



Global Prym–Torelli theorem for double coverings of elliptic curves

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ABSTRACT

The Prym variety for a branched double covering of a nonsingular projective curve is defined as a polarized abelian variety. We prove that any double covering of an elliptic curve which has more than four branch points is recovered from its Prym variety.

1. Introduction

Let C and C' be nonsingular projective curves, and let $\phi: C \rightarrow C'$ be a double covering branched at $2n$ points. In [Mum74], the Prym variety $P(\phi)$ for the double covering ϕ is defined as a polarized abelian variety of dimension $d = g' - 1 + n$, where g' is the genus of C' . Let $\mathcal{R} = \mathcal{R}_{g',2n}$ be the moduli space of such coverings, and let $\mathcal{A} = \mathcal{A}_d$ be the moduli space of polarized abelian varieties of dimension d . Then the construction of the Prym variety defines the Prym map $P: \mathcal{R} \rightarrow \mathcal{A}$, and the Prym–Torelli problem asks whether the Prym map is injective. If $g' = 0$, then it is injective by the classical Torelli theorem for hyperelliptic curves. We consider the case $g' > 0$ and $\dim \mathcal{R} \leq \dim \mathcal{A}$, where we note that $\dim \mathcal{R} = 3g' - 3 + 2n$ and $\dim \mathcal{A} = \frac{1}{2}(g' - 1 + n)(g' + n)$. The generical injectivity for the Prym map has been proved in most cases.

THEOREM 1.1. *The Prym map is generically injective in the following cases:*

- (i) (Friedman and Smith [FS82], Kanev [Kan83]) $n = 0$ and $\dim \mathcal{R} < \dim \mathcal{A}$
- (ii) (Marcucci and Pirola [MP12]) $g' > 1$, $n > 0$, and $\dim \mathcal{R} < \dim \mathcal{A} - 1$
- (iii) (Naranjo and Ortega [NO19]) $g' > 1$, $n > 0$ and $\dim \mathcal{R} = \dim \mathcal{A} - 1$
- (iv) (Marcucci and Naranjo [MN14]) $g' = 1$, $n > 0$ and $\dim \mathcal{R} \leq \dim \mathcal{A}$.

The Prym varieties for unramified coverings have been intensively studied because they are principally polarized abelian varieties. For ramified coverings, Nagaraj and Ramanan [NR95] proved the above Theorem 1.1(ii) for $n = 2$, and then Marcucci and Pirola [MP12] proved it for any $n > 0$. When $g' > 1$ and $\dim \mathcal{R} = \dim \mathcal{A}$, there are only two cases, $(g', n) = (6, 0), (3, 2)$. If $(g', n) = (6, 0)$, then the Prym map is generically finite of degree 27 (see [DS81]), and if $(g', n) = (3, 2)$, then it is generically finite of degree 3 (see [NR95, BCV95]).

Although the Prym map is not injective for many cases in Theorem 1.1 (see [Don81, Nar92, NR95, Ver04]), we prove the injectivity when $g' = 1$. The following is the main result of this paper, which improves Theorem 1.1(iv).

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THEOREM 1.2 (Theorem 3.1). *If $g' = 1$ and $n \geq 3$, then the Prym map is injective.*

We remark that $n \geq 3$ is equivalent to the condition $\dim \mathcal{R} \leq \dim \mathcal{A}$ when $g' = 1$. To prove this theorem, we use the Gauss map for the polarization divisor, which is a standard approach to Torelli problems. Let \mathcal{L} be an ample invertible sheaf which represents the polarization of the Prym variety $P = P(\phi)$. For a member $D \in |\mathcal{L}|$, we consider the Gauss map

$$\Psi_D: D \setminus D_{\text{sing}} \longrightarrow \mathbf{P}^{d-1} = \text{Grass}(d-1, H^0(P, \Omega_P^1)^\vee).$$

It is not difficult to show that there exists a member $D_0 \in |\mathcal{L}|$ such that the branch locus of Ψ_{D_0} recovers the original covering $\phi: C \rightarrow C'$, in a way similar to Andreotti's proof [And58] of the Torelli theorem for hyperelliptic curves. The essential part of our proof is to distinguish the special member $D_0 \in |\mathcal{L}|$. We study the restriction $\Psi_D|_{\text{Bs}|\mathcal{L}|}: \text{Bs}|\mathcal{L}| \setminus D_{\text{sing}} \rightarrow \mathbf{P}^{d-1}$ of the Gauss map to the base locus of the linear system $|\mathcal{L}|$. Although Ψ_D is difficult to compute, the restriction $\Psi_D|_{\text{Bs}|\mathcal{L}|}$ is rather simple for any member $D \in |\mathcal{L}|$. By using the image of $\Psi_D|_{\text{Bs}|\mathcal{L}|}$ and the branch locus of $\Psi_D|_{\text{Bs}|\mathcal{L}|}$, we can specify the member $D_0 \in |\mathcal{L}|$ which has the desired property.

In Section 2, we summarize some basic properties of bielliptic curves and their Prym varieties. In Section 3, we explain the strategy of the proof of Theorem 1.2 by using the key propositions in Section 6. In Section 4, we explicitly describe the base locus of the linear system of polarization divisors. In Section 5, we show that the restricted Gauss map $\Psi_D|_{\text{Bs}|\mathcal{L}|}$ is the same map as the restriction of the Gauss map for the theta divisor on the Jacobian variety of C . By giving a simple description for $\Psi_D|_{\text{Bs}|\mathcal{L}|}$, we prove some properties on the branch locus of $\Psi_D|_{\text{Bs}|\mathcal{L}|}$. In Section 6, we present key propositions, which are consequences of the results in Section 5.

In this paper, we work over an algebraically closed field k of characteristic different from 2.

2. Properties of bielliptic curves and Prym varieties

Let C be a nonsingular irreducible projective curve of genus g over k , and let σ be an involution on C . In this paper, we call the pair (C, σ) a bielliptic curve of genus g if $g > 1$ and the quotient $E = C/\sigma$ is a nonsingular curve of genus 1. We denote the quotient morphism by $\phi: C \rightarrow E$. First, we note the following.

LEMMA 2.1 ([Nar92, (3.3)]). *Let (C, σ) be a bielliptic curve of genus g . If $g > 3$, then C is not a hyperelliptic curve.*

Let $N: J(C) \rightarrow J(E)$ be the norm map of ϕ , which is a homomorphism on the Jacobian varieties.

LEMMA 2.2 ([Mum74]). *Let $\phi: C \rightarrow E$ be the covering defined from a bielliptic curve (C, σ) .*

- (i) *The map $\phi^*: \text{Pic}^0(E) \rightarrow \text{Pic}^0(C)$ is injective.*
- (ii) *The kernel P of the norm map $N: J(C) \rightarrow J(E)$ is reduced and connected.*

By Lemma 2.2, the kernel P of the norm map N is an abelian variety of dimension $n = g - 1$. Let P^\vee be the dual abelian variety of P , and let $\lambda_P: P \rightarrow P^\vee$ be the polarization isogeny which is defined as the restriction of the principal polarization on the Jacobian variety $J(C)$. Then the polarized abelian variety (P, λ_P) is called the Prym variety for the covering $\phi: C \rightarrow E$. We denote by $K(P) \subset P$ the kernel of the polarization $\lambda_P: P \rightarrow P^\vee$. An ample invertible sheaf \mathcal{L}

on P represents the polarization λ_P if the polarization isogeny λ_P is given by

$$\lambda_P: P \longrightarrow P^\vee = \text{Pic}^0(P), \quad x \longmapsto t_x^* \mathcal{L} \otimes \mathcal{L}^\vee,$$

where $t_x: P \rightarrow P$ denotes the translation by $x \in P$.

LEMMA 2.3 ([Mum74]). *Let (P, λ_P) be the Prym variety defined from a bielliptic curve (C, σ) , and let \mathcal{L} be an ample invertible sheaf which represents λ_P .*

- (i) *We have $K(P) = \phi^* J(E)_2 \subset J(C)$, where $J(E)_2$ denotes the set of points of order 2 on $J(E)$.*
- (ii) *$\deg \lambda_P = 4$ and $h^0(P, \mathcal{L}) = 2$.*

3. Proof of the main theorem

Since the condition $n \geq 3$ in Theorem 1.2 is equivalent to the condition $g > 3$ for bielliptic curves, we restate the main result in the following form.

THEOREM 3.1. *If $g > 3$, then the isomorphism class of a bielliptic curve of genus g is determined by the isomorphism class of its Prym variety.*

Let (P, λ_P) be the Prym variety of dimension $n \geq 3$ defined from a bielliptic curve (C, σ) of genus $g = n + 1$. We will recover the data $(E, e_1 + \cdots + e_{2n}, \eta)$ from the polarized abelian variety (P, λ_P) , where $E = C/\sigma$ is the quotient curve, $e_1 + \cdots + e_{2n}$ is the branch divisor of the covering $\phi: C \rightarrow E$, and $\eta \in \text{Pic}(E)$ is the invertible sheaf with $\phi^* \eta \cong \Omega_C^1$. We remark that $\eta^{\otimes 2} \cong \mathcal{O}_E(e_1 + \cdots + e_{2n})$, and η is the invertible sheaf which determines the double covering with branch divisor $e_1 + \cdots + e_{2n}$.

