the existence of their good moduli spaces (see Theorem 4.8). In this section, we always assume that X is a smooth projective CY 3-fold.

6.1 Ext-quivers

For a smooth projective CY 3-fold X, let us take a collection of objects in the derived category,

$$E_\bullet = (E_1, E_2, \ldots, E_k)\,, \quad E_i \in D^b(X)\,.$$

Here we recall the notion of Ext-quiver associated with the collection $E_\bullet$ and the construction of their convergent super-potentials. We will apply the construction to the collection of objects $E_\bullet$

associated with polystable objects.

For each $1 \leqslant i, j \leqslant k$, we fix a finite subset $E_{i,j} \subset \mathrm{Ext}^1(E_i, E_j)^\vee$ giving a basis of $\mathrm{Ext}^1(E_i, E_j)^\vee$. Let the quiver $Q_{E_\bullet}$ be defined as follows. The sets of vertices and edges are given by

$$V(Q_{E_\bullet}) := \{1, 2, \dots, k\}, \quad E(Q_{E_\bullet}) := \coprod_{1 \leqslant i,j \leqslant k} E_{i,j}.$$

The maps $s, t\colon E(Q_{E_\bullet}) \to V(Q_{E_\bullet})$ are given by $s|_{E_{i,j}} := i$ and $t|_{E_{i,j}} := j$. The quiver $Q_{E_\bullet}$ is called the *Ext quiver* associated with the collection $E_\bullet$. Note that we have $\mathbb{E}_{i,j} = \mathrm{Ext}^1(E_i, E_j)^\vee$, where $\mathbb{E}_{i,j}$ is defined as in (5.1).

For a map of sets $\psi\colon \{1, \dots, n+1\} \to \{1, \dots, k\}$, let m_n be the graded linear maps

$$\begin{aligned} m_n\colon\ & \mathrm{Ext}^*(E_{\psi(1)}, E_{\psi(2)}) \otimes \mathrm{Ext}^*(E_{\psi(2)}, E_{\psi(3)}) \otimes \cdots \\ & \cdots \otimes \mathrm{Ext}^*(E_{\psi(n)}, E_{\psi(n+1)}) \to \mathrm{Ext}^{*+2-n}(E_{\psi(1)}, E_{\psi(n+1)}) \end{aligned} \tag{6.1}$$

that give a minimal A_∞-category structure on the dg-category generated by $(E_1, \dots, E_k)$. Since X is a CY 3-fold, we can take the A_∞-structure (6.1) to be cyclic (see [Pol01]), that is, $\psi(1) = \psi(n+1)$, and for elements $a_i \in \mathrm{Ext}^*(E_{\psi(i)}, E_{\psi(i+1)})$ for $1 \leqslant i \leqslant n$, we have the relation

$$(m_{n-1}(a_1, \dots, a_{n-1}), a_n) = (m_{n-1}(a_2, \dots, a_n), a_1).$$

Here $(-,-)$ is the Serre duality pairing

$$(-,-)\colon\ \mathrm{Ext}^*(E_a, E_b) \times \mathrm{Ext}^{3-*}(E_b, E_a) \longrightarrow \mathrm{Ext}^3(E_a, E_a) \xrightarrow{\int_X \mathrm{tr}} \mathbb{C}.$$

Let $W_{E_\bullet} \in \mathbb{C}[\![Q_{E_\bullet}]\!]$ be defined by

$$W_{E_\bullet} := \sum_{n \geqslant 3} \sum_{\substack{\{1,\dots,n+1\} \xrightarrow{\psi} \{1,\dots,k\} \\ \psi(1)=\psi(n+1)}} \sum_{e_i \in E_{\psi(i),\psi(i+1)}} a_{\psi, e_\bullet} \cdot e_1 e_2 \dots e_n. \tag{6.2}$$

Here the coefficient $a_{\psi,e_\bullet}$ is given by

$$a_{\psi,e_\bullet} := \frac{1}{n}(m_{n-1}(e_1^\vee, e_2^\vee, \dots, e_{n-1}^\vee), e_n^\vee).$$

On the right-hand side, for $e \in E_{i,j}$, the element $e^\vee \in \mathrm{Ext}^1(E_i, E_j)$ is determined by $e^\vee(e) = 1$ and $e^\vee(e') = 0$ for any $e' \in E_{i,j}$ with $e \neq e'$. By taking the Dolbeaut model in defining the A_∞-products (6.1), the result of [Tod18, Lemma 4.1] (based on earlier works [Fuk03, Tu14]) shows that

$$W_{E_\bullet} \in \mathbb{C}\{Q_{E_\bullet}\} \subset \mathbb{C}[\![Q_{E_\bullet}]\!].$$

Here $\mathbb{C}\{Q_{E_\bullet}\}$ is given in Definition 5.1. Therefore, $W_{E_\bullet}$ determines a convergent super-potential of $Q_{E_\bullet}$.

A collection $E_\bullet$ is called a *simple collection* if we have

$$\mathrm{Ext}^{\leqslant 0}(E_i, E_j) = \mathbb{C} \cdot \delta_{ij}. \tag{6.3}$$

The right-hand side is concentrated on degree zero. In this case, the algebra $\mathbb{C}[\![Q_{E_\bullet}]\!]/(\partial W_{E_\bullet})$ gives a pro-representable hull of a non-commutative deformation functor associated with $E_\bullet$, developed in [Lau02, Eri10, Kaw15, BB15]. By taking the tensor product with the universal object, we have an equivalence of categories (see [Tod18, Corollary 6.7])

$$\Phi_{E_\bullet}\colon\ \mathrm{mod}_{\mathrm{nil}}\, \mathbb{C}[\![Q_{E_\bullet}]\!]/(\partial W_{E_\bullet}) \xrightarrow{\sim} \langle E_1, \dots, E_k \rangle_{\mathrm{ex}}. \tag{6.4}$$

Here the left-hand side is the category of nilpotent $\mathbb{C}[\![Q_{E_\bullet}]\!]/(\partial W_{E_\bullet})$-modules, and $\langle - \rangle_{\mathrm{ex}}$ on the right-hand side is the extension closure. This equivalence sends simple objects S_i for $1 \leqslant i \leqslant k$ corresponding to the vertex $i \in V(Q_{E_\bullet})$ to the object E_i.

6.2 Analytic neighborhood theorem

We return to the situation in Section 4. Namely, we take stability conditions

$$\sigma = (Z, \mathcal{A}) \in \mathrm{Stab}(X)\,, \quad \sigma^{\pm} = \left(Z^{\pm}, \mathcal{A}^{\pm}\right) \in \mathrm{Stab}(X)\,, \tag{6.5}$$

where $\sigma^{\pm}$ lie on adjacent chambers whose closures contain σ. We also take a primitive element $v \in \Gamma_X$ with $\Im Z(v) > 0$, the good moduli spaces of semistable objects $M_\sigma(v)$, $M_{\sigma^{\pm}}(v)$ and the wall-crossing diagram (4.7). As we mentioned in Subsection 4.2, a closed point $p \in M_\sigma(v)$ corresponds to a σ-polystable object $E \in \mathcal{A}$. The object E is of the form

$$E = \bigoplus_{i=1}^{k} V_i \otimes E_i\,, \tag{6.6}$$

where V_i is a finite-dimensional vector space, each $E_i \in \mathcal{A}$ is a σ-stable object with $\arg Z(E_i) = \arg Z(E_j)$, and $E_i \ncong E_j$ for $i \neq j$, satisfying

$$\sum_{i=1}^{k} \dim V_i \cdot \mathrm{ch}(E_i) = v\,. \tag{6.7}$$

Let $E_\bullet$ be the collection of σ-stable objects in (6.6):

$$E_\bullet = (E_1, E_2, \dots, E_k)\,. \tag{6.8}$$

Then $E_\bullet$ is a simple collection; that is, it satisfies the condition (6.3). Let $Q_{E_\bullet}$ be the Ext-quiver associated with the collection $E_\bullet$. We take data $\xi^{\pm} \in \mathcal{H}^k$ as in (5.7) for the Ext-quiver $Q_{E_\bullet}$ by setting

$$\xi^{\pm} := \left(\xi_i^{\pm}\right)_{1 \leqslant i \leqslant k}\,, \quad \xi_i^{\pm} = Z^{\pm}(E_i)\,, \quad 1 \leqslant i \leqslant k\,. \tag{6.9}$$

Then we have the associated $\xi^{\pm}$-stability condition on the abelian category of $Q_{E_\bullet}$-representations. Let $\vec{m} = (m_i)_{1 \leqslant i \leqslant k}$ be the dimension vector of $Q_{E_\bullet}$-representations given by

$$m_i = \dim V_i\,, \quad 1 \leqslant i \leqslant k\,, \tag{6.10}$$

where V_i is given in (6.6). Note that as $v \in \Gamma_X$ is primitive, the vector $\vec{m} \in \Gamma_{Q_{E_\bullet}}$ is also primitive by the identity (6.7).

The following *analytic neighborhood theorem* describes the wall-crossing diagram in terms of representations of quivers with convergent super-potentials, analytic-locally on the good moduli spaces.

Theorem 6.1. *For a primitive element $v \in \Gamma_X$ and stability conditions σ, $\sigma^{\pm}$ as in (6.5), let*

$$\begin{tikzcd} M_{\sigma^+}(v) \arrow[dr, "q_M^+"'] & & M_{\sigma^-}(v) \arrow[dl, "q_M^-"] \\ & M_\sigma(v) & \end{tikzcd} \tag{6.11}$$

be the wall-crossing diagram as in (4.7). For a closed point $p \in M_\sigma(v)$ corresponding to a σ-polystable object (6.6), let $Q = Q_{E_\bullet}$ be the associated Ext-quiver and $W = W_{E_\bullet}$ the convergent *super-potential as in* (6.2). *Then there exist analytic open neighborhoods*

$$p \in T \subset M_{\sigma}(v)\,, \quad 0 \in V \subset M_Q(\vec{m})\,,$$

where $\vec{m}$ *is the dimension vector* (6.10), *such that we have the commutative diagram of isomorphisms*

$$\begin{tikzcd} M^{\xi^{\pm}}_{(Q,\partial W)}(\vec{m})|_V \arrow[r, "\cong"] \arrow[d, "q^{\xi^{\pm}}_{(Q,\partial W)}"'] & (q^{\pm}_M)^{-1}(T) \arrow[d, "q^{\pm}_M"] \\ M_{(Q,\partial W)}(\vec{m})|_V \arrow[r, "\cong"] & T\,. \end{tikzcd}$$

Here the left vertical arrow is given in (5.17), *and the right vertical arrow is given in* (6.11) *pulled back to* T. *Moreover, the top isomorphism preserves d-critical structures, where the d-critical structure on the left-hand side is given in Lemma* 5.3 (*after changing* $\mathrm{tr}W$ *if necessary*) *and that on the right-hand side is given in Theorem* 4.1.

The above result is proved in [Tod18, Theorem 7.7] in the case of 1-dimensional (twisted) semistable sheaves, and the same argument also applies if we assume the existence of good moduli spaces for Bridgeland semistable objects (see Theorem 4.8). In Subsection 6.3, we will just give an outline of the proof. By Corollary 5.4 and Theorem 6.1, we have the following.

Corollary 6.2. *In the situation of Theorem* 6.1, *the diagram* (6.11) *is a d-critical* (*generalized*) *flip, flop, MFS at* $p \in M_{\sigma}(v)$ (*that corresponds to a polystable object* (6.6)) *if the diagram*

$$\begin{tikzcd} M^{\xi^+}_Q(\vec{m}) \arrow[dr, "q^{\xi^+}_Q"'] & & M^{\xi^-}_Q(\vec{m}) \arrow[dl, "q^{\xi^-}_Q"] \\ & M_Q(\vec{m}) & \end{tikzcd} \tag{6.12}$$

for the Ext-quiver $Q = Q_{E_{\bullet}}$, *dimension vector* $\vec{m}$ *in* (6.10) *and* $\xi^{\pm}$ *as in* (6.9) *is a* (*generalized*) *flip, flop, MFS, respectively.*

Using Corollary 6.2, we give the simplest case of d-critical flips and flops in the following example.

Example 6.3. In the situation of Corollary 6.2, suppose that $p \in M_{\sigma}(v)$ corresponds to a σ-polystable object E of the form $E = E_1 \oplus E_2$ with $E_1 \ncong E_2$, where E_1 and E_2 are σ-stable with $\arg Z(E_1) = \arg Z(E_2)$. Note that we have

$$\chi(E_1, E_2) := \sum_{i \in \mathbb{Z}} (-1)^i \dim \mathrm{Ext}^i(E_1, E_2) = \dim \mathrm{Ext}^1(E_2, E_1) - \dim \mathrm{Ext}^1(E_1, E_2)\,,$$

and the left-hand side is computed by the Riemann–Roch theorem. The Ext-quiver Q associated with $E_{\bullet} = (E_1, E_2)$ has two vertices, $V(Q) = \{1, 2\}$. Let us set

$$V^+ = \mathrm{Ext}^1(E_1, E_2)\,, \quad V^- = \mathrm{Ext}^1(E_2, E_1)\,,$$
$$U = \mathrm{Ext}^1(E_1, E_1) \oplus \mathrm{Ext}^1(E_2, E_2)\,.$$

The stack of Q-representations with dimension vector $\vec{m} = (1,1)$ is given by

$$\mathcal{M}_Q(\vec{m}) = \left[(V^+ \times V^-)/(\mathbb{C}^*)^2 \right] \times U\,.$$

Here the action of $(t_1, t_2) \in (\mathbb{C}^*)^2$ on $(\vec{x}, \vec{y}) \in V^+ \times V^-$ is given by

$$(t_1, t_2) \cdot (\vec{x}, \vec{y}) = \left(t_1 t_2^{-1} \cdot \vec{x}, t_1^{-1} t_2 \cdot \vec{y}\right).$$

We take $\sigma^\pm$ as in (6.5), so that $\arg Z^+(E_1) > \arg Z^+(E_2)$ and $\arg Z^-(E_1) < \arg Z^-(E_2)$ hold. It is easy to see that a point $(\vec{x}, \vec{y}, \vec{u}) \in \mathcal{M}_Q(\vec{m})$ is ξ^+-(semi)stable if and only if $\vec{x} \neq 0$ and that it is ξ^--(semi)stable if and only if $\vec{y} \neq 0$. Therefore, the moduli spaces of $\xi^\pm$-stable Q-representations are given by

$$M_Q^{\xi^+}(\vec{m}) = \mathrm{Tot}_{\mathbb{P}(V^+)}\left(\mathcal{O}_{\mathbb{P}(V^+)}(-1) \otimes V^-\right) \times U,$$
$$M_Q^{\xi^-}(\vec{m}) = \mathrm{Tot}_{\mathbb{P}(V^-)}\left(\mathcal{O}_{\mathbb{P}(V^-)}(-1) \otimes V^+\right) \times U.$$

It follows that the diagram (6.12) is a standard toric flip (respectively, flop) given in Example 3.8 if $\chi(E_1, E_2) < 0$ (respectively, $\chi(E_1, E_2) = 0$). Therefore, by Corollary 6.2, the diagram (6.11) is an analytic d-critical flip if $\chi(E_1, E_2) < 0$ (respectively, flop if $\chi(E_1, E_2) = 0$) at the point $p = [E_1 \oplus E_2] \in M_\sigma(v)$.

6.3 Outline of the proof of Theorem 6.1

Proof. Let us take a point $p \in M_\sigma(v)$ that corresponds to a σ-polystable object E as in (6.6). Note that for the Ext-quiver $Q = Q_{E_\bullet}$ associated with the collection (6.8), we have

$$\mathrm{Rep}_Q(\vec{m}) = \mathrm{Ext}^1(E, E), \quad G := \prod_{i=1}^{k} \mathrm{GL}(V_i) = \mathrm{Aut}(E),$$

where $\vec{m}$ is the dimension vector (6.10). Under these identifications, the G-action on $\mathrm{Rep}_Q(\vec{m})$ is compatible with the conjugate $\mathrm{Aut}(E)$-action on $\mathrm{Ext}^1(E, E)$.

Let $\mathcal{E}_i^\bullet \to E_i$ be a resolution of E_i by vector bundles, and set $\mathcal{E}^\bullet = \oplus_{i=1}^k V_i \otimes \mathcal{E}_i^\bullet$. We have linear maps of degree $1 - n$

$$I_n \colon \mathrm{Ext}^*(E, E)^{\otimes n} \longrightarrow \mathfrak{g}^*_{\mathcal{E}_\bullet} := A^{0,*}(\mathcal{H}om^*(\mathcal{E}^\bullet, \mathcal{E}^\bullet))$$

giving a A_∞ quasi-isomorphism between the minimal A_∞-algebra $\mathrm{Ext}^*(E, E)$ and the Dolbeault dg Lie algebra $\mathfrak{g}^*_{\mathcal{E}_\bullet}$. For $x \in \mathrm{Ext}^1(E, E)$, the formal sum

$$I_*(x) := \sum_{n \geqslant 1} I_n(x, \ldots, x) \tag{6.13}$$

has a convergent radius (see [Tod18, Lemma 4.1]). Let $\pi_Q \colon \mathrm{Rep}_Q(\vec{m}) \to M_Q(\vec{m})$ be the natural map to the GIT quotient, and take a sufficiently small analytic open subset $0 \in V \subset M_Q(\vec{m})$. The infinite sum (6.13) determines an $\mathrm{Aut}(E)$-equivariant analytic map

$$I_* \colon \pi_Q^{-1}(V) \longrightarrow \widehat{\mathfrak{g}}^1_{\mathcal{E}_\bullet}. \tag{6.14}$$

Here $\widehat{\mathfrak{g}}^1_{\mathcal{E}_\bullet}$ is a certain Sobolev completion of $\mathfrak{g}^1_{\mathcal{E}_\bullet}$ (see [Tod18, Lemma 5.1]). On the open subset $\pi_Q^{-1}(V) \subset \mathrm{Ext}^1(E, E)$, the Mauer–Cartan locus of the minimal A_∞-algebra $\mathrm{Ext}^*(E, E)$ is given by the critical locus of the analytic function

$$\mathrm{tr}\, W \colon \pi_Q^{-1}(V) \longrightarrow \mathbb{C}$$

given in (5.14) for the convergent super-potential $W = W_{E_\bullet}$ defined as in (6.2). Then the restriction of the map (6.14) to the Mauer–Cartan locus determines a smooth morphism of analytic stacks of relative dimension zero (see [Tod18, Proposition 4.3])

$$I_* \colon \mathcal{M}_{(Q, \partial W)}(\vec{m})|_V \longrightarrow \mathcal{M}. \tag{6.15}$$

Here $\mathcal{M}$ is the moduli stack of complexes given in Subsection 4.1.

By shrinking V if necessary, the image of the above morphism lies in the open substack $\mathcal{M}_\sigma(v) \subset \mathcal{M}$. Let $p_M : \mathcal{M}_\sigma(v) \to M_\sigma(v)$ be the morphism to the good moduli space. Then the argument of [Tod18, Proposition 5.4] shows that the map (6.15) induces the commutative diagram of isomorphisms

$$\begin{tikzcd} \mathcal{M}_{(Q,\partial W)}(\vec{m})|_V \arrow[r, "I_*", "\cong"'] \arrow[d, "p_Q"'] & p_M^{-1}(T) \arrow[d, "p_M"] \\ M_{(Q,\partial W)}(\vec{m})|_V \arrow[r, "\cong"'] & T \end{tikzcd} \tag{6.16}$$

for some analytic open neighborhood $T \subset M_\sigma(\beta, n)$ of p. In loc. cit., the claim is stated for moduli stacks of semistable sheaves and their good moduli spaces. But the argument can be generalized to the case of Bridgeland semistable objects, as the explicit construction of good moduli spaces is not needed in the proof of loc. cit. The properties we used for the good moduli spaces are their existence and the étale slice theorem. For the Bridgeland semistable objects, the former is given in [AHH21], and the latter for the map $p_M : \mathcal{M}_\sigma(v) \to M_\sigma(v)$ is given in [AHR20].

