Algebraic Geometry 9 (3) (2022) 311–363
doi:10.14231/AG-2022-010

Logarithmic resolution via weighted toroidal blow-ups

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Abstract

Let X be a fs logarithmic scheme which admits a strict closed embedding into a logarithmically smooth scheme Y over a field $\Bbbk$ of characteristic zero. We construct a simple and fast procedure to make a functorial logarithmic resolution of X, where the end result is, in particular, a stack-theoretic modification $X' \to X$ such that X' is logarithmically smooth over $\Bbbk$. In particular, if X is a finite-type $\Bbbk$-scheme embedded in a smooth $\Bbbk$-scheme Y, the procedure not only shares the same desirable features as the "dream resolution algorithm" of Abramovich–Temkin–Włodarczyk [Functorial embedded resolution via weighted blowings up, arxiv: 1906.07106] but also accounts for a key feature of Hironaka's Main Theorem I in [Resolution of singularities of an algebraic variety over a field of characteristic zero. I, Ann. of Math. 79 (1964), no. 1, 109–203] which was not addressed in the Abramovich–Temkin–Włodarczyk paper. As a consequence, we recover a different and simpler approach to Hironaka's resolution of singularities in characteristic zero.

1. Introduction

1.1 Statement of the main theorem

Consider a fs logarithmic scheme Y (see Definition B.1) which is logarithmically smooth over a field $\Bbbk$ of characteristic zero or, equivalently, a *toroidal* $\Bbbk$-scheme Y (see Definition B.6), as well as a reduced closed subscheme $X \subset Y$ of pure codimension c. More generally, we consider a reduced closed substack X of pure codimension c in a *toroidal Deligne–Mumford stack* Y *over* $\Bbbk$ (see Definition B.16). We will always regard X (without mention) as a logarithmic Deligne–Mumford stack over $\Bbbk$ by pulling the logarithmic structure of Y back to X. Such pairs $X \subset Y$ form the objects of a category, where a morphism between pairs $\big(\widetilde{X} \subset \widetilde{Y}\big) \to (X \subset Y)$ is a cartesian square

$$\begin{array}{ccc} \widetilde{X} = X \times_Y \widetilde{Y} & \hookrightarrow & \widetilde{Y} \\ \downarrow & & \downarrow f \\ X & \hookrightarrow & Y\,, \end{array}$$

where $f\colon \widetilde{Y} \to Y$ is logarithmically smooth and surjective. We refer to such morphisms as logarithmically smooth, surjective morphisms of pairs. Note, however, that in certain situations

Received 9 July 2020, accepted in final form 3 September 2021.
2020 Mathematics Subject Classification 14E15 (primary), 14A20, 14M25 (secondary).
Keywords: resolution of singularities, stacks, logarithmic geometry.

This research work was supported in part by funds from BSF grant 2014365 and NSF grant DMS-1759514.

given below, we do not demand surjectivity in our morphisms of pairs.

The goal of this paper is to define a *logarithmic embedded resolution functor* on the aforementioned category, which assigns to each pair $X \subset Y$ as above a proper birational morphism $\Pi\colon Y' \to Y$ such that both Y' and the proper transform $X' \subset X \times_Y Y'$ are toroidal. Moreover, Π satisfies two properties which resemble those in Hironaka's Main Theorem I in [Hir64a]:

(a) The morphism Π is an isomorphism over the logarithmically smooth locus $X^{\text{log-sm}}$ of X.

(b) We are able to control $\Pi^{-1}\big(X \setminus X^{\text{log-sm}}\big)$; namely, $\Pi^{-1}\big(X \setminus X^{\text{log-sm}}\big)$ will be contained in the toroidal divisor (see Remark B.5(ii)) of X'.

We will explicitly construct the proper birational morphism Π as a composition of stack-theoretic blow-ups along *toroidal centres* (which are the *weighted toroidal blow-ups* in the title of this paper). The notion of a *toroidal centre* will be defined in Section 3.2. A convenient tool for bookkeeping the information carried by toroidal centres is the notion of *idealistic exponents*, which we study in detail in Section 2. In Section 4.4, we will also explicate the charts of the weighted toroidal blow-ups appearing in Π.

In addition, for a point $p \in |Y|$, Section 6.1 defines a (logarithmic) *invariant* of $X \subset Y$ at p (motivated by the invariants in [ATW20a] and [ATW19]), denoted by $\operatorname{inv}_p(X \subset Y)$, which is an non-decreasing truncated sequence of non-negative rational numbers, but we allow for the last entry to be ∞. There is a total order on the set consisting of all such invariants of $X \subset Y$ at a point p by the lexicographic order $<$, which turns out to be a well-ordering (see Lemma 6.3(i)). There is a caveat here: our lexicographic order considers truncated sequences to be strictly larger. For example,

$$(0) < (1,1,2,2) < (1,1,3) < (1,1) < (1,2,5) < (1,3,4) < (1,\infty) < (1) < (\,),$$

where $(\,)$ is the empty sequence. The invariant satisfies the following properties:

(a) If $c \geqslant 1$, it detects logarithmic smoothness at any $p \in |X|$: $\operatorname{inv}_p(X \subset Y)$ is bounded below (via the lexicographic order) by the sequence $(1,\dots,1)$ of length c, and equality holds if and only if X is logarithmically smooth at p.

(a′) We have $\operatorname{inv}_p(X \subset Y) = (0)$ if and only if $p \notin |X|$.

(a″) If $c = 0$ (that is, $X = Y$), then $\operatorname{inv}_p(X \subset Y) = (\,)$ for all $p \in |Y| = |X|$.

(b) It is upper semi-continuous on Y.

(c) It is functorial for logarithmically smooth morphisms of pairs $X \subset Y$, whether or not surjective. See Lemma 6.3(iii) for a precise statement.

(d) The first term of $\operatorname{inv}_p(X \subset Y)$ is the logarithmic order (see Definition B.14) at p of the ideal $\mathcal{I}_{X\subset Y}$ of X embedded in Y. In particular, it is in $\mathbb{N} \cup \{\infty\}$.

This invariant is constructed via logarithmic analogues of the classical notions of maximal contact elements and coefficient ideals, which we study in Section 5. We set the *maximal invariant* of $X \subset Y$ to be $\max \operatorname{inv}(X \subset Y) = \max_{p\in|X|} \operatorname{inv}_p(X \subset Y)$; this is functorial for logarithmically smooth and surjective morphisms of pairs $X \subset Y$ and is equal to the sequence $(1,\dots,1)$ of length c if and only if X is toroidal. We can now state the main result.

Theorem 1.1. *There is a functor $F_{\text{log-ER}}$ associating with*

a reduced, closed substack X of pure codimension in a generically toroidal Deligne–Mumford stack Y over a field $\Bbbk$ of characteristic zero, such that X is not toroidal *a toroidal centre* $\overline{\mathscr{J}}$ *on* Y*, with weighted toroidal blow-up* $\pi\colon Y' \to Y$ *along* $\overline{\mathscr{J}}$ *and proper transform* $F_{\text{log-ER}}(X \subset Y) = X' \subset Y'$ *such that*

(i) Y' *is again a toroidal Deligne–Mumford stack over* $\Bbbk$*;*

(ii) $\max \operatorname{inv}(X' \subset Y') < \max \operatorname{inv}(X \subset Y)$*;*

(iii) π *is an isomorphism away from the closed locus consisting of the points* $p \in |X|$ *with* $\operatorname{inv}_p(X \subset Y) = \max \operatorname{inv}(X \subset Y)$*;*

(iv) *the exceptional divisor underlying* π *is contained in the toroidal divisor of* Y'*.*

Functoriality here is with respect to logarithmically smooth, surjective morphisms of pairs $X \subset Y$*, as described before the theorem.*

In particular, one stops at an integer $N \geqslant 1$ *where the iterated application* $(X_N \subset Y_N) = F_{\text{log-ER}}^{\circ N}(X \subset Y)$ *is accompanied by a sequence of weighted toroidal blow-ups* $\Pi\colon Y_N \xrightarrow{\pi_{N-1}} \cdots \xrightarrow{\pi_1} Y_1 \xrightarrow{\pi_0} Y_0 = Y$ *such that*

(1) X_N *and* Y_N *are both toroidal Deligne–Mumford stacks over* $\Bbbk$*;*

(2) Π *is an isomorphism over the logarithmically smooth locus* $X^{\text{log-sm}}$ *of* X*;*

(3) $\Pi^{-1}\big(X \setminus X^{\text{log-sm}}\big)$ *is contained in the toroidal divisor of* X_N*.*

This stabilized functor $F_{\text{log-ER}}^{\circ\infty}$*, together with the sequence of weighted toroidal blow-ups* Π *after removing empty blow-ups, is functorial for all logarithmically smooth morphisms of pairs* $X \subset Y$*, whether or not surjective.*

The toroidal centre $\overline{\mathscr{J}}$ associated with $X \subset Y$ in the first paragraph of Theorem 1.1 will be defined and studied in Section 6. In the words of [Kol07], Theorem 1.1 will be proven, seemingly by accident, via the logarithmic analogue of *principalization* (see Theorem 7.1). This is the content of Section 7.

Finally, we remark that parts (ii) and (iii) of Theorem 1.1 are precisely the two key features of the "dream resolution algorithm" in [ATW19]; namely, each step of that algorithm

(a) improves singularities immediately in a visible way

(b) and does so by blowing up the most singular locus.

It is well known that besides the case of curves, these two features are in general not plausible for Hironaka's resolution algorithm. There are plenty of examples corroborating this observation; see, for example, [ATW19, Section 1.7].

1.2 Recovering Hironaka's resolution of singularities

In this section, we recover [Hir64a, Main Theorem I] from Theorem 1.1 in three steps. The first step is to deduce *logarithmic resolution* from Theorem 1.1.

Theorem 1.2 (Logarithmic resolution). *There is a functor* $F_{\text{log-res}}$ *associating with*

> *a pure-dimensional, reduced, fs logarithmic Deligne–Mumford stack* X *of finite type over a field* $\Bbbk$ *of characteristic zero*

a proper and birational morphism $\Pi\colon F_{\text{log-res}}(X) \to X$ *such that*

(i) $F_{\text{log-res}}(X)$ *is a pure-dimensional, toroidal Deligne–Mumford stack over* $\Bbbk$*;*

(ii) Π *is an isomorphism over the logarithmically smooth locus* $X^{\text{log-sm}}$ *of* X*;*

(iii) $\Pi^{-1}\big(X \setminus X^{\text{log-sm}}\big)$ *is contained in the toroidal divisor of* $F_{\text{log-res}}(X)$*.*

Functoriality here is with respect to logarithmically smooth morphisms: if $\widetilde{X} \to X$ is a logarithmically smooth morphism, then $F_{\text{log-res}}(\widetilde{X}) = F_{\text{log-res}}(X) \times_X \widetilde{X}$.

See Section B.2 for the definition of "fs" in Theorem 1.2. We emphasize that the fibre product at the end of the theorem should be taken in the same category of the theorem. Note that this differs from the standard notation (for example, in [Ogu18, Section III.2.1]), where we would instead write $F_{\text{log-res}}(\widetilde{X}) = \big(F_{\text{log-res}}(X) \times_X \widetilde{X}\big)^{\text{sat}}$.

Proof. The proof of this theorem follows the strategy in [ATW19, Theorem 8.1.1], with minor modifications. Let X be as in Theorem 1.2. Since étale locally X can always be embedded in pure codimension in a toroidal $\Bbbk$-scheme, the theorem follows once we show the following:

> Given two strict closed embeddings of X into pure-dimensional, toroidal Deligne–Mumford stacks Y_i over $\Bbbk$ (where $i = 1, 2$), the logarithmic resolutions of X obtained from $F_{\text{log-ER}}(X \subset Y_i)$ (for $i = 1, 2$) coincide.

First assume $\dim(Y_1) = \dim(Y_2)$; in this case, the two embeddings are étale locally isomorphic. By functoriality, the logarithmic embedded resolutions $F^{\circ\infty}_{\text{log-ER}}(X \subset Y_i)$ (for $i = 1, 2$) are isomorphic, whence the resulting logarithmic resolutions of X coincide. In general, this reduces to the earlier case, by a repeated application of Lemma 1.3. □

Lemma 1.3 (Re-embedding principle [ATW20a, Proposition 2.9.3]). *Let X be a reduced, closed substack of pure codimension in a toroidal Deligne–Mumford stack Y over a field $\Bbbk$ of characteristic zero. Let Y_1 be the fibre product $Y \times_{\Bbbk} \mathbb{A}^1_{\Bbbk}$ in the category of logarithmic schemes, where $\mathbb{A}^1_{\Bbbk}$ and $\Bbbk$ are given the trivial logarithmic structure.*

(i) *For every $p \in |X|$, the invariant $\operatorname{inv}_p(X \subset Y_1)$ is the concatenation $(1, \operatorname{inv}_p(X \subset Y))$.*

(ii) *If $F_{\text{log-ER}}(X \subset Y) = (X' \subset Y')$ and $F_{\text{log-ER}}(X \subset Y_1) = (X'_1 \subset Y'_1)$, then Y' is canonically identified with the proper transform of $Y = Y \times \{0\} \subset Y_1$ in Y'_1, under which $X' = X'_1$.*

We prove Lemma 1.3 in Section 7.5. The second step is to deduce the following theorem from Theorem 1.2 via *resolution of toroidal singularities*.

Theorem 1.4 (Resolution). *There is a functor F_{res} associating with*

> *a pure-dimensional, reduced Deligne–Mumford stack X of finite type over a field $\Bbbk$ of characteristic zero*

a proper and birational morphism $\Pi\colon F_{\text{res}}(X) \to X$ such that

(i) *$F_{\text{res}}(X)$ is a pure-dimensional, smooth Deligne–Mumford stack over $\Bbbk$;*

(ii) *Π is an isomorphism over the smooth locus X^{sm} of X.;*

(iii) *$\Pi^{-1}(X \setminus X^{\text{sm}})$ is a simple normal crossing divisor on $F_{\text{res}}(X)$.*

Functoriality here is with respect to smooth morphisms: if $\widetilde{X} \to X$ is a smooth morphism, then $F_{\text{res}}(\widetilde{X}) = F_{\text{res}}(X) \times_X \widetilde{X}$.

Proof. Let X be as in the theorem, give X the trivial logarithmic structure, and apply Theorem 1.2 to obtain $F_{\text{log-res}}(X) \to X$. Note that $X^{\text{log-sm}} = X^{\text{sm}}$ in this case. Next, apply [Wło20, Theorem 6.5.1]: there is a projective birational morphism $\phi\colon X'' \to X' = F_{\text{log-res}}(X)$, where X'' is a pure-dimensional, smooth Deligne–Mumford stack over $\Bbbk$, the morphism ϕ is an isomorphism over the smooth locus $(X')^{\text{sm}}$ of X', the preimage of the toroidal divisor on X' under ϕ

is a simple normal crossing divisor, and ϕ is functorial with respect to strict, smooth morphisms of toroidal Deligne–Mumford stacks over $\Bbbk$. Take $\Pi\colon F_{\text{res}}(X) \to X$ to be the composition $X'' \xrightarrow{\phi} X' = F_{\text{log-res}}(X) \to X$. □

We remark that if X happens to be a scheme in Theorem 1.4, then $F_{\text{res}}(X)$ is, more often than not, a stack. Therefore, the final step involves *Bergh's destackification theorem*, which is as follows.

Theorem 1.5 (Coarse resolution). *There is a functor $F_{\text{c-res}}$ associating with*

a pure-dimensional, reduced Deligne–Mumford stack X of finite type over a field $\Bbbk$ of characteristic zero

a projective *and birational morphism $\Pi\colon F_{\text{c-res}}(X) \to X$ such that*

(i) *$F_{\text{c-res}}(X)$ is a pure-dimensional, smooth Deligne–Mumford stack over $\Bbbk$;*

(ii) *Π is an isomorphism over the smooth locus X^{sm} of X;*

(iii) *$\Pi^{-1}(X \setminus X^{\text{sm}})$ is a simple normal crossing divisor on $F_{\text{c-res}}(X)$.*

Functoriality here is with respect to smooth morphisms: if $\widetilde{X} \to X$ is a smooth morphism, then $F_{\text{c-res}}(\widetilde{X}) = F_{\text{c-res}}(X) \times_X \widetilde{X}$.

In particular, if we restrict to the full subcategory whose objects are pure-dimensional, reduced schemes of finite type over $\Bbbk$, we recover [Hir64a, Main Theorem I].

Proof. This proof follows verbatim the proof of [ATW19, Theorem 8.12]. Let X be as in the theorem, and apply [BR19, Theorem 7.1] to the standard pair $(F_{\text{res}}(X), D)$ (where D is the simple normal crossing divisor in Theorem 1.4(iii)) and $F_{\text{res}}(X) \to X \to \operatorname{Spec}(\Bbbk)$ (where the $F_{\text{res}}(X) \to X$ is given in Theorem 1.4). This provides a projective morphism $F_{\text{res}}(X)' \to F_{\text{res}}(X) \to X$, functorial for all smooth morphisms, such that the relative coarse moduli space $F_{\text{res}}(X)' \to \underline{F_{\text{res}}(X)'} \to X$ is projective over X and such that $F_{\text{res}}(X)'$ and $\underline{F_{\text{res}}(X)'}$ are smooth over $\Bbbk$. We then take $\Pi\colon F_{\text{c-res}}(X) \to X$ to be $\underline{F_{\text{res}}(X)'} \to X$. □

1.3 Adapting methods in [ATW19] to the logarithmic setting

We recall the set-up in [ATW19]. Let $\Bbbk$ be a field of characteristic zero (as before), but we instead consider a smooth $\Bbbk$-scheme Y and a reduced closed subscheme $X \subset Y$ of pure codimension c – or, more generally, a reduced closed substack X of pure codimension c in a smooth Deligne–Mumford stack Y over $\Bbbk$. Then [ATW19] proposes a faster and simpler approach to embedded resolution of singularities of X in Y, where each step immediately and visibly improves the singularities – by considering a *broader* notion of blow-up centres [ATW19, Section 2.4]. However, as the example in [ATW19, Section 8.3] demonstrates,

($*$) at each step of the resolution, the chosen blow-up centre does not necessarily have simple normal crossings with the exceptional loci obtained at that step,

and hence,

(†) the exceptional loci at subsequent steps of the resolution may not be simple normal crossing divisors.

Consequently, this does not address a key feature of [Hir64a, Main Theorem I]:

(♢) the preimage of the singular locus of X under the resolution in [ATW19] is not always a simple normal crossing divisor.

Our result (Theorem 1.1) on logarithmic embedded resolution can be seen as a resolution to the aforementioned issue as follows:

(a) Give Y (and hence X) the trivial logarithmic structure. At each step of the resolution, we will first encode the exceptional divisor obtained at that step into the logarithmic structure (cf. Theorem 1.1(iv)).

(b) With respect to these logarithmic structures, we then adapt the methods used in [ATW19, Section 5] to obtain a *toroidal* blow-up centre at each step.

We remark that statement (b) does not resolve ($*$) or (†): in fact, the Deligne–Mumford stacks Y_i obtained in this modified resolution $Y_N \to \cdots \to Y_1 \to Y_0 = Y$ may not even be smooth (unlike in [ATW19]). For an example of this, see Section 8.1. Nevertheless, statement (b) assures that the Y_i will be toroidal. Moreover, statements (a) and (b) will give us some control over the exceptional loci obtained in the process; namely, the exceptional loci at each step will always be contained in the toroidal divisor (Remark B.5(ii)) of the toroidal Deligne–Mumford stack Y_i at that step. Consequently, the preimage of the singular locus of X under this modified resolution $X_N = Y_N \times_Y X \to X$ is contained in the toroidal divisor of X_N (cf. Theorem 1.1(3)). One then resolves the issue in (♢) via *resolution of toroidal singularities* as in Theorem 1.4.

This justifies our need to work in the logarithmic setting as outlined in Section 1.1. We also remark that the above strategy of statements (a) and (b) was already pursued earlier in [ATW20a], although with respect to Hironaka's classical resolution algorithm.

2. Idealistic exponents

In this section, let $\Bbbk$ be a field. Unless otherwise stated, Y is usually a $\Bbbk$-variety[1] with field of fractions K. Let $\mathrm{ZR}(Y)$ denote the Zariski–Riemann space of Y, as defined in [ATW19, Section 2.1] or Appendix A of this paper. The space $\mathrm{ZR}(Y)$ is a locally ringed space whose elements are valuation rings R_ν of K containing $\Bbbk$ which possess a centre on Y; see [Har77, Exercise II.4.5]. We usually denote R_ν by its corresponding valuation $\nu\colon K^* \twoheadrightarrow G_\nu$, where G_ν is the value group of ν. The monoid of non-negative elements of G_ν is denoted by $(G_\nu)_+$. Let us fix some related notation for this chapter:

$\mathscr{O}_{\mathrm{ZR}(Y)}$ – sheaf of rings carried by $\mathrm{ZR}(Y)$ whose stalk at ν is R_ν,
Γ_Y – sheaf of ordered groups $K^*/\mathscr{O}^*_{\mathrm{ZR}(Y)}$ on $\mathrm{ZR}(Y)$ whose stalk at ν is G_ν,
$\Gamma_{Y,+}$ – subsheaf of Γ_Y consisting of the non-negative sections of Γ,
y_ν – the (unique) centre of ν on Y,
π_Y – the canonical morphism $\mathrm{ZR}(Y) \to Y$ which maps ν to y_ν.

See Appendix A for a self-contained exposition of the aforementioned notions.

2.1 Valuative ideals

Following [ATW19, Section 2.2], a *valuative ideal* over Y is defined to be a section

$$\gamma \in H^0(\mathrm{ZR}(Y), \Gamma_{Y,+}) .$$

A (coherent) ideal $0 \neq \mathcal{I} \subset \mathscr{O}_Y$ determines a valuative ideal $\gamma_\mathcal{I}$ over Y as follows. For every $\nu \in \mathrm{ZR}(Y)$, remember that y_ν denotes the centre of ν on Y, and let $f_\nu^\#\colon \mathscr{O}_{Y,y_\nu} \to R_\nu$ denote the

[1]Following [Har77], a $\Bbbk$-variety is an integral, separated scheme of finite type over $\Bbbk$.

corresponding local $\Bbbk$-homomorphism. We then set

$$\gamma_{\mathcal{I},\nu} := \min\{\nu(g)\colon g \text{ is a non-zero section of } \mathcal{I} \subset \mathscr{O}_Y \text{ over an open set containing } y_\nu\}\,,$$

where $\nu(g)$ is an abbreviation for $\nu(f_\nu^\#(g_{y_\nu}))$ (this abbreviation will persist in this paper). Note that since $\mathcal{I}$ is coherent, this minimum exists in $(G_\nu)_+$. Indeed, if $\mathcal{I}_{y_\nu}$ is generated by $g_1, \dots, g_r \in \mathscr{O}_{Y,y_\nu}$, then $\gamma_{\mathcal{I},\nu} = \min\{\nu(g_i)\colon 1 \leqslant i \leqslant r\}$.

Moreover, if we let $1 \leqslant j \leqslant r$ be such that $\nu(g_j) = \gamma_{\mathcal{I},\nu}$, then $f_\nu^\#(\mathcal{I}_{y_\nu})R_\nu$ is the principal ideal $(f_\nu^\#(g_j))R_\nu$ of R_ν. For

$$(\gamma_{\mathcal{I},\nu})_{\nu \in \mathrm{ZR}(Y)} \in \prod_{\nu \in \mathrm{ZR}(Y)} (G_\nu)_+$$

to define a valuative ideal $\gamma_{\mathcal{I}}$ over Y, we need to check that it is a compatible collection of germs of $\Gamma_{Y,+}$. Indeed, fix an arbitrary $\nu \in \mathrm{ZR}(Y)$, and assume that $g_1, \dots, g_r \in \mathscr{O}_{Y,y_\nu}$ generate $\mathcal{I}_{y_\nu}$, with $\nu(g_j) = \min\{\nu(g_i)\colon 1 \leqslant i \leqslant r\}$. There exists an affine open neighbourhood V_ν of y_ν in Y such that $g_1, \dots, g_r$ extend to sections of $\mathcal{I}$ over V_ν which generate the stalk of $\mathcal{I}$ at every point in V_ν. Then $U_\nu = \pi_Y^{-1}(V_\nu) \cap U\big(g_i/g_j\colon i \neq j\big)$ is an open neighbourhood of ν in $\mathrm{ZR}(Y)$ such that for all $\nu' \in U_\nu$, we have $\gamma_{\nu'} = \nu'(g_j)$.

In fact, the same argument shows that any valuative ideal over Y arising from an ideal on Y is locally represented by generators of that ideal.

Lemma 2.1. *Let the notation be as above, and let $\mathcal{I}$ be a non-zero ideal on Y. There exist*

(i) *a finite open affine cover $\mathcal{V} = \{V_\ell\colon 1 \leqslant \ell \leqslant m\}$ of Y;*
(ii) *for each $1 \leqslant \ell \leqslant m$, a finite open cover $\mathcal{U}_\ell = \{U_{\ell,j}\colon 1 \leqslant j \leqslant r_\ell\}$ of $\pi_Y^{-1}(V_\ell)$;*
(iii) *for each $1 \leqslant \ell \leqslant m$, sections $\{g_{\ell,j}\colon 1 \leqslant j \leqslant r_\ell\}$ of $\mathcal{I}$ over V_ℓ which generate $\mathcal{I}$ at every point of V_ℓ*

such that for each $1 \leqslant \ell \leqslant m$, each $1 \leqslant j \leqslant r_\ell$, and every $\nu \in U_{\ell,j}$, we have $\gamma_{\mathcal{I},\nu} = \nu(g_{\ell,j})$.

Proof. For every $y \in Y$, there exist $g_1, \dots, g_r \in \mathcal{I}_y$ and an open affine neighbourhood $y \in V_y \subset Y$ such that $g_1, \dots, g_r$ extend to sections of $\mathcal{I}$ over V_y generating $\mathcal{I}$ at every point of V_y. Since Y is quasi-compact, there exists a finite open subcover of $\{V_y\colon y \in Y\}$, say $\mathcal{V} = \{V_\ell\colon 1 \leqslant \ell \leqslant m\}$. For each ℓ, let $g_{\ell,1}, \dots, g_{\ell,r_\ell} \in \mathcal{I}(V_\ell)$ be the sections chosen earlier.

For each $1 \leqslant j \leqslant r_\ell$, let $U_{\ell,j} = \pi_Y^{-1}(V_\ell) \cap U\big(g_{\ell,i}/g_{\ell,j}\colon i \neq j\big)$. For all $\nu \in \pi_Y^{-1}(V_\ell)$, we have $y_\nu \in V_\ell$, whence $\mathcal{I}_{y_\nu}$ is generated by $\{g_{\ell,j}\colon 1 \leqslant j \leqslant r_\ell\}$, so $\gamma_{\mathcal{I},\nu} = \min\{\nu(g_{\ell,j})\colon 1 \leqslant j \leqslant r_\ell\}$. From this, it is immediate that $\pi_Y^{-1}(V_\ell) = \bigcup_{j=1}^{r_\ell} U_{\ell,j}$. The conclusion is also immediate. □

Definition 2.2 (Idealistic classes). Let Y be a $\Bbbk$-variety. A valuative ideal γ over Y associated with a non-zero ideal $\mathcal{I}$ on Y is called an *idealistic class* over Y.

Conversely, every valuative ideal γ over Y determines an ideal $\mathcal{I}_\gamma$ on Y: we let $\mathcal{I}_\gamma$ be the subsheaf of $\mathscr{O}_Y$ whose sections g over an open set U satisfy $\nu(g) \geqslant \gamma_\nu$ for every $\nu \in \pi_Y^{-1}(U) \subset \mathrm{ZR}(Y)$ (namely those ν such that $y_\nu \in U$).

Before moving on, we recall the following definition [Laz04, Section 9.6.A]: if I is an ideal of a ring A, the *integral closure* $\overline{I}$ of I in A consists of elements $x \in A$ which satisfy a "weighted integral equation"

$$x^n + a_1 x^{n-1} + \dots + a_{n-1}x + a_n = 0\,, \quad \text{where } a_i \in I^i\,.$$

We say that I is *integrally closed* in A if $I = \overline{I}$. Observe that $I \subset \overline{I} \subset \sqrt{I}$ (where $\sqrt{I}$ is the radical of I). In Section 2.2, we will prove

(a) that $\overline{I}$ is an ideal of A;

(b) that if $\mathcal{I}$ is an ideal on a $\Bbbk$-variety Y, the presheaf on Y given by $U \mapsto \overline{\mathcal{I}(U)}$ is a sheaf, denoted by $\overline{\mathcal{I}}$;

(c) and the following lemma, which was essentially observed in [Hir77].

Lemma 2.3. *Let the notation be as above.*

(i) *If γ is a valuative ideal over Y, then $\mathcal{I}_\gamma$ is integrally closed in $\mathscr{O}_Y$.*

(ii) *Let $0 \neq \mathcal{I} \subset \mathscr{O}_Y$ be an ideal, with associated idealistic class $\gamma = \gamma_{\mathcal{I}}$ over Y. Then $\mathcal{I}_\gamma = \overline{\mathcal{I}}$.*

Corollary 2.4. *Let Y be a $\Bbbk$-variety. Lemma 2.3 describes a one-to-one correspondence between non-zero, integrally closed ideals of $\mathscr{O}_Y$ and idealistic classes over Y.*

2.2 Rees algebras and valuative $\mathbb{Q}$-ideals

Definition 2.5 (Rees algebras). Given a scheme Y, a *Rees algebra* on Y is a finitely generated, quasi-coherent, graded $\mathscr{O}_Y$-subalgebra $\mathscr{R} = \bigoplus_{m\in\mathbb{N}} \mathcal{I}_m \cdot T^m \subset \mathscr{O}_Y[T]$ satisfying $\mathcal{I}_0 = \mathscr{O}_Y$ and $\mathcal{I}_m \supset \mathcal{I}_{m+1}$ for every $m \in \mathbb{N}$. We say that $\mathscr{R}$ is *non-zero* if $\mathcal{I}_m \neq 0$ for some $m \geqslant 1$.

Recall that we can associate a Rees algebra with every ideal $\mathcal{I}$ on Y, namely $\bigoplus_{m\in\mathbb{N}} \mathcal{I}^m \cdot T^m$. This sets up a one-to-one correspondence:

$$\{\text{ideals of } \mathscr{O}_Y\} \leftrightarrow \{\text{Rees algebras generated in degree } 1\}\,.$$

For the remainder of this section, let Y be a $\Bbbk$-variety Y. Accompanying the notion of a Rees algebra on Y is the notion of a *valuative $\mathbb{Q}$-ideal* over Y; see [ATW19, Section 2.2]. To define this notion, consider the sheaf of ordered groups $\Gamma_{Y,\mathbb{Q}} = \mathbb{Q} \otimes \Gamma_Y$. We denote the sheaf of monoids consisting of non-negative sections of $\Gamma_{Y,\mathbb{Q}}$ by $\Gamma_{Y,\mathbb{Q}+}$. A valuative $\mathbb{Q}$-ideal over Y is a section γ in $H^0(\mathrm{ZR}(Y), \Gamma_{Y,\mathbb{Q}+})$. Note that since γ is locally constant and $\mathrm{ZR}(Y)$ is quasi-compact, there exists a sufficiently large natural number $N \geqslant 1$ such that $N \cdot \gamma$ is a valuative ideal over Y.

A non-zero Rees algebra $\mathscr{R}$ on Y determines a valuative $\mathbb{Q}$-ideal $\gamma_{\mathscr{R}}$ over Y by

$$\gamma_{\mathscr{R}} := (\gamma_{\mathscr{R},\nu})_{\nu\in\mathrm{ZR}(Y)} \in \prod_{\nu\in\mathrm{ZR}(Y)} (\mathbb{Q}\otimes G_\nu)_+\,,$$

where $\gamma_{\mathscr{R},\nu}$ is defined as

$$\min\left\{\frac{1}{n}\cdot\nu(g)\colon gT^n \text{ is a non-zero section of } \mathscr{R} \text{ over an open set containing } y_\nu \text{ (for } n \geqslant 1)\right\}.$$

Again, we have to show (a) that this minimum exists in $(\mathbb{Q}\otimes G_\nu)_+$ and (b) that $(\gamma_{\mathscr{R},\nu})_{\nu\in\mathrm{ZR}(Y)}$ defines a compatible collection of germs and hence defines a valuative $\mathbb{Q}$-ideal over Y. Indeed, fix $\nu \in \mathrm{ZR}(Y)$, and suppose that $g_1T^{n_1}, \ldots, g_rT^{n_r}$ generate $\mathscr{R}_{y_\nu}$ as an $\mathscr{O}_{Y,y_\nu}$-algebra. Then we claim that

$$\gamma_{\mathscr{R},\nu} = \min\left\{\frac{1}{n_i}\cdot\nu(g_i)\colon 1 \leqslant i \leqslant r\right\},$$

from which statement (a) is immediate. Indeed, suppose $gT^n \in \mathcal{I}_{y_\nu}$. Then we can write

$$gT^n = \sum_{k_1n_1+\cdots+k_rn_r=n} a_{\vec{k}} \cdot \prod_{i=1}^{r} \left(g_iT^{n_i}\right)^{k_i} \quad \text{in } \mathscr{O}_{Y,y_\nu}[T]\,,$$

which means

$$g = \sum_{k_1 n_1 + \cdots + k_r n_r = n} a_{\vec{k}} \cdot \prod_{i=1}^{r} g_i^{k_i} \quad \text{in } \mathscr{O}_{Y,y_\nu} .$$

Consequently,

$$\begin{aligned} \frac{1}{n} \cdot \nu(g) &\geqslant \min \left\{ \frac{1}{n} \sum_{i=1}^{r} k_i n_i \cdot \left(\frac{1}{n_i} \cdot \nu(g_i) \right) : k_1 n_1 + \cdots + k_r n_r = n \right\} \\ &\geqslant \min \left\{ \left(\frac{1}{n} \sum_{i=1}^{r} k_i n_i \right) \cdot \min \left\{ \frac{1}{n_i} \cdot \nu(g_i) : 1 \leqslant i \leqslant r \right\} : k_1 n_1 + \cdots + k_r n_r = n \right\} \\ &= \min \left\{ \frac{1}{n_i} \cdot \nu(g_i) : 1 \leqslant i \leqslant r \right\} , \end{aligned}$$

as desired.

For statement (b), there exists an affine open neighbourhood V_ν of y_ν in Y such that $g_1 T^{n_1}, \ldots, g_r T^{n_r}$ extend to sections of $\mathscr{R}$ over V_ν which generate the stalk of $\mathscr{R}$ at every point in V_ν. Let $1 \leqslant j \leqslant r$ be such that $\gamma_{\mathscr{R},\nu} = (1/n_j) \cdot \nu(g_j)$. Then $U_\nu = \pi_Y^{-1}(V_\nu) \cap U\left(g_i^{n_j}/g_j^{n_i} : i \neq j\right)$ is an open neighbourhood of ν in $\mathrm{ZR}(Y)$ such that for all $\nu' \in U_\nu$, we have $\gamma_{\nu'} = (1/n_j) \cdot \nu'(g_j)$.