Proof of Theorem 3.1. Let \mathcal{L} be an ample invertible sheaf on P which represents the polarization λ_P . We denote the kernel of $\lambda_P: P \rightarrow P^\vee$ by $K(P)$. By Lemma 2.3, we have $\#K(P) = 4$ and $h^0(P, \mathcal{L}) = 2$. We define the subset $\Pi_{\mathcal{L}}$ in the linear pencil $|\mathcal{L}|$ by

$$\Pi_{\mathcal{L}} = \{D \in |\mathcal{L}| \mid t_x(D) = D \subset P \text{ for some } x \in K(P) \setminus \{0\}\},$$

where t_x is the translation by $x \in P$. By Lemma 4.8, the set $\Pi_{\mathcal{L}}$ has six members for any representative \mathcal{L} of the polarization λ_P . For a member $D \in |\mathcal{L}| \setminus \Pi_{\mathcal{L}}$, we consider the Gauss map

$$\Psi_D: D \setminus D_{\text{sing}} \longrightarrow \mathbf{P}^{n-1} = \text{Grass}(n-1, H^0(P, \Omega_P^1)^\vee),$$

where $\Psi_D(x)$ is defined by the inclusion $T_x(D) \subset T_x(P) \cong H^0(P, \Omega_P^1)^\vee$ of the tangent spaces at the point $x \in D \setminus D_{\text{sing}} \subset P$. We set $U_D = \text{Bs}|\mathcal{L}| \setminus D_{\text{sing}}$, where $\text{Bs}|\mathcal{L}| \subset P$ denotes the set of base points of the pencil $|\mathcal{L}|$. Let $X'_D = \overline{\Psi_D(U_D)} \subset \mathbf{P}^{n-1}$ be the Zariski closure of $\Psi_D(U_D) \subset \mathbf{P}^{n-1}$, and let $\nu_D: X_D \rightarrow X'_D$ be the normalization. By Lemma 6.1, the variety U_D is nonsingular; hence, there is a unique morphism $\psi_D: U_D \rightarrow X_D$ such that $\Psi_D|_{U_D} = \nu_D \circ \psi_D$. We consider the closed subset $Z_D = \overline{\psi_D(\text{Ram}(\psi_D))} \subset X_D$, where $\text{Ram}(\psi_D) \subset U_D$ denotes the ramification divisor of ψ_D . By Proposition 6.2, the divisor Z_D has a canonical decomposition $Z_D = \bigcup_{i=1}^{2n} Z_{D,i}$, and there is a unique hyperplane $H_{D,i} \subset \mathbf{P}^{n-1}$ such that $\nu_D(Z_{D,i}) \subset H_{D,i}$ for any $1 \leq i \leq 2n$. Then the effective divisor $\nu_D^* H_{D,i} - Z_{D,i}$ on X_D has two irreducible components for general $D \in |\mathcal{L}| \setminus \Pi_{\mathcal{L}}$, and these components coincide for special $D \in \Pi_{\mathcal{L}}$. We define the subset $\Pi'_{\mathcal{L}}$ in the linear pencil $|\mathcal{L}|$ by

$$\Pi'_{\mathcal{L}} = \{D \in |\mathcal{L}| \setminus \Pi_{\mathcal{L}} \mid \nu_D^* H_{D,i} - Z_{D,i} \text{ is irreducible for } 1 \leq i \leq 2n\}.$$

By Lemma 6.3, the set $\Pi'_\mathcal{L}$ has four members for any representative \mathcal{L} of the polarization λ_P . For a member $D \in \Pi'_\mathcal{L}$, we consider the dual variety $(X'_D)^\vee \subset (\mathbf{P}^{n-1})^\vee$ of $X'_D \subset \mathbf{P}^{n-1}$ and the dual variety $H_{D,i}^\vee \subset (\mathbf{P}^{n-1})^\vee$ of $H_{D,i} \subset \mathbf{P}^{n-1}$. By Proposition 6.4, the variety $H_{D,i}^\vee$ is a point on $(X'_D)^\vee$, and we have an isomorphism

$$(E, e_1 + \cdots + e_{2n}, \eta) \cong ((X'_D)^\vee, H_{D,1}^\vee + \cdots + H_{D,2n}^\vee, \mathcal{O}_{(\mathbf{P}^{n-1})^\vee}(1)|_{(X'_D)^\vee}). \quad \square$$

4. Pencil of polarization divisors

Let (C, σ) be a bielliptic curve of genus $g = n + 1 > 3$. For $\delta \in \text{Pic}^n(C)$, we define the divisor $W_\delta \subset J(C)$ by

$$W_\delta = \{L \in \text{Pic}^0(C) = J(C) \mid h^0(C, L \otimes \delta) > 0\}.$$

We remark that the singular locus of W_δ is given by

$$W_{\delta, \text{sing}} = \{L \in \text{Pic}^0(C) \mid h^0(C, L \otimes \delta) > 1\},$$

and $\dim W_{\delta, \text{sing}} = n - 3$ (see [AM67, Proposition 8]) because C is not a hyperelliptic curve by Lemma 2.1. Let $\lambda_C: J(C) \rightarrow J(C)^\vee$ be the homomorphism defined by

$$\lambda_C: J(C) \longrightarrow J(C)^\vee = \text{Pic}^0(J(C)), \quad x \longmapsto [t_x^* \mathcal{O}_C(W_\delta) \otimes \mathcal{O}_C(-W_\delta)],$$

which does not depend on the choice of $\delta \in \text{Pic}^n(C)$. For $q \in C$, let $\iota_q: C \rightarrow J(C)$ be the morphism defined by

$$\iota_q: C \longrightarrow \text{Pic}^0(C) = J(C), \quad q' \longmapsto [\mathcal{O}_C(q' - q)].$$

LEMMA 4.1. *We have*

$$x = \iota_q^*[\mathcal{O}_{J(C)}(W_\delta - W_{\delta+x})] \in \text{Pic}^0(C)$$

for any $q \in C$ and $x \in \text{Pic}^0(C)$.

Proof. The statement means that $(-1) \circ \lambda_C$ is the inverse of the homomorphism $\iota_q^*: J(C)^\vee \rightarrow J(C)$ defined by the pull-back $\iota_q^*: \text{Pic}^0(J(C)) \rightarrow \text{Pic}^0(C)$ of invertible sheaves. This is well known [Mil86, Lemma 6.9]. \square

Let P be the kernel of the norm map $N: J(C) \rightarrow J(E)$, and let $D_\delta \subset P$ the fiber of the restriction of the norm map $N|_{W_\delta}: W_\delta \rightarrow J(E)$ at $0 \in J(E)$. We denote the restriction of $\mathcal{O}_{J(C)}(W_\delta)$ to P by $\mathcal{L}_\delta = \mathcal{O}_P(D_\delta) = \mathcal{O}_{J(C)}(W_\delta)|_P$. Since W_δ is the theta divisor of $J(C)$, the ample invertible sheaf \mathcal{L}_δ represents the polarization λ_P .

LEMMA 4.2. *The divisor $D_{\delta+\phi^*s} \subset P$ is a member of the linear system $|\mathcal{L}_\delta|$ for any $s \in \text{Pic}^0(E)$.*

Proof. By Lemma 4.1, we have

$$\phi^*s = \iota_q^*[\mathcal{O}_{J(C)}(W_\delta - W_{\delta+\phi^*s})] \in \text{Pic}^0(C)$$

for $s \in \text{Pic}^0(E)$ and $q \in C$. We define $s' \in \text{Pic}^0(J(E))$ by $s = \iota_{\phi(q)}^*s'$, where $\iota_{\phi(q)}: E \rightarrow J(E)$ is the isomorphism determined by $\iota_{\phi(q)} \circ \phi = N \circ \iota_q$. Then we have

$$N^*s' = [\mathcal{O}_{J(C)}(W_\delta - W_{\delta+\phi^*s})] \in \text{Pic}^0(J(C))$$

because $\phi^*s = \iota_q^*N^*s'$ and $\iota_q^*: \text{Pic}^0(J(C)) \rightarrow \text{Pic}^0(C)$ is an isomorphism. Since $(N^*s')|_P = 0 \in \text{Pic}^0(P)$, we have

$$\mathcal{O}_P(D_{\delta+\phi^*s}) = \mathcal{O}_{J(C)}(W_{\delta+\phi^*s})|_P \cong \mathcal{O}_{J(C)}(W_\delta)|_P = \mathcal{L}_\delta. \quad \square$$

We denote the i th symmetric products of C by $C^{(i)}$, and points of $C^{(i)}$ are considered as effective divisors on C of degree i . For $\delta \in \text{Pic}^n(C)$, we define the morphism $\beta_\delta^i: C^{(n-2i)} \times E^{(i)} \rightarrow J(C)$ by

$$\begin{aligned} \beta_\delta^i: C^{(n-2i)} \times E^{(i)} &\longrightarrow J(C) = \text{Pic}^0(C), \\ (\mathbf{q}, \mathbf{p}) &\longmapsto \mathcal{O}_C(\mathbf{q}) \otimes \phi^* \mathcal{O}_E(\mathbf{p}) \otimes \delta^\vee. \end{aligned}$$

We remark that $W_\delta = \text{Image}(\beta_\delta^0)$, and we set

$$B_\delta^i = \begin{cases} \text{Image}(\beta_\delta^i), & 1 \leq 2i \leq n, \\ \emptyset, & 2i > n. \end{cases}$$

LEMMA 4.3. *We have $B_\delta^1 \not\subseteq W_{\delta, \text{sing}}$ and $B_\delta^2 \subset W_{\delta, \text{sing}}$.*

Proof. Let $B^1 \subset C^{(n)}$ be the image of the morphism

$$\beta^1: C^{(n-2)} \times E \longrightarrow C^{(n)}, \quad (\mathbf{q}, \phi(q)) \longmapsto \mathbf{q} + q + \sigma(q).$$