The diagram (6.16) in particular implies the isomorphism

$$I_* : p_Q^{-1}(0) \xrightarrow{\cong} p_M^{-1}(p)\,. \tag{6.17}$$

We show that this isomorphism restricts to the isomorphisms

$$I_* : p_Q^{-1}(0) \cap \mathcal{M}^{\xi^\pm}_{(Q,\partial W)}(\vec{m})|_V \xrightarrow{\cong} p_M^{-1}(p) \cap \mathcal{M}_{\sigma^\pm}(v)\,. \tag{6.18}$$

Indeed, if isomorphism (6.17) holds, then the argument of [Tod18, Theorem 6.8] shows that, after shrinking V and T if necessary, we have the isomorphism

$$I_* : \mathcal{M}^{\xi^\pm}_{(Q,\partial W)}(\vec{m})|_V \xrightarrow{\cong} p_M^{-1}(T) \cap \mathcal{M}_{\sigma^\pm}(v)\,.$$

By taking the associated isomorphism on good moduli spaces, we obtain the desired diagram (6.16). After changing $\mathrm{tr}W$ if necessary, the comparison of d-critical structures follows from the argument of [Tod22b, Proposition 5.3].

We show the isomorphism (6.18). Note that $\mathbb{C}$-valued points of $p_Q^{-1}(0)$ consist of nilpotent Q-representations with dimension vector $\vec{m}$, and those of $p_M^{-1}(p)$ consist of objects in the extension closure $\langle E_1, \ldots, E_k\rangle_{\mathrm{ex}}$ in $\mathcal{A}$ with Chern character v. Then the isomorphism (6.18) follows from Lemma 6.4 below. ☐

Lemma 6.4. *The isomorphism* (6.17) *is induced by the equivalence of categories given in* (6.4),

$$\Phi_{E_\bullet} : \mathrm{mod}_{\mathrm{nil}}\, \mathbb{C}[[Q]]/(\partial W) \xrightarrow{\sim} \langle E_1, \ldots, E_k\rangle_{\mathrm{ex}}\,. \tag{6.19}$$

Under this equivalence, a nilpotent Q-representation $\mathbb{V}$ is $\xi^\pm$-semistable if and only if $\Phi_{E_\bullet}(\mathbb{V})$ is $\sigma^\pm$-semistable in $\mathcal{A}$.

Proof. The compatibility of I_* with $\Phi_{E_\bullet}$ is due to [Tod18, Theorem 6.8], and the preservation of the stability follows from the argument of [Tod18, Lemma 7.8] without any modification. ☐

7. Representations of symmetric and extended quivers

By Corollary 6.2, in order to see whether a given wall-crossing diagram in a CY 3-fold is a d-critical flip or flop, it is enough to see whether a wall-crossing diagram in the Ext-quiver is a flip or flop. In this section, we study the latter problem in detail in the case of symmetric quivers and their extended version. The results in this section will be applied to geometric situations in later sections.

7.1 Some general facts on representations of quivers

Here we give some general facts on moduli spaces of representations of quivers. Below, we use the notation from Section 5. Let Q be a quiver, $\vec{m} \in \Gamma_{Q,>0}$ a dimension vector of Q and V_i for each $i \in V(Q)$ a vector space with dimension m_i. As in (5.5), we have the moduli stack of Q-representations with dimension vector $\vec{m}$

$$\mathcal{M}_Q(\vec{m}) = \left[\prod_{e \in E(Q)} \mathrm{Hom}(V_{s(e)}, V_{t(e)})/G\right], \quad G = \prod_{i \in V(Q)} \mathrm{GL}(V_i) \tag{7.1}$$

and its good moduli space $\mathcal{M}_Q(\vec{m}) \to M_Q(\vec{m})$. Let $\xi \in \mathcal{H}^{\sharp V(Q)}$ be a choice of a stability condition as in (5.7) and $M_Q^{\xi}(\vec{m})$ the good moduli space of ξ-semistable representations. As in (5.10), we have the natural morphism

$$q_Q^{\xi} \colon M_Q^{\xi}(\vec{m}) \longrightarrow M_Q(\vec{m})\,. \tag{7.2}$$

Let $M_Q^{s}(\vec{m}) \subset M_Q(\vec{m})$ be the simple part. If $M_Q^{s}(\vec{m}) \neq \emptyset$, then the morphism (7.2) is always birational.

A criterion for the condition $M_Q^{s}(\vec{m}) \neq \emptyset$ is given in [LP90]. In order to state this, we prepare some terminology. A full subquiver $Q' \subset Q$ is said to be *strongly connected* if each pair from its vertex set belongs to an oriented cycle. For $\vec{m} \in \Gamma_Q$, let $\mathrm{supp}(\vec{m})$ be the set of $i \in V(Q)$ with $m_i \neq 0$. Note that $\mathrm{supp}(\vec{m})$ is regarded as a full subquiver of Q. We also use the pairing on Γ_Q defined by

$$\langle \vec{m}, \vec{m}' \rangle := \sum_{i \in V(Q)} m_i \cdot m_i' - \sum_{e \in E(Q)} m_{s(e)} \cdot m'_{t(e)}\,. \tag{7.3}$$

We denote by $\widetilde{A}_n$ the extended Dynkin A_n-quiver; that is, $Q(\widetilde{A}_n) = \{1, 2, \dots, n\}$ with one arrow from i to $i+1$ for each $1 \leqslant i \leqslant n-1$ and one from n to 1. The result of [LP90] is stated as follows.

Theorem 7.1 ([LP90, Theorem 4]). *For $\vec{m} \in \Gamma_Q$, we have $M_Q^{s}(\vec{m}) \neq \emptyset$ if and only if either one of the following conditions holds:*

(i) *We have $m_i = 1$ for all $i \in \mathrm{supp}(\vec{m})$, and $\mathrm{supp}(\vec{m})$ is a quiver of type A_1 or $\widetilde{A}_n$ for $n \geqslant 1$.*

(ii) *The quiver $\mathrm{supp}(\vec{m})$ is not of the above type, is strongly connected, and*

$$\langle \vec{m}, \vec{i} \rangle \leqslant 0\,, \quad \langle \vec{i}, \vec{m} \rangle \leqslant 0 \quad \text{for all } i \in V(Q)\,. \tag{7.4}$$

Remark 7.2. Suppose that one of the conditions (7.4) fails; for example,

$$\langle \vec{m}, \vec{i} \rangle = m_i - \sum_{j \in V(Q),\, e \in E_{j,i}} m_j > 0$$

for some $i \in V(Q)$. Then for a Q-representation $\mathbb{V}$ as in (5.2), the natural map

$$\sum_{j \in V(Q),\, e \in E_{j,i}} u_e \colon \bigoplus_{e \in E_{j,i}} V_j \longrightarrow V_i$$

has a non-trivial cokernel. Therefore, we have a surjection $\mathbb{V} \twoheadrightarrow S_i$, and $M_Q^s(\vec{m}) = \emptyset$ holds. Similarly, if $\langle \vec{i}, \vec{m} \rangle > 0$, then there is an injection $S_i \hookrightarrow \mathbb{V}$. These facts will be used later.

Next, we consider canonical line bundles on moduli spaces of quiver representations. Suppose that $\vec{m}$ is primitive in Γ_Q, and let $M_Q^{\xi,s}(\vec{m}) \subset M_Q^{\xi}(\vec{m})$ be the ξ-stable part. Then by [Kin94, Proposition 5.3], there exist universal Q-representations

$$\mathcal{V}_i \longrightarrow M_Q^{\xi,s}(\vec{m}), \quad \mathbf{u}_e \colon \mathcal{V}_{s(e)} \longrightarrow \mathcal{V}_{t(e)}, \quad i \in V(Q), \quad e \in E(Q).$$

Here $\mathcal{V}_i$ is a vector bundle on $M_Q^{\xi,s}(\vec{m})$ whose fiber is V_i and $\mathbf{u}_e$ is a map of vector bundles, and for a point $p \in M_Q^{\xi,s}(\vec{m})$ corresponding to a Q-representation (5.2), we have $(\mathbf{u}_e)|_p = u_e$. In this case, we have the following lemma on the canonical line bundle of the smooth variety $M_Q^{\xi,s}(\vec{m})$.

Lemma 7.3. *In the above situation, we have*

$$\omega_{M_Q^{\xi,s}(\vec{m})} = \bigotimes_{e \in E(Q)} \det \mathcal{V}_{s(e)} \otimes \det \mathcal{V}_{t(e)}^{\vee}. \tag{7.5}$$

Proof. When $\vec{m}$ is primitive, the ξ-stable part of the stack $\mathcal{M}_Q(\vec{m})$ is a trivial $\mathbb{C}^*$-gerbe over $M_Q^{\xi,s}(\vec{m})$. By the description of the stack $\mathcal{M}_Q(\vec{m})$ in (7.1), the canonical line bundle of the stack $\mathcal{M}_Q(\vec{m})$ is induced by the 1-dimensional G-representation $G \to \mathbb{C}^*$ given by

$$g = (g_i)_{i \in V(Q)} \longmapsto \prod_{e \in E(Q)} \det g_{s(e)} \cdot (\det g_{t(e)})^{-1}.$$

Therefore, the identity (7.5) holds. □

7.2 Flops via representations of symmetric quivers

Here we investigate the morphism q_Q^{ξ} in (7.2) for a symmetric quiver Q, defined below.

Definition 7.4. A quiver Q is called *symmetric* if $\sharp E_{i,j} = \sharp E_{j,i}$ for any $i, j \in V(Q)$. Here $E_{i,j}$ is defined as in (5.1).

Remark 7.5. The symmetric condition for a quiver Q is equivalent to that the pairing (7.3) is symmetric.

Below for a symmetric quiver Q, we fix identifications $E_{i,j} = E_{j,i}$, so that Q and $Q^{\vee}$ are identified. In particular, for a Q-representation $\mathbb{V}$, its dual representation $\mathbb{V}^{\vee}$ given in (5.3) is also a Q-representation. We have the following lemma on the non-emptiness of the moduli spaces of stable Q-representations.

Lemma 7.6. *For a symmetric quiver Q and $\vec{m} \in \Gamma_Q$, we have $M_Q^{\xi,s}(\vec{m}) \neq \emptyset$ for some $\xi \in \mathcal{H}^{\sharp V(Q)}$ if and only if $M_Q^{\xi,s}(\vec{m}) \neq \emptyset$ for any $\xi \in \mathcal{H}^{\sharp V(Q)}$.*

Proof. It is enough to show that if $M_Q^{\xi,s}(\vec{m}) \neq \emptyset$ for some ξ, then $M_Q^{s}(\vec{m}) \neq \emptyset$. Suppose $M_Q^{\xi,s}(\vec{m}) \neq \emptyset$. We apply the criterion in Theorem 7.1 to show that $M_Q^{s}(\vec{m}) \neq \emptyset$. Since $\mathrm{supp}(\vec{m})$ is symmetric, it is of type A_1 or $\widetilde{A}_n$ only if $n = 1$ or $n = 2$. In A_1 and $\widetilde{A}_1$ cases, we have $M_Q^{\xi,s}(\vec{m}) = M_Q^{s}(\vec{m})$ for any dimension vector $\vec{m}$, and the statement is obvious.

Suppose that $\mathrm{supp}(\vec{m})$ is $\widetilde{A}_2$, and write its vertices as $\{1, 2\}$. We may assume $\arg Z_{\xi}(S_1) > \arg Z_{\xi}(S_2)$, where Z_{ξ} is given by (5.8). Let us write a Q-representation corresponding to a point in $M_Q^{\xi,s}(\vec{m})$ as

$$\mathbb{V} = \left(\begin{tikzcd} V_1 \arrow[r, shift left, "e_{12}"] & V_2 \arrow[l, shift left, "e_{21}"] \end{tikzcd} \right),$$

where the V_i are vector spaces of dimension m_i and e_{12}, e_{21} are linear maps. The ξ-stability implies that $\mathrm{Hom}(S_1, \mathbb{V}) = 0$ and $\mathrm{Hom}(\mathbb{V}, S_2) = 0$. These conditions imply that e_{12} is an isomorphism, so we can assume $V_1 = V_2 = V$ and $e_{12} = \mathrm{id}$. Let $v \in V$ be an eigenvector of e_{21} with eigenvalue λ. Then we have an injection of Q-representations

$$\left(\begin{tikzcd} \mathbb{C}v \arrow[r, shift left, "\mathrm{id}"] & \mathbb{C}v \arrow[l, shift left, "\lambda\cdot"] \end{tikzcd} \right) \hookrightarrow \mathbb{V}.$$

As $\mathbb{V}$ is ξ-stable, this injection must be an isomorphism. Therefore, we have $m_1 = m_2 = 1$, and $M_Q^{\xi,s}(\vec{m}) \neq \emptyset$ follows from Theorem 7.1.

Suppose that $\mathrm{supp}(\vec{m})$ is not of the above types. If $\mathrm{supp}(\vec{m})$ is not strongly connected, then as it is symmetric, it must be disconnected. Then $M_Q^{\xi,s}(\vec{m}) = \emptyset$ for any ξ, which gives a contradiction. Therefore, $\mathrm{supp}(\vec{m})$ is strongly connected. For $i \in V(Q)$, note that we have $\langle \vec{m}, \vec{i} \rangle = \langle \vec{i}, \vec{m} \rangle$ as Q is symmetric (see Remark 7.5). If $\langle \vec{m}, \vec{i} \rangle = \langle \vec{i}, \vec{m} \rangle > 0$, then by Remark 7.2, for any Q-representation $\mathbb{V}$ with dimension vector $\vec{m}$, there exist an injection $S_i \hookrightarrow \mathbb{V}$ and a surjection $\mathbb{V} \twoheadrightarrow S_i$. Such a representation $\mathbb{V}$ can never be ξ-stable for any choice of ξ, which gives a contradiction. Therefore, the criterion of Theorem 7.1 is satisfied, and $M_Q^s(\vec{m}) \neq \emptyset$ holds. $\square$

For a symmetric quiver Q and $\vec{m} \in \Gamma_Q$, let us take $\xi^{\pm} = \left(\xi_i^{\pm}\right)_{i \in V(Q)} \in \mathcal{H}^{\sharp V(Q)}$ satisfying the following:

$$\Re\left(\xi_i^+\right) = -\Re\left(\xi_i^-\right) \in \mathbb{Z}, \qquad \sum_{i \in V(Q)} m_i \cdot \Re\left(\xi_i^{\pm}\right) = 0. \tag{7.6}$$

We have the diagram

$$\begin{tikzcd} M_Q^{\xi^+}(\vec{m}) \arrow[dr, "q_Q^{\xi^+}"'] & & M_Q^{\xi^-}(\vec{m}) \arrow[dl, "q_Q^{\xi^-}"] \\ & M_Q(\vec{m}). & \end{tikzcd} \tag{7.7}$$

By Lemma 7.6, we have $M_Q^{\xi^+,s}(\vec{m}) \neq \emptyset$ if and only if $M_Q^{\xi^-,s}(\vec{m}) \neq \emptyset$.

Proposition 7.7. *Suppose that $\vec{m}$ is primitive and $M_Q^{\xi^{\pm},s}(\vec{m}) = M_Q^{\xi^{\pm}}(\vec{m}) \neq \emptyset$ hold. Then the diagram (7.7) is a generalized flop of smooth varieties $M_Q^{\xi^{\pm}}(\vec{m})$.*

Proof. Under the assumption, the morphisms $q_Q^{\xi^{\pm}}$ are projective birational morphisms from smooth varieties $M_Q^{\xi^{\pm}}(\vec{m})$. Moreover, the canonical divisors of $M_Q^{\xi^{\pm}}(\vec{m})$ are trivial by Lemma 7.3 and the symmetric condition for Q. We show that $q_Q^{\xi^{\pm}}$ are isomorphisms in codimension 1. By [Tod22b, Lemma 4.4], the maps $q_Q^{\xi^{\pm}}$ are semismall; that is, there is a stratification $\{S_\lambda\}_\lambda$ of $M_Q(\vec{m})$ such that for any $x \in S_\lambda$, we have the inequality

$$\dim \left(q_Q^{\xi^{\pm}}\right)^{-1}(x) \leqslant \tfrac{1}{2} \operatorname{codim} S_\lambda. \tag{7.8}$$

Moreover, from the proof of loc. cit. and [MR19, Theorem 1.4] (which is used in loc. cit., under the assumption $M_Q^s(\vec{m}) \neq \emptyset$, the equality holds in (7.8) only for the dense strata $S_\lambda = M_Q^s(\vec{m})$; that is, $q_Q^{\xi^\pm}$ are small maps. In particular, $q_Q^{\xi^\pm}$ are isomorphisms in codimension 1.

It remains to show that there exists a $q_Q^{\xi^+}$-ample divisor on $M_Q^{\xi^+}(\vec{m})$ whose strict transform to $M_Q^{\xi^-}(\vec{m})$ is $q_Q^{\xi^-}$-anti-ample. Let us consider the following characters of G:

$$g = (g_i)_{i \in V(Q)} \longmapsto (\det g_i)^{\Re(\xi_i^\pm)} . \tag{7.9}$$

They define G-equivariant line bundles on $\mathrm{Rep}_Q(\vec{m})$, hence on the stack $\mathcal{M}_Q(\vec{m})$, which we write as $\mathcal{L}_\pm$. Note that by the condition (7.6), the characters (7.9) are trivial on the diagonal torus $\mathbb{C}^* \subset G$. Therefore, the restrictions of $\mathcal{L}_\pm$ to $\mathcal{M}_Q^{\xi^+,s}(\vec{m})$, $\mathcal{M}_Q^{\xi^-,s}(\vec{m})$ descend to line bundles $(L_\pm)^+$, $(L_\pm)^-$ on $M_Q^{\xi^+,s}(\vec{m})$, $M_Q^{\xi^-,s}(\vec{m})$, respectively. By the GIT construction of $M_Q^{\xi^\pm}(\vec{m})$ (see [Kin94]), the line bundle $(L_+)^+$ is $q_Q^{\xi^+}$-ample, and $(L_-)^-$ is $q_Q^{\xi^-}$-ample. By the construction of these line bundles, the strict transform of $(L_+)^+$ on $M_Q^{\xi^+}(\vec{m})$ to $M_Q^{\xi^-}(\vec{m})$ is $(L_+)^- = ((L_-)^-)^\vee$, which is $q_Q^{\xi^-}$-anti-ample. Therefore, the diagram (7.7) is a generalized flop □

For a convergent super-potential W of Q, let us take an analytic open neighborhood $V \subset M_Q(\vec{m})$ of 0 as in (5.13). For two data $\xi^\pm$ as above, as in (5.19), we have the diagram

$$\begin{tikzcd} M^{\xi^+}_{(Q,\partial W)}(\vec{m})|_V \arrow[dr, "q^{\xi^+}_{(Q,\partial W)}"'] & & M^{\xi^-}_{(Q,\partial W)}(\vec{m})|_V \arrow[dl, "q^{\xi^-}_{(Q,\partial W)}"] \\ & M_{(Q,\partial W)}(\vec{m})|_V . & \end{tikzcd} \tag{7.10}$$

By Lemma 5.3 and Proposition 7.7, we obtain the following corollary.

Corollary 7.8. *Let (Q, W) be a symmetric quiver Q with a convergent super-potential W. Suppose that $\vec{m} \in \Gamma_Q$ is primitive, and take $\xi^\pm$ satisfying (7.6). If $M_Q^{\xi^\pm,s}(\vec{m}) = M_Q^{\xi^\pm}(\vec{m}) \neq \emptyset$ hold, then the diagram (7.10) is an analytic d-critical generalized flop.*

7.3 Flips via representations of extended quivers

Let Q be a symmetric quiver, and choose non-negative integers a_i, b_i for each $i \in V(Q)$ and another non-negative integer c. We construct an extended quiver $Q^\star$ in the following way: the set of vertices is $V(Q^\star) = \{0\} \sqcup V(Q)$. For $i, j \in V(Q) \subset V(Q^\star)$, the set of edges from i to j in $Q^\star$ is the same as that in Q. The numbers of other edges are given by

$$\sharp E_{0,i} = a_i\,, \quad \sharp E_{i,0} = b_i\,, \quad i \in V(Q)\,, \quad \sharp E_{0,0} = c\,.$$

For an example, see Figure 1. Note that $Q^\star$ contains Q as a subquiver. In particular, any Q-representation is regarded as a $Q^\star$-representation in a natural way.

For a dimension vector $\vec{m} \in \Gamma_Q$ of Q, we define the extended dimension vector $\vec{m}^\star \in \Gamma_{Q^\star}$ by

$$\vec{m}^\star := \vec{0} + \vec{m}\,;$$

that is, $(\vec{m}^\star)_0 = 1$ and $(\vec{m}^\star)_i = m_i$ for $i \in V(Q)$. The following lemma is obvious from the construction of $Q^\star$.