Note that if $\mathscr{R}$ is the Rees algebra of an ideal $0 \neq \mathcal{I} \subset \mathscr{O}_Y$, then $\gamma_{\mathscr{R}} = \gamma_{\mathcal{I}}$.

Lastly, imitating the proof of Lemma 2.1 yields the following analogous lemma.

Lemma 2.6. *Let the notation be as above, and let $\mathscr{R}$ be a non-zero Rees algebra on Y. There exist*

(i) *a finite open affine cover $\mathcal{V} = \{V_\ell : 1 \leqslant \ell \leqslant m\}$ of Y;*
(ii) *for each $1 \leqslant \ell \leqslant m$, a finite open cover $\mathcal{U}_\ell = \{U_{\ell,j} : 1 \leqslant j \leqslant r_\ell\}$ of $\pi_Y^{-1}(V_\ell)$;*
(iii) *for each $1 \leqslant \ell \leqslant m$, sections $\{g_{\ell,j} T^{n_{\ell,j}} : 1 \leqslant j \leqslant r_\ell\}$ of $\mathscr{R}$ over V_ℓ which generate $\mathscr{R}$ at every point of V_ℓ (as an $\mathscr{O}_{Y,y}$-algebra)*

such that for each $1 \leqslant \ell \leqslant m$, each $1 \leqslant j \leqslant r_\ell$, and every $\nu \in U_{\ell,j}$, we have $\gamma_{\mathscr{R},\nu} = (1/n_{\ell,j}) \cdot \nu(g_{\ell,j})$.

Definition 2.7 (Idealistic exponents, cf. [Hir77, Definition 3]). Let Y be a $\Bbbk$-variety. A valuative $\mathbb{Q}$-ideal γ over Y associated with some non-zero Rees algebra on Y is called an *idealistic exponent* over Y.

Conversely, let γ be a valuative $\mathbb{Q}$-ideal over Y. As in Section 2.1, the valuative ideal γ also determines an ideal $\mathcal{I}_\gamma$ on Y whose sections g over an open set U satisfy $\nu(g) \geqslant \gamma_\nu$ for every $\nu \in \pi_Y^{-1}(U) \subset \mathrm{ZR}(Y)$ (namely those ν such that $y_\nu \in U$). But γ also determines an $\mathscr{O}_Y$-subalgebra of $\mathscr{O}_Y[T]$

$$\mathscr{R}_\gamma = \bigoplus_{m \in \mathbb{N}} \mathcal{I}_{m \cdot \gamma} \cdot T^m \subset \mathscr{O}_Y[T] ,$$

where $\mathcal{I}_{m \cdot \gamma}$ is the ideal of $\mathscr{O}_Y$ associated with the multiple $m \cdot \gamma$ (which was just described). In general, $\mathscr{R}_\gamma$ is not a Rees algebra on Y, but Proposition 2.9 below says that $\mathscr{R}_\gamma$ is a Rees algebra on Y whenever γ is an idealistic exponent over Y. Note that $\mathscr{R}_\gamma$ contains the Rees algebra of $\mathcal{I}_\gamma$, but of course these are rarely equal; see Corollary 2.10.

Lemma 2.8. *Let the notation be as above, and let γ be a valuative $\mathbb{Q}$-ideal over Y. The corresponding $\mathscr{O}_Y$-subalgebra $\mathscr{R}_\gamma$ of $\mathscr{O}_Y[T]$ is integrally closed in $\mathscr{O}_Y[T]$.*

Proof (cf. [KKMS73, Chapter I, Lemma 1]). Since the integral closure of $\mathscr{R}_\gamma$ in $\mathscr{O}_Y[T]$ is a subring of $\mathscr{O}_Y[T]$, it suffices to show that whenever a non-zero homogeneous section gT^r of $\mathscr{O}_Y[T]$ over an open set $U \subset Y$ satisfies an equation of the form

$$\big(gT^r\big)^n + a_1\big(gT^r\big)^{n-1} + \cdots + a_{n-1}\big(gT^r\big) + a_n = 0\,, \quad a_i \in \mathscr{R}_\gamma(U)\,,$$

gT^r is a section of $\mathscr{R}_\gamma$ over U. By writing each a_i as a sum of homogeneous sections in $\mathscr{R}_\gamma(U)$ and comparing degrees, we may assume that each a_i is $\alpha_i T^{ir}$ for some $\alpha_i \in \mathcal{I}_{ir\cdot\gamma}(U)$. If $r = 0$, there is nothing to show. If $r > 0$, we have

$$g^n + \alpha_1 g^{n-1} + \cdots + \alpha_{n-1} g + \alpha_n = 0 \quad \text{in } \mathscr{O}_Y(U)\,.$$

Let $\nu \in \pi_Y^{-1}(U) \subset \mathrm{ZR}(Y)$. We claim that there must exist some $1 \leqslant j \leqslant n$ such that $j \cdot \nu(g) \geqslant \nu(\alpha_j)$. Indeed, if not, then $i \cdot \nu(g) < \nu(\alpha_i)$ for all $1 \leqslant i \leqslant n$, so $\nu(g^n) < \nu(\alpha_i g^{n-i})$ for all $1 \leqslant i \leqslant n$. This implies $g^n + \alpha_1 g^{n-1} + \cdots + \alpha_{n-1} g + \alpha_n \neq 0$, which gives a contradiction. Now our claim implies $\nu(g) \geqslant (1/j)\nu(\alpha_j) \geqslant r \cdot \gamma_\nu$, so $g \in \mathcal{I}_{r\cdot\gamma}(U)$. Since $\nu \in \pi_Y^{-1}(U)$ is arbitrary, $gT^r \in \mathscr{R}_\gamma(U)$. □

A special case of the next result is observed in [ATW19, Section 3.4].

PROPOSITION 2.9. *Let the notation be as above, and let $\gamma = \gamma_{\mathscr{R}}$ be the idealistic exponent over Y associated with a non-zero Rees algebra $\mathscr{R}$ on Y. Then $\mathscr{R}_\gamma$ is the integral closure of $\mathscr{R}$ in $\mathscr{O}_Y[T]$. In particular, $\mathscr{R}_\gamma$ is a finite $\mathscr{R}$-module and hence a Rees algebra on Y.*

Proof (cf. [Mat89, Theorem 10.4]). By Lemma 2.8, the algebra $\mathscr{R}_\gamma$ contains the integral closure of $\mathscr{R}$ in $\mathscr{O}_Y[T]$. We can check the converse on stalks. Let $y \in Y$; it suffices to show that whenever a homogeneous element gT^n of $\mathscr{O}_{Y,y}[T]$ is not integral over $\mathscr{R}_y$, that element is not in $(\mathscr{R}_\gamma)_y$. Fix a set of generators $g_1T^{n_1}, \ldots, g_rT^{n_r}$ of $\mathscr{R}_y$ as a $\mathscr{O}_{Y,y}$-algebra; then our goal is to find a $\nu \in \mathrm{ZR}(Y)$ whose centre y_ν on Y is y and such that

$$\frac{1}{n}\nu(g) < \min\left\{\frac{1}{n_i}\nu(g_i) \colon 1 \leqslant i \leqslant r\right\}.$$

Let $A = \mathscr{O}_{Y,y}\big[g_i^n/g^{n_i} \colon 1 \leqslant i \leqslant r\big]$; this is a subring of K containing $\Bbbk$. Let I be the ideal of A generated by $\big\{g_i^n/g^{n_i} \colon 1 \leqslant i \leqslant r\big\}$ and the maximal ideal $\mathfrak{m}_{Y,y}$ of $\mathscr{O}_{Y,y}$. We claim that $1 \notin I$. If not,

$$1 = \alpha + \sum_{\substack{J=(j_1,\ldots,j_r)\\ j_1+\cdots+j_r \geqslant 1}} \beta_J \prod_{i=1}^r \left(\frac{g_i^n}{g^{n_i}}\right)^{j_i},$$

where $\alpha \in \mathfrak{m}_{Y,y}$ and only finitely many $\beta_J \in \mathscr{O}_{Y,y}$ are non-zero. Since $1-\alpha$ is a unit in $\mathscr{O}_{Y,y}$, we may assume $\alpha = 0$. For each $1 \leqslant i \leqslant r$, let $t_i = \max\{j_i \colon \text{there exists a } J = (j_1, \ldots, j_r) \text{ such that } \beta_J \neq 0\}$. Multiplying the above equation throughout by $\prod_{i=1}^r (g^{n_i})^{t_i} = g^{\sum_{i=1}^r n_i t_i}$, we get

$$g^{\sum_{i=1}^r n_i t_i} = \sum_{\substack{J=(j_1,\ldots,j_r)\\ j_1+\cdots+j_r \geqslant 1}} \beta_J \prod_{i=1}^r \big(g_i^{nj_i} \cdot g^{n_i(t_i-j_i)}\big) = \sum_{\substack{J=(j_1,\ldots,j_r)\\ j_1+\cdots+j_r \geqslant 1}} \left(\beta_J \prod_{i=1}^r g_i^{nj_i}\right) \cdot g^{\sum_{i=1}^r n_i(t_i-j_i)}\,,$$

which implies

$$\big(gT^n\big)^{\sum_{i=1}^r n_i t_i} - \sum_{\substack{J=(j_1,\ldots,j_r)\\ j_1+\cdots+j_r \geqslant 1}} \left(\beta_J \prod_{i=1}^r \big(g_iT^{n_i}\big)^{nj_i}\right) \cdot \big(gT^n\big)^{\sum_{i=1}^r n_i(t_i-j_i)} = 0\,,$$

which is an integral equation for gT^n over $\mathscr{R}_y = \mathscr{O}_{Y,y}[g_iT^{n_i} : 1 \leqslant i \leqslant r]$; this gives a contradiction. Therefore, I is a proper ideal of A, so there exists a maximal ideal $\mathfrak{p}$ of A containing I. By [Mat89, Theorem 10.2], there exists[2] a $\nu \in \mathrm{ZR}(K, \Bbbk)$ such that $R_\nu \supset A$ and $\mathfrak{m}_\nu \cap A = \mathfrak{p}$. Consequently, $\{g_i^n/g^{n_i} : 1 \leqslant i \leqslant r\} \subset \mathfrak{p} \subset \mathfrak{m}_\nu$, whence $g^{n_i}/g_i^n \notin R_\nu$ for each $1 \leqslant i \leqslant r$. This means that for each $1 \leqslant i \leqslant r$,

$$\nu\left(\frac{g^{n_i}}{g_i^n}\right) < 0\,, \quad \text{which implies} \quad \frac{1}{n}\nu(g) < \frac{1}{n_i}\nu(g_i)\,,$$

as desired. Moreover, $\mathfrak{p} \cap \mathscr{O}_{Y,y} = \mathfrak{m}_{Y,y}$, so $\mathfrak{m}_\nu \cap \mathscr{O}_{Y,y} = \mathfrak{m}_{Y,y}$. Thus, the centre of ν on Y is necessarily y (in particular, $\nu \in \mathrm{ZR}(Y)$). □

Corollary 2.10. *Let $\mathcal{I}$ be a non-zero ideal on a $\Bbbk$-variety Y, with associated idealistic class $\gamma = \gamma_{\mathcal{I}}$ over Y. Then the Rees algebra $\mathscr{R}_\gamma$ associated with γ is the integral closure of the Rees algebra of $\mathcal{I}$ in $\mathscr{O}_Y[T]$.*

Proof. If $\mathscr{R}$ is the Rees algebra of $\mathcal{I}$, we noted earlier that $\gamma_{\mathscr{R}} = \gamma_{\mathcal{I}}$. Apply Proposition 2.9. □

Corollary 2.11. *Let Y be a $\Bbbk$-variety. The above describes a one-to-one correspondence between non-zero, integrally closed Rees algebras on Y and idealistic exponents over Y.*

Notation 2.12. In light of Corollary 2.11, the following notation in [ATW20a] makes sense, and we adopt it moving ahead. If $\mathscr{R}$ is the integral closure of a non-zero Rees algebra generated by sections $g_1^{a_1}T^{b_1}, \dots, g_r^{a_r}T^{b_r}$, we record $\mathscr{R}$ as $\mathscr{R} = (g_1^{q_1}, \dots, g_r^{q_r})$, where $q_i = a_i/b_i$ for $1 \leqslant i \leqslant r$. Note that since $\mathscr{R}$ is integrally closed, this expression is well defined, independent of the presentation of q_i as a quotient of two positive integers. Moreover, if we write $\mathscr{R} = (g_1^{q_1}, \dots, g_r^{q_r}, \mathcal{I}^q)$ for an ideal $\mathcal{I} \subset \mathscr{O}_Y$ and a positive rational number $q = a/b$, we mean that $\mathscr{R}$ is the integral closure of a Rees algebra generated by sections $g_1^{a_1}T^{b_1}, \dots, g_r^{a_r}T^{b_r}$ and $\{g^aT^b : g \text{ is a section of } \mathcal{I}\}$. For a positive rational number s, we write $\mathscr{R}^s$ to mean $(g_1^{q_1s}, \dots, g_r^{q_rs}, \mathcal{I}^{qs})$. By convention, we shall write $\mathscr{R}^0$ to mean the trivial Rees algebra $(1) = \mathscr{O}_Y[T]$.

Finally, let us tie up some loose ends from the end of Section 2.1. Note that if I is an ideal of a ring A, the Rees algebra of I is integrally closed in $A[T]$ if and only if I^r is integrally closed in A for all $r \geqslant 1$. In particular, $\overline{I}$ is the degree 1 part of the integral closure of the Rees algebra of I in $A[T]$, so it must be an ideal of A. This is assertion (a) before Lemma 2.3, and assertion (b) is proven similarly. We also deduce Lemma 2.3 from results in this section.

Proof of Lemma 2.3. Let γ be a valuative ideal over Y. By Lemma 2.8, the subalgebra $\mathscr{R}_\gamma$ is integrally closed in $\mathscr{O}_Y[T]$. Hence, $\mathcal{I}_\gamma^r$ is integrally closed in $\mathscr{O}_Y$ for all $r \geqslant 1$. In particular, we get part (i).

For part (ii), let $\mathscr{R}$ be the Rees algebra of $\mathcal{I}$, and apply Corollary 2.10: $\mathscr{R}_\gamma$ is the integral closure of $\mathscr{R}$ in $\mathscr{O}_Y[T]$. In particular, the degree 1 part of $\mathscr{R}_\gamma$ is $\overline{\mathcal{I}}$, so $\mathcal{I}_\gamma = \overline{\mathcal{I}}$. □

2.3 Functoriality with respect to dominant morphisms

Let $f : Y' \to Y$ be a dominant morphism of $\Bbbk$-varieties. In Section A.3, we note that f naturally induces a morphism $\mathrm{ZR}(f) : \mathrm{ZR}(Y') \to \mathrm{ZR}(Y)$ of locally ringed spaces, which induces a morphism of ordered groups $\Gamma_Y \to \mathrm{ZR}(f)_*\Gamma_{Y'}$ as well as a morphism of sheaves of monoids $\Gamma_{Y,+} \to \mathrm{ZR}(f)_*\Gamma_{Y',+}$. Tensoring with $\mathbb{Q}$, we also get a morphism of ordered groups $\Gamma_{Y,\mathbb{Q}} \to \mathrm{ZR}(f)_*\Gamma_{Y',\mathbb{Q}}$

[2]Recall that K denotes the field of fractions of Y, and an element $\nu \in \mathrm{ZR}(K, \Bbbk)$ is a valuation ring R_ν of K containing $\Bbbk$, as defined in Section A.1.

and a morphism of sheaves of monoids $\Gamma_{Y,\mathbb{Q}+} \to \mathrm{ZR}(f)_*\Gamma_{Y',\mathbb{Q}+}$. In particular, for every valuative ideal (or valuative $\mathbb{Q}$-ideal) γ over Y, we can consider the pullback of γ to Y', denoted by $\gamma\mathscr{O}_{Y'}$ (following [ATW19]). If $\gamma = \gamma_{\mathcal{I}}$ for some ideal $0 \neq \mathcal{I}$ on $\mathscr{O}_Y$, then $\gamma\mathscr{O}_{Y'}$ is simply $\gamma_{\mathcal{I}\mathscr{O}_{Y'}}$. Likewise, if $\gamma = \gamma_{\mathscr{R}}$ for some non-zero Rees algebra $\mathscr{R}$, then $\gamma\mathscr{O}_{Y'}$ is simply $\gamma_{\mathscr{R}\mathscr{O}_{Y'}}$. More generally, whenever $Y' \to Y$ is a morphism of $\Bbbk$-varieties with $\mathcal{I}\mathscr{O}_{Y'} \neq 0$ (respectively, $\mathscr{R}\mathscr{O}_{Y'}$ non-zero), the pullback $\gamma\mathscr{O}_Y$ of $\gamma = \gamma_{\mathcal{I}}$ (respectively, $\gamma = \gamma_{\mathscr{R}}$) is well defined.

3. Toroidal centres

3.1 Reminders

For the remainder of this paper, $\Bbbk$ denotes a field of characteristic zero. Although toroidal Deligne–Mumford stacks over $\Bbbk$ (see Definition B.16) are the main objects of study in our paper (as mentioned in Section 1.1), a significant portion of the paper instead deals with strict toroidal $\Bbbk$-schemes (see Definition B.6). There are two reasons for this:

(a) Étale locally, a toroidal Deligne–Mumford stack over $\Bbbk$ is a strict toroidal $\Bbbk$-scheme (see the paragraph after Definition B.16).

(b) The constructions and discussions in this paper are étale-local. This was hinted at in Theorem 1.1.

Henceforth, we shall assume that Y is a strict toroidal $\Bbbk$-scheme (with the exception of Section 6.4) and denote its logarithmic structure by $\alpha_Y\colon \mathscr{M}_Y \to \mathscr{O}_Y$. All ideals $\mathcal{I}$ of $\mathscr{O}_Y$ considered from here on are always assumed to be coherent. Let us recall the following notions from Appendix B:

- $\overline{\mathscr{M}}_Y$ – characteristic of $\mathscr{M}_Y$, defined as $\mathscr{M}_Y/\mathscr{O}_Y^*$,
- $\mathfrak{s}_y$ – logarithmic stratum through a point $y \in Y$,
- $\mathscr{D}_Y^1$ – logarithmic tangent sheaf of Y over $\Bbbk$,
- $\mathscr{D}_Y^{\leqslant n}$ – sheaf of logarithmic differential operators on Y of order at most n,
- $\mathscr{D}_Y^{\infty}$ – total sheaf of logarithmic differential operators on Y.

For an ideal $\mathcal{I}$ on Y, we also have the following notions:

- $\mathscr{D}_Y^{\leqslant n}(\mathcal{I})$ or $\mathscr{D}^{\leqslant n}(\mathcal{I})$ – ideal on Y generated by the image of $\mathcal{I}$ under $\mathscr{D}_Y^{\leqslant n}$,
- $\mathscr{D}_Y^{\infty}(\mathcal{I})$ or $\mathscr{D}^{\infty}(\mathcal{I})$ – ideal on Y generated by the image of $\mathcal{I}$ under $\mathscr{D}_Y^{\infty}$,
- $\mathscr{M}(\mathcal{I})$ – monomial saturation of $\mathcal{I}$,
- $\text{log-ord}_y(\mathcal{I})$ – logarithmic order of $\mathcal{I}$ at a point $y \in Y$.

These notions (and more) are discussed in Appendix B. In particular, we would also like to bring the reader's attention to the notion of logarithmic coordinates and parameters in Definition B.8, as well as Lemma B.9 and Theorem B.10. They will play a crucial role in the remainder of this paper.

3.2 Toroidal centres

In this section, we discuss the notion of toroidal centres on a strict toroidal $\Bbbk$-scheme Y. These are the "blow-up centres" for the resolution algorithm in this paper.

Definition 3.1 (Toroidal centres, cf. [ATW19, Section 2.4]). Fix a natural number $k \geqslant 1$ and a non-decreasing sequence

$$(a_1, \dots, a_k) \in \mathbb{Q}_{>0}^{k-1} \times (\mathbb{Q}_{>0} \cup \{\infty\}).$$

A *toroidal centre* $\mathscr{J}$ on Y, with *invariant*

$$\operatorname{inv}(\mathscr{J}) = (a_1, \ldots, a_k)\,,$$

is defined to be an integrally closed Rees algebra on Y (equivalently, an idealistic exponent over Y) such that at each point y in Y, there exists an (irreducible[3]) open affine neighbourhood $U_y \subset Y$ of y on which either

(i) $\mathscr{J}|_{U_y} = \mathscr{O}_Y[T]|_{U_y}$,

(ii) or there exist

(1) a choice of logarithmic parameters $\big((x_1^{(y)}, \ldots, x_{n_y}^{(y)}), M_y = \overline{\mathscr{M}}_{Y,y} \xrightarrow{\beta^{(y)}} H^0(U_y, \mathscr{M}_Y|_{U_y})\big)$ at y which defines a strict, smooth morphism $U_y \to \operatorname{Spec}(M_y \to \Bbbk[M_y \oplus \mathbb{N}^{n_y}])$ (as in Theorem B.10(ii));

(2) if $a_k = \infty$, a non-empty ideal Q_y of $M_y = \overline{\mathscr{M}}_{Y,y}$ whose image under $\beta^{(y)}$ generates a monomial ideal (see Definition B.12) $\mathscr{Q}_y$ on U_y

such that

$$\mathscr{J}|_{U_y} = \begin{cases} \big((x_1^{(y)})^{a_1}, \ldots, (x_k^{(y)})^{a_k}\big) & \text{if } a_k \in \mathbb{Q}_{>0}\,, \\ \big((x_1^{(y)})^{a_1}, \ldots, (x_{k-1}^{(y)})^{a_{k-1}}, \mathscr{Q}_y^r\big) & \text{if } a_k = \infty \end{cases}$$

for some positive rational number $r \in \mathbb{Q}_{>0}$ independent of y. (Note that, in particular, $k \leqslant n_y$ if $a_k \in \mathbb{Q}_{>0}$, and $k-1 \leqslant n_y$ if $a_k = \infty$.)

Given a toroidal centre $\mathscr{J}$ on Y, a choice of data as above for each $y \in Y$ is called a *presentation* of $\mathscr{J}$. We mimic the notation in [ATW19] and record the aforementioned presentation of $\mathscr{J}$ as

$$\mathscr{J} = \begin{cases} \big(x_1^{a_1}, \ldots, x_k^{a_k}\big) & \text{if } a_k \in \mathbb{Q}_{>0}\,, \\ \big(x_1^{a_1}, \ldots, x_{k-1}^{a_{k-1}}, (Q \subset M)^r\big) & \text{if } a_k = \infty\,. \end{cases}$$

By the *support*[4] of a toroidal centre $\mathscr{J}$, we mean the complement of the Zariski open subset of points $y \in Y$ such that $\mathscr{J}_y = \mathscr{O}_{Y,y}[T]$.

A toroidal centre $\mathscr{J}^{(y)}$ at a point $y \in Y$, with invariant $\operatorname{inv}\big(\mathscr{J}^{(y)}\big) = (a_1, \ldots, a_k)$, is an integrally closed Rees algebra on an open affine neighbourhood $U_y \subset Y$ of y that satisfies condition (ii) above.

Observe that we chose to drop the index y in the notation of a toroidal centre $\mathscr{J}$ on Y. This choice of notation would make more sense later: it is justified by the expectation that the resolution algorithm in this paper would be done locally around each $y \in Y$ and patched up afterwards. Some of our results later are written this way, that is, without making reference to the index y (see, for example, Section 4.4).

It is not immediate that the invariant of a toroidal centre is well defined, that is, independent of the choice of presentation of $\mathscr{J}$. This is the content of the next lemma.

Lemma 3.2. *The invariant* $\operatorname{inv}\big(\mathscr{J}^{(y)}\big)$ *of a toroidal centre* $\mathscr{J}^{(y)}$ *at a point* $y \in Y$ *is independent of choice of presentation for* $\mathscr{J}^{(y)}$ *and hence is well defined.*

We postpone the proof of Lemma 3.2 till Section 4.6.

[3]Recall that Y is a disjoint union of its irreducible components (Remark B.5(iii)), so the assertion that U_y is irreducible is equivalent to the assertion that U_y is contained in the component of Y containing y.

[4]Note that this is different from, and should not be confused with, the notion of the support of a Rees algebra defined in [Ryd13, Definition 5.1].

Remark 3.3. (i) Another equivalent definition of a toroidal centre on Y (respectively, at a point $y \in Y$) is an integrally closed Rees algebra on Y (respectively, on an open affine neighbourhood $U_y \subset Y$ of y) with a presentation $\big(x_1^{a_1}, \ldots, x_k^{a_k}, (Q \subset M)^r\big)$ as in Definition 3.1, but this time allowing Q to be the empty ideal of M. In this case, one defines the invariant as $(a_1, \ldots, a_k, \infty)$ if $Q \neq \emptyset$ and $(a_1, \ldots, a_k)$ if $Q = \emptyset$.

(ii) While the invariant of a toroidal centre is well defined, the positive rational number r appearing in the exponent of Q is evidently not. For example, by replacing Q with $m \cdot Q$ (or Q^m if the monoid M is written multiplicatively), one can replace r with r/m. In particular, one can always adjust Q so that $r = 1/N$ for some natural number $N \geqslant 1$.

DEFINITION 3.4 (Reduced toroidal centres). (i) A toroidal centre $\mathscr{J}$ on Y is *reduced* if the finite entries in $\operatorname{inv}(\mathscr{J})$ are $1/n_i$ for some positive integers n_i and the gcd of the n_i is 1.

(ii) Given a toroidal centre $\mathscr{J}$ on Y, let s be the unique positive rational number such that $\mathscr{J}^s$ is reduced. We denote $\mathscr{J}^s$ by $\overline{\mathscr{J}}$ and call it the *unique reduced toroidal centre associated with* $\mathscr{J}$.

One can also define the aforementioned notions for a toroidal centre $\mathscr{J}^{(y)}$ at a point $y \in Y$.

Remark 3.5. Akin to how one can adjust Q in Remark 3.3(ii), one can also adjust the x_i appearing in the presentation of a toroidal centre, without changing the toroidal centre. Let $y \in Y$, and let $\mathscr{J}^{(y)} = \big(x_1^{a_1}, \ldots, x_k^{a_k}, (Q \subset M)^r\big)$ be a toroidal centre at y, with $k \geqslant 1$. For each $1 \leqslant i \leqslant k$, replace x_i with

$$x_i' = (\lambda_{i,1}x_1 + \cdots + \lambda_{i,i-1}x_{i-1}) + x_i\,,$$

where the $\lambda_{i,j}$ are sections of $\mathscr{O}_{Y,y}$. Then we claim that $\mathscr{J}^{(y)} = \big((x_1')^{a_1}, (x_2')^{a_2}, \ldots, (x_k')^{a_k}, (Q \subset M)^r\big)$. While it is possible to prove this from the standpoint of integrally closed Rees algebras, we find it easier to tackle this from the equivalent standpoint of idealistic exponents, where this assertion is reduced to checking the following equality:

$$\begin{aligned}\min\big(\{a_i \cdot \nu(x_i) \colon 1 \leqslant i \leqslant k\} \cup \{r \cdot \nu(q) \colon q \in Q\}\big)\\ = \min\big(\{a_i \cdot \nu(x_i') \colon 1 \leqslant i \leqslant k\} \cup \{r \cdot \nu(q) \colon q \in Q\}\big)\,.\end{aligned}$$

More generally, note that one can replace each x_i with

$$x_i' = (\lambda_{i,1}x_1 + \cdots + \lambda_{i,i-1}x_{i-1}) + x_i + (\lambda_{i,i+1}x_{i+1} + \cdots + \lambda_{i,\ell}x_\ell)\,,$$

where $\ell = \max\{1 \leqslant j \leqslant k \colon a_j = a_i\}$ and, once again, the $\lambda_{i,j}$ are sections of $\mathscr{O}_{Y,y}$.

DEFINITION 3.6 (Admissibility). Let $\mathcal{I} \subset \mathscr{O}_Y$ be an ideal on Y, and let $y \in Y$.

(i) A toroidal centre $\mathscr{J}$ on Y is $\mathcal{I}$-*admissible* if $\mathscr{J}$ contains the Rees algebra of $\mathcal{I}$.

(ii) A toroidal centre $\mathscr{J}^{(y)}$ at y is $\mathcal{I}$-*admissible* if, after passing to a smaller affine neighbourhood of y on which $\mathscr{J}^{(y)}$ is defined, $\mathscr{J}^{(y)}$ contains the Rees algebra of $\mathcal{I}$.

Note that the support of an $\mathcal{I}$-admissible toroidal centre $\mathscr{J}$ is always contained in the vanishing locus $V(\mathcal{I})$ of $\mathcal{I}$: indeed, if $y \notin V(\mathcal{I})$, then $\mathcal{I}_y = \mathscr{O}_{Y,y}$, so $\mathscr{J}_y = \mathscr{O}_{Y,y}[T]$.

Before stating the next lemma, we revisit Definition 3.1. There we have that each U_y is a $\Bbbk$-variety, so by Section 2.2, the restriction $\mathscr{J}|_{U_y}$ defines an idealistic exponent over U_y, which we denote by $\gamma_{\mathscr{J}}^{(y)}$ and refer to as the idealistic exponent at y associated with $\mathscr{J}$ and the affine open neighbourhood U_y of y. We can express the notion of admissibility in terms of these idealistic exponents.

Lemma 3.7 (Valuative criterion for admissibility). *Let the notation be as above. Let $\mathscr{J}$ be a toroidal centre on Y. For a nowhere zero ideal $\mathcal{I}$ on Y, the following are equivalent:*

(i) *The toroidal centre $\mathscr{J}$ is $\mathcal{I}$-admissible.*

(ii) *For every $y \in Y$ and every open affine neighbourhood U_y of y as in Definition 3.1, we have $\gamma_{\mathscr{J}}^{(y)} \leqslant \gamma_{\mathcal{I}|_{U_y}}$.*

(iii) *For every $y \in Y$, there exists an open affine neighbourhood U_y of y as in Definition 3.1 such that $\gamma_{\mathscr{J}}^{(y)} \leqslant \gamma_{\mathcal{I}|_{U_y}}$.*

Proof. The toroidal centre $\mathscr{J}$ contains the Rees algebra of $\mathcal{I}$ if and only if $\mathscr{J}$ contains the integral closure of the Rees algebra of $\mathcal{I}$. Fix a choice of open affine neighbourhoods $(U_y)_{y\in Y}$ as in Definition 3.1. For every $y \in Y$, we deduce from Corollary 2.10 that $\mathscr{J}|_{U_y}$ contains the Rees algebra of $\mathcal{I}|_{U_y}$ if and only if $\mathscr{J}|_{U_y}$ contains the Rees algebra on U_y associated with $\gamma_{\mathcal{I}|_{U_y}}$. Passing to idealistic exponents over each U_y, we see that $\mathscr{J}$ is $\mathcal{I}$-admissible if and only if $\gamma_{\mathscr{J}}^{(y)} = \gamma_{\mathscr{J}|_{U_y}} \leqslant \gamma_{\mathcal{I}|_{U_y}}$ for every $y \in Y$. □

Fix a choice of affine open neighbourhoods $(U_y)_{y\in Y}$ as in Definition 3.1. Then $\big(\gamma_{\mathscr{J}}^{(y)}\big)_{y\in Y}$ is called the idealistic exponent over Y associated with $\mathscr{J}$ and $(U_y)_{y\in Y}$. We will only denote $\big(\gamma_{\mathscr{J}}^{(y)}\big)_{y\in Y}$ by $\gamma_{\mathscr{J}}$ whenever the discussion does not depend on the choice of $(U_y)_{y\in Y}$. The following is an example.

Notation 3.8. We write $\gamma_{\mathscr{J}} \leqslant \gamma_{\mathcal{I}}$ to mean either statement (ii) or statement (iii) in Lemma 3.7. Thus, $\mathscr{J}$ is $\mathcal{I}$-admissible if and only if $\gamma_{\mathscr{J}} \leqslant \gamma_{\mathcal{I}}$.

Given a toroidal centre $\mathscr{J}^{(y)}$ at y, let $\widehat{\mathscr{J}^{(y)}}$ denote the $\widehat{\mathscr{O}}_{Y,y}$-subalgebra of $\widehat{\mathscr{O}}_{Y,y}[T]$ generated by the image of $\mathscr{J}^{(y)}$ under

$$\mathscr{O}_Y[T] \to \mathscr{O}_{Y,y}[T] \to \widehat{\mathscr{O}}_{Y,y}[T]\,.$$

Equivalently, $\widehat{\mathscr{J}^{(y)}}$ is the completion $\varprojlim_k \mathscr{J}_y^{(y)}/\mathfrak{m}_{Y,y}^k\mathscr{J}_y^{(y)}$, where $\mathscr{J}_y^{(y)}$ is the stalk of $\mathscr{J}^{(y)}$ at y. The next lemma says that we can check admissibility by passing to completions.

Lemma 3.9. *Let the notation be as above. Let $\mathscr{J}^{(y)}$ be a toroidal centre at y. For an ideal $\mathcal{I}$ on Y, the toroidal centre $\mathscr{J}^{(y)}$ is $\mathcal{I}$-admissible if and only if $\widehat{\mathscr{J}^{(y)}}$ is $\widehat{\mathcal{I}}$-admissible.*

Proof. Indeed, $\mathscr{J}^{(y)}$ is $\mathcal{I}$-admissible if and only if the stalk of $\mathscr{J}^{(y)}$ at y contains the Rees algebra of $\mathcal{I}_y$. Since $\mathscr{O}_{Y,y}[T] \to \widehat{\mathscr{O}}_{Y,y}[T]$ is faithfully flat, the latter is equivalent to $\widehat{\mathscr{J}^{(y)}}$ being $\widehat{\mathcal{I}}$-admissible [Mat89, Theorem 7.5]. □

We conclude this section with some easy properties pertaining to admissibility.