Since C is not a hyperelliptic curve, we have $\dim(\beta_\delta^0)^{-1}(W_{\delta, \text{sing}}) = n - 2 < n - 1 = \dim B^1$; hence $B_\delta^1 = \beta_\delta^0(B^1) \not\subseteq W_{\delta, \text{sing}}$. To prove $B_\delta^2 \subset W_{\delta, \text{sing}}$, we assume $n \geq 4$ because $B_\delta^2 = \emptyset$ for $n = 3$. Then it is a consequence of the inequality

$$\begin{aligned} h^0(C, \mathcal{O}_C(\mathbf{q}) \otimes \phi^* \mathcal{O}_E(\mathbf{p})) &\geq h^0(C, \phi^* \mathcal{O}_E(\mathbf{p})) = h^0(E, \phi_* \mathcal{O}_C \otimes \mathcal{O}_E(\mathbf{p})) \\ &\geq h^0(E, \mathcal{O}_E(\mathbf{p})) = 2 \end{aligned}$$

for $(\mathbf{q}, \mathbf{p}) \in C^{(n-4)} \times E^{(2)}$. □

LEMMA 4.4. *For $s \in \text{Pic}^0(E)$, we have $D_\delta = D_{\delta + \phi^* s} \subset P$ if and only if $s = 0$ or $s = \eta - N(\delta)$.*

Proof. For $L \in P \subset \text{Pic}^0(C)$, we have $0 = \phi^* N(L) = L + \sigma^* L \in \text{Pic}^0(C)$. Hence we have $D_\delta = D_{\delta + \phi^*(\eta - N(\delta))}$ because

$$\begin{aligned} L \in D_\delta &\iff h^0(C, L \otimes \delta) > 0 \iff h^0(C, \sigma^* L \otimes \sigma^* \delta) > 0 \iff h^0(C, L^\vee \otimes \sigma^* \delta) > 0 \\ &\iff h^0(C, \Omega_C^1 \otimes L \otimes \sigma^* \delta^\vee) > 0 \iff L \in D_{[\Omega_C^1] - \sigma^* \delta} = D_{\delta + \phi^*(\eta - N(\delta))}. \end{aligned}$$

We assume $D_\delta = D_{\delta + \phi^* s}$ for $s \neq 0 \in \text{Pic}^0(E)$. By [Mum74, §5, Proof of the proposition, p. 338], we have the exact sequence

$$0 \longrightarrow s \longrightarrow \phi_*(\phi^* s \otimes L \otimes \delta) \longrightarrow s \otimes N(L \otimes \delta) \otimes \eta^\vee \longrightarrow 0$$

for $L \in D_\delta \setminus B_\delta^1$. By $h^0(E, s) = 0$ and $L \in D_\delta = D_{\delta + \phi^* s}$, we have

$$h^0(E, s \otimes N(L \otimes \delta) \otimes \eta^\vee) \geq h^0(E, \phi_*(\phi^* s \otimes L \otimes \delta)) > 0.$$

Since $s \otimes N(L \otimes \delta) \otimes \eta^\vee = s \otimes N(\delta) \otimes \eta^\vee \in \text{Pic}^0(E)$ is effective, we have $s \otimes N(\delta) \otimes \eta^\vee \cong \mathcal{O}_E$. □

Let $B_\delta \subset J(C)$ be the subset

$$B_\delta = \bigcap_{s \in \text{Pic}^0(E)} W_{\delta + \phi^* s}.$$

LEMMA 4.5. *We have $B_\delta \setminus W_{\delta, \text{sing}} = B_\delta^1 \setminus W_{\delta, \text{sing}}$.*

Proof. If $L \in B_\delta^1$, then $L \otimes \delta \cong \mathcal{O}_C(\mathbf{q} + q + \sigma(q))$ for some $\mathbf{q} \in C^{(n-2)}$ and $q \in C$. For $s \in \text{Pic}^0(E)$, there is a point $q' \in C$ such that $s = [\mathcal{O}_E(\phi(q') - \phi(q))]$. Since $L \otimes \delta \otimes \phi^* s \cong \mathcal{O}_C(\mathbf{q} + q' + \sigma(q'))$, we have $h^0(C, L \otimes \delta \otimes \phi^* s) > 0$ and $L \in W_{\delta + \phi^* s}$. Hence the inclusion $B_\delta^1 \subset B_\delta$ holds.

For $L \in B_\delta \setminus W_{\delta, \text{sing}}$, there is a unique $\mathbf{r} = r_1 + \cdots + r_n \in C^{(n)}$ such that $\Omega_C^1 \otimes L^\vee \otimes \delta^\vee \cong \mathcal{O}_C(\mathbf{r})$ because $h^0(C, \Omega_C^1 \otimes L^\vee \otimes \delta^\vee) = h^0(C, L \otimes \delta) = 1$. For $s \in \text{Pic}^0(E) \setminus \{0\}$, there is a divisor $\mathbf{q} = q_1 + \cdots + q_n \in C^{(n)}$ such that $L \otimes \delta \otimes \phi^*s \cong \mathcal{O}_C(\mathbf{q})$ because $L \in B_\delta \subset W_{\delta + \phi^*s}$. When $s \in \text{Pic}^0(E) \setminus \{0\}$ is general, we may assume $\sigma(q_n) \notin \{r_1, \dots, r_n\}$. This implies that $h^0(C, \Omega_C^1 \otimes L^\vee \otimes \delta^\vee \otimes \mathcal{O}_C(-\sigma(q_n))) = 0$ and $h^0(C, L \otimes \delta \otimes \mathcal{O}_C(\sigma(q_n))) = 1$. Let $\phi(q') \in E$ be the point determined by $s = [\mathcal{O}_E(\phi(q_n) - \phi(q'))]$. Then

$$L \otimes \delta \otimes \mathcal{O}_C(\sigma(q_n)) \cong \mathcal{O}_C(q_1 + \cdots + q_{n-1} + q' + \sigma(q')).$$

Since $s \neq 0$, we have $\sigma(q_n) \in \{q_1, \dots, q_{n-1}\}$ and $L \in B_\delta^1$. \square

LEMMA 4.6. *The map*

$$\text{Pic}^0(E) \longrightarrow |\mathcal{L}_\delta|, \quad s \longmapsto D_{\delta + \phi^*s}$$

is a double covering, and the base locus $\text{Bs } |\mathcal{L}_\delta|$ of the linear system $|\mathcal{L}_\delta|$ is $B_\delta \cap P$, which is of dimension $n - 2$.

Proof. The map is well defined by Lemma 4.2. Since $\dim |\mathcal{L}_\delta| = 1$, it is a double covering by Lemma 4.4. Hence we have

$$\text{Bs } |\mathcal{L}_\delta| = \bigcap_{s \in \text{Pic}^0(E)} D_{\delta + \phi^*s} = B_\delta \cap P.$$

By Lemma 4.3, the variety B_δ^1 is irreducible of dimension $n - 1$. Since the restriction of the norm map $N|_{B_\delta^1}: B_\delta^1 \rightarrow J(E)$ is surjective, we have $\dim B_\delta^1 \cap P = n - 2$, hence $\dim B_\delta \cap P = n - 2$ by Lemma 4.5. \square

LEMMA 4.7. *Let \mathcal{L} be an ample invertible sheaf which represents the polarization λ_P on P ; then there is a $\delta \in \text{Pic}^n(C)$ such that $N(\delta) = \eta$ and $\mathcal{L} \cong \mathcal{L}_\delta$.*

Proof. For any $\delta' \in \text{Pic}^n(C)$, we have $\mathcal{L} \otimes \mathcal{L}_{\delta'}^\vee \in \text{Pic}^0(P)$ because $\mathcal{L}_{\delta'}$ gives the same polarization as λ_P . Then $\mathcal{L} \cong t_x^* \mathcal{L}_{\delta'} \cong \mathcal{L}_{\delta' + x}$ for some $x \in P$. Let $s \in \text{Pic}^0(E)$ be a point with $2s = \eta - N(\delta' + x)$. For $\delta = \delta' + x + \phi^*s$, we have $N(\delta) = \eta$ and $\mathcal{L} \cong \mathcal{L}_\delta$. \square

For an ample invertible sheaf \mathcal{L} which represents the polarization λ_P , we define a subset in the linear system $|\mathcal{L}|$ by

$$\Pi_{\mathcal{L}} = \{D \in |\mathcal{L}| \mid t_x(D) = D \text{ for some } x \in K(P) \setminus \{0\}\},$$

where $K(P)$ is the kernel of the polarization λ_P .

LEMMA 4.8. *We have $\#\Pi_{\mathcal{L}} = 6$.*

Proof. By Lemma 4.7, there is a $\delta \in \text{Pic}^n(C)$ such that $N(\delta) = \eta$ and $\mathcal{L} \cong \mathcal{L}_\delta$. For any $D \in |\mathcal{L}_\delta|$, by Lemma 4.6, there is an $s \in \text{Pic}^0(E)$ such that $D = D_{\delta + \phi^*s}$. If $D_{\delta + \phi^*s} \in \Pi_{\mathcal{L}_\delta}$, then by Lemma 2.3, there is a $t \in J(E)_2 \setminus \{0\}$ such that $t_{\phi^*t}(D_{\delta + \phi^*s}) = D_{\delta + \phi^*s}$. Since $t_{\phi^*t}(D_{\delta + \phi^*s}) = D_{\delta + \phi^*s + \phi^*(-t)}$ and $t \neq 0$, by Lemma 4.4, we have $-t = \eta - N(\delta + \phi^*s)$, hence $t = 2s$. This means that

$$\Pi_{\mathcal{L}_\delta} = \{D_{\delta + \phi^*s} \in |\mathcal{L}_\delta| \mid s \in J(E)_4 \setminus J(E)_2\}.$$

Since $\#(J(E)_4 \setminus J(E)_2) = 12$ and $D_{\delta + \phi^*s} = D_{\delta - \phi^*s}$, we have $\#\Pi_{\mathcal{L}} = \#\Pi_{\mathcal{L}_\delta} = 6$. \square

5. Gauss maps

5.1 Gauss maps for the Jacobian and for the Prym variety

Let

$$\Psi_{J(C),\delta}: W_\delta \setminus W_{\delta,\text{sing}} \longrightarrow \mathbf{P}(H^0(C, \Omega_C^1)^\vee) = \text{Grass}(n, H^0(C, \Omega_C^1)^\vee)$$

be the Gauss map for the subvariety $W_\delta \subset J(C)$. We note that the projective space $\mathbf{P}(V)$ is the set of subspaces of codimension 1 in the vector space V . For $L \in W_\delta \setminus W_{\delta,\text{sing}}$, the tangent space $T_L(W_\delta)$ of W_δ at L defines the image $\Psi_{J(C),\delta}(L)$ by the natural identifications

$$T_L(W_\delta) \subset T_L(J(C)) \cong (\Omega_{J(C)}^1|_L)^\vee \cong H^0(J(C), \Omega_{J(C)}^1)^\vee \cong H^0(C, \Omega_C^1)^\vee.$$

LEMMA 5.1. *For $L \in W_\delta \setminus W_{\delta,\text{sing}}$, the image $\Psi_{J(C),\delta}(L)$ of the Gauss map is identified with the canonical divisor*

$$\mathbf{q} + \mathbf{q}' \in |\Omega_C^1| = \text{Grass}(1, H^0(C, \Omega_C^1)) \cong \mathbf{P}(H^0(C, \Omega_C^1)^\vee),$$

where $\mathbf{q} \in C^{(n)}$ and $\mathbf{q}' \in C^{(n)}$ are uniquely determined by $L \otimes \delta \cong \mathcal{O}_C(\mathbf{q})$ and $\Omega_C^1 \otimes L^\vee \otimes \delta^\vee \cong \mathcal{O}_C(\mathbf{q}')$.