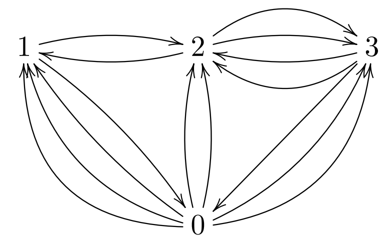

Figure 1. Picture of $Q^\star$ for $V(Q) = \{1, 2, 3\}$

Lemma 7.9. *Giving a $Q^\star$-representation $\mathbb{V}^\star$ with dimension vector $\vec{m}^\star$ is equivalent to giving a Q-representation*

$$\mathbb{V} = \{(V_i, u_e)\}_{i \in V(Q), e \in E(Q)} \tag{7.11}$$

with dimension vector $\vec{m}$, together with linear maps

$$\mathbb{E}_{0,i} \longrightarrow V_i\,, \quad \mathbb{E}_{i,0} \otimes V_i \longrightarrow \mathbb{C} \tag{7.12}$$

for each $i \in V(Q)$. Here $\mathbb{E}_{i,j}$ is the $\mathbb{C}$-vector space defined as in (5.1).

Let us take data $\xi^{\pm} = \left(\xi_i^{\pm}\right) \in \mathcal{H}^{\sharp V(Q^\star)}$ for $Q^\star$ satisfying

$$\xi_i^{\pm} = \sqrt{-1}\,, \quad i \in V(Q)\,, \quad \Re\left(\xi_0^{+}\right) < 0\,, \quad \Re\left(\xi_0^{-}\right) > 0\,. \tag{7.13}$$

The following lemma characterizes $\xi^{\pm}$-semistable $Q^\star$-representations.

Lemma 7.10. *Let $\mathbb{V}^\star$ be a $Q^\star$-representation with dimension vector $\vec{m}^\star$, given by a Q-representation $\mathbb{V}$ as in (7.11) together with linear maps (7.12).*

(i) *The object $\mathbb{V}^\star$ is ξ^+-semistable if and only if it is ξ^+-stable, which holds if and only if the images of the linear maps $\mathbb{E}_{0,i} \to V_i$ in (7.12) generate $\bigoplus_{i \in V(Q)} V_i$ as a $\mathbb{C}[Q]$-module.*

(ii) *The object $\mathbb{V}^\star$ is ξ^--semistable if and only if it is ξ^--stable, which holds if and only if the images of the linear maps $\mathbb{E}_{i,0} \to V_i^\vee$ induced by the right maps in (7.12) generate $\bigoplus_{i \in V(Q)} V_i^\vee$ as a $\mathbb{C}[Q]$-module. Here the $\mathbb{C}[Q]$-module structure on $\bigoplus_{i \in V(Q)} V_i^\vee$ is given by the dual $Q^\vee (= Q)$ representation $\mathbb{V}^\vee$ of $\mathbb{V}$.*

Proof. (i) By a choice of ξ^+, a $Q^\star$-representation $\mathbb{V}^\star$ with dimension vector $\vec{m}^\star$ is ξ^+-(semi)stable if and only if there is no surjection $\mathbb{V}^\star \twoheadrightarrow \mathbb{V}'$ as $Q^\star$-representations where $\mathbb{V}'$ is a non-zero Q-representation. The last condition is equivalent to that the images of linear maps $\mathbb{E}_{0,i} \to V_i$ generate $\bigoplus_{i \in V(Q)} V_i$ as a $\mathbb{C}[Q]$-module.

(ii) By a choice of ξ^-, a $Q^\star$-representation $\mathbb{V}^\star$ with dimension vector $\vec{m}^\star$ is ξ^--(semi)stable if and only if there is no injection $\mathbb{V}' \hookrightarrow \mathbb{V}^\star$ as $Q^\star$-representations where $\mathbb{V}'$ is a non-zero Q-representation. This is equivalent to that the dual representation $(\mathbb{V}^\star)^\vee$ does not admit a surjection to $\mathbb{V}''$ in the category of $(Q^\star)^\vee$-representations, where $\mathbb{V}''$ is a non-zero $Q^\vee (=Q)$-representation. Therefore, as for part (i), we conclude that part (ii) is true. $\square$

By Lemma 7.10, we have $M_{Q^\star}^{\xi^{\pm}, s}(\vec{m}^\star) = M_{Q^\star}^{\xi^{\pm}}(\vec{m}^\star)$, so they are smooth quasi-projective varieties. Let us consider the diagram

$$\begin{tikzcd} M_{Q^\star}^{\xi^+}(\vec{m}^\star) \arrow[dr, "q_{Q^\star}^{\xi^+}"'] & & M_{Q^\star}^{\xi^-}(\vec{m}^\star) \arrow[dl, "q_{Q^\star}^{\xi^-}"] \\ & M_{Q^\star}(\vec{m}^\star)\,. & \end{tikzcd} \tag{7.14}$$

Lemma 7.11. *Suppose $a_i > b_i$ for all $i \in V(Q)$. Then in the diagram (7.14), we have the following:*

(i) *The anti-canonical divisor of $M_{Q^\star}^{\xi^+}(\vec{m}^\star)$ is ample.*

(ii) *The canonical divisor of $M_{Q^\star}^{\xi^-}(\vec{m}^\star)$ is ample.*

Proof. For $i \in V(Q^\star)$ and $e \in E(Q^\star)$, let

$$\mathcal{V}_i^{\pm} \longrightarrow M_{Q^\star}^{\xi^\pm}(\vec{m}^\star)\,, \quad \mathbf{u}_e \colon \mathcal{V}_{s(e)}^{\pm} \longrightarrow \mathcal{V}_{t(e)}^{\pm}$$

be a universal $Q^\star$-representations. Note that such a universal representations exists as $\vec{m}^\star$ is primitive and $M_{Q^\star}^{\xi^\pm,s}(\vec{m}^\star) = M_{Q^\star}^{\xi^\pm}(\vec{m}^\star)$ hold. For $i=0$, the vector bundles $\mathcal{V}_0^{\pm}$ are line bundles by our choice of $\vec{m}^\star$, so by replacing $\mathcal{V}_i^{\pm}$ with $\mathcal{V}_i^{\pm} \otimes \left(\mathcal{V}_0^{\pm}\right)^{-1}$, we may assume that $\mathcal{V}_0^{\pm}$ are trivial line bundles. Then by (7.5), we have

$$\omega_{M_{Q^\star}^{\xi^\pm}(\vec{m}^\star)} = \bigotimes_{i \in V(Q)} \det(\mathcal{V}_i^{\pm})^{b_i - a_i}\,. \tag{7.15}$$

Let $\mathbf{E} \subset \mathbb{C}[Q]$ be the vector subspace generated by paths of the form $e_1 e_2 \dots e_n$ for $n \geqslant 1$ such that $s(e_1) = 0$, $t(e_1) \in V(Q)$ and $e_2, \dots, e_n \in E(Q)$. Then the compositions $\mathbf{u}_{e_n} \circ \dots \circ \mathbf{u}_{e_1}$ determine the morphism of sheaves

$$\mathbf{E} \otimes \mathcal{O}_{M_{Q^\star}^{\xi^+}(\vec{m}^\star)} \longrightarrow \bigoplus_{i \in V(Q)} \mathcal{V}_i^{+}\,.$$

Then Lemma 7.10(i) implies that this morphism is surjective. Therefore, each $\mathcal{V}_i^{+}$ is generated by its global sections. By (7.15) and the assumption $a_i > b_i$ for all $i \in V(Q)$, the line bundle $\left(\omega_{M_{Q^\star}^{\xi^+}(\vec{m}^\star)}\right)^{-1}$ is generated by its global sections. In order to show that it is ample, it is enough to show that it has positive degree on any projective curve on $M_{Q^\star}^{\xi^+}(\vec{m}^\star)$. Let C be a smooth projective curve, and take a non-constant map $h \colon C \to M_{Q^\star}^{\xi^+}(\vec{m}^\star)$. Note that each degree of $h^*\mathcal{V}_i^{+}$ is non-negative, as it is globally generated. Therefore, if the degree of $h^*\left(\omega_{M_{Q^\star}^{\xi^+}(\vec{m}^\star)}\right)^{-1}$ is non-positive, then each degree of $h^*\mathcal{V}_i^{+}$ must be zero. By Sublemma 7.12 below, in this case the vector bundle $h^*\mathcal{V}_i^{+}$ must be a direct sum of $\mathcal{O}_C$. Then the pull-back of the universal map $\mathbf{u}_e$ to C by the map h has to be constant. But this implies that the map h is constant, which gives a contradiction. Therefore, part (i) of the lemma holds. The result of part (ii) follows from Lemma 7.10(ii) and the dual argument of part (i). $\square$

We used the following sublemma.

Sublemma 7.12. *Let C be a smooth projective curve and $\mathcal{V}$ a vector bundle on it. Suppose that $\mathcal{V}$ is generated by its global sections and $\deg \mathcal{V} = 0$. Then $\mathcal{V}$ is isomorphic to a direct sum of $\mathcal{O}_C$.*

Proof. We set $W = H^0(C, \mathcal{V})$ and $r = \operatorname{rank} \mathcal{V}$. The natural surjection $W \otimes \mathcal{O}_C \twoheadrightarrow \mathcal{V}$ induces a morphism $\gamma \colon C \to \mathrm{Gr}(W, r) \hookrightarrow \mathbb{P}^N$. Here $N = \dim \bigwedge^r W$, and the right arrow is the Plücker embedding. Since $\deg \mathcal{V} = \deg \gamma^* \mathcal{O}(1)$, the assumption $\deg \mathcal{V} = 0$ implies that γ is a constant map. Also, the vector bundle $\mathcal{V}$ is a pull-back of the universal quotient bundle on $\mathrm{Gr}(W, r)$. Therefore, $\mathcal{V}$ is isomorphic to a direct sum of $\mathcal{O}_C$. $\square$

Using Lemma 7.11, we show the following proposition.

Proposition 7.13. *Suppose $a_i > b_i$ for all $i \in V(Q)$. Then in the diagram (7.14), either one of the following holds:*

(i) *We have $M^{\xi^-}_{Q^\star}(\vec{m}^\star) = \emptyset$, and the diagram (7.14) is a generalized MFS.*

(ii) *We have $M^{\xi^\pm}_{Q^\star}(\vec{m}^\star) \neq \emptyset$, and the diagram (7.14) is a generalized flip.*

Proof. If $M^{s}_{Q^\star}(\vec{m}^\star) \neq \emptyset$, then the maps in (7.14) are birational, so statement (ii) holds by Lemma 7.11. Below, we show that $M^{s}_{Q^\star}(\vec{m}^\star) = \emptyset$ implies that $M^{\xi^-}_{Q^\star}(\vec{m}^\star) = \emptyset$. Note that for $i \in V(Q)$, we have

$$\langle \vec{m}^\star, \vec{i} \rangle = \langle \vec{m}, \vec{i} \rangle - a_i\,, \quad \langle \vec{i}, \vec{m}^\star \rangle = \langle \vec{i}, \vec{m} \rangle - b_i\,.$$

We also have the identities

$$\langle \vec{m}^\star, \vec{0} \rangle = 1 - \sum_{i \in V(Q)} b_i m_i\,, \quad \langle \vec{0}, \vec{m}^\star \rangle = 1 - \sum_{i \in V(Q)} a_i m_i\,.$$

By our assumption $a_i > b_i$, we have the inequalities

$$\langle \vec{m}^\star, \vec{i} \rangle < \langle \vec{i}, \vec{m}^\star \rangle\,, \quad \langle \vec{0}, \vec{m}^\star \rangle < \langle \vec{m}^\star, \vec{0} \rangle\,. \tag{7.16}$$

If $\langle \vec{i}, \vec{m}^\star \rangle > 0$ for $i \in V(Q)$, then by Remark 7.2, any $Q^\star$-representation $\mathbb{V}^\star$ with dimension vector $\vec{m}^\star$ admits an injection $S_i \hookrightarrow \mathbb{V}^\star$. Therefore, we have $M^{\xi^-}_{Q^\star}(\vec{m}^\star) = \emptyset$. Similarly, if $\langle \vec{m}^\star, \vec{0} \rangle > 0$, then any such $\mathbb{V}^\star$ admits a surjection $\mathbb{V}^\star \twoheadrightarrow S_0$, which implies $M^{\xi^-}_{Q^\star}(\vec{m}^\star) = \emptyset$. Therefore, by the inequalities (7.16), we may assume $\langle \vec{m}^\star, \vec{j} \rangle \leqslant 0$ and $\langle \vec{j}, \vec{m}^\star \rangle \leqslant 0$ for any $j \in V(Q^\star)$.

By Theorem 7.1, the condition $M^{s}_{Q^\star}(\vec{m}^\star) = \emptyset$ implies that $\mathrm{supp}(\vec{m}^\star)$ is not strongly connected. Let $Q_1, \ldots, Q_l$ be the connected components of $\mathrm{supp}(\vec{m})$, which are subquivers of Q. As $\mathrm{supp}(\vec{m}^\star)$ is not strongly connected and Q is symmetric, there is a $1 \leqslant k \leqslant l$ such that we have $b_i = 0$ for any $i \in V(Q_k)$. This implies that any $Q^\star$-representation $\mathbb{V}^\star$ with dimension vector $\vec{m}^\star$ admits an injection $\mathbb{V}' \hookrightarrow \mathbb{V}^\star$ for a non-zero Q_k-representation $\mathbb{V}'$. Therefore, we have $M^{\xi^-}_{Q^\star}(\vec{m}^\star) = \emptyset$, and assertion (i) holds. $\square$

Example 7.14. Let Q be the symmetric quiver with one vertex and no loops, and write $V(Q) = \{1\}$. Then $Q^\star$ has two vertices $V(Q^\star) = \{0, 1\}$, with a_1 arrows from 0 to 1, b_1 arrows from 1 to 0, and c loops at 0. The dimension vector $\vec{m}^\star$ is written as $\vec{0} + m \cdot \vec{1}$ for $m \in \mathbb{Z}_{>0}$. Let V be a vector space with dimension m. Then by Lemma 7.9, the stack $\mathcal{M}_{Q^\star}(\vec{m}^\star)$ is written as

$$\mathcal{M}_{Q^\star}(\vec{m}^\star) = \left[\left(\mathrm{Hom}(\mathbb{E}_{0,1}, V) \times \mathrm{Hom}\left(\mathbb{E}_{1,0}, V^\vee\right)\right)/(\mathbb{C}^* \times \mathrm{GL}(V))\right] \times \mathbb{E}_{0,0}^\vee\,.$$

By Lemma 7.10, we see that

$$\begin{aligned} M^{\xi^+}_{Q^\star}(\vec{m}^\star) &= \mathrm{Tot}_{\mathrm{Gr}(\mathbb{E}_{0,1}, m)}\left(\mathcal{Q}_{0,1}^\vee \otimes \mathbb{E}_{1,0}^\vee\right) \times \mathbb{E}_{0,0}^\vee\,, \\ M^{\xi^-}_{Q^\star}(\vec{m}^\star) &= \mathrm{Tot}_{\mathrm{Gr}(\mathbb{E}_{1,0}, m)}\left(\mathcal{Q}_{1,0}^\vee \otimes \mathbb{E}_{0,1}^\vee\right) \times \mathbb{E}_{0,0}^\vee\,. \end{aligned}$$

Here $\mathcal{Q}_{i,j}$ is the universal quotient bundle on $\mathrm{Gr}(\mathbb{E}_{i,j}, m)$. In this case, the birational map $M^{\xi^+}_{Q^\star}(\vec{m}^\star) \dashrightarrow M^{\xi^-}_{Q^\star}(\vec{m}^\star)$ is a Grassmannian flip.

Let $W^\star$ be a convergent super-potential of $Q^\star$, and take an analytic neighborhood $V \subset M_{Q^\star}(\vec{m}^\star)$ of 0 as in (5.13). We have the diagram

$$\begin{tikzcd} M^{\xi^+}_{(Q^\star,\partial W^\star)}(\vec{m}^\star)|_V \arrow[dr, "q^{\xi^+}_{(Q^\star,\partial W^\star)}"'] & & M^{\xi^-}_{(Q^\star,\partial W^\star)}(\vec{m}^\star)|_V \arrow[dl, "q^{\xi^-}_{(Q^\star,\partial W^\star)}"] \\ & M_{(Q^\star,\partial W^\star)}(\vec{m}^\star)|_V . & \end{tikzcd} \tag{7.17}$$

By Lemma 5.3 and Proposition 7.13, we have the following corollary.

Corollary 7.15. *For the diagram* (7.17), *either one of the following holds:*

(i) *We have* $M^{\xi^-}_{Q^\star}(\vec{m}^\star) = M^{\xi^-}_{(Q^\star,\partial W^\star)}(\vec{m}^\star)|_V = \emptyset$, *and the diagram* (7.17) *is a d-critical generalized MFS.*

(ii) *The diagram* (7.17) *is a d-critical generalized flip.*

We also have the strictness (see Definition 3.14) of the diagram (7.17) under some conditions.

Lemma 7.16. *Suppose that* $W^\star$ *is minimal* (*see* (5.12)) *and* $a_i > m_i$ *for any* $i \in V(Q)$. *Then the diagram* (7.17) *is strict at* $0 \in M_{(Q^\star,\partial W^\star)}(\vec{m}^\star)|_V$.

Proof. We need to show that the map $q^{\xi^+}_{(Q^\star,\partial W^\star)}$ in the diagram (7.17) is not a finite morphism at $0 \in M_{(Q^\star,\partial W^\star)}(\vec{m}^\star)|_V$. We consider nilpotent $Q^\star$-representations $\mathbb{V}^\star$ given by Q-representations (7.11), where $u_e = 0$ for all $e \in E(Q)$, together with surjective linear maps $\mathbb{E}_{0,i} \to V_i$ and zero maps $\mathbb{E}_{i,0} \otimes V_i \to 0$ in (7.12). The isomorphism classes of such $Q^\star$-representations form the product of Grassmannians $\mathrm{Gr}(\mathbb{E}_{0,i}, m_i)$ for all $i \in V(Q)$. By Lemma 7.10, such $Q^\star$-representations are ξ^+-stable. They also satisfy the relation $\partial W^\star$ (see Remark 5.2) by the minimality of $W^\star$, so we have

$$\prod_{i \in V(Q)} \mathrm{Gr}(\mathbb{E}_{0,i}, m_i) \subset \left(q^{\xi^+}_{(Q^\star,\partial W^\star)}\right)^{-1}(0)\,.$$

Since the left-hand side is not zero-dimensional by the assumption $a_i > m_i$, the lemma holds. $\square$

8. D-critical flops of moduli spaces of 1-dimensional sheaves

In this section, we show that wall-crossing phenomena of 1-dimensional stable sheaves on CY 3-folds are described in terms of d-critical (generalized) flops. The proof of this result is related to the proof of the wall-crossing formula of Gopakumar–Vafa invariants given in [Tod22b].

8.1 Twisted semistable sheaves

For a smooth projective CY 3-fold X, let $\mathrm{Coh}_{\leqslant 1}(X) \subset \mathrm{Coh}(X)$ be the abelian subcategory of coherent sheaves E on X whose supports have dimensions less than or equal to 1 and

$$D^b_{\leqslant 1}(X) := D^b(\mathrm{Coh}_{\leqslant 1}(X)) \subset D^b(X)$$

its bounded derived category. Let $\Gamma_{\leqslant 1}$ be defined by

$$\Gamma_{\leqslant 1} := H_2(X, \mathbb{Z}) \oplus \mathbb{Z}\,. \tag{8.1}$$

The Chern character of an object in $D^b_{\leqslant 1}(X)$ takes its value in $\Gamma_{\leqslant 1}$ and is given by

$$\mathrm{ch}(E) = (\mathrm{ch}_2(E), \mathrm{ch}_3(E)) = ([E], \chi(E))\,. \tag{8.2}$$

Here $[E]$ is the fundamental 1-cycle associated with E.

