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Essential dimension of extensions of finite groups by tori

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Abstract

Let p be a prime, k be a p-closed field of characteristic different from p, and $1 \to T \to G \to F \to 1$ be an exact sequence of algebraic groups over k, where T is a torus and F is a finite p-group. In this paper, we study the essential dimension $\mathrm{ed}(G;p)$ of G at p. R. Lötscher, M. MacDonald, A. Meyer, and the first author showed that

$$\min \dim(V) - \dim(G) \leqslant \mathrm{ed}(G;p) \leqslant \min \dim(W) - \dim(G)\,,$$

where V and W range over the p-faithful and p-generically free k-representations of G, respectively. In the special case where $G = F$, one recovers the formula for $\mathrm{ed}(F;p)$ proved earlier by N. Karpenko and A. Merkurjev. In the case where $F = T$, one recovers the formula for $\mathrm{ed}(T;p)$ proved earlier by R. Lötscher et al. In both of these cases, the upper and lower bounds on $\mathrm{ed}(G;p)$ given above coincide. In general, there is a gap between them. Lötscher et al. conjectured that the upper bound is, in fact, sharp; that is, $\mathrm{ed}(G;p) = \min \dim(W) - \dim(G)$, where W ranges over the p-generically free representations. We prove this conjecture in the case where F is diagonalizable.

1. Introduction

Let p be a prime integer and k be a p-closed field of characteristic different from p. That is, the degree of every finite extension l/k is a power of p. Consider an algebraic group G defined over k which fits into the exact sequence

$$1 \longrightarrow T \longrightarrow G \xrightarrow{\ \pi\ } F \longrightarrow 1\,, \tag{1.1}$$

where T is a (not necessarily split) torus and F is a (not necessarily constant) finite p-group defined over k. We say that a linear representation $G \to \mathrm{GL}(V)$ is p-faithful if its kernel is a finite subgroup of G of order prime to p and p-generically free if the isotropy subgroup G_v is a finite group of order prime to p for $v \in V(\overline{k})$ in general position. We denote by $\eta(G)$ (respectively, $\rho(G)$) the smallest dimension of a p-faithful (respectively, p-generically free) representation of G defined over k. R. Lötscher, M. MacDonald, A. Meyer, and the first author [LMMR13b, Theorem 1.1]

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have shown that the essential p-dimension $\operatorname{ed}(G;p)$ of G over k satisfies the inequalities

$$\eta(G) - \dim(G) \leqslant \operatorname{ed}(G;p) \leqslant \rho(G) - \dim(G)\,. \tag{1.2}$$

For the definition of $\operatorname{ed}(G;p)$, see Section 4.

The inequalities (1.2) represent a common generalization of the formulas for the essential p-dimension of a finite constant p-group, due to N. Karpenko and A. Merkurjev [KM08, Theorem 4.1] (where $T = \{1\}$), and of an algebraic torus, due to R. Lötscher et al. [LMMR13a] (where $F = \{1\}$). In both of these cases, every p-faithful representation of G is p-generically free, and thus $\eta(G) = \rho(G)$. In general, $\eta(G)$ can be strictly smaller than $\rho(G)$. Lötscher et al. conjectured that the upper bound of (1.2) is, in fact, sharp.

Conjecture 1.1. Let p be a prime integer, k be a p-closed field of characteristic different from p, and G be an affine algebraic group defined over k. Assume that the connected component $G^0 = T$ is a k-torus and the component group $G/G^0 = F$ is a finite p-group. Then

$$\operatorname{ed}(G;p) = \rho(G) - \dim G\,,$$

where $\rho(G)$ is the minimal dimension of a p-generically free k-representation of G.

Informally speaking, the lower bound of (1.2) is the strongest lower bound on $\operatorname{ed}(G;p)$ one can hope to prove by the methods of [KM08, LMMR13a] and [LMMR13b]. In the case where the upper and lower bounds of (1.2) diverge, Conjecture 1.1 calls for a new approach.

Conjecture 1.1 appeared in print in [Rei10, Section 7.9] on the list of open problems in the theory of essential dimension. The only bit of progress since then has been a proof in the special case where G is a semi-direct product of a cyclic group $F = \mathbb{Z}/p\mathbb{Z}$ of order p and a split torus $T = \mathbb{G}_{\mathrm{m}}^n$, due to M. Huruguen (unpublished). Huruguen's argument relies on the classification of integral representations of $\mathbb{Z}/p\mathbb{Z}$ due to F. Diederichsen and I. Reiner [CR62, Theorem 74.3]. So far, this approach has resisted all attempts to generalize it beyond the case where $G \simeq \mathbb{G}_{\mathrm{m}}^n \rtimes (\mathbb{Z}/p\mathbb{Z})$.

Note that $\eta(G)$ is often accessible by cohomological and/or combinatorial techniques; see Section 6 and Lemma 9.3, as well as the remarks after this lemma. Computing $\rho(G)$ is usually a more challenging problem. The purpose of this paper is to establish Conjecture 1.1 in the case where F is a diagonalizable abelian p-group. Moreover, our main result also gives a way of computing $\rho(G)$ in this case.

Theorem 1.2. *Let p be a prime integer, k be a p-closed field of characteristic different from p, and G be an extension of a (not necessarily constant) diagonalizable p-group F by a (not necessarily split) torus T, as in (1.1).*

(a) *We have $\operatorname{ed}(G;p) = \rho(G) - \dim G$.*
(b) *Moreover, suppose that V is a p-faithful representation of G of minimal dimension, $\overline{k}$ is the algebraic closure of k, and $S_V \subset G_{\overline{k}}$ is a stabilizer in general position for the $G_{\overline{k}}$-action on $V_{\overline{k}}$. Then $\rho(G) = \eta(G) + \operatorname{rank}_p(S_V)$.*

Here $\operatorname{rank}_p(S_V)$ is the largest r such that S_V contains a subgroup isomorphic to μ_p^r. Note that S_V exists by Lemma 2.1. Most of the remainder of this paper (Sections 2–8) will be devoted to proving Theorem 1.2. A key ingredient in the proof is the resolution theorem (Theorem 7.2), which is based, in turn, on an old valuation-theoretic result of M. Artin and O. Zariski [Art86, Theorem 5.2]. In Section 9, we will use Theorem 1.2 to complete the computation of $\operatorname{ed}(N;p)$ initiated in [MR09] and [Mac11]. Here N is the normalizer of a split maximal torus in a split simple algebraic group.

2. Stabilizers in general position

In this section, we assume that the base field k is algebraically closed. Let G be a linear algebraic group defined over k. A G-variety X is called primitive if G transitively permutes the irreducible components of X.

Let X be a primitive G-variety. A subgroup $S \subset G$ is called a stabilizer in general position for the G-action on X if there exists an open G-invariant subset $U \subset X$ such that $\mathrm{Stab}_G(x)$ is conjugate to S for every $x \in U(k)$. Note that a stabilizer in general position does not always exist. See [PV94, Example 7.1.1] for an easy example where G is unipotent; further examples, with $G = \mathrm{SL}_n$, can be found in [Ric72, Section 12.4]. When a stabilizer in general position $S \subset G$ exists, it is unique up to conjugacy.

Lemma 2.1. *Let G be a linear algebraic group over k and X be a primitive quasi-projective G-variety. Assume that the connected component $T = G^0$ is a torus and the component group $F = G/G^0$ is finite of order prime to* $\mathrm{char}(k)$. *Then there exists a stabilizer in general position $S \subset G$.*

Proof. After replacing G with $\overline{G} := G/(K \cap T)$, where K is the kernel of the G-action on X, we may assume that the T-action on X is faithful and, hence, generically free. In other words, for $x \in X(k)$ in general position, $\mathrm{Stab}_G(x) \cap T = 1$; in particular, $\mathrm{Stab}_G(x)$ is a finite p-group. Since $\mathrm{char}(k) \neq p$, Maschke's theorem tells us that $\mathrm{Stab}_G(x)$ is linearly reductive. Hence, for $x \in X(k)$ in general position, $\mathrm{Stab}_G(x)$ is G-completely reducible; see [Jan04, Lemma 11.24]. The lemma now follows from [Mar15, Corollary 1.5]. □

Remark 2.2. The condition that X is quasi-projective can be dropped if $k = \mathbb{C}$; see [Ric72, Theorem 9.3.1]. With a bit more effort, this condition can also be removed for any algebraically closed base field k of characteristic different from p. Since we shall not need this more general variant of Lemma 2.1, we leave its proof as an exercise for the reader.

We define the (geometric) p-rank $\mathrm{rank}_p(G)$ of an algebraic group G to be the largest integer r such that G contains a subgroup isomorphic to $\mu_p^r = \mu_p \times \cdots \times \mu_p$ (r times).

Lemma 2.3. *Let X be a normal G-variety and $Y \subset X$ be a G-invariant prime divisor of X. Let S_X and S_Y be stabilizers in general position of the G-actions on X and Y, respectively. Assume that p is a prime and* $\mathrm{char}(k) \neq p$.

(a) *We have* $\mathrm{rank}_p(S_Y) \leqslant \mathrm{rank}_p(S_X) + 1$.

(b) *Assume that the G-action on X is p-faithful. Denote the kernel of the G-action on Y by N. Then there is a group homomorphism $\alpha\colon N \to \mathbb{G}_{\mathrm{m}}$ such that* $\mathrm{Ker}(\alpha)$ *does not contain a subgroup of order p.*

Proof. Let $U \subset X$ be a G-invariant dense open subset of X such that $\mathrm{Stab}_G(x)$ is conjugate to S_X for every $x \in U(k)$. If $Y \cap U \neq \emptyset$, then $S_Y = S_X$, and we are done. Thus we may assume that Y is contained in $Z = X \setminus U$. Since Y is a prime divisor in X, it is an irreducible component of Z. After removing all other irreducible components of Z from X, we may assume that $Z = Y$. Since X is normal, Y intersects the smooth locus of X non-trivially. Choose a k-point $y \in Y$ such that both X and Y are smooth at y and $\mathrm{Stab}_G(y)$ is conjugate to S_Y. After replacing S_Y with a conjugate, we may assume that $\mathrm{Stab}_G(y) = S_Y$. The group $\mathrm{Stab}_G(y)$ acts on the tangent spaces $T_y(X)$ and $T_y(Y)$, hence on the 1-dimensional normal space $T_y(X)/T_y(Y)$. This gives rise to a character $\alpha\colon S_Y \to \mathbb{G}_{\mathrm{m}}$.