Proof. This is a special case of Proposition 4.2 in [ACGH85, Chapter IV]. \square

LEMMA 5.2. *Let $K \in |\Omega_C^1|$ be an effective canonical divisor. If $q_1 + \sigma(q_1) \leq K$ for some $q_1 \in C$, then $K = \sum_{i=1}^n (q_i + \sigma(q_i))$ for some $q_2, \dots, q_n \in C$.*

Proof. We assume

$$K = \sum_{i=1}^m (q_i + \sigma(q_i)) + \sum_{j=1}^{2n-2m} r_j$$

and $\sigma(r_i) \neq r_j$ for $i \neq j$. By [Mum74, § 5, Proof of the proposition, p. 338], we have the exact sequence

$$0 \longrightarrow A \longrightarrow \phi_*(\phi^*A \otimes L) \longrightarrow A \otimes N(L) \otimes \eta^\vee \longrightarrow 0$$

for $A = \mathcal{O}_E(\sum_{i=1}^m \phi(q_i))$ and $L = \mathcal{O}_C(\sum_{j=1}^{2n-2m} r_j)$. By the assumption $m \geq 1$, we have $h^0(E, A) = m$, hence

$$\begin{aligned} h^0(E, A \otimes N(L) \otimes \eta^\vee) &\geq h^0(E, \phi_*(\phi^*A \otimes L)) - h^0(E, A) = n + 1 - m \\ &= \deg(A \otimes N(L) \otimes \eta^\vee) + 1. \end{aligned}$$

This implies that $m = n$ and $A \cong \eta$. \square

By the injective homomorphism

$$H^0(E, \eta) \longrightarrow H^0(C, \phi^*\eta) \cong H^0(C, \Omega_C^1),$$

we have the closed immersion

$$\iota: \mathbf{P}(H^0(E, \eta)^\vee) \longrightarrow \mathbf{P}(H^0(C, \Omega_C^1)^\vee).$$

LEMMA 5.3. *For $L \in B_\delta \setminus W_{\delta,\text{sing}}$, we have*

$$\Psi_{J(C),\delta}(L) \in \iota(\mathbf{P}(H^0(E, \eta)^\vee)) \subset \mathbf{P}(H^0(C, \Omega_C^1)^\vee).$$

Proof. For $L \in B_\delta \setminus W_{\delta,\text{sing}}$, by Lemma 5.1, the image $\Psi_{J(C),\delta}(L)$ of the Gauss map is given by

$$\mathbf{q} + \mathbf{q}' \in |\Omega_C^1| \cong \mathbf{P}(H^0(C, \Omega_C^1)^\vee),$$

where $\mathbf{q} = q_1 + \cdots + q_n \in C^{(n)}$ and $\mathbf{q}' \in C^{(n)}$ are uniquely determined by $L \otimes \delta \cong \mathcal{O}_C(\mathbf{q})$ and $\Omega_C^1 \otimes L^\vee \otimes \delta^\vee \cong \mathcal{O}_C(\mathbf{q}')$. Since $L \in B_\delta$, by Lemma 4.5, we have $\sigma(q_i) = q_j$ for some $i \neq j$. By Lemma 5.2, we have $\Psi_{J(C),\delta}(L) \in \iota(\mathbf{P}(H^0(E, \eta)^\vee))$. \square

Let

$$\Psi_{P,\delta}: D_\delta \setminus D_{\delta,\text{sing}} \rightarrow \mathbf{P}(H^0(P, \Omega_P^1)^\vee) = \text{Grass}(n-1, H^0(P, \Omega_P^1)^\vee)$$

be the Gauss map for the subvariety $D_\delta \subset P$. For $L \in D_\delta \setminus D_{\delta,\text{sing}}$, the tangent space $T_L(D_\delta)$ of D_δ at L defines the image $\Psi_{P,\delta}(L)$ by the natural identifications

$$T_L(D_\delta) \subset T_L(P) \cong (\Omega_P^1|_L)^\vee \cong H^0(P, \Omega_P^1)^\vee.$$

For $L \in P$, the tangent space $T_L(P)$ of P at L is naturally identified with the orthogonal complement

$$V_P = (\phi^* H^0(E, \Omega_E^1))^\perp \subset H^0(C, \Omega_C^1)^\vee \cong T_L(J(C))$$

of $\phi^* H^0(E, \Omega_E^1) \subset H^0(C, \Omega_C^1)$, and it corresponds to the ramification divisor $\text{Ram}(\phi) \in |\Omega_C^1|$ of the covering $\phi: C \rightarrow E$. We define the finite subset $\Sigma_\delta \subset J(C)$ by

$$\Sigma_\delta = \{\beta_\delta^0(\mathbf{r}) \in J(C) \mid \mathbf{r} \leq \text{Ram}(\phi), \mathbf{r} \in C^{(n)}\}.$$

LEMMA 5.4. *We have $D_{\delta,\text{sing}} = (W_{\delta,\text{sing}} \cup \Sigma_\delta) \cap D_\delta$.*

Proof. If $L \in D_\delta \cap W_{\delta,\text{sing}}$, then $L \in D_{\delta,\text{sing}}$. If $L \in D_\delta \setminus W_{\delta,\text{sing}}$, then by Lemma 5.1,

$$L \in D_{\delta,\text{sing}} \iff T_L(W_\delta) = T_L(P) \subset T_L(J(C)) \iff L \in \Sigma_\delta. \quad \square$$

LEMMA 5.5. *We have $(B_\delta \cap P) \setminus W_{\delta,\text{sing}} = (B_\delta \cap P) \setminus D_{\delta,\text{sing}}$.*

Proof. By Lemma 4.5, we have $(B_\delta \setminus W_{\delta,\text{sing}}) \cap \Sigma_\delta = (B_\delta^1 \setminus W_{\delta,\text{sing}}) \cap \Sigma_\delta$, and it is empty because $\text{Ram}(\phi)$ is reduced. Hence by Lemma 5.4, we have

$$W_{\delta,\text{sing}} \cap B_\delta \cap P = W_{\delta,\text{sing}} \cap B_\delta \cap D_\delta = D_{\delta,\text{sing}} \cap B_\delta = D_{\delta,\text{sing}} \cap B_\delta \cap P. \quad \square$$

We denote by

$$\begin{aligned} \pi: \mathbf{P}(H^0(C, \Omega_C^1)^\vee) \setminus \{V_P\} &\longrightarrow \mathbf{P}(H^0(P, \Omega_P^1)^\vee), \\ [V \subset H^0(C, \Omega_C^1)^\vee] &\longmapsto [V \cap V_P \subset V_P \cong H^0(P, \Omega_P^1)^\vee] \end{aligned}$$

the projection, where $V_P = (\phi^* H^0(E, \Omega_E^1))^\perp \subset H^0(C, \Omega_C^1)^\vee$ is the image of the dual of the restriction

$$H^0(C, \Omega_C^1) \cong H^0(J(C), \Omega_{J(C)}^1) \rightarrow H^0(P, \Omega_P^1).$$

LEMMA 5.6. *We have $\Psi_{P,\delta}(L) = \pi \circ \Psi_{J(C),\delta}(L)$ for $L \in D_\delta \setminus D_{\delta,\text{sing}}$.*

Proof. For $L \in D_\delta \setminus D_{\delta,\text{sing}}$, the tangent spaces at L satisfy

$$T_L(W_\delta) \cap T_L(P) = T_L(D_\delta) \subset T_L(J(C))$$

because $W_\delta \cap P = D_\delta \subset J(C)$. Since $T_L(P) \subset T_L(J(C))$ is identified with $V_P \subset H^0(C, \Omega_C^1)^\vee$ by $T_L(J(C)) \cong H^0(C, \Omega_C^1)^\vee$, we have $\Psi_{P,\delta}(L) = \pi \circ \Psi_{J(C),\delta}(L)$. \square

By Lemma 5.3, we have the morphism

$$\Psi_{J(C),\delta}^B: B_\delta \setminus W_{\delta,\text{sing}} \longrightarrow \mathbf{P}(H^0(E, \eta)^\vee)$$

satisfying $\iota \circ \Psi_{J(C),\delta}^B = \Psi_{J(C),\delta}$.