We denote by $\mathrm{Stab}_{\leqslant 1}(X)$ the space of Bridgeland stability conditions on $D^b_{\leqslant 1}(X)$ with respect to the Chern character map (8.2). Namely, a point $\sigma \in \mathrm{Stab}_{\leqslant 1}(X)$ is a pair

$$\sigma = (Z, \mathcal{A})\,, \quad \mathcal{A} \subset D^b_{\leqslant 1}(X)\,, \quad Z \colon \Gamma_{\leqslant 1} \longrightarrow \mathbb{C}\,,$$

where $\mathcal{A}$ is the heart of a bounded t-structure on $D^b_{\leqslant 1}(X)$ and Z is a group homomorphism, satisfying some conditions (see Appendix A). By Theorem A.4, the forgetful map $(Z, \mathcal{A}) \mapsto Z$ gives a local homeomorphism $\mathrm{Stab}_{\leqslant 1}(X) \to (\Gamma_{\leqslant 1})^{\vee}_{\mathbb{C}}$. Let $A(X)_{\mathbb{C}}$ be the complexified ample cone of X defined by

$$A(X)_{\mathbb{C}} := \big\{B + i\omega \in H^2(X, \mathbb{C}) \colon \omega \text{ is ample}\big\}\,.$$

For a given element $B + i\omega \in A(X)_{\mathbb{C}}$, let $Z_{B,\omega}$ be the group homomorphism defined by

$$Z_{B,\omega} \colon \Gamma_{\leqslant 1} \longrightarrow \mathbb{C}\,, \quad (\beta, n) \longmapsto -n + (B + i\omega)\beta\,.$$

Then the pair

$$\sigma_{B,\omega} := (Z_{B,\omega}, \mathrm{Coh}_{\leqslant 1}(X)) \tag{8.3}$$

determines a point in $\mathrm{Stab}_{\leqslant 1}(X)$. The map

$$A(X)_{\mathbb{C}} \longrightarrow \mathrm{Stab}_{\leqslant 1}(X)\,, \quad (B, \omega) \longmapsto \sigma_{B,\omega}$$

is a continuous injective map, whose image is denoted by

$$U(X) \subset \mathrm{Stab}_{\leqslant 1}(X)\,.$$

Remark 8.1. An object $E \in \mathrm{Coh}_{\leqslant 1}(X)$ is $\sigma_{B,\omega}$-stable (respectively, $\sigma_{B,\omega}$-semistable) if and only if for any subsheaf $0 \neq F \subsetneq E$, we have the inequality

$$\mu_{B,\omega}(F) < \mu_{B,\omega}(E) \quad (\text{respectively, } \mu_{B,\omega}(F) \leqslant \mu_{B,\omega}(E))\,.$$

Here $\mu_{B,\omega}(E) \in \mathbb{R} \cup \{\infty\}$ is defined by

$$\mu_{B,\omega}(E) := \frac{\chi(E) - B \cdot [E]}{\omega \cdot [E]} = -\frac{\Re Z_{B,\omega}(E)}{\Im Z_{B,\omega}(E)} \tag{8.4}$$

when $\omega \cdot [E] \neq 0$ and $\mu_{B,\omega}(E) = \infty$ when $\omega \cdot [E] = 0$.

8.2 Moduli spaces of 1-dimensional stable sheaves

Let us take

$$\beta \in H_2(X, \mathbb{Z})\,, \quad \sigma = (Z, \mathrm{Coh}_{\leqslant 1}(X)) \in U(X)\,,$$

where β is an effective curve class. We denote by $\mathcal{M}_\sigma(\beta)$ the moduli stack of σ-semistable $E \in \mathrm{Coh}_{\leqslant 1}(X)$ satisfying $\mathrm{ch}(E) = (\beta, 1)$. The stack $\mathcal{M}_\sigma(\beta)$ is an Artin stack locally of finite type, with a good moduli space

$$p_M \colon \mathcal{M}_\sigma(\beta) \longrightarrow M_\sigma(\beta)$$

for a projective scheme $M_\sigma(\beta)$ (see [Tod18, Lemma 7.4]). A closed point of $M_\sigma(\beta)$ corresponds to a σ-polystable sheaf, that is, a direct sum

$$E = \bigoplus_{i=1}^{k} V_i \otimes E_i\,, \tag{8.5}$$

where each V_i is a finite-dimensional vector space, $E_i \in \mathrm{Coh}_{\leqslant 1}(X)$ is a σ-stable sheaf with $\arg Z(E_i) = \arg Z(E)$, and $E_i \ncong E_j$ for $i \neq j$.

Remark 8.2. The stack $\mathcal{M}_\sigma(\beta)$ is a GIT quotient stack (see [Tod18, Lemma 7.4]), so the good moduli space $M_\sigma(\beta)$ exists without relying on [AHH21] (see Remark 4.9).

As we mentioned in Subsection 4.3, there is a wall-chamber structure on $\mathrm{Stab}_{\leqslant 1}(X)$. On the subset $U(X) \subset \mathrm{Stab}_{\leqslant 1}(X)$, each wall is given by

$$\{(Z, \mathrm{Coh}_{\leqslant 1}(X)) \in U(X) \colon Z(v_1) \in \mathbb{R}_{>0} Z(v_2)\}$$

for each decomposition $(\beta, 1) = v_1 + v_2$ with $v_i = (\beta_i, n_i) \in \Gamma_{\leqslant 1}$. Here β_i is an effective curve class. Suppose that $\sigma \in U(X)$ lies in one of these walls, and write $\sigma = \sigma_{B,\omega}$ as in (8.3) for $B + i\omega \in A(X)_{\mathbb{C}}$. Let us take other stability conditions $\sigma^{\pm} \in U(X)$ written as

$$\sigma^{\pm} = \sigma_{B \pm \varepsilon_B, \omega \pm \varepsilon_\omega} \in U(X) \tag{8.6}$$

for $\varepsilon_B + i\varepsilon_\omega \in H^2(X, \mathbb{C})$. We assume that $\varepsilon_B + i\varepsilon_\omega$ is sufficiently small and general so that both of $\sigma^{\pm}$ lie on chambers. Similarly to the diagram (6.11), we have the diagram

$$\begin{tikzcd} M_{\sigma^+}(\beta) \arrow[dr, "q_M^+"'] & & M_{\sigma^-}(\beta) \arrow[dl, "q_M^-"] \\ & M_\sigma(\beta)\,. & \end{tikzcd} \tag{8.7}$$

Note that as $(\beta, 1)$ is primitive in $\Gamma_{\leqslant 1}$ and $\sigma^{\pm}$ lie on chambers, both of $M_{\sigma^{\pm}}(\beta)$ consist of $\sigma^{\pm}$-stable sheaves. Therefore, by Theorem 4.1, they admit d-critical structures. Applying Corollary 6.2, we have the following.

Theorem 8.3. *The diagram (8.7) is an analytic d-critical generalized flop.*

Proof. For a point $p \in M_\sigma(\beta)$, suppose that it corresponds to a polystable sheaf E of the form (8.5). Since each E_i has at most 1-dimensional support, the Ext-quiver $Q = Q_{E_\bullet}$ associated with the collection $E_\bullet = (E_1, E_2, \ldots, E_k)$ is symmetric (see [Tod22b, Lemma 5.1]). We take data $\xi^{\pm} \in \mathcal{H}^k$ as in (5.7) for the quiver Q, given by

$$\xi_i^{\pm} = Z^{\pm}_{B \pm \varepsilon_B, \omega \pm \varepsilon_\omega}(E_i) = Z_{B,\omega}(E_i) \pm (\varepsilon_B + i\varepsilon_\omega) \cdot [E_i]\,.$$

As $\arg Z_{B,\omega}(E_i) = \arg Z_{B,\omega}(E)$, by taking rotations and scaling $\xi^{\pm}$, and also perturbing $\varepsilon_B + i\varepsilon_\omega$ if necessary, we may assume that $\xi^{\pm}$ satisfy the condition (7.6). Then the result follows from Corollaries 6.2 and 7.8. $\square$

8.3 Example: Elliptic CY 3-fold

Here we discuss an example of wall-crossing of 1-dimensional stable sheaves on an elliptic CY 3-fold. Let $S = \mathbb{P}^2$, and take general elements

$$u \in H^0(S, \mathcal{O}_S(-4K_S))\,, \quad v \in H^0(S, \mathcal{O}_S(-6K_S))\,.$$

Then as in [Tod12a, Section 6.4], we have a simply connected CY 3-fold X with a flat elliptic fibration $\pi_X\colon X \to S$ defined by the equation $zy^2 = uxz^2 + vz^3$ in the projective bundle

$$\mathbb{P}_S(\mathcal{O}_S(-2K_S) \oplus \mathcal{O}_S(-3K_S) \oplus \mathcal{O}_S) \longrightarrow S\,.$$

Here $[x:y:z]$ are the homogeneous coordinates of this projective bundle. Note that π_X admits a section $\iota\colon S \to X$ whose image $D := \iota(S)$ corresponds to the fiber point $[0:1:0]$. Let $H \subset X$ be the pull-back of a hyperplane in $\mathbb{P}^2$ to X by π_X. We have $H^2(X,\mathbb{R}) = \mathbb{R}[D] + \mathbb{R}[H]$. Let F be a fiber of π_X and $l \subset D$ a line. Then $[F]$ and $[l]$ span the Mori cone $\overline{NE}(X)$ of X. The intersection matrix is given by

$$\begin{pmatrix} H\cdot l & D\cdot l \\ H\cdot F & D\cdot F \end{pmatrix} = \begin{pmatrix} 1 & -3 \\ 0 & 1 \end{pmatrix}.$$

We fix an ample divisor ω_0 on X and write $d_1 = \omega_0 \cdot F > 0$, $d_2 = \omega_0 \cdot l > 0$. Let us take an effective curve class β and write it as $\beta = r[F] + k[l]$ with $r, k \in \mathbb{Z}_{\geqslant 0}$. We consider a wall-chamber structure on the subset of $U(X)$ given by the image of the map

$$H^2(X,\mathbb{R}) \longrightarrow U(X)\,, \quad B \longmapsto \sigma_{B,\omega_0}\,. \tag{8.8}$$

We identify the image of (8.8) with $H^2(X,\mathbb{R})$. For each decomposition in $\Gamma_{\leqslant 1}(X)$

$$(\beta, 1) = (\beta_1, n_1) + (\beta_2, n_2)\,, \quad \beta_i = r_i[F] + k_i[l]\,, \tag{8.9}$$

the wall is given by the equation $\mu_{B,\omega_0}(\beta_1, n_1) = \mu_{B,\omega_0}(\beta, 1)$. By writing $B = x[D] + y[H]$, a direct computation shows that the above condition is equivalent to

$$(3d_1 + d_2)x - d_1 y = \frac{r_1 d_1 + k_1 d_2 - n_1(r d_1 + k d_2)}{r k_1 - k r_1}\,. \tag{8.10}$$

It follows that every wall is proportional to the line $y = (3 + d_2/d_1)x$, so any two walls are disjoint if they do not coincide.

In the case $k = 1$, that is, $\beta = r[F] + [l]$, we have the decomposition

$$(r[F] + [l], 1) = (r[F], 1) + ([l], 0)\,. \tag{8.11}$$

We set

$$B_0 = \frac{d_2}{r(3d_1 + d_2)}[D]\,, \quad B_\pm = B_0 \pm \varepsilon[D]\,, \quad 0 < \varepsilon \ll 1\,.$$

This B_0 satisfies the equation (8.10) determined by the decomposition (8.11). Therefore, $\sigma_0 := \sigma_{B_0,\omega_0}$ lies on a wall, and $\sigma_\pm := \sigma_{B_\pm,\omega_0}$ lie on its adjacent chambers.

In the above $k = 1$ case, we can describe the wall-crossing diagram (8.7) for $\sigma = \sigma_0$ in terms of a d-critical simple flop. It is easy to see that σ_0 does not lie on a wall determined by a decomposition of the form (8.9), other than (8.11). Therefore, any point $p \in M_{\sigma_0}(\beta)$ that does not correspond to a σ_0-stable sheaf corresponds to a σ_0-polystable sheaf E of the form

$$E = E_1 \oplus E_2\,, \quad \mathrm{ch}(E_1) = (r[F], 1)\,, \quad \mathrm{ch}(E_2) = ([l], 0)\,.$$

Here E_1, E_2 are σ-stable sheaves. Then E_1 is a stable vector bundle on a fiber F with rank r and degree 1, and $E_2 = \mathcal{O}_l(-1)$ for some line $l \subset D$. If $F \cap l = \emptyset$, then

$$\mathrm{Ext}^1(E_1, E_2) = \mathrm{Ext}^1(E_2, E_1) = 0\,,$$

and we have $\left(q_M^{\pm}\right)^{-1}(U) = \emptyset$ for a small open subset $U \subset M_\sigma(\beta)$ containing p. This is an obvious case of a d-critical generalized flop (see Remark 3.11). Otherwise, $F \cap l$ is one point, and it is easy to check that

$$\mathrm{Ext}^1(E_1, E_1) = \mathbb{C}^3, \quad \mathrm{Ext}^1(E_2, E_2) = \mathbb{C}^2, \quad \mathrm{Ext}^1(E_1, E_2) = \mathrm{Ext}^1(E_2, E_1) = \mathbb{C}^r.$$

The associated Ext-quiver Q has two vertices $\{1, 2\}$ and r arrows from 1 to 2 and 2 to 1, and the numbers of loops at 1 and 2 are 3 and 2, respectively. In this case, the diagram (8.7) is a d-critical simple flop at $p \in M_\sigma(\beta)$, as we mentioned in Example 6.3.

8.4 D-critical flops under flops

For a CY 3-fold X and $\beta \in H_2(X, \mathbb{Z})$, we define

$$M_X(\beta) := M_{\sigma=\sigma_{(0,\omega)}}(\beta). \tag{8.12}$$

Here ω is an ample divisor on X. Note that $M_X(\beta)$ is independent of a choice of ω (see [MT18, Remark 3.2]). Moreover, $M_X(\beta)$ consists of stable sheaves, so it admits a d-critical structure by Theorem 4.1. The moduli space (8.12) was used in [MT18] in the definition of Gopakumar–Vafa invariants on CY 3-folds.

Suppose that we have a flop diagram of CY 3-folds

$$\begin{tikzcd} X \arrow[rr, dashed, "\phi"] \arrow[dr] & & X^\dagger \arrow[dl] \\ & Y. & \end{tikzcd} \tag{8.13}$$

Let $\phi_*\beta \in H_2(X^\dagger, \mathbb{Z})$ be defined by $\phi_*\beta \cdot D = \beta \cdot \phi_*^{-1}D$ for any divisor D on $X^\dagger$. Here $\phi_*^{-1}D$ is a strict transform of D to X. In the above situation, we have the following.

THEOREM 8.4. *The d-critical loci $M_X(\beta)$ and $M_{X^\dagger}(\phi_*\beta)$ are connected by a sequence of analytic d-critical generalized flops.*

Proof. Under a 3-fold flop (8.13), we have the commutative diagram (see [Bri02, Tod08])

$$\begin{tikzcd} D^b_{\leqslant 1}(X) \arrow[r, "\Phi"] \arrow[d, "\mathrm{ch}"'] & D^b_{\leqslant 1}(X^\dagger) \arrow[d, "\mathrm{ch}"] \\ \Gamma_{\leqslant 1} \arrow[r, "\Phi_\Gamma"'] & \Gamma^\dagger_{\leqslant 1}. \end{tikzcd}$$

Here Φ is an equivalence of derived categories, $\Gamma^\dagger_{\leqslant 1} = H_2(X^\dagger, \mathbb{Z}) \oplus \mathbb{Z}$ and Φ_Γ is an isomorphism that takes (β, n) to $(\phi_*\beta, n)$. The above equivalence induces the isomorphism

$$\Phi_* \colon \mathrm{Stab}_{\leqslant 1}(X) \xrightarrow{\cong} \mathrm{Stab}_{\leqslant 1}(X^\dagger).$$

Under this isomorphism, the closures of $\Phi_* U(X)$ and $U(X^\dagger)$ intersect (see [Tod22b, Lemma 6.4]). Let $\omega^\dagger$ be an ample divisor on $X^\dagger$. We take a path

$$\gamma \colon [0,1] \longrightarrow \Phi_*\overline{U}(X) \cup \overline{U}(X^\dagger)$$

such that $\gamma(0) = \Phi_*\sigma_{0,\omega}$ and $\gamma(1) = \sigma_{(0,\omega^\dagger)}$. By perturbing γ if necessary, we can assume that each wall-crossing of the path is given as in (8.6). Moreover, the intersection of $\Phi_*\overline{U}(X)$ and $\overline{U}(X^\dagger)$ is not a wall by [Tod22b, Lemma 6.7]. Therefore, applying Theorem 8.3 at each wall, we obtain the result. □

9. D-critical flips of moduli spaces of stable pairs

In this section, we show that wall-crossing phenomena of Pandharipande–Thomas stable pair moduli spaces [PT09], studied in [Bri11, Tod09a, Tod10b, Tod12a, Dia12], can be described in terms of d-critical birational geometry. The wall-crossing phenomena here were used in those references to show the rationality of the generating series of stable pair invariants, which we will review in Appendix C.

9.1 Stable pairs

Let X be a smooth projective CY 3-fold over $\mathbb{C}$. The notion of stable pairs introduced by Pandharipande–Thomas is given below.

Definition 9.1 ([PT09]). A *stable pair* consists of data (F, s) with $F \in \mathrm{Coh}_{\leqslant 1}(X)$ and $s\colon \mathcal{O}_X \to F$, where F is a pure 1-dimensional sheaf and s is a morphism of coherent sheaves that is surjective in dimension 1.

As in (8.1), we set $\Gamma_{\leqslant 1} = H_2(X, \mathbb{Z}) \oplus \mathbb{Z}$. For $(\beta, n) \in \Gamma_{\leqslant 1}$, let

$$P_n(X, \beta) \tag{9.1}$$

be the moduli space of stable pairs (F, s) such that $\mathrm{ch}(F) = (\beta, n)$. It is proved in [PT09, HT10] that the moduli space (9.1) is a projective scheme with a symmetric perfect obstruction theory. Indeed, the moduli space (9.1) is identified with the moduli space of 2-term complexes in the derived category (here $\mathcal{O}_X$ is located in degree zero)

$$I^{\bullet} = \left(\cdots \longrightarrow 0 \longrightarrow \mathcal{O}_X \stackrel{s}{\longrightarrow} F \to 0 \longrightarrow \cdots\right) \in D^b(X) \tag{9.2}$$

satisfying a certain stability condition on it (see [PT09, Tod10a]). It follows that by Theorem 4.1, there is a canonical d-critical structure on the moduli space of stable pairs (9.1).

9.2 Weak semistable objects

We study wall-crossing in the abelian subcategory in $D^b(X)$ defined below.

Definition 9.2. We define the subcategory $\mathcal{A}_X$ in $D^b(X)$ by

$$\mathcal{A}_X := \langle \mathcal{O}_X, \mathrm{Coh}_{\leqslant 1}(X)[-1] \rangle_{\mathrm{ex}} \subset D^b(X)\,. \tag{9.3}$$

Here $\langle * \rangle_{\mathrm{ex}}$ is the smallest extension-closed subcategory containing $*$.

The category (9.3) is an abelian subcategory of $D^b(X)$ (see [Tod10a, Lemma 6.2]). We have the Chern character map

$$\mathrm{cl}\colon K(\mathcal{A}_X) \longrightarrow \Gamma^{\star}_{\leqslant 1} := \mathbb{Z} \oplus \Gamma_{\leqslant 1}$$

sending $\mathcal{O}_X$ to $(1, 0)$ and $F \in \mathrm{Coh}_{\leqslant 1}(X)$ to $(0, \mathrm{ch}(F))$. We will be interested in certain rank 1 objects in $\mathcal{A}_X$. We have the following lemma describing rank 1 objects in $\mathcal{A}_X$; the proof is obvious.

Lemma 9.3. *An object $E \in D^b(X)$ with $\mathrm{rank}(E) = 1$ is an object in $\mathcal{A}_X$ if and only if there exist distinguished triangles in $D^b(X)$*

$$\begin{tikzcd}
0 = E_0 \arrow[rr] & & E_1 \arrow[rr] \arrow[dl] & & E_2 \arrow[rr] \arrow[dl] & & E_3 = E \arrow[dl] \\
& F_1 \arrow[ul, dotted, "{[1]}"] & & F_2 \arrow[ul, dotted, "{[1]}"] & & F_3 \arrow[ul, dotted, "{[1]}"] &
\end{tikzcd} \tag{9.4}$$

such that

$$F_1 \in \mathrm{Coh}_{\leqslant 1}(X)[-1]\,, \quad F_2 = \mathcal{O}_X\,, \quad F_3 \in \mathrm{Coh}_{\leqslant 1}(X)[-1]\,.$$

In this case, the top sequence of (9.4) *is a filtration in* $\mathcal{A}_X$.