(a) Assume the contrary: S_Y contains μ_p^{r+2}, where $r = \mathrm{rank}_p(S_X)$. Then the kernel of α contains a subgroup $\mu \simeq \mu_p^{r+1}$. By Maschke's theorem, the natural projection $T_y(X) \to T_y(X)/T_y(Y)$ is μ-equivariantly split. Equivalently, there exists a μ-invariant tangent vector $v \in T_y(X)$ which does not belong to $T_y(Y)$. By the Luna slice theorem,

$$T_y(X)^{\mu} = T_y(X^{\mu})\,. \tag{2.1}$$

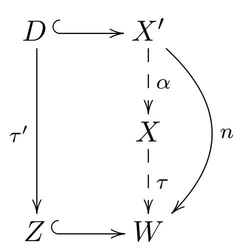
For a proof in characteristic 0, see [PV94, Section 6.5]. Generally speaking, Luna's theorem fails in prime characteristic, but (2.1) remains valid because μ is linearly reductive; see [BR85, Lemma 8.3]. Now observe that since μ does not fit into any conjugate of S_X, the subvariety X^{μ} is contained in $Y = X \setminus U$. Thus $v \in T_y(X)^{\mu} = T_y(X^{\mu}) \subset T_y(Y)$, which gives a contradiction.

(b) Let $y \in Y$ be a smooth k-point of X and Y, and $S_y = \mathrm{Stab}_G(y)$ as in part (a). Then N is contained in S_Y, and α restricts to a character $N \to \mathbb{G}_{\mathrm{m}}$. It suffices to show that the kernel of α in S_Y does not contain a subgroup of order p. Assume the contrary: a subgroup H of order p lies in the kernel of α. Then H fixes a smooth point y of X and acts trivially on both $T_y(Y)$ and $T_y(X)/T_y(Y)$ and hence (since H is linearly reductive) on $T_y(X)$. It is well known that this implies that H acts trivially on X; see, for example, the proof of [GR09, Lemma 4.1]. This contradicts our assumption that the G-action on X is p-faithful. □

3. Covers

Let k be an arbitrary field, and let G be a linear algebraic group defined over k. As usual, we will denote the algebraic closure of k by $\overline{k}$. A G-variety X is called primitive if the $G_{\overline{k}}$-variety $X_{\overline{k}}$ is primitive. A dominant G-equivariant rational map $X \dashrightarrow Y$ of primitive G-varieties is called a cover of degree d if $[k(X) : k(Y)] = d$. Here if $X_1, \ldots, X_n$ are the irreducible components of X, then $k(X)$ is defined as $k(X_1) \oplus \cdots \oplus k(X_n)$.

Lemma 3.1. *Let p be a prime integer, G be a smooth algebraic group such that G/G^0 is a finite p-group, W be an irreducible G-variety, $Z \subset W$ be an irreducible G-invariant divisor in W, and $\tau\colon X \dashrightarrow W$ be a G-equivariant cover of degree prime to p. Then there exists a commutative diagram of G-equivariant maps*



such that X' is normal, α is a birational isomorphism, D is an irreducible divisor in X', and τ' is a cover of Z of degree prime to p.

Proof. Let X' be the normalization of W in the function field $k(X)$. Since G acts compatibly on W and X, there is a G-action on X' such that the normalization map $n\colon X' \to W$ is G-equivariant. Over the dense open subset of W where τ is finite, n factors through X. Thus n factors into a composition of a birational isomorphism $\alpha\colon X' \dashrightarrow X$ and $\tau\colon X \dashrightarrow W$. This gives us the right column in the diagram.

To construct D, we argue as in the proof of [RY00, Proposition A.4]. Denote the irreducible components of the preimage of Z under n by $D_1, \ldots, D_r \subset X'$. These components are permuted by G. Denote the orbits of this permutation action by $\mathcal{O}_1, \ldots, \mathcal{O}_m$. After renumbering $D_1, \ldots, D_r$, we may assume that $D_i \in \mathcal{O}_i$ for $i = 1, \dots, m$. By the ramification formula (see, for example, [Lan02, XII, Corollary 6.3]),

$$d = \sum_{i=1}^{m} |\mathcal{O}_i| \cdot [D_i : Z] \cdot e_i \,,$$

where $[D_i : Z]$ denotes the degree of the cover $n_{|D_i} \colon D_i \to Z$ and e_i is the ramification index of n at the generic point of D_i. Since d is prime to p and each $|\mathcal{O}_i|$ is a power of p, we conclude that there exists an $i \in \{1, \dots, m\}$ such that $|\mathcal{O}_i| = 1$ (that is, D_i is G-invariant) and $[D_i : Z]$ is prime to p. We now set $D = D_i$ and $\tau' = n_{|D_i}$. □

Lemma 3.2. *Let G be a linear algebraic group over an algebraically closed field k, $p \neq \mathrm{char}(k)$ be a prime number, and $\tau \colon X \dashrightarrow W$ be a cover of G-varieties of degree d. Assume that stabilizers in general position for the G-actions on X and W exist; denote them by S_X and S_W, respectively. Assume that d is prime to p.*

(a) *If H is a finite p-subgroup of S_W, then S_X contains a conjugate of H.*

(b) *We have $\mathrm{rank}_p(S_X) = \mathrm{rank}_p(S_W)$.*

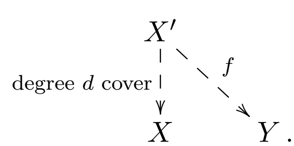
Proof. (a) After replacing W with a dense open subvariety, we may assume that the stabilizer of every point in W is a conjugate of S_W. Furthermore, after replacing X with the normal closure of W in $k(X)$, we may assume that τ is a finite morphism. We claim that $W^{S_W} \subset \tau(X^H)$. Indeed, suppose $w \in W^{S_W}$. Then H acts on $\tau^{-1}(w)$, which is a zero-cycle on X of degree d. Since H is a p-group, it fixes a k-point in $\tau^{-1}(w)$. Hence, $X^H \cap \tau^{-1}(w) \neq \emptyset$ or, equivalently, $w \in \tau\big(X^H\big)$. This proves the claim.

Since the stabilizer of every point of W is conjugate to S_W, we have $G \cdot W^{S_W} = W$. By the claim, $\tau\big(G \cdot X^H\big) = G \cdot \tau\big(X^H\big) = W$. Since G acts transitively on the irreducible components of X, this implies that $G \cdot X^H$ contains a dense open subset $X_0 \subset X$. In other words, the stabilizer of every point of X_0 contains a conjugate of H, and part (a) follows.

(b) Clearly $S_X \subset S_W$ and thus $\mathrm{rank}_p(S_X) \leqslant \mathrm{rank}_p(S_W)$. On the other hand, if S_W contains $H = \mu_p^r$ for some $r \geqslant 0$, then by part (a), the group S_X also contains a copy of μ_p^r. This proves the opposite inequality, $\mathrm{rank}_p(S_X) \geqslant \mathrm{rank}_p(S_W)$. □

4. Essential p-dimension

Let X and Y be G-varieties. Assume that X is primitive. By a G-equivariant correspondence $X \rightsquigarrow Y$ of degree d, we mean a diagram of rational maps



Here we require X' to be primitive. We say that this correspondence is dominant if f is dominant. A rational map may be viewed as a correspondence of degree 1.

The *essential dimension* $\mathrm{ed}(X)$ of a generically free G-variety X is the minimal value of $\dim(Y) - \dim(G)$, where the minimum is taken over all generically free G-varieties Y admitting a dominant rational map $X \dashrightarrow Y$. For a prime integer p, the essential dimension $\mathrm{ed}(X; p)$ of X at p is defined in a similar manner, as $\dim(Y) - \dim(G)$, where the minimum is taken over all generically free G-varieties X admitting a G-equivariant dominant correspondence $X \rightsquigarrow Y$ of degree prime to p. Note that these numbers depend on the base field k, which we assume to be fixed throughout.

It follows from [LMMR13b, Propositions 2.4 and 3.1] that this minimum does not change if we allow the G-action on Y to be p-generically free rather than generically free; we shall not need this fact. We will, however, need the following lemma.

Lemma 4.1. *Requiring Y to be projective in the above definitions does not change the values of* $\mathrm{ed}(X)$ *and* $\mathrm{ed}(X;p)$. *That is, for any primitive generically free G-variety X,*

(a) *there exists a G-equivariant dominant rational map $X \dashrightarrow Z$ where Z is projective, the G-action on Z is generically free, and* $\dim(Z) = \mathrm{ed}(X;G) + \dim(G)$*;*
(b) *there exists a G-equivariant dominant correspondence $X \rightsquigarrow Z'$ of degree prime to p where Z' is projective, the G-action on Z' is generically free, and* $\dim(Z') = \mathrm{ed}(X;p) + \dim(G)$.

Proof. Let Y be a generically free G-variety and V be a generically free linear representation of G. It is well known that the G-action on V is versal; see, for example, [Mer13, Proposition 3.10]. Consequently, there exist a G-invariant subvariety $Y_1 \subset V$ and a G-equivariant dominant rational map $Y \dashrightarrow Y_1$ such that the G-action on Y_1 is generically free. After replacing Y_1 with its Zariski closure Z in $\mathbb{P}(V \oplus k)$, where G acts trivially on k, we obtain a G-equivariant dominant rational map $\alpha\colon Y \dashrightarrow Z$ such that Z is projective and the G-action on Z is generically free.

To prove part (a), choose a dominant G-equivariant rational map $f\colon X \dashrightarrow Y$ such that the G-action on Y is generically free and $\dim(Y)$ is the smallest possible, that is, $\dim(Y) = \mathrm{ed}(X)+\dim(G)$. Now compose f with the map $\alpha\colon Y \dashrightarrow Z$ constructed above. By the minimality of $\dim(Y)$, we have $\dim(Z) = \dim(Y)$, and part (a) follows. The proof of part (b) is the same, except that the rational map f is replaced by a correspondence of degree prime to p. □

The essential dimension $\mathrm{ed}(G)$ (respectively, the essential dimension $\mathrm{ed}(G;p)$ at p) of the group G is the maximal value of $\mathrm{ed}(X)$ (respectively, of $\mathrm{ed}(X;p)$) taken over all generically free G-varieties X.