Lemma 3.10. *Fix a toroidal centre $\mathscr{J}$ on Y, let $\mathcal{I}$ and $\mathcal{I}_j$ be ideals on Y, and let r_j be positive rational numbers.*

(i) *The toroidal centre $\mathscr{J}$ is $\sum_j \mathcal{I}_j$-admissible if and only if $\mathscr{J}$ is $\mathcal{I}_j$-admissible for every j.*

(ii) *If $\mathscr{J}^{r_j}$ is $\mathcal{I}_j$-admissible for every j, then $\mathscr{J}^{\sum_j r_j}$ is $\prod_j \mathcal{I}_j$-admissible.*

(iii) *For an integer $\ell \geqslant 1$, the toroidal centre $\mathscr{J}$ is $\mathcal{I}$-admissible if and only if $\mathscr{J}^{\ell}$ is $\mathcal{I}^{\ell}$-admissible.*

Proof. Part (i) can be seen directly from Definition 3.6, and it is easier to deduce part (ii) using the criterion in Lemma 3.7 (after replacing Y with the support of $\mathcal{I}$): if $r_j \cdot \gamma_{\mathscr{J}} = \gamma_{\mathscr{J}^{r_j}} \leqslant \gamma_{\mathcal{I}_j}$ for each j, then $\sum_j r_j \cdot \gamma_{\mathscr{J}} \leqslant \sum_j \gamma_{\mathcal{I}_j} = \gamma_{\prod_j \mathcal{I}_j}$. Part (iii) is also clear using Lemma 3.7. □

Lemma 3.11. *Let $y \in Y$, and let $\mathscr{J}^{(y)} = \left(x_1^{a_1}, \dots, x_k^{a_k}, (Q \subset M)^r\right)$ be a toroidal centre at y, with $k \geqslant 1$. Let H be the hypersurface $x_1 = 0$ defined on a neighbourhood of y on which $\mathscr{J}^{(y)}$ is defined, and let $\mathcal{I}$ be an ideal on H. If the restriction of $\mathscr{J}^{(y)}$ to H, namely $\mathscr{J}_H^{(y)} = \left(x_2^{a_2}, \dots, x_k^{a_k}, (Q \subset M)^r\right)$, is $\mathcal{I}$-admissible, then $\mathscr{J}^{(y)}$ is $(\mathcal{I}\mathscr{O}_Y)$-admissible.*

Proof. This can also be verified using Definition 3.6. □

We remark that the ordinary parameters[5] $x_2, \dots, x_k$ appearing in the restricted toroidal centre $\mathscr{J}_H^{(y)}$ are, more precisely, the reduction of $x_2, \dots, x_k$ modulo $x_1 = 0$. Note that if $x_1, \dots, x_n$ is a system of ordinary parameters at y, then the reduction of $x_2, \dots, x_n$ modulo $x_1 = 0$ is a system of ordinary parameters on H at y.

4. Weighted toroidal blow-ups

4.1 Stack-theoretic Proj

Let Y be a scheme or, more generally, an algebraic stack, and let $\mathscr{R} = \bigoplus_{m \in \mathbb{N}} \mathscr{R}_m$ be a quasi-coherent sheaf of graded $\mathscr{O}_Y$-algebras on Y. In this paper, we will be using the construction $\underline{\mathscr{P}\mathrm{roj}}_Y(\mathscr{R})$ in [Ols16, Section 10.2.7], called the *stack-theoretic* (or *stacky*) *Proj* of $\mathscr{R}$ on Y. This construction was also recalled briefly in [ATW19, Section 3.1] and will be pursued in greater depth in [QR22]. For brevity, we will not repeat the full construction here but instead recall some of its properties which are relevant to this paper:

(a) When Y is a scheme, $\underline{\mathscr{P}\mathrm{roj}}_Y(\mathscr{R})$ is the quotient stack $[(\underline{\mathrm{Spec}}_Y(\mathscr{R}) \setminus S_0) \,/\, \mathbb{G}_m]$, where the grading on $\mathscr{R}$ defines a $\mathbb{G}_m$-action $(T, s) \mapsto T^m \cdot s$ for $s \in \mathscr{R}_m$ and the vertex S_0 is the closed subscheme defined by the irrelevant ideal $\bigoplus_{m \geqslant 1} \mathscr{R}_m$ of $\mathscr{R}$.

(b) When $\mathscr{R}_1$ is coherent and generates $\mathscr{R}$ over $\mathscr{R}_0$, this coincides with the construction in [Har77] following Example II.7.8.6 on p. 160.

(c) The stack $\underline{\mathscr{P}\mathrm{roj}}_Y(\mathscr{R})$ carries an invertible sheaf $\mathscr{O}_{\underline{\mathscr{P}\mathrm{roj}}_Y(\mathscr{R})}(1)$ corresponding to the graded $\mathscr{O}_Y$-algebra $\mathscr{R}(1)$. If $\mathscr{R}$ is a Rees algebra on Y, then the inclusion $\mathscr{O}_{\underline{\mathscr{P}\mathrm{roj}}_Y(\mathscr{R})}(1) \hookrightarrow \mathscr{O}_{\underline{\mathscr{P}\mathrm{roj}}_Y(\mathscr{R})}$ defines the ideal sheaf of a (possibly reducible) divisor $\mathscr{E}$ on $\underline{\mathscr{P}\mathrm{roj}}_Y(\mathscr{R})$, called the *exceptional divisor* in [QR22].

(d) When $\mathscr{R}$ is finitely generated as a $\mathscr{O}_Y$-algebra with coherent graded components, the resulting morphism $\underline{\mathscr{P}\mathrm{roj}}_Y(\mathscr{R}) \to Y$ is proper.

(e) If $f \colon Y' \to Y$ is a morphism of schemes (or algebraic stacks), $\underline{\mathscr{P}\mathrm{roj}}_{Y'}(f^*\mathscr{R}) = \underline{\mathscr{P}\mathrm{roj}}_Y(\mathscr{R}) \times_Y Y'$. If f is flat and $\mathscr{R}$ is a Rees algebra on Y, then $\underline{\mathscr{P}\mathrm{roj}}_{Y'}(\mathscr{R}\mathscr{O}_{Y'}) = \underline{\mathscr{P}\mathrm{roj}}_Y(\mathscr{R}) \times_Y Y'$.

4.2 Blow-up along a Rees algebra

If $\mathscr{R} = \bigoplus_{m \in \mathbb{N}} \mathcal{I}_m \cdot T^m \subset \mathscr{O}_Y[T]$ is a Rees algebra on Y, the blow-up $\mathrm{Bl}_Y(\mathscr{R})$ of Y along $\mathscr{R}$ is $\underline{\mathscr{P}\mathrm{roj}}_Y(\mathscr{R})$. If $\mathscr{R}$ is the Rees algebra of an ideal $\mathcal{I} \subset \mathscr{O}_Y$, then $\mathrm{Bl}_Y(\mathscr{R})$ is the usual blow-up of Y along the ideal $\mathcal{I}$ (see [Har77, Definition following Proposition II.7.1.2]).

4.3 Blow-up along an idealistic exponent

Let γ be an idealistic exponent over a reduced, separated scheme Y of finite type over $\Bbbk$, with associated Rees algebra $\mathscr{R}_\gamma$ on Y (Section 2.2). The blow-up $\mathrm{Bl}_Y(\gamma)$ of Y along γ is defined as $\mathrm{Bl}_Y(\mathscr{R}_\gamma)$.

[5] See Definition B.8 (also refer back to Definition 3.1).

4.4 Weighted toroidal blow-ups: Local charts and logarithmic structures

Consider a toroidal centre $\mathscr{J}^{(y)}$ at a point $y \in Y$ of the form $\mathscr{J} = \big(x_1^{1/n_1}, \dots, x_k^{1/n_k}, (Q \subset M)^{1/d}\big)$, where $n_i \geqslant 1$ and $d \geqslant 1$ are integers. For this section Section 4.4 only, we replace Y with the open affine neighbourhood of y on which $\mathscr{J}^{(y)}$ is defined and write $\mathscr{J} = \mathscr{J}^{(y)}$ (so $\mathscr{J}$ is now defined on Y). Unless otherwise stated, we do not assume that $\mathscr{J}$ is reduced, and we allow $Q = \emptyset$ (see Remark 3.3(i)). As in Definition 3.1,

(a) $x_1, \dots, x_k$ is part of a system of ordinary parameters $x_1, \dots, x_n$ on Y at y (with $n = \operatorname{codim}_{\mathfrak{s}_y} \overline{\{y\}} \geqslant k$),

(b) $M \to H^0(Y, \mathscr{M}_Y)$ is a chart which is neat at y,

and together they induce a morphism $Y \to \operatorname{Spec}(M \to \Bbbk[x_1, \dots, x_n, M])$ which is strict and smooth of relative dimension $\dim \overline{\{y\}}$ (as in Theorem B.10(ii)). It is notationally more convenient to identify the ideal $Q \subset M$ with its image of Q in $\mathscr{O}_Y$ and hence write Q multiplicatively.

In this section, we study the weighted toroidal blow-up $Y' = \mathrm{Bl}_Y(\mathscr{J}) \to Y$. Since $\mathscr{J}$ is the integral closure of the simpler Rees algebra generated by $\{x_1 T^{n_1}, \dots, x_k T^{n_k}\} \cup \{mT^d \colon m \in Q\}$, the blow-up Y' is covered by the $(x_i T^{n_i})$-charts (for $1 \leqslant i \leqslant k$) and the (mT^d)-charts (as m varies over a fixed finite set of generators for Q). Our first task is to explicate these charts.

Lemma 4.1. *The $(x_1 T^{n_1})$-chart of Y' is the pullback of the square*

$$\begin{array}{ccc} & & [U_{x_1} \,/\, \boldsymbol{\mu}_{n_1}] = [\operatorname{Spec}(M_{x_1} \to \Bbbk[x'_2, \dots, x'_n, M_{x_1}]) \,/\, \boldsymbol{\mu}_{n_1}] \\ & & \downarrow \\ Y & \xrightarrow{\text{smooth, strict}} & \operatorname{Spec}(M \to \Bbbk[x_1, \dots, x_n, M])\,, \end{array}$$

where

(i) $x_1 = u^{n_1}$;

(ii) $x'_i = x_i / u^{n_i}$ *for* $2 \leqslant i \leqslant k$;

(iii) $x'_i = x_i$ *for* $i > k$;

(iv) M_{x_1} *is the saturation of the submonoid of* $M \oplus \mathbb{Z} \cdot u$ *generated by* u, M, *and* $\{q' = q/u^d \colon q \in Q\}$;

(v) *the group* $\boldsymbol{\mu}_{n_1} = \langle \zeta_{n_1} \rangle$ *acts through* $\zeta_{n_1} \cdot u = \zeta_{n_1}^{-1} u$, $\zeta_{n_1} \cdot x'_i = \zeta_{n_1}^{n_i} x'_i$ *for* $2 \leqslant i \leqslant k$, *trivially on* x'_i *for* $i > k$, *and trivially on* M *(so* $\zeta_{n_1} \cdot q' = \zeta_{n_1}^{d} \cdot q'$ *for* $q \in Q$*).*

Proof. Since $Y \to \operatorname{Spec}(\Bbbk[x_1, \dots, x_n, M])$ is flat and stacky *Proj* commutes with pullbacks, it suffices to assume $Y = \operatorname{Spec}(\Bbbk[x_1, \dots, x_n, M])$. Set $y_1 = x_1 T^{n_1}$. The y_1-chart of Y' is the stack $\big[\operatorname{Spec}\big(\mathscr{J}[y_1^{-1}]\big) \,/\, \mathbb{G}_m\big]$. By [QR22, Lemma 1.3.1],

$$\mathscr{J}[y_1^{-1}] \to \mathscr{J}[y_1^{-1}]/(y_1 - 1)$$

of $(\mathbb{Z}/n_1\mathbb{Z})$-graded $\mathscr{O}_Y$-algebras induces an isomorphism of algebraic stacks

$$\left[\operatorname{Spec}\left(\frac{\mathscr{J}[y_1^{-1}]}{(y_1 - 1)}\right) \,/\, \boldsymbol{\mu}_{n_1}\right] \xrightarrow{\simeq} \big[\operatorname{Spec}\big(\mathscr{J}[y_1^{-1}]\big) \,/\, \mathbb{G}_m\big]\,. \tag{4.4.1}$$

We sketch the proof presented in loc. cit. On $W := \operatorname{Spec}\big(\mathscr{J}[y_1^{-1}]/(y_1 - 1)\big) \times \mathbb{G}_m$, there is the diagonal $\boldsymbol{\mu}_{n_1}$-action given by $(y, t) \cdot s = (ys, s^{-1} t)$, and there is also the $\mathbb{G}_m$-action on the second factor given by $(y, t) \cdot s = (y, ts)$. These two actions are free and commute with each other (and, hence, together they induce a free $(\boldsymbol{\mu}_{n_1} \times \mathbb{G}_m)$-action on W). Then the left-hand side of (4.4.1) is isomorphic to $[W \,/\, (\boldsymbol{\mu}_{n_1} \times \mathbb{G}_m)] = [(W \,/\, \boldsymbol{\mu}_{n_1}) \,/\, \mathbb{G}_m]$. One then checks that there is a natural $\mathbb{G}_m$-equivariant isomorphism from $(W \,/\, \boldsymbol{\mu}_{n_1})$ to $\operatorname{Spec}\left(\mathscr{J}\left[y_1^{-1}\right]\right)$, which yields the desired isomorphism

$$[(W \,/\, \boldsymbol{\mu}_{n_1}) \,/\, \mathbb{G}_m] \xrightarrow{\simeq} \left[\operatorname{Spec}\left(\mathscr{J}\left[y_1^{-1}\right]\right) \,/\, \mathbb{G}_m\right].$$

Thus, it remains to show that the left-hand side of (4.4.1) has the desired description. Since $\left(T^{-1}\right)^{n_1} = y_1^{-1}x_1 \in \mathscr{J}\left[y_1^{-1}\right]$ and $\mathscr{J}\left[y_1^{-1}\right]$ is integrally closed in $\mathscr{O}_Y\left[T, T^{-1}\right]$ (by Lemma 2.8), we see that $T^{-1} \in \mathscr{J}\left[y_1^{-1}\right]$. Let $u = T^{-1}$. Restricting to W_1, we get $u^{n_1} = x_1$, $x_i T^{n_i} = x_i/u^{n_i}$ for $2 \leqslant i \leqslant k$ and $qT^d = qu^{-d}$ for every $q \in Q$. Therefore, $\Bbbk[x_2', \ldots, x_n', M_{x_1}] \subset \mathscr{J}\left[y_1^{-1}\right]/(y_1 - 1)$ is a finite birational extension, and since both are integrally closed in $\mathscr{O}_Y\left[T, T^{-1}\right]$, that inclusion is actually an equality. □

A similar proof explicates the $\left(mT^d\right)$-charts of Y'. We first fix notation. Given a (multiplicative) monoid M with an element $m \in M$ and an integer $d > 1$, we write $M\left[m^{1/d}\right]$ for the pushout of the diagram

$$\begin{array}{ccc} \mathbb{N} & \xrightarrow{1 \mapsto m} & M \\ {\scriptstyle d\cdot}\downarrow & & \vdots \\ \mathbb{N} & \cdots\cdots\rightarrow & M\left[m^{1/d}\right] \end{array}$$

in the category of monoids or, equivalently, the monoid $M \oplus \mathbb{N}$ modulo the congruence generated by $(m, 0_{\mathbb{N}}) \sim (0_M, d)$. In the lemma below, we shall denote the image of 1 under the horizontal dotted arrow $\mathbb{N} \to M\left[m^{1/d}\right]$ by u (so $u^d = m$). Note that $M\left[m^{1/d}\right]$ may not be torsion-free in general, even if M is torsion-free.

Lemma 4.2. *The $\left(mT^d\right)$-chart of Y' is the pullback of the square*

$$\begin{array}{ccc} & & [U_m \,/\, \boldsymbol{\mu}_d] = [\operatorname{Spec}(M_m \to \Bbbk[x_1', x_2', \ldots, x_n', M_m]) \,/\, \boldsymbol{\mu}_d] \\ & & \downarrow \\ Y & \xrightarrow{\text{smooth, strict}} & \operatorname{Spec}(M \to \Bbbk[x_1, \ldots, x_n, M]), \end{array}$$

where

(i) *M_m is the saturation of the submonoid of $M\left[m^{1/d}\right]^{\mathrm{gp}}$ generated by $M\left[m^{1/d}\right]$ and $\left\{q' = q/m = q/u^d : q \in Q\right\}$;*

(ii) *$x_i' = x_i/u^{n_i}$ for $1 \leqslant i \leqslant k$;*

(iii) *$x_i' = x_i$ for $i > k$;*

(iv) *the group $\boldsymbol{\mu}_d = \langle \zeta_d \rangle$ acts through $\zeta_d \cdot u = \zeta_d^{-1} u$, $\zeta_d \cdot x_i' = \zeta_d^{n_i} x_i'$ for $1 \leqslant i \leqslant k$, trivially on x_i' for $i > k$, and trivially on M (so ζ_d also acts trivially on q' for $q \in Q$).*

Together, Lemmas 4.1 and 4.2 present a natural choice of an étale cover U of $Y' = \mathrm{Bl}_Y(\mathscr{J})$. Each $\left(x_i T^{n_i}\right)$-chart of $\mathrm{Bl}_Y(\mathscr{J})$ admits an étale cover from the pullback of U_{x_i} to Y, and each $\left(mT^d\right)$-chart of $\mathrm{Bl}_Y(\mathscr{J})$ admits an étale cover from the pullback of U_m to Y. For the remainder of this paper,

> U denotes the disjoint union of the pullbacks of U_{x_i} and U_m to Y (where $1 \leqslant i \leqslant k$, and m varies over a fixed finite set of generators for Q).

Note that the composition $U \to Y' = \mathrm{Bl}_Y(\mathscr{J}) \to Y$ is an alteration. In addition, the principal ideal $E = (u)$ on U descends to give the exceptional ideal $\mathscr{E}$ on $Y' = \mathrm{Bl}_Y(\mathscr{J})$.

In Lemmas 4.1 and 4.2, we have also specified logarithmic structures on the U_{x_i} and the U_m such that the exceptional ideal $E = (u)$ is encoded in the logarithmic structures (this should be compared to Theorem 1.1(iv)). These pull back, via the strict morphism $Y \to \operatorname{Spec}(M \to k[x_1, \ldots, x_n, M])$, to define a logarithmic structure on U, which manifests U as a strict toroidal $\Bbbk$-scheme.

Finally, the logarithmic structure on U descends to a logarithmic structure on $Y' = \operatorname{Bl}_Y(\mathscr{J})$ (see Section B.2). Observe (from the charts) that the étale cover U is a strict toroidal $\Bbbk$-scheme, whence Y' is a *toroidal Deligne–Mumford stack over* $\Bbbk$ (see Definition B.16). If $k = 0$, observe too that the morphism $Y' \to Y$ is logarithmically smooth (because $U \twoheadrightarrow Y' \to Y$ is logarithmically smooth). This is not true if $k \geqslant 1$.

Lemma 4.3. *Let the notation be as above, and let $\gamma = \gamma_{\mathscr{J}}$ be the idealistic exponent over Y associated with $\mathscr{J}$. Then $\gamma\mathscr{O}_U$ is the idealistic exponent over U associated with the exceptional ideal E; that is, $\gamma\mathscr{O}_U = \gamma_E$.*

Proof. This is a simple computation. For example, over U_{x_1}, we have for every $\nu \in \operatorname{ZR}(U_{x_1})$

(a) $\nu(u) = (1/n_1) \cdot \nu(x_1)$,
(b) $\nu(u) = (1/n_i) \cdot \nu(x_i) - (1/n_i) \cdot \nu(x_i') \leqslant (1/n_i)\nu(x_i)$ for $2 \leqslant i \leqslant k$,
(c) $\nu(u) = (1/d) \cdot \nu(q) - (1/d)\nu(q') \leqslant (1/d) \cdot \nu(q)$ for $q \in Q$.

Therefore, $\min\big(\big\{(1/n_i) \cdot \nu(x_i) \colon 1 \leqslant i \leqslant k\big\} \cup \big\{(1/d) \cdot \nu(q) \colon q \in Q\big\}\big) = \nu(u)$. This computation persists in the other U_{x_i} and U_m. □

Proposition 4.4. *Let $\mathcal{I} \subset \mathscr{O}_Y$ be a nowhere zero ideal on Y, and let $\mathscr{J} = \big(x_1^{1/n_1}, \ldots, x_k^{1/n_k}, (Q \subset M)^{1/d}\big)$ be a toroidal centre on Y, where $n_i, d \geqslant 1$ are integers. Let $\mathscr{E}$ be the exceptional ideal of the weighted toroidal blow-up $Y' = \operatorname{Bl}_Y(\mathscr{J}) \to Y$.*

(i) *If $\ell \geqslant 1$ is an integer such that $\mathscr{J}^{\ell}$ is $\mathcal{I}$-admissible, then $\mathcal{I}\mathscr{O}_{Y'}$ factors as $\mathscr{E}^{\ell} \cdot \mathcal{I}'$ for some ideal $\mathcal{I}'$ on $\mathscr{O}_{Y'}$.*
(ii) *The converse holds as well: if $\mathcal{I}\mathscr{O}_{Y'}$ factors as $\mathscr{E}^{\ell} \cdot \mathcal{I}'$ for some ideal $\mathcal{I}'$ on $\mathscr{O}_{Y'}$ and some integer $\ell \geqslant 1$, then $\mathscr{J}^{\ell}$ is $\mathcal{I}$-admissible.*

Proof. Let U be the étale cover of Y' defined earlier, with principal ideal $E = (u)$ on U. For part (i), use Lemma 4.3: we have $\gamma_{\mathscr{J}}\mathscr{O}_U = \gamma_E$, so $\gamma_{\mathscr{J}^{\ell}}\mathscr{O}_U = \gamma_{E^{\ell}}$. But $\mathscr{J}^{\ell}$ is $\mathcal{I}$-admissible, so $\gamma_{\mathscr{J}^{\ell}} \leqslant \gamma_{\mathcal{I}}$, whence $\gamma_{\mathcal{I}\mathscr{O}_U} = \gamma_{\mathcal{I}}\mathscr{O}_U \geqslant \gamma_{\mathscr{J}^{\ell}}\mathscr{O}_U = \gamma_{E^{\ell}}$. But U is normal (Remark B.5(iii)), so Lemma A.1, coupled with the inequality $\gamma_{\mathcal{I}\mathscr{O}_U} \geqslant \gamma_{E^{\ell}}$, implies that the fractional ideal $E^{-\ell}(\mathcal{I}\mathscr{O}_U)$ is an ideal I' on $\mathscr{O}_U$. Moreover, since E is principal, $\mathcal{I}\mathscr{O}_U = E^{\ell} \cdot I'$. By descent, we get $\mathcal{I}\mathscr{O}_{Y'} = \mathscr{E}^{\ell} \cdot \mathcal{I}'$ for some ideal $\mathcal{I}'$ on $\mathscr{O}_{Y'}$.

For part (ii), the hypothesis says $\gamma_{\mathcal{I}}\mathscr{O}_U = \gamma_{\mathcal{I}\mathscr{O}_U} \geqslant \gamma_{E^{\ell}} = \gamma_{\mathscr{J}^{\ell}}\mathscr{O}_U$. Pulling idealistic exponents back to $\mathscr{O}_U$ is order-preserving, whence $\gamma_{\mathcal{I}} \geqslant \gamma_{\mathscr{J}^{\ell}}$. Thus, $\mathscr{J}^{\ell}$ is $\mathcal{I}$-admissible. □

Definition 4.5. Take $\ell = \max\{n \in \mathbb{N} \colon \mathscr{J}^n \text{ is } \mathcal{I}\text{-admissible}\}$ in Proposition 4.4(i). The corresponding ideal $\mathcal{I}'$ is called the weak (or birational) transform of $\mathcal{I}$ under the weighted toroidal blow-up $Y' = \operatorname{Bl}_Y(\mathscr{J}) \to Y$.

By considering the charts in Lemmas 4.1 and 4.2, we get part (i) of the following lemma.

Lemma 4.6. *Let $\mathscr{J} = \big(x_1^{1/n_1}, \ldots, x_k^{1/n_k}, (Q \subset M)^{1/d}\big)$ be a toroidal centre on Y, where $n_i, d \geqslant 1$ are integers. Fix a natural number $c \geqslant 1$, and set $\widetilde{\mathscr{J}} = \mathscr{J}^{1/c} = \big(x_1^{1/cn_1}, \ldots, x_k^{1/cn_k}, (Q \subset M)^{1/cd}\big)$.*

(i) *If $Y' \to Y$ and $\widetilde{Y}' \to Y$ are weighted toroidal blow-ups corresponding to $\mathscr{J}$ and $\widetilde{\mathscr{J}}$, with respective exceptional ideals $\mathscr{E}$ and $\widetilde{\mathscr{E}}$, then $\widetilde{Y}' = Y'(\sqrt[c]{\mathscr{E}})$ is the root stack of Y' along $\mathscr{E}$.*

(ii) *Assume $k \geqslant 1$. Write H for the hypersurface $x_1 = 0$ on Y, and let $\overline{H}' \to H$ be the weighted toroidal blow-up along the reduced toroidal centre $\overline{\mathscr{J}}_H$ associated with the restricted toroidal centre $\mathscr{J}_H = (x_2^{1/n_2}, \dots, x_k^{1/n_k}, \mathscr{Q}^{1/d})$, with exceptional ideal $\overline{\mathscr{E}}_H$. Then the proper transform $\widetilde{H}' \to H$ of H via the weighted toroidal blow-up along $\widetilde{\mathscr{J}}$ is the root stack $\overline{H}'(\sqrt[(cc')]{\overline{\mathscr{E}}_H})$ of $\overline{H}'$ along $\overline{\mathscr{E}}_H \subset \overline{H}'$, where $c' = \gcd(n_2, \dots, n_k)$. In other words, $\widetilde{H}' \to H$ is the weighted toroidal blow-up along $\overline{\mathscr{J}}_H^{1/(cc')}$.*

Proof of part (ii). Let $H' \to H$ be the weighted toroidal blow-up along $\mathscr{J}_H$, with exceptional ideal $\mathscr{E}_H$. By part (i), we have $H' = \overline{H}'(\sqrt[c']{\overline{\mathscr{E}}_H})$. Next, note that $H' \to H$ coincides with the proper transform of H via the weighted toroidal blow-up along $\mathscr{J}$ – this can be seen from the charts in Lemmas 4.1 and 4.2. Now apply part (i) again: it says that $\widetilde{H}' = H'(\sqrt[c]{\mathscr{E}_H})$. Combining our observations, we get $\widetilde{H}' = \overline{H}'(\sqrt[(cc')]{\overline{\mathscr{E}}_H})$, as desired. □

4.5 Admissibility of toroidal centres: Further results

Lemma 4.4 provides a convenient method to verify more intricate results on the admissibility of toroidal centres. Before doing so, we state a key lemma.

Lemma 4.7. *Let $\mathscr{J} = (x_1^{1/n_1}, \dots, x_k^{1/n_k}, (Q \subset M)^{1/d})$ be a toroidal centre on Y, where $k \geqslant 1$ and $n_i, d \geqslant 1$ are integers. Let $\mathscr{E}$ be the exceptional ideal of the weighted toroidal blow-up $Y' = \mathrm{Bl}_Y(\mathscr{J}) \to Y$. For an ideal $\mathcal{I}$ on Y, we have $\mathscr{D}_Y^{\leqslant 1}(\mathcal{I})\mathscr{O}_{Y'} \subset \mathscr{E}^{-n_1} \cdot \mathscr{D}_{Y'}^{\leqslant 1}(\mathcal{I}\mathscr{O}_{Y'})$.*

Proof. We can check the lemma over a point $y \in Y$, and hence it suffices to assume that $\mathscr{J}$ is a toroidal centre at a fixed $y \in Y$ (as in the beginning of Section 4.4). We shall also work on the étale cover U of $Y' = \mathrm{Bl}_Y(\mathscr{J})$ defined before Lemma 4.3, where $\mathscr{E}$ pulls back to the principal ideal $E = (u)$ on U. We can also pass to completion at y, that is, work in $\widehat{\mathscr{O}}_{Y,y} \simeq \kappa[\![x_1, \dots, x_n, M]\!]$, where $x_1, \dots, x_n$ are ordinary parameters at y, $M = \overline{\mathscr{M}}_{Y,y}$, and $\kappa = \kappa(y)$ is the residue field at y. Extend $x_1, \dots, x_n$ to ordinary coordinates $x_1, \dots, x_N$ (Definition B.8) at y, and fix a basis $m_1, \dots, m_r \in M$ for M^{gp}. Let $u_i = \exp(m_i)$ (Definition B.8) for $1 \leqslant i \leqslant r$. By Lemma B.9, the stalk $\mathscr{D}^1_{Y,y}$ admits an $\mathscr{O}_{Y,y}$-basis given by $\partial/\partial x_1, \dots, \partial/\partial x_N, u_1\partial/\partial u_1, \dots, u_r\partial/\partial u_r$. For a point y' in the $(x_1T^{n_1})$-chart U_{x_1} over y, Lemma B.9 says that $\mathscr{D}^1_{U_{x_1},y'}$ admits a basis given by $\partial/\partial x'_2, \dots, \partial/\partial x'_k, \partial/\partial x_{k+1}, \dots, \partial/\partial x_N, u\partial/\partial u, u_1\partial/\partial u_1, \dots, u_r\partial/\partial u_r$ (where $x_i = u^{n_i}x'_i$ for $2 \leqslant i \leqslant k$). For $f = f(x_1, \dots, x_n, u_1, \dots, u_r) \in \widehat{\mathcal{I}} \subset \widehat{\mathscr{O}}_{Y,y}$, we compute, on U_{x_1}, the following equations:

(a) For $2 \leqslant i \leqslant k$,

$$\begin{aligned}&\frac{\partial}{\partial x'_i}\big(f(u^{n_1}, u^{n_2}x'_2, \dots, u^{n_k}x'_k, x'_{k+1}, \dots, x'_n, u_1, \dots, u_r)\big)\\ &\quad = \frac{\partial f}{\partial x_i}(u^{n_1}, u^{n_2}x'_2, \dots, u^{n_k}x'_k, x'_{k+1} \dots, x'_n, u_1, \dots, u_r) \cdot u^{n_i}\,.\end{aligned}$$

(b)

$$\begin{aligned}&\left(u\frac{\partial}{\partial u}\right)\left(f\left(u^{n_1}, u^{n_2}x_2', \ldots, u^{n_k}x_k', x_{k+1}', \ldots, x_n'\right)\right)\\&\quad=\frac{\partial f}{\partial x_1}\left(u^{n_1}, u^{n_2}x_2', \ldots, u^{n_k}x_k', x_{k+1}', \ldots, x_n'\right)\cdot\left(n_1u^{n_1}\right)\\&\qquad+\sum_{i=2}^{k}\frac{\partial f}{\partial x_i}\left(u^{n_1}, u^{n_2}x_2', \ldots, u^{n_k}x_k', x_{k+1}', \ldots, x_n'\right)\cdot\left(n_iu^{n_i}x_i'\right).\end{aligned}$$

Rewriting the equation in part (b), we get

$$\begin{aligned}&\frac{\partial f}{\partial x_1}\left(u^{n_1}, u^{n_2}x_2', \ldots, u^{n_k}x_k', x_{k+1}', \ldots, x_n'\right)\\&\quad=\frac{1}{n_1}\cdot u^{-n_1}\Bigg(\left(u\frac{\partial}{\partial u}\right)\left(f\left(u^{n_1}, u^{n_2}x_2', \ldots, u^{n_k}x_k', x_{k+1}', \ldots, x_n'\right)\right)\\&\qquad-\sum_{i=2}^{k}n_i\cdot x_i'\cdot\frac{\partial}{\partial x_i'}\left(f\left(u^{n_1}, u^{n_2}x_2', \ldots, u^{n_k}x_k', x_{k+1}', \ldots, x_n'\right)\right)\Bigg).\end{aligned}$$

Since $n_1 \geqslant n_2 \geqslant \cdots \geqslant n_k$, these equations suffice to show the lemma on the $\left(x_1T^{n_1}\right)$-chart. This computation persists for points y' over y in the remaining $\left(x_iT^{n_i}\right)$-charts, as well as the $\left(mT^d\right)$-charts (see Lemma 4.2). □

The key proposition in this section is the following.

Proposition 4.8. *Let $\mathscr{J} = \left(x_1^{a_1}, \ldots, x_k^{a_k}, (Q \subset M)^r\right)$ be a toroidal centre on Y, where $k \geqslant 1$. Let $\mathcal{I}$ be a nowhere zero ideal on Y. If $\mathscr{J}$ is $\mathcal{I}$-admissible, then we have the following statements:*

(i) *If $a_1 \geqslant 1$, then $\mathscr{J}^{(a_1-1)/a_1}$ is $\mathscr{D}^{\leqslant 1}(\mathcal{I})$-admissible.*
(ii) *The toroidal centre $\mathscr{J}^{(a_1+1)/a_1}$ is $(x_1\mathcal{I})$-admissible.*

Proof. Before delving into the proof, let us fix some notation. Let N be a natural number such that $\widetilde{\mathscr{J}} = \mathscr{J}^{1/N}$ is of the form $\left(x_1^{1/n_1}, \ldots, x_k^{1/n_k}, (Q \subset M)^{r/N}\right)$ for positive integers n_i. Note that, in particular, $N = a_1n_1$. By replacing Q with some multiple $m \cdot Q$ (or Q^m if the monoid is written multiplicatively), we may assume $r/N = 1/d$ for some integer $d \geqslant 1$. Let $Y' \to Y$ be the weighted toroidal blow-up along $\widetilde{\mathscr{J}}$, with exceptional ideal $\mathscr{E}$. By Proposition 4.4(i), since $\widetilde{\mathscr{J}}^N = \mathscr{J}$ is $\mathcal{I}$-admissible, $\mathcal{I}\mathscr{O}_{Y'}$ factors as $\mathscr{E}^N \cdot \mathcal{I}' = \mathscr{E}^{a_1n_1} \cdot \mathcal{I}'$ for some ideal $\mathcal{I}'$ on $\mathscr{O}_{Y'}$. If we can show that $\mathscr{D}^{\leqslant 1}(\mathcal{I})\mathscr{O}_{Y'} = \mathscr{E}^{(a_1-1)n_1} \cdot \mathcal{I}_1$ for some ideal $\mathcal{I}_1$ on $\mathscr{O}_{Y'}$, part (i) follows from Proposition 4.4(ii). This is Lemma 4.9 below.

Lemma 4.9. *Assume the hypotheses of Proposition 4.8, and adopt the set-up above. Then $\mathscr{D}_Y^{\leqslant 1}(\mathcal{I})\mathscr{O}_{Y'}$ factors as $\mathscr{E}^{(a_1-1)n_1} \cdot \mathcal{I}_1$ for some ideal $\mathcal{I}_1$ on Y', with $\mathcal{I}_1 \subset \mathscr{D}_{Y'}^{\leqslant 1}(\mathcal{I}')$.*

Proof. We use the product rule to obtain

$$\begin{aligned}\mathscr{D}_{Y'}^{\leqslant 1}(\mathcal{I}\mathscr{O}_{Y'}) = \mathscr{D}_{Y'}^{\leqslant 1}\left(\mathscr{E}^N \cdot \mathcal{I}'\right) &\subset \mathscr{D}_{Y'}^{\leqslant 1}\left(\mathscr{E}^N\right) \cdot \mathcal{I}' + \mathscr{E}^N \cdot \mathscr{D}_{Y'}^{\leqslant 1}(\mathcal{I}')\\&= \mathscr{E}^N \cdot \mathcal{I}' + \mathscr{E}^N \cdot \mathscr{D}_{Y'}^{\leqslant 1}(\mathcal{I}') = \mathscr{E}^N \cdot \mathscr{D}_{Y'}^{\leqslant 1}(\mathcal{I}').\end{aligned}$$

Next, Lemma 4.7 says $\mathscr{D}_Y^{\leqslant 1}(\mathcal{I})\mathscr{O}_{Y'} \subset \mathscr{E}^{-n_1} \cdot \mathscr{D}_{Y'}^{\leqslant 1}(\mathcal{I}\mathscr{O}_{Y'})$. Combining this with the above computation, we obtain $\mathscr{D}_Y^{\leqslant 1}(\mathcal{I})\mathscr{O}_{Y'} \subset \mathscr{E}^{N-n_1} \cdot \mathscr{D}_{Y'}^{\leqslant 1}(\mathcal{I}') = \mathscr{E}^{(a_1-1)n_1} \cdot \mathscr{D}_{Y'}^{\leqslant 1}(\mathcal{I}')$. The fractional ideal $\mathscr{E}^{-(a_1-1)n_1} \cdot \mathscr{D}_Y^{\leqslant 1}(\mathcal{I})\mathscr{O}_{Y'}$ is contained in $\mathscr{D}_{Y'}^{\leqslant 1}(\mathcal{I}')$ and hence is an ideal $\mathcal{I}_1$ on Y'. Since $\mathscr{E}$ is a principal ideal, we get the desired factorization $\mathscr{D}_Y^{\leqslant 1}(\mathcal{I})\mathscr{O}_{Y'} = \mathscr{E}^{(a_1-1)n_1} \cdot \mathcal{I}_1$, with $\mathcal{I}_1 \subset \mathscr{D}_{Y'}^{\leqslant 1}(\mathcal{I}')$. □

Proof of Proposition 4.8, continued. For part (ii), adopt the set-up at the beginning of this proof. Then x_1 factors as $u^{n_1} \cdot x_1'$ in $\mathscr{O}_{Y'}$ (cf. Lemmas 4.1 and 4.2), whence $(x_1\mathcal{I})\mathscr{O}_{Y'} = \mathscr{E}^{(a_1+1)n_1} \cdot (x_1'\mathcal{I}')$. Once again, an application of Proposition 4.4(ii) completes the proof. □

We can now prove a generalization of Lemma 4.9.