Remark 9.4. A 2-term complex $I^\bullet$ in (9.2) associated with a stable pair is an object in $\mathcal{A}_X$ as it fits into the exact sequence in $\mathcal{A}_X$

$$0 \longrightarrow F[-1] \longrightarrow I^\bullet \longrightarrow \mathcal{O}_X \longrightarrow 0\,.$$

Similarly, the derived dual $\mathbb{D}(I^\bullet)$ for $\mathbb{D}(*) = \mathbf{R}\mathcal{H}\mathrm{om}(*, \mathcal{O}_X)$ is an object in $\mathcal{A}_X$ as it fits into the exact sequence in $\mathcal{A}_X$

$$0 \longrightarrow \mathcal{O}_X \longrightarrow \mathbb{D}(I^\bullet) \longrightarrow F^\vee[-1] \longrightarrow 0\,,$$

where $F^\vee := \mathcal{E}\mathrm{xt}^2_{\mathcal{O}_X}(F, \mathcal{O}_X)$.

In what follows, we fix an ample divisor ω on X. For $t \in \mathbb{R}$, let $\mu_t^\star$ be the slope function on the abelian category $\mathcal{A}_X$ defined, for $E \in \mathcal{A}_X$, by

$$\mu_t^\star(E) := \begin{cases} t & \text{if } E \notin \mathrm{Coh}_{\leqslant 1}(X)[-1]\,, \\ \mu_\omega(E) = \chi(E)/\omega \cdot [E] & \text{if } E \in \mathrm{Coh}_{\leqslant 1}(X)[-1]\,. \end{cases}$$

Here $\mu_\omega := \mu_{0,\omega}$ is the slope function on 1-dimensional sheaves defined as in (8.4) for $B = 0$. This slope function $\mu_t^\star$ on $\mathcal{A}_X$ satisfies the weak see-saw property and defines the weak stability condition on $\mathcal{A}_X$ (see [Tod10b, Tod12a]).

Definition 9.5. An object $E \in \mathcal{A}_X$ is $\mu_t^\star$*-stable* (respectively, $\mu_t^\star$*-semistable* if for any exact sequence $0 \to F \to E \to G \to 0$ in $\mathcal{A}_X$, we have the inequality

$$\mu_t^\star(F) < \mu_t^\star(G) \quad (\text{respectively, } \mu_t^\star(F) \leqslant \mu_t^\star(G)$$

Remark 9.6. The above $\mu_t^\star$-stability condition on $\mathcal{A}_X$ can also be formulated in terms of a Bridgeland-type weak stability condition introduced in [Tod10a], by taking the filtration $\{0\} \oplus \Gamma_{\leqslant 1} \subset \Gamma^\star_{\leqslant 1}$. See [Tod12a] for details.

9.3 Moduli spaces of weak semistable objects

Let $\mathcal{M}$ be the moduli stack of objects in $D^b(X)$ considered in Subsection 4.1. For $(\beta, n) \in \Gamma_{\leqslant 1}$ and $t \in \mathbb{R}$, we have the substack

$$\mathcal{M}_t^\star(\beta, n) \subset \mathcal{M} \tag{9.5}$$

consisting of the $\mu_t^\star$-semistable objects $E \in \mathcal{A}_X$ satisfying $\mathrm{cl}(E) = (1, -\beta, -n) \in \Gamma^\star_{\leqslant 1}$. The substack (9.5) is an open substack of $\mathcal{M}$, which is an Artin stack of finite type over $\mathbb{C}$ (see [Tod10b, Proposition 3.17], [Tod12a, Proposition 5.4]). As in Theorem 4.8, the result of [AHH21] is applied to the stack $\mathcal{M}_t^\star(\beta, n)$. So it admits a good moduli space

$$p_t \colon \mathcal{M}_t^\star(\beta, n) \longrightarrow M_t^\star(\beta, n)\,,$$

where $M_t^\star(\beta, n)$ is a separated algebraic space of finite type. A closed point of $M_t^\star(\beta, n)$ corresponds to a $\mu_t^\star$-polystable object $E \in \mathcal{A}_X$ written as

$$E = \bigoplus_{i=0}^{k} V_i \otimes E_i\,, \quad E_i \in \mathcal{A}_X\,. \tag{9.6}$$

Here each V_i is a finite-dimensional vector space with $V_0 = \mathbb{C}$, the object $E_0 \in \mathcal{A}_X$ is a rank 1 $\mu_t^\star$-stable object, and each E_i, for $1 \leqslant i \leqslant k$, is isomorphic to $F_i[-1]$ for a μ_ω-semistable sheaf $F_i \in \mathrm{Coh}_{\leqslant 1}(X)$ with $\mu_\omega(F_i) = t$ such that $F_i \ncong F_j$ for $i \neq j$. The moduli space $M_t^\star(\beta, n)$ is related to stable pair moduli spaces as follows.

Proposition 9.7 ([Tod10b, Theorem 3.21], [Tod12a, Proposition 5.4]). *For $|t| \gg 0$, the moduli space $M_t^\star(\beta, n)$ consists of $\mu_t^\star$-stable objects. Moreover, we have isomorphisms*

$$P_n(X, \beta) \xrightarrow{\cong} M_t^\star(\beta, n)\,, \quad t \gg 0\,, \tag{9.7}$$

$$P_{-n}(X, \beta) \xrightarrow{\cong} M_t^\star(\beta, n)\,, \quad t \ll 0\,. \tag{9.8}$$

The isomorphism (9.7) sends a stable pair (F, s) to the 2-term complex (9.2), and the isomorphism (9.8) sends (F, s) to the derived dual of (9.2) (see Remark 9.4).

Proof. For $|t| \gg 0$, it is proved in [Tod10b, Theorem 3.21], [Tod12a, Proposition 5.4] that the stack $\mathcal{M}_t^\star(\beta, n)$ consists of $\mu_t^\star$-stable objects, and we have the isomorphism

$$[P_{\pm n}(X, \beta)/\mathbb{C}^*] \xrightarrow{\cong} \mathcal{M}_t^\star(\beta, n)\,, \quad \pm t \gg 0\,.$$

Here $\mathbb{C}^*$ acts on $P_{\pm n}(X, \beta)$ trivially, and the above isomorphisms are defined as in the statement of the proposition. By taking the good moduli spaces of both sides, we obtain the proposition. □

Remark 9.8. By the isomorphism (9.8), an object $E \in \mathcal{A}_X$ is isomorphic to $\mathbb{D}(I^\bullet)$ for a stable pair (9.2) as in Remark 9.4 if and only if E fits into an exact sequence in $\mathcal{A}_X$

$$0 \longrightarrow \mathcal{O}_X \longrightarrow E \longrightarrow F'[-1] \longrightarrow 0$$

for some $F' \in \mathrm{Coh}_{\leqslant 1}(X)$ such that $\mathrm{Hom}(F''[-1], E) = 0$ for any $F'' \in \mathrm{Coh}_{\leqslant 1}(X)$. This fact will be not used later in this paper.

9.4 Wall-crossing of weak semistable objects

For $t \in \mathbb{R}$, we set $t^\pm := t \pm \varepsilon$ with $0 < \varepsilon \ll 1$. Then we have open immersions of stacks

$$\mathcal{M}_{t^+}^\star(\beta, n) \subset \mathcal{M}_t^\star(\beta, n) \supset \mathcal{M}_{t^-}^\star(\beta, n)\,. \tag{9.9}$$

We define $\mathcal{W} \subset \mathbb{R}$ to be the subset of $t \in \mathbb{R}$ where at least one of the open immersions in (9.9) is not an isomorphism; that is, $\mathcal{W}$ is the set of walls for the $\mu_t^\star$-stability for $t \in \mathbb{R}$.

Lemma 9.9. *The subset $\mathcal{W} \subset \mathbb{R}$ is a finite set.*

Proof. An element $t \in \mathcal{W}$ is of the form $t = n'/(\omega \cdot \beta')$ for $0 < \beta' < \beta$ and $n' \in \mathbb{Z}$. In particular, $\mathcal{W}$ is locally finite. It is also bounded by Proposition 9.7; hence $\mathcal{W}$ is a finite set. □

We write $\mathcal{W}$ as

$$\mathcal{W} = \{\infty = t_0 > t_1 > t_2 > \cdots > t_l > t_{l+1} = -\infty \colon t_1, \ldots, t_l \in \mathbb{R}\}\,. \tag{9.10}$$

Note that each $\mathcal{M}_t^\star(\beta, n)$ is constant if t lies in a connected component of $\mathbb{R} \setminus \mathcal{W}$ but may change if t crosses one of t_i. For $t \in \mathcal{W}$, the open immersions (9.9) induce the diagram of good moduli spaces

$$\begin{tikzcd} M_{t^+}^\star(\beta, n) \arrow[dr, "q_{M^\star}^+"'] & & M_{t^-}^\star(\beta, n) \arrow[dl, "q_{M^\star}^-"] \\ & M_t^\star(\beta, n)\,. & \end{tikzcd} \tag{9.11}$$

Note that $M_{t^{\pm}}(\beta, n)$ consists of $\mu_{t^{\pm}}$-stable objects; hence they admit d-critical structures by Theorem 4.1. Below, we investigate the d-critical birational geometry of the diagram (9.11).

Let us take a point $p \in M_t^{\star}(\beta, n)$ in the diagram (9.11) and suppose that it corresponds to a $\mu_t^{\star}$-polystable object (9.6). Let $Q^{\star} = Q_{E_{\bullet}^{\star}}$ be the Ext-quiver associated with the collection

$$E_{\bullet}^{\star} = (E_0, E_1, \dots, E_k). \tag{9.12}$$

On the other hand, let $Q = Q_{E_{\bullet}}$ be the Ext-quiver associated with the collection of shifts of 1-dimensional sheaves

$$E_{\bullet} = (E_1, E_2, \dots, E_k). \tag{9.13}$$

As we observed in the proof of Theorem 8.3, the quiver Q is symmetric. Then we have $V(Q^{\star}) = \{0\} \sqcup V(Q)$, and the numbers of arrows from 0 to $i \in V(Q)$, from i to 0, and of loops at 0 are

$$a_i := \mathrm{ext}^1(E_0, E_i), \quad b_i := \mathrm{ext}^1(E_i, E_0), \quad c := \mathrm{ext}^1(E_0, E_0), \tag{9.14}$$

respectively. Therefore, $Q^{\star}$ is obtained from Q as in the construction of Subsection 7.3. We also have the convergent super-potential

$$W^{\star} := W_{E_{\bullet}^{\star}} \in \mathbb{C}\{Q^{\star}\}/[\mathbb{C}\{Q^{\star}\}, \mathbb{C}\{Q^{\star}\}]$$

of $Q^{\star}$ associated with the collection (9.12) as in the construction of (6.2). The following lemma (which will be used in Lemma 9.23) is obvious from the constructions, and we leave the details to the reader.

LEMMA 9.10. *Let $\mathbb{C}\{Q^{\star}\} \twoheadrightarrow \mathbb{C}\{Q\}$ be the linear map sending a path $e_1 e_2 \dots e_n$ to itself if each $e_i \in E(Q)$ and to zero otherwise. This map induces the linear map*

$$\mathbb{C}\{Q^{\star}\}/[\mathbb{C}\{Q^{\star}\}, \mathbb{C}\{Q^{\star}\}] \twoheadrightarrow \mathbb{C}\{Q\}/[\mathbb{C}\{Q\}, \mathbb{C}\{Q\}], \quad f \longmapsto f|_Q.$$

Under this linear map, we have $W^{\star}|_Q = W$, where $W = W_{E_{\bullet}}$ is the convergent super-potential of Q associated with the collection (9.13).

Let $\vec{m}^{\star}$ be the dimension vector of $Q^{\star}$ given by

$$(\vec{m}^{\star})_i = \dim V_i, \quad 0 \leqslant i \leqslant k, \tag{9.15}$$

where V_i is given in (9.6). Note that we have $\vec{m}^{\star} = \vec{0} + \vec{m}$, where $\vec{m}$ is the dimension vector of Q given by $m_i = \dim V_i$, for $1 \leqslant i \leqslant k$. Let us also take

$$\xi^{\pm} = \left(\xi_i^{\pm}\right)_{0 \leqslant i \leqslant k} \in \mathcal{H}^{\sharp V(Q^{\star})}$$

as in (7.13). Similarly to Theorem 6.1, we have the local description of the morphisms in (9.11) in terms of $Q^{\star}$.

THEOREM 9.11. *For a closed point $p \in M_t^{\star}(\beta, n)$ corresponding to a $\mu_t^{\star}$-polystable object (9.6), let $Q^{\star} = Q_{E_{\bullet}^{\star}}$ be the Ext-quiver associated with $E_{\bullet}^{\star}$ and $W^{\star} = W_{E_{\bullet}^{\star}}$ the convergent super-potential of $Q^{\star}$ constructed in (6.2). Then there exist analytic open neighborhoods*

$$p \in T \subset M_t^{\star}(\beta, n), \quad 0 \in V \subset M_{Q^{\star}}(\vec{m}^{\star}),$$

where $\vec{m}^{\star}$ is the dimension vector (9.15), such that we have the commutative diagram of isomor-

phisms

$$\begin{tikzcd} M^{\xi^{\pm}}_{(Q^\star,\partial W^\star)}(\vec{m}^\star)|_V \arrow[r, "\cong"] \arrow[d, "q^{\xi^{\pm}}_{(Q^\star,\partial W^\star)}"'] & \left(q^{\pm}_{M^\star}\right)^{-1}(T) \arrow[d, "q^{\pm}_{M^\star}"] \\ M_{(Q^\star,\partial W^\star)}(\vec{m}^\star)|_V \arrow[r, "\cong"] & T\,. \end{tikzcd}$$

Here the left vertical arrow is given in (7.17)*, the right vertical arrow is given in* (9.11) *pulled back to T. Moreover, the top isomorphism preserves d-critical structures, where the d-critical structure on the left-hand side is given in Lemma* 5.3 *and that on the right-hand side is given in Theorem* 4.1*.*

Proof. The proof is almost identical to that of Theorem 6.1. The only required modification is the proof of the preservation of stability in Lemma 6.4, as we need to compare Bridgeland stability on $Q^\star$-representations given by $\xi^{\pm}$ with weak stability $\mu^\star_{t^{\pm}}$ on $\mathcal{A}_X$. In Lemma 9.12 below, we give an ad-hoc proof for this using characterizations of semistable objects in both sides. $\square$

Lemma 9.12. *For the collection* (9.12)*, let*

$$\Phi_{E^\star_\bullet}\colon \operatorname{mod}_{\mathrm{nil}} \mathbb{C}[[Q^\star]]/(\partial W^\star) \xrightarrow{\sim} \langle E_0, E_1, \dots, E_k\rangle_{\mathrm{ex}}$$

be the equivalence of categories given in (6.4)*. Under the above equivalence, a nilpotent $Q^\star$-representation $\mathbb{V}$ is $\xi^{\pm}$-semistable if and only if $\Phi_{E^\star_\bullet}(\mathbb{V})$ is $\mu^\star_{t^{\pm}}$-semistable in $\mathcal{A}_X$.*

Proof. Let S_i, for $0 \leqslant i \leqslant k$, be the simple $Q^\star$-representation corresponding to the vertex $i \in V(Q^\star)$. As in the proof of Lemma 7.10, a $Q^\star$-representation $\mathbb{V}$ is ξ^+-semistable if and only if there is no non-trivial surjection $\mathbb{V} \twoheadrightarrow \mathbb{V}'$ of $Q^\star$-representations such that $\mathbb{V}'$ is an object in $\langle S_1, \dots, S_k\rangle_{\mathrm{ex}}$.

For an object $F \in \langle E_0, E_1, \dots, E_k\rangle_{\mathrm{ex}}$, we claim that it is $\mu^\star_{t_+}$-semistable if and only if there is no non-trivial surjection $F \twoheadrightarrow F'$ in $\langle E_0, E_1, \dots, E_k\rangle_{\mathrm{ex}}$ such that $F' \in \langle E_1, \dots, E_k\rangle_{\mathrm{ex}}$. The "only if" direction is obvious from the definition of $\mu^\star_{t_+}$-stability. In order to show the "if" direction, suppose that F is not $\mu^\star_{t_+}$-semistable in $\mathcal{A}_X$. Then there exists an exact sequence

$$0 \longrightarrow A \longrightarrow F \longrightarrow B \longrightarrow 0 \qquad (9.16)$$

in $\mathcal{A}_X$ such that $\mu^\star_{t_+}(A) > \mu^\star_{t_+}(B)$. By taking the limit $t_+ \to t$, we have $\mu^\star_t(A) \geqslant \mu^\star_t(B)$. On the other hand, since F is $\mu^\star_t$-semistable, we have $\mu^\star_t(A) \leqslant \mu^\star_t(B)$. It follows that $\mu^\star_t(A) = \mu^\star_t(B)$; therefore, both A and B are $\mu^\star_t$-semistable. By the uniqueness of Jordan–Hölder factors, the exact sequence (9.16) is an exact sequence in $\langle E_0, E_1, \dots, E_k\rangle_{\mathrm{ex}}$. Then by the inequality $\mu^\star_{t_+}(A) > \mu^\star_{t_+}(B)$, we have $B \in \langle E_1, \dots, E_k\rangle_{\mathrm{ex}}$. Therefore, the "if" direction is also proved.

The lemma for the plus sign now holds by the above descriptions of ξ^+-semistable $Q^\star$-representations and $\mu^\star_{t_+}$-semistable objects in $\mathcal{A}_X$, together with the fact that the equivalence (6.19) sends S_i to E_i. The result for the minus sign holds in a similar way. $\square$

Using Theorem 9.11, we can describe the diagram (9.11) in terms of d-critical birational transformations.

Theorem 9.13. *For $t \in \mathcal{W}$ with $t > 0$, the diagram* (9.11) *is an analytic d-critical generalized flip at any $p \in \operatorname{Im} q^-_{M^\star}$ and an analytic d-critical generalized MFS at any $p \in M^\star_t(\beta, n) \setminus \operatorname{Im} q^-_{M^\star}$.*

Moreover, for each effective curve class β, there is a $t(\beta) > 0$ (which is independent of n) such that if $t > t(\beta)$, then the diagram (9.11) *is strict. There is also an $n(\beta) > 0$ such that if*

$n > n(\beta)$, then any $t \in \mathcal{W}$ satisfies $t > t(\beta)$. Therefore, in this case, the diagram (9.11) is always strict.

Proof. Suppose that $p \in M_t^\star(\beta, n)$ corresponds to a $\mu_t^\star$-polystable object (9.6), and take $1 \leqslant i \leqslant k$. Let a_i, b_i be as in (9.14). Since $\mu_\omega(E_i) = t > 0$ and $E_i \in \mathrm{Coh}_{\leqslant 1}(X)[-1]$, we have $\chi(E_i) < 0$. By the Riemann–Roch theorem and $\hom(E_0, E_i) = \hom(E_i, E_0) = 0$, we have

$$0 > \chi(E_i) = \chi(E_0, E_i) = -a_i + b_i\,.$$

Therefore, we have $a_i > b_i$, and the first statement follows from Corollary 7.15 and Theorem 9.11.

We now show the second strictness statement. Again suppose that p corresponds to a $\mu_t^\star$-polystable object (9.6), but now assume that (9.6) is not $\mu_t^\star$-stable, that is, $k \geqslant 1$. We write

$$\mathrm{cl}(E_0) = (1, -\beta_0, -n_0)\,, \quad \mathrm{ch}(E_i) = -(\beta_i, n_i)$$

for $1 \leqslant i \leqslant k$. Then we have $\beta_0 \geqslant 0$, $\beta_i > 0$ for $1 \leqslant i \leqslant k$, and

$$\beta_0 + m_1\beta_1 + \cdots + m_k\beta_k = \beta\,, \tag{9.17}$$

$$n_0 + m_1 n_1 + \cdots + m_k n_k = n\,, \tag{9.18}$$

where $m_i = \dim V_i$ for $1 \leqslant i \leqslant k$. By the identity (9.17), for a fixed $\beta \geqslant 0$, there is only a finite number of possibilities for k, β_i and m_i. If we have $n_i \leqslant m_i$ for some $1 \leqslant i \leqslant k$, then we have

$$t = \frac{n_i}{\omega \cdot \beta_i} \leqslant \frac{m_i}{\omega \cdot \beta_i}\,,$$

and the right-hand side is bounded above. Therefore, there is a $t(\beta) > 0$ such that if $t > t(\beta)$, then $n_i > m_i$ for any $1 \leqslant i \leqslant k$. But then $-n_i = \chi(E_i) = -a_i + b_i$, which implies $a_i \geqslant n_i > m_i$. Therefore, the diagram (9.11) is strict at p by Lemma 7.16.