5. The groups G_n

Let G be an algebraic group over k such that the connected component $T = G^0$ is a torus and the component group $F = G/T$ is a finite p-group, as in (1.1). By [LMMR13b, Lemma 5.3], there exists a finite p-subgroup $F' \subset G$ such that $\pi|_{F'} : F' \to F$ is surjective. We will refer to F' as a "quasi-splitting subgroup" for G. We will denote the subgroup generated by F' and $T[n]$ by G_n. Here $T[n]$ denotes the n-torsion subgroup of T, that is, the kernel of the homomorphism $T \xrightarrow{\times\, n} T$. Note that our definition of G_n depends on the choice of the quasi-splitting subgroup F'. We will assume that F' is fixed throughout. We will be particularly interested in the subgroups

$$G_1 \subset G_p \subset G_{p^2} \subset G_{p^3} \subset \cdots . \tag{5.1}$$

Informally speaking, we will show that these groups approximate "p-primary behavior" of G in various ways; see Lemma 5.2 and Proposition 6.2(b) below.

From here on, we denote the center of G by $Z(G)$.

Lemma 5.1. (a) *Let $z \in Z(G)(\overline{k})$ be a central element of G of order p^n for some $n \geqslant 0$. Then $z \in G_{p^m}(\overline{k})$ for $m \gg 0$.*

(b) *For every $n \geqslant 0$, we have $Z(G)[p^n] = Z(G_{p^r})[p^n]$ as group schemes for all $r \gg 0$.*

Proof. (a) By the definition of F', there exist a $g \in F'(\overline{k})$ and a $t \in T(\overline{k})$ such that $g = zt$. Since F' is a p-group, $g^N = 1$, where N is a sufficiently high power of p. Taking $N \geqslant p^n$, we also have $z^N = 1$. Since z is central, $1 = g^N = (zt)^N = z^N t^N = t^N$. Thus $t \in T[N](\overline{k}) \subset G_N(\overline{k})$, and, consequently, $z = gt^{-1}$ is a $\overline{k}$-point of $F' \cdot T[N] = G_N$.

(b) Let $n \geqslant 0$ be fixed. Since both $Z(G)[p^n]$ and G_{p^r} are finite p-groups and we are assuming that $\mathrm{char}(k) \neq p$, part (a) tells us that there exists an $m \geqslant 0$ such that $Z(G)[p^n] \subset Z(G_{p^r})[p^n]$ as group schemes for all $r \geqslant m$.

Let $r \geqslant 0$, and let $x \in Z(G_{p^r})[p^n](\overline{k})$. Let $f_x \colon T_{\overline{k}} \to T_{\overline{k}}$ be the homomorphism of conjugation by x. Passing to character lattices, we obtain a homomorphism $\langle x \rangle \to \mathrm{GL}_d(\mathbb{Z})$, where $d = \operatorname{rank} X(T_{\overline{k}})$. By a theorem of Jordan, in $\mathrm{GL}_d(\mathbb{Z})$ there are at most finitely many finite subgroups up to conjugacy. In particular, we may find an integer $N \gg 0$ such that the restriction of $\mathrm{GL}_d(\mathbb{Z}) \to \mathrm{GL}_d(\mathbb{Z}/p^N\mathbb{Z})$ to every finite subgroup is injective.

Thus, if $r \geqslant N$, then f_x is the identity for every $x \in Z(G_{p^r})[p^n](\overline{k})$. Since F' is contained in G_{p^r}, every $x \in Z(G_{p^r})[p^n](\overline{k})$ commutes with F'. Since G^0 and F' generate G, we deduce that $x \in Z(G)[p^n](\overline{k})$. This shows that $Z(G_{p^r})[p^n] \subset Z(G)[p^n]$ for $r \geqslant N$. We conclude that for $r \geqslant \max(N, m)$, we have $Z(G_{p^r})[p^n] = Z(G)[p^n]$. □

Lemma 5.2. *Let K be a p-closed field containing k. Then every class $\alpha \in H^1(K, G)$ lies in the image of the map $H^1(K, G_{p^r}) \to H^1(K, G)$ for sufficiently high r.*

Proof. Let $\alpha \in H^1(K, G)$. Consider the commutative diagram with exact rows

$$\begin{array}{ccccccccc} 1 & \longrightarrow & T[n] & \longrightarrow & G_n & \longrightarrow & F & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \| & & \\ 1 & \longrightarrow & T & \longrightarrow & G & \longrightarrow & F & \longrightarrow & 1 \end{array}$$

and the associated diagram in Galois cohomology. Let $\overline{\alpha} \in H^1(K, F)$ be the image of α under the natural morphism $H^1(K, G) \to H^1(K, F)$. Since T is abelian, the conjugation actions of G on T and of G_n on $T[n]$ descend to F. Twisting the bottom sequence by $\overline{\alpha}$ and setting $U = {}^{\overline{\alpha}}T$, we see that the fiber of $\overline{\alpha}$ equals the image of $H^1(K, U)$; see [Ser97, Section I.5.5]. Similarly twisting the top sequence by $\overline{\alpha}$, we see that the fiber of $H^1(K, G_n) \to H^1(K, F)$ over $\overline{\alpha}$ equals the image of $H^1(K, U[n])$. Here n is a power of p. Thus it suffices to prove the following:

Claim. *Let K be a p-closed field and U be a torus defined over K. Then the natural map $H^1(K, U[p^r]) \to H^1(K, U)$ is surjective for r sufficiently large.*

To prove the claim, note that since K is p-closed, the torus U is split by an extension L/K of degree n, where n is a power of p. By a restriction-corestriction argument, it follows that $H^1(K, U)$ is n-torsion. Now consider the short exact sequence

$$1 \longrightarrow U[n] \longrightarrow U \xrightarrow{\times\ n} U \longrightarrow 1\,.$$

The associated exact cohomology sequence

$$H^1(K, U[n]) \longrightarrow H^1(K, U) \xrightarrow{\times\ n} H^1(K, U)$$

shows that $H^1(K, U[n])$ surjects onto $H^1(K, U)$. This completes the proof of the claim and thus of the Lemma 5.2. □

6. The index

Let μ be a diagonalizable abelian p-group and

$$1 \longrightarrow \mu \longrightarrow G \longrightarrow \overline{G} \longrightarrow 1 \tag{6.1}$$

be a central exact sequence of affine algebraic groups defined over k. This sequence gives rise to the exact sequence of pointed sets

$$H^1(K,G) \longrightarrow H^1\big(K,\overline{G}\big) \xrightarrow{\partial_K} H^2(K,\mu)$$

for any field extension K of the base field k. Any character $x\colon \mu \to \mathbb{G}_{\mathrm{m}}$ induces a homomorphism $x_*\colon H^2(K,\mu) \to H^2(K,\mathbb{G}_{\mathrm{m}})$. We define $\operatorname{ind}^x(G,\mu)$ as the maximal index of $x_* \circ \partial_K(E) \in H^2(K,\mu)$, where the maximum is taken over all field extensions K/k and over all $E \in H^1\big(K,\overline{G}\big)$. This number is finite for every character $x\colon \mu \to \mathbb{G}_{\mathrm{m}}$; see [Mer13, Theorem 6.1].

Remark 6.1. Since μ is a finite p-group, the index of $x_* \circ \partial_K(E)$ does not change when K is replaced by a finite extension K'/K whose degree is prime to p and E is replaced by its image under the natural restriction map $H^1\big(K,\overline{G}\big) \to H^1\big(K',\overline{G}\big)$. Equivalently, we may replace K with its p-closure $K^{(p)}$. In other words, the maximal value of $x_* \circ \partial_K(E)$ will be attained if we only allow K to range over p-closed fields extensions of k.

Set $\operatorname{ind}(G,\mu) := \min \sum_{i=1}^r \operatorname{ind}^{x_i}(G,\mu)$, where the minimum is taken over all generating sets $x_1,\dots,x_r$ of the group $X(\mu)$ of characters of μ.

Now suppose that $G^0 = T$ is a torus and $G/G^0 = F$ is a p-group, as in (1.1). In this case, there is a particularly convenient choice of $\mu \subset G$. Following [LMMR13b, Section 4], we denote this central subgroup of G by $C(G)$. If k is algebraically closed, $C(G)$ is simply the p-torsion subgroup of the center of G, that is, $C(G) = Z(G)[p]$. If k is only assumed to be p-closed, then we set $C(G) = \operatorname{Split}_k(Z(G)[p])$ to be the largest k-split subgroup of $Z(G)[p]$ in the sense of [LMMR13a, Section 2].

Proposition 6.2. *Let G be as in (1.1). Denote by $\eta(G)$ the smallest dimension of a p-faithful G-representation.*

(a) *We have $\operatorname{ind}(G,C(G)) = \eta(G)$.*

(b) *If r is sufficiently large, then $\eta(G) = \eta(G_{p^r}) = \operatorname{ed}(G_{p^r}) = \operatorname{ed}(G_{p^r};p)$.*

Proof. (a) Let $\operatorname{Rep}^x(G)$ be the set of irreducible G-representations $\nu\colon G \to \operatorname{GL}(V)$ such that $\nu(z) = x(z)\operatorname{Id}_V$ for every $z \in \mu(\overline{k})$. By the index formula [Mer13, Theorem 6.1], we have $\operatorname{ind}^x(G) = \gcd\dim(\nu)$, where ν ranges over $\operatorname{Rep}^x(G)$ and gcd stands for the greatest common divisor. By [LMMR13b, Proposition 4.2], the dimension $\dim(\nu)$ is a power of p for every irreducible representation ν of G defined over k. Thus one can replace $\gcd\dim(\nu)$ with $\min\dim(\nu)$ in the index formula. Decomposing an arbitrary representation of G as a direct sum of irreducible subrepresentations, we see that $\operatorname{ind}(G,C(G))$ equals the minimal dimension of a k-representation $\nu\colon G \to \operatorname{GL}(V)$ such that the restriction $\nu_{|C(G)}\colon C(G) \to \operatorname{GL}(V)$ is faithful. Finally, by [LMMR13b, Proposition 4.3], the restriction $\nu_{|C(G)}$ is faithful if and only if ν is p-faithful.

