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Differentiability of non-archimedean volumes and non-archimedean Monge–Ampère equations

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With an appendix by Robert Lazarsfeld

Abstract

Let X be a normal projective variety over a complete discretely valued field and L a line bundle on X. We denote by X^{an} the analytification of X in the sense of Berkovich and equip the analytification L^{an} of L with a continuous metric $\| \ \|$. We study non-archimedean volumes, a tool which allows us to control the asymptotic growth of small sections of big powers of L. We prove that the non-archimedean volume is differentiable at a continuous semipositive metric and that the derivative is given by integration with respect to a Monge–Ampère measure. Such a differentiability formula had been proposed by M. Kontsevich and Y. Tschinkel. In residue characteristic zero, it implies an orthogonality property for non-archimedean plurisubharmonic functions which allows us to drop an algebraicity assumption in a theorem of S. Boucksom, C. Favre and M. Jonsson about the solution to the non-archimedean Monge–Ampère equation. The appendix by R. Lazarsfeld establishes the holomorphic Morse inequalities in arbitrary characteristic.

1. Introduction

1.1. Monge–Ampère equations. Let (X, ω) be a compact Kähler manifold of dimension n, normalized by $\int \omega^{\wedge n} = 1$. For a probability measure μ on X which is induced by a smooth volume form, E. Calabi conjectured that the *Monge–Ampère equation* $\eta^{\wedge n} = \mu$ has a unique solution by a real smooth $(1,1)$-form η in the same de Rham class as ω. The uniqueness was proven by E. Calabi [Cal54, Cal57], and the existence of solutions of the Monge–Ampère equation was settled by S. T. Yau [Yau78].

Now we consider a field K endowed with a discretely valued complete absolute value. Let L be a line bundle on an n-dimensional projective variety X over K. For a continuous semipositive metric $\| \ \|$ on L^{an}, A. Chambert-Loir has introduced the Monge–Ampère measure $c_1(L, \| \ \|)^{\wedge n}$ on the analytification X^{an} as a Berkovich space (see Section 2 for details). Then $c_1(L, \| \ \|)^{\wedge n}$ is

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a positive Radon measure of total mass equal to the degree of X with respect to L. Assume that X is smooth and L is ample. In the non-archimedean analogue of the Calabi–Yau problem, there is a positive Radon measure μ of total mass $\deg_L(X)$ given on X^{an}, and we ask for a continuous semipositive metric $\| \ \|$ on L^{an} with $\mu = c_1(L, \| \ \|)^{\wedge n}$.

The uniqueness of the metric $\| \ \|$ up to scaling was shown by X. Yuan and S. Zhang [YZ17, Corollary 1.2]. In [BFJ15, BFJ16], S. Boucksom, C. Favre and M. Jonsson have proved the existence assuming that the residue field k of K has characteristic zero, that μ is supported on the dual complex of some SNC model of X (see Subsection 1.6) and that X satisfies the algebraicity condition (†). The latter means that X is defined over the function field of a curve over k having K as its completion at a closed point. Condition (†) is essential in their proof, allowing them to use global methods on the model to prove the existence of solutions of the non-archimedean Monge–Ampère equation. However, this global hypothesis is quite strong as a variety over a field as $\mathbb{C}((t))$ is usually not defined over a function field of a curve over $\mathbb{C}$.

The main motivation of the present work is to remove condition (†), following a strategy outlined in unpublished notes by M. Kontsevich and Y. Tschinkel [KT02]. To this end, we need some local volumes to replace the global methods used in [BFJ15, BFJ16].

1.2. Volumes of line bundles on algebraic varieties. Let k be an algebraically closed field and Y a projective variety over k of dimension n. For a line bundle L on Y, the *volume*

$$\operatorname{vol}(L) \coloneqq \limsup_m \frac{h^0(Y, L^{\otimes m})}{m^n/n!}$$

is in $\mathbb{R}_{\geqslant 0}$ (see [Laz04a]). Outside the nef cone, we have Siu's inequality [Laz04a, Example 2.2.47] in terms of algebraic intersection numbers: if L, M are nef, then $\operatorname{vol}\left(L \otimes M^{-1}\right) \geqslant L^n - nL^{n-1} \cdot M$. It is also known that the function vol is differentiable on the big cone [BFJ09].

For $i \in \mathbb{N}$, A. Küronya [Kür06] has introduced *asymptotic cohomological functions*

$$\widehat{h}^i(Y, L) \coloneqq \limsup_m \frac{h^i(Y, L^{\otimes m})}{m^n/n!} .$$

In particular, $\widehat{h}^0 = \operatorname{vol}$. For L nef and $i > 0$, one has $\widehat{h}^i(Y, L) = 0$ (see [Laz04a, Theorem 1.4.40]) and the main difficulty is again to understand $\widehat{h}^i$ outside of the nef cone. For L and M nef line bundles on Y, the *asymptotic holomorphic Morse inequalities* give

$$\widehat{h}^i\left(Y, L \otimes M^{-1}\right) \leqslant \binom{n}{i} L^{n-i} \cdot M^i . \tag{1.1}$$

First, an analytic proof of these inequalities was given by J.-P. Demailly [Dem85]. Later, F. Angelini [Ang96] gave an algebraic proof in characteristic zero. For our applications in this paper, we need the volume and the asymptotic cohomological functions for projective schemes over an arbitrary field k. In the appendix by R. Lazarsfeld, there is an algebraic proof of (1.1) which works for a projective scheme Y over any field.

We will study cohomological functions in Section 3. More precisely, we generalize classical results about the asymptotic behavior of the dimension of the higher cohomology of a coherent sheaf $\mathcal{F}$ on a projective variety twisted by a family of divisors $D_1, \dots, D_m$ (see Proposition 3.5.1) and show that the asymptotic is uniform in $D_1, \dots, D_m$. These results might be of independent interest and have been used already in [BN19]. In Subsection 3.6, we consider the more general case of a projective scheme Y over a noetherian ring since we need this for Sections 4 and 5. Then Y is allowed to be non-reduced or non-irreducible.

1.3. Arithmetic volumes of line bundles. A. Moriwaki [Mor09] has introduced an arithmetic analogue of the volume in the setting of Arakelov theory. Let F be a number field, Y a projective variety over F of dimension n and L a line bundle on Y. For each place v of F, let F_v be the completion of F at v and Y_v^{an} the associated analytic space (either as a complex analytic space or as a Berkovich space). Assume that we are given, for each place v, a continuous metric $\| \ \|_v$ on the analytic line bundle L_v^{an} over Y_v^{an} determined by L. We assume also that almost all metrics $\| \ \|_v$ are determined by a model of (Y, L) over some open subset of $\operatorname{Spec} \mathcal{O}_K$. Write $\overline{L} = (L, \{\| \ \|_v\}_v)$ for the line bundle and the metrics. Then the arithmetic volume of $\overline{L}$ is defined as

$$\widehat{\mathrm{vol}}(\overline{L}) \coloneqq \limsup_m \frac{\log \#\big\{s \in H^0(Y, L^{\otimes m}) \mid \|s\|_v^{\otimes m} \leqslant 1 \ \forall\, v\big\}}{m^{n+1}/(n+1)!} .$$

A. Moriwaki [Mor09] has shown that the arithmetic volume is continuous. H. Chen [Che08] has proved that the arithmetic volume is in fact a limit as in the classical case.

The χ-*arithmetic volume* is a variant of the arithmetic volume, which is also known as the *logarithm of the sectional capacity*. Its definition is recalled in Remark 4.1.7. In contrast to the arithmetic volume, the χ-arithmetic volume can also take negative values. Both volumes agree when $\overline{L}$ is (arithmetically) nef. X. Yuan [Yua08] has proved an analogue of Siu's inequality for the χ-arithmetic volume and used it to prove a very general equidistribution result.

1.4. Volumes of balls of bounded sections. Let us now assume that X is a projective variety over a local field K. We also fix a line bundle L on X. We consider a continuous metric $\| \ \|$ on L^{an} and study the asymptotic behavior of the volume of the sets

$$\widehat{H}^0\big(X, L^{\otimes m}, \| \ \|^{\otimes m}\big) \coloneqq \big\{s \in \Gamma\big(X, L^{\otimes m}\big) \,\big|\, \|s\|_{\sup} \leqslant 1\big\}$$

with respect to a Haar measure μ_m on $\Gamma(X, L^{\otimes m})$, where $\|s\|_{\sup} = \sup_{p \in X^{\mathrm{an}}} \|s(p)\|^{\otimes m}$. However, μ_m is well defined only up to multiplication by a positive constant. To bypass this ambiguity, one fixes a continuous reference metric $\| \cdot \|_0$ on L^{an} and introduces the local volume

$$\mathrm{vol}(L, \| \ \|, \| \ \|_0) \coloneqq \limsup_m \frac{n!}{m^{n+1}} \cdot \log\left(\frac{\mu_m\big(\widehat{H}^0\big(X, L^{\otimes m}, \| \ \|^{\otimes m}\big)\big)}{\mu_m\big(\widehat{H}^0\big(X, L^{\otimes m}, \| \ \|_0^{\otimes m}\big)\big)}\right). \tag{1.2}$$

These *local volumes* will be called *archimedean* or *non-archimedean* depending on the nature of the local ground field K. If F is a number field, K is the completion of F at a non-archimedean place v and L is ample, then we will show in Remark 4.1.7 that the local volume at v is a local version of the χ-arithmetic volume obtained by choosing fixed metrics at the other places.

Non-archimedean volumes were introduced by M. Kontsevich and Y. Tschinkel in [KT02]. Furthermore, differentiability for this local volume was proposed [KT02, § 7.2].

In the archimedean context, R. Berman and S. Boucksom have introduced and studied in [BB10] a variant of the archimedean volume. For an ample line bundle, they introduce an energy functional on the space of continuous metrics. They prove that the archimedean volume of two metrics agrees with the relative energy of the two metrics (see [BB10, Theorem A]) and that the energy satisfies a differentiability property (see [BB10, Theorem B]).

A variant of local volumes has been studied by H. Chen and C. Maclean [CM15]. They work over $\mathbb{C}$ or over any non-archimedean field and associate with $(L, \| \ \|, \| \ \|_0)$ a sequence of logarithmic ratios between determinants of sup norms on graded linear systems associated with L. Their main result implies conditions when the lim sup in (1.2) is a limit (see Remark 4.1.8).

1.5. Differentiability of non-archimedean volumes. Let us now turn back to the non-archimedean situation and explain the main results of this paper. We fix a complete discretely valued field K with discrete valuation ring K° and a normal projective variety X over K equipped with a line bundle L. In this context, a non-archimedean analogue of a smooth hermitian metric is an *algebraic metric* associated with a K° model $(\mathscr{X}, \mathscr{L})$ of (X, L). The algebraic metric is called *semipositive* if $\mathscr{L}|_{\mathscr{X}_s}$ is nef. A metric is called a *semipositive model metric* if a suitable positive tensor power is a semipositive algebraic metric. We call $\| \ \|$ on L^{an} a *continuous semipositive metric* if it is a uniform limit of semipositive model metrics. Such metrics were first considered by S. Zhang [Zha95b]. A construction of A. Chambert-Loir [Cha06] gives an associated Monge–Ampère measure $c_1(L, \| \ \|)^{\wedge n}$ on X^{an} which is important for arithmetic equidistribution theorems. For details, we refer to Section 2.

Given two continuous metrics $\| \ \|_1$ and $\| \ \|_2$ on L^{an}, we define $\mathrm{vol}(L, \| \ \|_1, \| \ \|_2)$ as in (1.2). However, since fields such as $\mathbb{C}((t))$ are not locally compact, we use the length of the virtual K°-module $\widehat{H}^0(X, L, \| \ \|_1)/\widehat{H}^0(X, L, \| \ \|_2)$ instead of the quotient of the Haar measures (for details, see Subsection 4.1).

In Theorem 4.2.3 we prove the following non-archimedean analogue of [BB10, Theorem A].

THEOREM A. *If $\| \ \|_1$ and $\| \ \|_2$ are two continuous semipositive metrics on L^{an}, then*

$$\mathrm{vol}(L, \| \ \|_1, \| \ \|_2) = \frac{1}{n+1} \sum_{j=0}^{n} \int_{X^{\mathrm{an}}} -\log \frac{\| \ \|_1}{\| \ \|_2} c_1(L, \| \ \|_1)^{\wedge(n-j)} \wedge c_1(L, \| \ \|_2)^{\wedge j}.$$

From the proof of this equation, we deduce that for continuous semipositive metrics, the lim sup in the definition of $\mathrm{vol}(L, \| \ \|_1, \| \ \|_2)$ is actually a limit. S. Boucksom and D. Eriksson told us that they have a proof of Theorem A using different methods.[1] Our proof is based on a study of non-archimedean volumes and on the results of Section 3.

Our main result (following from Theorem 5.4.3) is the *differentiability of the non-archimedean volume* over any discretely valued complete field K.

THEOREM B. *Let $\| \ \|$ be a continuous semipositive metric on L^{an} and $f\colon X^{\mathrm{an}} \to \mathbb{R}$ a continuous function. Then if we consider everything fixed except $\varepsilon \in \mathbb{R}$, one has*

$$\mathrm{vol}\left(L, \| \ \| e^{-\varepsilon f}, \| \ \|\right) = \varepsilon \int_{X^{\mathrm{an}}} f c_1(L, \| \ \|)^{\wedge n} + o(\varepsilon) \tag{1.3}$$

for $\varepsilon \to 0$. Equivalently, the function $t \in \mathbb{R} \mapsto \mathrm{vol}(L, \| \ \| e^{-tf}, \| \ \|)$ is differentiable at $t = 0$ and

$$\left.\frac{d}{dt}\right|_{t=0} \mathrm{vol}\left(L, \| \ \| e^{-tf}, \| \ \|\right) = \int_{X^{\mathrm{an}}} f c_1(L, \| \ \|)^{\wedge n}.$$

This formula is the exact non-archimedean analogue of [BB10, Theorem B] and was proposed by M. Kontsevich and Y. Tschinkel [KT02, § 7.2].

Section 5 is devoted to the proof of Theorem B. The proof of Theorem B is similar to the proof of Theorem A, but additional problems arise from leaving the nef cone. Our arguments were inspired by the techniques of A. Abbes and T. Bouche [AB95] and X. Yuan [Yua08]. In fact, the differentiability of the non-archimedean volume in Theorem B is related to the differentiability of the χ-arithmetic volume shown by X. Yuan (see [Yua08] and [Che11, § 4.4]) as follows. If K

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[1] See [BE18, Theorem 8.5], which holds for any non-archimedean field K under the additional assumptions that X is smooth and L is ample.

is a completion of a number field F at a non-archimedean place, if X, L are defined over F and if L is ample, then Yuan proves the differentiability of the χ-arithmetic volume. Using the relation between the χ-arithmetic and the non-archimedean volume explained in Remark 4.1.7, this implies Theorem B under the above assumptions on X and L. Conversely, Theorem B implies the differentiability of the χ-arithmetic volume in the direction of a non-archimedean metric change.

The proof of the differentiability of the arithmetic volume in the global case can be made in two steps. The first one is to prove only the inequality "$\geqslant$" in equation (1.3) for each place of the global field ([Yua08, Lemma 3.3] and its non-archimedean analogue). In the non-archimedean situation, this inequality is obtained by controlling the size of certain groups of global sections. The second step is to prove that the arithmetic volume is log concave (see [Yua09, Theorem B]). As explained in [Che11, Proposition 4.1], these two ingredients are enough to prove the differentiability.[2]

In the non-archimedean local case, we use a different strategy. Instead of proving only the inequality "$\geqslant$" in equation (1.3), we prove directly the full equality (1.3). To this end, instead of controlling only the size of certain H^0-groups, we need to control also the size of certain first cohomology groups. This control is achieved through the use of the holomorphic Morse inequalities and the results on the asymptotic growth of algebraic volumes obtained in Section 3.

1.6. Orthogonality and Monge–Ampère equations. We keep the assumptions on K from Subsection 1.5. Although we are able to establish the differentiability of the local non-archimedean volume in arbitrary characteristic, this is not yet enough to solve the non-archimedean Monge–Ampère equation. One important ingredient which is still missing is the existence of the continuous semipositive envelope $P(\|\ \|)$ for an arbitrary continuous metric $\|\ \|$ on a line bundle L^{an}. Given a continuous metric $\|\ \|$, one defines its *semipositive envelope* $P(\|\ \|)$ as the pointwise infimum of all metrics $\|\ \|_1$ on L^{an} such that $\|\ \|_1$ is a semipositive model metric on L^{an} with $\|\ \| \leqslant \|\ \|_1$. It is a priori not clear that $P(\|\ \|)$ is a continuous semipositive metric on L^{an}.

From now on, we assume that the characteristic of the residue field $\tilde{K}$ of K is zero and that L is ample. Then the regularization theorem of S. Boucksom, C. Favre and M. Jonsson [BFJ16, Theorem 8.3] ensures that $P(\|\ \|)$ is a continuous semipositive metric. Using a local approach to semipositivity as in [GK19, GM19], we find $\widehat{H}^0(X, L, \|\ \|) = \widehat{H}^0(X, L, P(\|\ \|))$ for any continuous metric $\|\ \|$ and its semipositive envelope $P(\|\ \|)$, hence

$$\mathrm{vol}(\|\ \|, P(\|\ \|)) = 0\,. \tag{1.4}$$

In Corollary 6.2.2, we will deduce from (1.4) that the lim sup in the definition of the non-archimedean volume is a lim. Theorem B and (1.4) yield the following *orthogonality property.*

THEOREM C. *We assume* $\mathrm{char}(\tilde{K}) = 0$. *Let L be an ample line bundle on a smooth projective variety X over K, let $n := \dim(X)$, and let $\|\ \|$ be a continuous metric on L^{an}. Then*

$$\int_{X^{\mathrm{an}}} \log \frac{P(\|\ \|)}{\|\ \|} c_1(L, P(\|\ \|))^{\wedge n} = 0\,.$$

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[2] A referee suggested that a similar strategy might be used in the local non-archimedean case. First one proves the inequality "$\geqslant$" in equation (1.3) by controlling the size of certain H^0-groups. Second, using Okounkov bodies as explained in [CM15, proof of Theorem 4.5], one writes $\mathrm{vol}(L, \|\ \|_1, \|\ \|_2)$ as the difference of two log concave quantities, one depending on $\|\ \|_1$ and the other on $\|\ \|_2$. This decomposition will depend on the choice of a regular K-rational point (whose existence is assumed), a system of parameters around that point and a monomial order.