LEMMA 5.7. *The restriction $\Psi_{P,\delta}|_{(B_\delta \cap P) \setminus D_{\delta,\text{sing}}}$ of the Gauss map $\Psi_{P,\delta}$ is identified with the restriction $\Psi_{J(C),\delta}^B|_{(B_\delta \cap P) \setminus D_{\delta,\text{sing}}}$ of $\Psi_{J(C),\delta}^B$ by the isomorphism*

$$\pi \circ \iota: \mathbf{P}(H^0(E, \eta)^\vee) \xrightarrow{\cong} \mathbf{P}(H^0(P, \Omega_P^1)^\vee).$$

Proof. Since the composition

$$H^0(E, \eta) \hookrightarrow H^0(C, \Omega_C^1) \cong H^0(J(C), \Omega_{J(C)}^1) \rightarrow H^0(P, \Omega_P^1)$$

is an isomorphism, the result is a consequence of Lemma 5.6. \square

5.2 Description for the restricted Gauss maps

Let $\gamma_\delta: E^{(n-2)} \times E \rightarrow J(E)$ be the morphism defined by

$$\begin{aligned} \gamma_\delta: E^{(n-2)} \times E &\longrightarrow \text{Pic}^0(E) = J(E), \\ (\mathbf{p}, p) &\longmapsto \mathcal{O}_E(\mathbf{p} + 2p) \otimes (N(\delta))^\vee. \end{aligned}$$

Let $X_\delta \subset E^{(n-2)} \times E$ be the fiber of γ_δ at $0 \in J(E)$, and let $Y_\delta \subset C^{(n-2)} \times E$ be the fiber of the composition

$$C^{(n-2)} \times E \xrightarrow{\phi^{(n-2)} \times \text{id}_E} E^{(n-2)} \times E \xrightarrow{\gamma_\delta} J(E)$$

at $0 \in J(E)$. We denote by $\psi_\delta: Y_\delta \rightarrow X_\delta$ the morphism induced by $\phi^{(n-2)} \times \text{id}_E$. Let $\nu_\delta: X_\delta \rightarrow |\eta| \cong \mathbf{P}(H^0(E, \eta)^\vee)$ be the morphism defined by

$$\nu_\delta: X_\delta \longrightarrow |\eta|, \quad (\mathbf{p}, p) \longmapsto \mathbf{p} + p + t_{\eta-N(\delta)}(p),$$

where $t_{\eta-N(\delta)}(p) \in E$ is the point determined by

$$[\mathcal{O}_E(t_{\eta-N(\delta)}(p))] = [\mathcal{O}_E(p)] + \eta - N(\delta) \in \text{Pic}(E).$$

We remark that $\beta_\delta^1(Y_\delta) = B_\delta^1 \cap P \subset J(C)$, and we set

$$Y_\delta^\circ = (\beta_\delta^1)^{-1}((B_\delta^1 \cap P) \setminus D_{\delta,\text{sing}}) = (\beta_\delta^1)^{-1}((B_\delta \cap P) \setminus D_{\delta,\text{sing}}).$$

LEMMA 5.8. *The diagram*

$$\begin{array}{ccc} Y_\delta^\circ & \xrightarrow{\beta_\delta^1} & (B_\delta \cap P) \setminus D_{\delta,\text{sing}} \\ \psi_\delta \downarrow & & \downarrow \Psi_{J(C),\delta}^B \\ X_\delta & \xrightarrow{\nu_\delta} & \mathbf{P}(H^0(E, \eta)^\vee) \end{array}$$

is commutative.

Proof. Let $L \in \text{Pic}^0(C)$ be the invertible sheaf which represents the point $\beta_\delta^1(y) \in J(C)$ for $y = (\mathbf{q}, \phi(q)) \in Y_\delta^\circ$. Then $\mathbf{q} + q + \sigma(q) \in C^{(n)}$ is the unique effective divisor with $L \cong \mathcal{O}_C(\mathbf{q} + q + \sigma(q)) \otimes \delta^\vee$. Since

$$\nu_\delta \circ \psi_\delta(y) = \phi^{(n-2)}(\mathbf{q}) + \phi(q) + t_{\eta-N(\delta)}(\phi(q)) \in |\eta|,$$

we have

$$\mathbf{q} + \sigma(\mathbf{q}) + q + \sigma(q) + r + \sigma(r) \in |\Omega_C^1|,$$

where $r \in C$ is given by $\phi(r) = t_{\eta-N(\delta)}(\phi(q))$. Then $\sigma(\mathbf{q}) + r + \sigma(r) \in C^{(n)}$ is the unique effective divisor with $\mathcal{O}_C(\sigma(\mathbf{q}) + r + \sigma(r)) \cong \Omega_C^1 \otimes L^\vee \otimes \delta^\vee$, and by Lemma 5.1, the image $\Psi_{J(C),\delta}^B(L)$ is equal to $\nu_\delta \circ \psi_\delta(y)$. \square

LEMMA 5.9. *The projective varieties X_δ and Y_δ are nonsingular.*

Proof. We fix a point $p_0 \in E$. Let $\gamma_{\delta,p_0}: E^{(n-2)} \rightarrow J(E)$ be the morphism defined by

$$\begin{aligned} \gamma_{\delta,p_0}: E^{(n-2)} &\longrightarrow \mathrm{Pic}^0(E) = J(E), \\ \mathbf{p} &\longmapsto \mathcal{O}_E(\mathbf{p} + 2p_0) \otimes (N(\delta))^\vee. \end{aligned}$$

Then $\psi_\delta: Y_\delta \rightarrow X_\delta$ is the base change of $\phi^{(n-2)}: C^{(n-2)} \rightarrow E^{(n-2)}$ by the étale covering of degree 4:

$$\begin{array}{ccccc} Y_\delta & \xrightarrow{\psi_\delta} & X_\delta & \xrightarrow{\mathrm{pr}_2} & E \\ \mathrm{pr}_1 \downarrow & \square & \downarrow \mathrm{pr}_1 & \square & \downarrow (-2)_{J(E)} \circ \iota_{p_0} \\ C^{(n-2)} & \xrightarrow{\phi^{(n-2)}} & E^{(n-2)} & \xrightarrow{\gamma_{\delta,p_0}} & J(E), \end{array}$$

where $(-2)_{J(E)} \circ \iota_{p_0}: E \rightarrow J(E)$ is given by

$$(-2)_{J(E)} \circ \iota_{p_0}: E \longrightarrow \mathrm{Pic}^0(E) = J(E), \quad p \longmapsto \mathcal{O}_E(2p_0 - 2p). \quad \square$$

LEMMA 5.10. *The map $\beta_\delta^1|_{Y_\delta^\circ}: Y_\delta^\circ \rightarrow (B_\delta \cap P) \setminus D_{\delta,\mathrm{sing}}$ is an isomorphism. In particular, $(B_\delta \cap P) \setminus D_{\delta,\mathrm{sing}}$ is a nonsingular variety.*

Proof. By Lemma 5.5, the image $\beta_\delta^1(Y_\delta^\circ) = (B_\delta \cap P) \setminus D_{\delta,\mathrm{sing}}$ is a closed subset in $W_\delta \setminus W_{\delta,\mathrm{sing}}$. We show that $\beta_\delta^1|_{Y_\delta^\circ}: Y_\delta^\circ \rightarrow W_\delta \setminus W_{\delta,\mathrm{sing}}$ is a closed immersion. Let $\beta^1: C^{(n-2)} \times E \rightarrow C^{(n)}$ be the morphism given in the proof of Lemma 4.3. Since $\beta_\delta^1 = \beta_\delta^0 \circ \beta^1$ and $\beta_\delta^0: C^{(n)} \rightarrow J(C)$ induces the isomorphism

$$\beta_\delta^0: C^{(n)} \setminus (\beta_\delta^0)^{-1}(W_{\delta,\mathrm{sing}}) \xrightarrow{\cong} W_\delta \setminus W_{\delta,\mathrm{sing}},$$

it is enough to show that the finite morphism

$$\beta^1: (C^{(n-2)} \times E) \setminus (\beta_\delta^1)^{-1}(W_{\delta,\mathrm{sing}}) \longrightarrow C^{(n)} \setminus (\beta_\delta^0)^{-1}(W_{\delta,\mathrm{sing}})$$

is a closed immersion. We remark that it is injective by Lemma 4.3. For $y = (\mathbf{q}, \phi(q_0)) \in (C^{(n-2)} \times E) \setminus (\beta_\delta^1)^{-1}(W_{\delta,\mathrm{sing}})$, we prove that the homomorphism

$$d\beta^1: T_y(C^{(n-2)} \times E) \longrightarrow T_{\beta^1(y)}(C^{(n)})$$

on the tangent spaces is injective. If $y = (\mathbf{q}, \phi(q_0))$ does not satisfy $\phi(q_0) \leq \phi(\mathbf{q})$, then the point $y' = (\mathbf{q}, q_0 + \sigma(q_0)) \in C^{(n-2)} \times C^{(2)}$ is not contained in the ramification divisor of the natural covering $C^{(n-2)} \times C^{(2)} \rightarrow C^{(n)}$. Since the morphism

$$E \longrightarrow C^{(2)}, \quad \phi(q) \longmapsto q + \sigma(q)$$

is a closed immersion, the homomorphism

$$T_y(C^{(n-2)} \times E) \hookrightarrow T_{y'}(C^{(n-2)} \times C^{(2)}) \cong T_{\beta^1(y)}(C^{(n)})$$

is injective. When $y = (\mathbf{q}, \phi(q_0))$ satisfies $\phi(q_0) \leq \phi(\mathbf{q})$, we can prove the injectivity of $d\beta^1$ in a similar way, by using the fact that the morphism

$$C \times E \longrightarrow C^{(3)}, \quad (q', \phi(q)) \longmapsto q' + q + \sigma(q)$$

is a closed immersion. □

Let $X'_\delta = \overline{\Psi_{J(C),\delta}^B((B_\delta \cap P) \setminus D_{\delta,\mathrm{sing}})}$ be the Zariski closure of the image of the restricted Gauss map $\Psi_{J(C),\delta}^B|_{(B_\delta \cap P) \setminus D_{\delta,\mathrm{sing}}}$ in $\mathbf{P}(H^0(E, \eta)^\vee)$.