Lemma 4.10. *Assume the hypotheses of Proposition 4.8, and adopt the set-up above. Then for every integer $1 \leqslant j \leqslant a_1$, the sheaf $\mathscr{D}_Y^{\leqslant j}(\mathcal{I})\mathscr{O}_{Y'}$ factors as $\mathscr{E}^{(a_1-j)n_1} \cdot \mathcal{I}_j$ for some ideal $\mathcal{I}_j$ on Y', with $\mathcal{I}_j \subset \mathscr{D}_{Y'}^{\leqslant j}(\mathcal{I}')$.*

Proof. We have already shown the case $j = 1$ in Lemma 4.9. In general, induct on j. We assume that Lemma 4.10 is known for some $1 \leqslant j < a_1$, and we prove the lemma for $j+1$. Applying Proposition 4.8(i) repeatedly, we see that $\mathscr{J}^{(a_1-j)/a_1}$ is $\mathscr{D}^{\leqslant j}(\mathcal{I})$-admissible, and the induction hypothesis says that $\mathscr{D}^{\leqslant j}(\mathcal{I})\mathscr{O}_{Y'} = \mathscr{E}^{(a_1-j)n_1} \cdot \mathcal{I}_j$ for an ideal $\mathcal{I}_j$ on Y', with $\mathcal{I}_j \subset \mathscr{D}^{\leqslant j}(\mathcal{I}')$. Applying the case $j = 1$ with

(a) $\mathscr{J}$ replaced by $\mathscr{J}^{(a_1-j)/a_1} = \overline{\mathscr{J}}^{n_1(a_1-j)}$,
(b) $\mathcal{I}$ replaced by $\mathscr{D}^{\leqslant j}(\mathcal{I})$,

we see that $\mathscr{D}^{\leqslant j+1}(\mathcal{I})\mathscr{O}_{Y'} = \mathscr{D}^{\leqslant 1}\big(\mathscr{D}^{\leqslant j}(\mathcal{I})\big)\mathscr{O}_{Y'}$ factors as $\mathscr{E}^{(a_1-j-1)n_1} \cdot \mathcal{I}_{j+1}$ for some ideal $\mathcal{I}_{j+1}$ on Y', with $\mathcal{I}_{j+1} \subset \mathscr{D}^{\leqslant 1}(\mathcal{I}_j) \subset \mathscr{D}^{\leqslant 1}\big(\mathscr{D}^{\leqslant j}(\mathcal{I}')\big) = \mathscr{D}^{\leqslant j+1}(\mathcal{I}')$, as desired. □

Proposition 4.8(i) provides us with the first piece of information about the invariant of an $\mathcal{I}$-admissible toroidal centre at a point $y \in Y$.

Corollary 4.11. *Let $\mathcal{I}$ be an ideal on Y, and fix a $y \in Y$ such that $\mathcal{I}_y \neq 0$. If $\operatorname{log-ord}_y(\mathcal{I}) = b_1 < \infty$ and $\mathscr{J}^{(y)} = \big(x_1^{a_1}, \ldots, x_k^{a_k}, (Q \subset M)^r\big)$ is a $\mathcal{I}$-admissible toroidal centre at y with $k \geqslant 1$, then $a_1 \leqslant b_1$.*

Proof. Suppose for a contradiction that $a_1 > b_1$. We apply Proposition 4.8(i) repeatedly to conclude that $\big(\mathscr{J}^{(y)}\big)^{(a_1-b_1)/a_1}$ is $\mathscr{D}^{\leqslant b_1}(\mathcal{I})$-admissible. By restricting to a smaller open affine neighbourhood U_y of y on which $\mathscr{J}^{(y)}$ is defined, we may arrange for $\mathscr{D}^{\leqslant b_1}(\mathcal{I})$ to be $\mathscr{O}_Y$ when restricted to U_y. Replacing Y with U_y, we see that $\big(\mathscr{J}^{(y)}\big)^{(a_1-b_1)/a_1}$ is $\mathscr{O}_Y$-admissible, but

$$\big(\mathscr{J}^{(y)}\big)^{\frac{a_1-b_1}{a_1}} = \Big(x_1^{a_1-b_1}, \ldots, x_k^{\frac{(a_1-b_1)a_k}{a_1}}, (Q \subset M)^{\frac{(a_1-b_1)r}{a_1}}\Big),$$

with $a_1 - b_1 > 0$, which gives a contradiction. □

4.6 The invariant of a toroidal centre is well defined

We prove Lemma 3.2 in this section. The main ingredient of the proof is Corollary 4.11, but we will need two lemmas.

Lemma 4.12. *Let $y \in Y$, let $\kappa(y)$ be the residue field at y, and let $\mathfrak{s}_y$ be the logarithmic stratum of Y at y. Let $\mathscr{J}^{(y)} = \big(x_1^{a_1}, \ldots, x_k^{a_k}, (Q \subset M)^r\big)$ be a toroidal centre at y, with $k \geqslant 1$. For a homogeneous section $fT^\ell \in \mathscr{J}^{(y)}$, write the image of f under $\mathscr{O}_{Y,y} \twoheadrightarrow \mathscr{O}_{\mathfrak{s}_y,y} \to \widehat{\mathscr{O}}_{\mathfrak{s}_y,y} \simeq \kappa(y)[\![x_1, \ldots, x_n]\!]$ as $\sum_{\vec{\alpha}} c_{\vec{\alpha}} \cdot x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ for some $c_{\vec{\alpha}} \in \kappa(y)$. Then $\sum_{i=1}^k \alpha_i/a_i \geqslant \ell$ whenever $c_{\vec{\alpha}} \neq 0$.*

Proof. We may replace Y with $\mathfrak{s}_y$ and reduce to the case where Y is a smooth $\Bbbk$-variety with trivial logarithmic structure and $\mathscr{J}^{(y)} = \big(x_1^{a_1}, \ldots, x_k^{a_k}\big)$ with $k \geqslant 1$. Replacing $\mathscr{J}^{(y)}$ with $\big(\mathscr{J}^{(y)}\big)^\ell$, we may assume $\ell = 1$. This is the same exact situation as in [ATW20a, Lemma 5.2.1]. We recall its proof. Consider the following valuation in $\mathrm{ZR}\left(\widehat{\mathscr{O}}_{Y,y}\right)$:

$$\nu_{\mathscr{J}}\left(\sum_{\vec{\alpha}} c_{\vec{\alpha}} \cdot x_1^{\alpha_1} \cdots x_n^{\alpha_n}\right) := \min_{\substack{\vec{\alpha} \\ c_{\vec{\alpha}} \neq 0}} \left(\sum_{i=1}^{k} \frac{\alpha_i}{a_i}\right).$$

The hypothesis implies that $\mathscr{J}^{(y)}$ is (f)-admissible, and hence, by Lemma 3.7, we have the inequality $\gamma_{\mathscr{J}^{(y)}} \leqslant \gamma_{(f)}$. Therefore,

$$\min_{\substack{\vec{\alpha} \\ c_{\vec{\alpha}} \neq 0}} \left(\sum_{i=1}^{k} \frac{\alpha_i}{a_i}\right) = \nu_{\mathscr{J}}(f) = \gamma_{(f),\nu_{\mathscr{J}}} \geqslant \gamma_{\mathscr{J}^{(y)},\nu_{\mathscr{J}}} = \min\{a_i \cdot \nu_{\mathscr{J}}(x_i) \colon 1 \leqslant i \leqslant k\} = 1\,.$$

This completes the proof. □

Lemma 4.13 (Exchange). *Let $y \in Y$, and let $\mathscr{J}^{(y)} = \left(x_1^{a_1}, \ldots, x_k^{a_k}, (Q \subset M)^r\right)$ be a toroidal centre at y, with $k \geqslant 1$. Suppose that $x_1', x_2, \ldots, x_n$ is also a system of ordinary parameters at y, and suppose that $\mathscr{J}^{(y)}$ is $\left((x_1')^{a_1}\right)$-admissible. After possibly passing to a smaller affine neighbourhood of y on which $\mathscr{J}^{(y)}$ is defined, we have $\mathscr{J}^{(y)} = \left((x_1')^{a_1}, x_2^{a_2}, \ldots, x_k^{a_k}, (Q \subset M)^r\right)$.*

Proof. The hypothesis says that $\mathscr{J}^{(y)}$ contains $(\mathscr{J}')^{(y)} = \left((x_1')^{a_1}, x_2^{a_2}, \ldots, x_n^{a_n}, (Q \subset M)^r\right)$. This is necessarily an equality near y, as can be checked by passing to completion at y and seeing that the $\kappa(y)$-dimensions of each T^N-graded piece on both sides match. □

We can now prove Lemma 3.2.

Proof of Lemma 3.2. Suppose that $\mathscr{J}^{(y)}$ admits the following presentations:

$$\left(x_1^{a_1}, \ldots, x_k^{a_k}, (Q \subset M)^r\right) = \mathscr{J}^{(y)} = \left((x_1')^{b_1}, \ldots, (x_\ell')^{b_\ell}, (Q' \subset M)^s\right).$$

Note that $k = 0$ if and only if $\ell = 0$, in which case $\mathrm{inv}\left(\mathscr{J}^{(y)}\right) = (\infty)$. Henceforth, assume $k \geqslant 1$ and hence $\ell \geqslant 1$. By replacing $\mathscr{J}^{(y)}$ with some power of itself, we may assume that a_1 and b_1 are integers. Observe that, in particular, $\left(x_1^{a_1}, \ldots, x_k^{a_k}, (Q \subset M)^r\right)$ is $\left((x_1')^{b_1}\right)$-admissible. Using Corollary 4.11, we see that $a_1 \leqslant b_1$. Reversing the roles, we get $b_1 \leqslant a_1$, whence $a_1 = b_1$. Applying Proposition 4.8(i) repeatedly, we see that $(\mathscr{J}^{(y)})^{1/a_1} = \left(x_1, x_2^{a_2/a_1}, \ldots, x_k^{a_k/a_1}, (Q \subset M)^{r/a_1}\right)$ is (x_1')-admissible. Extending $x_1, \ldots, x_k$ to a system of ordinary parameters $x_1, \ldots, x_n$ at y and passing to completion at y, write the image of x_1' under $\mathscr{O}_{Y,y} \twoheadrightarrow \mathscr{O}_{\mathfrak{s}_y,y} \to \widehat{\mathscr{O}}_{\mathfrak{s}_y,y} \simeq \kappa(y)[\![x_1, \ldots, x_n]\!]$ as $\sum_{\vec{\alpha}} c_{\vec{\alpha}} x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ for some $c_{\vec{\alpha}} \in \kappa(y)$. Applying Lemma 4.12, we see that whenever $c_{\vec{\alpha}} \neq 0$, we have $\alpha_1 + \sum_{i=2}^{k} \alpha_i/(a_i/a_1) \geqslant 1$. Consequently, if we let $k_0 = \max\{1 \leqslant i \leqslant k \colon a_i = a_1\}$, which is at least 1, the image of x_1' in $\mathscr{O}_{\mathfrak{s}_y,y}$ lies in $(x_1, \ldots, x_{k_0}) + \mathfrak{m}_{\mathfrak{s}_y,y}^2$, where $\mathfrak{m}_{\mathfrak{s}_y,y}$ is the maximal ideal of $\mathscr{O}_{\mathfrak{s}_y,y}$. Therefore, after possibly reordering $x_1, \ldots, x_{k_0}$, we may replace x_1 with an x_1' such that $(x_1', x_2, \ldots, x_n)$ is a system of ordinary parameters at y. It is essential to note that the reordering does not mess up the presentation of $\mathscr{J}^{(y)} = \left(x_1^{a_1}, \ldots, x_k^{a_k}, (Q \subset M)^r\right)$ since $a_1 = \cdots = a_{k_0}$. Applying Lemma 4.13, we obtain

$$\left((x_1')^{a_1}, x_2^{a_2}, \ldots, x_k^{a_k}, (Q \subset M)^r\right) = \mathscr{J}^{(y)} = \left((x_1')^{a_1}, (x_2')^{b_2}, \ldots, (x_\ell')^{b_\ell}, (Q' \subset M)^s\right).$$

We now restrict to the hypersurface H containing y given by $x_1' = 0$, and we get

$$\left(x_2^{a_2}, \ldots, x_k^{a_k}, (Q \subset M)^r\right) = \mathscr{J}_H^{(y)} = \left((x_2')^{b_2}, \ldots, (x_\ell')^{b_\ell}, (Q' \subset M)^s\right).$$

By the induction hypothesis, we obtain $k = \ell$ and $a_i = b_i$ for $2 \leqslant i \leqslant k = \ell$, as desired. Moreover, $Q \neq \emptyset$ if and only if $Q' \neq \emptyset$. However, we remind the reader (Remark 3.3(ii)) that Q may be different from Q' and r may be not equal to s. □

5. Coefficient ideals

As always, assume that Y is a strict toroidal $\Bbbk$-scheme. Fix an ideal $\mathcal{I} \subset \mathscr{O}_Y$.

5.1 Maximal contact element

In this section, we assume $1 \leqslant a = \max \text{log-ord}(\mathcal{I}) < \infty$. Following [Kol07, Definition 3.79], the *maximal contact ideal* of $\mathcal{I}$ is defined as

$$\mathrm{MC}(\mathcal{I}) = \mathscr{D}^{\leqslant a-1}(\mathcal{I})\,.$$

For a point $y \in Y$ with $a = \text{log-ord}_y(\mathcal{I})$, a *maximal contact element* of $\mathcal{I}$ at y is a section of $\mathrm{MC}(\mathcal{I})$ over a neighbourhood of y in Y which can be extended to a system of ordinary parameters at y (or, equivalently, has logarithmic order 1 at y). Maximal contact elements at such points $y \in Y$ always exist because we are in characteristic zero. The vanishing locus of a maximal contact element of $\mathcal{I}$ at y is called a *hypersurface of maximal contact* for $\mathcal{I}$ through y. It is well known that hypersurfaces of maximal contact play a crucial role in the resolution of singularities in characteristic zero, in the sense that they allow for induction on dimension: namely, one passes to a hypersurface of maximal contact in the induction step.

Following [Kol07, Definition 3.53], we say that $\mathcal{I} \subset \mathscr{O}_Y$ is *MC-invariant* if

$$\mathrm{MC}(\mathcal{I}) \cdot \mathscr{D}^{\leqslant 1}(\mathcal{I}) \subset \mathcal{I}\,.$$

The reason why we care about such a property is reflected in the theorem below.

Theorem 5.1 (Invariance of maximal contact for MC-invariant ideals). *Assume that $\mathcal{I}$ is MC-invariant. For every $y \in Y$ such that $\text{log-ord}_y(\mathcal{I}) = a \geqslant 1$ and every pair of maximal contact elements x and x' of $\mathcal{I}$, there exist strict and étale morphisms*

$$\widetilde{U} \underset{\phi_{x'}}{\overset{\phi_x}{\rightrightarrows}} Y$$

from a strict toroidal $\Bbbk$-scheme $\widetilde{U}$ into Y, and a point $\widetilde{y}$ of $\widetilde{U}$ such that $\phi_x(\widetilde{y}) = y = \phi_{x'}(\widetilde{y})$, satisfying the following properties:

(i) *We have $\phi_x^*(\mathcal{I}) = \phi_{x'}^*(\mathcal{I})$.*

(ii) *We have $\phi_x^*(x) = \phi_{x'}^*(x')$ in $\mathrm{MC}\left(\widetilde{\mathcal{I}}\right)$, where $\widetilde{\mathcal{I}}$ denotes the ideal in part* (i).

The statement (and proof) of Theorem 5.1 follows [Wło05, Lemma 3.5.5], [ATW20a, Lemma 5.3.3], and [Kol07, Theorem 3.92] closely. See Appendix C for a proof.

5.2 Coefficient ideals

In this section, we recall the method of taking coefficient ideals. This originates from Hironaka [Hir64a, Hir64b] and has been studied extensively in the papers of Bierstone–Milman ([BM08], etc.), Encinas–Villamayor ([EV00], etc.), Włodarczyk [Wło05], and many others. Our treatment closely follows [ATW19], which studies coefficient ideals from the Rees algebra approach of [EV07].

For an integer $a \geqslant 1$, consider the graded $\mathscr{O}_Y$-subalgebra $\mathscr{G}_\bullet(\mathcal{I}, a) \subset \mathscr{O}_Y[T]$ generated by $\mathscr{O}_Y$ and $\mathscr{D}^{\leqslant j}(\mathcal{I}) \cdot T^{a-j}$ for every $0 \leqslant j < a$. Its graded pieces are

$$\mathscr{G}_s(\mathcal{I}, a) := \left(\prod_{j=0}^{a-1} \left(\mathscr{D}^{\leqslant j}(\mathcal{I})\right)^{c_j} : c_j \in \mathbb{N},\ \sum_{j=0}^{a-1} (a-j)c_j \geqslant s \right) \subset \mathscr{O}_Y \quad \text{for } s \geqslant 1\,.$$

The main reason for putting $\mathscr{D}^{\leqslant j}(\mathcal{I})$ in degree $a-j$ is the following lemma.

Lemma 5.2. *Let $y \in Y$, and let $a \geqslant 1$ be an integer. If $\operatorname{log-ord}_y(\mathcal{I}) \geqslant a$, then we have $\operatorname{log-ord}_y(\mathscr{G}_s(\mathcal{I},a)) \geqslant s$ for every $s \geqslant 1$.*

Proof. Each term $\prod_{j=0}^{a-1}\left(\mathscr{D}^{\leqslant j}(\mathcal{I})\right)^{c_j}$ in $\mathscr{G}_s(\mathcal{I},a)$ has logarithmic order at y given by

$$\sum_{j=0}^{a-1} c_j \cdot \operatorname{log-ord}_y\left(\mathscr{D}^{\leqslant j}(\mathcal{I})\right) = \sum_{j=0}^{a-1} c_j(\operatorname{log-ord}_y(\mathcal{I})-j) \geqslant \sum_{j=0}^{a-1} c_j(a-j) \geqslant s\,,$$

whence $\operatorname{log-ord}_y(\mathscr{G}_s(\mathcal{I},a)) \geqslant s$. □

Remark 5.3. Since the formation of $\mathscr{D}^{\leqslant 1}$, as well as taking products and sums of ideals, is functorial for logarithmically smooth morphisms, the formation of $\mathscr{G}_s(-,a)$ is also functorial for logarithmically smooth morphisms; that is, if $\widetilde{Y} \to Y$ is a logarithmically smooth morphism of toroidal $\Bbbk$-schemes, then $\mathscr{G}_s(\mathcal{I},a)\mathscr{O}_{\widetilde{Y}} = \mathscr{G}_s(\mathcal{I}\mathscr{O}_{\widetilde{Y}},a)$.

The graded pieces satisfy the following standard properties.

Lemma 5.4 (cf. [Kol07, Proposition 3.99]). *Let $\mathscr{G}_\bullet = \mathscr{G}_\bullet(\mathcal{I},a)$ be as above, and assume $1 \leqslant a = \max \operatorname{log-ord}(\mathcal{I}) < \infty$.*

(i) *We have $\mathscr{G}_{s+1} \subset \mathscr{G}_s$ for every s.*
(ii) *We have $\mathscr{G}_s \cdot \mathscr{G}_t \subset \mathscr{G}_{s+t}$ for every s, t.*
(iii) *We have $\mathscr{D}^{\leqslant 1}(\mathscr{G}_{s+1}) = \mathscr{G}_s$ for every s.*
(iv) *We have $\mathscr{D}^{\leqslant s-1}(\mathscr{G}_s) = \mathscr{G}_1 = \mathrm{MC}(\mathcal{I})$ for every s. In particular, $s = \max \operatorname{log-ord}(\mathscr{G}_s)$.*
(v) *For every s, the ideal $\mathscr{G}_s$ is MC-invariant.*
(vi) *We have $\mathscr{G}_s \cdot \mathscr{G}_t = \mathscr{G}_{s+t}$ whenever $t \geqslant (a-1)\cdot \operatorname{lcm}(2,\dots,a)$ and s is a multiple of $\operatorname{lcm}(2,\dots,m)$. In fact, the same holds if $t \geqslant a!$.*
(vii) *We have $(\mathscr{G}_s)^j = \mathscr{G}_{js}$ whenever $s = r\cdot\operatorname{lcm}(2,\dots,a)$ for some $r \geqslant a-1$. In fact, the same holds for $s = a!$.*
(viii) *We have $\left(\mathscr{D}^{\leqslant i}(\mathscr{G}_s)\right)^s \subset \mathscr{G}_s^{s-i}$ whenever $s = r\cdot\operatorname{lcm}(2,\dots,a)$ for some $r \geqslant a-1$, and $0 \leqslant i < s$. In fact, the same holds for $s = a!$.*

Proof. Even though we are in the logarithmic case, the proof for [Kol07, Proposition 3.99] works verbatim, but one should be aware of an inconsequential but noteworthy difference: for the inclusion $\mathscr{G}_s \subset \mathscr{D}^{\leqslant 1}(\mathscr{G}_{s+1})$ in part (iii), the proof utilizes a maximal contact element x of $\mathcal{I}$ at a point, which in the logarithmic case is an ordinary parameter, and hence the corresponding logarithmic derivation is still $\partial/\partial x$. □

Corollary 5.5. *Assume $1 \leqslant a = \max \operatorname{log-ord}(\mathcal{I}) < \infty$, and let $y \in Y$. If $\operatorname{log-ord}_y(\mathcal{I}) = a$, then $\operatorname{log-ord}_y(\mathscr{G}_s(\mathcal{I},a)) = s$ for every $s \geqslant 1$. Moreover, if x is a maximal contact element for $\mathcal{I}$ at y, then x is also a maximal contact element for $\mathscr{G}_s(\mathcal{I},a)$ at y.*

Proof. This is a consequence of Lemmas 5.2 and 5.4(iv). □

With the exception of part (viii), all the properties in Lemma 5.4 are self-explanatory. For example, Lemma 5.4(vii) says that the $(a!)$-Veronese subalgebra $\mathscr{G}_{a!\bullet}(\mathcal{I},a)$ of $\mathscr{G}_\bullet(\mathcal{I},a)$ is generated in degree 1; that is, it is the Rees algebra of the coefficient ideal of $(\mathcal{I},a)$ defined below.

Definition 5.6 (Coefficient ideal). Let $\mathcal{I}$ be an ideal on Y, and assume

$$1 \leqslant a = \max \operatorname{log-ord}(\mathcal{I}) < \infty\,.$$

The *coefficient ideal* of the marked ideal $(\mathcal{I}, a)$ is

$$\mathscr{C}(\mathcal{I}, a) := \mathscr{G}_{a!}(\mathcal{I}, a) \subset \mathscr{O}_Y\,.$$

Historically, the coefficient ideal provides a method to enrich an ideal with its higher derivatives which retains information that would otherwise be lost when one restricts the original ideal (as opposed to the coefficient ideal) to a hypersurface of maximal contact.

Finally, let us explicate the property in Lemma 5.4(viii). Following [Kol07, Definition 3.83], we say that an ideal $\mathcal{I}$ on Y, with $1 \leqslant a = \max \operatorname{log-ord}(\mathcal{I}) < \infty$, is *$\mathscr{D}$-balanced* (in the logarithmic sense) if

$$\mathscr{D}^{\leqslant i}(\mathcal{I})^a \subset \mathcal{I}^{a-i} \quad \text{for} \quad 0 \leqslant i < a\,.$$

In particular, Lemma 5.4(viii) says that if $a = \max \operatorname{log-ord}(\mathcal{I}) < \infty$, the coefficient ideal $\mathscr{C}(\mathcal{I}, a)$ is $\mathscr{D}$-balanced.

The "$\mathscr{D}$-balanced" property plays a subtle role in our paper. Namely, let x be a maximal contact element of $\mathcal{I}$ at some point $y \in Y$, and denote the corresponding hypersurface of maximal contact by H. If one extends x to a system of ordinary parameters at y, one easily sees that $\mathscr{D}^{\leqslant 1}(\mathcal{I}|_H) \subset \mathscr{D}^{\leqslant 1}(\mathcal{I})|_H$. Note, however, that the reverse inclusion does not hold in general. As noted in [Kol07, paragraph before Definition 3.83], the "$\mathscr{D}$-balanced" property provides a partial remedy to this issue.

Let us be more precise about this by stating the issue in terms of admissibility of toroidal centres. Namely, let $\mathscr{J}^{(y)}$ be a toroidal centre at y, and assume that the restriction of $\mathscr{J}^{(y)}$ to H, denoted by $\mathscr{J}_H^{(y)}$, is $\mathcal{I}|_H$-admissible. Then a repeated application of Proposition 4.8(i) tells us that after we replace $\mathscr{J}_H^{(y)}$ with some power of itself, $\mathscr{J}_H^{(y)}$ is $\mathscr{D}^{\leqslant i}(\mathcal{I}|_H)$-admissible. Unfortunately, $\mathscr{D}^{\leqslant i}(\mathcal{I}|_H)$-admissibility does not imply $\mathscr{D}^{\leqslant i}(\mathcal{I})|_H$-admissibility. However, if one assumes that $\mathcal{I}$ is $\mathscr{D}$-balanced (with $a = \max \operatorname{log-ord}(\mathcal{I})$), then $\big(\mathscr{D}^{\leqslant i}(\mathcal{I})|_H\big)^a \subset (\mathcal{I}|_H)^{a-i}$, so that applying Lemma 3.10(iii) twice gives the following chain of implications:

$$\begin{aligned}\mathscr{J}_H^{(y)} \text{ is } \mathcal{I}|_H\text{-admissible} &\Rightarrow \big(\mathscr{J}_H^{(y)}\big)^{a-i} \text{ is } (\mathcal{I}|_H)^{a-i}\text{-admissible} \\ &\Rightarrow \big(\mathscr{J}_H^{(y)}\big)^{a-i} \text{ is } (\mathscr{D}^{\leqslant i}(\mathcal{I})|_H)^{a}\text{-admissible} \\ &\Rightarrow \big(\mathscr{J}_H^{(y)}\big)^{(a-i)/a} \text{ is } \mathscr{D}^{\leqslant i}(\mathcal{I})|_H\text{-admissible}\,.\end{aligned}$$

In Section 6.2, it turns out that this strategy works out very well (see the proof of Theorem 6.5(i)).

5.3 Formal decomposition

Let $y \in Y$, and assume $\operatorname{log-ord}_y(\mathcal{I}) = a$ (where $a \geqslant 1$ is an integer). Let x_1 be a maximal contact element of $\mathcal{I}$ at a point $y \in Y$. Extending it to a system of ordinary parameters $x_1, \ldots, x_n$ at y, we have

$$\widehat{\mathscr{O}}_{Y,y} \simeq \kappa[\![x_1, x_2 \ldots, x_n, M]\!]\,, \quad \text{where } \kappa = \kappa(y) \text{ and } M = \overline{\mathscr{M}}_{Y,y}\,.$$

For integers $s \geqslant 1$,

(a) let $\widehat{\mathscr{G}}_s(\mathcal{I}, a) = \mathscr{G}_s(\mathcal{I}, a)\widehat{\mathscr{O}}_{Y,y}$,

(b) let $\overline{\mathscr{C}}_s(\mathcal{I}, a)$ denote the ideal generated by the image of $\widehat{\mathscr{G}}_s(\mathcal{I}, a)$ under the reduction homomorphism $\widehat{\mathscr{O}}_{Y,y} = \kappa[\![x_1, x_2, \ldots, x_n, M]\!] \twoheadrightarrow \kappa[\![x_2, \ldots, x_n, M]\!]$,

(c) and let $\widetilde{\mathscr{C}}_s(\mathcal{I}, a) = \overline{\mathscr{C}}_s(\mathcal{I}, a)\kappa[\![x_1, x_2, \ldots, x_n, M]\!] = \overline{\mathscr{C}}_s(\mathcal{I}, a)\widehat{\mathscr{O}}_{Y,y}$.

PROPOSITION 5.7 (Formal decomposition, cf. [ATW19, Proposition 4.4.1]). *After passing to the completion at y, we have*

$$\widehat{\mathscr{G}}_s(\mathcal{I}, a) = \left(x_1^s\right) + \left(x_1^{s-1}\right)\widetilde{\mathscr{C}}_1(\mathcal{I}, a) + \cdots + (x_1)\widetilde{\mathscr{C}}_{s-1}(\mathcal{I}, a) + \widetilde{\mathscr{C}}_s(\mathcal{I}, a)\,, \quad \text{where } s \geqslant 1\,.$$

In particular,

$$\widehat{\mathscr{C}}(\mathcal{I}, a) = \left(x_1^{a!}\right) + \left(x_1^{a!-1}\right)\widetilde{\mathscr{C}}_1(\mathcal{I}, a) + \cdots + (x_1)\widetilde{\mathscr{C}}_{a!-1}(\mathcal{I}, a) + \widetilde{\mathscr{C}}_{a!}(\mathcal{I}, a)\,.$$

Proof. We shall prove the result by induction on s, with the case $s = 1$ being clear. For integers $N \geqslant s$, we have the ideals $\left(x_1^{N+1}\right) \subset \widehat{\mathscr{G}}_s(\mathcal{I}, a)$, which are stable under the linear operator $x_1\partial/\partial x_1$. Thus, $x_1\partial/\partial x_1$ descends to a linear operator on $\widehat{\mathscr{G}}_s(\mathcal{I}, a)/\left(x_1^{N+1}\right)$ and decomposes it into a direct sum of m-eigenspaces for integers $0 \leqslant m \leqslant N$. These m-eigenspaces are independent of the choice of $N \geqslant m$. Therefore, we can write the m-eigenspace as $x_1^m \cdot \widehat{\mathscr{G}}_s^{(m)}(\mathcal{I}, a)$ for subspaces $\widehat{\mathscr{G}}_s^{(m)}(\mathcal{I}, a) \subset \kappa[\![x_2, \ldots, x_n, M]\!]$, so that

$$\widehat{\mathscr{G}}_s(\mathcal{I}, a)/\left(x_1^{N+1}\right) = \bigoplus_{m=0}^{N} x_1^m \cdot \widehat{\mathscr{G}}_s^{(m)}(\mathcal{I}, a)\,.$$

This implies that

$$\widehat{\mathscr{G}}_s(\mathcal{I}, a) = \left(x_1^m \cdot \widehat{\mathscr{G}}_s^{(m)}(\mathcal{I}, a) \colon 0 \leqslant m \leqslant N\right) + \left(x_1^{N+1}\right)\,. \tag{5.3.1}$$

Next, we explicate the terms in the above equation. The simple terms are $\widehat{\mathscr{G}}_s^{(0)}(\mathcal{I}, a) = \overline{\mathscr{C}}_s(\mathcal{I}, a)$ and $\widehat{\mathscr{G}}_s^{(m)}(\mathcal{I}, a) = \kappa[\![x_2, \ldots, x_n, M]\!]$ for $m \geqslant s$. For integers $0 < m < s$, we have

$$\begin{aligned}\widehat{\mathscr{G}}_s^{(m)}(\mathcal{I}, a) = \frac{\partial^m}{\partial x_1^m}\left(x_1^m \cdot \widehat{\mathscr{G}}_s^{(m)}(\mathcal{I}, a)\right) &\subset \mathscr{D}^{\leqslant m}\left(\widehat{\mathscr{G}}_s(\mathcal{I}, a)\right) \cap \kappa[\![x_2, \ldots, x_n, M]\!] \\ &= \widehat{\mathscr{G}}_{s-m}(\mathcal{I}, a) \cap \kappa[\![x_2, \ldots, x_n, M]\!] \subset \overline{\mathscr{C}}_{s-m}(\mathcal{I}, a)\,,\end{aligned}$$

where the equality in the second line follows from Lemma 5.4(iii). Substituting these into equation (5.3.1) with $N = s$, we get

$$\widehat{\mathscr{G}}_s(\mathcal{I}, a) \subset \widetilde{\mathscr{C}}_s(\mathcal{I}, a) + (x_1)\widetilde{\mathscr{C}}_{s-1}(\mathcal{I}, a) + \cdots + \left(x_1^{s-1}\right)\widetilde{\mathscr{C}}_1(\mathcal{I}, a) + \left(x_1^s\right)\,.$$

The induction hypothesis gives

$$(x_1)\widetilde{\mathscr{C}}_{s-1}(\mathcal{I}, a) + \cdots + \left(x_1^{s-1}\right)\widetilde{\mathscr{C}}_1(\mathcal{I}, a) + \left(x_1^s\right) = (x_1)\widehat{\mathscr{G}}_{s-1}(\mathcal{I}, a) \subset \widehat{\mathscr{G}}_s(\mathcal{I}, a)\,.$$

Since $\widetilde{\mathscr{C}}_s(\mathcal{I}, a) \subset \widehat{\mathscr{G}}_s(\mathcal{I}, a)$ as well, the proposition follows. □

6. Invariants and toroidal centres associated with ideals

6.1 Defining invariants and toroidal centres at points

To an ideal $\mathcal{I}$ on a strict toroidal $\Bbbk$-scheme Y and $y \in Y$, we shall first assign some preliminary data, namely

(a) a finite sequence of natural numbers $(b_1, \ldots, b_k) \in \mathbb{N}^k$,
(b) a finite sequence of ordinary parameters $x_1, \ldots, x_k$ at y,
(c) and an ideal Q of $M = \overline{\mathscr{M}}_{Y,y}$.

We do this by induction, which terminates only once Q is defined. For the base case, we consider the following:

Case 1a: If $\operatorname{log-ord}_y(\mathcal{I}) = 0$ (that is, $\mathcal{I}_y = (1)$), then set $k := 1$, $b_1 := 0$, and $Q := \emptyset$. Let x_1 be any ordinary parameter at p.