(b) Since G_{p^r} is a (not necessarily constant) finite p-group and k is p-closed, the identities $\eta(G_{p^r}) = \operatorname{ed}(G_{p^r}) = \operatorname{ed}(G_{p^r};p)$ follow from [LMMR13a, Theorem 7.1]. It thus remains to show that

$$\eta(G) = \eta(G_{p^r}) \quad \text{for } r \gg 0\,. \tag{6.2}$$

By Lemma 5.1(b), we have $Z(G)[p] = Z(G_{p^r})[p]$ and thus $C(G) = C(G_{p^r})$ for $r \gg 0$. In view of part (a), the identity (6.2) is thus equivalent to

$$\operatorname{ind}(G, C(G)) = \operatorname{ind}(G_{p^r}, C(G)) \quad \text{for } r \gg 0\,. \tag{6.3}$$

Let h be the natural projection $G \to \overline{G} = G/C(G)$. Note that the group $\overline{G}$ is of the same type as G. That is, the connected component $\overline{G}^0$ is the torus $\overline{T} := h(T)$, and since the homomorphism $F = G/T \to \overline{G}/\overline{T}$ is surjective, $\overline{F} := \overline{G}/\overline{G}^0$ is a p-group. Moreover, if F' is a quasi-splitting subgroup for G (as defined at the beginning of Section 5), then $\overline{F}' := h(F')$ is a quasi-splitting subgroup for $\overline{G}$. We will use this subgroup to define the finite subgroups $\overline{G}_n$ of $\overline{G}$ for every integer n in the same way as we defined G_n:

$$\overline{G}_n \text{ is the subgroup of } \overline{G} \text{ generated by } \overline{F}' \text{ and the torsion subgroup } \overline{T}[n]\,.$$

Now observe that since $C(G)$ is p-torsion in G, we have $h(T[n]) \subset \overline{T}[n] \subset h(T[pn])$ and thus

$$h(G_n) \subset \overline{G}_n \subset h(G_{pn}) \tag{6.4}$$

for every n. We now proceed with the proof of (6.3). Consider the diagram of natural maps

$$\begin{array}{ccccccccc}
1 & \longrightarrow & C(G) & \longrightarrow & G & \longrightarrow & \overline{G} & \longrightarrow & 1 \\
 & & \| & & \uparrow{\scriptstyle i} & & \uparrow{\scriptstyle \bar{i}} & & \\
1 & \longrightarrow & C(G) & \longrightarrow & G_{p^r} & \longrightarrow & h(G_{p^r}) & \longrightarrow & 1
\end{array}$$

and the induced diagram in Galois cohomology

$$\begin{array}{ccccc}
H^1(K, G) & \longrightarrow & H^1\big(K, \overline{G}\big) & \xrightarrow{\partial_K} & H^2(K, C(G)) \\
\uparrow{\scriptstyle i_*} & & \uparrow{\scriptstyle \bar{i}_*} & & \| \\
H^1(K, G_{p^r}) & \longrightarrow & H^1(K, h(G_{p^r})) & \xrightarrow{\overline{\partial}_K} & H^2(K, C(G))\,.
\end{array}$$

In view of Remark 6.1, for the purpose of computing $\operatorname{ind}(G, C(G))$ and $\operatorname{ind}(G_{p^r}, C(G))$, we may assume that K is a p-closed field. We claim that for $r \gg 0$, the vertical map $\bar{i}_*\colon H^1(K, h(G_{p^r})) \to H^1\big(K, \overline{G}\big)$ is surjective for every p-closed field K/k. If we can prove this claim, then for $r \gg 0$, the image of $\overline{\partial}_K$ in $H^2(K, C(G))$ is the same as the image of ∂_K. Thus $\operatorname{ind}^x(G)$ and $\operatorname{ind}^x(G_{p^r})$ are the same for every $x \in X(C(G))$, and (6.3) will follow.

To prove the claim, note that $\overline{G}_{p^r} \subset h(G_{p^{r+1}})$ by (6.4). Consider the composition

$$H^1\big(K, \overline{G}_{p^{r-1}}\big) \longrightarrow H^1(K, h(G_{p^r})) \xrightarrow{\bar{i}_*} H^1\big(K, \overline{G}\big)\,.$$

By Lemma 5.2, the map $H^1\big(K, \overline{G}_{p^{r-1}}\big) \to H^1\big(K, \overline{G}\big)$ is surjective for $r \gg 0$. Hence, so is $\bar{i}_*$. This completes the proof of the claim and thus of (6.3) and of Proposition 6.2. □

7. A resolution theorem for rational maps

The following lemma is a minor variant of [BRV18, Lemma 2.1]. For the sake of completeness, we supply a self-contained proof.

Lemma 7.1. *Let $K \subset L$ be a field extension and $v\colon L^\times \to \mathbb{Z}$ be a discrete valuation. Assume that $v|_{K^\times}$ is non-trivial, and denote the residue fields of v and $v|_{K^\times}$ by L_v and K_v, respectively. Then $\operatorname{trdeg}_K L \geqslant \operatorname{trdeg}_{K_v} L_v$.*

Proof. Let $\overline{x}_1, \dots, \overline{x}_m \in L_v$. For every i, let x_i be a preimage of $\overline{x}_i$ in the valuation ring $\mathcal{O}_L$. It suffices to show that if $\overline{x}_1, \dots, \overline{x}_m$ are algebraically independent over K_v, then $x_1, \dots, x_m$ are algebraically independent over K. To prove this, we argue by contradiction. Suppose that there exists a non-zero polynomial $f \in K[t_1, \dots, t_m]$ such that $f(x_1, \dots, x_m) = 0$. Multiplying f by a suitable power of a uniformizing parameter for $v|_{K^\times}$, we may assume that $f \in \mathcal{O}_K[x_1, \dots, x_m]$ and that at least one coefficient of f has valuation equal to 0. Reducing modulo the maximal ideal of the valuation ring $\mathcal{O}_K$, we see that $\overline{x}_1, \dots, \overline{x}_m$ are algebraically dependent over K_v, which leads to a contradiction. □

Recall that if X_1 is normal and X_2 is complete, any rational map $f\colon X_1 \dashrightarrow X_2$ is regular in codimension 1. It follows that if $D \subset X_1$ is a prime divisor of X_1, the closure of the image $\overline{f(D)} \subset X_2$ is well defined.

Theorem 7.2. *Let G be a smooth linear algebraic group over k and $f\colon X \dashrightarrow Y$ be a dominant rational map of G-varieties. Assume that X is normal, Y is normal and complete, $D \subset X$ is a prime divisor, and $\overline{f(D)} \neq Y$. Then there exist a commutative diagram of G-equivariant dominant rational maps*

$$\begin{array}{ccc} & & Y' \\ & \overset{f'}{\nearrow} & \downarrow \pi \\ X & \overset{}{\underset{f}{\dashrightarrow}} & Y \end{array}$$

and a divisor $E \subset Y'$ such that Y' is complete, $\pi\colon Y' \to Y$ is a birational morphism, and $\overline{f'(D)} = E$.

Proof. Let $v\colon k(X)^\times \to \mathbb{Z}$ be the valuation given by the order of vanishing or pole along D. Since X is normal and Y is complete, f restricts to a rational map $D \dashrightarrow Y$. Denote the Zariski closure of the image of this map by C, and set

$$w\colon k(Y)^\times \xrightarrow{f^*} k(X)^\times \xrightarrow{v} \mathbb{Z}\,.$$

We claim that w is non-zero; that is, w is a discrete valuation on $k(Y)$. Indeed, choose $\varphi \in k(Y)^\times$ so that φ is regular in an open neighborhood U of the generic point of C and $\varphi|_{U\cap C} = 0$. It follows that $\varphi \circ f$ is zero on D, hence $w(f) = v(\varphi \circ f) > 0$. This proves the claim.

Since D maps dominantly onto C, we have an inclusion of local rings $f^*\colon \mathcal{O}_{Y,C} \hookrightarrow \mathcal{O}_{X,D}$. It follows that if $\varphi \in \mathcal{O}_{Y,C}$, then $w(\varphi) = v(\varphi \circ f) \geqslant 0$; that is, $\mathcal{O}_{Y,C}$ is contained in the valuation ring of w. In other words, C is the center of w.

Denote by $k(Y)_w$ the residue field of w. By Lemma 7.1, we have

$$\operatorname{trdeg}_k k(X) - \operatorname{trdeg}_k k(Y) \geqslant \operatorname{trdeg}_k k(D) - \operatorname{trdeg}_k k(Y)_w\,.$$

Since $\operatorname{trdeg}_k k(D) = \operatorname{trdeg}_k k(X) - 1$, this can be rewritten as

$$\operatorname{trdeg}_k k(Y)_w \geqslant \operatorname{trdeg}_k k(Y) - 1\,.$$

On the other hand, we have $\operatorname{trdeg}_k k(Y)_w \leqslant \operatorname{trdeg}_k k(Y) - 1$ by the Zariski–Abhyankar inequality [Bou89, Section VI.10.3, Corollary 1], hence

$$\operatorname{trdeg}_k k(Y)_w = \operatorname{trdeg}_k k(Y) - 1\,.$$

By [Art86, Theorem 5.2], there exists a sequence of proper birational morphisms

$$Y' = Y_n \to Y_{n-1} \to \cdots \to Y_1 \to Y_0 = Y$$

such that each $Y_{i+1} \to Y_i$ is a blow-up at the center of w on Y_i, the center E' of w on Y' is a prime divisor, and Y' is normal at the generic point of E'. Since C is G-invariant, by the universal property of the blow-up, the G-action on Y lifts to every Y_i, and the maps $Y_{i+1} \to Y_i$ are G-equivariant.

We let $\pi\colon Y' \to Y$ be the composition of the maps $Y_{i+1} \to Y_i$ and $f'\colon X \dashrightarrow Y'$ be the composition of f with the birational inverse of π. By construction, f' is G-equivariant. It suffices to show that $\overline{f'(D)} = E$. Since the center of w is the divisor $E \subset Y'$, the valuation w is given by the order of vanishing or pole along E. If we identify $k(Y)$ with $k(Y')$ via π, we also have $w = (f')^*v$. It follows that $\varphi \in k(Y')^\times$ is regular and vanishes at the generic point of E if and only if $w(\varphi) > 0$ if and only if $v(\varphi \circ f') > 0$ if and only if φ vanishes at the generic point of $f'(D)$. We conclude that $\overline{f'(D)} = E$, as desired.