LEMMA 5.11. We have $\nu_\delta(X_\delta) = X'_\delta$.

Proof. By Lemma 5.8, we have

$$\Psi_{J(C),\delta}^B((B_\delta \cap P) \setminus D_{\delta,\text{sing}}) = \nu_\delta(\psi_\delta(Y_\delta^\circ)) \subset \nu_\delta(X_\delta),$$

hence $X'_\delta = \overline{\nu_\delta(\psi_\delta(Y_\delta^\circ))} \subset \nu_\delta(X_\delta)$ and $Y_\delta^\circ \subset (\nu_\delta \circ \psi_\delta)^{-1}(X'_\delta)$. Since Y_δ° is dense in Y_δ , we have $Y_\delta \subset (\nu_\delta \circ \psi_\delta)^{-1}(X'_\delta)$ and $\nu_\delta(X_\delta) = (\nu_\delta \circ \psi_\delta)(Y_\delta) \subset X'_\delta$. \square

LEMMA 5.12. If $N(\delta) - \eta \notin J(E)_2 \setminus \{0\}$, then $\nu_\delta: X_\delta \rightarrow X'_\delta$ is the normalization of X'_δ .

Proof. Let $U \subset E^{(n)}$ be the dense open subset whose complement $E^{(n)} \setminus U$ is the set of the point $\mathbf{p} \in E^{(n)}$ satisfying the condition

$$\begin{cases} \mathbf{p} \geq 2p + t_{\eta-N(\delta)}(p), \\ \mathbf{p} \geq 2p + t_{N(\delta)-\eta}(p), \\ \mathbf{p} \geq p + t_{\eta-N(\delta)}(p) + t_{N(\delta)-\eta}(p), \\ \text{or } \mathbf{p} \geq p + t_{\eta-N(\delta)}(p) + p' + t_{\eta-N(\delta)}(p') \end{cases} \quad \text{for some } p, p' \in E.$$

By the natural inclusion $X'_\delta \subset |\eta| \subset E^{(n)}$, the subset $X'_\delta \cap U$ is dense open in X'_δ . We show that the morphism $\nu_\delta: \nu_\delta^{-1}(U) \rightarrow U$ is a closed immersion. For $\mathbf{p} \in X'_\delta \cap U$, we assume

$$\mathbf{p} \geq p + t_{\eta-N(\delta)}(p) \quad \text{and} \quad \mathbf{p} \geq p' + t_{\eta-N(\delta)}(p')$$

for some $p \neq p' \in E$. Since $\mathbf{p} \in U$, we have

$$t_{\eta-N(\delta)}(p) = p' \quad \text{and} \quad p = t_{\eta-N(\delta)}(p'),$$

hence $N(\delta) - \eta \in J(E)_2 \setminus \{0\}$. This means that $\nu_\delta: \nu_\delta^{-1}(U) \rightarrow X'_\delta \cap U$ is bijective if $N(\delta) - \eta \notin J(E)_2 \setminus \{0\}$. In the following, we prove that the homomorphism

$$d\nu_\delta: T_x(X_\delta) \longrightarrow T_{\nu_\delta(x)}(E^{(n)})$$

on the tangent spaces is injective for $x \in \nu_\delta^{-1}(U)$. Let $\tilde{\nu}_\delta: E^{(n-2)} \times E \rightarrow E^{(n-2)} \times E^{(2)}$ be the morphism defined by

$$\begin{aligned} \tilde{\nu}_\delta: E^{(n-2)} \times E &\longrightarrow E^{(n-2)} \times E^{(2)}, \\ (\mathbf{p}, p) &\longmapsto (\mathbf{p}, p + t_{\eta-N(\delta)}(p)). \end{aligned}$$

If $N(\delta) - \eta \notin J(E)_2 \setminus \{0\}$, then the morphism $\tilde{\nu}_\delta$ is a closed immersion. For $x \in \nu_\delta^{-1}(U)$, the image $\tilde{\nu}_\delta(x)$ is not contained in the ramification divisor of the natural covering $E^{(n-2)} \times E^{(2)} \rightarrow E^{(n)}$. Hence the homomorphism

$$T_x(X_\delta) \subset T_x(E^{(n-2)} \times E) \longleftarrow T_{\tilde{\nu}_\delta(x)}(E^{(n-2)} \times E^{(2)}) \cong T_{\nu_\delta(x)}(E^{(n)})$$

is injective. By Lemma 5.9, the finite birational morphism $\nu_\delta: X_\delta \rightarrow X'_\delta$ gives the normalization of X'_δ . \square

Remark 5.13. If $N(\delta) - \eta \in J(E)_2 \setminus \{0\}$, then $\nu_\delta: X_\delta \rightarrow X'_\delta$ is a covering of degree 2.

5.3 The branch locus of the restricted Gauss maps

Let $R_\delta \subset Y_\delta$ be the divisor defined by

$$R_\delta = \{(q_1 + \cdots + q_{n-2}, p) \in Y_\delta \mid q_i = \sigma(q_j) \text{ for some } i \neq j\}.$$

LEMMA 5.14. We have $\beta_\delta^1(R_\delta) \subset W_{\delta,\text{sing}}$.

Proof. This is a consequence of Lemma 4.3 because $\beta_\delta^1(R_\delta) \subset B_\delta^2$. \square

Let $S_{\delta,r} \subset Y_\delta$ be the divisor defined by

$$S_{\delta,r} = \{(\mathbf{q}, p) \in Y_\delta \mid \mathbf{q} \geq r\}$$

for $r \in \text{Ram}(\phi)$. Then the ramification divisor of $\psi_\delta: Y_\delta \rightarrow X_\delta$ is

$$\text{Ram}(\psi_\delta) = R_\delta \cup \bigcup_{r \in \text{Ram}(\phi)} S_{\delta,r}.$$

LEMMA 5.15. *We have $\beta_\delta^1(S_{\delta,r}) \not\subset W_{\delta,\text{sing}}$ and, moreover, $\beta_\delta^1(S_{\delta,r}) \cap W_{\delta,\text{sing}} = \emptyset$ if $n = 3$.*

Proof. Let $W_{\delta,r}^1 \subset J(C)$ be the subvariety defined by

$$W_{\delta,r}^1 = \{L \in \text{Pic}^0(C) \mid h^0(C, \mathcal{L} \otimes \mathcal{O}_C(-r) \otimes \delta) > 1\},$$

and let $T_{\delta,r} \subset J(C)$ be the image of the morphism

$$\begin{aligned} C^{(n-3)} \times E &\longrightarrow \text{Pic}^0(C) = J(C), \\ (\mathbf{q}, \phi(q)) &\longmapsto \mathcal{O}_C(\mathbf{q} + r + q + \sigma(q)) \otimes \delta^\vee. \end{aligned}$$

Since C is not a hyperelliptic curve, by Martens's theorem [Mar67, Theorem 1], we have $\dim W_{\delta,r}^1 \leq n - 4$ and $T_{\delta,r} \not\subset W_{\delta,r}^1$, hence $\dim T_{\delta,r} = n - 2$. We remark that $W_{\delta,r}^1 = \emptyset$ in the case when $n = 3$. Since $\beta_\delta^1(S_{\delta,r}) = T_{\delta,r} \cap P$ is the fiber of the composition

$$T_{\delta,r} \subset J(C) \xrightarrow{N} J(E)$$

at $0 \in J(E)$, we have $\dim \beta_\delta^1(S_{\delta,r}) = n - 3$. Let $T_{\delta,2r} \subset T_{\delta,r}$ be the image of the morphism

$$\begin{aligned} C^{(n-4)} \times E &\longrightarrow \text{Pic}^0(C) = J(C), \\ (\mathbf{q}, \phi(q)) &\longmapsto \mathcal{O}_C(\mathbf{q} + 2r + q + \sigma(q)) \otimes \delta^\vee. \end{aligned}$$

Since $2r = r + \sigma(r)$, we have $T_{\delta,2r} \subset B_\delta^2 \subset W_{\delta,\text{sing}}$ by Lemma 4.3. For $L \in (T_{\delta,r} \cap W_{\delta,\text{sing}}) \setminus T_{\delta,2r}$, there is an element $(q_1 + \cdots + q_{n-3}, \phi(q)) \in C^{(n-3)} \times E$ such that $L = \mathcal{O}_C(q_1 + \cdots + q_{n-3} + r + q + \sigma(q)) \otimes \delta^\vee$ and $r \notin \{q_1, \dots, q_{n-3}\}$. If

$$q'_1 + \cdots + q'_{n-i} + ir \in |\Omega_C^1(-q_1 - \cdots - q_{n-3} - r - q - \sigma(q))| = |\Omega_C^1 \otimes (L \otimes \delta)^\vee|$$

and $r \notin \{q'_1, \dots, q'_{n-i}\}$, then by Lemma 5.2, the number i is odd. In the same way, any member in the linear system $|\Omega_C^1(-q'_1 - \cdots - q'_{n-i} - ir)| = |L \otimes \delta|$ has an odd multiplicity at r . This implies that $h^0(C, L \otimes \mathcal{O}_C(-r) \otimes \delta) = h^0(C, L \otimes \delta) > 1$. Hence we have

$$T_{\delta,r} \cap W_{\delta,\text{sing}} = T_{\delta,2r} \cup (T_{\delta,r} \cap W_{\delta,r}^1).$$

When $n = 3$, this implies that $T_{\delta,r} \cap W_{\delta,\text{sing}} = \emptyset$. When $n \geq 4$,

$$\beta_\delta^1(S_{\delta,r}) \cap W_{\delta,\text{sing}} = (T_{\delta,2r} \cap P) \cup (\beta_\delta^1(S_{\delta,r}) \cap W_{\delta,r}^1)$$

is a proper closed subset of $\beta_\delta^1(S_{\delta,r}) = T_{\delta,r} \cap P$ because $\dim(T_{\delta,2r} \cap P) \leq n - 4$ and $\dim W_{\delta,r}^1 \leq n - 4$. \square