Case 1b: If $\operatorname{log-ord}_y(\mathcal{I}) = \infty$ (that is, $\mathscr{M}(\mathcal{I})_y \neq (1)$), do not define any b_i or x_i (that is, set $k := 0$), and define Q by passing the stalk of $\alpha_Y^{-1}(\mathscr{M}(\mathcal{I}))$ at y to $\overline{\mathscr{M}}_{Y,y}$; we denote the result by $\overline{\alpha_Y^{-1}(\mathscr{M}(\mathcal{I}))_y}$. Note that it may happen that $\mathcal{I}_y = \mathscr{M}(\mathcal{I})_y = 0$, in which case $Q := \emptyset$.

Case 2: If $\operatorname{log-ord}_y(\mathcal{I})$ is not 0 or ∞, set $b_1 := \operatorname{log-ord}_y(\mathcal{I}) \in \mathbb{N}_{\geqslant 1}$, and let x_1 be a maximal contact element of $\mathcal{I}$ at y.

In case 2, set $\mathcal{I}[1] = \mathcal{I}$; we shall define the remaining b_i, x_i, and Q by means of induction. Assuming that $\mathcal{I}[i]$, b_i, x_i are defined for $i \leqslant \ell$, we set

$$\mathcal{I}[\ell+1] := \mathscr{C}(\mathcal{I}[\ell], b_\ell)|_{V(x_1,\ldots,x_\ell)}\,.$$

In what follows, we pull back the logarithmic structure $\mathscr{M}_Y$ on Y to define a logarithmic structure $\alpha_{V(x_1,\ldots,x_\ell)} \colon \mathscr{M}_{V(x_1,\ldots,x_\ell)} \to \mathscr{O}_{V(x_1,\ldots,x_k)}$ on $V(x_1, \ldots, x_\ell)$. Note that since $x_1, \ldots, x_\ell$ are ordinary parameters at y, the vanishing locus $V(x_1, \ldots, x_\ell)$ is a strict toroidal $\Bbbk$-scheme under this logarithmic structure.

Case A: If $\mathscr{M}(\mathcal{I}[\ell+1])_y \neq (1)$ (that is, $\operatorname{log-ord}_y(\mathcal{I}[\ell+1]) = \infty$), no further b_i or x_i are defined. Define Q to be the preimage of $\overline{\alpha^{-1}_{V(x_1,\ldots,x_\ell)}(\mathscr{M}(\mathcal{I}[\ell+1]))_y}$ under the canonical isomorphism $\overline{\mathscr{M}}_{Y,y} \xrightarrow{\simeq} \overline{\mathscr{M}}_{V(x_1,\ldots,x_k),y}$.

Case B: If $\mathscr{M}(\mathcal{I}[\ell+1])_y = (1)$, set $b_{\ell+1} := \operatorname{log-ord}_y(\mathcal{I}[\ell+1]) \in \mathbb{N}_{\geqslant 1}$, and define $x_{\ell+1}$ to be a lifting to $\mathscr{O}_Y$ of the maximal contact element of $\mathcal{I}[\ell+1]$ at y.

This concludes the induction. Although different choices of ordinary parameters x_i can be made above, the next lemma shows that the b_i and Q are well defined.

Lemma 6.1. *The b_i and Q are independent of the choices of ordinary parameters x_i above.*

Proof. We proceed by induction on k, the number of b_i. The case $k = 0$ occurs if and only if $\mathscr{M}(\mathcal{I})_y \neq (1)$, in which case there are no b_i and the definition of Q does not require choices. Henceforth, consider $k \geqslant 1$ (that is, $\operatorname{log-ord}_y(\mathcal{I}) < \infty$). Evidently, the integer $b_1 = \operatorname{log-ord}_y(\mathcal{I})$ requires no choices. Next, suppose that we are presented with two choices of maximal contact elements x and x' of $\mathcal{I}$ at y. We can replace Y with a neighbourhood of y so that $\max \operatorname{log-ord}(\mathcal{I}) = b_1$; then $\mathscr{C}(\mathcal{I}, b_1)$ is MC-invariant (see Lemma 5.4(v)), and x and x' are still maximal contact elements of $\mathscr{C}(\mathcal{I}, b_1)$ at y (see Corollary 5.5). Therefore, we can apply Theorem 5.1 to $\mathscr{C}(\mathcal{I}, b_1)$: we get strict and étale morphisms $\phi_{x,x'} \colon \widetilde{U} \rightrightarrows Y$ and a point $\widetilde{y} \in \widetilde{U}$ such that $\phi_x(\widetilde{y}) = y = \phi_{x'}(\widetilde{y})$. Moreover, $\phi_x^*(\mathscr{C}(\mathcal{I}, b_1)) = \phi_{x'}^*(\mathscr{C}(\mathcal{I}, b_1))$ (call this ideal $\widetilde{\mathcal{I}}$) and $z = \phi_x^*(x) = \phi_{x'}^*(x') \in \widetilde{I}$. Letting $\mathcal{I}[2] = \mathscr{C}(\mathcal{I}, b_1)|_{V(x)}$ and $\mathcal{I}[2'] = \mathscr{C}(\mathcal{I}, b_1)|_{V(x')}$, we have

$$\phi_x^*(\mathcal{I}[2]) = \widetilde{\mathcal{I}}|_{V(z)} = \phi_{x'}^*(\mathcal{I}[2'])\,. \tag{6.1.1}$$

If $k = 1$, we are in case A above. By [Ogu18, Proposition IV.3.1.6] and Lemma B.13(iii),

$$\phi_x^*\big(\mathscr{M}(\mathcal{I}[2])\big) = \mathscr{M}\big(\widetilde{\mathcal{I}}|_{V(z)}\big) = \phi_{x'}^*\big(\mathscr{M}(\mathcal{I}[2'])\big)\,. \tag{6.1.2}$$

Since ϕ_x is strict, $\phi_x^\flat\colon \phi_x^*(\mathscr{M}_Y) \to \mathscr{M}_{\widetilde{U}}$ is an isomorphism. We therefore get isomorphisms

$$\overline{\mathscr{M}}_{V(x),y} \xleftarrow{\simeq} \overline{\mathscr{M}}_{Y,y} \xrightarrow{\simeq} \overline{\phi_x^*(\mathscr{M}_Y)}_{\widetilde{y}} \xrightarrow{\simeq} \overline{\mathscr{M}}_{\widetilde{U},\widetilde{y}}\,,$$

which map $\overline{\alpha_{V(x)}^{-1}\big(\mathscr{M}(\mathcal{I}[2])\big)_y}$ on the left, isomorphically, onto $\overline{\alpha_{\widetilde{U}}^{-1}\big(\phi_x^*\big(\mathscr{M}(\mathcal{I}[2])\big)\big)_{\widetilde{y}}}$ on the right. The same statement holds with $V(x)$ replaced by $V(x')$, ϕ_x replaced by $\phi_{x'}$, and $\mathcal{I}[2]$ replaced by $\mathcal{I}[2']$. Combining this and (6.1.2), one concludes that Q is also independent of choices.

On the other hand, if $k \geqslant 2$, we are in case B above. Then (6.1.1) implies

$$\text{log-ord}_y(\mathcal{I}[2]) = \text{log-ord}_{\widetilde{y}}\,\big(\widetilde{\mathcal{I}}|_{V(z)}\big) = \text{log-ord}_y(\mathcal{I}[2'])\,.$$

Thus, b_2 is independent of choices. By the induction hypothesis, the remaining $b_3, b_4, \ldots$ and Q are independent of choices. □

We are now ready to define the key invariant associated with an ideal at a point.

Definition 6.2 (Invariant of an ideal at a point). Let $\mathcal{I}$ be an ideal on Y, and fix $y \in Y$. The (logarithmic) *invariant* of $\mathcal{I}$ at y is defined as

$$\operatorname{inv}_y(\mathcal{I}) := \begin{cases} \left(b_1, \dfrac{b_2}{(b_1-1)!}, \dfrac{b_3}{(b_1-1)!\cdot(b_2-1)!}, \ldots, \dfrac{b_k}{\prod_{i=1}^{k-1}(b_i-1)!}\right) & \text{if } Q = \emptyset\,, \\ \left(b_1, \dfrac{b_2}{(b_1-1)!}, \dfrac{b_3}{(b_1-1)!\cdot(b_2-1)!}, \ldots, \dfrac{b_k}{\prod_{i=1}^{k-1}(b_i-1)!}, \infty\right) & \text{if } Q \neq \emptyset\,, \end{cases}$$

where $(b_1, \ldots, b_k)$ and Q are defined for $\mathcal{I}$ at y as before. We will denote the finite entries of $\operatorname{inv}_y(\mathcal{I})$ by a_i so, in particular, $a_1 = b_1$. We also set $\max\operatorname{inv}(\mathcal{I}) := \max_{y\in Y}\operatorname{inv}_y(\mathcal{I})$.

Observe that $\operatorname{inv}_y(\mathcal{I})$ is the empty sequence () if and only if $\mathcal{I}_y = 0$ (that is, $y \notin \operatorname{Supp}(\mathcal{I})$). Moreover, $\operatorname{inv}_y(\mathcal{I}) = (0)$ if and only if $\mathcal{I}_y = (1)$ (that is, $y \notin V(\mathcal{I})$), while $\operatorname{inv}_y(\mathcal{I}) = (a_1)$ for an integer $a_1 \geqslant 1$ if and only if $\mathcal{I}_y = (x_1^{a_1})$. Finally, $\operatorname{inv}_y(\mathcal{I}) = (\infty)$ if and only if $\mathscr{M}(\mathcal{I}_y) \neq (1)$ (that is, $y \in V(\mathscr{M}(\mathcal{I}))$).

Lemma 6.3. *The invariant* inv_y *satisfies the following properties:*

(i) *If* $\text{log-ord}_y(\mathcal{I}) = a_1 < \infty$ *and* x_1 *is a maximal contact element of* $\mathcal{I}$ *at* y, *then* $\operatorname{inv}_y(\mathcal{I})$ *is the concatenation* $\big(a_1, \operatorname{inv}_y(\mathscr{C}(\mathcal{I}, a_1)|_{x_1=0})/(a_1-1)!\big)$.

(ii) *The invariant* $\operatorname{inv}_y(\mathcal{I})$ *is upper semi-continuous on* Y (*with respect to the lexicographic order which was described in Section* 1.1).

(iii) *If* $\widetilde{Y} \to Y$ *is a logarithmically smooth morphism of strict toroidal* $\Bbbk$*-schemes which maps* $\widetilde{y} \in \widetilde{Y}$ *to* $y \in Y$, *then* $\operatorname{inv}_{\widetilde{y}}(\mathcal{I}\mathscr{O}_{\widetilde{Y}}) = \operatorname{inv}_y(\mathcal{I})$. *If* $\widetilde{Y} \to Y$ *is moreover surjective, then* $\max\operatorname{inv}(\mathcal{I}\mathscr{O}_{\widetilde{Y}}) = \max\operatorname{inv}(\mathcal{I})$.

Proof. Part (i) is evident from Definition 6.2, while part (iii) follows from Lemma B.15(iv) and Remark 5.3. For part (ii), fix some non-decreasing truncated sequence of non-negative rational numbers $(a_1, \ldots, a_k)$ whose last entry could possibly be ∞. We need to show that the locus Z of points $y \in Y$ such that $\operatorname{inv}_y(\mathcal{I}) \geqslant (a_1, \ldots, a_k)$ is closed in Y. We do so by induction on k. If $k = 0$, then $Z = Y \setminus \operatorname{Supp}(\mathcal{I})$. Since Y is a disjoint union of its irreducible components (Remark B.5(iii)), the support $\operatorname{Supp}(\mathcal{I})$ is a union of some of the irreducible components of Y, whence it is open (and closed) in Y, so Z is closed in Y. Now assume $k \geqslant 1$. If $a_1 = 0$, then $Z = Y$. If $a_1 \in \mathbb{Q}_{\geqslant 0} \setminus \mathbb{Z}_{\geqslant 0}$, then $Z = V\big(\mathscr{D}^{\leqslant \lceil a_1 \rceil - 1}(\mathcal{I})\big)$ by Lemma B.15(i). If $a_1 = \infty$, then $k = 1$ and $Z = V(\mathscr{M}(\mathcal{I}))$ by Lemma B.15(ii). Finally, consider $a_1 \in \mathbb{Z}_{>0}$. By Lemma B.15(i), the locus W of points $y \in Y$ with $\text{log-ord}_y(\mathcal{I}) > a_1$ is $V\big(\mathscr{D}^{\leqslant a_1}(\mathcal{I})\big)$. By part (i) of this lemma and the induction hypothesis, the locus W' of points $y \in V(x_1)$ such that $\text{inv}_y(\mathscr{C}(\mathcal{I}, a_1)|_{x_1=0}) \geqslant (a_1 - 1)! \cdot (a_2, \dots, a_k)$ is closed in $V(x_1)$ (and hence in Y). Note that if $y \in W'$, then $\text{log-ord}_y(\mathcal{I}) \geqslant a_1$ (if not, the stalk of $\mathscr{C}(\mathcal{I}, a_1)$ at y is (1), whence $\text{inv}_y(\mathscr{C}(\mathcal{I}, a_1)|_{x_1=0}) = (0) < (a_1 - 1)! \cdot (a_2, \dots, a_k)$). By part (i) of this lemma again, $Z = W \cup W'$, so Z is closed in Y, as desired. $\square$

Definition 6.4 (Toroidal centre associated with an ideal at a point). Let $\mathcal{I}$ be a ideal on Y, and fix a $y \in Y$ such that $\mathcal{I}_y \neq 0$. For a choice of ordinary parameters $x_1, \dots, x_k$ associated with $\mathcal{I}$ at y as above, the corresponding toroidal centre $\mathscr{J}^{(y)}(\mathcal{I})$ at y associated with $\mathcal{I}$ is defined as

$$\mathscr{J}^{(y)}(\mathcal{I}) := \begin{cases} \left(x_1^{b_1}, x_2^{\frac{b_2}{(b_1-1)!}}, x_3^{\frac{b_3}{(b_1-1)!\cdot(b_2-1)!}}, \dots, x_k^{\frac{b_k}{\prod_{i=1}^{k-1}(b_i-1)!}} \right) & \text{if } Q = \emptyset\,, \\ \left(x_1^{b_1}, x_2^{\frac{b_2}{(b_1-1)!}}, x_3^{\frac{b_3}{(b_1-1)!\cdot(b_2-1)!}}, \dots, x_k^{\frac{b_k}{\prod_{i=1}^{k-1}(b_i-1)!}}, (Q \subset M)^{\frac{1}{\prod_{i=1}^{k}(b_i-1)!}} \right) & \text{if } Q \neq \emptyset\,, \end{cases}$$

where $(b_1, \dots, b_k)$ and Q are defined for $\mathcal{I}$ at y as before. (We use the convention that $x_1^0 := 1$.) Observe that it has invariant equal to $\text{inv}_y(\mathcal{I})$. For the remainder of this paper, we denote $\mathscr{J}^{(y)}(\mathcal{I})$ by $\big(x_1^{a_1}, \dots, x_k^{a_k}, (Q \subset M)^{1/d}\big)$, where Q could be $\emptyset$ and d is always the positive integer $\prod_{i=1}^{k}(b_i - 1)!$.

We will show later in Corollary 6.6 that $\mathscr{J}^{(y)}(\mathcal{I})$ does not actually depend on the choice of ordinary parameters $x_1, \dots, x_k$ associated with $\mathcal{I}$ at y, which justifies the notation.

6.2 The associated toroidal centre is uniquely admissible

The goal of this subsection is to show the following.

Theorem 6.5 (Unique admissibility). *Let $\mathcal{I}$ be an ideal on Y, and fix a $y \in Y$ such that $\mathcal{I}_y \neq 0$.*

(i) *For any choice of ordinary parameters x_i at y, the toroidal centre $\mathscr{J}^{(y)}(\mathcal{I})$ at y is $\mathcal{I}$-admissible.*

(ii) *Every $\mathcal{I}$-admissible toroidal centre $\mathscr{J}^{(y)}$ at y has invariant $\text{inv}\big(\mathscr{J}^{(y)}\big) \leqslant \text{inv}_y(\mathcal{I})$ (where $<$ refers to the lexicographic order which was described in Section 1.1). Consequently, we have the characterization*

$$\text{inv}_y(\mathcal{I}) = \max_{\mathscr{J}^{(y)}\ \mathcal{I}\text{-admissible}} \text{inv}\big(\mathscr{J}^{(y)}\big)\,.$$

(iii) *Let $\mathscr{J}^{(y)} = \big((x_1')^{a_1}, \dots, (x_k')^{a_k}, (Q' \subset M)^r\big)$ be a $\mathcal{I}$-admissible toroidal centre at y, with invariant $\text{inv}\big(\mathscr{J}^{(y)}\big) = \text{inv}_y(\mathcal{I})$. For any choice of ordinary parameters $x_1, \dots, x_k$ associated with $\mathcal{I}$ at y, we have $\mathscr{J}^{(y)} = \big(x_1^{a_1}, \dots, x_k^{a_k}, (Q' \subset M)^r\big)$ after possibly passing to a smaller affine neighbourhood of y on which $\mathscr{J}^{(y)}$ is defined.*

Before proving the theorem, let us note an immediate consequence of Theorem 6.5(iii).

Corollary 6.6. *Let $\mathcal{I}$ be an ideal on Y, and fix $y \in Y$ such that $\mathcal{I}_y \neq 0$. Then the stalk of $\mathscr{J}^{(y)}(\mathcal{I})$ at y does not depend on the choice of ordinary parameters x_i associated with $\mathcal{I}$ at y.*

We shall divide the proof of Theorem 6.5 into two parts. In the proof of both parts, we will need the following lemma for the induction step.

Lemma 6.7. *Let $\mathcal{I}$ be an ideal on Y, and let $y \in Y$ be such that $\mathcal{I}_y \neq 0$. Let $\mathscr{J}^{(y)} = \big(x_1^{a_1}, \dots, x_k^{a_k}, (Q \subset M)^r\big)$ be a toroidal centre at y, where $k \geqslant 1$ and $a_1 \geqslant 1$ is an integer.*

(i) *Suppose that $\mathscr{J}^{(y)}$ is $\mathcal{I}$-admissible. Then for any integer $1 \leqslant m \leqslant a_1$ and $s \geqslant 1$, the toroidal centre $\big(\mathscr{J}^{(y)}\big)^{s/m}$ is $\mathscr{G}_s(\mathcal{I}, m)$-admissible.*

(ii) *Conversely, if $\big(\mathscr{J}^{(y)}\big)^{(m-1)!}$ is $\mathscr{C}(\mathcal{I}, m)$-admissible for some integer $1 \leqslant m \leqslant a_1$, then $\mathscr{J}^{(y)}$ is $\mathcal{I}$-admissible.*

In particular, for any integer $1 \leqslant m \leqslant a_1$, the toroidal centre $\mathscr{J}^{(y)}$ is $\mathcal{I}$-admissible if and only if $\big(\mathscr{J}^{(y)}\big)^{(m-1)!}$ is $\mathscr{C}(\mathcal{I}, m)$-admissible.

Proof. If $\mathscr{J}^{(y)}$ is $\mathcal{I}$-admissible, iterating Proposition 4.8(i) tells us that for all $0 \leqslant j \leqslant m-1$, the toroidal centre $\big(\mathscr{J}^{(y)}\big)^{(a_1-j)/a_1}$ is $\mathscr{D}^{\leqslant j}(\mathcal{I})$-admissible. For natural numbers $c_0, \dots, c_{m-1}$, Lemma 3.10(ii) implies that $\big(\mathscr{J}^{(y)}\big)^{\sum_{j=0}^{m-1} ((a_1-j)/a_1)c_j}$ is $\big(\prod_{j=0}^{m-1} (\mathscr{D}^{\leqslant j}(\mathcal{I}))^{c_j}\big)$-admissible. Since $m \leqslant a_1$, we have $(m-j)/m \leqslant (a_1-j)/a_1$, so $\big(\mathscr{J}^{(y)}\big)^{\sum_{j=0}^{m-1} ((m-j)/m)c_j}$ is $\big(\prod_{j=0}^{m-1} \big(\mathscr{D}^{\leqslant j}(\mathcal{I})\big)^{c_j}\big)$-admissible. For $(c_0, \dots, c_{m-1}) \in \mathbb{N}^m$ satisfying $\sum_{j=0}^{m-1} (m-j)c_j \geqslant s$, we have the inequality $\sum_{j=0}^{m-1} \big((m-j)/m\big)c_j \geqslant s/m$, and hence $\big(\mathscr{J}^{(y)}\big)^{s/m}$ is $\big(\prod_{j=0}^{m-1} \big(\mathscr{D}^{\leqslant j}(\mathcal{I})\big)^{c_j}\big)$-admissible. By Lemma 3.10(i), the toroidal centre $\big(\mathscr{J}^{(y)}\big)^{s/m}$ is $\mathscr{G}_s(\mathcal{I}, m)$-admissible. This proves part (i).

Conversely, if $\big(\mathscr{J}^{(y)}\big)^{(m-1)!}$ is $\mathscr{C}(\mathcal{I}, m)$-admissible, then $\big(\mathscr{J}^{(y)}\big)^{(m-1)!}$ is $\mathcal{I}^{(m-1)!}$-admissible. By Lemma 3.10(iii), the toroidal centre $\mathscr{J}^{(y)}$ is $\mathcal{I}$-admissible. This proves part (ii). □

We can now prove Theorem 6.5(i).

Proof of Theorem 6.5(i). Write $\mathscr{J}^{(y)} = \mathscr{J}^{(y)}(\mathcal{I})$ in this proof. We proceed by induction on the length L of $\operatorname{inv}_y(\mathcal{I}) = \operatorname{inv}\big(\mathscr{J}^{(y)}\big)$. The base case is $L = 1$. The case $\operatorname{inv}\big(\mathscr{J}^{(y)}\big) = (a_1)$, with $a_1 < \infty$, is evident. If $\operatorname{inv}\big(\mathscr{J}^{(y)}\big) = (\infty)$, then $\mathscr{J}^{(y)}$ is $\mathcal{I}$-admissible because $\mathscr{M}(\mathcal{I})_y \supset \mathcal{I}_y$. Henceforth, assume $L \geqslant 2$ so, in particular, the first entry in $\operatorname{inv}\big(\mathscr{J}^{(y)}\big)$ is an integer $a_1 \geqslant 1$. By Lemma 6.7, we may replace $\mathcal{I}$ with $\mathscr{C} = \mathscr{C}(\mathcal{I}, a_1)$ and replace $\mathscr{J}^{(y)}$ with $\big(\mathscr{J}^{(y)}\big)^{(a_1-1)!}$. By Lemma 3.9, we may pass to completion at y and instead show that $\big(\widehat{\mathscr{J}^{(y)}}\big)^{(a_1-1)!}$ is $\widehat{\mathscr{C}}$-admissible. By Proposition 5.7, we can decompose $\widehat{\mathscr{C}}$ as

$$\widehat{\mathscr{C}} = \big(x_1^{a_1!}\big) + \big(x_1^{a_1!-1}\big)\widetilde{\mathscr{C}}_1 + \cdots + (x_1)\widetilde{\mathscr{C}}_{a_1!-1} + \widetilde{\mathscr{C}}_{a_1!}\,, \quad \text{where } \widetilde{\mathscr{C}}_{a_1!-i} = \widetilde{\mathscr{C}}_{a_1!-i}(\mathcal{I}, a_1)\,,$$

and therefore by Lemma 3.10(i), it remains to show that $\big(\widehat{\mathscr{J}^{(y)}}\big)^{(a_1-1)!}$ is $\big((x_1^i)\widetilde{\mathscr{C}}_{a_1!-i}\big)$-admissible for $0 \leqslant i \leqslant a_1!$. The case $i = a_1!$ is straightforward.

For the remaining i with $0 \leqslant i < a_1!$, let H denote the hypersurface of maximal contact $x_1 = 0$, and let $\mathscr{J}_H^{(y)}$ be the restricted toroidal centre $\mathscr{J}^{(y)}|_{x_1=0}$. By Lemma 6.3(i), as well as the induction hypothesis (applied to $\mathscr{C}(\mathcal{I}, a_1)|_{x_1=0}$), the toroidal centre $\big(\mathscr{J}_H^{(y)}\big)^{(a_1-1)!}$ is $\mathscr{C}|_H$-admissible. Since (after restricting to a neighbourhood U of y on which $\max \operatorname{log-ord}(\mathcal{I}|_U) = a_1$), the ideal $\mathscr{C}$ is $\mathscr{D}$-balanced by Lemma 5.4(viii), we have $\big(\mathscr{D}^{\leqslant i}(\mathscr{C})|_H\big)^{a_1!} \subset (\mathscr{C}|_H)^{a_1!-i}$. By Lemma 3.10(iii), we see that $\big(\mathscr{J}_H^{(y)}\big)^{(a_1-1)! \cdot (a_1!-i)}$ is $(\mathscr{C}|_H)^{(a_1!-i)}$-admissible and hence $\big(\mathscr{D}^{\leqslant i}(\mathscr{C})|_H\big)^{a_1!}$-admissible. Consequently, Lemma 3.11 implies that $\big(\mathscr{J}^{(y)}\big)^{(a_1-1)! \cdot (a_1!-i)}$ is $\big(\mathscr{D}^{\leqslant i}(\mathscr{C})|_H \mathscr{O}_Y\big)^{a_1!}$-admissible. By a repeated application of Proposition 4.8(ii), we see that $\left(\mathscr{J}^{(y)}\right)^{(a_1-1)!\cdot a_1!}$ is $\left((x_1^{ia_1!})\left(\mathscr{D}^{\leqslant i}(\mathscr{C})|_H\mathscr{O}_Y\right)^{a_1!}\right)$-admissible. By another application of Lemma 3.10(iii), we see that the toroidal centre $\left(\mathscr{J}^{(y)}\right)^{(a_1-1)!}$ is $\left((x_1^i)\left(\mathscr{D}^{\leqslant i}(\mathscr{C})|_H\mathscr{O}_Y\right)\right)$-admissible. Recall that we have $\mathscr{D}^{\leqslant i}(\mathscr{C}) = \mathscr{G}_{a_1!-i}$ by Lemma 5.4(iii). Hence, passing to completion at y, we obtain that $\left(\widehat{\mathscr{J}}^{(y)}\right)^{(a_1-1)!}$ is $\left((x_1^i)\widetilde{\mathscr{C}}_{a_1!-i}\right)$-admissible. This completes the proof. □

Next, we prove the remaining two parts of Theorem 6.5. The proof of these two parts should be compared to the proof of Lemma 3.2 in Section 4.6.

Proof of Theorem 6.5(ii),(iii). We prove both parts by induction on the length L of $\operatorname{inv}_y(\mathcal{I})$. Consider the base case $L = 1$. If $\operatorname{inv}_y(\mathcal{I}) = (\infty)$, there is nothing to show. On the other hand, if $\operatorname{inv}_y(\mathcal{I}) = (a_1)$ with $a_1 < \infty$, then $\mathcal{I}_y = \left(x_1^{a_1}\right)$ for some ordinary parameter x_1 at y, and both parts are immediate. Henceforth, assume $L \geqslant 2$. Let $\mathscr{J}^{(y)} = \left((x_1')^{b_1}, \dots, (x_\ell')^{b_\ell}, (Q' \subset M)^r\right)$ be an $\mathcal{I}$-admissible toroidal centre at y. Since $L \geqslant 2$, the first entry in $\operatorname{inv}_y(\mathcal{I})$ is an integer $a_1 \geqslant 1$, where $a_1 = \operatorname{log-ord}_y(\mathcal{I}) < \infty$. Consequently, $\ell \geqslant 1$. Applying Corollary 4.11, we find $b_1 \leqslant a_1$. If $b_1 < a_1$, then $\operatorname{inv}\left(\mathscr{J}^{(y)}\right) \leqslant \operatorname{inv}_y(\mathcal{I})$ follows. Thus, assume $b_1 = a_1 < \infty$ for the remainder of this proof.

Let x_1 be a maximal contact element for $\mathcal{I}$ at y. Applying Proposition 4.8(i) repeatedly, we see that $\left(\mathscr{J}^{(y)}\right)^{1/a_1} = \left(x_1', (x_2')^{b_2/a_1}, \dots, (x_\ell')^{b_\ell/a_1}, (Q' \subset M)^{r/a_1}\right)$ is $\mathscr{D}^{\leqslant a_1-1}(\mathcal{I})$-admissible and hence (x_1)-admissible. Extending $x_1', \dots, x_\ell'$ to a system of ordinary parameters $x_1', \dots, x_n'$ at y and passing to completion at y, we can write the image of x_1 under $\mathscr{O}_{Y,y} \twoheadrightarrow \mathscr{O}_{\mathfrak{s}_y,y} \to \widehat{\mathscr{O}}_{\mathfrak{s}_y,y} \simeq \kappa(y)[\![x_1', \dots, x_n']\!]$ as $\sum_{\vec{\alpha}} c_{\vec{\alpha}}(x_1')^{\alpha_1}\cdots(x_n')^{\alpha_n}$ for some $c_{\vec{\alpha}} \in \kappa(y)$. By Lemma 4.12, we have $\sum_{i=1}^k \alpha_i/(b_i/a_1) \geqslant 1$ whenever $c_{\vec{\alpha}} \neq 0$. Consequently, if we let $\ell_0 = \max\{1 \leqslant i \leqslant \ell \colon b_i = a_1\} \geqslant 1$, then the image of x_1 in $\mathscr{O}_{\mathfrak{s}_y,y}$ lies in $(x_1', \dots, x_{\ell_0}') + \mathfrak{m}_{\mathfrak{s}_y,y}^2$, where $\mathfrak{m}_{\mathfrak{s}_y,y}^2$ is the maximal ideal of $\mathscr{O}_{\mathfrak{s}_y,y}$. Therefore, after possibly reordering $x_1', \dots, x_{\ell_0}'$, we may replace x_1' with an x_1 such that $(x_1, x_2', \dots, x_n')$ is a system of ordinary parameters at y. Note that any such reordering does not mess up the presentation of $\mathscr{J}^{(y)} = \left((x_1')^{b_1}, \dots, (x_\ell')^{b_\ell}, (Q \subset M)^r\right)$ since $a_1 = b_1 = \cdots = b_{\ell_0}$. Applying Lemma 4.13 gives

$$\mathscr{J}^{(y)} = \left(x_1^{a_1}, (x_2')^{b_2}, \dots, (x_k')^{b_\ell}, (Q' \subset M)^r\right).$$

The next natural step is to pass to the induction step.

Let $\mathscr{C} = \mathscr{C}(\mathcal{I}, a_1)$. By Lemma 6.7, the toroidal centre $\left(\mathscr{J}^{(y)}\right)^{(a_1-1)!}$ is $\mathscr{C}$-admissible. Let H denote the hypersurface of maximal contact given by $x_1 = 0$, and let $\mathscr{J}_H^{(y)}$ denote the restricted toroidal centre $\mathscr{J}^{(y)}|_{x_1=0}$. Then $\left(\mathscr{J}_H^{(y)}\right)^{(a_1-1)!}$ is $\mathscr{C}|_H$-admissible. By the induction hypothesis (for Theorem 6.5(ii)) applied to $\mathscr{C}|_H$, we see that $\operatorname{inv}\left(\mathscr{J}_H^{(y)}\right)^{(a_1-1)!} \leqslant \operatorname{inv}_y(\mathscr{C}|_H)$, so $\operatorname{inv}\left(\mathscr{J}_H^{(y)}\right) \leqslant 1/(a_1-1)! \cdot \operatorname{inv}_y(\mathscr{C}|_H)$. Applying Lemma 6.3(i), we obtain $\operatorname{inv}\left(\mathscr{J}^{(y)}\right) = \left(a_1, \operatorname{inv}\left(\mathscr{J}_H^{(y)}\right)\right) \leqslant \left(a_1, \operatorname{inv}_y(\mathscr{C}|_H)/(a_1-1)!\right) = \operatorname{inv}_y(\mathcal{I})$, proving Theorem 6.5(ii).

If $\operatorname{inv}\left(\mathscr{J}^{(y)}\right) = \operatorname{inv}_y(\mathcal{I})$, then $\operatorname{inv}\left(\mathscr{J}_H^{(y)}\right)^{(a_1-1)!} = \operatorname{inv}_y(\mathscr{C}|_H)$, so that $\ell = k$ and $b_i = a_i$ for $1 \leqslant i \leqslant k = \ell$. Let $x_1, x_2, \dots, x_k$ be ordinary parameters associated with $\mathcal{I}$ at y (where x_1 was arbitrarily chosen earlier), as in Section 6.1. By the induction hypothesis (for Theorem 6.5(iii)) applied to the $\mathscr{C}|_H$-admissible toroidal centre $\left(\mathscr{J}_H^{(y)}\right)^{(a_1-1)!}$ at y, we have

$$\mathscr{J}_H^{(y)} = \left((x_2')^{a_2}, \dots, (x_k')^{a_k}, (Q' \subset M)^r\right) = \left(x_2^{a_2}, \dots, x_k^{a_k}, (Q' \subset M)^r\right).$$

In the above expression, x'_i is more precisely the reduction of x'_i modulo $x_1 = 0$ and, similarly, x_i is the reduction of x_i modulo $x_1 = 0$. We claim that this implies

$$\mathscr{J}^{(y)} = \left(x_1^{a_1}, (x'_2)^{a_2}, \ldots, (x'_k)^{a_k}, (Q' \subset M)^r\right) = \left(x_1^{a_1}, x_2^{a_2}, \ldots, x_k^{a_k}, (Q' \subset M)^r\right).$$

This follows by the same method illustrated in Remark 3.5, that is, by checking that both sides are equal as idealistic $\mathbb{Q}$-exponents and hence as integrally closed Rees algebras. This proves Theorem 6.5(iii). □

Corollary 6.8. *Let $\mathcal{I}$ be an ideal on Y, and fix a $y \in Y$ such that $\mathcal{I}_y \neq 0$.*

(i) *We have $\operatorname{inv}_y(\mathcal{I}^m) = m \cdot \operatorname{inv}_y(\mathcal{I})$.*

(ii) *Assume $\operatorname{log-ord}_y(\mathcal{I}) = a_1 < \infty$. For any integer $1 \leqslant m \leqslant a_1$, we have $\operatorname{inv}_y(\mathscr{C}(\mathcal{I}, m)) = (m-1)! \cdot \operatorname{inv}_y(\mathcal{I})$.*

Proof. Apply Theorem 6.5(ii) in conjunction with Lemmas 3.10(iii) and 6.7. □

6.3 Compatibility of associated toroidal centres

In Section 6.1, we defined the toroidal centre associated with an ideal $\mathcal{I} \subset \mathscr{O}_Y$ at a point $y \in Y$. The next theorem glues toroidal centres at points $y \in Y$ with invariant $\operatorname{inv}_y(\mathcal{I}) = \max \operatorname{inv}(\mathcal{I})$.