We show this in Theorem 6.3.2. This orthogonality property was proven in [BFJ15, Theorem A.6] assuming that X satisfies the algebraicity condition (†) mentioned in Subsection 1.1. It follows from the variational method of S. Boucksom, C. Favre and M. Jonsson that the orthogonality property yields the existence of solutions in the non-archimedean Calabi–Yau problem (see [BFJ15, Theorem 8.2]) and hence Theorem C implies the following.

THEOREM D. *We assume* $\mathrm{char}(\tilde{K}) = 0$ *and that* L *is an ample line bundle on the smooth projective variety* X *over* K. *Let* μ *be a positive Radon measure on* X^{an} *with* $\mu(X^{\mathrm{an}}) = \deg_L(X)$ *and supported on the dual complex of an SNC model of* X. *Then there is a continuous semipositive metric* $\| \ \|$ *on* L^{an} *with* $c_1(L, \| \ \|)^{\wedge n} = \mu$.

Here, an *SNC model* is a regular projective variety $\mathscr{X}$ over the valuation ring K° with generic fiber X such that the special fiber, which is not assumed to be reduced, agrees as a closed subset with a simple normal crossing divisor D of $\mathscr{X}$. The *dual complex* $\Delta_{\mathscr{X}}$ of $\mathscr{X}$ is defined as the dual complex of D and can be realized as a canonical compact subset of X^{an} (see [BFJ16, § 3] for details).

Recall that uniqueness up to scaling was proven by X. Yuan and S. Zhang [YZ17, Corollary 1.2] without any assumptions on the residue characteristic. For a more general existence result in terms of plurisubharmonic functions, we refer to Corollary 6.3.4.

1.7. Notation and conventions. Let X be a scheme. A *divisor* on X is always a Cartier divisor on X. We denote by $\mathrm{Div}(X)$ the group of Cartier divisors on X and put $\mathrm{Div}(X)_{\mathbb{Q}} = \mathrm{Div}(X) \otimes_{\mathbb{Z}} \mathbb{Q}$ and $\mathrm{Div}(X)_{\mathbb{R}} = \mathrm{Div}(X) \otimes_{\mathbb{Z}} \mathbb{R}$.

Let k be a field. A *variety* X *over* k is an integral k-scheme X which is separated and of finite type. A curve is a variety of dimension one. For X a variety and D a Cartier divisor on X, we will sometimes write $h^i(D)$ or $h^i(X, D)$ for $h^i(X, \mathcal{O}_X(D))$. We also write $H^i(X, D)$ for $H^i(X, \mathcal{O}_X(D))$. If $\mathcal{F}$ is a coherent sheaf on a scheme X and $D \in \mathrm{Div}(X)$, we write $\mathcal{F}(D)$ for $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_X(D)$.

Let M be a module over a commutative ring A with unit. Then $\ell_A(M)$ denotes the length of the A-module M. We write $\ell(M)$ if A is clear from the context.

Let X be a noetherian scheme over a noetherian base scheme S. For an n-cycle Z on X with support proper over a zero-dimensional subscheme of S and line bundles $L_1, \dots, L_n$ on X, there is an intersection number $L_1 \cdots L_n \cdot Z \in \mathbb{Z}$. A definition of such intersection numbers is given in [Kol96, Appendix VI.2] for coherent sheaves $\mathcal{F}$ instead of Z; hence, we may apply it for $\mathcal{F} \coloneqq \mathcal{O}_Z$ in case of a prime cycle, and we extend it by linearity to all cycles of the above form. These intersection numbers are multilinear and satisfy a projection formula; hence, they agree with the usual intersection numbers as given in [Ful98] in case $S = \mathrm{Spec}(R)$ with R a field or a discrete valuation ring. Indeed, functoriality and multilinearity yield that this can be checked for a prime cycle in projective space over a field, and hence it follows easily from [Kol96, Theorem 2.8].

If $L_i = \mathcal{O}(D_i)$ for Cartier divisors $D_1, \dots, D_n$ on X, then we set

$$D_1 \cdots D_n \cdot Z = \mathcal{O}(D_1) \cdots \mathcal{O}(D_n) \cdot Z\,. \tag{1.5}$$

This is multilinear and symmetric in $D_1, \dots, D_n$. If Z is the fundamental cycle of X, then we simply write $D_1 \cdots D_n$ for the intersection product in (1.5).

If $\{M_1, \dots, M_s\} = \{L_1, \dots, L_n\}$, then we write $M_1^{n_1} \cdots M_s^{n_s} \cdot Z \coloneqq L_1 \cdots L_n \cdot Z$ if M_j occurs n_j-times in the intersection number. We will always use $M_j^{n_j}$ in this way; this should not be mixed up with the tensor power $M^{\otimes n}$ of a line bundle M.

2. Preliminaries on semipositive metrics, envelopes and measures

The aim of this section is to recall the central notions for our paper following the terminology in [BFJ15, BFJ16]. In this section, let K be a complete discretely valued field with valuation ring K°, uniformizer π and residue class field $\tilde{K} = K^\circ/(\pi)$. We normalize the absolute value on K in such a way that $-\log|\pi| = 1$.

2.1. Models, analytification and reduction. Let X be a proper variety over K. Let $S = \operatorname{Spec} K^\circ$. A *model of* X is a proper, flat scheme $\mathscr{X}$ over S together with a fixed isomorphism h between X and the generic fiber $\mathscr{X}_\eta$ of the S-scheme $\mathscr{X}$. Usually, we read h as an identification. The special fiber $\mathscr{X} \otimes_{K^\circ} \tilde{K}$ of $\mathscr{X}$ over S is denoted by $\mathscr{X}_s$.

Let X be a variety over K. We denote by X^{an} the analytification of X over K in the sense of Berkovich [Ber90, Theorem 3.4.1]. The K-analytic space X^{an} consists of a locally compact Hausdorff topological space together with a sheaf $\mathcal{O}_{X^{\mathrm{an}}}$ of regular analytic functions. The space X^{an} is compact if X is proper over K.

Let X be a proper variety over K. For a model $\mathscr{X}$ of X over K° with special fiber $\mathscr{X}_s$, there is a canonical *reduction map* $\mathrm{red}\colon X^{\mathrm{an}} \longrightarrow \mathscr{X}_s$ which is surjective. If the model $\mathscr{X}$ is normal, then for an irreducible component V of $\mathscr{X}_s$, its generic point ξ_V has a unique preimage x_V in X^{an} (see [BPS14, Proposition 1.3.3]) called the *divisorial point determined by* V.

2.2. Metrics, model metrics and model functions. In this subsection, we study metrics on a line bundle L of a proper variety X over K.

2.2.1. A *continuous metric* $\|\ \|$ on L^{an} associates with each section $s \in \Gamma(U, L)$ on some Zariski open subset U of X a continuous function $\|s\|\colon U^{\mathrm{an}} \to [0, \infty)$ such that $\|f \cdot s\| = |f| \cdot \|s\|$ holds for each $f \in \mathcal{O}_X(U)$. We further require that $\|s\| > 0$ if s is an invertible section of L. Given a continuous metric $\|\ \|$ on L^{an}, we define

$$\widehat{H}^0(X, L, \|\ \|) \coloneqq \left\{ s \in H^0(X, L) \,\middle|\, \|s(p)\| \leqslant 1 \text{ for all } p \in X^{\mathrm{an}} \right\}. \tag{2.1}$$

Observe that $\widehat{H}^0(X, L, \|\ \|)$ is a free K°-module of rank $r \coloneqq \dim_K H^0(X, L)$. To see this, pick a K-basis $s_1, \ldots, s_r$ of $H^0(X, L)$ and remark that for an integer $\alpha > 0$ big enough, we have $\pi^\alpha \langle s_1 \ldots s_r \rangle_{K^\circ} \subseteq \widehat{H}^0(X, L, \|\ \|) \subseteq \pi^{-\alpha} \langle s_1 \ldots s_r \rangle_{K^\circ}$ because two vector space norms on $H^0(X, L)$ are equivalent [Bos14, Appendix A, Theorem 1].

Given a continuous reference metric $\|\ \|_0$ on L^{an}, any other continuous metric on L^{an} is of the form $\|\ \| = \|\ \|_0 \, e^{-\varphi}$ for some $\varphi \in C^0(X^{\mathrm{an}})$. We obtain the class of *singular metrics on* L^{an} if we allow arbitrary functions $\varphi\colon X^{\mathrm{an}} \to \mathbb{R} \cup \{-\infty\}$.

2.2.2. The space of continuous metrics on L^{an} is a metric space for the distance

$$d(\|\ \|_1, \|\ \|_2) = \sup_{X^{\mathrm{an}}} \left| \log \frac{\|\ \|_1}{\|\ \|_2} \right|. \tag{2.2}$$

Convergence for this distance is called *uniform convergence of metrics on* L^{an}.

2.2.3. Let L be a line bundle on the proper variety X. A *model of* (X, L) or briefly a *model of* L consists of a model $(\mathscr{X}, h)$ of X together with a line bundle $\mathscr{L}$ on $\mathscr{X}$ and an isomorphism h' between L and $h^*(\mathscr{L}|_{\mathscr{X}_\eta})$. Usually, we read h' as an identification.

Let $(\mathscr{X}, \mathscr{L})$ be a model of $(X, L^{\otimes m})$ for some $m \in \mathbb{N}_{>0}$. There is a unique metric $\|\ \|_{\mathscr{L}}$ on L^{an} over X^{an} such that the following holds: given a frame t of $\mathscr{L}$ over some open subset $\mathscr{U}$ of $\mathscr{X}$ and a section s of L over $U = X \cap \mathscr{U}$ such that $s^{\otimes m} = ht$ for some regular function h on U, we have $\|s\| = \sqrt[m]{|h|}$ on $U^{\mathrm{an}} \cap \mathrm{red}^{-1}(\mathscr{U}_s)$. Such a metric on L^{an} is called a *model metric* (*determined on* $\mathscr{X}$). A model metric is called *algebraic* if we can choose $m = 1$ in the construction above. Note that model metrics are continuous.

Lemma 2.2.4. *Let X be a normal proper variety over K and $\mathscr{X}$ a normal model of X. For a model $\mathscr{L}$ of L over $\mathscr{X}$, we have $\Gamma(\mathscr{X}, \mathscr{L}) = \widehat{H}^0(X, L, \|\ \|_{\mathscr{L}})$.*

Proof. The inclusion $\subseteq$ is obvious. Note that every $s \in \Gamma(X, L)$ extends uniquely to a meromorphic section $\widetilde{s}$ of $\mathscr{L}$. It remains to show that $\|s\|_{\mathscr{L}} \leqslant 1$ yields that $\widetilde{s}$ is a global section of $\mathscr{L}$. Since $\mathscr{X}$ is normal, it is equivalent to show that the Weil divisor associated with $\widetilde{s}$ is effective. Let ξ_i be the generic point of the irreducible component E_i of the special fiber $\mathscr{X}_s$. The local ring $\mathcal{O}_{\mathscr{X},\xi_i}$ is a valuation ring, and we may normalize the corresponding valuation v_i such that it extends the given valuation v on K. Then the multiplicity of the Weil divisor associated with $D := \mathrm{div}(\widetilde{s})$ in E_i is equal to $v_i(\gamma_i)$, where γ_i is a local equation of D in ξ_i. Let x_i be the divisorial point of X^{an} corresponding to E_i. Then it is clear from our assumptions that $v_i(\gamma_i) = -\log|\gamma_i(x_i)| \geqslant 0$. Since the restriction s of $\widetilde{s}$ to the generic fiber X is a global section anyway, this proves that the Weil divisor associated with D is effective. □

2.2.5. Each model metric $\|\ \|$ on $\mathcal{O}_{X^{\mathrm{an}}}$ induces a continuous real function $f = -\log\|1\|$ on X^{an}. The space of *model functions*

$$\mathscr{D}(X) = \left\{ f\colon X^{\mathrm{an}} \to \mathbb{R} \mid f = -\log\|1\| \text{ for a model metric } \|\ \| \text{ on } \mathcal{O}_X \right\}$$

has a natural structure of a $\mathbb{Q}$-vector space. We write $\mathscr{D}(X)_{\mathbb{R}} = \mathscr{D}(X) \otimes_{\mathbb{Q}} \mathbb{R}$. It is shown in [Gub98, Theorem 7.12] that the space of model functions $\mathscr{D}(X)$ is dense in the space $C^0(X^{\mathrm{an}})$ for the topology of uniform convergence. A model function $f = -\log\|1\|$ on X^{an} which comes from an algebraic metric $\|\ \|$ on $\mathcal{O}_{X^{\mathrm{an}}}$ is called a *$\mathbb{Z}$-model function.*

Let $\mathscr{X}$ be a model of X. We say that a model function $f = -\log\|1\|$ is *determined on* $\mathscr{X}$ if the model metric $\|\ \|$ is determined on $\mathscr{X}$. Let $\mathrm{Div}_0(\mathscr{X})$ denote the subgroup of $\mathrm{Div}(\mathscr{X})$ of vertical Cartier divisors on the model $\mathscr{X}$. Each $D \in \mathrm{Div}_0(\mathscr{X})$ determines a model $\mathcal{O}(D)$ of $\mathcal{O}_X$ and an associated model function $\varphi_D := -\log\|1\|_{\mathcal{O}(D)}$.

Proposition 2.2.6. *Let D be a vertical Cartier divisor on the model $\mathscr{X}$ of X. If D is effective, then $\varphi_D \geqslant 0$. The converse holds if $\mathscr{X}$ is normal.*

Proof. If D is an effective Cartier divisor, then it follows easily from the definition of $\|\ \|_{\mathcal{O}(D)}$ that $\varphi_D \geqslant 0$. Conversely, if $\varphi_D \geqslant 0$, then the multiplicity formula (2.5) in Lemma 2.4.2 below shows that the Weil divisor associated with D is effective. Since $\mathscr{X}$ is normal, D has to be an effective Cartier divisor [Har77, Proposition II.6.3.A]. □

Remark 2.2.7. We note that Lemma 2.2.4 and hence Proposition 2.2.6 hold also for a non-complete discretely valued field F. The proof of Lemma 2.2.4 has to be slightly changed: Working on the base change $\mathscr{X}' := \mathscr{X} \otimes_{F^\circ} K^\circ$, where K is the completion of F, and using $\|s\|_{\mathscr{L}} \leqslant 1$, it follows from [Gub98, Proposition 6.5] that $\tilde{s}$ induces an effective Weil divisor on $\mathscr{X}'$. Since the special fibers of $\mathscr{X}$ and $\mathscr{X}'$ agree, it follows that the Weil divisor on $\mathscr{X}$ associated with $\tilde{s}$ is effective. By the normality of $\mathscr{X}$, we conclude again that $s \in \Gamma(\mathscr{X}, \mathscr{L})$.

2.3. Closed (1,1)-forms and semipositive metrics.

We consider a model $\mathscr{X}$ of a proper variety X over K.

2.3.1. The finite-dimensional real vector space $N^1(\mathscr{X}/S)$ is defined as the quotient of $\operatorname{Pic}(\mathscr{X})_{\mathbb{R}} := \operatorname{Pic}(\mathscr{X}) \otimes \mathbb{R}$ by the subspace generated by classes of line bundles $\mathscr{L}$ such that $\mathscr{L} \cdot C = 0$ for each closed curve C in $\mathscr{X}_s$. An element $\alpha \in N^1(\mathscr{X}/S)$ is called *nef* if $\alpha \cdot C \geqslant 0$ for all closed curves C in $\mathscr{X}_s$. We call a line bundle $\mathscr{L}$ on $\mathscr{X}$ *nef* if the class of $\mathscr{L}$ in $N^1(\mathscr{X}/S)$ is nef. The *space of closed* $(1,1)$*-forms on* X is defined as

$$\mathcal{Z}^{1,1}(X) := \varinjlim N^1(\mathscr{X}/S)\,, \tag{2.3}$$

where $\mathscr{X}$ runs over the isomorphism classes of models of X.

Let L be a line bundle on X. Let $\|\ \|$ be a model metric on L^{an} which is determined on $\mathscr{X}$ by a model $\mathscr{L}$ of $L^{\otimes m}$. The class of $m^{-1}\mathscr{L}$ in $N^1(\mathscr{X}/S)$ determines a well-defined class $c_1(L,\|\ \|) \in \mathcal{Z}^{1,1}(X)$ called the *curvature form* $c_1(L,\|\ \|)$ *of* $(L,\|\ \|)$.

2.3.2. We denote by $N^1(X)$ the real vector space $\operatorname{Pic}(X) \otimes \mathbb{R}$ modulo numerical equivalence. A class in $N^1(X)$ is called *ample* if it is an $\mathbb{R}_{>0}$-linear combination of classes induced by ample line bundles on X. The restriction maps $N^1(\mathscr{X}/S) \to N^1(X)$, $[\mathscr{L}] \mapsto [\mathscr{L}|_X]$ induce a linear map $\{\ \}\colon \mathcal{Z}^{1,1}(X) \longrightarrow N^1(X)$, $\theta \mapsto \{\theta\}$.

2.3.3. A closed $(1,1)$-form θ is called *semipositive* if it is represented by a nef element $\theta_{\mathscr{X}} \in N^1(\mathscr{X}/S)$ for some model $\mathscr{X}$ of X. We say that a model metric $\|\ \|$ on L^{an} for a line bundle L on X is *semipositive* if the same holds for the curvature form $c_1(L,\|\ \|)$.

2.3.4. Let L be a line bundle on X. Following S. Zhang [Zha95b], we say that a continuous metric $\|\ \|$ on L^{an} is *continuous semipositive* if it is a uniform limit of semipositive model metrics on L^{an}.

Remark 2.3.5. Let L be a line bundle on X which admits a continuous semipositive metric. Then the line bundle L is nef (use [BFJ16, Lemma 1.2] or [GM19, § 4.8]). This implies in particular that the generic fiber $L = \mathscr{L}|_X$ of a nef line bundle $\mathscr{L}$ on some model $\mathscr{X}$ of X is nef.

2.4. Chambert-Loir measures and energy.

Throughout this subsection, X denotes a normal proper K-variety of dimension n.