Suppose that $t \in \mathcal{W}$ satisfies $t \leqslant t(\beta)$. Applying Lemma 9.14 below for β_0 and all $0 < t \leqslant t(\beta)$, we can find an $n'(\beta_0) \in \mathbb{Z}$ such that $n_0 \leqslant n'(\beta_0)$ holds. By (9.18) and using $t = n_i/\omega \cdot \beta_i$, we obtain

$$t \geqslant \frac{n - n'(\beta_0)}{\sum_{i=1}^k m_i(\omega \cdot \beta_i)}\,.$$

There is an $n(\beta) > 0$ such that for $n > n(\beta)$, the right-hand side is bigger than $t(\beta)$. Therefore, for such an $n(\beta)$, the desired statement holds. □

We have used the following lemma.

Lemma 9.14. *For each effective curve class β and $t \in \mathbb{R}$, there is an $n_t(\beta) > 0$ such that $\mathcal{M}_t^\star(\beta, n) = \emptyset$ for $|n| > n_t(\beta)$.*

Proof. The lemma is proved in the proof of [Tod10b, Lemma 4.4]. □

In the following example, we see that Grassmannian flips appear as relative d-critical charts.

Example 9.15. In Theorem 9.13, suppose that $p \in M_t^\star(\beta, n)$ corresponds to a $\mu_t^\star$-polystable object $E \in \mathcal{A}_X$ of the form $E = E_0 \oplus (V \otimes F[-1])$, where $E_0 \in \mathcal{A}_X$ is a rank 1 $\mu_t^\star$-stable object, V is a finite-dimensional vector space, and F is an object of $\mathrm{Coh}_{\leqslant 1}(X)$ satisfying $\mathrm{Ext}^1(F, F) = 0$, for example, $F = \mathcal{O}_C(k)$ for a rational curve $\mathbb{P}^1 = C \subset X$ with $N_{C/X} = \mathcal{O}_C(-1)^{\oplus 2}$. In this case, the Ext-quiver $Q^\star$ for $\{E_0, F[-1]\}$ is the same one considered in Example 7.14, and the birational map

$$M_{Q^\star}^{\xi^+}(\vec{m}^\star) \dashrightarrow M_{Q^\star}^{\xi^-}(\vec{m}^\star)$$

is a Grassmannian flip as in Example 7.14. For example, the wall-crossing diagrams in local $\mathbb{P}^1$ studied in [NN11] are described as d-critical Grassmannian flips as above.

Remark 9.16. In Theorem 9.13, it is possible that the diagram (9.11) is a strict analytic *d*-critical generalized MFS at $p \in M_t^{\star}(\beta, n) \setminus \operatorname{Im} q_{M^{\star}}^{-}$, but in the notation of Theorem 9.11, we have $M_{Q^{\star}}^{\xi-}(\vec{m}^{\star}) = \emptyset$, and the morphism

$$q_{Q^{\star}}^{\xi+} \colon M_{Q^{\star}}^{\xi+}(\vec{m}^{\star}) \longrightarrow M_{Q^{\star}}(\vec{m}^{\star})$$

is birational.

Indeed, suppose that there exist disjoint smooth curves $C_1, C_2 \subset X$ such that $C_1 \cong \mathbb{P}^1$ and $\omega \cdot C_2 = (\omega \cdot C_1) \cdot m$ for some integer $m \geqslant 2$. Then for $t = 1/(\omega \cdot C_1)$, $\beta = [C_1] + [C_2]$ and $n = m + 1$, we have the point $p \in M_t^{\star}(\beta, n)$ corresponding to the $\mu_t^{\star}$-polystable object

$$\mathcal{O}_X \oplus \mathcal{O}_{C_1}[-1] \oplus \mathcal{O}_{C_2}(D)[-1],$$

where D is a divisor on C_2 with $\deg D = m + g(C_2) - 1$. Suppose that $h^1(\mathcal{O}_{C_2}(D)) \neq 0$. Then at the point p, it is easy to see that we have the situation mentioned above.

9.5 PT/L correspondence via d-critical MMP

Let $l \in \mathbb{Z}$ be the number of elements in $\mathcal{W}$ given in (9.10). For each $1 \leqslant i \leqslant l+1$, we set

$$M_i := M_{t_i^+}^{\star}(\beta, n) = M_{t_{i-1}^-}^{\star}(\beta, n), \quad A_i := M_{t_i}^{\star}(\beta, n)$$

with d-critical structure s_i on M_i given in Theorem 4.1. We also define $L_n^{\pm}(X, \beta)$ to be

$$L_n^{\pm}(X, \beta) := M_{\pm\varepsilon}^{\star}(\beta, n), \quad 0 < \varepsilon \ll 1.$$

Let us take unique $1 \leqslant l' \leqslant l+1$ satisfying $t_{l'-1} > 0 \geqslant t_{l'}$. We have the following zigzag diagram that connects $M_1 \cong P_n(X, \beta)$ and $M_{l'} = L_n^+(X, \beta)$:

$$\begin{tikzcd}[column sep=small] M_1 \arrow[dr, "\pi_1^+"'] & & M_2 \arrow[dl, "\pi_1^-"] \arrow[dr, "\pi_2^+"'] & & \cdots \arrow[dl] \arrow[dr] & & M_{l'} \arrow[dl, "\pi_{l'-1}^-"] \\ & A_1 & & A_2 & & A_{l'-1}. & \end{tikzcd} \tag{9.19}$$

The wall-crossing diagrams at $t \leqslant 0$ are given by the following zigzag diagram, which connects $M_{l+1} \cong P_{-n}(X, \beta)$ and $M_{l'} = L_n^+(X, \beta)$:

$$\begin{tikzcd}[column sep=small] M_{l+1} \arrow[dr, "\pi_l^-"'] & & M_l \arrow[dl, "\pi_l^+"] \arrow[dr, "\pi_{l-1}^-"'] & & \cdots \arrow[dl] \arrow[dr] & & M_{l'} \arrow[dl, "\pi_{l'}^+"] \\ & A_l & & A_{l-1} & & A_{l'}. & \end{tikzcd} \tag{9.20}$$

Corollary 9.17. *The diagrams (9.19) and (9.20) are d-critical MMP. In particular, we have the inequalities of virtual canonical line bundles*

$$(M_1, s_1) \geqslant_K (M_2, s_2) \geqslant_K \cdots \geqslant_K (M_{l'}, s_{l'}),$$
$$(M_{l+1}, s_{l+1}) \geqslant_K (M_l, s_l) \geqslant_K \cdots \geqslant_K (M_{l'}, s_{l'}).$$

Moreover, for each effective curve class β, there is an $n(\beta) > 0$ such that the diagrams (9.19) and (9.20) are strict and $M_{l'} = \emptyset$ if $|n| > n(\beta)$ holds. In this case, the morphisms

$$\pi_{l'-1}^+ \colon M_{l'-1} \longrightarrow A_{l'-1}, \quad \pi_{l'}^- \colon M_{l'+1} \longrightarrow A_{l'} \tag{9.21}$$

are d-critical generalized MFS that are strict at any point in $A_{l'-1}$ and $A_{l'}$, respectively.

Proof. The inequalities of virtual canonical line bundles for the diagram (9.19) are immediate from Theorem 9.13. The same argument also applies to the diagram (9.20).

If we take $n(\beta) > 0$ sufficiently large, then the diagrams (9.19) and (9.20) are strict as in the argument of Theorem 9.13. Moreover, we have $M_{l'} = \emptyset$ by Lemma 9.14. Then any point in $A_{l'-1}$ or $A_{l'}$ does not correspond to a $\mu^{\star}_{t_{l'-1}}$-stable object or $\mu^{\star}_{t_{l'}}$-stable object, respectively (as otherwise $M_{l'} \neq \emptyset$). Then as in the proof of Theorem 9.13, the morphisms in (9.21) are strict at any point in $A_{l'-1}$ and $A_{l'}$, respectively. $\square$

If $0 \in \mathcal{W}$, that is, $t_{l'} = 0$, then the wall-crossing at $t = 0$ is given by the diagram

$$\begin{tikzcd} M_{l'} = L_n^+(X,\beta) \arrow[dr] & & L_n^-(X,\beta) = M_{l'+1} \arrow[dl] \\ & M^{\star}_{t=0}(\beta, n)\,. & \end{tikzcd} \tag{9.22}$$

COROLLARY 9.18. *The diagram (9.22) is an analytic d-critical generalized flop. In particular, we have $L_n^+(X,\beta) =_K L_n^-(X,\beta)$.*

Proof. For a point $p \in M^{\star}_{t=0}(\beta, n)$, let a_i, b_i be as in (9.14). Then as in the proof of Theorem 9.13, we have $a_i = b_i$. This implies that the Ext-quiver Q^* associated with the collection (9.12) is symmetric. Therefore, as in Theorem 8.3, the diagram (9.22) is an analytic d-critical generalized flop. $\square$

In the next subsection, we discuss the case when β is irreducible in detail. Here we give some explicit examples for non-irreducible curve classes, discussed in [Tod09a, Section 5].

EXAMPLE 9.19. Let $X \to Y$ be a birational contraction with exceptional locus $C = C_1 \cup C_2$, where each C_i is isomorphic to $\mathbb{P}^1$, $C_1 \cap C_2 = \{p\}$, and $N_{C_i/X} = \mathcal{O}_{C_i}(-1)^{\oplus 2}$. We set $d_i := C_i \cdot \omega$ and assume $d_1 > d_2 > 0$. Let us consider the diagram (9.19) in the case $(\beta, n) = ([C], 2)$. In this case, we have two walls:

$$\mathcal{W} = \left\{ \infty > t_1 = \frac{1}{d_1} > t_2 = \frac{2}{d_1 + d_2} > 0 \right\}.$$

The reduced part of the diagram (9.19) becomes (see [Tod09a, Section 5.2])

$$\begin{tikzcd} C \arrow[dr, "\pi_1^+"'] & & C_1 \arrow[dl, "\pi_1^-"] \arrow[dr, "\pi_2^+"'] & & \emptyset \arrow[dl, "\pi_2^-"] \\ & C_1 & & \mathrm{pt}\,. & \end{tikzcd} \tag{9.23}$$

The map π_1^+ contracts $C_2 \subset C$ to the point $p \in C_1$, and $\pi_1^- = \mathrm{id}$. The point p corresponds to the $\mu^{\star}_{t_1}$-polystable object $I_{C_1} \oplus \mathcal{O}_{C_2}[-1]$, where I_{C_1} is the ideal sheaf of C_1. The Ext-quiver $Q^{\star}$ of $\{I_{C_1}, \mathcal{O}_{C_2}[-1]\}$ is of the form

$$Q^{\star} = \left(\begin{tikzcd} 0 \arrow[r, shift left=2] \arrow[r] & 1 \arrow[l, shift left=2] \end{tikzcd} \right).$$

Therefore, locally at p, the $\pi_1^{\pm}$-relative d-critical charts are given by a diagram of the form

$$\begin{tikzcd} \widehat{\mathbb{C}}^2 \arrow[r, "f^+"] \arrow[dr, "w^+"'] & \mathbb{C}^2 \arrow[d, "g"] & \mathbb{C}^2 \arrow[l, "\mathrm{id}"'] \arrow[dl, "w^-"] \\ & \mathbb{C}\,, & \end{tikzcd}$$

where $f^{\dagger}$ is the blow-up at the origin. It seems likely that $g(u,v)=u^2$, and the scheme structure of $P_2(X,\beta)$ at $p \in C = P_2^{\mathrm{red}}(X,\beta)$ is given by the critical locus of $\mathbb{C}^2 \to \mathbb{C}$, $(x,y) \mapsto x^2y^2$. In particular, the left diagram of (9.23) is an analytic d-critical divisorial contraction. Similarly, the right diagram of (9.23) is a d-critical MFS.

Example 9.20. Let $X \to Y$ be a birational contraction with $C \cong \mathbb{P}^1$ and$N_{C/X} = \mathcal{O}_C(-1)^{\oplus 2}$. We set $d := C \cdot \omega$. Let us consider the diagram (9.19) in the case $(\beta, n) = (2[C], 4)$. In this case, we have two walls:

$$\mathcal{W} = \left\{ \infty > t_1 = \frac{3}{d} > t_2 = \frac{2}{d} > 0 \right\}.$$

The reduced part of the diagram (9.19) becomes (see [Tod09a, Section 5.3] and [PT09, Section 4.1] for $P_4^{\mathrm{red}}(X,\beta) = \mathbb{P}^3$)

$$\begin{tikzcd} \mathbb{P}^3 \arrow[dr, "\pi_1^+"'] & & \mathrm{pt} \arrow[dl, "\pi_1^-"] \arrow[dr, "\pi_2^+"'] & & \emptyset \arrow[dl, "\pi_2^-"] \\ & \mathrm{pt} & & \mathrm{pt}\,. & \end{tikzcd} \tag{9.24}$$

The map π_1^+ contracts $\mathbb{P}^3$ to a point, corresponding to the $\mu_{t_1}^{\star}$-polystable object $I_C \oplus \mathcal{O}_C(2)[-1]$. The Ext-quiver $Q^{\star}$ of $\{I_C, \mathcal{O}_C(2)[-1]\}$ is of the form

$$Q^{\star} = \left(\begin{tikzcd} 0 \arrow[r, shift left=4] \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] & 1 \arrow[l, shift left=4] \end{tikzcd} \right).$$

Therefore, the $\pi_1^{\pm}$-relative d-critical charts are given by a diagram of the form

$$\begin{tikzcd} \widehat{\mathbb{C}}^4 \arrow[r, "f^+"] \arrow[dr, "w^+"'] & \mathbb{C}^4 \arrow[d, "g"] & \mathbb{C}^4 \arrow[l, "\mathrm{id}"'] \arrow[dl, "w^-"] \\ & \mathbb{C}\,, & \end{tikzcd}$$

where $f^{\dagger}$ is the blow-up at the origin. It seems likely that $g(u,v,t,s) = uv + ts$, and the scheme structure on $P_4(X,\beta)$ is non-reduced along the quadric in $\mathbb{P}^3 = P_4^{\mathrm{red}}(X,\beta)$. In particular, the left diagram of (9.24) is an analytic d-critical divisorial contraction. Similarly, the right diagram of (9.24) is a d-critical MFS.

9.6 The case of an irreducible curve class

Suppose that β is an irreducible curve class; that is, it is not written as $\beta_1 + \beta_2$ for effective curve classes $\beta_i > 0$. Let $\mathcal{M}_n(X,\beta)$ be the moduli stack of μ_ω-semistable 1-dimensional sheaves F on X with $\mathrm{ch}(F) = (\beta, n)$ and

$$\mathcal{M}_n(X,\beta) \longrightarrow M_n(X,\beta) \tag{9.25}$$

be its good moduli space for a projective scheme $M_n(X,\beta)$.

Remark 9.21. When β is irreducible, a 1-dimensional sheaf F with $[F] = \beta$ is μ_ω-semistable if and only if it is a pure 1-dimensional sheaf. In particular, any point in $\mathcal{M}_n(X,\beta)$ corresponds to a μ_ω-stable sheaf, and the morphism (9.25) is a $\mathbb{C}^*$-gerbe.

As β is irreducible, by Remark 9.21, we have the following diagram:

$$\begin{tikzcd} P_n(X,\beta) \arrow[dr, "q_P^+"'] & & P_{-n}(X,\beta) \arrow[dl, "q_P^-"] \\ & M_n(X,\beta)\,. & \end{tikzcd} \tag{9.26}$$

Here the morphisms $q_P^{\pm}$ are given by $q_P^+(F,s) = F$ and $q_P^-(F',s') = \mathcal{E}xt^2_{\mathcal{O}_X}(F', \mathcal{O}_X)$. The diagram (9.26) appeared in [PT10] to show the BPS rationality of the generating series of stable pair invariants with irreducible curve classes. The above diagram is indeed a special case of the wall-crossing in the previous subsection, and we have the following.

Theorem 9.22. (i) *Suppose $n > 0$. Then at a point $p = [F] \in M_n(X,\beta)$, the diagram (9.26) is an*

$$\begin{cases} \text{analytic d-critical flip} & \text{if } h^1(F) > 1\,, \\ \text{analytic d-critical divisorial contraction} & \text{if } h^1(F) = 1\,, \\ \text{analytic d-critical MFS} & \text{if } h^1(F) = 0\,. \end{cases} \tag{9.27}$$

(ii) *Suppose $n = 0$. Then at a point $p = [F] \in M_n(X,\beta)$, the diagram (9.26)*

$$\begin{cases} \text{is an analytic d-critical flop} & \text{if } h^1(F) > 1\,, \\ \text{consists of isomorphisms} & \text{if } h^1(F) = 1\,, \\ \text{consists of empty sets} & \text{if } h^1(F) = 0\,. \end{cases}$$

Proof. If β is irreducible, there is only one wall (see [Tod09a, Section 5.1]):

$$\mathcal{W} = \left\{ t_1 = \frac{n}{\omega \cdot \beta} \right\}.$$

As in the previous subsection, for $(\beta, n) \in \Gamma_{\leqslant 1}$, we have the wall-crossing diagram

$$\begin{tikzcd} M^{\star}_{t_1^+}(\beta,n) \arrow[dr, "q_{M^\star}^+"'] & & M^{\star}_{t_1^-}(\beta,n) \arrow[dl, "q_{M^\star}^-"] \\ & M^{\star}_{t_1}(\beta,n)\,. & \end{tikzcd} \tag{9.28}$$

The algebraic space $M^{\star}_{t_1}(\beta,n)$ parametrizes $\mu^{\star}_{t_1}$-polystable objects E of the form

$$E = E_0 \oplus E_1\,, \quad E_0 = \mathcal{O}_X\,, \quad E_1 = F[-1]\,, \tag{9.29}$$

where F is a pure 1-dimensional sheaf satisfying $\mathrm{ch}(F) = (\beta, n)$ (see Remark 9.21). Let us take a $p \in M^{\star}_{t_1}(\beta,n)$ that corresponds to a $\mu^{\star}_t$-polystable object (9.29). The Ext-quiver $Q^{\star}$ associated with the collection $\{E_0, E_1\}$ has two vertices $\{0,1\}$, the numbers of arrows from 0 to 1, 1 to 0 and of loops at 1, 0 are

$$a := h^0(F)\,, \quad b := h^1(F)\,, \quad c := \mathrm{ext}^1(F,F)\,, \quad 0 = \mathrm{ext}^1(\mathcal{O}_X, \mathcal{O}_X)\,,$$

respectively. Note that $a - b = n$ by the Riemann–Roch theorem. Let us set

$$V^+ = H^0(F)\,, \quad V^- = H^1(F)^{\vee}\,, \quad U = \mathrm{Ext}^1(F,F)\,.$$

Let $\vec{m}^{\star} = (1,1)$ be the dimension vector of $Q^{\star}$. Similarly to the argument in Example 6.3, we have

$$M^{\xi+}_{Q^\star}(\vec{m}^\star) = \mathrm{Tot}_{\mathbb{P}(V^+)}\big(\mathcal{O}_{\mathbb{P}(V^+)}(-1) \otimes V^-\big) \times U\,,$$
$$M^{\xi-}_{Q^\star}(\vec{m}^\star) = \mathrm{Tot}_{\mathbb{P}(V^-)}\big(\mathcal{O}_{\mathbb{P}(V^-)}(-1) \otimes V^+\big) \times U\,.$$

Therefore, for $n > 0$, the diagram

$$\begin{tikzcd} M^{\xi^+}_{Q^\star}(\vec{m}^\star) \arrow[dr, "q^{\xi^+}_{Q^\star}"'] & & M^{\xi^-}_{Q^\star}(\vec{m}^\star) \arrow[dl, "q^{\xi^-}_{Q^\star}"] \\ & M_{Q^\star}(\vec{m}^\star) & \end{tikzcd}$$

is a standard toric flip if $b > 1$, a divisorial contraction (indeed, a blow-up of a smooth variety at a smooth center) if $b = 1$ and MFS if $b = 0$ (see Example 3.8). By Theorem 9.11, it follows that the diagram (9.28) satisfies the condition (9.27).