Theorem 6.9 (gluing). *Let $\mathcal{I}$ be a nowhere zero ideal on Y, and define*

$$\max \operatorname{inv}(\mathcal{I}) := \max_{y \in Y} \operatorname{inv}_y(\mathcal{I}).$$

There exists a unique $\mathcal{I}$-admissible toroidal centre $\mathscr{J} = \mathscr{J}(\mathcal{I})$ on Y such that for all $y \in Y$, there exists an open affine neighbourhood U_y of y on which the following hold:

(i) *If $\operatorname{inv}_y(\mathcal{I}) = \max \operatorname{inv}_y(\mathcal{I})$, then $\mathscr{J}|_{U_y}$ is the toroidal centre $\mathscr{J}^{(y)}(\mathcal{I})$ at y.*

(ii) *If $\operatorname{inv}_y(\mathcal{I}) < \max \operatorname{inv}(\mathcal{I})$, then $\mathscr{J}|_{U_y} = \mathscr{O}_Y[T]|_{U_y}$.*

Definition 6.10 (Toroidal centre associated with an ideal). Let $\mathcal{I}$ be a nowhere zero ideal on Y. The toroidal centre associated with $\mathcal{I}$ is $\mathscr{J}(\mathcal{I})$ in Theorem 6.9.

Proof of Theorem 6.9. Since $\operatorname{inv}_y(\mathcal{I})$ is upper semi-continuous (Lemma 6.3(ii)), the locus V of points $y \in Y$ where $\operatorname{inv}_y(\mathcal{I}) < \max \operatorname{inv}(\mathcal{I})$ is open. We claim that we can glue

- $\mathscr{O}_Y[T]|_V$
- for each $y \in Y$ with $\operatorname{inv}_y(\mathcal{I}) = \max \operatorname{inv}(\mathcal{I})$, the toroidal centre $\mathscr{J}^{(y)}(\mathcal{I})$ restricted to an appropriately chosen open affine neighbourhood U_y of y

to obtain a toroidal centre $\mathscr{J}$ on Y. This toroidal centre would have the desired properties.

First fix $y \in Y$ with $\operatorname{inv}_y(\mathcal{I}) = \max \operatorname{inv}(\mathcal{I})$. Let $\mathscr{J}^{(y)}(\mathcal{I}) = \left(x_1^{a_1}, \ldots, x_k^{a_k}, (Q \subset M)^{1/d}\right)$ be defined on an open affine neighbourhood U_y of y in Y. Recall that the x_i are choices of ordinary parameters associated with $\mathcal{I}$ at y and Q is the preimage of $\overline{\alpha^{-1}_{V(x_1,\ldots,x_k)}(\mathscr{M}(\mathcal{I}[k+1]))_y}$ under the canonical isomorphism $M = \overline{\mathscr{M}}_{Y,y} \xrightarrow{\simeq} \overline{\mathscr{M}}_{V(x_1,\ldots,x_k),y}$, where $\mathcal{I}[k+1]$ is the ideal on $V(x_1, \ldots, x_k)$ which was defined inductively in Section 6.1. Moreover, we have a chart $\beta \colon M \to H^0(U_y, \mathscr{M}_Y|_{U_y})$ which is neat at y. Then our claim in the preceding paragraph amounts to showing that after possibly shrinking U_y, one has, for each $y' \in U_y$, the following statements:

(a) If $\operatorname{inv}_{y'}(\mathcal{I}) = \max \operatorname{inv}(\mathcal{I})$, then the stalks of $\mathscr{J}^{(y)}(\mathcal{I})$ and $\mathscr{J}^{(y')}(\mathcal{I})$ at y' coincide.

(b) If $\operatorname{inv}_{y'}(\mathcal{I}) < \max \operatorname{inv}(\mathcal{I})$, then the stalk of $\mathscr{J}^{(y)}(\mathcal{I})$ at y' is $\mathscr{O}_{Y,y'}[T]$.

For part (a), the parameters $x_1, \dots, x_k$ are also ordinary parameters associated with $\mathcal{I}$ at y'. By unique admissibility (Theorem 6.5(iii)), we have $\mathscr{J}^{(y')}(\mathcal{I}) = \big(x_1^{a_1}, \dots, x_k^{a_k}, (Q' \subset M')^{1/d}\big)$, and Lemma 6.1 says that Q' is equal to the preimage of $\overline{\alpha^{-1}_{V(x_1,\dots,x_k)}(\mathscr{M}(\mathcal{I}[k+1]))_{y'}}$ under the canonical isomorphism $M' = \overline{\mathscr{M}}_{Y,y'} \xrightarrow{\simeq} \overline{\mathscr{M}}_{V(x_1,\dots,x_k),y'}$. On the other hand, Lemma 6.11(ii) says that the ideal of $\mathscr{M}_{V(x_1,\dots,x_k)}|_{U_y \cap V(x_1,\dots,x_k)}$ generated by the image of $\overline{\alpha^{-1}_{V(x_1,\dots,x_k)}(\mathscr{M}(\mathcal{I}[k+1]))_y}$ under the chart $\overline{\beta}$ defined by

$$\begin{array}{ccc} M = \overline{\mathscr{M}}_{Y,y} & \xrightarrow{\quad\beta\quad} & H^0(U_y, \mathscr{M}_Y|_{U_y}) \\ \downarrow{\scriptstyle\simeq} & & \downarrow \\ \overline{\mathscr{M}}_{V(x_1,\dots,x_k),y} & \xrightarrow{\overline{\beta}} & H^0(U_y \cap V(x_1,\dots,x_k), \mathscr{M}_{V(x_1,\dots,x_k)}|_{U_y \cap V(x_1,\dots,x_k)}) \end{array}$$

is equal to $\alpha^{-1}_{V(x_1,\dots,x_k)}(\mathscr{M}(\mathcal{I}[k+1]))|_{U_y \cap V(x_1,\dots,x_k)}$, from which part (a) follows.

For part (b), let $\operatorname{inv}_{y'}(\mathcal{I}) = (a'_1, \dots, a'_\ell) < \max \operatorname{inv}(\mathcal{I})$. First consider the case when there exists a j with $1 \leqslant j \leqslant k$ such that $a'_j < a_j$. Let $j_0 = \min\{1 \leqslant j \leqslant k \colon a'_j < a_j\}$. By Lemma B.15(i) and unique admissibility (Theorem 6.5(iii)), we may adjust x_{j_0}, if needed, so that x_{j_0} is a unit in $\mathscr{O}_{Y,y'}$, which yields part (b) for this first case.

The other case occurs when $a'_i = a_i$ for all $1 \leqslant i \leqslant k$. In this case, $x_1, \dots, x_k$ are also ordinary parameters associated with $\mathcal{I}$ at y' (as in part (a)). Let us always rule out the case $Q = \emptyset$ (which occurs if and only if $\mathcal{I}[k+1]_y = 0$ or, equivalently, $\operatorname{inv}_y(\mathcal{I}) = (a_1, \dots, a_k)$), by shrinking U_y so that $\mathcal{I}[k+1]_{y'} = 0$ (and hence $\operatorname{inv}_{y'}(\mathcal{I}) = \operatorname{inv}_y(\mathcal{I})$) for every $y' \in U_y \cap V(x_1, \dots, x_k)$. On the other hand, if $Q' \neq \emptyset$, then $a'_{k+1} < \infty$, and Lemma B.15(i) implies $\mathscr{M}(\mathcal{I}[k+1])_{y'} = (1)$. Combining that with Lemma 6.11(ii) as in part (a) completes the proof for part (b) in the second case. □

Lemma 6.11. *Let Y be a fs Zariski logarithmic scheme, and let $\beta\colon M \to H^0(Y, \mathscr{M}_Y)$ be a chart for $\mathscr{M}_Y$ which is neat at some $y \in Y$ (so we shall identify $M = \overline{\mathscr{M}}_{Y,y}$ in the statements below). For an ideal $\mathcal{Q} \subset H^0(Y, \mathscr{M}_Y)$, we have*

- (i) *$\beta^{-1}(\mathcal{Q}) = \overline{\mathcal{Q}_y}$ (where the latter denotes the passage of the stalk of $\mathcal{Q}$ at y to $\overline{\mathscr{M}}_{Y,y}$);*
- (ii) *the image of $\overline{\mathcal{Q}_y} \subset \overline{\mathscr{M}}_{Y,y} = M$ under β generates $\mathcal{Q}$.*

In particular, $\mathcal{Q} = \emptyset$ if and only if $\overline{\mathcal{Q}}_y = \emptyset$.

Proof. For part (i), the composition $\gamma\colon \overline{\mathscr{M}}_{Y,y} \xrightarrow{\beta} H^0(Y, \mathscr{M}_Y) \to \mathscr{M}_{Y,y}$ defines a splitting $\mathscr{M}_{Y,y} \simeq \overline{\mathscr{M}}_{Y,y} \oplus \mathscr{O}^*_{Y,y}$. The splitting allows us to write every $m \in \mathscr{M}_{Y,y}$ as $(\overline{m}, u_m)$ for unique $\overline{m} \in \overline{\mathscr{M}}_{Y,y}$ and $u_m \in \mathscr{O}^*_{Y,y}$. Under this notation, we have $\beta^{-1}(\mathcal{Q}) = \gamma^{-1}(\mathcal{Q}_y) = \big\{\overline{m} \in \overline{\mathscr{M}}_{Y,y} \colon (\overline{m}, 0) \in \mathcal{Q}_y\big\} = \big\{\overline{m} \in \overline{\mathscr{M}}_{Y,y} \colon m = (\overline{m}, u_m) \in \mathcal{Q}_y\big\} = \overline{\mathcal{Q}_y}$, as desired.

For part (ii), the chart β factors as $M \overset{\iota}{\hookrightarrow} M \oplus \mathscr{O}^*_Y \overset{\pi}{\twoheadrightarrow} \mathscr{M}_Y$. It then suffices to show that $\iota\big(\overline{\mathcal{Q}_y}\big)$ generates $\pi^{-1}(\mathcal{Q})$. But part (i) implies $\overline{\mathcal{Q}_y} = \iota^{-1}\big(\pi^{-1}(\mathcal{Q})\big) = \big\{m \in M \colon (m, 0) = \iota(m) \in \pi^{-1}(\mathcal{Q})\big\}$; its image under ι evidently generates $\pi^{-1}(\mathcal{Q})$. □

6.4 The case of toroidal Deligne–Mumford stacks over $\Bbbk$

The goal of this section is to extend the definition of associated toroidal centres and associated invariants to toroidal $\Bbbk$-schemes or, more generally, *toroidal Deligne–Mumford stacks over* $\Bbbk$ (Definition B.16).

Lemma 6.12 (Functoriality of associated toroidal centres). *Let $f\colon \widetilde{Y} \to Y$ be a logarithmically smooth morphism of strict toroidal* $\Bbbk$*-schemes, which maps $\widetilde{y} \in Y$ to $y \in Y$. For an ideal $\mathcal{I}$ on Y satisfying $\mathcal{I}_y \neq 0$, we have $\mathscr{J}^{(y)}(\mathcal{I})\mathscr{O}_{\widetilde{Y}} = \mathscr{J}^{(\widetilde{y})}(\mathcal{I}\mathscr{O}_{\widetilde{Y}})$ on an affine open neighbourhood of $\widetilde{y}$. If $\mathcal{I}$ is a nowhere zero ideal on Y and f is moreover surjective, then $\mathscr{J}(\mathcal{I})\mathscr{O}_{\widetilde{Y}} = \mathscr{J}(\mathcal{I}\mathscr{O}_{\widetilde{Y}})$.*

Proof. For the first assertion, we may replace Y with an affine open neighbourhood of y on which $\mathscr{J}^{(y)}(\mathcal{I})$ is defined. Firstly observe that if $\text{log-ord}_y(\mathcal{I}) = \infty$, then $\mathscr{M}(\mathcal{I})\mathscr{O}_{\widetilde{Y}} = \mathscr{M}(\mathcal{I}\mathscr{O}_{\widetilde{Y}})$ (by Lemma B.13(iii)), and the lemma is immediate. On the other hand, if $\text{log-ord}_y(\mathcal{I}) = b_1 < \infty$, then any maximal contact element x_1 of $\mathcal{I}$ at y is also a maximal contact element of $\mathcal{I}\mathscr{O}_{\widetilde{Y}}$ at $\widetilde{y}$. Let $V_Y(x_1)$ (respectively, $V_{\widetilde{Y}}(x_1)$) be the hypersurface on Y (respectively, $\widetilde{Y}$) given by $x_1 = 0$. We restrict to the logarithmically smooth morphism $V_{\widetilde{Y}}(x_1) \to V_Y(x_1)$. By Remark 5.3, we have $\mathscr{C}(\mathcal{I}\mathscr{O}_{\widetilde{Y}}, b_1)|_{V_{\widetilde{Y}}(x_1)} = \mathscr{C}(\mathcal{I}, b_1)\mathscr{O}_{\widetilde{Y}}|_{V_{\widetilde{Y}}(x_1)} = \mathscr{C}(\mathcal{I}, b_1)|_{V_Y(x_1)}\mathscr{O}_{V_{\widetilde{Y}}(x_1)}$. The first assertion is then proven by applying the induction hypothesis to the ideal $\mathscr{C}(\mathcal{I}, b_1)|_{V_Y(x_1)}$ on $V_Y(x_1)$. The second assertion follows from the first since the surjectivity of f implies that $\max\text{inv}(\mathcal{I}\mathscr{O}_{\widetilde{Y}}) = \max\text{inv}(\mathcal{I})$ (see Lemma 6.3(iii)). □

Corollary 6.13. *Let Y be a toroidal Deligne–Mumford stack over* $\Bbbk$*, and fix an atlas $p_{1,2}\colon Y_1 \rightrightarrows Y_0$ of Y by schemes such that Y_0 is a strict toroidal* $\Bbbk$*-scheme. Let $y \in |Y|$, and let $\mathcal{I}$ be an ideal on Y such that $\mathcal{I}_y \neq 0$.*

(i) *If $y_1, y_2 \in Y_0$ are points over y, then $\text{inv}_{y_1}(\mathcal{I}\mathscr{O}_{Y_0}) = \text{inv}_{y_2}(\mathcal{I}\mathscr{O}_{Y_0})$.*
(ii) *If y_1 is a point over y, the toroidal centre $\mathscr{J}^{(y_1)}(\mathcal{I}\mathscr{O}_{Y_0})$ descends to a toroidal centre $\mathscr{J}^{(y)}(\mathcal{I})$ on an open substack of Y containing y. (One can extend the definition of toroidal centres to toroidal Deligne–Mumford stacks over* $\Bbbk$*, which we have opted not to state explicitly.)*

If $\mathcal{I}$ is a nowhere zero ideal on Y, then the toroidal centre $\mathscr{J}(\mathcal{I}\mathscr{O}_{Y_0})$ descends to a toroidal centre $\mathscr{J}(\mathcal{I})$ on Y.

Because of Corollary 6.13(i), we can define the invariant $\text{inv}_y(\mathcal{I})$ of $\mathcal{I}$ at y to be $\text{inv}_{y_1}(\mathcal{I}\mathscr{O}_{Y_0})$ for any point $y_1 \in Y_0$ above y.

Proof. Let $(y_1, y_2) \in Y_1$ denote the point mapping to y_i via p_i for $i = 1, 2$. Since p_1 and p_2 are both strict and étale, Lemma 6.3(iii) implies $\text{inv}_{y_1}(\mathcal{I}\mathscr{O}_{Y_0}) = \text{inv}_{(y_1,y_2)}(\mathcal{I}\mathscr{O}_{Y_1}) = \text{inv}_{y_2}(\mathcal{I}\mathscr{O}_{Y_0})$, so part (i) follows. If that invariant is equal to $\max\text{inv}(\mathcal{I}\mathscr{O}_{Y_0})$, then Lemma 6.12 implies $p_1^*\mathscr{J}^{(y_1)}(\mathcal{I}\mathscr{O}_{Y_0}) = \mathscr{J}^{(y_1,y_2)}(\mathcal{I}\mathscr{O}_{Y_1}) = p_2^*\mathscr{J}^{(y_2)}(\mathcal{I}\mathscr{O}_{Y_0})$. If not, evidently the same equality holds. Therefore, we obtain the desired descent in the final statement. Part (ii) is a consequence of the final statement, as can be seen by replacing Y with an invariant open affine neighbourhood of y_1 on which $\mathscr{J}^{(y_1)}(\mathcal{I}\mathscr{O}_{Y_0})$ is defined. □

7. Logarithmic principalization

7.1 Statement of theorem

The goal of this section is to prove the following.

Theorem 7.1 (Logarithmic principalization). *There is a functor $F_{\text{log-pr}}$ associating with*

> *a nowhere zero, proper ideal $\mathcal{I}$ on a toroidal Deligne–Mumford stack Y over a field* $\Bbbk$ *of characteristic zero*

an $\mathcal{I}$-admissible toroidal centre $\mathscr{J} = \mathscr{J}(\mathcal{I})$ with reduced toroidal centre $\overline{\mathscr{J}}$, weighted toroidal blow-up $Y' = \text{Bl}_Y(\overline{\mathscr{J}}) \to Y$, and weak transform $F_{\text{log-pr}}(\mathcal{I} \subsetneq \mathscr{O}_Y) = (\mathcal{I}' \subset \mathscr{O}_{Y'})$ such that

$\max \operatorname{inv}(\mathcal{I}') < \max \operatorname{inv}(\mathcal{I})$. *Functoriality here is with respect to logarithmically smooth, surjective morphisms.*

In particular, there is an integer $N \geqslant 1$ such that the iterated application $(\mathcal{I}_N \subset \mathscr{O}_{Y_N}) = F_{\text{log-pr}}^{\circ N}(\mathcal{I} \subsetneq \mathscr{O}_Y)$ of $F_{\text{log-pr}}$ has $\mathcal{I}_N = (1)$. This stabilized functor $F_{\text{log-pr}}^{\circ\infty}$ is functorial for all logarithmically smooth morphisms, whether or not surjective.

We prove the principalization theorem as a consequence of the results in Section 7.2. We remind the reader that the notion of weak transform was introduced in Definition 4.5.

7.2 The invariant drops

Let $\mathcal{I}$ be an ideal on a strict toroidal $\Bbbk$-scheme Y, and let $y \in Y$ be such that $\mathcal{I}_y \neq 0$. Let $\mathscr{J} = \mathscr{J}^{(y)}(\mathcal{I}) = \big(x_1^{a_1}, \dots, x_k^{a_k}, (Q \subset M)^{1/d}\big)$ be the toroidal centre associated with $\mathcal{I}$ at y, which has invariant $\operatorname{inv}(\mathscr{J}) = \operatorname{inv}_y(\mathcal{I})$. Let $\overline{\mathscr{J}}$ be the reduced toroidal centre associated with $\mathscr{J}$, so that $\overline{\mathscr{J}} = \mathscr{J}^{1/(a_1 n_1)} = \big(x_1^{1/n_1}, \dots, x_k^{1/n_k}, (Q \subset M)^{1/(a_1 n_1 d)}\big)$ if $k \geqslant 1$ and $\overline{\mathscr{J}} = \mathscr{J}$ if $k = 0$.

For this section only, let us work locally at y and replace Y with the open affine neighbourhood of y on which $\mathscr{J}$ is defined. For any integer $c \geqslant 1$, we shall write $Y_c' \to Y$ for the weighted toroidal blow-up along $\overline{\mathscr{J}}^{1/c}$, with exceptional ideal $\mathscr{E}_c$. By Proposition 4.4, there exists an ideal $\mathcal{I}_c'$ on Y_c' such that $\mathcal{I}\mathscr{O}_{Y_c'}$ factors as $\mathscr{E}_c^{a_1 n_1 c} \cdot \mathcal{I}_c'$ if $k \geqslant 1$ and as $\mathscr{E}_c^{c} \cdot \mathcal{I}_c'$ if $k = 0$. The goal of this section is to show the following.

Theorem 7.2 (The invariant drops). *Let the notation be as above, and assume $\mathcal{I}_y \neq (1)$. For every integer $c \geqslant 1$ and every point $y' \in |Y_c'|$ over y, we have $\operatorname{inv}_{y'}(\mathcal{I}_c') < \operatorname{inv}_y(\mathcal{I})$.*

Of course, we are only interested in the theorem for the case $c = 1$. We prove it for all integers $c \geqslant 1$ so that induction can take place. Let us first deal with the special case $k = 0$.

Lemma 7.3 (Cleaning up, cf. [ATW20a, Proposition 2.2.1]). *Let the notation be as above. Assume $k = 0$, that is, $\operatorname{inv}_y(\mathcal{I}) = (\infty)$, and write $\mathscr{J} = (Q \subset M)$, where $M = \overline{\mathscr{M}}_{Y,y}$ and $Q = \overline{\alpha_Y^{-1}(\mathscr{M}(\mathcal{I}))_y}$. Then Theorem 7.2 holds.*

Proof. We use an $\big(mT^c\big)$-chart as in Lemma 4.2, where m belongs to a fixed finite set of generators for Q. In this case, Y_c' is the pullback of the logarithmically smooth morphism $\operatorname{Spec}(M_m \to \Bbbk[x_1, \dots, x_n, M_m]) \to \operatorname{Spec}(M \to \Bbbk[x_1, \dots, x_n, M])$ to Y, via the strict and smooth morphism $Y \to \operatorname{Spec}(M \to \Bbbk[x_1, \dots, x_n, M])$, where M_m is the saturation of the submonoid of $M\big[m^{1/c}\big]^{\mathrm{gp}}$ generated by $M\big[m^{1/c}\big]$ and $\{q' = q/m = q/u^c : q \in Q\}$. Since $Y_c' \to Y$ is logarithmically smooth, Lemma B.13(iii) implies that $\mathscr{M}(\mathcal{I}\mathscr{O}_{Y_c'}) = \mathscr{M}(\mathcal{I})\mathscr{O}_{Y_c'}$, so $\mathscr{M}(\mathcal{I}\mathscr{O}_{Y_c'})_{y'} = \mathscr{M}(\mathcal{I})_y\mathscr{O}_{Y_c',y'}$. Since every $q \in Q$ factors as $q' \cdot u^c$ in M_m, we have $\mathscr{M}(\mathcal{I})_y\mathscr{O}_{Y_c',y'} = (\mathscr{E}_c)_{y'}^c$. Therefore, $(\mathcal{I}\mathscr{O}_{Y_c'})_{y'} = (\mathscr{E}_c)_{y'}^c \cdot (\mathcal{I}_c')_{y'} = \mathscr{M}(\mathcal{I}\mathscr{O}_{Y_c'})_{y'} \cdot (\mathcal{I}_c')_{y'}$. Applying Lemma B.13(iv), one sees that $\mathscr{M}(\mathcal{I}_c')_{y'} = (1)$, whence $\operatorname{log-ord}_{y'}(\mathcal{I}_c') < \infty$. Thus, $\operatorname{inv}_y(\mathcal{I}_c') < (\infty) = \operatorname{inv}_y(\mathcal{I})$. ∎

For the case $k \geqslant 1$, the next lemma (and its corollary) shows that we can replace $\mathcal{I}$ with the coefficient ideal $\mathscr{C}(\mathcal{I}, a_1)$.

Lemma 7.4 (cf. [BM08, Lemma 3.3]). *Let the notation be as above. Assume $k \geqslant 1$, so that $a_1 = \operatorname{log-ord}_y(\mathcal{I}) < \infty$, and let $\mathscr{C} = \mathscr{C}(\mathcal{I}, a_1)$. For every integer $c \geqslant 1$, factorize $\mathscr{C}\mathscr{O}_{Y_c'} = \mathscr{E}_c^{a_1! n_1 c} \cdot \mathscr{C}_c'$ for some ideal $\mathscr{C}_c'$ on Y_c', as in Proposition 4.4. Then we have the inclusions $(\mathcal{I}_c')^{(a_1-1)!} \subset \mathscr{C}_c' \subset \mathscr{C}(\mathcal{I}_c', a_1)$.*

Proof. We have

$$\mathscr{C}\mathscr{O}_{Y_c'} = \left(\prod_{j=0}^{a_1-1} \left(\mathscr{D}_Y^{\leqslant j}(\mathcal{I})\mathscr{O}_{Y_c'}\right)^{c_j} : c_j \in \mathbb{N},\ \sum_{j=0}^{a_1-1} (a_1 - j)c_j \geqslant a_1! \right).$$

Applying Lemma 4.10, we see that for every $1 \leqslant j < a_1$,

$$\mathscr{D}^{\leqslant j}(\mathcal{I})\mathscr{O}_{Y_c'} \subset \mathscr{E}_c^{(a_1-j)n_1c} \cdot \mathscr{D}^{\leqslant j}(\mathcal{I}_c').$$

We also have $\mathcal{I}\mathscr{O}_{Y_c'} = \mathscr{E}_c^{a_1n_1c} \cdot \mathcal{I}_c'$. Plugging this into the first equation yields

$$\mathscr{C}\mathscr{O}_{Y_c'} \subset \mathscr{E}_c^{a_1!n_1c} \cdot \left(\prod_{j=0}^{a_1-1} \left(\mathscr{D}^{\leqslant j}(\mathcal{I}_c')\right)^{c_j} : c_j \in \mathbb{N},\ \sum_{j=0}^{a_1-1} (a_1 - j)c_j \geqslant a_1! \right) = \mathscr{E}_c^{a_1!n_1c} \cdot \mathscr{C}(\mathcal{I}_c', a_1).$$

Thus, we get the second inclusion $\mathscr{C}_c' \subset \mathscr{C}(\mathcal{I}_c', a_1)$. The first inclusion follows from the inclusion $\mathscr{C}\mathscr{O}_{Y_c'} \supset (\mathcal{I}\mathscr{O}_{Y_c'})^{(a_1-1)!} = \mathscr{E}_c^{a_1!n_1c} \cdot (\mathcal{I}_c')^{(a_1-1)!}$. □

Corollary 7.5. *Assume that the hypotheses of Lemma 6.7 hold. For every point $y' \in |Y_c'|$ over y, we have*

(i) $\mathrm{inv}_{y'}(\mathscr{C}_c') = (a_1 - 1)! \cdot \mathrm{inv}_{y'}(\mathcal{I}_c')$,
(ii) $\mathrm{inv}_{y'}(\mathcal{I}_c') < \mathrm{inv}_y(\mathcal{I})$ *if and only if* $\mathrm{inv}_{y'}(\mathscr{C}_c') < \mathrm{inv}_y(\mathscr{C})$.

Proof. By Lemma 7.4, we have

$$\mathrm{inv}_{y'}\left((\mathcal{I}_c')^{(a_1-1)!}\right) \geqslant \mathrm{inv}_{y'}(\mathscr{C}_c') \geqslant \mathrm{inv}_{y'}(\mathscr{C}(\mathcal{I}_c', a_1 - 1)),$$

but Corollary 6.8(ii) implies $\mathrm{inv}_{y'}(\mathscr{C}(\mathcal{I}_c', a_1)) = (a_1-1)! \cdot \mathrm{inv}_{y'}(\mathcal{I}_c') = \mathrm{inv}_{y'}\left((\mathcal{I}_c')^{(a_1-1)!}\right)$. This forces equality throughout, yielding part (i). Part (ii) follows from part (i) and Corollary 6.8(ii). □

Proof of Theorem 7.2. We induct on the length L of $\mathrm{inv}_y(\mathcal{I})$. First consider the base case $L = 1$. The sub-case $\mathrm{inv}_y(\mathcal{I}) = (\infty)$ is settled in Lemma 7.3. On the other hand, if $\mathrm{inv}_y(\mathcal{I}) = (a_1)$ with $a_1 < \infty$, then $\mathcal{I}_y = (x_1^{a_1})$, with weak transform $(\mathcal{I}_c')_{y'} = (1)$. Henceforth, assume $L \geqslant 2$. In particular, $k \geqslant 1$, so Corollary 7.5 says that we can replace[6] $\mathcal{I}$ with $\mathscr{C} = \mathscr{C}(\mathcal{I}, a_1)$ and show that the invariant drops for $\mathscr{C}$.

Let us first outline the set-up for induction. Let H be the hypersurface of maximal contact for $\mathcal{I}$ through y given by $x_1 = 0$, and let $\mathscr{J}_H = \left(x_2^{a_2}, \dots, x_k^{a_k}, (Q \subset M)^{1/d}\right)$ be the restriction of $\mathscr{J}$ to H. Let $\overline{\mathscr{J}}_H$ denote the reduced toroidal centre associated with $\mathscr{J}_H$, so $\overline{\mathscr{J}}_H = (\mathscr{J}_H)^{c'/(a_1n_1)}$, where $c' = \gcd(n_2, \dots, n_k)$. Note that $\mathscr{J}^{(y)}(\mathscr{C}|_H) = \mathscr{J}_H^{(a_1-1)!} = \overline{\mathscr{J}}_H^{(a_1!n_1)/c'}$, so $\overline{\mathscr{J}}_H$ is the reduced toroidal centre associated with $\mathscr{J}^{(y)}(\mathscr{C}|_H)$. Since the length of $\mathrm{inv}_y(\mathscr{C}|_H)$ is less than L, the induction hypothesis implies, in particular, that the invariant of $\mathscr{C}|_H$ at y drops after the weighted toroidal blow-up along $\overline{\mathscr{J}}_H^{1/(cc')}$. But the weighted toroidal blow-up along $\overline{\mathscr{J}}_H^{1/(cc')}$ coincides with the proper transform $H_c' \to H$ of H via the weighted toroidal blow-up along $\overline{\mathscr{J}}^{1/c}$ (see Lemma 4.6(ii)).

Therefore, to leverage on the preceding paragraph, we consider the following two cases:

(a) y' is in the $\left(x_1T^{n_1c}\right)$-chart of Y_c';
(b) y' is in the proper transform H_c', in which case y' is in the other charts of Y_c'.

[6] Note that $\mathscr{J}^{(y)}(\mathscr{C}) = \mathscr{J}^{(y)}(\mathcal{I})^{(a_1-1)!} = \overline{\mathscr{J}}^{a_1!n_1}$, so $\overline{\mathscr{J}}$ is also the reduced toroidal centre associated with $\mathscr{J}^{(y)}(\mathscr{C})$.

For case (a), the local section $x_1^{a_1!}$ of $\mathscr{C}$ factors as $x_1^{a_1!} = u^{a_1!n_1c} \cdot 1$ in $\mathscr{C}\mathscr{O}_{Y'_c} = u^{a_1!n_1c} \cdot \mathscr{C}'_c$, where u is the equation for $\mathscr{E}_c$. Therefore, $(\mathscr{C}'_c)_{y'} = (1)$, that is, $\mathrm{inv}_{y'}(\mathscr{C}'_c) = (0) < \mathrm{inv}_y(\mathscr{C})$, as desired.

For case (b), we saw earlier that the induction hypothesis implies

$$\mathrm{inv}_{y'}(\mathscr{C}'_c|_{H'_c}) < \mathrm{inv}_y(\mathscr{C}|_H)\,. \tag{7.2.1}$$

Moreover, the local section $x_1^{a_1!}$ of $\mathscr{C}$ now factors as $x_1^{a_1!} = u^{a_1!n_1c} \cdot (x'_1)^{a_1!}$ in $\mathscr{C}\mathscr{O}_{Y'_c} = u^{a_1!n_1c} \cdot \mathscr{C}'_c$, where u is the equation for $\mathscr{E}_c$ and x'_1 is the equation for H'_c. Thus, $(x'_1)^{a_1!} \subset (\mathscr{C}'_c)_{y'}$, so that $\text{log-ord}_{y'}(\mathscr{C}'_c) \leqslant a_1!$. Let us now consider two sub-cases of case (b):

(bi) If $\text{log-ord}_{y'}(\mathscr{C}'_c) < a_1!$, then a fortiori $\mathrm{inv}_{y'}(\mathscr{C}'_c) < \mathrm{inv}_y(\mathscr{C})$.

(bii) On the other hand, if $\text{log-ord}_{y'}(\mathscr{C}'_c) = a_1!$, then x'_1 is a maximal contact element for $\mathscr{C}'_c$ at y', so H'_c is a hypersurface of maximal contact for $\mathscr{C}'_c$ through y'. Therefore,

$$\begin{aligned}
\mathrm{inv}_{y'}(\mathscr{C}'_c) &= \left(a_1!, \frac{\mathrm{inv}_{y'}(\mathscr{C}(\mathscr{C}'_c, a_1!)|_{H'_c})}{(a_1!-1)!}\right) && \text{by Lemma 6.3(i)}\\
&\leqslant \left(a_1!, \frac{\mathrm{inv}_{y'}(\mathscr{C}(\mathscr{C}'_c|_{H'_c}, a_1!))}{(a_1!-1)!}\right) && \text{since } \mathscr{C}(\mathscr{C}'_c, a_1!)|_{H'_c} \supset \mathscr{C}(\mathscr{C}'_c|_{H'_c}, a_1!)\\
&= \big(a_1!, \mathrm{inv}_{y'}(\mathscr{C}'_c|_{H'_c})\big) && \text{by Corollary 6.8(ii)}\\
&< \big(a_1!, \mathrm{inv}_y(\mathscr{C}|_H)\big) && \text{by (7.2.1)}\\
&= (a_1-1)! \cdot \mathrm{inv}_y(\mathcal{I}) && \text{by Lemma 6.3(i)}\\
&= \mathrm{inv}_y(\mathscr{C}) && \text{by Corollary 6.8(ii)}\,,
\end{aligned}$$

as desired.

This completes the proof of the induction step. □

7.3 Proof of the logarithmic principalization

Proof of Theorem 7.1. For the first paragraph of the theorem, let $\mathscr{J} = \mathscr{J}(\mathcal{I})$ be as in Section 6. Following the notation in Definition 6.4, write $\mathscr{J} = \big(x_1^{a_1}, \ldots, x_k^{a_k}, (Q \subset M)^{1/d}\big)$, and write $\overline{\mathscr{J}} = \mathscr{J}^{1/(a_1n_1)} = \big(x_1^{1/n_1}, \ldots, x_k^{1/n_k}, (Q \subset M)^{1/(a_1n_1d)}\big)$. Let $Y' = \mathrm{Bl}_Y\left(\overline{\mathscr{J}}\right) \to Y$ be as in the theorem. By Proposition 4.4(i), the ideal $\mathcal{I}\mathscr{O}_{Y'}$ factors as $\mathscr{E}^{a_1n_1} \cdot \mathcal{I}'$. By Theorem 6.5(ii), the ideal $\mathcal{I}'$ is the weak transform of $\mathcal{I}$. By Theorem 7.2, we have $\max\mathrm{inv}(\mathcal{I}') < \max\mathrm{inv}(\mathcal{I})$. The functoriality with respect to logarithmically smooth surjective morphisms follows from Lemma 6.12.

The second paragraph of the theorem is now immediate by a standard argument. Namely, if $Y_n \to \cdots \to Y$ is the logarithmic principalization of $\mathcal{I} \subsetneq \mathscr{O}_Y$ and $\widetilde{Y} \to Y$ is a logarithmically smooth morphism of toroidal Deligne–Mumford stacks over $\Bbbk$ with $\widetilde{\mathcal{I}} = \mathcal{I}\mathscr{O}_{\widetilde{Y}}$, then the logarithmic principalization of $\widetilde{\mathcal{I}} \subsetneq \mathscr{O}_{\widetilde{Y}}$ is obtained from the pullback of $Y_n \to \cdots \to Y$ to $\widetilde{Y}$ by removing empty blow-ups. □

7.4 Proof of the logarithmic embedded resolution

Proof of Theorem 1.1. This proceeds in the same way as the proof of [ATW19, Theorem 1.1.1]. For the first paragraph in the theorem, one applies Theorem 7.1 to the ideal $\mathcal{I} = \mathcal{I}_X$ defining X in Y and replaces the weak transform $\mathcal{I}'$ with the proper transform $\mathcal{I}_{X'} \supset \mathcal{I}'$. This implies part (ii) of the theorem, that is, $\max\mathrm{inv}(\mathcal{I}_{X'}) \leqslant \max\mathrm{inv}(\mathcal{I}') < \max\mathrm{inv}(\mathcal{I}_X)$. Parts (i) and (iv) of the theorem were observed in the paragraphs between Lemmas 4.2 and 4.3, while part (iii) follows from the fact that the chosen toroidal centre $\mathscr{J} = \mathscr{J}(\mathcal{I})$ is $\mathcal{I}$-admissible (Theorem 6.5(i)).