2.4.1. Let $\mathscr{X}$ be a normal model of X. For line bundles $\mathscr{L}_1, \ldots, \mathscr{L}_n$ on the model $\mathscr{X}$, A. Chambert-Loir [Cha06] introduced the discrete signed measure

$$c_1(\mathscr{L}_1) \wedge \ldots \wedge c_1(\mathscr{L}_n) := \sum_V \ell_{\mathcal{O}_{\mathscr{X}_s,\xi_V}}(\mathcal{O}_{\mathscr{X}_s,\xi_V})(\mathscr{L}_1 \cdots \mathscr{L}_n \cdot V)\delta_{x_V} \tag{2.4}$$

on X^{an}, where V runs over the irreducible components of the special fiber $\mathscr{X}_s$ of our model, ξ_V is the generic point of V, the point x_V is the divisorial point in X^{an} determined by V and δ_{x_V} is the Dirac measure supported in the point x_V.

Let $\mathscr{L}_1, \ldots, \mathscr{L}_n$ be nef on $\mathscr{X}$ with $L_i := \mathscr{L}_i|_X$. Then the measure (2.4) is positive of total mass $L_1 \cdots L_n \cdot X$.

Lemma 2.4.2. *Let E be a vertical Cartier divisor on a normal model $\mathscr{X}$ of X with model function φ_E. For an irreducible component V of $\mathscr{X}_s$ with divisorial point $x_V \in X^{\mathrm{an}}$, let b_V and c_V be the multiplicities of $\mathscr{X}_s$ and E, respectively, in V. Then we have*

$$c_V = \varphi_E(x_V) \cdot b_V\,. \tag{2.5}$$

Moreover, for line bundles $\mathscr{L}_1, \ldots, \mathscr{L}_n$ on $\mathscr{X}$, we have

$$\mathscr{L}_1 \cdots \mathscr{L}_n \cdot E = \int_{X^{\mathrm{an}}} \varphi_E c_1(\mathscr{L}_1) \wedge \cdots \wedge c_1(\mathscr{L}_n)\,. \tag{2.6}$$

Proof. Denote by ξ_V the generic point of V. Since $\mathscr{X}$ is normal, it is regular in codimension one. Thus there exists a local equation γ for V at ξ_V. Then γ^{c_V} is a local equation for E. By [BPS14, Proposition 1.3.3], the seminorm associated with x_V is precisely the one which comes from the valuation of $\mathcal{O}_{\mathscr{X},\xi_V}$. For a uniformizer π of K°, we get

$$1 = v(\pi) = -\log|\gamma^{b_V}(x_V)| = -b_V \log|\gamma(x_V)|\,.$$

This implies

$$\varphi_E(x_V) = -\log||1(x_V)||_{\mathcal{O}(E)} = -c_V \log|\gamma(x_V)| = c_V/b_V\,,$$

which proves (2.5). From the first part and (2.4), we deduce (2.6). □

2.4.3. For continuous semipositive metrized line bundles $(L_1, \|\ \|_1), \dots, (L_n, \|\ \|_n)$ on X, there exists a unique positive Radon measure $c_1(L_1, \|\ \|_1) \wedge \cdots \wedge c_1(L_n, \|\ \|_n)$ of total mass $L_1 \cdots L_n \cdot X$ on X^{an} with the following properties (see [Cha06, Gub07]):

(i) The map $((L_1, \|\ \|_1), \dots, (L_n, \|\ \|_n)) \mapsto c_1(L_1, \|\ \|_1) \wedge \cdots \wedge c_1(L_n, \|\ \|_n)$ is multilinear and symmetric.

(ii) If the metrics on $(L_1, \|\ \|_1), \dots, (L_n, \|\ \|_n)$ are induced by line bundles $\mathscr{L}_1, \dots, \mathscr{L}_n$ on a model $\mathscr{X}$ of X, then $c_1(L_1, \|\ \|_1) \wedge \cdots \wedge c_1(L_n, \|\ \|_n)$ agrees with (2.4).

(iii) If each metric $\|\ \|_i$ is a uniform limit of continuous semipositive metrics $(\|\ \|_{ij})_{j\in\mathbb{N}}$ on L_i^{an}, then the measures $(c_1(L_1, \|\ \|_{1j}) \wedge \cdots \wedge c_1(L_n, \|\ \|_{nj}))_{j\in\mathbb{N}}$ on X^{an} converge weakly to the measure $c_1(L_1, \|\ \|_1) \wedge \cdots \wedge c_1(L_n, \|\ \|_n)$.

(iv) Given a morphism $f\colon X' \to X$ of normal proper K-varieties over K of dimension n, we have for $\overline{L_i} \coloneqq (L_i, \|\ \|_i)$ the projection formula

$$f_*\big(c_1(f^*\overline{L_1}) \wedge \cdots \wedge c_1(f^*\overline{L_n})\big) = \deg(f)\, c_1(\overline{L_1}) \wedge \cdots \wedge c_1(\overline{L_n})\,,$$

where $\deg(f)$ is the degree of the finite function field extension $K(X')/K(X)$ if f is dominant and zero otherwise.

We call $c_1(L_1, \|\ \|_1) \wedge \cdots \wedge c_1(L_n, \|\ \|_n)$ the *Chambert-Loir measure for* $\overline{L_1}, \dots, \overline{L_n}$.

Definition 2.4.4. For continuous semipositive metrics $\|\ \|_1$ and $\|\ \|_2$ on a line bundle L over X, the *energy* is defined as

$$E(L, \|\ \|_1, \|\ \|_2) \coloneqq \frac{1}{n+1} \sum_{j=0}^{n} \int_{X^{\mathrm{an}}} -\log \frac{\|\ \|_1}{\|\ \|_2} c_1(L, \|\ \|_1)^{\wedge j} \wedge c_1(L, \|\ \|_2)^{\wedge(n-j)} \in \mathbb{R}\,. \tag{2.7}$$

This energy is denoted by $E_\theta(\varphi)$ with $\theta = c_1(L, \|\ \|_1)$ and $\varphi = -\log(\|\ \|_1/\|\ \|_2)$ in [BFJ15, § 6].

2.4.5. If $\|\ \|_1$ and $\|\ \|_2$ are algebraic metrics induced by models $\mathscr{L}_1$ and $\mathscr{L}_2$ of L on a normal model $\mathscr{X}$ of X, then we can write $\mathscr{L}_1 = \mathscr{L}_2(D)$ for some vertical Cartier divisor D on $\mathscr{X}$, and (2.6) yields the explicit formula

$$E\big(L, \|\ \|_{\mathscr{L}_1}, \|\ \|_{\mathscr{L}_2}\big) = \frac{1}{n+1} \sum_{j=0}^{n} \mathscr{L}_1^{j} \cdot \mathscr{L}_2^{n-j} \cdot D\,. \tag{2.8}$$

2.5. The semipositive envelope. Let X be a normal projective variety over K, let L be a line bundle on X, and let $\|\ \|$ be a continuous metric on L^{an}.

Definition 2.5.1. The *semipositive envelope of the metric* $\| \ \|$ is the singular metric

$$P(\| \ \|) \coloneqq \inf\bigl\{\| \ \|_1 \bigm| \| \ \|_1 \text{ is a semipositive model metric on } L^{\mathrm{an}} \text{ with } \| \ \| \leqslant \| \ \|_1\bigr\}$$

on L^{an} with the infimum taken pointwise on X^{an}.

Remark 2.5.2. (i) By definition, we have $P(\| \ \|^{\otimes m}) = P(\| \ \|)^{\otimes m}$ for all $m \in \mathbb{Z}$.

(ii) Assume that the semipositive envelope $P(\| \ \|)$ is a continuous metric. Using that the minimum of two semipositive model metrics is a semipositive model metric [GM19, Propositions 3.11 and 3.12], we see that $P(\| \ \|)$ is the infimum of a decreasing family of semipositive model metrics, and hence it follows from Dini's theorem that $P(\| \ \|)$ is a continuous semipositive metric.

For the rest of this subsection, we assume that $\tilde{K}$ has characteristic zero and that L is an ample line bundle on a smooth projective variety X over K. In [BFJ16], the envelope was introduced in terms of θ-psh functions. To compare, let us fix a model metric $\| \ \|_0$ on L^{an} for reference and consider $\theta \coloneqq c_1(L, \| \ \|_0)$. The function $-\log(P(\| \ \|)/\| \ \|)$ is the θ-psh envelope of the continuous function $-\log(\| \ \|/\| \ \|_0)$ on X^{an} as defined in [BFJ16, Definition 8.1], and [BFJ16, Theorem 8.3] gives the following.

Theorem 2.5.3 (Boucksom, Favre, Jonsson). *Assume* $\mathrm{char}(\tilde{K}) = 0$ *and that* L *is an ample line bundle on a smooth projective variety over* K. *Then the semipositive envelope* $P(\| \ \|)$ *is a continuous semipositive metric on* L^{an}.

3. Asymptotic formulas for algebraic volumes

The goal of this section is to study the asymptotics of $h^i(Y, m_1D_1 + \cdots + m_rD_r)$ for fixed divisors $D_1, \ldots, D_r$ on a projective variety Y over any field k. Our main result is Proposition 3.5.1. Its consequences from Subsection 3.6 will be applied in Sections 4 and 5. In these applications, we will need to consider non-reduced projective schemes Y over a non-reduced basis as $R = K^\circ/(\pi^\alpha)$ for a uniformizer π of a discrete valuation ring K° and a non-zero α. Note that R is not necessarily an algebra over the residue field. Therefore, we will develop much of the theory over any noetherian ring R in the spirit of the appendix in [Kol96, § VI.2].

Let us recall that the canonical morphism $\mathrm{Div}(Y) \to \mathrm{Pic}\,(Y)$ is surjective if the scheme Y is projective over the noetherian scheme $S = \mathrm{Spec}(R)$ [GD67, Corollary 21.3.5]. This means that we can switch freely between the language of Cartier divisors and the language of line bundles. In this section, we have a slight preference to the former.

3.1. Infinitesimal perturbations. In this subsection, let $S = \mathrm{Spec}(R)$ for any noetherian ring R, and consider a projective scheme Y over S. We fix a coherent $\mathcal{O}_Y$-module $\mathcal{F}$ on Y with support over a zero-dimensional closed subset of $S = \mathrm{Spec}(R)$. The dimension of the support of $\mathcal{F}$ is denoted by n. We note that the cohomology $H^q(Y, \mathcal{F})$ is an R-module of finite length, and we set

$$h^q(Y, \mathcal{F}) \coloneqq \ell_R\bigl(H^q(Y, \mathcal{F})\bigr)\,.$$

Lemma 3.1.1. *Let T be a finite subset of Y, let D be a Cartier divisor on Y, and let A be an ample divisor on Y. Then there exists a sufficiently large $m \in \mathbb{N}$ such that the Cartier divisors mA and $D + mA$ are linearly equivalent to effective Cartier divisors E and F, respectively, with the property that the supports of E and F are disjoint from T.*

Proof. Recall that regular global sections are precisely those global sections which correspond to effective Cartier divisors. In [SP16, Tag 0AYL], it is explained that a global section is regular if and only if it does not vanish in the associated points of Y.

Step 1: There is an $m_0 \geqslant 0$ such that for any integer $m \geqslant m_0$, there exists a global section s of $\mathcal{O}(D + mA)$ with $s(t) \neq 0$ for all $t \in T$.

Since Y is noetherian, the closure of any point $t \in T$ contains a closed point t_0 by [SP16, Tag 02IL]. If s is a global section of $\mathcal{O}(D+mA)$ with $s(t_0) \neq 0$, then $s(t) \neq 0$, and hence we can replace t by t_0. So we may assume that the points in T are closed.

We consider T as a reduced closed subscheme $t \colon T \to Y$ By restriction of regular functions to T, we get a short exact sequence of coherent $\mathcal{O}_Y$-modules

$$0 \longrightarrow \mathcal{K} \longrightarrow \mathcal{O}_Y \longrightarrow t_*\mathcal{O}_T \longrightarrow 0\,.$$

We twist by $\mathcal{O}(D + mA)$ and consider the associated long exact cohomology sequence. Since A is ample, Serre's vanishing theorem [Har77, Theorem III 5.2] yields a surjection

$$\Gamma(Y, \mathcal{O}(D + mA)) \longrightarrow \Gamma(T, t^*\mathcal{O}(D + mA)) \longrightarrow 0$$

for $m \gg 0$. Since T is discrete, $t^*\mathcal{O}(D + mA)$ is trivial, and we find a nowhere vanishing section $s_0 \in \Gamma(T, t^*\mathcal{O}(D+mA))$. The above surjection allows us to lift s_0 to a section $s \in \Gamma(Y, \mathcal{O}(D+mA))$ which does not vanish at any $t \in T$. This proves the first step.

Step 2: There is an $m_0 \geqslant 0$ such that for any integer $m \geqslant m_0$, there exists an effective Cartier divisor F linearly equivalent to $D + mA$ with $\operatorname{supp}(F) \cap T = \emptyset$.

We first enlarge T to include the finitely many associated points of Y. Then we apply the first step to get a global section s of $\mathcal{O}(D + mA)$ with $s(t) \neq 0$ for all $t \in T$. As explained at the beginning, such a global section has to be regular, and hence the associated effective Cartier divisor does the job in Step 2.

Lemma 3.1.1 follows by applying Step 2 twice, once for $D = 0$ and once for D. □

It is well known (see [Laz04a, Example 1.2.33] if the base is a field) that for every integer q,

$$h^q(Y, \mathcal{F}(mD)) = O(m^n)\,. \tag{3.1}$$

We need the following easy generalization. We will fix line bundles $M_1, \dots, M_r$ and $P_1, \dots, P_s$ on Y. For $\mathbf{m} = (m_1, \dots, m_r) \in \mathbb{N}^r$ and $\mathbf{p} = (p_1, \dots, p_s) \in \mathbb{N}^s$, with $r, s \geqslant 0$, we set

$$\mathcal{F}(\mathbf{m}, \mathbf{p}) \coloneqq \mathcal{F} \otimes M_1^{\otimes m_1} \otimes \cdots \otimes M_r^{\otimes m_r} \otimes P_1^{\otimes p_1} \otimes \cdots \otimes P_s^{\otimes p_s}\,.$$

Proposition 3.1.2. *There is constant $C \in \mathbb{R}$ (depending on the isomorphism classes of $\mathcal{F}, M_1, \dots, M_r, P_1, \dots, P_s$) such that for all $m_1, \dots, m_r, p_1, \dots, p_s \in \mathbb{N} \setminus \{0\}$, we have*

$$|h^q(Y, \mathcal{F}(\mathbf{m}, \mathbf{p})) - h^q(Y, \mathcal{F}(\mathbf{0}, \mathbf{p}))| \leqslant C \cdot m(m + p)^{n-1}\,,$$

where $m \coloneqq \sum_{i=1}^r m_i$ and $p \coloneqq \sum_{j=1}^s p_j$.

Proof. We prove the claim by induction on $n = \dim(\operatorname{supp}(\mathcal{F}))$. If the support is empty, then $\mathcal{F} = 0$ and the claim holds for $n = -\infty$. So we may assume $n \geqslant 0$. As a first step, we will show the existence of a constant C' depending only on the isomorphism classes of $\mathcal{F}, M_1, \dots, M_r$ and of a line bundle L such that

$$|h^q(Y, \mathcal{F}(\mathbf{m}) \otimes L) - h^q(Y, \mathcal{F}(\mathbf{m}))| \leqslant C' m^{n-1} \tag{3.2}$$

for all $\mathbf{m} \in (\mathbb{N} \setminus \{0\})^r$, where $\mathcal{F}(\mathbf{m}) := \mathcal{F} \otimes M_1^{\otimes m_1} \otimes \cdots \otimes M_r^{\otimes m_r}$. By Lemma 3.1.1, there are effective Cartier divisors E and F of Y such that $\mathcal{O}(E-F) \simeq L$ and such that the supports of E and F both do not contain a generic point of $\operatorname{supp}(\mathcal{F})$. This means that the support of $\mathcal{F}(\mathbf{m})|_E$ has dimension at most $n-1$. The same also holds for the restriction of $\mathcal{F}(\mathbf{m}, E) := \mathcal{F}(\mathbf{m}) \otimes \mathcal{O}(E)$ to E and for the restrictions to F. Then we have the short exact sequence

$$0 \longrightarrow \mathcal{F}(\mathbf{m}) \overset{\otimes s_E}{\longrightarrow} \mathcal{F}(\mathbf{m}, E) \longrightarrow \mathcal{F}(\mathbf{m}, E)|_E \longrightarrow 0\,, \tag{3.3}$$

where s_E is the canonical global section of $\mathcal{O}(E)$. By induction on n, we have

$$h^q(E, \mathcal{F}(\mathbf{m}, E)|_E) \leqslant C_{n-1} \cdot m^{n-1} \tag{3.4}$$

for a $C_{n-1} \in \mathbb{R}_{\geqslant 0}$ depending only on the isomorphism classes of $\mathcal{F}, M_1, \ldots, M_r$ and $\mathcal{O}(E)$. Using the long exact cohomology sequence associated with (3.3), we deduce

$$-h^{q-1}(E, \mathcal{F}(\mathbf{m}, E)|_E) \leqslant h^q(Y, \mathcal{F}(\mathbf{m}, E)) - h^q(Y, \mathcal{F}(\mathbf{m})) \leqslant h^q(E, \mathcal{F}(\mathbf{m}, E)|_E)\,.$$

Using these inequalities and (3.4), we get

$$|h^q(Y, \mathcal{F}(\mathbf{m}, E)) - h^q(Y, \mathcal{F}(\mathbf{m}))| \leqslant C_{n-1} \cdot m^{n-1}\,. \tag{3.5}$$

We apply (3.5) to $\mathcal{F}' := \mathcal{F}(E-F)$ instead of $\mathcal{F}$ and F instead of E. We get a $C'_{n-1} \in \mathbb{R}_{\geqslant 0}$ depending only on the isomorphism classes of $\mathcal{F}, M_1, \ldots, M_r, \mathcal{O}(E)$ and $\mathcal{O}(F)$ such that

$$\big|h^q(Y, \mathcal{F}'(\mathbf{m}, F)) - h^q(Y, \mathcal{F}'(\mathbf{m}))\big| \leqslant C'_{n-1} \cdot m^{n-1}\,. \tag{3.6}$$

Given that $\mathcal{F}'(\mathbf{m}) \simeq \mathcal{F}(\mathbf{m}) \otimes L$ and that $\mathcal{F}'(\mathbf{m}, F) \simeq \mathcal{F}(\mathbf{m}, E)$, the inequality (3.2) follows easily from (3.5) and (3.6) with the constant $C' := C_{n-1} + C'_{n-1}$.