Since G is smooth, the G-action on Y' lifts to the normalization $(Y')^{\mathrm{norm}}$, so that the normalization map $(Y')^{\mathrm{norm}} \to Y'$ is G-equivariant. After replacing Y' with $(Y')^{\mathrm{norm}}$ and E' with its preimage in $(Y')^{\mathrm{norm}}$, we may assume that Y' is normal everywhere (and not just at the generic point of E'). □

8. Proof of Theorem 1.2

Let G be an algebraic group as in (1.1). Let $\nu\colon G \to \mathrm{GL}(V)$ be a p-faithful representation of G of minimal dimension $\eta(G)$. By Lemma 2.1, there exists a stabilizer in general position S_V for the $G_{\overline{k}}$-action on $V_{\overline{k}}$. Since $V(k)$ is dense in V, we may assume without loss of generality that S_V is the stabilizer of a k-point of V. In particular, we may assume that S_V is a closed subgroup of G defined over k. Since T acts p-faithfully on V, we have $S_V \cap T = \{1\}$.

Reduction 8.1. To prove Theorem 1.2, it suffices to construct a G-representation V' such that $\dim(\tilde{V}) = \mathrm{rank}_p(S_V)$, $W := V \oplus \tilde{V}$ is p-generically free, and

$$\mathrm{ed}(W;p) = \dim(W) - \dim(G)\,. \tag{8.1}$$

Here when we write $\mathrm{ed}(W;p)$, we are viewing W as a generically free $G/\operatorname{Ker}(\varphi)$-variety, where $\varphi\colon G \to \mathrm{GL}(W)$ denotes the representation of G on W. The kernel $\operatorname{Ker}(\varphi)$ of this representation is a finite normal subgroup of G of order prime to p.

Proof. Suppose that we manage to construct $\tilde{V}$ so that (8.1) holds. Then

$$\mathrm{ed}(W;p) \overset{\text{(i)}}{=} \mathrm{ed}(G/\operatorname{Ker}(\varphi);p) \overset{\text{(ii)}}{=} \mathrm{ed}(G;p) \overset{\text{(iii)}}{\leqslant} \rho(G) - \dim(G) \overset{\text{(iv)}}{\leqslant} \dim(W) - \dim(G)\,,$$

where

(i) follows from the fact that W is a versal $G/\operatorname{Ker}(\varphi)$-variety; see, for example, [Mer13, Propositions 3.10 and 3.11];

(ii) follows by [LMMR13b, Proposition 2.4];

(iii) is the right-hand side of (1.2); and

(iv) is immediate from the definition of $\rho(G)$.

If we know that (8.1) holds, then the inequalities (iii) and (iv) are, in fact, equalities. Equality in (iii) yields Theorem 1.2(a). On the other hand, since

$$\dim(W) = \dim(V) + \dim(\tilde{V}) = \eta(G) + \mathrm{rank}_p(S_V)\,,$$

equality in (iv) tells us that $\eta(G) + \mathrm{rank}_p(S_V) = \rho(G)$, thus proving Theorem 1.2(b). □

We now proceed with the construction of W. From now on, we replace G with $\overline{G} = G/\operatorname{Ker}(\nu)$. All other G-actions we will construct (including the linear G-action on W) will factor through $\overline{G}$. In the end, we will show that $\operatorname{ed}(W;p) = \operatorname{ed}\left(\overline{G};p\right)$; once again, this is enough because $\operatorname{ed}(G;p) = \eta(G) = \eta(\overline{G}) = \operatorname{ed}\left(\overline{G};p\right)$ by [LMMR13b, Proposition 2.4]. In other words, from now on we may (and will) assume that the G-action on V is faithful.

Recall that S_V denotes the stabilizer in general position for the G-action on V and that we have chosen S_V (which is a priori a closed subgroup of $G_{\overline{k}}$ defined up to conjugacy) so that it is defined over k. Since T is a torus and T acts faithfully on V, this action is automatically generically free. That is, $S_V \cap T = 1$ or, equivalently, the natural projection $\pi|_{S_V} \colon S_V \to F$ is injective. In particular, $\pi(S_V)$ is diagonalizable. By our assumption, F is isomorphic to the product $\mu_{p^{i_1}} \times \cdots \times \mu_{p^{i_R}}$ for some integers $R \geqslant 0$ and $i_1, \ldots, i_R \geqslant 1$. Moreover, this isomorphism can be chosen so that

$$\pi(S_V) = \mu_{p^{j_1}} \times \cdots \times \mu_{p^{j_r}}$$

for some $0 \leqslant r \leqslant R$ and some integers j_m with $1 \leqslant j_m \leqslant i_m$ for every $m = 1, \ldots, r$. Let χ_m be the composition of $\pi \colon G \to F$ with the projection map $F \to \mu_{p^{i_m}}$ to the mth component and V_m be a 1-dimensional vector space on which G acts by χ_m. Set $W_d = V$ and $W_{d+m} = V \oplus V_1 \oplus \cdots \oplus V_m$ for $m = 1, \ldots, r$. A stabilizer in general position for the G-action on W_{d+m} is

$$S_{W_{d+m}} = S_V \cap \operatorname{Ker}(\chi_1) \cap \cdots \cap \operatorname{Ker}(\chi_m)\,;$$

equivalently,

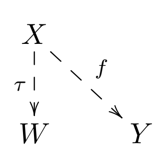
$$S_{W_{d+m}} \simeq \pi(S_{W_{d+m}}) = \{1\} \times \cdots \times \{1\} \times \mu_{p^{j_{m+1}}} \times \cdots \times \mu_{p^{j_r}} \qquad (8.2)$$

for any $0 \leqslant m \leqslant r$. In particular, $S_{W_{d+r}} = \{1\}$; in other words, the G-action on W_{d+r} is generically free. We now set

$$W = W_{d+r} = V \oplus V_1 \oplus \cdots \oplus V_r\,.$$

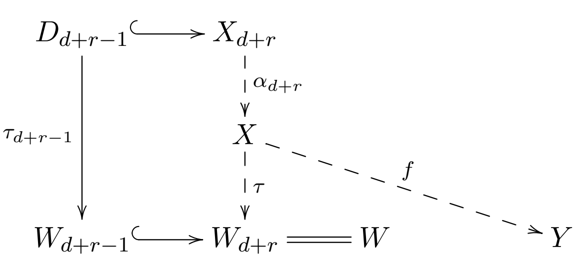
Having defined W, we now proceed with the proof of (8.1). In view of Lemma 4.1(b), it suffices to establish the following.

Proposition 8.2. *Let W be as above. Consider a dominant G-equivariant correspondence*



of degree prime to p, where Y is a p-generically free projective G-variety. Then $\dim(Y) = \dim(W) = d + r$.

We now proceed with the proof of the proposition. By Lemma 3.1 (with $Z = W_{d+r-1}$), there exists a commutative diagram of G-equivariant maps



such that X_{d+r} is normal, α_{d+r} is a birational isomorphism, D_{d+r-1} is an irreducible divisor in X_{d+r}, and τ_{d+r-1} is a cover of W_{d+r-1} of degree prime to p. Let $S_{D_{d+r-1}} \subset G$ be a stabilizer in general position for the G-action on D_{d+r-1}; it exists by Lemma 2.1. In view of (8.2), Lemma 3.2 tells us that

$$\operatorname{rank}_p(S_{D_{d+r-1}}) = 1\,. \tag{8.3}$$

On the other hand, by our assumption, the G-action on Y is p-generically free. Thus the restriction[1] of the dominant rational map $f \circ \alpha_{d+r} \colon X_{d+r} \dashrightarrow Y$ to D_{d+r-1} cannot be dominant, and Theorem 7.2 applies: there exists a commutative diagram

$$\begin{array}{ccc} X_{d+r} & \overset{f_{d+r}}{\dashrightarrow} & Y_{d+r} \\ {\scriptstyle \alpha_{d+r}} \downarrow & & \downarrow {\scriptstyle \sigma_{d+r}} \\ X & \overset{f}{\dashrightarrow} & Y \end{array}$$

of dominant G-equivariant rational maps, where σ_{d+r} is a birational morphism, Y_{d+r} is normal and complete, and f_{d+r} restricts to a dominant G-equivariant rational map $D_{d+r-1} \dashrightarrow E_{d+r-1}$ for some G-invariant irreducible divisor E_{d+r-1} of Y_{d+r}. We will denote this dominant rational map by $f_{d+r-1} \colon D_{d+r-1} \dashrightarrow E_{d+r-1}$. We now iterate this construction with f_{d+r} replaced by f_{d+r-1}.

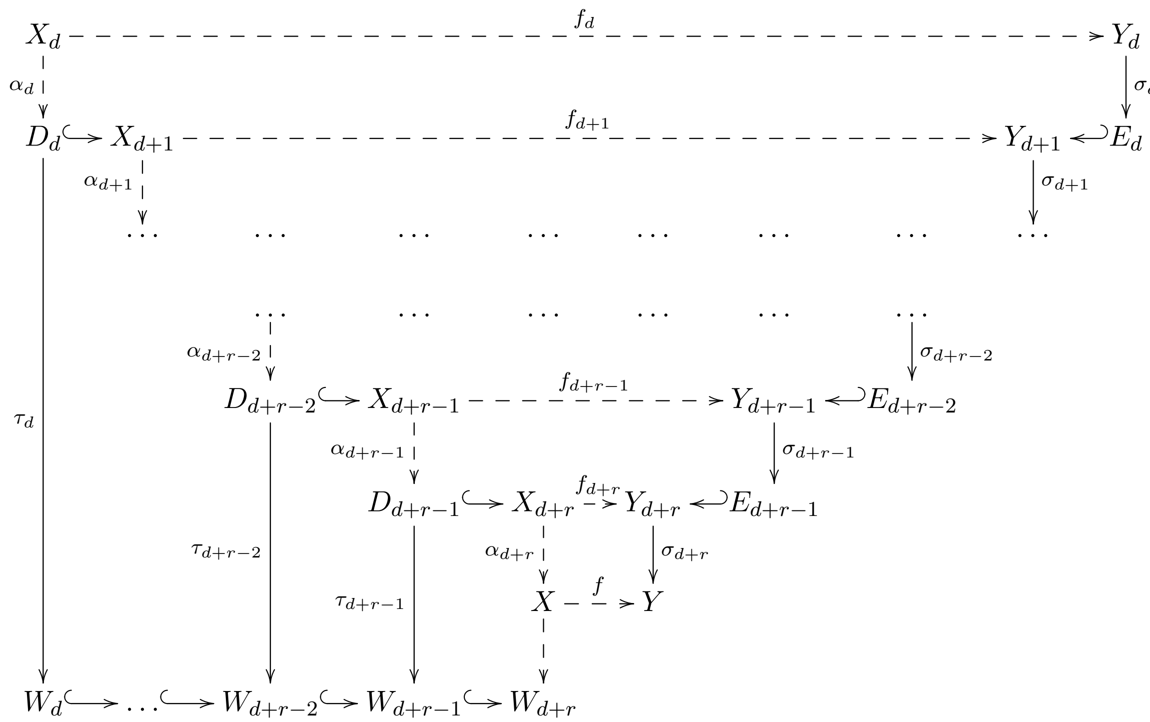
By Lemma 3.1, there exists a commutative diagram of G-equivariant maps

$$\begin{array}{ccc} D_{d+r-2} & \hookrightarrow & X_{d+r-1} \\ & & \downarrow {\scriptstyle \alpha_{d+r-1}} \\ {\scriptstyle \tau_{d+r-2}} \downarrow & & D_{d+r-1} \\ & & \downarrow {\scriptstyle \tau_{d+r-1}} \\ W_{d+r-2} & \hookrightarrow & W_{d+r-1} \end{array}$$

such that X_{d+r-1} is normal, α_{d+r-1} is a birational isomorphism, D_{d+r-2} is an irreducible divisor in X_{d+r-1}, and τ_{d+r-2} is a cover of W_{d+r-2} of degree prime to p.