The second paragraph of the theorem is just a repeated application of the first paragraph. One stops at the point where $\max \mathrm{inv}(\mathcal{I}_{X_N})$ is the sequence $(1, \dots, 1)$ of length c (where c is the codimension of X in Y): at this point, the toroidal centre $\mathscr{J}_N$, whose support is contained in X_N, is everywhere of the form $(x_1, \dots, x_c)$ for ordinary parameters x_i, and hence the support of $\mathscr{J}_N$ is in particular toroidal. Since $\mathrm{inv}_p(\mathcal{I}_{X_N}) = (1, \dots, 1)$ at a point p at which X_N is logarithmically smooth [ATW20b, Lemma 5.1.2], the support of $\mathscr{J}_N$ contains a dense open in X_N, whence they coincide and X_N is toroidal. This gives part (1), while parts (2) and (3) are immediate from parts (iii) and (iv). □

7.5 Proof of the re-embedding principle

Proof of Lemma 1.3. We may assume that Y is a strict toroidal $\Bbbk$-scheme. Let $\mathcal{I}_{X\subset Y}$ denote the ideal of X in Y, and write $\mathbb{A}^1_{\Bbbk}$ as $\mathrm{Spec}(\Bbbk[x_0])$. Then the ideal $\mathcal{I}_{X\subset Y_1}$ of X in Y_1 is $(x_0)+\mathcal{I}_{X\subset Y}$. Then $\mathscr{D}^{\leqslant 1}_{Y}(\mathcal{I}_{X\subset Y_1})_p = (1)$ with maximal contact element x_0 everywhere, so that $\mathcal{I}[2] = \mathcal{I}_{X\subset Y_1}|_{V(x_0)=Y} = \mathcal{I}_{X\subset Y}$. Therefore, part (i) follows by the definition of the invariant (Section 6.1).

For part (ii), first note that if $\mathscr{J}(\mathcal{I}_{X\subset Y}) = \big(x_1^{a_1}, \dots, x_k^{a_k}, (Q\subset M)^{1/d}\big)$, then $\mathscr{J}(\mathcal{I}_{X\subset Y_1}) = \big(x_0, x_1^{a_1}, \dots, x_k^{a_k}, (Q\subset M)^{1/d}\big)$. Then the fact that Y' is identified with the proper transform $V(x_0')$ of $Y = V(x_0) \subset Y_1$ in Y_1' follows from Lemma 4.6(ii). Moreover, if $\mathcal{I}'_{X\subset Y}$ (respectively, $\mathcal{I}'_{X\subset Y_1}$) denotes the underlying ideal of $X' \subset Y'$ (respectively, $X_1' \subset Y_1'$), then $\mathcal{I}'_{X\subset Y_1} = (x_0') + \mathcal{I}'_{X\subset Y}$, and hence part (ii) follows. □

8. An example

Consider the set-up in Section 1.3. We show, by way of example, that the toroidal Deligne–Mumford stacks Y_i obtained in our logarithmic embedded resolution algorithm $Y_N \to \cdots \to Y_1 \to Y_0 = Y$ are not necessarily smooth over $\Bbbk$ and the proper transform $X_N = Y_N \times_Y X_N$ is not necessarily smooth over $\Bbbk$. This necessitates a *resolution of toroidal singularities*, as outlined in Theorem 1.4.

8.1 A resolution of toroidal singularities is necessary

We revisit the following singular surface in [ATW19, Section 8.3]:

$$X = V(\mathcal{I}) = V\big(x^2yz + y^4z\big) \subset Y = \mathbb{A}^3_{\Bbbk}\,.$$

While Y_1 and Y_2 for this example are smooth over $\Bbbk$, we will see below that Y_3 is not. We do this by focusing on a particular chart at each step of our logarithmic resolution algorithm.

Step 1. Since $\mathscr{D}^{\leqslant 4}(\mathcal{I}) = (x, y, z)$, we have $\max \mathrm{inv}(X \subset Y) = \mathrm{inv}_{(0,0,0)}(X \subset Y) = (4, 4, 4)$ and $\mathscr{J}(\mathcal{I}) = \big(x^4, y^4, z^4\big)$. Rescaling, the first step in our logarithmic resolution algorithm involves the blow-up $Y_1 \to Y$ along $\overline{\mathscr{J}}(\mathcal{I}) = (x, y, z)$. Here Y_1 is a priori a strict toroidal $\Bbbk$-scheme, but in fact it is also smooth over $\Bbbk$. This can be seen by examining the x-, y-, and z-charts. For example, the z-chart $Y_1^{(z)}$ of Y_1 is given by the following strict toroidal $\Bbbk$-scheme which is also smooth over $\Bbbk$:

$$Y_1^{(z)} = \mathrm{Spec}\left(\mathbb{N}^1 \to \Bbbk[x_1, y_1, \underline{z}_1]\right),$$

where $x = x_1\underline{z}_1$, $y = y_1\underline{z}_1$, and $z = \underline{z}_1$ is the equation of the exceptional divisor. Here we underline $\underline{z}_1$ to indicate that it is the image of the standard basis vector e_1 of $\mathbb{N}^1$ under the logarithmic structure $\mathbb{N}^1 \to \Bbbk[x_1, y_1, \underline{z}_1]$ (as given by Lemma 4.1). In this chart, the equation $\big(x^2yz + y^4z\big)$

of $X \subset Y$ becomes $\underline{z}_1^4\big(x_1^2y_1 + y_1^4\underline{z}_1\big)$, with proper transform

$$X_1^{(z)} = V\big(\mathcal{I}_1^{(z)}\big) = V\big(x_1^2y_1 + y_1^4\underline{z}_1\big) \subset Y_1^{(z)} = \operatorname{Spec}\big(\mathbb{N}^1 \to \Bbbk[x_1, y_1, \underline{z}_1]\big)\,.$$

Step 2. Next, we have $\mathscr{D}^{\leqslant 1}\big(\mathcal{I}_1^{(z)}\big) = \big(x_1y_1, x_1^2 + 4y_1^3\underline{z}_1, y_1^4\underline{z}_1\big)$ and $\mathscr{D}^{\leqslant 2}\big(\mathcal{I}_1^{(z)}\big) = (x_1, y_1)$, whence $\mathscr{C}\big(\mathcal{I}_1^{(z)}, 3\big)|_{y_1=0} = \big(x_1^6\big)$. Therefore,

$$\operatorname{max\,inv}\big(X_1^{(z)} \subset Y_1^{(z)}\big) = (3,3) < (4,4,4) = \operatorname{max\,inv}(X \subset Y)\,,$$

and $\mathscr{J}\big(\mathcal{I}_1^{(z)}\big) = \big(x_1^3, y_1^3\big)$. Rescaling, the second step in our logarithmic resolution algorithm for the z-chart involves the blow-up $Y_2^{(z)} \to Y_1^{(z)}$ along $\overline{\mathscr{J}}\big(\mathcal{I}_1^{(z)}\big) = (x_1, y_1)$. As in step 1, the z-chart $Y_2^{(z)}$ is a strict toroidal $\Bbbk$-scheme which is smooth over $\Bbbk$. For example, the y_1-chart $Y_2^{(z,y_1)}$ of $Y_2^{(z)}$ is given by the following strict toroidal $\Bbbk$-scheme which is also smooth over $\Bbbk$:

$$Y_2^{(z,y_1)} = \operatorname{Spec}\big(\mathbb{N}^2 \to \Bbbk[x_2, \underline{y}_2, \underline{z}_2]\big)\,,$$

where $x_1 = \underline{y}_2x_2$, $\underline{z}_1 = \underline{z}_2$, and $y_1 = \underline{y}_2$ is the equation of the exceptional divisor. Once again, we underline $\underline{y}_2$ and $\underline{z}_2$ to indicate that they are the respective images of the standard basis vectors e_1 and e_2 of $\mathbb{N}^2$ under the logarithmic structure $\mathbb{N}^2 \to \Bbbk[x_2, \underline{y}_2, \underline{z}_2]$ on $Y_2^{(z,y_1)}$ (as given by Lemma 4.1). In this chart, the equation $\big(x_1^2y_1 + y_1^4\underline{z}_1\big)$ of $X_1^{(z)} \subset Y_1^{(z)}$ becomes $\underline{y}_2^3\big(x_2^2 + \underline{y}_2\underline{z}_2\big)$, with proper transform

$$X_2^{(z,y_1)} = V\big(\mathcal{I}_2^{(z,y_1)}\big) = V\big(x_2^2 + \underline{y}_2\underline{z}_2\big) \subset Y_2^{(z,y_2)} = \operatorname{Spec}\big(\mathbb{N}^2 \to \Bbbk[x_2, \underline{y}_2, \underline{z}_2]\big)\,.$$

Step 3. Finally, we have $\mathscr{D}^{\leqslant 1}\big(\mathcal{I}_2^{(z,y_1)}\big) = (x_2, \underline{y}_2\underline{z}_2)$, whence $\operatorname{max\,inv}\big(X_2^{(z,y_1)} \subset Y_2^{(z,y_1)}\big) = (2,\infty) < (3,3) = \operatorname{max\,inv}\big(X_1^{(z)} \subset Y_1^{(z)}\big)$. Since $\mathscr{C}\big(\mathcal{I}_2^{(z,y_1)}, 2\big)|_{x_2=0} = (\underline{y}_2\underline{z}_2)$, we also have $\mathscr{J}\big(\mathcal{I}_2^{(z,y_1)}\big) = \big(x_2^2, \underline{y}_2\underline{z}_2\big)$. Rescaling, the third step in our logarithmic resolution algorithm for the y_1-chart involves the blow-up $Y_3^{(z,y_1)} \to Y_2^{(z,y_1)}$ along $\overline{\mathscr{J}}\big(\mathcal{I}_2^{(z,y_1)}\big) = \big(x_2, (\underline{y}_2\underline{z}_2)^{1/2}\big)$. A priori, $Y_3^{(z,y_1)}$ is a toroidal Deligne–Mumford stack over $\Bbbk$, but this time the x_2-chart $Y_3^{(z,y_1,x_2)}$ of $Y_3^{(z,y_1)}$ is no longer smooth over $\Bbbk$:

$$Y_3^{(z,y_1,x_2)} = \operatorname{Spec}\left(\frac{\mathbb{N}^4}{\langle e_2 + e_3 \sim e_4 + 2e_1\rangle} \to \frac{\Bbbk[\underline{x}_3, \underline{y}_3, \underline{z}_3, \underline{w}_3]}{(\underline{y}_3\underline{z}_3 - \underline{w}_3\underline{x}_3^2)}\right),$$

where $\underline{y}_2 = \underline{y}_3$, $\underline{z}_2 = \underline{z}_3$, $\underline{y}_2\underline{z}_2 = \underline{w}_3x_2^2$, and $x_2 = \underline{x}_3$ is the equation of the exceptional divisor. As before, we underline $\underline{x}_3$, $\underline{y}_3$, $\underline{z}_3$, and $\underline{w}_3$ to indicate that they are the respective images of the standard basis vectors e_1, e_2, e_3, and e_4 of $\mathbb{N}^4$ under the logarithmic structure $\mathbb{N}^4/\langle e_2 + e_3 \sim e_4 + 2e_1\rangle \to \Bbbk[\underline{x}_3, \underline{y}_3, \underline{z}_3, \underline{w}_3]/\big(\underline{y}_3\underline{z}_3 - \underline{w}_3\underline{x}_3^2\big)$ on $Y_3^{(z,y_1,x_2)}$ (as given by Lemma 4.1). In this chart, the equation $\big(x_2^2 + \underline{y}_2\underline{z}_2\big)$ of $X_2^{(z,y_1)} \subset Y_2^{(z,y_1)}$ becomes $\underline{x}_3^2(1 + \underline{w}_3)$, with proper transform

$$X_3^{(z,y_1,x_2)} = V(1 + \underline{w}_3) \subset \operatorname{Spec}\left(\frac{\mathbb{N}^4}{\langle e_2 + e_3 \sim e_4 + 2e_1\rangle} \to \frac{\Bbbk[\underline{x}_3, \underline{y}_3, \underline{z}_3, \underline{w}_3]}{(\underline{y}_3\underline{z}_3 - \underline{w}_3\underline{x}_3^2)}\right).$$

Note that $\operatorname{max\,inv}\big(X_3^{(z,y_1,x_2)} \subset Y_3^{(z,y_1,x_2)}\big) = (1) < (2,\infty) = \operatorname{max\,inv}\big(X_2^{(z,y_1)} \subset Y_2^{(z,y_1)}\big)$, so our logarithmic embedded resolution algorithm stops here (for this chart). In other words, $X_3^{(z,y_1,x_2)}$

is toroidal. However, as a scheme,

$$X_3^{(z,y_1,x_2)} \simeq \operatorname{Spec}\left(\frac{\Bbbk[x_3,y_3,z_3]}{(x_3^2+y_3z_3)}\right)$$

is not smooth over $\Bbbk$.

Appendix A. The Zariski–Riemann space

In this appendix, fix an algebraic function field K over a ground field $\Bbbk$.

A.1 The Zariski–Riemann space of $K/\Bbbk$

The Zariski–Riemann space of $K/\Bbbk$, which we shall describe shortly, was originally called the Riemann manifold of $K/\Bbbk$ by Zariski in his proof of the resolution of singularities of $\Bbbk$-varieties[7] of dimensions 2 and 3. This notion is implicit in Hironaka's work on the resolution of singularities for all dimensions in characteristic zero. It also plays an essential role in [ATW19], as well as this paper. We shall describe this space in steps:

Step 1. As a set,

$$\mathrm{ZR}(K,\Bbbk) := \{\text{valuation rings } R \text{ of } K \text{ containing } \Bbbk\}.$$

We usually denote an element R of $\mathrm{ZR}(K,\Bbbk)$ by its corresponding valuation $\nu\colon K^* \twoheadrightarrow G$ instead, where $G = \{xR\colon x \in K^*\}$ is the value group of ν. In that case, we write R_ν for R and G_ν for G. We denote the unique maximal ideal of R_ν by $\mathfrak{m}_\nu$ and its residue field by $\kappa_\nu = R_\nu/\mathfrak{m}_\nu$.

Step 2. As a topological space, $\mathrm{ZR}(K,\Bbbk)$ has a basis of open sets given by

$$\mathscr{F} = \{U(x_1,\dots,x_n)\colon n \geqslant 0 \text{ and } x_i \in K^*\},$$

where $U(x_1,\dots,x_n) = \{\nu \in \mathrm{ZR}(K,\Bbbk)\colon R_\nu \supset \Bbbk[x_1,\dots,x_n]\}$.

Step 3. Finally, as a locally ringed space, $\mathrm{ZR}(K,\Bbbk)$ is equipped with a sheaf of rings $\mathscr{O} = \mathscr{O}_{\mathrm{ZR}(K,\Bbbk)}$ described by

$$\mathscr{O}(U) := \bigcap_{\nu \in U} R_\nu, \quad \text{where } U \subset \mathrm{ZR}(K,\Bbbk) \text{ is open}.$$

In particular, $\mathscr{O}(U(x_1,\dots,x_n))$ is the integral closure of $\Bbbk[x_1,\dots,x_n]$ in K; see [Mat89, Theorem 10.4]. Then $\mathscr{O}$ is a subsheaf of the constant sheaf K on $\mathrm{ZR}(K,\Bbbk)$, and the stalk of $\mathscr{O}$ at ν is R_ν. Note that $\mathrm{ZR}(K,\Bbbk)$ also carries a sheaf of ordered groups $\Gamma = K^*/\mathscr{O}^*$, whose sections over an open set U are

$$\left\{(s_\nu)_{\nu\in U} \in \prod_{\nu \in U} G_\nu\colon \begin{array}{l}\forall \nu \in U,\ \exists \text{ open set } \nu \in V \subset U \\ \text{and } \exists x \in K^* \text{ such that } \forall \nu' \in V,\ s_{\nu'} = \nu'(x)\end{array}\right\}$$

and whose stalk at ν is G_ν, with a morphism of sheaves of ordered groups $\mathrm{val}\colon K^* \twoheadrightarrow \Gamma$. The image $\mathrm{val}(\mathscr{O} \setminus \{0\}) \subset \Gamma$ is the sheaf of monoids consisting of non-negative sections of Γ, denoted

[7] See footnote 1 at the start of Section 2.

by Γ_+. Explicitly, its sections over an open set U are

$$\left\{(s_\nu)_{\nu\in U} \in \prod_{\nu\in U} G_\nu \colon \forall\nu\in U,\ \exists \text{ open set } \nu\in V\subset U \text{ and } \exists\, 0\neq x\in\mathscr{O}(V) \text{ such that } \forall\nu'\in V,\ s_{\nu'}=\nu'(x)\right\}.$$

Two remarks are in order. Firstly, $\mathrm{ZR}(K,\Bbbk)$ is quasi-compact. For a proof, see [Mat89, Theorem 10.5]. Secondly, $\mathrm{ZR}(K,\Bbbk)$ can be characterized as an inverse limit of projective models of $K/\Bbbk$ in the category of locally ringed spaces. Let us expound on this further. By a projective model Y of $K/\Bbbk$, we mean that Y is a projective $\Bbbk$-variety whose field of functions $K(Y)$ is isomorphic to K. For every $\nu\in\mathrm{ZR}(K,\Bbbk)$, there exists a unique dotted arrow making the triangles in the diagram below commute:

$$\begin{array}{ccc}
\mathrm{Spec}(K) & \xrightarrow{\text{generic pt.}} & Y \\
\downarrow & \overset{f_\nu}{\nearrow} & \downarrow \\
\mathrm{Spec}(R_\nu) & \longrightarrow & \mathrm{Spec}(\Bbbk)\,.
\end{array}$$

The composition $\mathrm{Spec}(\kappa_\nu)=\mathrm{Spec}(R_\nu/\mathfrak{m}_\nu)\to\mathrm{Spec}(R_\nu)\xrightarrow{f_\nu} Y$ demarcates a point y_ν on Y, which is called the centre of R_ν on Y; see [Har77, Exercise II.4.5]. This gives an injective local $\Bbbk$-homomorphism $f_\nu^\#\colon \mathscr{O}_{Y,y_\nu}\to R_\nu$ of local rings whose field of fractions is K, in which case we say that R_ν dominates $\mathscr{O}_{Y,y_\nu}$ via f_ν (cf. [Har77, Lemma II.4.4]).

The projective models of $K/\Bbbk$ form an inverse system as follows: An arrow from a projective model Y_a to another Y_b is a birational morphism $\varphi_{a\to b}\colon Y_b\to Y_a$. For every ν in $\mathrm{ZR}(K,\Bbbk)$, the morphism $\varphi_{a\to b}$ necessarily maps the centre $y_{b,\nu}$ of R_ν on Y_b to the centre $y_{a,\nu}$ of R_ν on Y_a. In other words, φ_a^b induces a local homomorphism $\varphi_{a\to b}^\#$ of local rings with field of fractions K, which makes the diagram below commute:

$$\begin{array}{ccc}
\mathscr{O}_{Y_a,y_{a,\nu}} & \xrightarrow{\varphi_{a\to b}^\#} & \mathscr{O}_{Y_b,y_{b,\nu}} \\
 & {\scriptstyle f_{a,\nu}^\#}\searrow \quad \swarrow{\scriptstyle f_{b,\nu}} & \\
 & R_\nu\,. &
\end{array}$$

The join Y_c of two projective models Y_a and Y_b admits birational morphisms $Y_c\to Y_a$ and $Y_c\to Y_b$, whence this is indeed an inverse system.

Step 1: The space $\mathrm{ZR}(K,\Bbbk)$ is the set-theoretic inverse limit. As shown above, a point $\nu\in\mathrm{ZR}(K,\Bbbk)$ determines a collection of points $\{y_\nu\in Y\colon Y \text{ is a projective model of } K/\Bbbk\}$ – which is (by definition) preserved by arrows in the inverse system – and hence determines a point in the inverse limit.

Conversely, a point in the inverse limit is a collection of points $\Sigma=\{y_\Sigma\in Y\colon Y \text{ is a projective model of } K/\Bbbk\}$ which is preserved by arrows in the inverse system. Let R be the direct limit of the system whose objects are the local rings $\mathscr{O}_{Y,y_\Sigma}$ and whose arrows are given by the local $\Bbbk$-homomorphisms $\varphi_{a\to b}^\#$ of local rings with field of fractions K (where $a\to b$ is an arrow in the inverse system of projective models of $K/\Bbbk$).

Since R is the direct limit of a system of local rings with local homomorphisms, R is a local ring with maximal ideal $(\mathfrak{m}_{Y,y_\Sigma}\colon Y \text{ is a projective model for } K/\Bbbk)$. By [Har77, Theorem I.6.1A], the ring R is a valuation ring of K containing $\Bbbk$ and hence determines a point $\nu\in\mathrm{ZR}(K,\Bbbk)$.

For each projective model Y of $K/\Bbbk$, the local ring $\mathscr{O}_{Y,y_\Sigma}$ must be the unique local ring of Y dominated by R, whence the centre of R on Y is y_Σ. One can also use [Har77, Theorem I.6.1A] to show that given a point $\nu \in \mathrm{ZR}(K, \Bbbk)$, the ring R_ν is the direct limit of the system of local rings $\mathscr{O}_{Y,y_\nu}$. This establishes the desired one-to-one correspondence of points.

Step 2: The space $\mathrm{ZR}(K, \Bbbk)$ is the topological inverse limit. The inverse limit topology on $\mathrm{ZR}(K, \Bbbk)$ is the coarsest topology such that the projection map $\pi_Y \colon \mathrm{ZR}(K, \Bbbk) \to Y$ (where Y is a projective model for $K/\Bbbk$), which sends $\nu \in \mathrm{ZR}(K, \Bbbk)$ to the centre y_ν of R_ν on Y, is continuous.

Let Y be a projective model for $K/\Bbbk$. Let $U = \mathrm{Spec}(A) \subset Y$ be an open affine subset. Then $\pi_Y^{-1}(\mathrm{Spec}(A))$ consists of the $\nu \in \mathrm{ZR}(K, \Bbbk)$ such that there exists a dotted arrow filling in the diagram

$$\begin{array}{ccc} K & \longleftarrow & A \\ \uparrow & \swarrow & \uparrow \\ R_\nu & \longleftarrow & \Bbbk\,. \end{array}$$

Since A is a finitely generated $\Bbbk$-algebra, we can write $A = \Bbbk[x_1, \ldots, x_n]$ for $x_i \in K^*$. Then $U(x_1, \ldots, x_n) = \pi_Y^{-1}(\mathrm{Spec}(A))$. Conversely, given $x_1, \ldots, x_n \in K^*$, we can find $x_{n+1}, \ldots, x_m$ in K^* such that $A = \Bbbk[x_1, \ldots, x_m]$ has fraction field K. The projection $T_i \mapsto x_i$ gives a presentation of A as $A \simeq \Bbbk[T_1, \ldots, T_m]/\mathfrak{p}$, where $\mathfrak{p}$ is a prime ideal of the polynomial ring $\Bbbk[T_1, \ldots, T_m]$. We can homogenize the prime ideal $\mathfrak{p} \subset \Bbbk[T_1, \ldots, T_m]$ to a homogeneous prime ideal $\mathfrak{P} \subset \Bbbk[T_1, \ldots, T_m, T_{m+1}]$. Then $U = \mathrm{Spec}(A)$ is an open affine subset of $Y = \mathrm{Proj}(\Bbbk[T_1, \ldots, T_{m+1}]/\mathfrak{P})$, which is a projective model of $K/\Bbbk$ with $\pi_Y^{-1}(\mathrm{Spec}(A)) = U(x_1, \ldots, x_m) \subset U(x_1, \ldots, x_n)$. Since open affines form a basis for Zariski topology, we are done.

Step 3: The space $\mathrm{ZR}(K, \Bbbk)$ is the inverse limit in the category of locally ringed spaces. Set $\mathscr{O}' := \varinjlim_Y \pi_Y^{-1}\mathscr{O}_Y$, where the direct limit is taken over the projective models Y of $K/\Bbbk$. This is the correct sheaf of rings on $\mathrm{ZR}(K, \Bbbk)$ as the inverse limit in the category of locally ringed spaces (see for example [Gil08, Theorem 4 and Corollary 5]). It remains to note that $\mathscr{O}' = \mathscr{O}_{\mathrm{ZR}(K,\Bbbk)}$. For this, observe that there are morphisms $\pi_Y^{-1}\mathscr{O}_Y \to \mathscr{O}_{\mathrm{ZR}(K,\Bbbk)}$ (adjoint to the canonical morphisms $\mathscr{O}_Y \to (\pi_Y)_*\mathscr{O}_{\mathrm{ZR}(K,\Bbbk)}$) for each projective model Y of $K/\Bbbk$, culminating in a morphism $\mathscr{O}' \to \mathscr{O}_{\mathrm{ZR}(K,\Bbbk)}$ which we can see is an isomorphism by checking it on stalks.

Note that $\mathrm{ZR}(K, \Bbbk)$ is also the inverse limit of a similar system of proper models of $K/\Bbbk$ (proper $\Bbbk$-varieties whose field of fractions is isomorphic to K), in which the projective models of $K/\Bbbk$ form a cofinal subsystem (by Chow's lemma [Har77, Exercise II.4.10]).

A.2 The Zariski–Riemann space of a $\Bbbk$-variety

More generally, we can define the Zariski–Riemann space for a $\Bbbk$-variety Y. Let K be the field of fractions $K(Y)$ of Y. Since Y is separated but not necessarily proper, not every $\nu \in \mathrm{ZR}(K, \Bbbk)$ possesses a centre y_ν on Y, but if it does, the centre y_ν is unique. Therefore, we set

$$\mathrm{ZR}(Y) := \{\nu \in \mathrm{ZR}(K, \Bbbk) \colon \nu \text{ has a centre on } Y\} \subset \mathrm{ZR}(K, \Bbbk)\,.$$

This agrees with the notation in [ATW19]. If Y is a proper model of $K/\Bbbk$, then $\mathrm{ZR}(Y)$ is simply the space $\mathrm{ZR}(K, \Bbbk)$ defined in Section A.1. We let $\mathrm{ZR}(Y)$ inherit its topology, sheaf of rings $\mathscr{O}_{\mathrm{ZR}(Y)}$, and sheaf of ordered groups Γ_Y from $\mathrm{ZR}(K, \Bbbk)$. As before, $\mathrm{ZR}(Y)$ is the inverse limit of the system of modifications $Y' \to Y$ in the category of morphisms of locally ringed spaces into Y. We write π_Y for the morphism $\mathrm{ZR}(Y) \to Y$ sending ν to the centre of ν on Y.

Note that $\mathrm{ZR}(Y)$ is quasi-compact and open in $\mathrm{ZR}(K, \Bbbk)$. This can be seen as follows.

First suppose that $Y = \mathrm{Spec}(A)$ is an affine $\Bbbk$-variety, with A generated as a $\Bbbk$-algebra by $x_1, \ldots, x_n \in K$. In this case, we have seen earlier that $\mathrm{ZR}(Y)$ is $U(x_1, \ldots, x_n)$ and is quasi-compact (by [Mat89, Theorem 10.5]). In general, since Y is covered by finitely many affine opens, one deduces that $\mathrm{ZR}(Y)$ is quasi-compact and open in $\mathrm{ZR}(K, \Bbbk)$. We conclude this section with a noteworthy fact.

Lemma A.1. *Let Y be a $\Bbbk$-variety, with morphism $\pi_Y\colon \mathrm{ZR}(Y) \to Y$. If Y is normal, then the morphism $\pi_Y^\#\colon \mathscr{O}_Y \to (\pi_Y)_*\mathscr{O}_{\mathrm{ZR}(Y)}$ is an isomorphism of sheaves on Y.*

Proof. Since open affines form a basis for the Zariski topology on Y, it suffices to check this isomorphism on open affines $U = \mathrm{Spec}(A) \subset Y$. Since Y is normal, $\mathscr{O}_Y(U) = A$ is normal, whence, by [Mat89, Theorem 10.4],

$$\mathscr{O}_Y(U) = \bigcap_{\substack{\nu \in \mathrm{ZR}(K,\Bbbk) \\ R_\nu \supseteq A}} R_\nu \,.$$

But the set of $\nu \in \mathrm{ZR}(K, \Bbbk)$ such that $R_\nu \supseteq A$ is precisely the set of $\nu \in \mathrm{ZR}(Y)$ which have a centre on $U = \mathrm{Spec}(A) \subset Y$. Therefore, $\mathscr{O}_Y(U) = \bigcap_{\nu \in \pi_Y^{-1}(U)} R_\nu = \mathscr{O}_{\mathrm{ZR}(K,\Bbbk)}\big(\pi_Y^{-1}(U)\big)$. □

A.3 Functoriality with respect to dominant morphisms

If $f\colon Y' \to Y$ is a dominant morphism of $\Bbbk$-varieties, f induces a morphism $\mathrm{ZR}(f)\colon \mathrm{ZR}(Y') \to \mathrm{ZR}(Y)$ of locally ringed spaces, which maps R_ν to $R_\nu \cap K(Y)$. The morphism $\mathscr{O}_{\mathrm{ZR}(Y)} \to \mathrm{ZR}(f)_*\mathscr{O}_{\mathrm{ZR}(Y')}$ is given by the inclusion $\bigcap_{\nu \in U} R_\nu \hookrightarrow \bigcap_{\eta \in \mathrm{ZR}(f)^{-1}(U)} R_\eta$ over an open set U and is stalk-wise given by the local homomorphism $R_\nu \cap K(Y) \hookrightarrow R_\nu$. This morphism $\mathscr{O}_{\mathrm{ZR}(Y)} \to \mathrm{ZR}(f)_*\mathscr{O}_{\mathrm{ZR}(Y')}$ descends to a morphism of sheaves of ordered groups $\Gamma_Y \to \mathrm{ZR}(f)_*\Gamma_{Y'}$, as well as a morphism of sheaves of monoids $\Gamma_{Y,+} \to \mathrm{ZR}(f)_*\Gamma_{Y',+}$.

Appendix B. Toroidal geometry

In this appendix, we briefly mention some preliminaries on logarithmic geometry pertinent to this paper. Most of the notation and language here follows [Ogu18] closely. Other relevant references include [Kat89, Kat94, Niz06, AT17, ATW20a, ATW20b, GR18].

B.1 Toroidal $\Bbbk$-schemes

In this section, Y denotes a logarithmic scheme, and we denote its underlying scheme by $\underline{Y}$ and its underlying logarithmic structure by $\alpha_Y\colon \mathscr{M}_Y \to \mathscr{O}_Y$. Occasionally, we also use the letter $\underline{Y}$ to denote the logarithmic scheme given by the scheme $\underline{Y}$ equipped with the trivial logarithmic structure.

Definition B.1. We say that Y is *fs* if

> Y admits a covering $\mathcal{U}$ (in the Zariski or étale topology, depending on if $\mathscr{M}_Y$ is Zariski or not) such that the pullback of $\mathscr{M}_Y$ to each U in $\mathcal{U}$ admits a chart subordinate to a fs (= *fine* and *saturated*) monoid M – or, equivalently, U admits a strict morphism $U \to \mathrm{Spec}(M \to \mathbb{Z}[M])$ for a fs monoid M.

Remark B.2. Let Y be a fs logarithmic scheme. In what follows, y always denotes a point in Y, while $\overline{y}$ denotes a geometric point over y. Then one can show the following:

(i) The group $\overline{\mathscr{M}}^{\mathrm{gp}}_{Y,\overline{y}}$ is free abelian of finite rank $r(y)$. Note that $r(y)$ is independent of the choice of $\overline{y}$ over y. See [Ogu18, Proposition I.1.3.5(2)].

(ii) The rank $r(y) = \operatorname{rank}\left(\overline{\mathscr{M}}^{\mathrm{gp}}_{Y,\overline{y}}\right)$ is upper semi-continuous on Y; that is, for each $n \in \mathbb{N}$,

$$Y^{(n)} := \left\{y \in Y \colon \operatorname{rank}\left(\overline{\mathscr{M}}^{\mathrm{gp}}_{Y,\overline{y}}\right) \leqslant n\right\}$$

is Zariski open in Y [Ogu18, Corollary II.2.16]. In particular, $Y^* := \{y \in Y \colon \mathscr{M}_{Y,\overline{y}} := \mathscr{O}^*_{Y,\overline{y}}\}$ is Zariski open in Y, called the *locus of triviality* of Y.

(iii) For each $n \in \mathbb{N}$,

$$Y(n) := \left\{y \in Y \colon \operatorname{rank}\left(\overline{\mathscr{M}}_{Y,\overline{y}}\right) = n\right\} \subset Y^{(n)}$$

is a Zariski closed subscheme of $Y^{(n)}$ and has the following étale-local description: for all $\overline{y} \in Y(n)$, we have $\mathscr{O}_{Y(n),\overline{y}} = \mathscr{O}_{Y,\overline{y}}/I(\overline{y})$, where $I(\overline{y})$ is the ideal of $\mathscr{O}_{Y,\overline{y}}$ generated by the image of the unique maximal ideal $\mathscr{M}^+_{Y,\overline{y}}$ of $\mathscr{M}_{Y,\overline{y}}$ under $\alpha_{Y,\overline{y}} \colon \mathscr{M}_{Y,\overline{y}} \to \mathscr{O}_{Y,\overline{y}}$. See [AT17, Section 2.2.10].

(iv) After replacing Y by an étale neighbourhood of $\overline{y}$, the scheme Y admits a fine chart $M \to H^0(Y, \mathscr{M}_Y)$ which is neat at $\overline{y}$; that is, the composition $M \to H^0(Y, \mathscr{M}_Y) \to \mathscr{M}_{Y,\overline{y}} \to \overline{\mathscr{M}}_{Y,\overline{y}}$ is an isomorphism. See [Ogu18, Proposition III.1.2.7]. In particular, étale locally, every logarithmic scheme Y is a Zariski logarithmic scheme.

If $\mathscr{M}_Y$ is Zariski, all statements apply with $\overline{y}$ replaced by the scheme-theoretic point $y \in Y$, and statement (iv) holds after replacing Y by a Zariski neighbourhood of y.

Definition B.3 (Logarithmic stratification). Let Y be a fs logarithmic scheme. The *logarithmic stratification* of Y is the stratification given by $\{Y(n) \colon n \in \mathbb{N}\}$ in Remark B.2(iii). For each $y \in Y$, we set $\mathfrak{s}_y = Y(n)$ for $n = \operatorname{rank}\left(\overline{\mathscr{M}}^{\mathrm{gp}}_{Y,\overline{y}}\right)$, and $\mathfrak{s}_y$ is called the logarithmic stratum through y.