To prove Proposition 3.1.2, we apply (3.2) for any $\mathbf{k} \in \mathbb{N}^r$ with $k = \sum_{j=1}^r k_j$ to get

$$|h^q(Y, \mathcal{F}(\mathbf{k}, \mathbf{p}) \otimes L) - h^q(Y, \mathcal{F}(\mathbf{k}, \mathbf{p})| \leqslant C(k+p)^{n-1} \tag{3.7}$$

for any $L \in \{M_1, \ldots, M_r\}$, with $C \in \mathbb{R}_{\geqslant 0}$ depending only on the isomorphism classes of $\mathcal{F}, M_1, \ldots, M_r, P_1, \ldots, P_s$. The claim follows from an m-fold application of (3.7). ☐

3.2. Dévissage and non-reduced schemes. In this subsection, we work over $S = \operatorname{Spec}(R)$ for a noetherian ring R. The goal is to generalize the following classical fact from [Deb01, Proof of Theorem 1.5 and Proposition 1.31] to the situation over the base scheme S.

Lemma 3.2.1. *Let Y be an n-dimensional projective variety over an arbitrary field k, and let $q \in \mathbb{N}$. Let $D_1, \ldots, D_r$ be Cartier divisors and $\mathcal{F}$ a coherent sheaf on Y. Then for $m_1, \ldots, m_r \in \mathbb{N} \setminus \{0\}$ and $m = \sum_{i=1}^r m_i$, we have*

$$h^q\left(Y, \mathcal{F}\left(\sum_{i=1}^r m_i D_i\right)\right) = \operatorname{rank}(\mathcal{F}) h^q\left(Y, \mathcal{O}_Y\left(\sum_{i=1}^r m_i D_i\right)\right) + O\big(m^{n-1}\big)\,,$$

where $\operatorname{rank}(\mathcal{F})$ is the dimension of the $\mathcal{O}_{Y,\xi}$-vector space $\mathcal{F}_\xi$ at the generic point ξ of Y.

We need the following dévissage result for coherent sheaves.

Lemma 3.2.2. *For a coherent sheaf $\mathcal{F}$ on a noetherian scheme Y, there is a filtration*

$$0 = \mathcal{F}_0 \subset \mathcal{F}_1 \subset \cdots \subset \mathcal{F}_s = \mathcal{F} \tag{3.8}$$

by coherent subsheaves, closed integral subschemes $\iota_j \colon Z_j \hookrightarrow Y$ and coherent sheaves of ideals $\mathcal{I}_j \subset \mathcal{O}_{Z_j}$ with $\operatorname{supp}(\mathcal{I}_j) = Z_j$ and $\mathcal{F}_j/\mathcal{F}_{j-1} \simeq \iota_{j,*}(\mathcal{I}_j)$ for $j = 1, \ldots, s$.

Proof. This can be found in [SP16, Tag 01YC] except for the precise statement for the support of the $\mathcal{I}_j$. The latter follows immediately from the argument in loc. cit. □

We have the following generalization of Lemma 3.2.1.

Lemma 3.2.3. *Let Y be a projective scheme over S, and let $\mathcal{F}$ be a coherent sheaf on Y with support over a zero-dimensional subscheme of S. We denote by $\{E_i\}_{i\in I}$ the set of irreducible components of $\operatorname{supp}(\mathcal{F})$ of maximal dimension $n \coloneqq \dim(\operatorname{supp}(\mathcal{F}))$. Let $D_1, \dots, D_r$ be some Cartier divisors and $q \in \mathbb{N}$. Then for $m_1, \dots, m_r \in \mathbb{N} \setminus \{0\}$, we have*

$$h^q\left(Y, \mathcal{F}\left(\sum_{j=1}^r m_j D_j\right)\right) \leqslant \sum_{i\in I} \ell_{\mathcal{O}_{Y,\xi_i}}(\mathcal{F}_{\xi_i}) h^q\left(E_i, \mathcal{O}_Y\left(\sum_{j=1}^r m_j D_j\right)\Big|_{E_i}\right) + O\big(m^{n-1}\big)\,, \tag{3.9}$$

where $m = \sum_{j=1}^r m_j$ and where ξ_i is the generic point of E_i.

Proof. We proceed by induction on the length s of a dévissage of $\mathcal{F}$ as in (3.8). The case $s = 0$ means that $\mathcal{F} = 0$ and the claim is obvious. So we may assume $s \geqslant 1$. The corresponding dévissage (3.8) leads to the short exact sequence

$$0 \longrightarrow \mathcal{G}\left(\sum_{j=1}^r m_j D_j\right) \longrightarrow \mathcal{F}\left(\sum_{j=1}^r m_j D_j\right) \longrightarrow \mathcal{H}\left(\sum_{j=1}^r m_j D_j\right) \longrightarrow 0$$

for $\mathcal{G} \coloneqq \mathcal{F}_{s-1}$ and $\mathcal{H} \coloneqq \mathcal{F}/\mathcal{F}_{s-1}$. The long exact sequence in cohomology yields

$$h^q\left(Y, \mathcal{F}\left(\sum_{j=1}^r m_j D_j\right)\right) \leqslant h^q\left(Y, \mathcal{G}\left(\sum_{j=1}^r m_j D_j\right)\right) + h^q\left(Y, \mathcal{H}\left(\sum_{j=1}^r m_j D_j\right)\right). \tag{3.10}$$

By the definition of the dévissage, $\mathcal{H} \simeq \varphi_*(\mathcal{I})$, where $\varphi\colon Z \to Y$ is an integral closed subscheme of Y and $\mathcal{I} \subset \mathcal{O}_Z$ is a coherent sheaf of ideals with $\operatorname{supp}(\mathcal{I}) = Z$. By the projection formula [Har77, Exercise II.5.1(d)] and by [Har77, Lemma III.2.10], we deduce

$$H^q\left(Y, \mathcal{H}\left(\sum_{j=1}^r m_j D_j\right)\right) \simeq H^q\left(Z, \varphi^*\left(\mathcal{O}_Y\left(\sum_{j=1}^r m_j D_j\right)\right) \otimes \mathcal{I}\right). \tag{3.11}$$

Case 1. If $\dim(Z) < n$, then $h^q(Y, \mathcal{H}(\sum_{j=1}^r m_j D_j)) = O\big(m^{n-1}\big)$ by Proposition 3.1.2; hence, (3.10) yields

$$h^q\left(Y, \mathcal{F}\left(\sum_{j=1}^r m_j D_j\right)\right) \leqslant h^q\left(Y, \mathcal{G}\left(\sum_{j=1}^r m_j D_j\right)\right) + O\big(m^{n-1}\big)\,. \tag{3.12}$$

Since $\mathcal{H}$ is the push-forward of $\mathcal{I}$ from Z, the assumption in Case 1 yields $\mathcal{H}_{\xi_i} = 0$ for all $i \in I$. Since the length is additive, we deduce $\ell_{\mathcal{O}_{Y,\xi_i}}(\mathcal{F}_{\xi_i}) = \ell_{\mathcal{O}_{Y,\xi_i}}(\mathcal{G}_{\xi_i})$ for all $i \in I$. Hence, the result follows from (3.12) by the induction hypothesis applied to $\mathcal{G}$.

Case 2. If $\dim Z = n$, then $Z = E_{i_0}$ for some $i_0 \in I$. Then the stalk I_ξ at the generic point ξ of Z is a non-zero ideal in the field $\mathcal{O}_{Z,\xi}$ and hence equal to this field. Since ξ is in the support of $\mathcal{F}$, it lies over a closed point η in the base scheme S, and hence Z may be viewed as a variety over the residue field of η. So we may apply Lemma 3.2.1 to the right-hand side of (3.11) with

$\operatorname{rank}(\mathcal{I}) = 1$ to get

$$h^q\left(Y, \mathcal{G}\left(\sum_{j=1}^r m_j D_j\right)\right) = h^q\left(E_{i_0}, \left(\mathcal{O}_Y\left(\sum_{j=1}^r m_j D_j\right)\right)\Big|_{E_{i_0}}\right) + O\big(m^{n-1}\big)\,. \tag{3.13}$$

Using the additivity of the length, we have $\ell_{\mathcal{O}_{Y,\xi_i}}(\mathcal{F}_{\xi_i}) = \ell_{\mathcal{O}_{Y,\xi_i}}(\mathcal{G}_{\xi_i})$ for $i \neq i_0$ and $\ell_{\mathcal{O}_{Y,\xi}}(\mathcal{F}_\xi) = \ell_{\mathcal{O}_{Y,\xi}}(\mathcal{G}_\xi) + 1$. Hence, the result follows from (3.10) and (3.13) using the induction hypothesis applied to $\mathcal{G}$.

□

3.3. Volumes and asymptotic cohomological functions. In this subsection, we assume that Y is a projective variety over a field k. We will recall the volume of a Cartier divisor and its higher cohomological analogues. We fix a Cartier divisor D on Y.

3.3.1. The *volume* of D or of the corresponding line bundle $L = \mathcal{O}(D)$ is defined by

$$\operatorname{vol}(D) := \operatorname{vol}(L) := \limsup_m \frac{h^0(Y, \mathcal{O}_Y(mD))}{m^n/n!}\,.$$

Since $h^0(Y, \mathcal{O}_Y(mD)) = O(m^n)$, one easily gets that $\operatorname{vol}(D) \in \mathbb{R}_{\geqslant 0}$. Actually, the lim sup is a lim. This follows from Fujita's approximation theorem when k is algebraically closed (cf. [Laz04b, Example 11.4.7] for characteristic zero and use [Tak07] in characteristic $p > 0$). For arbitrary fields, we refer to [Cut14, Theorem 8.1].

Remark 3.3.2. If D is nef, then $\operatorname{vol}(D) = D^n$ (cf. [Laz04a, Corollary 1.4.41]).

A. Küronya has introduced and studied the following higher volume-type invariants, called *asymptotic cohomological functions*, in [Kür06].

Definition 3.3.3. For $0 \leqslant i \leqslant n$, the *asymptotic cohomological function* $\widehat{h}^i(Y, D)$ is defined by

$$\widehat{h}^i(Y, D) := \limsup_m \frac{h^i(Y, \mathcal{O}_Y(mD))}{m^n/n!}\,. \tag{3.14}$$

For $i = 0$, we get the volume. For $i > 0$, it seems to be unknown whether lim sup is a limit. In the case $k = \mathbb{C}$, A. Küronya showed that $\widehat{h}^i(Y, D)$ is homogeneous in D and extends uniquely to a continuous homogeneous function $N^1(Y) \to \mathbb{R}_{\geqslant 0}$. In fact, the arguments work for every algebraically closed base field k. We will prove in Subsection 3.4 a weaker continuity property which holds over any field k.

3.4. Asymptotic cohomological functions for real divisors. In this subsection, we assume that Y is an n-dimensional projective scheme over a field k. As promised in Subsection 3.3, we will extend A. Küronya's asymptotic cohomological functions to $\operatorname{Div}_{\mathbb{R}}(Y) := \operatorname{Div}(Y) \otimes_{\mathbb{Z}} \mathbb{R}$, and we will characterize them by homogenity and continuity. Note that A. Küronya proved stronger results in the special case of a projective variety over an algebraically closed field (see Definition 3.3.3).

Definition 3.4.1. Let $D \in \operatorname{Div}_{\mathbb{R}}(Y)$. Then we have $D = \sum_{i=1}^r a_i D_i$ for suitable $a_i \in \mathbb{R}$ and $D_i \in \operatorname{Div}(Y)$. We call this a *decomposition* $\mathcal{D}$ of D. We define the *round-up of D* with respect to $\mathcal{D}$ to be

$$\lceil D \rceil_{\mathcal{D}} := \sum_{i=1}^r \lceil a_i \rceil D_i \in \operatorname{Div}(Y)\,,$$

and for $q \in \mathbb{N}$, we set $h^q(D)_{\mathcal{D}} := h^q(Y, \mathcal{O}_Y(\lceil D \rceil_{\mathcal{D}}))$.

Remark 3.4.2. The above definitions indeed depend on the choice of a given decomposition $\mathcal{D}$. Similar methods are used in [FKL16, Theorem 3.5(i)]. One can also define canonical round-downs and round-ups for $\mathbb{R}$-Weil divisors [Laz04b, § 9.1].

LEMMA 3.4.3. *Let V be a finitely generated $\mathbb{Z}$-module, and let $x \in V \otimes_{\mathbb{Z}} \mathbb{R}$. We consider two decompositions $x = \sum_{i=1}^{p} x_i v_i = \sum_{j=1}^{q} y_j w_j$ with $x_i, y_j \in \mathbb{R}$ and $v_i, w_j \in V$. Then the set $\mathcal{S} := \left\{ \sum_{i=1}^{p} \lceil m x_i \rceil v_i - \sum_{j=1}^{q} \lceil m y_j \rceil w_j \mid m \in \mathbb{Z} \right\}$ is finite.*

Proof. Let us put a Euclidean norm $\| \ \|$ on $V_{\mathbb{R}} := V \otimes_{\mathbb{Z}} \mathbb{R}$. For all $m \in \mathbb{N}$, we have

$$\left\| \left(\sum_{i=1}^{p} \lceil m x_i \rceil v_i \right) - m x \right\| \leqslant K_1 := \sum_{i=1}^{p} \|v_i\| .$$

Similarly, there exists a $K_2 \in \mathbb{R}$ for the second decomposition, and hence we get

$$\left\| \left(\sum_{i=1}^{p} \lceil m x_i \rceil v_i \right) - \left(\sum_{j=1}^{q} \lceil m y_j \rceil w_j \right) \right\| \leqslant K$$

for $K := K_1 + K_2$. On the other hand, $\left(\sum_{i=1}^{p} \lceil m x_i \rceil v_i \right) - \left(\sum_{j=1}^{q} \lceil m y_j \rceil w_j \right) \in V$. Since a given ball in $V_{\mathbb{R}}$ contains only finitely many points in the lattice $\operatorname{im}(V \to V_{\mathbb{R}})$, we deduce that the image of $\mathcal{S}$ in $V_{\mathbb{R}}$ is finite. The claim follows from the fact that the kernel of the map $V \to V_{\mathbb{R}}$ is the group of torsion elements, which is finite as V is finitely generated. □

In the following, we will use linear equivalence $D \sim E$ for real divisors $D, E \in \operatorname{Div}(Y)_{\mathbb{R}}$, meaning that D and E have the same image in $\operatorname{Pic}(Y) \otimes_{\mathbb{Z}} \mathbb{R}$.

LEMMA 3.4.4. *Let $D, E \in \operatorname{Div}(Y)_{\mathbb{R}}$ be real Cartier divisors with decompositions $\mathcal{D}$ and $\mathcal{E}$. If $D \sim E$, then there exists a $C > 0$ such that for all $m, q \in \mathbb{N}$,*

$$\left| h^q(mD)_{\mathcal{D}} - h^q(mD)_{\mathcal{E}} \right| \leqslant C m^{n-1} .$$

Proof. Let $D = \sum_{i=1}^{r} a_i D_i$ be the decomposition $\mathcal{D}$, and let $E = \sum_{j=1}^{s} b_j E_j$ be the decomposition $\mathcal{E}$. The images of $D_1, \dots, D_r, E_1, \dots, E_s$ in $\operatorname{Pic}(Y)$ generate a subgroup V. Let $\pi\colon \operatorname{Div}(Y) \to \operatorname{Pic}(Y)$ be the canonical homomorphism. Using $\sum_{i=1}^{r} a_i \pi(D_i) = \sum_{j=1}^{s} b_j \pi(E_j)$ in $V_{\mathbb{R}}$ and Lemma 3.4.3, we see that

$$\mathcal{S} := \left\{ \sum_{i=1}^{r} \lceil m a_i \rceil \pi(D_i) - \sum_{j=1}^{s} \lceil m b_j \rceil \pi(E_j) \,\middle|\, m \in \mathbb{N} \right\}$$

is a finite subset of $\operatorname{Pic}(Y)$. We fix representatives $G \in \operatorname{Div}(Y)$ of the elements in $\mathcal{S}$. Then (3.2) yields a constant C_G such that for all $m \in \mathbb{N}$,

$$\left| h^q\left(Y, \mathcal{O}_Y\left(\sum_{j=1}^{s} \lceil m b_j \rceil E_j + G \right) \right) - h^q\left(Y, \mathcal{O}_Y\left(\sum_{j=1}^{s} \lceil m b_j \rceil E_j \right) \right) \right| \leqslant C_G \left(1 + \sum_{j=1}^{s} \lceil m b_j \rceil \right)^{n-1} .$$

Using $h^q(mD)_{\mathcal{D}} = h^q\big(Y, \mathcal{O}_Y\big(\sum_{j=1}^{s} \lceil m b_j \rceil E_j + G \big)\big)$ for a suitable representative G and the finiteness of $\mathcal{S}$, we easily deduce the claim. □

Remark 3.4.5. We are interested in the asymptotics of $h^q(m_1 D_1 + \cdots + m_r D_r)_{\mathcal{D}}$ for real divisors $D_1, \dots, D_r$ with respect to decompositions $\mathcal{D}_k$ of D_k and $\mathcal{D} := \coprod_{k=1}^{r} \mathcal{D}_k$. An obvious generalization of Lemma 3.4.4 shows that this function depends only on the linear equivalence classes of $D_1, \dots, D_r$ and is independent of the choice of the decompositions $\mathcal{D}_k$ up to an error term of the form $O\big(m^{n-1}\big)$ for $m := \sum_{k=1}^r m_k$. We use the notation $h^q(m_1 D_1 + \cdots + m_r D_r)$, which is a well-defined function in $(m_1, \dots, m_r)$ up to $O\big(m^{n-1}\big)$.

DEFINITION 3.4.6. For $D \in \mathrm{Div}(Y)_{\mathbb{R}}$ and $0 \leqslant q \leqslant n$, we define

$$\widehat{h}^q(Y, D) := \limsup_m \frac{h^q(Y, mD)}{m^n/n!}\,.$$

By Remark 3.4.5, the value of $\widehat{h}^q(Y, D)$ depends only on the linear equivalence class of D and is independent of the decomposition chosen to calculate $h^q(Y, mD)$.