Denote a stabilizer in general position for the G-action on E_{d+r-1} by $S_{E_{d+r-1}}$. Recall that the G-action on Y (and thus Y_{d+r}) is p-generically free. Since E_{d+r-1} is a G-invariant hypersurface in Y_{d+r}, Lemma 2.3(a) tells us that $\operatorname{rank}_p(S_{E_{d+r-1}}) \leqslant 1$. On the other hand, since X_{d+r-1} maps dominantly to E_{d+r-1}, the group $S_{E_{d+r-1}}$ contains (a conjugate of) $S_{X_{d+r-1}}$ and thus $\operatorname{rank}_p(S_{E_{d+r-1}}) \geqslant \operatorname{rank}_p(S_{X_{d+r-1}})$, where $\operatorname{rank}_p(S_{X_{d+r-1}}) = 1$ by (8.3). We conclude that $\operatorname{rank}_p(S_{E_{d+r-1}}) = 1$. Now observe that since $\operatorname{rank}_p(S_{E_{d+r-1}}) = 1$ and $\operatorname{rank}_p(S_{X_{d+r-2}}) = 2$ (see (8.2)), the image of X_{d+r-2} under f_{d+r-1} cannot be Zariski dense in E_{d+r-1}. Consequently, Theorem 7.2 can be applied to $f_{d+r-1} \colon X_{d+r-1} \dashrightarrow E_{d+r-1}$. It yields a birational morphism $\sigma_{d+r-1} \colon Y_{d+r-1} \to E_{d+r-1}$ such that Y_{d+r-1} is normal and complete, and the composition $\sigma_{d+r-1}^{-1} \circ f_{d+r-1}$ restricts to a dominant G-equivariant rational map $f_{d+r-2} \colon D_{d+r-2} \dashrightarrow E_{d+r-2}$ for some G-invariant prime divisor E_{d+r-2} of Y_{d+r-1}. Proceeding recursively, we obtain a commutative diagram of G-equivariant maps

[1] The restriction of $f \circ \alpha_{d+r}$ to D_{d+r-1} is well defined because X_{d+r} is normal and Y is complete.



such that for every m,

(i) D_{d+m} is an irreducible divisor in X_{d+m+1} and E_{d+m} is an irreducible divisor in Y_{d+m+1};

(ii) the vertical maps $\alpha_{d+m}\colon X_{d+m} \dashrightarrow D_{d+m}$ and $\sigma_{d+m}\colon Y_{d+m} \to E_{d+m}$ are birational isomorphisms;

(iii) X_{d+m} and Y_{d+m} are normal, and Y_{d+m} is complete;

(iv) $\operatorname{rank}_p(S_{X_{d+m}}) = \operatorname{rank}_p(S_{Y_{d+m}}) = r - m$;

(v) the vertical morphism $\tau_{d+m}\colon D_{d+m} \to W_{d+m}$ is a cover of degree prime to p.

Note that the subscripts are chosen so that $\dim(X_{d+m}) = \dim(W_{d+m}) = d + m$ for each $m = 0, \ldots, r$. We will eventually show that $\dim(Y_{d+m}) = d + m$ for each m as well, but we do not know what $\dim(Y_{d+m})$ is at this point.

LEMMA 8.3. *The G-action on Y_{d+m} (or, equivalently, on E_{d+m}) is p-faithful for every $m = 0, \ldots, r$.*

Assume, for a moment, that this lemma is established. By our construction, f_d may be viewed as a dominant G-equivariant correspondence $W_d \rightsquigarrow Y_d$ of degree prime to p. Now recall that $W_d = V$ is a p-faithful representation of G of minimal possible dimension $\eta(G)$. By Lemma 8.3, the G-action of Y_d is p-faithful. Restricting to the p-subgroup $G_n \subset G$, where n is a power of p, we obtain a dominant G_n-equivariant correspondence $f_d\colon V \rightsquigarrow Y_d$ of degree prime to p, where the G_n-action on Y is faithful. Thus $\dim(Y_d) \geqslant \operatorname{ed}(G_n; p)$. When n is a sufficiently high power of p, Proposition 6.2 tells us that

$$\operatorname{ed}(G_n; p) = \eta(G_n) = \eta(G) = \dim(V) = d\,.$$

By conditions (i) and (ii) above, $\dim(Y_{d+m+1}) = \dim(E_{d+m}) + 1 = \dim(Y_{d+m}) + 1$ for each $m = 0, 1, \ldots, r$. Thus $\dim(Y) = \dim(Y_{d+r}) = \dim(Y_d) + r = \dim(V) + r = d + r = \dim(W)$, as desired. This will complete the proof of Proposition 8.2 and thus of Theorem 1.2.

Proof of Lemma 8.3. For the purpose of this proof, we may replace k with its algebraic closure $\overline{k}$ and thus assume that k is algebraically closed. We argue by reverse induction on m. For the base case, where $m = r$, note that by our assumption, the G-action on Y is p-generically free and hence p-faithful. Since Y_{d+r} is birationally isomorphic to Y, the same is true of the G-action on Y_{d+r}.

For the induction step, assume that the G-action on Y_{d+m+1} is p-faithful for some m with $0 \leqslant m \leqslant r-1$. Our goal is to show that the G-action on Y_{d+m} is also p-faithful. Let N be the kernel of the G-action on Y_{d+m}. Recall that by Lemma 2.3(b), there is a homomorphism

$$\alpha\colon N \to \mathbb{G}_{\mathrm{m}} \tag{8.4}$$

where $\mathrm{Ker}(\alpha)$ has no elements of order p. Since $\mathrm{Ker}(\alpha)$ is a subgroup of G and we are assuming that $G^0 = T$ is a torus and $G/G^0 = F$ is a finite p-group, we conclude that

$$\mathrm{Ker}(\alpha) \text{ is a finite subgroup of } T \text{ of order prime to } p\,. \tag{8.5}$$

It remains to show that $\alpha(N)$ is a finite group of order prime to p. Assume the contrary: $\alpha(N)$ contains $\mu_p \subset \mathbb{G}_{\mathrm{m}}$.

Claim. *There exists a subgroup $\mu_p \simeq N_0 \subset N$ such that N_0 is central in G.*

First we observe that in order to prove the claim, it suffices to show that there exists a subgroup $\mu_p \simeq N_0 \subset N$ such that N_0 is normal in G. Indeed, since $G^0 = T$ is a torus and $G/G^0 = F$ is a p-group, if $N_0 \simeq \mu_p$ is normal in G, then the conjugation map $G \to \mathrm{Aut}(\mu_p) \simeq \mathbb{Z}/(p-1)\mathbb{Z}$ is trivial, so N_0 is automatically central. Now consider two cases.

Case 1: $G^0 = T$ does not act p-faithfully on Y_{d+m}. Then $\mu_p \subset N \cap T \triangleleft G$. In view of (8.4) and (8.5), the intersection $N \cap T$ contains exactly one copy of μ_p. This implies that μ_p is characteristic in $N \cap T$ and, hence, normal in G, as desired.

Case 2: The intersection $N \cap T$ does not contain μ_p; that is, $N \cap T$ is a finite group of order prime to p. Examining the exact sequence

$$1 \to N \cap T \to N \to F = G/T\,,$$

we see that N is a finite group of order pm, where m is prime to p. Let $\mathrm{Syl}_p(N)$ be the set of Sylow p-subgroups of N. By Sylow's theorem, we have[2] $|\mathrm{Syl}_p(N)| \equiv 1 \pmod{p}$. The group G acts on $\mathrm{Syl}_p(N)$ by conjugation. Clearly T acts trivially, and the p-group $F = G/T$ fixes a subgroup $N_0 \in \mathrm{Syl}_p$. In other words, $N_0 \simeq \mu_p$ is normal in G. This proves the claim.

We are now ready to finish the proof of Lemma 8.3. Let $S_{Y_{d+m}} \subset G$ be a stabilizer in general position for the G-action on Y_{d+m}, N be the kernel of this action, and N_0 be the central subgroup of N isomorphic to μ_p, as in the claim. Clearly $N_0 \subset N \subset S_{Y_{d+m}}$. Since $f_{d+m}\colon X_{d+m} \dashrightarrow Y_{d+m}$ is a dominant G-equivariant rational map, $S_{Y_{d+m}}$ contains (a conjugate of) $S_{X_{d+m}}$. By condition (iv),

$$\mathrm{rank}_p(S_{Y_{d+m}}) = r - m = \mathrm{rank}_p(S_{X_{d+m}})\,. \tag{8.6}$$

In particular, $S_{X_{d+m}}$ contains a subgroup A isomorphic to μ_p^{r-m}. Since $N_0 \simeq \mu_p$ is central in G, it has to be contained in A; otherwise, $S_{Y_{d+m}}$ would contain a subgroup isomorphic to $A \times \mu_p = (\mu_p)^{r-m+1}$, contradicting (8.6). Thus $\mu_p \simeq N_0 \subset S_{X_{d+m}}$. Moreover, since N_0 is central in G, it is contained in every conjugate of $S_{X_{d+m}}$. This implies that N_0 stabilizes every point

[2] Recall that we are assuming that k is an algebraically closed field of characteristic different from p. If $\mathrm{char}(k)$ does not divide $|N|$, then $\mathrm{Syl}_p(N)$ is the set of Sylow subgroups of the finite group $N(k)$. If $\mathrm{char}(k)$ divides $|N|$, then elements of $\mathrm{Syl}_p(N)$ can be identified with Sylow p-subgroups of the finite group $N_{\mathrm{red}}(k)$.

of X_{d+m}. In other words, N_0 acts trivially on X_{d+m}. Tracing to the above diagram, we see that N_0 acts trivially on D_{d+m-1}, hence on X_{d+m-1}, hence on D_{d+m-2}, etc. Finally, we conclude that N_0 acts trivially on X_d and hence on $\tau_d(X_d) = W_d = V$, contradicting our assumption that G acts p-faithfully on $W_d = V$.