On the other hand, by Proposition 9.7, we have the isomorphisms $P_{\pm n}(X, \beta) \xrightarrow{\cong} M^\star_{t_1^\pm}(\beta, n)$. We also have the morphism of stacks $\mathcal{M}_n(X, \beta) \to \mathcal{M}^\star_{t_1}(\beta, n)$ sending a flat family of 1-dimensional sheaves $\mathcal{F}$ on $X \times S$ over a $\mathbb{C}$-scheme S to the family of $\mu^\star_t$-semistable objects $\mathcal{O}_{X\times S} \oplus \mathcal{F}[-1]$ over S. By the universality of good moduli spaces, we have the induced morphism

$$\gamma\colon M_n(X, \beta) \longrightarrow M^\star_{t_1}(\beta, n)\,. \tag{9.30}$$

The morphism γ is bijective on closed points, by the description of $\mu^\star_t$-polystable objects in (9.29). We can also show that γ is a closed immersion (see Lemma 9.23 below), so $M^\star_{t_1}(\beta, n)$ is a nilpotent thickening of $M_n(X, \beta)$. Therefore, by comparing the diagram (9.26) with (9.28), the result for $n > 0$ follows. The result for $n = 0$ also holds by a similar argument. □

We have used the following lemma.

Lemma 9.23. *The morphism* (9.30) *is a closed immersion.*

Proof. It is enough to show the claim analytic-locally at any point $p \in M^\star_{t_1}(\beta, n)$. Let p correspond to a $\mu^\star_{t_1}$-polystable object (9.29), $Q^\star$ be the Ext-quiver associated with $\{E_0, E_1\} = \{\mathcal{O}_X, F[-1]\}$, where F is a pure 1-dimensional sheaf with $[F] = \beta$. Let Q be the sub-Ext-quiver of $Q^\star$ associated with $\{E_1\} = \{F[-1]\}$. Let

$$W^\star \in \mathbb{C}\{Q^\star\}\,, \quad W \in \mathbb{C}\{Q\}$$

be the convergent super-potentials defined as in (6.2) for the collections $\{E_0, E_1\}$ and $\{E_1\}$, respectively. Let us take an analytic open neighborhood $V \subset M_{Q^\star}(\vec{m}^\star)$ of 0 for $\vec{m}^\star = (1, 1)$. As in (5.14), for $G = (\mathbb{C}^*)^2$, we have the G-invariant analytic function

$$\mathrm{tr}\, W^\star\colon \pi^{-1}_{Q^\star}(V) \longrightarrow \mathbb{C}\,, \tag{9.31}$$

where $\pi_{Q^\star}\colon \mathrm{Rep}_{Q^\star}(\vec{m}^\star) \to M_{Q^\star}(\vec{m}^\star)$ is the quotient morphism. By Theorem 9.11, the local analytic structure of $M^\star_{t_1}(\beta, n)$ at p is isomorphic to the analytic closed subspace in V defined by the ideal

$$(d\,\mathrm{tr}\, W^\star)^G \subset (\mathcal{O}_{\pi^{-1}_{Q^\star}(V)})^G = \mathcal{O}_V\,. \tag{9.32}$$

Let $\vec{x} = (x_1, \ldots, x_a)$, $\vec{y} = (y_1, \ldots, y_b)$, and $\vec{z} = (z_1, \ldots, z_c)$ be the coordinates of $\mathbb{E}_{0,1}$, $\mathbb{E}_{1,0}$, $\mathbb{E}_{1,1}$, respectively, corresponding to the basis $E_{i,j} \subset \mathbb{E}_{i,j}$ (see the notation in Subsection 5.1). Since

$W^{\star}|_Q = W$ by Lemma 9.10, the function (9.31) is of the form

$$\operatorname{tr} W^{\star}(\vec{x}, \vec{y}, \vec{z}) = \operatorname{tr} W(\vec{z}) + \sum_{i,j} f_{ij}(\vec{z}) x_i y_j + \sum_{i,i',j,j'} f_{ii'jj'}(\vec{z}) x_i x_{i'} y_j y_{j'} + \cdots .$$

Here $f_{\bullet}(\vec{z})$ is an analytic function on an open neighborhood of $0 \in \mathbb{E}_{1,1}$. Since each $x_i y_j$ is G-invariant, the above description of $\operatorname{tr} W^{\star}$ implies that

$$(d \operatorname{tr} W^{\star})^G \subset (d \operatorname{tr} W(\vec{z}), x_i y_j \colon 1 \leqslant i \leqslant a, 1 \leqslant j \leqslant b) .$$

Since we have

$$M_{Q^{\star}}(\vec{m}^{\star}) = \operatorname{Spec} \mathbb{C}[x_i y_j \colon 1 \leqslant i \leqslant a, 1 \leqslant j \leqslant b] \times \mathbb{E}_{1,1} ,$$

it follows that the analytic closed subspace in V defined by the ideal (9.32) contains the analytic closed subspace defined by $d \operatorname{tr} W(\vec{z}) = 0$ on $V \cap (\{0\} \times \mathbb{E}_{1,1})$. Since the latter is analytic-locally isomorphic to $M_n(X, \beta)$ at $[F] \in M_n(X, \beta)$ (see Theorem 6.1), the lemma holds. □

In the following example, we see that the classical diagrams on symmetric products of curves appear as special cases of the diagram (9.26).

Example 9.24. Suppose that X contains a smooth projective curve C of genus g that is super-rigid in X, that is, $H^0(N_{C/X}) = 0$, and set $\beta = [C]$. Suppose that $C \subset X$ is the unique curve on X with $[C] = \beta$. For example, we can take a local model (see Remark 9.25)

$$X = \operatorname{Tot}_C(L_1 \oplus L_2) \tag{9.33}$$

for general line bundles L_1, L_2 on C satisfying $\deg L_i = g - 1$ and $L_1 \otimes L_2 \cong \omega_C$. Here X contains C as a zero section. By setting $\beta = [C]$, we have the isomorphism

$$S^{n+g-1}(C) \xrightarrow{\cong} P_n(X, \beta) , \quad Z \longmapsto (\mathcal{O}_C(Z), s) ,$$

where s is the section of $\mathcal{O}_C(Z)$ that vanishes at Z. Therefore, in this case, the diagram (9.26) coincides with (3.11), and the statement in Example 3.9 follows from Theorem 9.22.

Remark 9.25. A subtlety of using the local model (9.33) is that it is non-compact. Let us compactify X to the $\mathbb{P}^2$-bundle $\overline{X} \to C$. Still $\overline{X}$ is not CY3, and also $H^1(\mathcal{O}_{\overline{X}}) \neq 0$, so we need to modify the argument. Let $E_0 = \mathcal{O}_{\overline{X}}$ and $E_1 = F[-1]$, where F is a line bundle on C pushed forward to $\overline{X}$ by the zero section of $X \to C$. Then for $E = E_0 \oplus E_1$, we replace the A_∞-structure on $\operatorname{Ext}^*(E, E)$ with the L_∞-structure on its traceless part $\operatorname{Ext}^*(E, E)_0$. Then the latter is cyclic (though the former is not), and an argument similar to Theorem 9.11 shows that the diagram (3.11) satisfies the desired property in Example 3.9.

Appendix A. Review of Bridgeland stability conditions

Here we recall basic definitions on Bridgeland stability conditions on triangulated categories [Bri07].

A.1 Stability conditions on abelian categories

Let $\mathcal{A}$ be an abelian category and $K(\mathcal{A})$ its Grothendieck group.

Definition A.1. A *stability condition* on an abelian category $\mathcal{A}$ is a group homomorphism $Z \colon K(\mathcal{A}) \to \mathbb{C}$ satisfying the following:

(i) (Positivity property): For any non-zero $0 \neq E \in \mathcal{A}$, we have

$$Z(E) \in \{z \in \mathbb{C} \colon \Im z > 0\} \cup \mathbb{R}_{<0} .$$

A non-zero object $E \in \mathcal{A}$ is called *Z-stable* (respectively, *Z-semistable*) if for any non-zero subobject $0 \neq F \subsetneq E$ in $\mathcal{A}$, we have the inequality in $(0, \pi]$

$$\arg Z(F) < \arg Z(E) \quad (\text{respectively, } \arg Z(F) \leqslant \arg Z(E)).$$

(ii) (Harder–Narasimhan property): For any $E \in \mathcal{A}$, there exists a filtration (called *Harder–Narasimhan filtration*)

$$0 = E_0 \subset E_1 \subset \cdots \subset E_n = E$$

such that each $F_i := E_i/E_{i-1}$ is Z-semistable with $\arg Z(F_i) > \arg Z(F_{i+1})$ for all $1 \leqslant i \leqslant n-1$.

A.2 Stability conditions on triangulated categories

Let $\mathcal{D}$ be a $\mathbb{C}$-linear triangulated category and $K(\mathcal{D})$ its Grothendieck group. We fix a finitely generated abelian group Γ together with a group homomorphism

$$\mathrm{cl} \colon K(\mathcal{D}) \longrightarrow \Gamma . \tag{A.1}$$

Remark A.2. A choice of (Γ, cl) corresponds to a choice of a Chern character map. For example, if $\mathcal{D} = D^b(X)$ for a smooth projective variety X, we can take $\Gamma = \Gamma_X$, where Γ_X is the image of the Chern character map $\mathrm{ch} \colon K(X) \to H^{2*}(X, \mathbb{Q})$ and $\mathrm{cl} = \mathrm{ch}$.

Definition A.3 ([Bri07]). A *Bridgeland stability condition* on $\mathcal{D}$ consists of data

$$\sigma = (Z, \mathcal{A}), \quad \mathcal{A} \subset \mathcal{D}, \quad Z \colon \Gamma \longrightarrow \mathbb{C} .$$

Here $\mathcal{A}$ is the heart of a bounded t-structure on $\mathcal{D}$, and Z is a group homomorphism (called *central charge*) such that $Z \circ \mathrm{cl}$ is a stability condition on $\mathcal{A}$ as in Definition A.1. An object $E \in \mathcal{D}$ is called *σ-(semi)stable* if $E[k] \in \mathcal{A}$ for some $k \in \mathbb{Z}$ and it is Z-(semi)stable in $\mathcal{A}$.

A.3 The space of Bridgeland stability conditions

Let $\mathrm{Stab}_\Gamma(\mathcal{D})$ be the set of Bridgeland stability conditions on $\mathcal{D}$ with respect to the group homomorphism (A.1) satisfying the following condition (called *support property*):

$$\sup \left\{ \frac{\|\mathrm{cl}(E)\|}{|Z(E)|} \colon E \text{ is } \sigma\text{-semistable} \right\} < \infty .$$

Here $\|{*}\|$ is a fixed norm on $\Gamma_\mathbb{R}$. The following is the main result in [Bri07].

Theorem A.4 ([Bri07, Theorem 1.2]). *The set* $\mathrm{Stab}_\Gamma(\mathcal{D})$ *has the structure of a complex manifold such that the map* $\mathrm{Stab}_\Gamma(\mathcal{D}) \to \mathrm{Hom}(\Gamma, \mathbb{C})$ *sending* $(Z, \mathcal{A})$ *to* Z *is a local isomorphism.*

Let X be a smooth projective variety, and take $\mathcal{D} = D^b(X)$. By setting Γ to be the image of the Chern character map as in Remark A.2, we have the complex manifold

$$\mathrm{Stab}(X) := \mathrm{Stab}_{\Gamma_X}\left(D^b(X)\right) .$$

Remark A.5. Note that we have the natural $\mathbb{C}^*$-action on $\mathrm{Hom}(\Gamma, \mathbb{C})$ by multiplication. This action lifts to a $\mathbb{C}$-action on $\mathrm{Stab}_\Gamma(\mathcal{D})$ via the universal cover $\mathbb{C} \to \mathbb{C}^*$, which does not change semistable objects. See [Bri09, Section 3.3].

Appendix B. Other examples

In this section, we discuss some other examples of wall-crossing in CY 3-folds in terms of d-critical birational geometry. We just describe the results without details since the arguments are similar to those of Theorem 1.1.

B.1 DT/PT correspondence

Let X be a smooth projective CY 3-fold. For $(\beta, n) \in \Gamma_{\leqslant 1}$, let

$$I_n(X, \beta) \tag{B.1}$$

be the moduli space of subschemes $C \subset X$ such that $[C] = \beta$ and $\chi(\mathcal{O}_C) = n$. The moduli space (B.1) is identified with the moduli space of rank 1 torsion-free sheaves I with Chern character $(1, 0, -\beta, -n)$. In particular, it has a canonical d-critical structure.

The moduli spaces (B.1) and (9.1) are related by wall-crossing phenomena with respect to certain Bridgeland-type weak stability conditions in the derived category (see [Tod10a]). The above wall-crossing is relevant in showing the DT/PT correspondence conjecture [PT09] (see Subsection C.3). As in Section 9, we have the diagram

$$\begin{tikzcd} I_n(X,\beta) \arrow[dr, "q_I"'] & & P_n(X,\beta) \arrow[dl, "q_P"] \\ & T_n(X,\beta), & \end{tikzcd} \tag{B.2}$$

where $T_n(X, \beta)$ is an algebraic space that parametrizes objects of the form

$$I_C \oplus \left(\bigoplus_{i=1}^{k} V_i \otimes \mathcal{O}_{x_i}[-1] \right),$$

where I_C is an ideal sheaf of a pure 1-dimensional subscheme $C \subset X$ and $x_i \neq x_j$ for $i \neq j$, satisfying $\chi(\mathcal{O}_C) + \sum_{i=1}^{k} \dim V_i = n$. We have the following.

Theorem B.1. *The diagram (B.2) is an analytic d-critical generalized flip at any point in* $\operatorname{Im} q_P$ *and an analytic d-critical generalized MFS at any point in* $T_n(X, \beta) \setminus \operatorname{Im} q_P$*. In particular, we have* $I_n(X, \beta) \geqslant_K P_n(X, \beta)$.

Proof. The proof is the same as that of Theorem 9.13. □

Here is an example of DT/PT correspondence for a fixed smooth curve.

Example B.2. In the situation of Example 9.24, we have

$$\mathrm{Quot}(I_C, n+g-1) \xrightarrow{\cong} I_n(X, \beta).$$

Here the left-hand side is the Quot scheme parameterizing quotients $I_C \twoheadrightarrow Q$ such that Q is a zero-dimensional sheaf with length $n + g - 1$, and the above isomorphism is given by taking the kernel of $I_C \twoheadrightarrow Q$. By setting $m = n + g - 1$, the diagram (B.2) is

$$\begin{tikzcd} \mathrm{Quot}(I_C, m) \arrow[dr, "q_I"'] & & S^m(C) \arrow[dl, "q_P"] \\ & S^m(X). & \end{tikzcd}$$

Here q_I sends $I_C \twoheadrightarrow Q$ to the support of Q, and q_P is induced by $C \subset X$. Note that $S^m(C)$ is always smooth while $\mathrm{Quot}(I_C, m)$ can be singular. By Theorem B.1, this diagram is an analytic d-critical generalized flip at any point in $\mathrm{Im}\, q_P$ and an analytic d-critical generalized MFS at any point in $S^m(X) \setminus \mathrm{Im}\, q_P$.

B.2 Wall-crossing in local $K3$ surfaces

Let S be a smooth projective $K3$ surface over $\mathbb{C}$ and Γ_S its Mukai lattice: $\Gamma_S := \mathbb{Z} \oplus \mathrm{NS}(S) \oplus \mathbb{Z}$. Let $v(-)$ be the Mukai vector

$$v\colon K(S) \longrightarrow \Gamma_S\,, \quad E \longmapsto \mathrm{ch}(E) \cdot \sqrt{\mathrm{td}}_X\,. \tag{B.3}$$

Then we have the space of Bridgeland stability conditions on $D^b(S)$ with respect to group homomorphism (B.3), denoted by $\mathrm{Stab}(S)$. The structure of the space $\mathrm{Stab}(S)$ is studied in [Bri08], and moduli spaces of Bridgeland stable objects on S are studied in [BM14b].

Let X be the non-compact CY 3-fold defined by

$$X := \mathrm{Tot}_S(K_S) = S \times \mathbb{C}\,.$$

We consider moduli spaces of semistable objects on the triangulated category

$$D^b_c(X) := \left\{E \in D^b(X)\colon \mathrm{Supp}(E) \text{ is compact}\right\}.$$

Let $p_S\colon X \to S$ be the projection. We have the group homomorphism

$$\mathrm{cl}\colon K\big(D^b_c(X)\big) \longrightarrow \Gamma_S\,, \quad E \longmapsto v(p_{S*}E)\,. \tag{B.4}$$

Let $\mathrm{Stab}_c(X)$ be the space of Bridgeland stability conditions on $D^b_c(X)$ with respect to the group homomorphism (B.4). Then we have the isomorphism (see [Tod09b, Theorem 6.5, Lemma 5.3])

$$p_{S*}\colon \mathrm{Stab}_c(X) \xrightarrow{\cong} \mathrm{Stab}(S)\,,$$

which is identity on central charges. Under this isomorphism, an object $E \in D^b_c(X)$ is σ-(semi)stable for $\sigma \in \mathrm{Stab}_c(X)$ if and only if $p_{S*}E \in D^b(S)$ is $p_{S*}\sigma$-(semi)stable.

For $\sigma = (Z, \mathcal{A}) \in \mathrm{Stab}_c(X)$ and $v \in \Gamma_S$ with $\Im Z(v) > 0$, let $\mathcal{M}_\sigma(v)$ be the moduli stack of σ-semistable objects $E \in \mathcal{A}$ such that $\mathrm{cl}(E) = v$, and

$$\mathcal{M}_\sigma(v) \longrightarrow M_\sigma(v)$$

be its good moduli space. Note that if v is primitive and σ is general, then $\mathcal{M}_\sigma(v)$ consists of σ-stable objects of the form $i_{c*}F$ for $c \in \mathbb{C}$, where $F \in D^b(S)$ is $p_{S*}\sigma$-stable and $i_c\colon S \times \{c\} \hookrightarrow X$ is the inclusion. Therefore, we have

$$M_\sigma(v) = M_{p_{S*}\sigma}(v) \times \mathbb{C}\,,$$

where $M_{p_{S*}\sigma}(v)$ is the moduli space of $p_{S*}\sigma$-stable objects in $D^b(S)$ with Mukai vector v. By [BM14b], the moduli space $M_{p_{S*}\sigma}(v)$ is a projective holomorphic symplectic manifold of dimension $2 + (v, v)$, where $(-,-)$ is the Mukai product on Γ_S.

Suppose that $v \in \Gamma_S$ is primitive and $\sigma \in \mathrm{Stab}_c(X)$ lies on a wall with respect to v. If $\sigma^\pm \in \mathrm{Stab}_c(X)$ lie on its adjacent chambers, we obtain the diagram

$$\begin{tikzcd} M_{\sigma^+}(v) \arrow[dr, "q_M^+"'] & & M_{\sigma^-}(v) \arrow[dl, "q_M^-"] \\ & M_\sigma(v)\,. & \end{tikzcd} \tag{B.5}$$

By noting the Ext-quiver associated with any collection in $D^b(S)$ is symmetric, we have the following.

Theorem B.3. *The diagram* (B.5) *is an analytic d-critical generalized flop.*

Appendix C. Wall-crossing formula of DT-type invariants

Here we review the previous works [Bri11, Tod10a, Tod09a, Tod10b, Tod12a, Dia12] where wall-crossing phenomena in this paper were applied to show several properties on DT invariants.