LEMMA 3.4.7. *Fix $D_1 \sim D_1', \dots, D_r \sim D_r', E_1, \dots, E_s \in \mathrm{Div}(Y)_{\mathbb{R}}$ and $q \in \mathbb{N}$. There exists a $C \in \mathbb{R}$ (depending on the linear equivalence classes of $D_1, \dots, D_r, E_1, \dots, E_s$) such that for all $m_1, \dots, m_r, p_1, \dots, p_s \in \mathbb{R}_{\geqslant 0}$ and for $m = \sum_{i=1}^r m_i$ and $p = \sum_{j=1}^s p_j$, we have*

$$\left| h^q\left(Y, \sum_{i=1}^r m_i D_i + \sum_{j=1}^s p_j E_j\right) - h^q\left(Y, \sum_{i=1}^r m_i D_i'\right)\right| \leqslant Cp(m+p)^{n-1} + O\big(d^{n-1}\big) \tag{3.15}$$

for $d := m + p + 1$ and

$$\left| \widehat{h}^q\left(Y, \sum_{i=1}^r m_i D_i + \sum_{j=1}^s p_j E_j\right) - \widehat{h}^q\left(Y, \sum_{i=1}^r m_i D_i'\right)\right| \leqslant n!Cp(m+p)^{n-1}\,. \tag{3.16}$$

Proof. The bound (3.15) follows directly from Proposition 3.1.2 after choosing decompositions of D_i and E_j for all i, j. Then (3.16) is an asymptotic consequence of (3.15). □

PROPOSITION 3.4.8. *For any $q \in \mathbb{N}$, the function $\widehat{h}^q$ is homogeneous of degree n on $\mathrm{Div}(Y)_{\mathbb{R}}$ and continuous on every finite-dimensional $\mathbb{R}$-subspace with respect to any norm.*

Proof. To prove the homogenity, we choose $\lambda > 0$. For every non-zero $m \in \mathbb{N}$, there are $k_m \in \mathbb{N}$ and $r_m \in \mathbb{R}$ with $m\lambda = k_m + r_m$ and $0 \leqslant r_m \leqslant 1$. By (3.15), we have

$$\big|h^q(Y, m\lambda D) - h^q(Y, k_m D)\big| \leqslant Cr_m(k_m + r_m)^{n-1} + O\big(m^{n-1}\big) = O\big(m^{n-1}\big)\,. \tag{3.17}$$

Dividing (3.17) by $m^n/n! = (k_m)^n/(n!\lambda^n) + O\big(m^{n-1}\big)$ and passing to the lim sup, we get

$$\widehat{h}^q(Y, \lambda D) \leqslant \lambda^n \widehat{h}^q(Y, D)\,.$$

Replacing D by $\lambda^{-1}D$, we get the reversed inequality for $\mu := \lambda^{-1}$ instead of λ. This proves the homogenity. The continuity on finite-dimensional subspaces follows from (3.16). □

Remark 3.4.9. If Y is a projective variety over the field k, we call $\widehat{h}^0(Y, D)$ the *volume* of $D \in \mathrm{Div}(Y)_{\mathbb{R}}$, extending the classical notion from Paragraph 3.3.1 to real Cartier divisors. Then we claim that the lim sup in the definition of vol is actually a limit, thus

$$\mathrm{vol}(D) = \lim_{m \to \infty} \frac{h^0(mD)}{m^n/n!}\,. \tag{3.18}$$

Proof. For $D \in \mathrm{Div}(Y)$, this follows from a result of S. D. Cutkosky [Cut14, Theorem 8.1]. For $D \in \mathrm{Div}(Y)_{\mathbb{Q}}$, there is a non-zero $e \in \mathbb{N}$ with eD represented by a Cartier divisor D' on Y. Applying the previous case to D' and using (3.15), we deduce that

$$\mathrm{vol}(D') = \lim_{k \to \infty} \frac{h^0(kD')}{k^n/n!} = \lim_{k \to \infty} \frac{h^0(kD' + rD)}{k^n/n!} = e^n \lim_{k \to \infty} \frac{h^0((ke+r)D)}{(ke+r)^n/n!}$$

for $r = 0, \dots, e-1$. By the homogenity of the volume, we get (3.18) for $D \in \mathrm{Div}(Y)_{\mathbb{Q}}$.

To prove the claim for $D \in \mathrm{Div}(Y)_{\mathbb{R}}$, we choose a finite-dimensional real subspace W which has a basis $D_1, \dots, D_r$ in $\mathrm{Div}(Y)_{\mathbb{Q}}$ and is such that $D \in W$. For $\varepsilon > 0$, pick $D' \in \mathrm{Div}(Y)_{\mathbb{Q}}$ with distance to D in W bounded by ε. By (3.15), there is a $C \in \mathbb{R}_{\geqslant 0}$ independent of ε and m with $h^0(Y, mD) - h^0(mD') \leqslant C\varepsilon m^n$. Then (3.18) for D' yields (3.18) for D. ☐

3.5. Asymptotic formulas for families of real divisors. In this subsection, Y is a projective variety over a field k. We will use the continuity of the asymptotic cohomological functions in Proposition 3.4.8 to derive asymptotic estimates for real divisors. Since we are using the asymptotic cohomological functions, we obtain only estimates up to $o\big(m^n\big)$ and not up to $O\big(m^{n-1}\big)$, but these will be enough for our applications.

PROPOSITION 3.5.1. *For $D_1, \dots, D_r \in \mathrm{Div}(Y)_{\mathbb{R}}$, there is a $\rho\colon \mathbb{N} \to \mathbb{R}_{\geqslant 0}$ with $\rho(m) = o\big(m^n\big)$ for $m \to \infty$ such that for all non-zero $m_1, \dots, m_r \in \mathbb{N}$ and $m := \sum_{i=1}^r m_i$, we have*

$$h^q\left(Y, \sum_{i=1}^r m_i D_i\right) \leqslant \frac{m^n}{n!}\widehat{h}^q\left(Y, \sum_{i=1}^r \frac{m_i}{m} D_i\right) + \rho(m) \tag{3.19}$$

and for $q = 0$, we even have $\left|h^0\big(Y, \sum_{i=1}^r m_i D_i\big) - (1/n!)\,\mathrm{vol}\,\big(\sum_{i=1}^r m_i D_i\big)\right| \leqslant \rho(m)$.

Proof. Let us prove the proposition by contradiction. Then there are $\alpha > 0$ and some sequences $(m_{i,k})_{k\in\mathbb{N}}$ in $\mathbb{N} \setminus \{0\}$ for $i = 1, \dots, r$ such that $m_k := \sum_{i=1}^r m_{i,k} \to \infty$ and

$$h^q\left(Y, \sum_{i=1}^r m_{i,k} D_i\right) - \frac{m_k^n}{n!}\widehat{h}^q\left(Y, \sum_{i=1}^r \frac{m_{i,k}}{m_k} D_i\right) \geqslant \alpha m_k^n\,. \tag{3.20}$$

When $q = 0$, we replace the left side by its absolute value. Since for each i and k, we get $m_{i,k}/m_k \in [0,1]$, by compactness and up to considering subsequences, we may assume $\lim\limits_{k\to\infty} m_{i,k}/m_k = c_i \in [0,1]$. For $k \gg 0$, the continuity of $\widehat{h}^q$ given in (3.16) yields

$$h^q\left(Y, \sum_{i=1}^r m_{i,k} D_i\right) - \frac{m_k^n}{n!}\widehat{h}^q\left(Y, \sum_{i=1}^r c_i D_i\right) > \frac{\alpha}{2} m_k^n\,. \tag{3.21}$$

When $q = 0$, this holds again with the absolute value of the left-hand side. Using that $m_{i,k} = m_k c_i + (m_{i,k} - m_k c_i)$, Lemma 3.4.7 gives a $C \geqslant 0$ such that for all $k \in \mathbb{N}$,

$$\left|h^q\left(Y, \sum_{i=1}^r m_{i,k} D_i\right) - h^q\left(Y, \sum_{i=1}^r m_k c_i D_i\right)\right| \leqslant C\left(\sum_{i=1}^r |m_{i,k} - m_k c_i|\right) \cdot m_k^{n-1} + O\big(m_k^{n-1}\big)\,.$$

Since $m_{i,k}/m_k \xrightarrow[k]{} c_i$, it follows that $\sum_{i=1}^r |m_{i,k} - m_k c_i| = o(m_k)$ always for $k \to \infty$. Hence,

$$\left|h^q\left(Y, \sum_{i=1}^r m_{i,k} D_i\right) - h^q\left(Y, \sum_{i=1}^r m_k c_i D_i\right)\right| = o\big(m_k^n\big) \tag{3.22}$$

for $k \to \infty$. By Definition 3.4.6 for $\widehat{h}^q$ and using $\sum_{i=1}^r m_k c_i D_i = m_k\big(\sum_{i=1}^r c_i D_i\big)$, we get

$$h^q\left(Y, \sum_{i=1}^r m_k c_i D_i\right) - \frac{m_k^n}{n!}\widehat{h}^q\left(Y, \sum_{i=1}^r c_i D_i\right) \leqslant o\big(m_k^n\big) \tag{3.23}$$

for $k \to \infty$. In case $q = 0$, the $\limsup$ in the definition of $\mathrm{vol} = \widehat{h}^0$ is a limit (see Remark 3.4.9), and then (3.23) holds with the absolute value of the left side. Combining (3.22) with (3.23), we get a contradiction to (3.21). This proves the proposition. ☐

3.6. Asymptotic formulas in the non-reduced case. We fix the following notation for this subsection. The base is $S = \operatorname{Spec}(R)$ for a noetherian ring R, and Y is a projective scheme over S. We consider a coherent sheaf $\mathcal{F}$ on Y with support over a zero-dimensional subscheme of S. Let $n \coloneqq \dim(\operatorname{supp}(\mathcal{F}))$, and let $\{E_i\}_{i\in I}$ be the set of n-dimensional irreducible components of $\operatorname{supp}(\mathcal{F})$. For each $i \in I$, let $\ell_i \coloneqq \ell_{\mathcal{O}_{Y,\xi_i}}(\mathcal{F}_{\xi_i})$, where ξ_i is the generic point of E_i.

We also fix Cartier divisors $D_1, \ldots, D_r$. For $i_1, \ldots, i_n \in \{0, \ldots, r\}$, we will use the intersection numbers

$$D_{i_1} \cdots D_{i_n} \cdot \mathcal{F} = \sum_{i\in I} \ell_i D_{i_1} \cdots D_{i_n} \cdot E_i \tag{3.24}$$

from [Kol96, § VI.2]. We start with an asymptotic formula for the Euler characteristic χ.

Proposition 3.6.1. *With the above notation, we have*

$$\chi\left(Y, \mathcal{F}\left(\sum_{i=1}^r m_i D_i\right)\right) = \frac{1}{n!}\left(\sum_{i=1}^r m_i D_i\right)^n \cdot \mathcal{F} + O\big(m^{n-1}\big)\,.$$

Proof. This follows from [Kol96, Theorem VI.2.13] using the definition of intersection numbers in [Kol96, Definition VI.2.6]. ☐

Proposition 3.6.2. *For $q \in \mathbb{N}$, there is a $\rho \colon \mathbb{N} \to \mathbb{R}_{\geqslant 0}$ with $\rho(m) = o(m^n)$ such that for all $m_1, \ldots, m_r \in \mathbb{N} \setminus \{0\}$ and $m \coloneqq \sum_{j=1}^r m_j$, we have*

$$h^q\left(Y, \mathcal{F}\left(\sum_{j=1}^r m_j D_j\right)\right) \leqslant \frac{1}{n!} \sum_{i\in I} \ell_i \widehat{h}^q\left(E_i, \mathcal{O}\left(\sum_{j=1}^r m_j D_j\right)\bigg|_{E_i}\right) + \rho(m)\,.$$

Proof. By assumption, E_i lies over a closed point x_i of S, and hence we may view E_i as a projective variety over the residue field of x_i. The result now follows from Lemma 3.2.3 and Proposition 3.5.1. ☐

Corollary 3.6.3. *If $D_1, \ldots, D_r$ are nef and $q \geqslant 1$, then there are functions $\rho_i \colon \mathbb{N} \to \mathbb{R}_{\geqslant 0}$ $(i = 1, 2)$ with $\rho_i(m) = o(m^n)$ such that for all $m_1, \ldots, m_r \in \mathbb{N} \setminus \{0\}$ and $m \coloneqq \sum_{j=1}^r m_j$, we have*

$$h^q\left(Y, \mathcal{F}\left(\sum_{j=1}^r m_j D_j\right)\right) = \rho_1(m)$$

and

$$h^0\left(Y, \mathcal{F}\left(\sum_{j=1}^r m_j D_j\right)\right) = \frac{1}{n!}\left(\sum_{j=1}^r m_j D_j\right)^n \cdot \mathcal{F} + \rho_2(m)\,.$$

Proof. Again, we may view any E_i as a projective variety over a suitable field. Note that the asymptotic Riemann–Roch formula in [Kol96, Theorem VI.2.15] yields $\widehat{h}^q(E_i, D) = 0$ for any nef divisor D on E_i, and hence the first claim follows from Proposition 3.6.2. The second claim follows from the first claim and Proposition 3.6.1. ☐

4. Non-archimedean volumes and energy

In this section, K is a discretely valued complete field with $-\log(|\pi|) = 1$ for a uniformizer π. We consider a projective variety X over K of dimension n with a line bundle L. All metrics on line bundles are assumed to be continuous. The length of a K°-module M is denoted by $\ell(M)$. We will use the algebraic volume $\mathrm{vol}(L)$ from Paragraph 3.3.1.

4.1. Non-archimedean volumes

DEFINITION 4.1.1. If V is a finite-dimensional K-vector space, a *lattice of V* is a free K°-submodule of $\Lambda \subset V$ with K-span V. If $\Lambda_2 \subset \Lambda_1 \subset V$ are lattices of V, then $\ell(\Lambda_1/\Lambda_2)$ is finite since Λ_1/Λ_2 is a finitely generated torsion K°-module. If Λ_1 and Λ_2 are any lattices of V, we choose a lattice Λ_3 contained in both Λ_1 and Λ_2, and we set

$$\ell(\Lambda_1/\Lambda_2) = \ell(\Lambda_1/\Lambda_3) - \ell(\Lambda_2/\Lambda_3) \in \mathbb{Z}\,.$$

This is independent of the choice of Λ_3. Observe that $\ell(\Lambda_1/\Lambda_2)$ might become negative. Recall from Paragraph 2.2.1 that $\widehat{H}^0(X, L, \|\ \|) := \left\{s \in H^0(X, L) \mid \|s\|_{\sup} \leqslant 1\right\}$ is a lattice of $H^0(X, L)$.

DEFINITION 4.1.2. If $\|\ \|_1$ and $\|\ \|_2$ are two metrics on L^{an}, we define the *non-archimedean volume of L with respect to $\|\ \|_1$ and $\|\ \|_2$* by

$$\mathrm{vol}(L, \|\ \|_1, \|\ \|_2) = \limsup_{m\to\infty} \frac{n!}{m^{n+1}} \cdot \ell\left(\frac{\widehat{H}^0(X, L^{\otimes m}, \|\ \|_1^{\otimes m})}{\widehat{H}^0(X, L^{\otimes m}, \|\ \|_2^{\otimes m})}\right).$$

Often, we will write $\mathrm{vol}(\|\ \|_1, \|\ \|_2)$ instead of $\mathrm{vol}(L, \|\ \|_1, \|\ \|_2)$. For the following result, recall that we have $|\pi|^{-1} = \exp(1)$ by our normalization of the valuation on K.

LEMMA 4.1.3. *For $t \in \mathbb{R}$, we have*

$$\mathrm{vol}\left(L, e^{-t}\|\ \|_1, \|\ \|_2\right) = \mathrm{vol}\left(L, \|\ \|_1, e^{t}\|\ \|_2\right) = t\,\mathrm{vol}(L) + \mathrm{vol}(L, \|\ \|_1, \|\ \|_2)\,.$$

Proof. Note that $M_m := \widehat{H}^0\big(X, L^{\otimes m}, \|\ \|_1^{\otimes m}\big)$ and $M'_m := \widehat{H}^0\big(X, L^{\otimes m}, \|\ \|_2^{\otimes m}\big)$ are free K°-modules of the same rank $h^0\big(X, L^{\otimes m}\big)$. We first assume $t = k \in \mathbb{Z}$. Then the additivity of the length and $\widehat{H}^0\big(X, L^{\otimes m}, e^{-km}\|\ \|_1^{\otimes m}\big) = \pi^{-km} M_m$ show that

$$\ell\left(\widehat{H}^0\big(X, L^{\otimes m}, e^{-km}\|\ \|_1^{\otimes m}\big)/M'_m\right) = kmh^0\big(X, L^{\otimes m}\big) + \ell(M_m/M'_m)\,. \tag{4.1}$$

By Paragraph 3.3.1, we have

$$\mathrm{vol}(L) = \lim_{m\to\infty} \frac{h^0(X, L^{\otimes m})}{m^n/n!}\,,$$

and $\mathrm{vol}\left(L, e^{-k}\|\ \|_1, \|\ \|_2\right) = k\,\mathrm{vol}(L) + \mathrm{vol}(L, \|\ \|_1, \|\ \|_2)$ follows from (4.1) and the definition of the non-archimedean volumes. Similarly, we prove the other equality.

If $t \notin \mathbb{Z}$, then $\pi^{-\lfloor tm \rfloor} M_m \subset \widehat{H}^0\big(X, L^{\otimes m}, e^{-tm}\|\ \|_1^{\otimes m}\big) \subset \pi^{-\lceil tm \rceil} M_m$, and the claim follows from a sandwich argument similarly as above. □

PROPOSITION 4.1.4. *For metrics $\|\ \|_1$ and $\|\ \|_2$ on L^{an}, we have the following properties:*

(a) *The volume $\mathrm{vol}(\|\ \|_1, \|\ \|_2)$ is monotone decreasing in $\|\ \|_1$ and monotone increasing in $\|\ \|_2$.*

(b) *The volume $\mathrm{vol}(\|\ \|_1, \|\ \|_2)$ is finite and continuous in $(\|\ \|_1, \|\ \|_2)$.*

Proof. Property (a) is obvious. The finiteness in (b) and the inequality

$$\left|\operatorname{vol}(\| \;\|_1', \| \;\|_2) - \operatorname{vol}(\| \;\|_1, \| \;\|_2)\right| \leqslant \operatorname{vol}(L) d(\| \;\|_1, \| \;\|_1') \tag{4.2}$$

for any metric $\| \;\|_1'$ on L^{an} follow from an easy sandwich argument based on property (a) and Lemma 4.1.3, where d is the distance from Paragraph 2.2.2. Just as in (4.2), the absolute difference $|\operatorname{vol}(\| \;\|_1', \| \;\|_2) - \operatorname{vol}(\| \;\|_1, \| \;\|_2)|$ is bounded by $\operatorname{vol}(L) d(\| \;\|_2, \| \;\|_2')$, and hence the continuity in property (b) follows. □

Lemma 4.1.5. *Let L and M be line bundles on X. Then we have*

$$\limsup_{m\to\infty} \left| \frac{n!}{m^n} \cdot \ell\left(\frac{\widehat{H}^0(X, M \otimes L^{\otimes m}, \| \;\|_1 \otimes \| \;\|^{\otimes m})}{\widehat{H}^0(X, M \otimes L^{\otimes m}, \| \;\|_2 \otimes \| \;\|^{\otimes m})} \right) \right| \leqslant \operatorname{vol}(L) d(\| \;\|_1, \| \;\|_2)$$

for any metrics $\| \;\|$ on L^{an} and $\| \;\|_1, \| \;\|_2$ on M^{an}.