Let $Z_{\delta,r} = \psi_\delta(S_{\delta,r})$ be the image of $S_{\delta,r}$ by $\psi_\delta: Y_\delta \rightarrow X_\delta$. Then

$$Z_{\delta,r} = \{(\mathbf{p}, p) \in X_\delta \mid \mathbf{p} \geq \phi(r)\}.$$

LEMMA 5.16. *If $n \geq 4$, then $Z_{\delta,r}$ is irreducible of dimension $n - 3$. If $n = 3$, then $X_\delta \cong E$ and $Z_{\delta,r} \subset X_\delta$ is a $J(X_\delta)_2$ -orbit by the natural action of $J(X_\delta)$ on the curve X_δ of genus 1.*

Proof. Let $\gamma_{\delta,p_0}: E^{(n-2)} \rightarrow J(E)$ be the morphism given in the proof of Lemma 5.9 for fixed $p_0 \in E$, and let $i_r: E^{(n-3)} \rightarrow E^{(n-2)}$ be the morphism defined by

$$i_r: E^{(n-3)} \longrightarrow E^{(n-2)}, \quad \mathbf{p} \longmapsto \mathbf{p} + \phi(r).$$

If $n \geq 4$, then $Z_{\delta,r}$ is a \mathbf{P}^{n-4} -bundle over E by the base change

$$\begin{array}{ccccc} Z_{\delta,r} & \longrightarrow & X_\delta & \xrightarrow{\text{pr}_2} & E \\ \downarrow & \square & \downarrow \text{pr}_1 & \square & \downarrow (-2)_{J(E) \circ \iota_{p_0}} \\ E^{(n-3)} & \xrightarrow{i_r} & E^{(n-2)} & \xrightarrow{\gamma_{\delta,p_0}} & J(E) \end{array}$$

of the \mathbf{P}^{n-4} -bundle $\gamma_{\delta,p_0} \circ i_r: E^{(n-3)} \rightarrow J(E)$; hence $Z_{\delta,r}$ is irreducible of dimension $n - 3$. If $n = 3$, then $\text{pr}_2: X_\delta \rightarrow E$ is an isomorphism and

$$Z_{\delta,r} \cong \{p \in E \mid \mathcal{O}_E(\phi(r) + 2p) \cong N(\delta)\}$$

is an orbit by the $J(E)_2$ -action. □

We denote the ramification divisor of $\psi_\delta^\circ = \psi_\delta|_{Y_\delta^\circ}: Y_\delta^\circ \rightarrow X_\delta$ by $\text{Ram}(\psi_\delta^\circ) \subset Y_\delta^\circ$.

LEMMA 5.17. *We have*

$$\overline{\psi_\delta^\circ(\text{Ram}(\psi_\delta^\circ))} = \bigcup_{r \in \text{Ram}(\phi)} Z_{\delta,r}.$$

Proof. Since $\text{Ram}(\psi_\delta) = R_\delta \cup \bigcup_{r \in \text{Ram}(\phi)} S_{\delta,r}$, by Lemma 5.14 we have the equality $\text{Ram}(\psi_\delta^\circ) = \bigcup_{r \in \text{Ram}(\phi)} S_{\delta,r} \cap Y_\delta^\circ$. By Lemmas 5.5 and 5.15, we have $S_{\delta,r} \cap Y_\delta^\circ \neq \emptyset$ for $n \geq 3$ and $S_{\delta,r} \cap Y_\delta^\circ = S_{\delta,r}$ for $n = 3$. Since ψ_δ is a finite morphism, $\psi_\delta^\circ(S_{\delta,r} \cap Y_\delta^\circ)$ is of dimension $n - 3$. By Lemma 5.16, we have $\overline{\psi_\delta^\circ(S_{\delta,r} \cap Y_\delta^\circ)} = \psi_\delta(S_{\delta,r}) = Z_{\delta,r}$. □

Let $H_r \subset \mathbf{P}(H^0(E, \eta)^\vee)$ be the hyperplane corresponding to the subspace

$$H^0(E, \eta \otimes \mathcal{O}_E(-\phi(r))) \subset H^0(E, \eta).$$

LEMMA 5.18. *The hyperplane H_r is the unique hyperplane with the property $\nu_\delta(Z_{\delta,r}) \subset H_r$.*

Proof. The inclusion $\nu_\delta(Z_{\delta,r}) \subset H_r$ is obvious. We prove the uniqueness of the hyperplane H_r . Let $z = (p_1 + \cdots + p_{n-3} + \phi(r), p) \in Z_{\delta,r}$ satisfy $p_i \neq t_{\eta-N(\delta)}(p_j)$ for $i \neq j$. We take a point $p' \in E \setminus \{p\}$ such that $\mathcal{O}_E(2p') \cong \mathcal{O}_E(2p)$ and $\mathcal{O}_E(p' - p) \not\cong \eta \otimes (N(\delta))^\vee$. Then $z' = (p_1 + \cdots + p_{n-3} + \phi(r), p')$ is contained in $Z_{\delta,r}$, and $\nu_\delta(z) \neq \nu_\delta(z')$. This implies the uniqueness in the case when $n = 3$. When $n \geq 4$, we show that $\nu_\delta(Z_{\delta,r}) \subset H_r$ is a nonlinear hypersurface in H_r . Let $l \subset H_r$ be the line containing the two points $\nu_\delta(z), \nu_\delta(z') \in \nu_\delta(Z_{\delta,r})$. Then the line $l \subset \mathbf{P}(H^0(E, \eta)^\vee)$ corresponds to the linear pencil

$$|\eta(-p_1 - \cdots - p_{n-3} - \phi(r))| \subset |\eta| \cong \mathbf{P}(H^0(E, \eta)^\vee).$$

For a point $p_0 \in E$, there is a unique point $p'_0 \in E$ such that $p_0 + p'_0 \in |\eta(-p_1 - \cdots - p_{n-3} - \phi(r))|$. If $\mathcal{O}_E(2p_0) \not\cong \mathcal{O}_E(2p)$, $\mathcal{O}_E(2p'_0) \not\cong \mathcal{O}_E(2p)$, and

$$p_0, p'_0 \notin \{t_{\eta-N(\delta)}(p_1), \dots, t_{\eta-N(\delta)}(p_{n-3}), t_{N(\delta)-\eta}(p_1), \dots, t_{N(\delta)-\eta}(p_{n-3})\},$$

then the point $p_1 + \cdots + p_{n-3} + \phi(r) + p_0 + p'_0 \in |\eta|$ on the line l is not contained in $\nu_\delta(Z_{\delta,r})$. □

LEMMA 5.19. *The pull-back of the divisor H_r on $\mathbf{P}(H^0(E, \eta)^\vee)$ by $\nu_\delta: X_\delta \rightarrow \mathbf{P}(H^0(E, \eta)^\vee)$ is*

$$\nu_\delta^* H_r = Z_{\delta,r} + M_{\delta,r} + M'_{\delta,r},$$

where $M_{\delta,y}$ and $M'_{\delta,y}$ are the irreducible divisors on X_δ defined by

$$\begin{aligned} M_{\delta,r} &= \{(\mathbf{p}, p) \in X_\delta \mid p = \phi(r)\}, \\ M'_{\delta,r} &= \{(\mathbf{p}, p) \in X_\delta \mid p = t_{N(\delta)-\eta}(\phi(r))\}. \end{aligned}$$

Proof. Let I_r be the irreducible divisor on $E^{(n)}$ defined by

$$I_r = \{\mathbf{p} \in E^{(n)} \mid \mathbf{p} \geq \phi(r)\},$$

and let Z_r, M_r, M'_r be the irreducible divisors on $E^{(n-2)} \times E$ defined by

$$\begin{aligned} Z_r &= \{(\mathbf{p}, p) \in E^{(n-2)} \times E \mid \mathbf{p} \geq \phi(r)\}, \\ M_r &= \{(\mathbf{p}, p) \in E^{(n-2)} \times E \mid p = \phi(r)\}, \\ M'_r &= \{(\mathbf{p}, p) \in E^{(n-2)} \times E \mid p = t_{N(\delta)-\eta}(\phi(r))\}. \end{aligned}$$

Then the pull-back of the divisor I_r by the morphism

$$\begin{aligned} E^{(n-2)} \times E &\longrightarrow E^{(n)}, \\ (\mathbf{p}, p) &\longmapsto \mathbf{p} + p + t_{\eta-N(\delta)}(p) \end{aligned}$$

is the divisor $Z_r + M_r + M'_r$ on $E^{(n-2)} \times E$. Since the restriction of I_r to $|\eta| \subset E^{(n)}$ is the divisor H_r on $\mathbf{P}(H^0(E, \eta)^\vee) \cong |\eta|$, the pull-back $\nu_\delta^* H_r$ is the restriction of $Z_r + M_r + M'_r$ to X_δ . \square

COROLLARY 5.20. *The divisor $\nu_\delta^* H_r - Z_{\delta,r}$ on X_δ is irreducible if and only if $N(\delta) = \eta$.*

We consider the dual variety $(\Phi_{|\eta|}(E))^\vee \subset \mathbf{P}(H^0(E, \eta)^\vee)$ of the image of the closed immersion $\Phi_{|\eta|}: E \rightarrow \mathbf{P}(H^0(E, \eta)^\vee)$ which is defined by the linear system $|\eta|$.