Definition B.4. We say that a fs logarithmic scheme Y is *logarithmically regular* at a point $y \in Y$ if for some (and hence any) geometric point $\overline{y}$ over y,

$$\mathfrak{s}_y \text{ is regular at } \overline{y} \text{ and the equality } \dim(\mathscr{O}_{Y,\overline{y}}) = \operatorname{rank}\left(\overline{\mathscr{M}}^{\mathrm{gp}}_{Y,\overline{y}}\right) + \dim(\mathscr{O}_{\mathfrak{s}_y,\overline{y}}) \text{ holds}.$$

If Y is a fs Zariski logarithmic scheme, we say that Y is logarithmically regular at $y \in Y$ if the same statement holds with $\overline{y}$ replaced by the scheme-theoretic point y throughout. We say that Y is logarithmically regular if Y is logarithmically regular at every point $y \in Y$.

Remark B.5. Let Y be a fs logarithmic scheme.

(i) In general, for every $y \in Y$, we have $\dim(\mathscr{O}_{Y,\overline{y}}) \leqslant \operatorname{rank}\left(\overline{\mathscr{M}}^{\mathrm{gp}}_{Y,\overline{y}}\right) + \dim(\mathscr{O}_{\mathfrak{s}_y,\overline{y}})$; see [Kat94, Lemma 2.3].

(ii) Let $U = Y^*$ be the triviality locus of Y, with open embedding into Y denoted by j. If Y is logarithmically regular, then $\alpha_Y \colon \mathscr{M}_Y \to \mathscr{O}_Y$ is injective, and the image of α_Y is $j_*(\mathscr{O}^*_U) \cap \mathscr{O}_Y$. If $D = Y \setminus U$ is non-empty, then D is a divisor on Y, called the toroidal divisor of Y. See [Kat94, Theorem 3.2.4] and [Niz06, Proposition 2.6].

If $\mathscr{M}_Y$ is Zariski, then the above statements hold with $\overline{y}$ replaced by the scheme-theoretic point $y \in Y$. In addition, the following hold:

(iii) If Y is logarithmically regular, $\underline{Y}$ is Cohen–Macaulay and normal [Kat94, Theorem 4.1]. In particular, $\underline{Y}$ is reduced, and if Y is locally Noetherian, $\underline{Y}$ is a disjoint union of its irreducible components. Moreover, $\underline{Y}$ is catenary, so each non-empty logarithmic stratum $Y(n)$ of Y has pure codimension n.

(iv) If Y is logarithmically regular at all closed points in Y, then Y is logarithmically regular [Kat94, Proposition 7.1].

We collate the aforementioned properties in the following definition.

Definition B.6 (Toroidal $\Bbbk$-schemes [AT17, Section 2.3.4]). Let $\Bbbk$ be a field of characteristic zero. A *toroidal* $\Bbbk$*-scheme* is a fs logarithmic $\Bbbk$-scheme Y which is logarithmically regular, such that $\underline{Y}$ is of finite type over $\Bbbk$. If, moreover, $\mathscr{M}_Y$ is a Zariski logarithmic structure, then we say that Y is a strict toroidal $\Bbbk$-scheme.

Note that every regular $\Bbbk$-scheme is a toroidal $\Bbbk$-scheme when we equip it with the trivial logarithmic structure. The remark in [AT17, Remark 2.3.5] deserves mention here: if $\Bbbk = \overline{\Bbbk}$, strict toroidal $\Bbbk$-varieties correspond to the toroidal embeddings without self-intersections in [KKMS73]. More generally, toroidal $\Bbbk$-varieties correspond to general toroidal embeddings, possibly with self-intersections.

Remark B.7. Let Y be a strict toroidal $\Bbbk$-scheme.

(i) For every $y \in Y$, fix $x_1, \ldots, x_n \in \mathscr{O}_{Y,y}$ which reduce to a regular system of parameters $x_1, \ldots, x_n$ of $\mathscr{O}_{\mathfrak{s}_y,y}$, fix a local fs chart $\beta\colon M \to H^0(U, \mathscr{M}_Y|_U)$ at y which is neat at y, and fix a coefficient field κ for $\widehat{\mathscr{O}}_{Y,y}$. Then the induced surjective homomorphism

$$\kappa[\![X_1, \ldots, X_n, M = \overline{\mathscr{M}}_{Y,y}]\!] \to \widehat{\mathscr{O}}_{Y,y}\,, \quad X_i \mapsto x_i$$

is an isomorphism [Kat94, Theorem 3.2(1)].

(ii) Endow Spec($\Bbbk$) with the trivial logarithmic structure. Then Y is logarithmically smooth over $\Bbbk$. Moreover, if $\widetilde{Y}$ is a fs Zariski logarithmic $\Bbbk$-scheme which admits a logarithmically smooth morphism $f\colon \widetilde{Y} \to Y$ to a strict toroidal $\Bbbk$-scheme Y, then $\widetilde{Y}$ is also a strict toroidal $\Bbbk$-scheme. See [Kat94, Proposition 8.3].

Étale locally, every toroidal $\Bbbk$-scheme is a strict toroidal $\Bbbk$-scheme (Remark B.2(iv)). Therefore, if we want to understand the étale-local structure of toroidal $\Bbbk$-schemes, it suffices to explicate the local structure of strict toroidal $\Bbbk$-schemes. We shall do this via a choice of logarithmic coordinates and parameters.

Definition B.8 (Logarithmic coordinates and parameters [ATW20a, Section 3.1.2]). Let Y be a strict toroidal $\Bbbk$-scheme, and let $y \in Y$. Set $n = \operatorname{codim}_{\mathfrak{s}_y} \overline{\{y\}}$, $N = \dim(\mathfrak{s}_y)$, and $M = \overline{\mathscr{M}}_{Y,y}$. By a system of *logarithmic coordinates* at y, we mean the following data:

(i) sections $x_1, \ldots, x_N$ of $\mathscr{O}_{Y,y}$ whose images under $\mathscr{O}_{Y,y} \twoheadrightarrow \mathscr{O}_{\mathfrak{s}_y,y} \xrightarrow{d} \Omega^1_{\underline{\mathfrak{s}_y},y}$ reduce to a $\kappa(y)$-basis for $\Omega^1_{\underline{\mathfrak{s}_y}}(y)$, and such that the images of the first n sections $x_1, \ldots, x_n$ in $\mathscr{O}_{\mathfrak{s}_y,y}$ form a regular system of parameters of $\mathscr{O}_{\mathfrak{s}_y,y}$;

(ii) and a local fs chart $\beta\colon M \to H^0(U, \mathscr{M}_Y|_U)$ at y which is neat at y.

We usually denote this data by

$$\big((x_1, \ldots, x_N),\ M = \overline{\mathscr{M}}_{Y,y} \xrightarrow{\beta} H^0(U, \mathscr{M}_Y|_U)\big)\,.$$

We call $\{x_1, \ldots, x_N\}$ a system of *ordinary coordinates* at y, and we call the subset $\{x_1, \ldots, x_n\}$ a system of *ordinary parameters* at y. The sub-data

$$\big((x_1, \ldots, x_n), M = \overline{\mathscr{M}}_{Y,y} \xrightarrow{\beta} H^0(U, \mathscr{M}_Y|_U)\big)$$

is called a system of logarithmic parameters at y.

The elements of $\alpha_Y(\beta(M\setminus\{0\}))$ are called *monomial parameters* at y. For an element $m\in M\setminus\{0\}$, we generally use the same letter m for $\beta(m)$ and write $\exp(m)$ for the monomial parameter $\alpha_Y(\beta(m))$. If (d,D) denotes the universal logarithmic derivation $\mathscr{O}_Y\oplus\mathscr{M}_Y\to\Omega^1_Y$, this notation should remind you of the "exponential rule" in calculus: $d(\exp(m))=\exp(m)\cdot Dm$.

In what follows, we denote the logarithmic tangent sheaf of a strict toroidal $\Bbbk$-scheme Y over $\Bbbk$ by $\mathscr{D}^1_Y$ (instead of the usual T^1_Y or $T^1_{Y/\Bbbk}$).

Lemma B.9. *Let Y be a strict toroidal $\Bbbk$-scheme. Let $y\in Y$, and fix a system of logarithmic parameters $\big((x_1,\dots,x_N),\ M=\overline{\mathscr{M}}_{Y,y}\xrightarrow{\beta}H^0(U,\mathscr{M}_Y|_U)\big)$ at y.*

(i) *The universal logarithmic derivation $(d,D)\colon\mathscr{O}_Y\oplus\mathscr{M}_Y\to\Omega^1_Y$ induces a natural isomorphism $\Omega^1_{Y,y}\xleftarrow{\simeq}\big(\bigoplus_{i=1}^N\mathscr{O}_{Y,y}\cdot dx_i\big)\oplus\big(\mathscr{O}_{Y,y}\otimes\overline{\mathscr{M}}^{\mathrm{gp}}_{Y,y}\big)$. In particular, $\Omega^1_{Y,y}$ is generated as a $\mathscr{O}_{Y,y}$-module by dx_i for $1\leqslant i\leqslant N$, as well as $D(M)$.*

(ii) *For every element L of $\operatorname{Hom}\big(\overline{\mathscr{M}}^{\mathrm{gp}}_{Y,y},\mathscr{O}_{Y,y}\big)$, there exists a unique derivation $(\mathbb{D}_L,L)\in\mathscr{D}^1_{Y,y}$ such that $\mathbb{D}_L(\exp(m))=\exp(m)\cdot L(m)$ for every monomial parameter $\exp(m)$ and $\mathbb{D}_L(x_i)=0$ for every $1\leqslant i\leqslant N$. This defines an isomorphism $\mathscr{D}^1_{Y,y}\xleftarrow{\simeq}\big(\bigoplus_{i=1}^N\mathscr{O}_{Y,y}\cdot\partial/\partial x_i\big)\oplus\operatorname{Hom}\big(\overline{\mathscr{M}}^{\mathrm{gp}}_{Y,y},\mathscr{O}_{Y,y}\big)$, where $\partial/\partial x_i$ is the derivation dual to x_i.*

(iii) *Fix a basis $m_1,\dots,m_r\in M$ for $M^{\mathrm{gp}}=\overline{\mathscr{M}}^{\mathrm{gp}}_{Y,y}$, and write $u_i=\exp(m_i)$ for $1\leqslant i\leqslant r$. Then $\Omega^1_{Y,y}$ is a free $\mathscr{O}_{Y,y}$-module with basis $dx_1,\dots,dx_N,du_1/u_1,\dots,du_r/u_r$, and $\mathscr{D}^1_{Y,y}$ is a free $\mathscr{O}_{Y,y}$-module with dual basis $\partial/\partial x_1,\dots,\partial/\partial x_N,u_1\partial/\partial u_1,\dots,u_r\partial/\partial u_r$.*

Sketch of a proof. Let $\mathbb{A}_M=\operatorname{Spec}(M\to\Bbbk[M])$. Adapting the diagram in the proof of [Ogu18, Theorem IV.3.3.3], one can deduce the split short exact sequence

$$0\to\kappa(y)\otimes\overline{\mathscr{M}}^{\mathrm{gp}}_{Y,y}\simeq\Omega^1_{\mathbb{A}_M/\Bbbk}(y)\to\Omega^1_{Y/\Bbbk}(y)\to\Omega^1_{Y/\underline{Y}}(y)\simeq\Omega^1_{\underline{\mathfrak{s}_y}}(y)\to0\,,$$

from which part (i) follows by Nakayama's lemma. Part (ii) is the dual of part (i), and part (iii) follows from parts (i) and (ii). (An alternative proof can be found in [ATW20a, Lemma 3.34].) □

We can now explicate the local structure of strict toroidal $\Bbbk$-schemes.

Theorem B.10. *Let Y be a strict toroidal $\Bbbk$-scheme. Fix $y\in Y$, and set $M=\overline{\mathscr{M}}_{Y,y}$. Then the following statements hold:*

(i) *After replacing Y with a Zariski neighbourhood of y, the scheme Y admits a strict morphism $f\colon Y\to\operatorname{Spec}(M\to\Bbbk[M])$.*

(ii) *After replacing Y with a Zariski neighbourhood of y, the morphism f admits a factorization*

$$U\xrightarrow{f_1}\operatorname{Spec}(M\to\Bbbk[M\oplus\mathbb{N}^n])\xrightarrow{g_1}\operatorname{Spec}(M\to\Bbbk[M])\,,$$

where $n=\operatorname{codim}_{\mathfrak{s}_y}\overline{\{y\}}$, f_1 is strict and smooth of relative dimension $\dim\overline{\{y\}}$, f_1 maps y to the vertex of $\operatorname{Spec}(M\to\Bbbk[M\oplus\mathbb{N}^n])$, and g_1 is induced by the inclusion $M\hookrightarrow M\oplus\mathbb{N}^n$.

(iii) *After replacing Y with a Zariski neighbourhood of y, the morphism f_1 admits a factorization*

$$U\xrightarrow{f_2}\operatorname{Spec}\big(M\to\Bbbk\big[M\oplus\mathbb{N}^N\big]\big)\xrightarrow{g_2}\operatorname{Spec}\big(M\to\Bbbk\big[M\oplus\mathbb{N}^n\big]\big)\,,$$

where $N=\dim(\mathfrak{s}_y)$, f_2 is strict and étale, and g_2 is induced by the inclusion $\mathbb{N}^n\hookrightarrow\mathbb{N}^N$ into the first n coordinates.

Sketch of a proof. Part (i) follows from Remark B.2(iv). The remaining parts follow from Lemma B.9 and [Ogu18, Theorem IV.3.2.3(2) and Proposition IV.3.16]. □

The remainder of this section reviews some notions developed in [ATW20a, Section 3] which are pertinent to this paper.

Definition B.11 (Logarithmic differential operators). Let Y be a strict toroidal $\Bbbk$-scheme.

(i) For each natural number $n \geqslant 1$, let $\mathscr{D}_Y^{\leqslant n}$ be the $\mathscr{O}_Y$-submodule of the total sheaf $\mathscr{D}_{\underline{Y}}^{\infty}$ of differential operators on $\underline{Y}$ generated by $\mathscr{O}_Y$ and the images of $\left(\mathscr{D}_Y^1\right)^{\otimes i}$ for $1 \leqslant i \leqslant n$. The submodule $\mathscr{D}_Y^{\leqslant n}$ is called the sheaf of *logarithmic differential operators on Y of order at most n*.

(ii) The direct limit $\bigcup_{n\in\mathbb{N}} \mathscr{D}_Y^{(\leqslant n)} \subset \mathscr{D}_{\underline{Y}}^{\infty}$ is called the *total sheaf of logarithmic differential operators* of Y and is denoted by $\mathscr{D}_Y^{\infty}$.

(iii) Given an ideal $\mathcal{I}$ on Y, let $\mathscr{D}_Y^{\leqslant n}(\mathcal{I})$ (respectively, $\mathscr{D}_Y^{\infty}(\mathcal{I})$) denote the ideal on Y generated by the image of $\mathcal{I}$ under $\mathscr{D}_Y^{\leqslant n}$ (respectively, under $\mathscr{D}_Y^{\infty}$).

When Y is clear from context, we usually write $\mathscr{D}_Y^{\leqslant n}$ as $\mathscr{D}^{\leqslant n}$ (likewise for $\mathscr{D}_Y^{\infty}$). We caution the reader that the definition in Definition B.11 only makes sense for $\mathrm{char}(\Bbbk) = 0$.

Definition B.12 (Monomial ideals and saturation). Let Y be a strict toroidal $\Bbbk$-scheme, and let $\mathcal{I}$ be an ideal on Y.

(i) We say that $\mathcal{I}$ is a *monomial ideal* if it is generated by the image of an ideal $\mathcal{Q} \subset \mathscr{M}_Y$ under $\alpha_Y : \mathscr{M}_Y \to \mathscr{O}_Y$.

(ii) The *monomial saturation* of $\mathcal{I}$, denoted by $\mathscr{M}(\mathcal{I})$, is defined to be the intersection of the collection of all monomial ideals on Y containing $\mathcal{I}$.

Evidently, $\mathscr{M}(\mathcal{I})$ contains $\mathcal{I}$, and $\mathcal{I}$ is monomial if and only if $\mathcal{I} = \mathscr{M}(\mathcal{I})$.

Lemma B.13. *Let Y be a strict toroidal $\Bbbk$-scheme. The following statements hold for an ideal $\mathcal{I}$ on Y:*

(i) *The ideal $\mathcal{I}$ is monomial if and only if $\mathscr{D}_Y^{\leqslant 1}(\mathcal{I}) = \mathcal{I}$.*

(ii) $\mathscr{D}_Y^{\infty}(\mathcal{I}) = \mathscr{M}(\mathcal{I})$.

(iii) *If $f : \widetilde{Y} \to Y$ is a logarithmically smooth morphism of strict toroidal $\Bbbk$-schemes, then $\mathscr{D}_{\widetilde{Y}}^{\leqslant n}(\mathcal{I}\mathscr{O}_{\widetilde{Y}}) = \mathscr{D}_Y^{\leqslant n}(\mathcal{I})\mathscr{O}_{\widetilde{Y}}$ for all natural numbers $n \geqslant 1$, and $\mathscr{M}(\mathcal{I}\mathscr{O}_{\widetilde{Y}}) = \mathscr{M}(\mathcal{I})\mathscr{O}_{\widetilde{Y}}$.*

(iv) *If $\mathscr{Q}$ is a monomial ideal on Y, then $\mathscr{D}_Y^{\leqslant n}(\mathscr{Q} \cdot \mathcal{I}) = \mathscr{Q} \cdot \mathscr{D}_Y^{\leqslant n}(\mathcal{I})$ for all natural numbers $n \geqslant 1$.*

Proof. This is [ATW20a, Corollary 3.3.12, Theorem 3.4.2, and Lemma 3.5.2]. □

Definition B.14 (Logarithmic order). Let Y be a strict toroidal $\Bbbk$-scheme. If $\mathcal{I}$ is an ideal on Y, the *logarithmic order* of $\mathcal{I}$ at a point $y \in Y$ is defined as

$$\text{log-ord}_y(\mathcal{I}) = \mathrm{ord}_y(\mathcal{I}|_{\mathfrak{s}_y}) \in \mathbb{N} \cup \{\infty\},$$

where ord_y refers to the usual order of an ideal at a point (see, for example, [Kol07, Definition 3.47]). The maximal logarithmic order of $\mathcal{I}$ is $\max \text{log-ord}(\mathcal{I}) = \max_{y\in Y} \text{log-ord}_y(\mathcal{I})$.

Lemma B.15. *Let Y be a strict toroidal $\Bbbk$-scheme. The following statements hold for an ideal $\mathcal{I}$ on Y and a point $y \in Y$:*

(i) *We have* $\operatorname{log-ord}_y(\mathcal{I}) = \min\{n \in \mathbb{N} \colon \mathscr{D}_Y^{\leqslant n}(\mathcal{I})_y = \mathscr{O}_{Y,y}\}$, *where we take* $\min(\emptyset) = \infty$ *by convention.*

(ii) *We have* $\operatorname{log-ord}_y(\mathcal{I}) = \infty$ *if and only if* $y \in V(\mathscr{M}(\mathcal{I}))$.

(iii) *We have* $\mathscr{M}(\mathcal{I}) = (1)$ *if and only if* $\max \operatorname{log-ord}(\mathcal{I}) < \infty$.

(iv) *If $f \colon \widetilde{Y} \to Y$ is a logarithmically smooth morphism of strict toroidal $\Bbbk$-schemes and $\widetilde{y} \in \widetilde{Y}$ maps to $y \in Y$, then* $\operatorname{log-ord}_{\widetilde{y}}(\mathcal{I}\mathscr{O}_{\widetilde{Y}}) = \operatorname{log-ord}_y(\mathcal{I})$.

Note that parts (i) and (ii) say that $\operatorname{log-ord}_y(\mathcal{I})$ is upper semi-continuous on Y: (i) for a natural number n, the vanishing locus $V\big(\mathscr{D}_Y^{\leqslant n}(\mathcal{I})\big)$ is the locus of points $y \in Y$ satisfying $\operatorname{log-ord}_y(\mathcal{I}) > n$; (ii) the vanishing locus $V(\mathscr{M}(\mathcal{I}))$ is the locus of points $y \in Y$ satisfying $\operatorname{log-ord}_y(\mathcal{I}) = \infty$.

Proof. This is [ATW20a, Lemmas 3.6.3 and 3.6.5, Corollary 3.66 and Lemma 3.6.8]. □

B.2 Toroidal Deligne-Mumford stacks over $\Bbbk$

Let $\Bbbk$ be a field of characteristic zero. Before defining the notion of a toroidal Deligne–Mumford stack over $\Bbbk$, we recall some preliminaries from [ATW20b, Section 3.3]. A logarithmic structure $\mathscr{M}_Y$ on a Deligne–Mumford stack Y is a sheaf of monoids on the étale site $Y_{\text{ét}}$ and a homomorphism $\alpha_Y \colon \mathscr{M}_Y \to \mathscr{O}_{Y_{\text{ét}}}$ inducing an isomorphism $\mathscr{M}_Y^* \xrightarrow{\simeq} \mathscr{O}_{Y_{\text{ét}}}^*$. The pair $(Y, \mathscr{M}_Y)$ is called a logarithmic Deligne–Mumford stack. If $p_{1,2} \colon Y_1 \rightrightarrows Y_0$ is an atlas of Y by schemes, then a logarithmic structure $\mathscr{M}_Y$ on Y is equivalent to logarithmic structures $\mathscr{M}_{Y_i}$ on Y_i (for $i = 0, 1$) such that $p_1^*\mathscr{M}_{Y_0} = \mathscr{M}_{Y_1} = p_2^*\mathscr{M}_{Y_0}$. We say that a logarithmic Deligne–Mumford stack is *fs* if for some (and hence any) atlas $p_{1,2} \colon Y_1 \rightrightarrows Y_0$ of Y by schemes, $(Y_0, \mathscr{M}_{Y_0})$ is fs.

Definition B.16 (Toroidal DM stacks [ATW20b, Section 3.3.3]). A *toroidal Deligne–Mumford stack over* $\Bbbk$ is a fs logarithmic Deligne–Mumford stack $(Y, \mathscr{M}_Y)$ over $\Bbbk$ admitting an atlas $p_{1,2} \colon Y_1 \rightrightarrows Y_0$ by schemes such that $(Y_0, \mathscr{M}_{Y_0})$ is a toroidal $\Bbbk$-scheme.

If Y is a toroidal Deligne–Mumford stack over $\Bbbk$, then $(Y_0, \mathscr{M}_{Y_0})$ is a toroidal $\Bbbk$-scheme for every atlas $p_{1,2} \colon Y_1 \rightrightarrows Y_0$ of Y by schemes. This follows from [GR18, Proposition 12.5.46]. Moreover, since étale locally every toroidal $\Bbbk$-scheme is a strict toroidal $\Bbbk$-scheme, we may choose the atlas in Definition B.16 such that $(Y_0, \mathscr{M}_{Y_0})$ is a strict toroidal $\Bbbk$-scheme. In this case, Y_1 is also a strict toroidal $\Bbbk$-scheme.

Appendix C. Proof of Theorem 5.1

The proof of Theorem 5.1 follows ideas from both [ATW20a, Lemma 5.3.3] and [Kol07, Theorem 3.92]. In particular, we need to make a modification to [Kol07, Proposition 3.94]. Let us first fix some notation: let $\Bbbk$ be a field of characteristic zero, $\kappa/\Bbbk$ be a field extension, and M be a sharp monoid (written multiplicatively), and consider the logarithmic $\Bbbk$-algebra $M \to \kappa[\![\mathbb{N}^n \oplus M]\!] = \kappa[\![x_1, \dots, x_n, M]\!] = R$, with maximal ideal $\mathfrak{m} = (x_1, \dots, x_n, M \setminus \{1\})$. For a proper ideal $J \subset \mathfrak{m}$ of R, we say that an automorphism ψ of R is of the form $\mathbb{1} + J$ if ψ maps each x_i to $x_i + f_i$ for some $f_i \in J$ and fixes M. For an ideal $I \subset R$, we have

$$\mathscr{D}^{\leqslant 1}(I) = I + \left(\frac{\partial f}{\partial x_i} \colon f \in I,\ 1 \leqslant i \leqslant n\right),$$

and, inductively, we have $\mathscr{D}^{\leqslant \ell}(I) = \mathscr{D}\big(\mathscr{D}^{\leqslant \ell - 1}(I)\big)$ for all $\ell \geqslant 2$.

Lemma C.1. *Let the notation be as above, and let $I \subset R$ be an ideal. The following statements are equivalent:*

(i) *We have $\psi(I) = I$ for every automorphism ψ of the form $\mathbb{1} + J$.*
(ii) *We have $J \cdot \mathscr{D}^{\leqslant 1}(I) \subset I$.*
(iii) *We have $J^\ell \cdot \mathscr{D}^{\leqslant \ell}(I) \subset I$ for every $\ell \geqslant 1$.*

Proof. This proof proceeds in the same way as that of [Kol07, Proposition 3.94], with minor modifications.

Assume statement (iii). Let ψ be an automorphism of the form $\mathbb{1} + J$, and for all $1 \leqslant i \leqslant n$, let $b_i \in J$ be such that $\psi(x_i) = x_i + b_i$. Then Taylor expansion gives us

$$\psi(f) = f + \sum_{i=1}^{n} b_i \frac{\partial f}{\partial x_i} + \frac{1}{2} \sum_{i,j=1}^{n} b_i b_j \frac{\partial^2 f}{\partial x_i \partial x_j} + \cdots .$$

For any $\ell \geqslant 1$, we get

$$\psi(f) \in I + J \cdot \mathscr{D}^{\leqslant 1}(I) + \cdots + J^\ell \cdot \mathscr{D}^{\leqslant \ell}(I) + \mathfrak{m}^{\ell+1} \subset I + \mathfrak{m}^{\ell+1} .$$

By Krull's intersection theorem, this implies $\psi(f) \in I$, so we get statement (i).

Next, assume statement (i). Let $b \in J$, and let $1 \leqslant i \leqslant n$. For general $\lambda \in \mathbb{k}$, the endomorphism on R which maps $(x_1, \dots, x_n)$ to $(x_1, \dots, x_{i-1}, x_i + \lambda b, x_{i+1}, \dots, x_n)$ and fixes M is an automorphism of R of the form $\mathbb{1} + J$. Therefore, for every $f \in I$ and every $\ell \geqslant 1$,

$$\left(f + \lambda b \frac{\partial f}{\partial x_i} + \cdots + (\lambda b)^\ell \frac{\partial^\ell f}{\partial x_i^\ell} \right) \in \psi(f) + \mathfrak{m}^{\ell+1} \subset I + \mathfrak{m}^{\ell+1} .$$

For $\ell + 1$ general elements $\lambda = \lambda_0, \dots, \lambda_\ell$ in $\mathbb{k}$, the column vector obtained from

$$\begin{pmatrix} 1 & \lambda_0 & \lambda_0^2 & \cdots & \lambda_0^\ell \\ 1 & \lambda_1 & \lambda_1^2 & \cdots & \lambda_1^\ell \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \lambda_\ell & \lambda_\ell^2 & \cdots & \lambda_\ell^\ell \end{pmatrix} \cdot \begin{pmatrix} f \\ b \frac{\partial f}{\partial x_i} \\ \vdots \\ b^\ell \frac{\partial^\ell f}{\partial x_i^\ell} \end{pmatrix}$$

has entries in $I + \mathfrak{m}^{\ell+1}$, and the Vandermonde determinant (λ_i^j) is invertible. Therefore, $b \cdot \partial f / \partial x_i \in I + \mathfrak{m}^{\ell+1}$. By Krull's intersection theorem again, $b \cdot \partial f / \partial x_i \in I$. Since $J \cdot \mathscr{D}^{\leqslant 1}(I)$ is generated by elements of the form $b \cdot f$ or $b \cdot \partial f / \partial x_i$ for $b \in J$, $f \in I$, and $1 \leqslant i \leqslant n$, this proves statement (ii).

Finally, assume statement (ii). We prove by induction that $J^\ell \cdot \mathscr{D}^{\leqslant \ell}(I) \subset I$ for every $\ell \geqslant 1$. The ideal $J^{\ell+1} \cdot \mathscr{D}^{\leqslant \ell+1}(I)$ is generated by elements of the form $b_0 \cdots b_\ell \cdot \mathscr{D}^{\leqslant 1}(g)$ for $g \in \mathscr{D}^{\leqslant \ell}(I)$. The product rule says

$$\begin{aligned} b_0 \cdots b_\ell \cdot \mathscr{D}^{\leqslant 1}(g) &= b_0 \cdot \mathscr{D}^{\leqslant 1}(b_1 \cdots b_\ell \cdot g) - \sum_{i=1}^{\ell} \mathscr{D}^{\leqslant 1}(b_i) \cdot \big(b_0 \cdots \widehat{b_i} \cdots b_\ell \cdot g\big) \\ &\in J \cdot \mathscr{D}^{\leqslant 1}\big(J^\ell \cdot \mathscr{D}^{\leqslant \ell}(I)\big) + J^\ell \cdot \mathscr{D}^{\leqslant \ell}(I) \subset J \cdot \mathscr{D}^{\leqslant 1}(I) + J^\ell \cdot \mathscr{D}^{\leqslant \ell}(I) \subset I , \end{aligned}$$

where the last two inclusions hold by the induction hypothesis. This proves statement (iii). □

Proof of Theorem 5.1. Let $n = \operatorname{codim}_{\mathfrak{s}_y} \overline{\{y\}}$ and $M = \overline{\mathscr{M}}_{Y,y}$ be as in Definition B.8. There exist $x_2, \dots, x_n \in \mathscr{O}_{\mathfrak{s}_y, y}$ such that both $x, x_2, \dots, x_n$ and $x', x_2, \dots, x_n$ form regular systems of parameters of $\mathscr{O}_{\mathfrak{s}_y,y}$. By Remark B.7(ii), we have

$$\kappa[\![x, x_2 \ldots, x_n, M]\!] \simeq \widehat{\mathscr{O}}_{Y,y} \simeq \kappa[\![x', x_2, \ldots, x_n, M]\!], \quad \text{where } \kappa = \kappa(y).$$

Consider the endomorphism ψ of $\widehat{\mathscr{O}}_{Y,y}$ which maps $(x, x_2, \ldots, x_n)$ to $(x' = x + (x'-x), x_2, \ldots, x_n)$ and fixes M. Since $x', x_2, \ldots, x_n$ are linearly independent modulo $\mathfrak{m}_{Y,y}^2$ (where $\mathfrak{m}_{Y,y}$ is the maximal ideal of $\mathscr{O}_{Y,y}$), the endomorphism ψ is an automorphism of $\widehat{\mathscr{O}}_{Y,y}$. Moreover, since x and x' are maximal contact elements at y, we have $x' - x \in \widehat{\mathrm{MC}(\mathcal{I})} = \mathrm{MC}\,(\widehat{\mathcal{I}})$ (note that logarithmic derivatives commute with completions), whence ψ is an automorphism of $\widehat{\mathscr{O}}_{Y,y}$ of the form $\mathbb{1} + \mathrm{MC}\,(\widehat{\mathcal{I}})$. Finally, since $\mathcal{I}$ is MC-invariant, we have $\mathrm{MC}\,(\widehat{\mathcal{I}}) \cdot \mathscr{D}^{\leqslant 1}(\widehat{\mathcal{I}}) \subset \widehat{\mathcal{I}}$, whence Lemma C.1 implies $\psi(\widehat{\mathcal{I}}) = \widehat{\mathcal{I}}$.

Our goal now is to realize this automorphism ψ on $\widehat{\mathscr{O}}_{Y,y}$ on some strict, étale neighbourhood $\widetilde{U}$ of y. We first extend both $(x, x_2, \ldots, x_n)$ and $(x', x_2, \ldots, x_n)$ to systems of logarithmic coordinates at y (Definition B.8):

$$\big((x, x_2 \ldots, x_N), M \xrightarrow{\beta} H^0(U, \mathscr{M}_Y|_U)\big) \quad \text{and} \quad \big((x', x_2 \ldots, x_N), M \xrightarrow{\beta} H^0(U, \mathscr{M}_Y|_U)\big),$$

where $N = \dim(\mathfrak{s}_y)$. We then apply Theorem B.10: after shrinking U if necessary, U admits strict and étale morphisms

$$U \underset{\tau_{x'}}{\overset{\tau_x}{\rightrightarrows}} \mathrm{Spec}(M \to \Bbbk[X_1, \ldots, X_N, M])$$

induced by

(a) morphisms $U \rightrightarrows \mathbb{A}^N_{\Bbbk}$ induced by ring morphisms $\Bbbk[X_1, \ldots, X_n] \rightrightarrows \Gamma(U, \mathscr{O}_U)$ mapping $(X_1, X_2, \ldots, X_N)$ to $(x, x_2, \ldots, x_N)$ and $(X_1, X_2, \ldots, X_N)$ to $(x', x_2, \ldots, x_N)$, respectively;

(b) the chart $M = \overline{\mathscr{M}}_{Y,y} \xrightarrow{\beta} H^0(U, \mathscr{M}_Y|_U)$.

Finally, we obtain $\widetilde{U}$ in the statement of Theorem 5.1 by forming the following cartesian square (in the category of fs logarithmic schemes):

$$\begin{array}{ccc} \widetilde{U} & \xrightarrow{\phi_{x'}} & U \\ \downarrow{\scriptstyle \phi_x} & & \downarrow{\scriptstyle \tau_{x'}} \\ U & \xrightarrow{\tau_x} & \mathrm{Spec}(M \to \Bbbk[X_1, \ldots, X_N, M]). \end{array}$$

Since both τ_x and $\tau_{x'}$ are strict and étale, ϕ_x and $\phi_{x'}$ are also strict and étale. Moreover, $\phi_x^*(x) = \phi_x^*(\tau_x^*(X_1)) = \phi_{x'}^*(\tau_{x'}^*(X_1)) = \phi_{x'}^*(x')$. Note that τ_x and $\tau_{x'}$ maps y to the same point in $\mathrm{Spec}(M \to \Bbbk[X_1, \ldots, X_N, M])$, so there is a point $\widetilde{y} = (y, y) \in \widetilde{U}$ which is mapped to y via ϕ_x and $\phi_{x'}$. Finally, the completion of $\widetilde{U}$ at $\widetilde{y} = (y, y)$ is the graph of the automorphism ψ on $\widehat{\mathscr{O}}_{Y,y}$, and since $\psi(\widehat{\mathcal{I}}) = \widehat{\mathcal{I}}$, it follows (after shrinking $\widetilde{U}$ if necessary) that $\phi_x^*(\mathcal{I}) = \phi_{x'}^*(\mathcal{I})$. □

Acknowledgements

I would like to express my gratitude to my advisor Dan Abramovich for suggesting this project and for his continued patience and guidance. I would also like to thank Jonghyun Lee, David Rydh, Michael Temkin, Jarosław Włodarczyk, and the referee for clarifications, corrections, comments, and questions, as well as Tangli Ge and Stephen Obinna for discussions during early stages of this project.

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