Proof. This is a twisted variant of (4.2) which follows along the same lines. □

Remark 4.1.6. Let L be a line bundle on X which is not big. By definition, this means that $\operatorname{vol}(L) = 0$. It follows easily from Lemma 4.1.3, Proposition 4.1.4 and a sandwich argument that $\operatorname{vol}(L, \| \;\|_1, \| \;\|_2) = 0$ for all continuous metrics $\| \;\|_1$ and $\| \;\|_2$ on L^{an}.

Remark 4.1.7. Let us describe how the *non-archimedean volume* is related to the *χ-arithmetic volume* which is studied in Arakelov theory. The precise relation is given in formula (4.4) below. We assume in this remark that F is a number field with ring of integers $\mathcal{O}_F$ and with set of places M_F.

Let L be a line bundle on an n-dimensional projective variety X over F endowed with an adelic metric, which means that we have a continuous metric $\| \;\|_w$ on $L \otimes_F F_w$ for the completion F_w of any $w \in M_F$ and that we assume that there is a finite set S of $\operatorname{Spec}(\mathcal{O}_F)$ such that the metric $\| \;\|_w$ is induced by a single model of (X, L) over $\operatorname{Spec}(\mathcal{O}_F) \setminus S$ for all non-archimedean places $w \notin S$. We denote the resulting metrized line bundle by $\overline{L}$, and we set $E := H^0(X, L)$. For $w \in M_F$, let B_w be the unit ball in $E \otimes_F F_w = H^0(X \otimes_F F_w, L \otimes_F F_w)$ with respect to the sup norm. Observe that B_w is a finitely generated F_w°-module. We note that $\Lambda := \bigcap_{w \text{ finite}} B_w \cap E$ is a lattice in $E \otimes_{\mathbb{Q}} \mathbb{R} = \prod_{w|\infty} H^0(X \otimes_F F_w, L \otimes_F F_w)$ (see [BG06, Proposition C.2.6]), and we set

$$\chi(X, \overline{L}) := \log\left(\frac{\operatorname{vol}\left(\prod_{w|\infty} B_w\right)}{\operatorname{covol}(\Lambda)} \right),$$

where the volume and the covolume are computed with respect to the same Haar measure on $E \otimes_{\mathbb{Q}} \mathbb{R}$. If the adelic metric is induced by a normal $\mathcal{O}_F$-model $(\mathscr{X}, \mathscr{L})$ of (X, L), then $\Lambda = H^0(\mathscr{X}, \mathscr{L})$ (see Lemma 2.2.4 and Remark 2.2.7).

Now we assume that L is ample. Then we have the *χ-arithmetic volume*

$$\widehat{\operatorname{vol}}_\chi(X, \overline{L}) := \limsup_{m\to\infty} \frac{(n+1)!}{m^{n+1}} \chi\left(X, \overline{L}^{\otimes m}\right)$$

considered in Arakelov theory. It agrees with the *logarithm of the sectional capacity* studied in the book of R. Rumely, C. F. Lau and R. Varley [RLV00]. It follows from [RLV00, Theorem B] that the lim sup in the definition is actually a limit. S. Zhang's extension [Zha95a, Theorem 1.4] of the arithmetic Hilbert–Samuel formula of Gillet–Soulé shows that $\widehat{\operatorname{vol}}_\chi(X, \overline{L})$ is finite in the case of a semipositive adelic metric, and hence the continuity argument in [CT09, § 5] shows that $\widehat{\operatorname{vol}}_\chi(X, \overline{L}) \in \mathbb{R}$ is finite for any adelic metric on the ample line bundle L.

Let us now fix a non-archimedean place v of F. We consider two continuous metrics $\| \;\|_v$ and $\| \;\|_v'$ on $L \otimes_F F_v$ at the fixed non-archimedean place v inducing unit balls B_v and B_v' in

$H^0(X \otimes_F F_v, L \otimes_F F_v)$ with respect to the sup norms. We extend the metrics to adelically metrized line bundles $\overline{L}$ and $\overline{L}'$ using the same metrics $\| \ \|_w$ for all places $w \neq v$. From Arakelov theory on the arithmetic curve $\operatorname{Spec}(\mathcal{O}_F)$, we get the formula

$$\chi(X, \overline{L}) - \chi(X, \overline{L}') = \log\left(\#\widetilde{F_v}\right) \cdot \ell_{F_v^\circ}\left(B_v / B_v'\right), \tag{4.3}$$

which holds without assuming L ample and which can be deduced from the Riemann–Roch formula given in [Gau08] before Lemma 4.2. If L is ample, then we apply (4.3) for $\overline{L}^{\otimes m}$ and $\overline{L}'^{\otimes m}$, multiply it with $(n+1)!/m^{n+1}$ and pass to the limit. This proves the formula

$$\widehat{\operatorname{vol}}_\chi(X, \overline{L}) - \widehat{\operatorname{vol}}_\chi(X, \overline{L}') = (n+1) \log\left(\#\widetilde{F_v}\right) \cdot \operatorname{vol}\left(L, \| \ \|_v, \| \ \|_v'\right), \tag{4.4}$$

which describes the non-archimedean volume in the number field case as a *localized* χ-arithmetic volume.

Remark 4.1.8. We conjecture that the lim sup in the definition of $\operatorname{vol}(L, \| \ \|_1, \| \ \|_2)$ is always a limit. In the case of a non-archimedean completion K of a number field F, with X and L defined over F and with L ample, this follows from the argument deducing (4.4) from (4.3). In Theorem 4.2.3 and in Corollary 6.2.2, we will prove special cases of the conjecture.

A referee pointed out that a result of H. Chen and C. Maclean [CM15, Corollary 4.6] proves this conjecture if the projective variety X contains a K-rational regular point. Indeed, the arguments in Remark 4.1.6 show that we may assume L big. By [CM15, Footnote 10 on p. 388], the conditions (a)–(c) in [CM15, § 4.3, p. 385] are satisfied for the complete graded linear system induced by $\left(H^0\left(X, L^{\otimes m}\right)\right)_{m \in \mathbb{N}}$ and hence we may apply [CM15, Corollary 4.6] to get the existence of the limit in the definition of $\operatorname{vol}(L, \| \ \|_1, \| \ \|_2)$. For this last step, one has to ensure that the considered sequence in [CM15, Corollary 4.6] is asymptotically equal to

$$\left(\frac{1}{m h^0(X, L^{\otimes m})} \cdot \ell\left(\frac{\widehat{H}^0\left(X, L^{\otimes m}, \| \ \|_1^{\otimes m}\right)}{\widehat{H}^0\left(X, L^{\otimes m}, \| \ \|_2^{\otimes m}\right)}\right)\right)_{m \in \mathbb{N} \setminus \{0\}},$$

which follows from [BE18, Proposition 2.21]. To apply the latter, we note that the determinant norms in [CM15, § 3] and in [BE18, § 2] agree as all ultrametric norms on a finite-dimensional K-vector space are diagonalizable [BE18, Example 1.12] and we have the same concrete formula for diagonalizable norms. We thank the referee for pointing us to the reference [CM15, Corollary 4.6].

Note also that the existence of the limit in [CM15, Corollary 4.6] holds over any not necessarily discretely valued complete non-archimedean field K.

4.2. Volumes and semipositive metrics.

In this subsection, we consider a normal projective variety X over the complete discretely valued field K.

If M is a K°-module and $a \in K^\circ$, we set

$$M_{a-\mathrm{tor}} = \{m \in M \mid am = 0\}.$$

Lemma 4.2.1. *Let M be a K°-module of finite type. For any $\alpha \in \mathbb{N}$, we have*

$$\ell(M_{\pi^\alpha-\mathrm{tor}}) \leqslant \ell\left(M/\pi^\alpha M\right).$$

Proof. This follows from the classification of modules of finite type over a principal ideal domain. □

Recall from (2.7) that we have defined the energy $E(L, \| \ \|_1, \| \ \|_2)$ of continuous semipositive metrics $\| \ \|_1$ and $\| \ \|_2$ on a line bundle L over X. The following proposition is our key point to interpret the energy as a non-archimedean volume.

Proposition 4.2.2. *Let L be a line bundle on X, and let $\mathscr{X}$ be a normal model of X. We consider nef models $\mathscr{L}_1$ and $\mathscr{L}_2$ of L, and we write $\mathscr{L}_1 \otimes \mathscr{L}_2^{-1} = \mathcal{O}(D)$ for some vertical Cartier divisor D on $\mathscr{X}$. In addition, let $\mathcal{M}$ be a line bundle on $\mathscr{X}$ with generic fiber $M := \mathcal{M}_{|X}$. Then we have*

$$E(L, \| \ \|_{\mathscr{L}_1}, \| \ \|_{\mathscr{L}_2}) = \lim_{m\to 0} \frac{n!}{m^{n+1}}\, \ell\left(\frac{\widehat{H}^0\big(X, M\otimes L^{\otimes m}, \| \ \|_{\mathcal{M}} \otimes \| \ \|_{\mathscr{L}_1}^{\otimes m}\big)}{\widehat{H}^0\big(X, M\otimes L^{\otimes m}, \| \ \|_{\mathcal{M}} \otimes \| \ \|_{\mathscr{L}_2}^{\otimes m}\big)}\right).$$

Proof. First, we reduce the claim to the case when D is an effective vertical Cartier divisor. There is a $k \in \mathbb{N}$ such that $D' := \operatorname{div}(\pi^k) + D$ is an effective Cartier divisor and for $\mathscr{L}_1' := \mathscr{L}_1\big(\operatorname{div}(\pi^k)\big) \simeq \mathscr{L}_1$, we get $\mathcal{O}(D') = \mathscr{L}_1' \otimes \mathscr{L}_2^{-1}$. Note that $\mathscr{L}_1'$ is still nef and $\| \ \|_{\mathscr{L}_1'} = |\pi|^k \| \ \|_{\mathscr{L}_1}$. Using the definition of the energy and property 2.4.3(i), we get

$$E\big(L, \| \ \|_{\mathscr{L}_1'}, \| \ \|_{\mathscr{L}_2}\big) = kL^n + E\big(L, \| \ \|_{\mathscr{L}_1}, \| \ \|_{\mathscr{L}_2}\big).$$

The same argument as for (4.1) and then Paragraph 3.3.1 and Remark 3.3.2 yield

$$\ell\left(\frac{\widehat{H}^0\big(X, M\otimes L^{\otimes m}, \| \ \|_{\mathcal{M}} \otimes \| \ \|_{\mathscr{L}_1'}^{\otimes m}\big)}{\widehat{H}^0\big(X, M\otimes L^{\otimes m}, \| \ \|_{\mathcal{M}} \otimes \| \ \|_{\mathscr{L}_1}^{\otimes m}\big)}\right) = kmh^0\big(X, M\otimes L^{\otimes m}\big) \underset{m\to+\infty}{\sim} k\frac{m^{n+1}}{n!}L^n.$$

Hence, the claim for D' implies the claim for D, and we can replace D by D'.

So we may assume that D is an effective vertical Cartier divisor. Let $s_D \in \Gamma(\mathscr{X}, \mathcal{O}(D))$ denote the canonical global section of $\mathcal{O}(D)$. Note that $\operatorname{div}(s_D) = D$. Let φ_D denote the model function associated with D. For $j \in \{0, \dots, m\}$, we use the notation

$$\mathcal{F}_j^{(m)} := \mathcal{M} \otimes \mathscr{L}_1^{\otimes j} \otimes \mathscr{L}_2^{\otimes m-j}. \tag{4.5}$$

For $j \in \{1, \dots, m\}$, we consider the short exact sequence

$$0 \to \mathcal{F}_{j-1}^{(m)} \xrightarrow{\otimes s_D} \mathcal{F}_j^{(m)} \longrightarrow \mathcal{F}_j^{(m)}\big|_D \longrightarrow 0. \tag{4.6}$$

The associated long exact sequence in cohomology gives

$$0 \longrightarrow \Gamma\big(\mathscr{X}, \mathcal{F}_{j-1}^{(m)}\big) \xrightarrow{\otimes s_D} \Gamma\big(\mathscr{X}, \mathcal{F}_j^{(m)}\big) \longrightarrow \Gamma\big(D, \mathcal{F}_j^{(m)}\big) \longrightarrow H^1\big(\mathscr{X}, \mathcal{F}_{j-1}^{(m)}\big) \longrightarrow \cdots. \tag{4.7}$$

Let us pick $\alpha \in \mathbb{N}$ such that $0 \leqslant \varphi_D \leqslant \alpha$. Because $\mathscr{X}$ is normal, Proposition 2.2.6 yields $\pi^\alpha \in \mathcal{J}_D$, where $\mathcal{J}_D$ is the ideal sheaf of the closed subscheme D. Hence, D is in a natural way a scheme of finite type over $S := \operatorname{Spec}(K^\circ/\pi^\alpha K^\circ)$. The K°-module $\Gamma(D, \mathcal{F}_j^{(m)})$ is π^α-torsion as $\pi^\alpha \in \mathcal{J}_D$. Since the restrictions of $\mathscr{L}_1$ and $\mathscr{L}_2$ to $\mathscr{X}_S := \mathscr{X} \times_{K^\circ} S$ are nef, Corollary 3.6.3 yields that

$$\ell\big(H^1\big(\mathscr{X}_S, \mathcal{F}_{j-1}^{(m)}\big)\big) = o\big(m^n\big). \tag{4.8}$$

From the short exact sequence

$$0 \longrightarrow \mathcal{F}_{j-1}^{(m)} \xrightarrow{\cdot\pi^\alpha} \mathcal{F}_{j-1}^{(m)} \longrightarrow \mathcal{F}_{j-1}^{(m)}\big|_{\mathscr{X}_S} \longrightarrow 0,$$

we get the exact sequence

$$H^1\big(\mathscr{X}, \mathcal{F}_{j-1}^{(m)}\big) \xrightarrow{\cdot\pi^\alpha} H^1\big(\mathscr{X}, \mathcal{F}_{j-1}^{(m)}\big) \longrightarrow H^1\big(\mathscr{X}_S, \mathcal{F}_{j-1}^{(m)}\big),$$

and hence the induced homomorphism

$$H^1\big(\mathscr{X}, \mathcal{F}_{j-1}^{(m)}\big)/\pi^\alpha H^1\big(\mathscr{X}, \mathcal{F}_{j-1}^{(m)}\big) \hookrightarrow H^1\big(\mathscr{X}_S, \mathcal{F}_{j-1}^{(m)}\big)$$

is injective. Together with Lemma 4.2.1 and (4.8), this shows that

$$\ell\big(H^1\big(\mathscr{X}, \mathcal{F}_{j-1}^{(m)}\big)_{\pi^\alpha-\text{tors}}\big) = o\big(m^n\big). \tag{4.9}$$

Then (4.7) and (4.9) show that

$$\ell\left(\Gamma(\mathscr{X},\mathcal{F}_j^{(m)})/\Gamma(\mathscr{X},\mathcal{F}_{j-1}^{(m)})\right)=\ell\left(\Gamma(D,\mathcal{F}_j^{(m)})\right)+o(m^n)\,. \tag{4.10}$$

Let D_1 be a Cartier divisor with $\mathscr{L}_1=\mathcal{O}(D_1)$ and $D_2:=D_1-D$. Observing (3.24), Corollary 3.6.3 gives

$$\ell\left(\Gamma(D,\mathcal{F}_j^{(m)})\right)=\frac{m^n}{n!}\left(\frac{j}{m}D_1+\left(1-\frac{j}{m}\right)D_2\right)^n\cdot D+o(m^n)\,. \tag{4.11}$$

It follows from Lemma 2.2.4 that $\widehat{H}^0(X,M\otimes L^{\otimes m},\|\ \|_{\mathcal{M}}\otimes\|\ \|_{\mathscr{L}_1}^{\otimes m})=\Gamma(\mathscr{X},\mathcal{F}_m^{(m)})$ and $\widehat{H}^0(X,M\otimes L^{\otimes m},\|\ \|_{\mathcal{M}}\otimes\|\ \|_{\mathscr{L}_2}^{\otimes m})=\Gamma(\mathscr{X},\mathcal{F}_0^{(m)})$. Hence, we have to show that

$$\frac{1}{n!}E(L,\|\ \|_{\mathscr{L}_1},\|\ \|_{\mathscr{L}_2})=\lim_{m\to\infty}\frac{1}{m^{n+1}}\,\ell\left(\Gamma(\mathscr{X},\mathcal{F}_m^{(m)})/\Gamma(\mathscr{X},\mathcal{F}_0^{(m)})\right). \tag{4.12}$$

The additivity of the length, (4.10) and (4.11) yield

$$\frac{1}{m^{n+1}}\,\ell\left(\Gamma(\mathscr{X},\mathcal{F}_m^{(m)})/\Gamma(\mathscr{X},\mathcal{F}_0^{(m)})\right)=\frac{1}{n!\,m}\sum_{j=1}^{m}\left(\frac{j}{m}D_1+\left(1-\frac{j}{m}\right)D_2\right)^n\cdot D+o(1)\,.$$

The limit for $m\to\infty$ exists and is given by the sum of Riemann integrals

$$\frac{1}{n!}\int_0^1\left(tD_1+(1-t)D_2\right)^n\cdot D\,dt=\frac{1}{n!}\sum_{k=0}^{n}\binom{n}{k}\int_0^1 t^k(1-t)^{n-k}dt\,D_1^k\cdot D_2^{n-k}\cdot D\,.$$

Using the identity $\int_0^1(1-t)^kt^{n-k}dt=\left((n+1)\binom{n}{k}\right)^{-1}$, we get

$$\lim_{m\to\infty}\frac{1}{m^{n+1}}\,\ell\left(\Gamma(\mathscr{X},\mathcal{F}_m^{(m)})/\Gamma(\mathscr{X},\mathcal{F}_0^{(m)})\right)=\frac{1}{n!}\frac{1}{n+1}\sum_{k=0}^{n}D_1^k\cdot D_2^{n-k}\cdot D\,,$$

and hence (4.12) follows from (2.8). ☐

THEOREM 4.2.3. *Let L be a line bundle on the normal projective variety X, and let $\|\ \|_1$ and $\|\ \|_2$ be continuous semipositive metrics on L^{an}. Then we have*

$$\mathrm{vol}\left(L,\|\ \|_1,\|\ \|_2\right)=E\left(L,\|\ \|_1,\|\ \|_2\right). \tag{4.13}$$

Furthermore, under our assumptions, the lim sup *in the definition of the non-archimedean volume is a limit.*

Proof. In the following, let $\varphi:=-\log\left(\|\ \|_1/\|\ \|_2\right)$. We first prove the claim for semipositive model metrics. Then there exist an integer $k\in\mathbb{N}$ and nef models $\mathcal{N}_1$ and $\mathcal{N}_2$ of the line bundle $N:=L^{\otimes k}$ such that $\|\ \|_1^{\otimes k}=\|\ \|_{\mathcal{N}_1}$ and $\|\ \|_2^{\otimes k}=\|\ \|_{\mathcal{N}_2}$. We fix some $r\in\{0,\dots,k-1\}$ which will play the role of the remainder in the Euclidean division by k. Moreover, we fix a model $\mathcal{M}$ of $L^{\otimes r}$. To have all our models of line bundles defined on the same normal model $\mathscr{X}$, we pass to a common finer model. There is now a vertical Cartier divisor D on $\mathscr{X}$ such that $\mathcal{O}(D)=\mathcal{N}_1\otimes\mathcal{N}_2^{-1}$. Note that we have $\varphi_D=k\varphi$.