This contradiction shows that our assumption that $\alpha(N)$ contains μ_p was false. Returning to (8.4) and (8.5), we deduce that the kernel N of the G-action on Y_{d+m} is a finite group of order prime to p. In other words, the G-action on Y_{d+m} is p-faithful. This completes the proof of Lemma 8.3 and thus of Proposition 8.2 and Theorem 1.2. □

Remark 8.4. Our proof of Theorem 1.2 goes through even if F is not abelian, provided that the stabilizer in general position S_V projects isomorphically to $F/[F,F]$. (If F is abelian, this is always the case.)

Remark 8.5. Theorem 1.2 implies that if V and V' are p-faithful representations of G of minimal dimension $\eta(G)$, then the stabilizers in general position, S_V and $S_{V'}$, have the same p-rank:

$$\mathrm{rank}_p(S_V) = \mathrm{rank}_p(S_{V'}) = \rho(G) - \eta(G) = \mathrm{ed}(G;p) + \dim(G) - \eta(G)\,.$$

In our proof of Theorem 1.2, this number is denoted by r.

9. Normalizers of maximal tori in split simple groups

In this section, Γ will denote a split simple algebraic group over k, T will denote a k-split maximal torus of Γ, N will denote the normalizer of T in Γ, and $W = N/T$ will denote the Weyl group. These groups fit into an exact sequence

$$1 \longrightarrow T \longrightarrow N \xrightarrow{\pi} W \longrightarrow 1\,. \tag{9.1}$$

A. Meyer and the first author [MR09] have computed $\mathrm{ed}(N;p)$ in the case where $\Gamma = \mathrm{PGL}_n$ for every prime number p. M. MacDonald [Mac11] subsequently found the exact value of $\mathrm{ed}(N;p)$ for most other split simple groups Γ. One reason this is of interest is that

$$\mathrm{ed}(N;p) \geqslant \mathrm{ed}(\Gamma;p)\,;$$

see, for example, [Mer13, Section 10a]. Let W_p denote a Sylow p-subgroup of W and N_p denote the preimage of W_p in N. Then

$$\mathrm{ed}(N;p) = \mathrm{ed}(N_p;p)\,;$$

see [MR09, Lemma 4.1]. The exact sequence

$$1 \longrightarrow T \longrightarrow N_p \xrightarrow{\pi} W_p \longrightarrow 1$$

is of the form of (1.1), and thus the inequalities (1.2) apply to N_p. MacDonald computed the exact value of $\mathrm{ed}(N;p) = \mathrm{ed}(N_p;p)$ for most split simple linear algebraic groups Γ by showing that the left-hand side and right-hand side of the inequalities (1.2) for N_p coincide. There are two families of groups Γ where the exact value of $\mathrm{ed}(N;p)$ remained inaccessible by this method, $\Gamma = \mathrm{SL}_n$ and $\Gamma = \mathrm{SO}_{4n}$.[3] As an application of Theorem 1.2, we will now compute $\mathrm{ed}(N;p)$ in these two remaining cases. Our main results are Theorems 9.1 and 9.2 below.

[3] The omission of SL_n from [Mac11, Remark 5.11] is an oversight; we are grateful to Mark MacDonald for clarifying this point for us.

Theorem 9.1. *Let $n \geqslant 1$ be an integer, and let N be the normalizer of a k-split maximal torus T in SL_n. Then*

(a) $\mathrm{ed}(N;p) = n/p + 1$ *if $p \geqslant 3$ and n is divisible by p,*
(b) $\mathrm{ed}(N;p) = n/2 + 1$ *if $p = 2$ and n is divisible by 4,*
(c) $\mathrm{ed}(N;p) = \lfloor n/p \rfloor$ *if $p \geqslant 3$ and n is not divisible by p,*
(d) $\mathrm{ed}(N;p) = \lfloor n/2 \rfloor$ *if $p = 2$ and n is not divisible by 4.*

Theorem 9.2. *Let k be a field of characteristic different from 2 and $n \geqslant 1$ be an integer. Let N be the normalizer of a k-split maximal torus of SO_{4n}. Then $\mathrm{ed}(N;2) = 4n$.*

Our proofs of these theorems will rely on the following simple lemma, which is implicit in [MR09] and [Mac11]. Let F be a finite discrete p-group, and let M be an F-lattice. The symmetric p-rank of M is the minimal cardinality d of a finite H-invariant p-spanning subset $\{x_1, \ldots, x_d\} \subset M$. Here "$p$-spanning" means that the index of the $\mathbb{Z}$-module spanned by $x_1, \ldots, x_d$ in M is finite and prime to p. Following MacDonald, we will denote the symmetric p-rank of M by $\mathrm{SymRank}(M;p)$.

Lemma 9.3. *Consider an exact sequence $1 \to T \to G \to F \to 1$ of algebraic groups over k, as in* (1.1)*. Assume further that T is a split torus and F is a constant finite p-group. Denote the character lattice of T by $X(T)$, we will view it as an F-lattice. Then $\eta(G) \geqslant \mathrm{SymRank}(X(T);p)$.*

Here $\eta(G)$ denotes the minimal dimension of a p-faithful representation of G, as defined in the introduction, and $X(T)$ is viewed as an F-lattice. If we further assume that the sequence (1.1) in Lemma 9.3 is split, then, in fact, $\eta(G) = \mathrm{SymRank}(X(T);p)$. We shall not need this equality, so we leave its proof as an exercise for the reader.

Proof of Lemma 9.3. Let V be a p-faithful representation of G of minimal dimension $r = \eta(G)$. As a T-representation, V decomposes as the direct sum of characters $\chi_1, \ldots, \chi_r$. A simple calculation shows that the F-action permutes the χ_i. Let $S \subset \mathrm{GL}(V)$ be the diagonal torus generated by the images of the χ_i. By construction, the kernel of the F-equivariant homomorphism

$$(\chi_1, \ldots, \chi_r) \colon T \to S$$

is finite and of order prime to p. Passing to character lattices, we obtain an F-equivariant homomorphism $X(S) \to X(T)$ whose cokernel is finite and of order prime to p. In other words, the images of the χ_i in $X(T)$ form a p-spanning subset of $X(T)$ of size r. We conclude that $\mathrm{SymRank}(X(T);p) \leqslant r = \eta(G)$, as claimed. □

For the proof of Theorem 9.1 we will also need the following lemma. Let $\Gamma = \mathrm{SL}_n$, T be the diagonal maximal torus, N be the normalizer of T in SL_n, H be a subgroup of the Weyl group $W = N/T \simeq \mathrm{S}_n$, and N' be the preimage of H in N. Restricting (9.1) to N', we obtain an exact sequence

$$1 \longrightarrow T \longrightarrow N' \xrightarrow{\pi} H \longrightarrow 1\,.$$

Lemma 9.4. *Let V_n be the natural n-dimensional representation of SL_n and S be the stabilizer in general position for the restriction of this representation to N'. Then* (a) *$S \cap T = 1$ and* (b) *$\pi(S) = H \cap \mathrm{A}_n$.*

Here, as usual, A_n denotes the alternating subgroup of S_n.

Proof. Part (a) follows from the fact that the T-action on V_n is generically free. To prove part (b), note that $\pi(S)$ is the kernel of the action of H on V_n/T, where V_n/T is the rational quotient of V_n by the action of T; see, for example, the proof of [LMMR13b, Proposition 7.2]. Consider the dense open subset $\mathbb{G}_{\rm m}^n \subset V_n$ consisting of vectors of the form $(x_1, x_2, \dots, x_n)$, where $x_i \neq 0$ for any $i = 1, \dots, n$. We can identify $\mathbb{G}_{\rm m}^n$ with the diagonal maximal torus in GL_n. Now

$$V_n/T \overset{\cong}{\dashleftarrow\dashrightarrow} (\mathbb{G}_{\rm m})^n/T \xrightarrow[\det]{\simeq} \mathbb{G}_{\rm m}\,,$$

where S_n acts on $\mathbb{G}_{\rm m}$ by $\sigma \cdot t = \mathrm{sign}(\sigma)t$. Thus the kernel of the H-action on V_n/T is $H \cap \mathrm{A}_n$, as claimed. □

Proof of Theorem 9.1. We will assume that $\Gamma = \mathrm{SL}_n$ and T is the diagonal torus in Γ. The inequalities

$$\left\lfloor \frac{n}{p} \right\rfloor \leqslant \mathrm{ed}(N;p) \leqslant \left\lfloor \frac{n}{p} \right\rfloor + 1\,; \tag{9.2}$$

are known for every n and p; see [Mac11, Section 5.4]. We will write V_n for the natural n-dimensional representation of SL_n (which we will sometimes restrict to N or subgroups of N).

(a) Assume that p is an odd prime and n is divisible by p. Let $H \simeq (\mathbb{Z}/p\mathbb{Z})^{n/p}$ be the subgroup of $W = N/T \simeq \mathrm{S}_n$ generated by the commuting p-cycles $(1\ 2\ \dots\ p)$, $(p+1\ p+2\ \dots\ 2p)$, $\dots$, $(n-p+1\ \dots\ n)$. Since H is a p-group, it lies in a Sylow p-subgroup W_p of S_n. Denote the preimage of H in N by N'. Then N' is a subgroup of N of finite index, so

$$\mathrm{ed}(N;p) \geqslant \mathrm{ed}(N';p)\,; \tag{9.3}$$

see [BRV10, Lemma 2.2]. It thus suffices to show that $\mathrm{ed}(N';p) = n/p + 1$.

CLAIM. $\eta(N') = n$.