C.1 Product expansion formula of PT invariants

For a CY 3-fold X and an element $(\beta, n) \in \Gamma_{\leqslant 1}$, the moduli space of stable pairs $P_n(X, \beta)$ admits a zero-dimensional virtual class [PT09]. The PT invariant $P_{n,\beta} \in \mathbb{Z}$ is defined by the integration of the virtual class. It also coincides with weighted Euler characteristics

$$P_{n,\beta} = \int_{[P_n(X,\beta)]} \chi_B \, de := \sum_{m \in \mathbb{Z}} m \cdot e\big(\chi_B^{-1}(m)\big)\,.$$

Here χ_B is the Behrend constructible function [Beh09] on $P_n(X, \beta)$. The generating series of PT invariants satisfies the product expansion formula

$$1 + \sum_{\beta>0, n\in\mathbb{Z}} P_{n,\beta} q^n t^\beta = \exp\left(\sum_{\beta>0,n>0} (-1)^{n-1} n N_{n,\beta} q^n t^\beta\right) \cdot \left(\sum_{\beta>0,n\in\mathbb{Z}} L_{n,\beta} q^n t^\beta\right). \tag{C.1}$$

Here $N_{n,\beta}$ and $L_{n,\beta}$ are as follows:

- The invariant $N_{n,\beta} \in \mathbb{Q}$ is the generalized DT invariant [JS12] counting 1-dimensional semistable sheaves on X with Chern character $(\beta, n) \in \Gamma_{\leqslant 1}$. It satisfies the symmetric property $N_{n,\beta} = N_{-n,\beta}$ and the periodicity property $N_{n,\beta} = N_{n,\beta+\omega\cdot\beta}$ for any ample divisor ω on X.
- The invariant $L_{n,\beta} \in \mathbb{Z}$ is a DT-type invariant counting certain stable objects $E \in D^b(X)$ satisfying $\mathrm{ch}(E) = (1, 0, -\beta, -n)$ (see (C.3) below). It satisfies the symmetric property $L_{n,\beta} = L_{-n,\beta}$ and the vanishing $L_{n,\beta} = 0$ for $|n| \gg 0$.

The formula (C.1) is proved using wall-crossing phenomena in Section 9, which led to the proof of the rationality conjecture of the generating series of PT invariants [MNOP06, PT09].

C.2 Wall-crossing formula of PT invariants

Below we recall how to derive the formula (C.1) from the wall-crossing in Section 9. For $t \in \mathbb{R}$, let us consider the moduli stack $\mathcal{M}_t^\star(\beta, n)$ and its good moduli space $M_t^\star(\beta, n)$ as in Subsection 9.3. Let $\mathcal{W} \subset \mathbb{R}$ be the set of walls (9.10), and take $t \notin \mathcal{W}$. By integrating the Behrend function on $M_t^\star(\beta, n)$, we have the invariant

$$L_{n,\beta}^t := \int_{M_t^\star(\beta,n)} \chi_B \, de \in \mathbb{Z}\,. \tag{C.2}$$

By Proposition 9.7, we have the identities

$$L_{n,\beta}^t = \begin{cases} P_{n,\beta}\,, & t \gg 0\,, \\ P_{-n,\beta}\,, & t \ll 0\,. \end{cases}$$

For $t_i \in \mathcal{W}$, the wall-crossing formula of the invariants (C.2) is given by (see [Tod12a, Theorem 5.7])

$$\lim_{\varepsilon\to+0}\sum_{\beta>0,n\in\mathbb{Z}} L^{t_i+\varepsilon}_{n,\beta}q^n t^\beta = \exp\left(\sum_{n/\omega\cdot\beta=t_i}(-1)^{n-1}nN_{n,\beta}q^n t^\beta\right)\cdot\left(\lim_{\varepsilon\to+0}\sum_{\beta>0,n\in\mathbb{Z}} L^{t_i-\varepsilon}_{n,\beta}q^n t^\beta\right).$$

Applying this formula from $t\to\infty$ to $t\to+0$, using the identity (9.7) and setting

$$L_{n,\beta} := L^t_{n,\beta}\,, \quad 0<t\ll 1\,, \tag{C.3}$$

we obtain the formula (C.1).

C.3 DT/PT correspondence

Let $I_n(X,\beta)$ be the moduli space defined in (B.1). The rank 1 DT invariants

$$I_{n,\beta} := \int_{I_n(X,\beta)}\chi_B\,de$$

are related to PT invariants by the identity

$$\sum_{n\in\mathbb{Z}} I_{n,\beta}q^n = \prod_{n\geqslant 1}\left(1-q^n\right)^{-n\cdot e(X)}\sum_{n\in\mathbb{Z}} P_{n,\beta}q^n\,.$$

This formula, called the DT/PT correspondence, was conjectured in [PT09] and proved in [Bri11, Tod10a] via wall-crossing phenomena in the derived category discussed in Subsection B.1.

Acknowledgements

The author is grateful to Daniel Halpern-Leistner for explaining the work [AHH21] on the existence of good moduli spaces for Bridgeland semistable objects. The author is also grateful to Chen Jiang and Dominic Joyce for valuable discussions. The author thanks to the referee for carefully checking the manuscript and for several useful comments, especially Remark 4.3.

References

ABCH13 D. Arcara, A. Bertram, I. Coskun and J. Huizenga, *The minimal model program for the Hilbert scheme of points on $\mathbb{P}^2$ and Bridgeland stability*, Adv. Math. **235** (2013), 580–626; doi:10.1016/j.aim.2012.11.018.

AC21 M. Anel and D. Calaque, *Shifted symplectic reduction of derived critical loci*, 2021, arXiv:2106.06625.

AHH21 J. Alper, D. Halpern-Leistner and J. Heinloth, *Existence of moduli spaces for algebraic stacks*, 2021, arXiv:1812.01128.

AHR20 J. Alper, J. Hall and D. Rydh, *A Luna étale slice theorem for algebraic stacks*, Ann. of Math. **191** (2020), no. 3, 675–738; doi:10.4007/annals.2020.191.3.1.

Alp13 J. Alper, *Good moduli spaces for Artin stacks*, Ann. Inst. Fourier (Grenoble) **63** (2013), no. 6, 2349–2402; doi:10.5802/aif.2833.

BB15 A. Bodzenta and A. Bondal, *Flops and spherical functors*, 2015, arXiv:1511.00665.

BBBJ15 O. Ben-Bassat, C. Brav, V. Bussi and D. Joyce, *A 'Darboux theorem' for shifted symplectic structures on derived Artin stacks, with applications*, Geom. Topol. **19** (2015), no. 3, 1287–1359; doi:10.2140/gt.2015.19.1287.

BCHM10 C. Birkar, P. Cascini, C. D. Hacon and J. McKernan, *Existence of minimal models for varieties of log general type*, J. Amer. Math. Soc. **23** (2010), no. 2, 405–468; doi:10.1090/S0894-0347-09-00649-3.

Beh09 K. Behrend, *Donaldson–Thomas type invariants via microlocal geometry*, Ann. of Math. **170** (2009), no. 3, 1307–1338; doi:10.4007/annals.2009.170.1307.

BM14a A. Bayer and E. Macrì, *MMP for moduli of sheaves on K3s via wall-crossing: nef and movable cones, Lagrangian fibrations*, Invent. Math. **198** (2014), no. 3, 505–590; doi:10.1007/s00222-014-0501-8.

BM14b ———, *Projectivity and birational geometry of Bridgeland moduli spaces*, J. Amer. Math. Soc. **27** (2014), no. 3, 707–752; doi:10.1090/S0894-0347-2014-00790-6.

BMS16 A. Bayer, E. Macrì and P. Stellari, *The space of stability conditions on abelian threefolds, and on some Calabi–Yau threefolds*, Invent. Math. **206** (2016), no. 3, 869–933; doi:10.1007/s00222-016-0665-5.

BMT14 A. Bayer, E. Macrì, and Y. Toda, *Bridgeland stability conditions on threefolds I: Bogomolov–Gieseker type inequalities*, J. Algebraic Geom. **23** (2014), no. 1, 117–163; doi:10.1090/S1056-3911-2013-00617-7.

BO95 A. Bondal and D. Orlov, *Semiorthogonal decomposition for algebraic varieties*, 1995, arXiv:alg-geom/9506012.

Bri02 T. Bridgeland, *Flops and derived categories*, Invent. Math. **147** (2002), no. 3, 613–632; doi:10.1007/s002220100185.

Bri07 ———, *Stability conditions on triangulated categories*, Ann. of Math. **166** (2007), no. 2, 317–345; doi:10.4007/annals.2007.166.317.

Bri08 ———, *Stability conditions on $K3$ surfaces*, Duke Math. J. **141** (2008), no. 2, 241–291; doi:10.1215/S0012-7094-08-14122-5.

Bri09 ———, *Spaces of stability conditions*, Algebraic Geometry – Seattle 2005, Part 1, Proc. Sympos. Pure Math., vol. 80 (Amer. Math. Soc., Providence, RI, 2009), 1–21; doi:10.1090/pspum/080.1/2483930.

Bri11 ———, *Hall algebras and curve-counting invariants*, J. Amer. Math. Soc. **24** (2011), no. 4, 969–998; doi:10.1090/S0894-0347-2011-00701-7.

Dia12 D. E. Diaconescu, *Moduli of ADHM sheaves and the local Donaldson–Thomas theory*, J. Geom. Phys. **62** (2012), no. 4, 763–799; doi:10.1016/j.geomphys.2011.12.018.

EG95 G. Ellingsrud and L. Göttsche, *Variation of moduli spaces and Donaldson invariants under change of polarization*, J. reine angew. Math. **467** (1995), 1–49; doi:10.1515/crll.1995.467.1.

Eri10 E. Eriksen, *Computing noncommutative deformations of presheaves and sheaves of modules*, Canad. J. Math. **62** (2010), no. 3, 520–542; doi:10.4153/CJM-2010-015-6.

FQ95 R. Friedman and Z. Qin, *Flips of moduli spaces and transition formulas for Donaldson polynomial invariants of rational surfaces*, Comm. Anal. Geom. **3** (1995), no. 1-2, 11–83; doi:10.4310/CAG.1995.v3.n1.a2.

Fuk03 K. Fukaya, *Deformation theory, homological algebra and mirror symmetry*, Geometry and Physics of Branes (Como, 2001), Ser. High Energy Phys. Cosmol. Gravit. (IOP, Bristol, 2003), 121–209.

Gre15 D. Greb, *Complex-analytic quotients of algebraic G-varieties*, Math. Ann. **363** (2015), no. 1–2, 77–100; doi:10.1007/s00208-014-1163-y.

HL97 D. Huybrechts and M. Lehn, *The geometry of moduli spaces of sheaves*, Aspects Math., vol. E31 (Friedr. Vieweg & Sohn, Braunschweig, 1997); doi:10.1007/978-3-663-11624-0.

HMP98 P. Heinzner, L. Migliorini and M. Polito, *Semistable quotients*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) **26** (1998), no. 2, 233–248.

HT10 D. Huybrechts and R. P. Thomas, *Deformation-obstruction theory for complexes via Atiyah and Kodaira–Spencer classes*, Math. Ann. **346** (2010), no. 3, 545–569; doi:10.1007/s00208-009-0397-6.

Ina02 M. Inaba, *Toward a definition of moduli of complexes of coherent sheaves on a projective scheme*, J. Math. Kyoto Univ. **42** (2002), no. 2, 317–329; doi:10.1215/kjm/1250283873.

Joy13 D. Joyce, *Shifted symplectic geometry, Calabi–Yau moduli spaces, and generalizations of Donaldson–Thomas theory: our current and future research*, 2013, Talks given at a workshop for EPSRC Programme Grant research group, Oxford, October 2013; available at https://people.maths.ox.ac.uk/joyce/PGhandout.pdf.

Joy15 ———, *A classical model for derived critical loci*, J. Differential Geom. **101** (2015), no. 2, 289–367; doi:10.4310/jdg/1442364653.

JS12 D. Joyce and Y. Song, *A theory of generalized Donaldson–Thomas invariants*, Mem. Amer. Math. Soc. **217** (2012), no. 1020; doi:10.1090/S0065-9266-2011-00630-1.

JS19 D. Joyce and P. Safronov, *A Lagrangian neighbourhood theorem for shifted symplectic derived schemes*, Ann. Fac. Sci. Toulouse Math. **28** (2019), no. 5, 831–908; doi:10.5802/afst.1616.

Kaw02 Y. Kawamata, *D-equivalence and K-equivalence*, J. Differential Geom. **61** (2002), no. 1, 147–171; doi:10.4310/jdg/1090351323.

Kaw08 ———, *Flops connect minimal models*, Publ. Res. Inst. Math. Sci. **44** (2008), no. 2, 419–423; doi:10.2977/prims/1210167332.

Kaw15 ———, *On multi-pointed non-commutative deformations and Calabi–Yau threefolds*, 2015, arXiv:1512.06170.

Kaw18 ———, *Birational geometry and derived categories*, in *Surveys in Differential Geometry 2017. Celebrating the 50th anniversary of the Journal of Differential Geometry*, Surv. Differ. Geom., vol. 22 (Int. Press, Somerville, MA, 2018), 291–317; doi:10.4310/SDG.2017.v22.n1.a11.

Kin94 A. D. King, *Moduli of representations of finite-dimensional algebras*, Quart. J. Math. Oxford Ser. (2) **45** (1994), no. 180, 515–530; doi:10.1093/qmath/45.4.515.

KM98 J. Kollár and S. Mori, *Birational geometry of algebraic varieties*, Cambridge Tracts in Math., vol. 134 (Cambridge Univ. Press, Cambridge, 1998); doi:10.1017/CBO9780511662560.

KS08 M. Kontsevich and Y. Soibelman, *Stability structures, motivic Donaldson–Thomas invariants and cluster transformations*, 2008, arXiv:0811.2435.

Lau02 O. A. Laudal, *Noncommutative deformations of modules*, Homology Homotopy Appl. **4** (2002), no. 2, part 2, 357–396; doi:10.4310/hha.2002.v4.n2.a17.

LP90 L. Le Bruyn and C. Procesi, *Semisimple representations of quivers*, Trans. Amer. Math. Soc. **317** (1990), no. 2, 585–598; doi:10.2307/2001477.

Lie06 M. Lieblich, *Moduli of complexes on a proper morphism*, J. Algebraic Geom. **15** (2006), no. 1, 175–206; doi:10.1090/S1056-3911-05-00418-2.

MNOP06 D. Maulik, N. Nekrasov, A. Okounkov, and R. Pandharipande, *Gromov–Witten theory and Donaldson–Thomas theory. I*, Compos. Math. **142** (2006), no. 5, 1263–1285; doi:10.1112/S0010437X06002302.

MP15 A. Maciocia and D. Piyaratne, *Fourier–Mukai transforms and Bridgeland stability conditions on abelian threefolds*, Algebr. Geom. **2** (2015), no. 3, 270–297; doi:10.14231/AG-2015-012.

MR19 S. Meinhardt and M. Reineke, *Donaldson–Thomas invariants versus intersection cohomology of quiver moduli*, J. reine angew. Math. **754** (2019), 143–178; doi:10.1515/crelle-2017-0010.

MT18 D. Maulik and Y. Toda, *Gopakumar–Vafa invariants via vanishing cycles*, Invent. Math. **213** (2018), no. 3, 1017–1097; doi:10.1007/s00222-018-0800-6.

MW97 K. Matsuki and R. Wentworth, *Mumford–Thaddeus principle on the moduli space of vector bundles on an algebraic surface*, Internat. J. Math. **8** (1997), no. 1, 97–148; doi:10.1142/S0129167X97000068.

NN11 K. Nagao and H. Nakajima, *Counting invariant of perverse coherent sheaves and its wall-crossing*, Int. Math. Res. Not. **2011** (2011), no. 17, 3885–3938; doi:10.1093/imrn/rnq195.

NY11 H. Nakajima and K. Yoshioka, *Perverse coherent sheaves on blow-up. II. Wall-crossing and Betti numbers formula*, J. Algebraic Geom. **20** (2011), no. 1, 47–100; doi:10.1090/S1056-3911-10-00534-5.

Pol01 A. Polishchuk, *Homological mirror symmetry with higher products*, Winter School on Mirror Symmetry, Vector Bundles and Lagrangian Submanifolds (Cambridge, MA, 1999), AMS/IP Stud. Adv. Math., vol. 23 (Amer. Math. Soc., Providence, RI, 2001), 247–259.

PT09 R. Pandharipande and R. P. Thomas, *Curve counting via stable pairs in the derived category*, Invent. Math. **178** (2009), no. 2, 407–447; doi:10.1007/s00222-009-0203-9.

PT10 ———, *Stable pairs and BPS invariants*, J. Amer. Math. Soc. **23** (2010), no. 1, 267–297; doi:10.1090/S0894-0347-09-00646-8.

PT19 D. Piyaratne and Y. Toda, *Moduli of Bridgeland semistable objects on 3-folds and Donaldson–Thomas invariants*, J. reine angew. Math. **747** (2019), 175–219; doi:10.1515/crelle-2016-0006.

PTVV13 T. Pantev, B. Toën, M. Vaquié and G. Vezzosi, *Shifted symplectic structures*, Publ. Math. Inst. Hautes Études Sci. **117** (2013), 271–328; doi:10.1007/s10240-013-0054-1.

Rei92 M. Reid, *What is a flip*, Colloquium talk, University of Utah, 1992, available at https://homepages.warwick.ac.uk/~masda/3folds/what_flip.pdf.

Rei08 M. Reineke, *Moduli of representations of quivers*, in *Trends in Representation Theory of Algebras and Related Topics*, EMS Ser. Congr. Rep. (Eur. Math. Soc., Zürich, 2008), 589–637; doi:10.4171/062-1/14.

Tho00 R. P. Thomas, *A holomorphic Casson invariant for Calabi–Yau 3-folds, and bundles on* $K3$ *fibrations*, J. Differential Geom. **54** (2000), no. 2, 367–438; doi:10.4310/jdg/1214341649.

Toë14 B. Toën, *Derived algebraic geometry and deformation quantization*, Proc. Interat. Congr. Math. – Seoul 2014, Vol. II (Kyung Moon Sa, Seoul, 2014), 769–792.

Tod08 Y. Toda, *Birational Calabi–Yau threefolds and BPS state counting*, Commun. Number Theory Phys. **2** (2008), no. 1, 63–112; doi:10.4310/CNTP.2008.v2.n1.a2.

Tod09a ———, *Limit stable objects on Calabi–Yau 3-folds*, Duke Math. J. **149** (2009), no. 1, 157–208; doi:10.1215/00127094-2009-038.

Tod09b ———, *Stability conditions and Calabi–Yau fibrations*, J. Algebraic Geom. **18** (2009), no. 1, 101–133; doi:10.1090/S1056-3911-08-00511-0.

Tod10a ———, *Curve counting theories via stable objects I. DT/PT correspondence*, J. Amer. Math. Soc. **23** (2010), no. 4, 1119–1157; doi:10.1090/S0894-0347-10-00670-3.

Tod10b ———, *Generating functions of stable pair invariants via wall-crossings in derived categories*, New Developments in Algebraic Geometry, Integrable Systems and Mirror Symmetry (RIMS, Kyoto, 2008), Adv. Stud. Pure Math., vol. 59 (Math. Soc. Japan, Tokyo, 2010), 389–434; doi:10.2969/aspm/05910389.

Tod12a ———, *Stability conditions and curve counting invariants on Calabi–Yau 3-folds*, Kyoto J. Math. **52** (2012), no. 1, 1–50; doi:10.1215/21562261-1503745.

Tod12b ———, *Stable pairs on local* $K3$ *surfaces*, J. Differential Geom. **92** (2012), no. 2, 285–371; doi:10.4310/jdg/1352297809.

Tod14 ———, *Stability conditions and birational geometry of projective surfaces*, Compos. Math. **150** (2014), no. 10, 1755–1788; doi:10.1112/S0010437X14007337.

Tod18 ———, *Moduli stacks of semistable sheaves and representations of Ext-quivers*, Geom. Topol. **22** (2018), no. 5, 3083–3144; doi:10.2140/gt.2018.22.3083.

Tod19 ———, *Categorical Donaldson–Thomas theory for local surfaces*, 2019, arXiv:1907.09076.

Tod21 ———, *Semiorthogonal decompositions of stable pair moduli spaces via d-critical flips*, J. Eur. Math. Soc. (JEMS) **23** (2021), no. 5, 1675–1725; doi:10.4171/JEMS/1041.

Tod22a ______, *Categorical Donaldson–Thomas theory for local surfaces: $\mathbb{Z}/2$-periodic version*, Int. Math. Res. Not., published online on 7 June 2022, to appear in print; doi:10.1093/imrn/rnac142.

Tod22b ______, *Gopakumar–Vafa invariants and wall-crossing*, J. Differential Geom. (to appear), arXiv:1710.01843.

Tu14 J. Tu, *Homotopy L-infinity spaces*, 2014, arXiv:1411.5115.

Vak06 R. Vakil, *Murphy's law in algebraic geometry: badly-behaved deformation spaces*, Invent. Math. **164** (2006), no. 3, 569–590; doi:10.1007/s00222-005-0481-9.

Yukinobu Toda yukinobu.toda@ipmu.jp
Kavli Institute for the Physics and Mathematics of the Universe (WPI), University of Tokyo, 5-1-5 Kashiwanoha, Kashiwa, 277-8583, Japan