Then it is enough to study the arithmetic progression made of the integers m of the form $m=kq+r$ for $q\in\mathbb{N}$. By Lemma 4.1.5, we note that both

$$\ell\left(\frac{\widehat{H}^0(X,L^{\otimes m},\|\ \|_1^{\otimes m})}{\widehat{H}^0(X,L^{\otimes r}\otimes L^{\otimes kq},\|\ \|_{\mathcal{M}}\otimes\|\ \|_1^{\otimes kq})}\right)\quad\text{and}\quad\ell\left(\frac{\widehat{H}^0(X,L^{\otimes r}\otimes L^{\otimes kq},\|\ \|_{\mathcal{M}}\otimes\|\ \|_2^{\otimes kq})}{\widehat{H}^0(X,L^{\otimes m},\|\ \|_2^{\otimes m})}\right)$$

equal $O(q^n)$. Using this together with the additivity of the length and $\| \ \|_i^{\otimes k} = \| \ \|_{\mathcal{N}_i}$, we get

$$\ell\left(\frac{\widehat{H}^0\big(X, L^{\otimes m}, \| \ \|_1^{\otimes m}\big)}{\widehat{H}^0\big(X, L^{\otimes m}, \| \ \|_2^{\otimes m}\big)}\right) = \ell\left(\frac{\widehat{H}^0\big(X, L^{\otimes r} \otimes L^{\otimes kq}, \| \ \|_{\mathcal{M}} \otimes \| \ \|_{\mathcal{N}_1}^{\otimes q}\big)}{\widehat{H}^0\big(X, L^{\otimes r} \otimes L^{\otimes kq}, \| \ \|_{\mathcal{M}} \otimes \| \ \|_{\mathcal{N}_2}^{\otimes q}\big)}\right) + O(q^n)\,.$$

By Proposition 4.2.2, the equality $\varphi_D = k\varphi$ and the homogeneity of the energy, we deduce

$$\begin{aligned}\ell\left(\frac{\widehat{H}^0\big(X, L^{\otimes m}, \| \ \|_1^{\otimes m}\big)}{\widehat{H}^0\big(X, L^{\otimes m}, \| \ \|_2^{\otimes m}\big)}\right) &= \frac{q^{n+1}}{n!} E\big(L^{\otimes k}, \| \ \|_{\mathcal{N}_1}, \| \ \|_{\mathcal{N}_2}\big) + o\big((kq)^{n+1}\big) \\ &= \frac{q^{n+1}k^{n+1}}{n!} E(L, \| \ \|_1, \| \ \|_2) + o\big((kq)^{n+1}\big) \\ &= \frac{m^{n+1}}{n!} E(L, \| \ \|_1, \| \ \|_2) + o\big(m^{n+1}\big)\end{aligned}$$

along the arithmetic progression $(m = kq + r)_{q \in \mathbb{N}}$. This proves the claim for model metrics.

Arbitrary continuous semipositive metrics on L^{an} are uniform limits of semipositive model metrics on L^{an}. Then the formula in the theorem follows from the first case as both the non-archimedean volume and the Chambert-Loir measure are continuous in $(\| \ \|_1, \| \ \|_2)$ (see Proposition 4.1.4 and Paragraph 2.4.3).

It remains to see that the lim sup in the definition of the non-archimedean volume is a limit. We choose a rational number $\varepsilon > 0$. For $i = 1, 2$, there is a semipositive model metric $\| \ \|_i'$ on L^{an} with distance to $\| \ \|_i$ bounded by ε and hence $e^{-\varepsilon}\| \ \|_i' \leqslant \| \ \|_i \leqslant e^{\varepsilon}\| \ \|_i'$. As the $e^{\pm\varepsilon}\| \ \|_i'$ are semipositive model metrics, we deduce easily from a sandwich argument, from the first case and using $\varepsilon \to 0$ that the lim sup is a limit. □

Remark 4.2.4. As Sébastien Boucksom pointed out to us, in the proof of [DEL00, Lemma 3.5], one can find arguments involving remainders in Euclidean divisions which are similar to some arguments in the proof of Theorem 4.2.3.

The kind of use of Riemann sums made at the end of the proof of Proposition 4.2.2 already appeared in the literature on algebraic volumes. See, for instance, [Laz04a, Example 2.3.6] and [ELM$^+$05, Example 2.2].

There is a description of the non-archimedean volume in terms of the energy for arbitrary continuous metrics if the residue characteristic of K is zero and if X is a smooth projective variety. Moreover, the lim sup in the definition of the non-archimedean volume is again a limit. These results will be shown in Corollary 6.2.2.

5. Differentiability

As usual, K is a complete discretely valued field with valuation ring K°. Recall that we normalized our absolute value such that $-\log|\pi| = 1$ for a uniformizer π. Let X be a projective variety over K of dimension n. In this section, we consider projective K°-models $\mathscr{X}$ of X. The special fiber will be denoted by $\mathscr{X}_s$. This is a scheme of finite type over the residue field $\tilde{K}$ but not necessarily reduced. We denote the irreducible components of $\mathscr{X}_s$ by $(E_i)_{i \in I}$ and let b_i denote the multiplicity of $\mathscr{X}_s$ in E_i.

5.1. Upper bounds for the first cohomology group. In the following, we will use the notation introduced in Paragraph 2.3.2. Given Cartier divisors $D_1, \ldots, D_n$ on a model $\mathscr{X}$ of X, we denote by $\{D_1\} \cdots \{D_n\}$ the algebraic intersection number in the generic fiber.

Lemma 5.1.1. *Let D, M_1, M_2 be nef divisors, and let $\mathcal{N}$ be any line bundle on $\mathscr{X}$. There exists a function $\rho\colon \mathbb{N} \to \mathbb{R}$ with $\rho(m) = o(m^n)$ as $m \to \infty$ such that*

$$\dim_{\tilde{K}}\big(H^1\big(\mathscr{X}, \mathcal{N}(mD + j(M_1 - M_2))\big) \otimes_{K^\circ} \tilde{K}\big) \leqslant \frac{m^n}{n!} n\{D + M_1\}^{n-1} \cdot \{M_2\} + \rho(m)$$

holds for all $m \in \mathbb{N}$ and all $j \in \{0, \ldots, m\}$.

Proof. We will use the notation $\mathcal{F}_{j,m} := \mathcal{N}(mD + j(M_1 - M_2))$. Let π be a uniformizer of the discrete valuation ring K°, and let $M := M_1 - M_2$. The short exact sequence

$$0 \longrightarrow \mathcal{F}_{j,m} \xrightarrow{\cdot\pi} \mathcal{F}_{j,m} \longrightarrow \mathcal{F}_{j,m}|_{\mathscr{X}_s} \longrightarrow 0$$

yields the long exact sequence

$$\cdots \longrightarrow H^1(\mathscr{X}, \mathcal{F}_{j,m}) \xrightarrow{\cdot\pi} H^1(\mathscr{X}, \mathcal{F}_{j,m}) \longrightarrow H^1(\mathscr{X}_s, \mathcal{F}_{j,m}|_{\mathscr{X}_s}) \longrightarrow \cdots .$$

Forming the cokernel of the first map, we obtain an injection

$$H^1(\mathscr{X}, \mathcal{F}_{j,m}) \otimes_{K^\circ} \tilde{K} \simeq H^1(\mathscr{X}, \mathcal{F}_{j,m})/\pi H^1(\mathscr{X}, \mathcal{F}_{j,m}) \hookrightarrow H^1(\mathscr{X}_s, \mathcal{F}_{j,m}|_{\mathscr{X}_s}) .$$

By Proposition 3.6.2, we have

$$h^1(\mathscr{X}_s, \mathcal{F}_{j,m}|_{\mathscr{X}_s}) \leqslant \frac{(m+j)^n}{n!}\left(\sum_{i\in I} b_i \widehat{h}^1\left(E_i, \mathcal{O}\left(\frac{m}{m+j}D + \frac{j}{m+j}M\right)\bigg|_{E_i}\right)\right) + o\big((m+j)^n\big) .$$

For the cycle $\operatorname{cyc}(\mathscr{X}_s)$ associated with $\mathscr{X}_s$, we have $\operatorname{cyc}(\mathscr{X}_s) = \sum_{i\in I} b_i E_i$. Now the holomorphic Morse inequalities in Theorem A.2 applied on every component E_i and the above inequality show that $h^1(\mathscr{X}_s, \mathcal{F}_{j,m}|_{\mathscr{X}_s})$ is bounded above by

$$\frac{(m+j)^n}{n!}\left(n\left(\frac{m}{m+j}D + \frac{j}{m+j}M_1\right)^{n-1} \cdot \frac{j}{m+j}M_2 \cdot \operatorname{cyc}(\mathscr{X}_s)\right) + o\big((m+j)^n\big) .$$

By the flatness of $\mathscr{X}$ over K°, the degrees of the special fiber $\mathscr{X}_s$ and of the generic fiber X of $\mathscr{X}$ with respect to n line bundles on $\mathscr{X}$ are equal (cf. [Kol96, Proposition 2.10]). Hence, the above upper bound is equal to

$$\frac{m^n}{n!} n\left\{D + \frac{j}{m}M_1\right\}^{n-1} \cdot \left\{\frac{j}{m}M_2\right\} + o\big((m+j)^n\big) \leqslant \frac{m^n}{n!} n\{D + M_1\}^{n-1} \cdot \{M_2\} + o\big(m^n\big) ,$$

where we have used that D, M_1, M_2 are nef and $j \leqslant m$. This proves the claim. □

Corollary 5.1.2. *Let π be a uniformizer of K°, let D, M_1, M_2 be nef divisors, and let $\mathcal{N}$ be any line bundle on $\mathscr{X}$. There exists a function $\rho\colon \mathbb{N} \to \mathbb{R}$ with $\rho(m) = o(m^n)$ as $m \to \infty$ such that for all $a \in \mathbb{N}$, $m \in \mathbb{N}$ and all $j \in \{1, \ldots, m\}$, we get*

$$\ell\left(H^1\big(\mathscr{X}, \mathcal{N}(mD + j(M_1 - M_2))\big)_{\pi^a-\mathrm{tors}}\right) \leqslant \frac{m^n}{n!} an\{D + M_1\}^{n-1} \cdot \{M_2\} + a\rho(m) .$$

Proof. Since $\mathscr{X}$ is projective, $H^1\big(\mathscr{X}, \mathcal{N}(mD + j(M_1 - M_2))\big)$ is a finitely generated K°-module. Since $\ell(M_{\pi^a\text{-tors}}) \leqslant a \dim_{\tilde{K}}\big(M \otimes_{K^\circ} \tilde{K}\big)$ holds for any finitely generated K°-module M, the claim follows from Lemma 5.1.1. □

5.2. Bounds for the zeroth cohomology group. We continue working with the setup from the beginning of Section 5. Let E be an effective vertical Cartier divisor on $\mathscr{X}$ and s the canonical global section of $\mathcal{O}(E)$. We write the Weil divisor corresponding to E as $\sum_{i\in I} c_i E_i$. We define $\alpha_i := c_i/b_i$ and $\alpha := \max_{i\in I} \alpha_i$.

Let D, M_1, M_2 be nef Cartier divisors on $\mathscr{X}$. We consider the sum

$$\delta_D(M_1, M_2) = \sum_{a,b,c} \{D\}^a \cdot \{M_1\}^b \cdot \{M_2\}^c \tag{5.1}$$

of intersection numbers on X, where $(a, b, c) \in \mathbb{N}^3$ with $a + b + c = n$ and $a \neq n$. By [Kol96, Proposition 2.10], we have

$$\delta_D(M_1, M_2) = \sum_{a,b,c} D^a \cdot M_1^b \cdot M_2^c \cdot \operatorname{cyc}(\mathscr{X}_s)\,.$$

This is non-negative and will be used in the error terms of asymptotic estimates. Note that the definition of $\delta_D(M_1, M_2)$ can be extended to the case when M_1 and M_2 are $\mathbb{Q}$-divisors and

$$\delta_D(\varepsilon M_1, \varepsilon M_2) = O(\varepsilon) \tag{5.2}$$

for $\varepsilon \to 0$ in $\mathbb{Q}_{\geqslant 0}$. Let further $\mathscr{N}$ be an arbitrary line bundle on $\mathscr{X}$.

LEMMA 5.2.1. *There is an explicit constant $C_n > 0$ depending only on n such that for all X, $\mathscr{X}$, D, E, M_1, M_2, $\mathscr{N}$ as above, there exists a function $\rho \colon \mathbb{N} \to \mathbb{R}$ with $\rho(m) = o(1)$ as $m \to \infty$ such that for all $m \in \mathbb{N}$ and all $j \in \{0, \ldots, m\}$, we have*

$$\left| \frac{n!}{m^n} h^0\big(E, \mathscr{N}(mD + j(M_1 - M_2))|_E\big) - D^n \cdot E \right| \leqslant C_n \delta_D(M_1, M_2)\alpha + \rho(m)\,.$$

Proof. We argue similarly as in the proof of Lemma 5.1.1. For all $q \geqslant 0$, it follows from Proposition 3.6.2 and the holomorphic Morse inequalities of Theorem A.2 that

$$h^q(E, \mathscr{N}(mD + j(M_1 - M_2))|_E) \leqslant \frac{m^n}{n!}\binom{n}{q}\Big(D + \frac{j}{m}M_1\Big)^{n-q} \cdot \Big(\frac{j}{m}M_2\Big)^q \cdot E + \tilde{\rho}(m + j) \tag{5.3}$$

for some function $\tilde{\rho} \colon \mathbb{N} \to \mathbb{R}$ with $\tilde{\rho}(m) = o(m^n)$ as $m \to \infty$. Using that D, M_1, M_2 are nef and using that the Weil divisor $\operatorname{cyc}(E)$ associated with E satisfies $\operatorname{cyc}(E) \leqslant \alpha \cdot \operatorname{cyc}(\mathscr{X}_s)$, we may replace E in the bound (5.3) by $\alpha \cdot \operatorname{cyc}(\mathscr{X}_s)$. As before, since the model $\mathscr{X}$ is flat, the degree of the special fiber $\mathscr{X}_s$ with respect to line bundles on $\mathscr{X}$ agrees with the corresponding degree of the generic fiber X. For all $q \geqslant 1$, we deduce from (5.3) and $j/m \leqslant 1$ that there is an explicit constant C'_n depending only on n such that

$$h^q\big(E, \mathscr{N}(mD + j(M_1 - M_2))|_E\big) \leqslant \frac{m^n}{n!}\alpha C'_n \delta_D(M_1, M_2) + \rho'(m) \tag{5.4}$$

holds for all $m \in \mathbb{N}$ and $j \in \{1, \ldots, m\}$ with $\rho'(m) := \max\{\tilde{\rho}(m + i) \mid 1 \leqslant i \leqslant m\}$.

By Proposition 3.6.1, the Euler characteristic $\chi(E, \mathscr{N}(mD + j(M_1 - M_2))|_E)$ equals

$$\frac{m^n}{n!}\sum_{q=0}^{n}(-1)^q\binom{n}{q}\left(D + \frac{j}{m}M_1\right)^{n-q} \cdot \left(\frac{j}{m}M_2\right)^q \cdot E + O\big(m^{n-1}\big)\,. \tag{5.5}$$

Expanding (5.5), bounding all terms involving at least one M_i by $C''_n \delta_D(M_1, M_2)\alpha$ as above, using again $\operatorname{cyc}(E) \leqslant \alpha \cdot \operatorname{cyc}(\mathscr{X}_s)$ and (5.4), we get the claim. □

5.3. A filtration argument. We consider a normal projective variety X over K with a normal projective model $\mathscr{X}$ over K°. Let f be a $\mathbb{Z}$-model function determined on $\mathscr{X}$ by a vertical Cartier divisor $V \in \operatorname{Div}_0(\mathscr{X})$. In this situation, we will write $\mathcal{O}(f) := \mathcal{O}(V)$.

Since $\mathscr{X}$ is projective, we can write $\mathcal{O}(f) = \mathcal{O}(M_1 - M_2)$ for nef Cartier divisors M_1 and M_2 on $\mathscr{X}$. We consider a nef Cartier divisor D on $\mathscr{X}$, and we will use again $\delta_D(M_1, M_2)$ from Subsection 5.2 to bound error terms.

In the following result, we assume $f \leqslant 0$. Then Proposition 2.2.6 yields that the Cartier divisor $E := -V$ is effective, and we denote the canonical global section of $\mathcal{O}(E)$ by s. We consider also an arbitrary line bundle $\mathscr{N}$ on $\mathscr{X}$.