LEMMA 5.21. *The projective curve $\Phi_{|\eta|}(E) \subset \mathbf{P}(H^0(E, \eta)^\vee)$ is reflexive. In particular, $\Phi_{|\eta|}(E) = ((\Phi_{|\eta|}(E))^\vee)^\vee \subset \mathbf{P}(H^0(E, \eta)^\vee)$.*

Proof. If $1 \leq i < n$, then $h^0(E, \eta \otimes \mathcal{O}_E(-ip)) = n - i$ for any $p \in E$. If $n = 3$, then $h^0(E, \eta \otimes \mathcal{O}_E(-3p)) = 0$ for general $p \in E$. Hence $h^0(E, \eta \otimes \mathcal{O}_E(-2p)) > h^0(E, \eta \otimes \mathcal{O}_E(-3p))$ for general $p \in E$. Then there is a hyperplane $H \subset \mathbf{P}(H^0(E, \eta)^\vee)$ which intersects $\Phi_{|\eta|}(E)$ at $\Phi_{|\eta|}(p)$ with the multiplicity 2. By [HK85, (3.5)], the projective curve $\Phi_{|\eta|}(E) \subset \mathbf{P}(H^0(E, \eta)^\vee)$ is reflexive because the characteristic of the base field k is not equal to 2. \square

LEMMA 5.22. *If $N(\delta) = \eta$, then the dual variety of $X'_\delta \subset \mathbf{P}(H^0(E, \eta)^\vee)$ is $\Phi_{|\eta|}(E) \subset \mathbf{P}(H^0(E, \eta)^\vee)$.*

Proof. By Lemma 5.21, we show that the dual variety of $\Phi_{|\eta|}(E)$ is X'_δ . For $L \in (B_\delta \cap P) \setminus D_{\delta, \text{sing}} \subset \text{Pic}^0(C)$, there is a unique effective divisor $\mathbf{q} + q + \sigma(q) \in C^{(n)}$ such that $L \otimes \delta \cong \mathcal{O}_C(\mathbf{q} + q + \sigma(q))$. Since $L \in P$, we have

$$\eta = N(\delta) = [\mathcal{O}_E(\phi^{(n-2)}(\mathbf{q}) + 2\phi(q))] \in \text{Pic}^n(E),$$

hence

$$\Omega_C^1 \otimes L^\vee \otimes \delta^\vee \cong \phi^* \eta \otimes L^\vee \otimes \delta^\vee \cong \mathcal{O}_C(\sigma(\mathbf{q}) + \sigma(q) + q),$$

and $\Psi_{J(C), \delta}^B(L) \in \mathbf{P}(H^0(E, \eta)^\vee)$ is defined by the effective divisor

$$\phi^{(n-2)}(\mathbf{q}) + 2\phi(q) \in |\eta| \cong \mathbf{P}(H^0(E, \eta)^\vee).$$

This means that the hyperplane in $\mathbf{P}(H^0(E, \eta)^\vee)$ corresponding to $\Psi_{J(C), \delta}^B(L)$ is tangent to the image $\Phi_{|\eta|}(E)$. Hence we have $\Psi_{J(C), \delta}^B((B_\delta \cap P) \setminus D_{\delta, \text{sing}}) \subset (\Phi_{|\eta|}(E))^\vee$. Since $(\Phi_{|\eta|}(E))^\vee$ and $\Psi_{J(C), \delta}^B((B_\delta \cap P) \setminus D_{\delta, \text{sing}})$ are irreducible hypersurfaces in $\mathbf{P}(H^0(E, \eta)^\vee)$, we have $X'_\delta = (\Phi_{|\eta|}(E))^\vee$. \square

6. Key propositions

Let \mathcal{L} be an ample invertible sheaf on P which represents the polarization λ_P .

LEMMA 6.1. *The variety $U_D = \text{Bs}|\mathcal{L}| \setminus D_{\text{sing}}$ is nonsingular for any $D \in |\mathcal{L}|$.*

Proof. Since $D = D_\delta$ for some $\delta \in \text{Pic}^n(C)$, this is a consequence of Lemmas 4.6 and 5.10. \square

Consider the Gauss map

$$\Psi_D: D \setminus D_{\text{sing}} \longrightarrow \mathbf{P}^{n-1} = \text{Grass}(n-1, H^0(P, \Omega_P^1)^\vee)$$

for $D \in |\mathcal{L}|$, and let $\nu_D: X_D \rightarrow X'_D$ be the normalization of $X'_D = \overline{\Psi_D(U_D)} \subset \mathbf{P}^{n-1}$. Then by Lemma 6.1, there is a unique morphism $\psi_D: U_D \rightarrow X_D$ such that $\Psi_D|_{U_D} = \nu_D \circ \psi_D$. Let $Z_D = \overline{\psi_D(\text{Ram}(\psi_D))} \subset X_D$ be the Zariski closure of the image of the ramification divisor of ψ_D .

PROPOSITION 6.2. *Let $D \subset P$ be a member in $|\mathcal{L}| \setminus \Pi_{\mathcal{L}}$, where $\Pi_{\mathcal{L}} \subset |\mathcal{L}|$ is the subset in Lemma 4.8.*

- (i) *If $n = 3$, then X_D is a nonsingular projective curve of genus 1 and Z_D is a disjoint union of six orbits $Z_{D,1}, \dots, Z_{D,6}$ by the $J(X_D)_2$ -action.*
- (ii) *If $n \geq 4$, then Z_D has $2n$ irreducible components $Z_{D,1}, \dots, Z_{D,2n}$.*
- (iii) *For any subset $Z_{D,i} \subset Z_D$ in parts (i) and (ii), there is a unique hyperplane $H_{D,i} \subset \mathbf{P}^{n-1}$ such that $\nu_D(Z_{D,i}) \subset H_{D,i}$.*

Proof. By Lemma 4.7, there is a $\delta \in \text{Pic}^n(C)$ such that $N(\delta) = \eta$ and $\mathcal{L} \cong \mathcal{L}_\delta$. By Lemma 4.6, there is an $s \in \text{Pic}^0(E)$ such that $D = D_{\delta+\phi^*s}$. By the proof of Lemma 4.8, we have that $D \notin \Pi_{\mathcal{L}}$ implies $s \notin J(E)_4 \setminus J(E)_2$. By Lemma 5.7, the Gauss map $\Psi_D|_{U_D}: U_D \rightarrow \mathbf{P}^{n-1}$ is identified with $\Psi_{J(C), \delta+\phi^*s}|_{U_D}: U_D \rightarrow \mathbf{P}(H^0(E, \eta)^\vee)$. Since $N(\delta + \phi^*s) - \eta = 2s \notin J(E)_2 \setminus \{0\}$, by Lemma 5.12, the normalization of $X'_{\delta+\phi^*s} = X'_D$ is given by $\nu_{\delta+\phi^*s}: X_{\delta+\phi^*s} \rightarrow X'_{\delta+\phi^*s}$, and by Lemmas 5.8 and 5.10, the map $\psi_D: U_D \rightarrow X_D$ is identified with $\psi_{\delta+\phi^*s}^\circ: Y_{\delta+\phi^*s}^\circ \rightarrow X_{\delta+\phi^*s}$. Hence the statements (i), (ii), and (iii) are consequences of Lemmas 5.16, 5.17, and 5.18. \square

We define the subset $\Pi'_{\mathcal{L}}$ in the linear pencil $|\mathcal{L}|$ by

$$\Pi'_{\mathcal{L}} = \{D \in |\mathcal{L}| \setminus \Pi_{\mathcal{L}} \mid \nu_D^* H_{D,i} - Z_{D,i} \text{ is irreducible for } 1 \leq i \leq 2n\}.$$

LEMMA 6.3. *We have $\#\Pi'_{\mathcal{L}} = 4$.*

Proof. We use the same identification for Gauss maps as in the proof of Proposition 6.2. Then by Corollary 5.20,

$$D = D_{\delta+\phi^*s} \in \Pi'_{\mathcal{L}} \iff N(\delta + \phi^*s) = \eta \iff s \in J(E)_2,$$

and by Lemma 4.4, we have $\#\Pi'_{\mathcal{L}} = \#J(C)_2 = 4$. \square

Let $e_1 + \dots + e_{2n}$ be the branch divisor of the original covering $\phi: C \rightarrow E$, and let $\eta \in \text{Pic}(E)$ be the invertible sheaf with $\phi^*\eta \cong \Omega_C^1$.

PROPOSITION 6.4. *For any member $D \in \Pi'_{\mathcal{L}}$, there is an isomorphism*

$$(E, e_1 + \dots + e_{2n}, \eta) \cong ((X'_D)^\vee, H_{D,1}^\vee + \dots + H_{D,2n}^\vee, \mathcal{O}_{(\mathbf{P}^{n-1})^\vee}(1)|_{(X'_D)^\vee}),$$

where $H_{D,i}^\vee \in (\mathbf{P}^{n-1})^\vee$ is the point corresponding to the hyperplane $H_{D,i}$ and $(X'_D)^\vee \subset (\mathbf{P}^{n-1})^\vee$ is the dual variety of $X'_D \subset \mathbf{P}^{n-1}$.

Proof. We use the same identification for Gauss maps as in the proof of Proposition 6.2. When $D \in \Pi'_{\mathcal{L}}$, we may assume $D = D_{\delta}$ and $N(\delta) = \eta$ by Corollary 5.20. Then the point $H_{D,i}^{\vee}$ is identified with the point $H_r^{\vee} = \Phi_{|\eta|}(\phi(r))$ for $r \in \text{Ram}(\phi)$, and $(X'_D)^{\vee}$ is identified with $(X'_{\delta})^{\vee}$, which coincides with $\Phi_{|\eta|}(E) \subset \mathbf{P}(H^0(E, \eta))$ by Lemma 5.22. \square

Remark 6.5. For a member $D \in \Pi'_{\mathcal{L}}$, the Gauss map $\Psi_D: D \setminus D_{\text{sing}} \rightarrow \mathbf{P}^{n-1}$ is a covering of degree 2^n , and $X'_D \cup \bigcup_{i=1}^{2^n} H_{D,i}$ is the branch locus of Ψ_D . But for $D \notin \Pi'_{\mathcal{L}}$, the Gauss map Ψ_D is not easy to compute.

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