Suppose that the claim is established. Then V_n is a p-faithful representation of N' of minimal dimension. Since p is odd, H lies in the alternating group A_n. By Lemma 9.4(a), the stabilizer in general position for the N'-action on V is isomorphic to H. By Theorem 1.2,

$$\mathrm{ed}(N';p) = \dim(V_n) + \mathrm{rank}(H) - \dim(N') = n + \frac{n}{p} - (n-1) = \frac{n}{p} + 1\,,$$

and we are done.

To prove the claim, note that N' has a faithful representation V_n of dimension n. Hence, $\eta(N') \leqslant n$. To prove the opposite inequality, $\eta(N') \geqslant n$, it suffices to show that

$$\mathrm{SymRank}(X(T);p) \geqslant n\,; \tag{9.4}$$

see Lemma 9.3. Here we view $X(T)$ as an H-lattice. By definition, $\mathrm{SymRank}(X(T);p)$ is the minimal cardinality of a finite H-invariant p-spanning subset $\{x_1, \dots, x_d\} \subset X(T)$. The H-action on $\{x_1, \dots, x_d\}$ gives rise to a permutation representation $\varphi\colon H \to \mathrm{S}_d$.

The permutation representation φ is necessarily faithful. Indeed, assume the contrary: $1 \neq h$ lies in the kernel of φ. Then $x_1, \dots, x_d$ lie in $X(T)^h$. On the other hand, it is easy to see that $X(T)^h$ is of infinite index in $X(T)$. Hence, $\{x_1, \dots, x_d\}$ cannot be a p-spanning subset of $X(T)$. This contradiction shows that φ is faithful.

Now [AG89, Theorem 2.3(b)] tells us that the order of any abelian p-subgroup of S_d is at most $p^{d/p}$. In particular, $|H| \leqslant p^{d/p}$. In other words, $p^{n/p} \leqslant p^{d/p}$ or, equivalently, $n \leqslant d$. This completes the proof of (9.4) and thus of the claim and of part (a).

(b) When $p = 2$ and n is even, the argument in part (a) does not work as stated because it is no longer true that H lies in the alternating group A_n. However, when n is divisible by 4, we can redefine H as

$$H_1 \times \cdots \times H_{n/4} \hookrightarrow \underbrace{\mathrm{A}_4 \times \cdots \times \mathrm{A}_4}_{(n/4 \text{ times})} \hookrightarrow A_n\,,$$

where $H_i \simeq (\mathbb{Z}/2\mathbb{Z})^2$ is the unique normal subgroup of order 4 in the ith copy of A_4. With H defined this way, $H \simeq (Z/2\mathbb{Z})^{n/2}$ is a subgroup of A_n, and the rest of the proof of part (a) goes through unchanged.

(c) Write $n = pq+r$, where $1 \leqslant r \leqslant p-1$. The subgroup of S_n consisting of the permutations σ such that $\sigma(i) = i$ for any $i > pq$, is naturally identified with S_{pq}. Let P_{pq} be a p-Sylow subgroup of S_{pq}, and let N' be the preimage of P_{pq} in N. Then $[N : N'] = [\mathrm{S}_n : P_{pq}]$ is prime to p; hence, it suffices to show that $\mathrm{ed}(N'; p) = \lfloor n/p \rfloor$. In view of (9.2), it is enough to show that $\mathrm{ed}(N'; p) \leqslant \lfloor n/p \rfloor$. Since $r \geqslant 1$, as an N'-representation, V_n splits as $k^{pq} \oplus k^r$ in the natural way. Let us now write k^r as $k^{r-1} \oplus k$ and combine k^{r-1} with k^{pq}. This yields a decomposition $V_n = k^{n-1} \oplus k$, where the action of N' on k^{n-1} is faithful. Now recall that P_{pq} has a faithful q-dimensional representation; see, for example, the proof of [MR09, Lemma 4.2]. Denote this representation by V'. Viewing V' as a q-dimensional representation of N' via the natural projection $N' \to P_{pq}$, we obtain a generically free representation $k^{n-1} \oplus V'$ of N'. Thus

$$\mathrm{ed}(N'; p) \leqslant \dim\left(k^{n-1} \oplus V'\right) - \dim(N') = (n-1) + q - (n-1) = q = \left\lfloor \frac{n}{p} \right\rfloor,$$

as desired.

(d) The argument of part (c) is valid for any prime. In particular, if $p = 2$, it proves part (d) in the case where n is odd. Thus we may assume without loss of generality that $n \equiv 2 \pmod 4$. Let N' be the preimage of P_n in N, where P_n is a Sylow 2-subgroup of S_n. Then the index $[N : N'] = [\mathrm{S}_n : P_n]$ is finite and odd; hence, $\mathrm{ed}(N; 2) = \mathrm{ed}(N'; 2)$. In view of (9.2), it suffices to show that $\mathrm{ed}(N'; 2) \leqslant n/2$.

Since $n \equiv 2 \pmod 4$, we have $P_n = P_{n-2} \times P_2$, where $P_2 \simeq \mathrm{S}_2$ is the subgroup of S_n of order 2 generated by the 2-cycle $(n-1,\ n)$. Let V' be a faithful representation of P_{n-2} of dimension $(n-2)/2$. We may view V' as a representation of N' via the projection $N' \to P_n \to P_{n-2}$.

Claim. *The action of N' on $V_n \oplus V'$ is generically free.*

If this claim is established, then

$$\mathrm{ed}(N') \leqslant \dim(V_n \oplus V') - \dim(N') = n + \frac{n-2}{2} - (n-1) = \frac{n}{2}\,,$$

and we are done.

To prove the claim, let S the stabilizer in general position for the action of N' on V_n. Denote the natural projection $N' \to P_n$ by π. By Lemma 9.4(a), we have $S \cap T = 1$. In other words, π is an isomorphism between S and $\pi(S)$. Since $P_n = P_{n-2} \times P_2$, the kernel of the P_n-action on V' is P_2. It now suffices to show that S acts faithfully on V', that is, $\pi(S) \cap P_2 = 1$.

By Lemma 9.4, we have $\pi(S) \subset \mathrm{A}_n$; that is, every permutation in $\pi(S)$ is even. On the other hand, the non-trivial element of P_2, namely the transposition $(n-1,\ n)$, is odd. This shows that $\pi(S) \cap P_2 = 1$, as desired. □

Proof of Theorem 9.2. By [Mac11, Section 5.7], we have $\mathrm{ed}(N; 2) \leqslant 4n$. Thus it suffices to show that $\mathrm{ed}(N; 2) \geqslant 4n$.

Recall that a split maximal torus T of SO_{4n} is isomorphic to $(\mathbb{G}_\mathrm{m})^{2n}$ and the Weyl group W is a semi-direct product $A \rtimes \mathrm{S}_{2n}$. Here $A \simeq (\mathbb{Z}/2\mathbb{Z})^{2n-1}$ is the multiplicative group of $2n$-tuples $\epsilon = (\epsilon_1, \ldots, \epsilon_{2n})$, where each ϵ_i is ± 1 and $\epsilon_1\epsilon_2 \cdots \epsilon_{2n} = 1$. The symmetric group S_{2n} acts on A by permuting $\epsilon_1, \ldots, \epsilon_{2n}$. The action of W on $(t_1, \ldots, t_{2n}) \in T$ is as follows: S_{2n} permutes $t_1, \ldots, t_{2n}$, and ϵ takes each t_i to $t_i^{\epsilon_i}$. The normalizer N of T in SO_{4n} is the semidirect product of T and W with respect to this action.

Let H be the subgroup of W generated by elements $(\epsilon_1, \ldots, \epsilon_{2n}) \in A$ with $\epsilon_1 = \epsilon_2$, $\epsilon_3 = \epsilon_4$, $\ldots$, $\epsilon_{2n-1} = \epsilon_{2n}$ and the n disjoint 2-cycles $(1, 2), (3, 4), \ldots, (2n-1, 2n)$ in S_{2n}. It is easy to see that these generators are of order 2 and commute with each other, so that $H \simeq (\mathbb{Z}/2\mathbb{Z})^n$. Let N' be the preimage of H in N.

Note that H arises as a stabilizer in general position of the natural $4n$-representation V_{4n} of N (restricted from SO_{4n}). Here $(t_1, \ldots, t_{2n}) \in T$ acts on $(x_1, \ldots, x_{2n}, y_1, \ldots, y_{2n}) \in V_{4n}$ by $x_i \mapsto t_i x_i$ and $y_i \mapsto t_i^{-1} y_i$ for each i. The symmetric group S_{2n} simultaneously permutes $x_1, \ldots, x_{2n}$ and $y_1, \ldots, y_{2n}$; the $2n$-tuple $\epsilon \in A$ leaves x_i and y_i invariant if $\epsilon_i = 1$ and switches them if $\epsilon_i = -1$.

Note that N' is a subgroup of finite index in N. Hence, $\mathrm{ed}(N; 2) \geqslant \mathrm{ed}(N'; 2)$, and it suffices to show that $\mathrm{ed}(N'; 2) \geqslant 4n$.

Claim. $\eta(N') = 4n$.

Suppose for a moment that the claim is established. Then V_{4n} is a 2-faithful representation of N' of minimal dimension. As we mentioned above, a stabilizer in general position for this representation is isomorphic to H. By Theorem 1.2,

$$\mathrm{ed}(N'; 2) = \dim(V_{4n}) + \mathrm{rank}(H) - \dim(N') = 4n + 2n - 2n = 4n\,,$$

and we are done.

To prove the claim, note that $\eta(N') \leqslant 4n$ since N' has a faithful representation V_{4n} of dimension $4n$. By Lemma 9.4, in order to establish the opposite inequality, $\eta(N') \geqslant 4n$, it suffices to show that $\mathrm{SymRank}(X(T); 2) \geqslant 4n$. To prove this last inequality, we will use the same argument as in the proof of Theorem 9.1(a). Recall that $\mathrm{SymRank}(X(T); 2)$ is the minimal size of an H-invariant 2-generating set $x_1, \ldots, x_d$ of $X(T)$. The H-action on $x_1, \ldots, x_d$ induces a permutation representation $\varphi\colon H \to \mathrm{S}_d$. Once again, this representation has to be faithful. By [AG89, Theorem 2.3(b)], we have $|H| \leqslant 2^{d/2}$. In other words, $2^{2n} \leqslant 2^{d/2}$ or, equivalently, $d \geqslant 4n$, as claimed. ☐

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