Lemma 5.3.1. *There is an explicit constant $C_n > 0$ depending only on n such that for every X, $\mathscr{X}$, D, $f \leqslant 0$, M_1, M_2, $\mathscr{N}$ as above, there exists a function $\rho\colon \mathbb{N} \to \mathbb{R}$ with $\rho(m) = o(1)$ as $m \to \infty$ such that*

$$\left| \frac{n!}{m^n} \ell \left(\frac{\Gamma(\mathscr{X}, \mathcal{F}_{j+1,m})}{\Gamma(\mathscr{X}, \mathcal{F}_{j,m})} \right) - \int_{X^{\mathrm{an}}} f c_1(\mathcal{O}(D))^{\wedge n} \right| \leqslant C_n \delta_D(M_1, M_2) \cdot \lceil |f|_{\sup} \rceil + \rho(m)$$

holds for all $m \in \mathbb{N}$ and all $j \in \{0, \ldots, m-1\}$, where $\mathcal{F}_{j,m} := \mathscr{N}(mD + j(M_1 - M_2))$.

Proof. Recall that $\int_{X^{\mathrm{an}}} f c_1(\mathcal{O}(D))^{\wedge n}$ was introduced in Subsection 2.4. By Lemma 2.4.2, we have

$$\int_{X^{\mathrm{an}}} (-f) c_1(\mathcal{O}(D))^{\wedge n} = D^n \cdot E\,. \tag{5.6}$$

The section s determines a short exact sequence of coherent sheaves on $\mathscr{X}$:

$$0 \longrightarrow \mathcal{F}_{j+1,m} \xrightarrow{\otimes s} \mathcal{F}_{j,m} \longrightarrow \mathcal{F}_{j,m}|_E \longrightarrow 0\,. \tag{5.7}$$

The associated long exact sequence in cohomology is

$$0 \to \Gamma(\mathscr{X}, \mathcal{F}_{j+1,m}) \xrightarrow{\otimes s} \Gamma(\mathscr{X}, \mathcal{F}_{j,m}) \xrightarrow{\phi_j} \Gamma(E, \mathcal{F}_{j,m}) \xrightarrow{\psi_j} H^1(\mathscr{X}, \mathcal{F}_{j+1,m}) \to \cdots\,. \tag{5.8}$$

We have to compute $\ell(\mathrm{im}(\phi_j)) = \ell\left(\Gamma(\mathscr{X}, \mathcal{F}_{j,m})/\Gamma(\mathscr{X}, \mathcal{F}_{j+1,m})\right)$. Using the obvious relation $\ell(\Gamma(E, \mathcal{F}_{j,m})) = \ell(\ker(\psi_j)) + \ell(\mathrm{im}(\psi_j))$ and $\mathrm{im}(\phi_j) = \ker(\psi_j)$, we deduce that

$$\ell(\mathrm{im}(\phi_j)) = \ell(\Gamma(E, \mathcal{F}_{j,m})) - \ell(\mathrm{im}(\psi_j))\,. \tag{5.9}$$

Using the notation from Subsection 5.2, we have $\alpha_i = -f(x_i)$; hence, Lemma 5.2.1 and (5.6) give

$$\left| \frac{n!}{m^n} \ell(\Gamma(E, \mathcal{F}_{j,m})) - \int_{X^{\mathrm{an}}} (-f) c_1(\mathcal{O}(D))^{\wedge n} \right| \leqslant C_n \delta_D(M_1, M_2) \cdot |f|_{\sup} + \rho(m)\,. \tag{5.10}$$

For $a := \lceil |f|_{\sup} \rceil$, the model function associated with the Cartier divisor $\mathrm{div}(\pi^a) - E$ equals $a + f \geqslant 0$, and hence Proposition 2.2.6 shows that $\mathrm{div}(\pi^a) - E$ is an effective Cartier divisor on $\mathscr{X}$. We deduce that $\mathcal{O}_E$ is π^a-torsion and thus

$$\mathrm{im}(\psi_j) \subset H^1(\mathscr{X}, \mathcal{F}_{j+1,m})_{\pi^a\text{-tors}}\,.$$

This allows us to bound $\ell(\mathrm{im}(\psi_j))$ using Corollary 5.1.2. With (5.9) and (5.10), we get

$$\left| \frac{n!}{m^n} \ell(\mathrm{im}(\phi_j)) - \int_{X^{\mathrm{an}}} (-f) c_1(\mathcal{O}(D))^{\wedge n} \right| \leqslant C_n \delta_D(M_1, M_2) \cdot a + \rho(m) \tag{5.11}$$

for larger C_n and ρ. By $\ell(\mathrm{im}(\phi_j)) = \ell\left(\Gamma(\mathscr{X}, \mathcal{F}_{j,m})/\Gamma(\mathscr{X}, \mathcal{F}_{j+1,m})\right)$, we get the claim. □

5.4. From model metrics to continuous semipositive metrics. In this subsection, X is a normal projective variety of dimension n over K with a line bundle L. We will generalize the result from Subsection 5.3 to a continuous semipositive metric $\|\ \|$ on L^{an} (cf. Subsection 2.3). Let $\overline{L} = (L, \|\ \|)$ be the corresponding metrized line bundle. We will use the notation

$$\|\ \|_g := e^{-g} \|\ \|$$

for any continuous function $g\colon X^{\mathrm{an}} \to \mathbb{R}$. If f is a $\mathbb{Z}$-model function, if $\mathscr{L}$ is a model of L and if $\overline{L} = (L, \|\ \|_{\mathscr{L}})$, then $\|\ \|_{\mathscr{L},f} = \|\ \|_{\mathscr{L}(f)}$ for $\mathscr{L}(f) = \mathscr{L} \otimes \mathcal{O}(f)$.

5.4.1. Let $\mathscr{X}$ be a projective K°-model of X, and let f be a model function on X^{an} determined on $\mathscr{X}$. Choose some non-zero $k \in \mathbb{N}$ such that kf is a $\mathbb{Z}$-model function determined on $\mathscr{X}$. In much the same way as before, there is a decomposition $\mathcal{O}(kf) = \mathcal{O}(kM_1 - kM_2)$ for nef $\mathbb{Q}$-Cartier divisors M_1 and M_2 on $\mathscr{X}$ such that kM_1 and kM_2 belong to $\mathrm{Div}_0(\mathscr{X})$.

Since $\| \ \|$ is a continuous semipositive metric on L^{an}, it follows from [BFJ16, Lemma 1.2] that L is nef. Using algebraic intersection numbers on X, we have

$$\delta_L(M_1, M_2) \coloneqq \sum_{a,b,c} L^a \cdot \{M_1\}^b \cdot \{M_2\}^c \geqslant 0\,,$$

where (a, b, c) ranges over $\mathbb{N}^3$ with $a + b + c = n$ and $a \neq n$. Note that in the setup of (5.1), we have $\delta_D(M_1, M_2) = \delta_L(M_1, M_2)$ for $L \coloneqq \mathcal{O}(D)|_X$.

PROPOSITION 5.4.2. *There is an explicit constant C_n depending only on n such that for all X, L, f, M_1, M_2 as above and any continuous semipositive metric $\| \ \|$ on L^{an}, we have*

$$\left|\mathrm{vol}(L, \| \ \|_f, \| \ \|) - \int_{X^{\mathrm{an}}} f c_1(L, \| \ \|)^{\wedge n}\right| \leqslant C_n \delta_L(M_1, M_2) |f|_{\sup}\,.$$

Proof. We first prove the claim under the assumptions $f \leqslant 0$ and that $\| \ \|$ is a semipositive model metric. We will proceed as in the proof of Theorem 4.2.3. We first choose a non-zero $k \in \mathbb{N}$ such that kf is a $\mathbb{Z}$-model function with $|kf|_{\sup} \in \mathbb{N}$, the divisors kM_1 and kM_2 are Cartier divisors on $\mathscr{X}$ and $\| \ \|^{\otimes k}$ is an algebraic metric. As we may always pass to a finer model (which does not change the quantities involved), we may assume that we have $\| \ \|^{\otimes k} = \| \ \|_{\mathscr{L}}$ for a nef line bundle $\mathscr{L}$ on $\mathscr{X}$ with $\mathscr{L}|_X = L^{\otimes k}$. We fix some $r \in \{0, \ldots, k-1\}$, and we consider the arithmetic progression $(m = kq + r)_{q \in \mathbb{N}}$. By passing to a finer model, we may assume that $L^{\otimes r}$ has a model $\mathcal{M}$ on $\mathscr{X}$ and that $\mathscr{X}$ is normal. By an argument similar to that in the proof of Theorem 4.2.3, we deduce from Lemma 4.1.5 that

$$\ell\left(\frac{\widehat{H}^0(X, L^{\otimes m}, \| \ \|_f^{\otimes m})}{\widehat{H}^0(X, L^{\otimes m}, \| \ \|^{\otimes m})}\right) = \ell\left(\frac{\widehat{H}^0(X, L^{\otimes r} \otimes L^{\otimes kq}, \| \ \|_{\mathcal{M}} \otimes \| \ \|^{\otimes q}_{\mathscr{L}(kf)})}{\widehat{H}^0(X, L^{\otimes r} \otimes L^{\otimes kq}, \| \ \|_{\mathcal{M}} \otimes \| \ \|^{\otimes q}_{\mathscr{L}})}\right) + O(q^n)\,,$$

along the arithmetic progression $(m = kq + r)_{q \in \mathbb{N}}$.

By Lemma 2.2.4, the first summand on the right-hand side is equal to

$$\ell\left(\frac{\Gamma(\mathscr{X}, \mathcal{M} \otimes \mathscr{L}(kf)^{\otimes q})}{\Gamma(\mathscr{X}, \mathcal{M} \otimes \mathscr{L}^{\otimes q})}\right) = \ell\left(\frac{\Gamma(\mathscr{X}, \mathcal{F}_{q,q})}{\Gamma(\mathscr{X}, \mathcal{F}_{0,q})}\right) = \sum_{j=0}^{q-1} \ell\left(\frac{\Gamma(\mathscr{X}, \mathcal{F}_{j+1,q})}{\Gamma(\mathscr{X}, \mathcal{F}_{j,q})}\right) \tag{5.12}$$

for any decreasing filtration $\mathcal{M} \otimes \mathscr{L}^{\otimes q} = \mathcal{F}_{0,q} \supset \mathcal{F}_{1,q} \supset \cdots \supset \mathcal{F}_{q,q} = \mathcal{M} \otimes \mathscr{L}(kf)^{\otimes q}$ into coherent $\mathcal{O}_{\mathscr{X}}$-submodules $\mathcal{F}_{j,q}$ of $\mathcal{M} \otimes \mathscr{L}^{\otimes q}$. We will now apply Lemma 5.3.1 with q, $\mathscr{L}$, kf, kM_1, kM_2, $\mathcal{M}$ instead of m, $\mathcal{O}(D)$, f, M_1, M_2, $\mathcal{N}$; hence, we use the filtration $\mathcal{F}_{j,q} \coloneqq \mathcal{M} \otimes \mathscr{L}^{\otimes q} \otimes \mathcal{O}(j(kM_1 - kM_2))$. Then Lemma 5.3.1 shows that

$$\left|\frac{n!}{q^n} \ell\left(\frac{\Gamma(\mathscr{X}, \mathcal{F}_{j+1,q})}{\Gamma(\mathscr{X}, \mathcal{F}_{j,q})}\right) - \int_{X^{\mathrm{an}}} kf c_1(\mathscr{L})^{\wedge n}\right| \leqslant C_n \delta_{\mathscr{L}}(kM_1, kM_2) \cdot |kf|_{\sup} + o(1)\,. \tag{5.13}$$

Now the claim in the special case can be easily deduced from (5.12) and (5.13).

Next, we skip the above assumption $f \leqslant 0$. Note that $C \coloneqq |f|_{\sup} \in \mathbb{Q}$ and hence C is the model function of a numerically trivial $\mathbb{Q}$-Cartier divisor E_1 on $\mathscr{X}$. The $\mathbb{Q}$-Cartier divisor $M_1' \coloneqq M_1 - E_1$ is nef. Replacing k by a suitable multiple, we may assume that kM_1' is also a Cartier divisor on $\mathscr{X}$. The decomposition $\mathcal{O}(k(f - C)) = \mathcal{O}(kM_1' - kM_2)$ follows from Paragraph 5.4.1. An application of the above special case to $f - C \leqslant 0$ gives

$$\left| \operatorname{vol}(L, \| \ \|_{(f-C)}, \| \ \|) - \int_{X^{\mathrm{an}}} (f - C) c_1(L, \| \ \|)^{\wedge n} \right| \leqslant C_n \delta_L(M_1', M_2) |f - C|_{\sup} .$$

We have $\operatorname{vol}(L, \| \ \|_{(f-C)}, \| \ \|) = \operatorname{vol}(L, \| \ \|_f, \| \ \|) - CL^n$ by Remark 3.3.2 and Lemma 4.1.3. Now Paragraph 2.4.3, the equality $\delta_L(M_1, M_2) = \delta_L(M_1', M_2)$ and $|f - C|_{\sup} \leqslant 2|f|_{\sup}$ yield

$$\left| \operatorname{vol}(L, \| \ \|_f, \| \ \|) - \int_{X^{\mathrm{an}}} f c_1(L, \| \ \|)^{\wedge n} \right| \leqslant 2C_n \delta_L(M_1, M_2) |f|_{\sup} .$$

This proves the claim for a semipositive model metric.

Finally, we prove the claim for any continuous semipositive metric $\| \ \|$. By definition, $\| \ \|$ is a uniform limit of semipositive model metrics on L^{an}, and hence the claim follows from the continuity of the non-archimedean volume in Proposition 4.1.4 and of the Chambert-Loir measure in Paragraph 2.4.3. □

THEOREM 5.4.3. *Let $\| \ \|$ be a continuous semipositive metric on L^{an}, and let f be a continuous function on X^{an}. Then if we consider everything fixed except $\varepsilon \in \mathbb{R}$, one has*

$$\operatorname{vol}(L, \| \ \|_{\varepsilon f}, \| \ \|) = \varepsilon \int_{X^{\mathrm{an}}} f c_1(L, \| \ \|)^{\wedge n} + o(\varepsilon) \tag{5.14}$$

for $\varepsilon \to 0$. In the special case of a model function f on X^{an}, the formula (5.14) holds even after replacing $o(\varepsilon)$ with $O(\varepsilon^2)$.

Proof. It is enough to prove the claim for $\varepsilon > 0$. In the following, all ε are assumed to be positive. We choose the same setup as in Paragraph 5.4.1. For $\varepsilon \in \mathbb{Q}_{>0}$, Proposition 5.4.2 yields

$$\left| \operatorname{vol}(L, \| \ \|_{\varepsilon f}, \| \ \|) - \varepsilon \int_{X^{\mathrm{an}}} f c_1(L, \| \ \|)^{\wedge n} \right| \leqslant C_n \delta_L(\varepsilon M_1, \varepsilon M_2) |\varepsilon f|_{\sup} . \tag{5.15}$$

Using Proposition 4.1.4, this inequality and the equality $\delta_L(\varepsilon M_1, \varepsilon M_2) = O(\varepsilon)$ from (5.2) can be continuously extended to all $\varepsilon \in \mathbb{R}_{>0}$, and hence (5.14) follows for model functions.

To prove the case of a continuous function f, we argue by contradiction. Then either

$$\liminf_{\varepsilon \to 0} \frac{1}{\varepsilon} \operatorname{vol}(L, \| \ \|_{\varepsilon f}, \| \ \|) < \int_{X^{\mathrm{an}}} f c_1(L, \| \ \|)^{\wedge n} \tag{5.16}$$

or a reverse strict inequality with the lim sup holds. We will prove that (5.16) leads to a contradiction; the case of the lim sup is similar.

Let $\delta > 0$. By the density of model functions [Gub98, Theorem 7.12], there is a model function f_δ with $f - \delta \leqslant f_\delta \leqslant f$. By (5.16), we can choose $\delta > 0$ so small that

$$\liminf_{\varepsilon \to 0} \frac{1}{\varepsilon} \operatorname{vol}(L, \| \ \|_{\varepsilon f}, \| \ \|) < \int_{X^{\mathrm{an}}} (f - \delta) c_1(L, \| \ \|)^{\wedge n} \leqslant \int_{X^{\mathrm{an}}} f_\delta c_1(L, \| \ \|)^{\wedge n} .$$

By the model case, the right-hand side equals $\liminf_{\varepsilon \to 0} \varepsilon^{-1} \operatorname{vol}(L, \| \ \|_{\varepsilon f_\delta}, \| \ \|)$. This contradicts the monotonicity of the volume as we have $\| \ \|_{\varepsilon f} \leqslant \| \ \|_{\varepsilon f_\delta}$ since $\varepsilon > 0$. □

Remark 5.4.4. We note here that only the use of the holomorphic Morse inequalities from Theorem A.2 and our considerations about the asymptotic growth of algebraic volumes in Section 3, applied in the proofs of Lemmas 5.1.1 and 5.2.1, allowed us to prove equality in (5.14). Without using the holomorphic Morse inequalities, we can still prove "$\geqslant$" in (5.14), as we explain below. This would have been enough for our applications to orthogonality in Section 6 and for the proof of Theorem D.

The following result is a non-archimedean analogue of the main result in X. Yuan's paper [Yua08, Theorem 2.2]. It makes the lower bound in Proposition 5.4.2 very explicit and leads to "$\geqslant$" in (5.14) with the same arguments as in the proof of Theorem 5.4.3.

PROPOSITION 5.4.5. *Let f be a model function on X^{an} with $f \leqslant 0$, and let $\| \ \|$ be a continuous semipositive metric on L^{an}. Then $f = -\log(\| \ \|_1/\| \ \|_2)$ for semipositive model metrics $\| \ \|_1$ and $\| \ \|_2$ of a line bundle M on X. For any such presentation, we have*

$$\mathrm{vol}(L, \| \ \| e^{-f}, \| \ \|) \geqslant \int_{X^{\mathrm{an}}} f\left(c_1(L, \| \ \|) + c_1(M, \| \ \|_1)\right)^{\wedge n} . \tag{5.17}$$

Proof. The existence of the presentation is equivalent to that of a decomposition $\mathcal{O}(kf) = \mathcal{O}(kM_1 - kM_2)$ as in Paragraph 5.4.1, and so the existence follows from Paragraph 5.4.1.

To prove (5.17), we need to review some of the results of this section. Under the same assumptions as in Lemma 5.2.1, we get the explicit upper bound

$$\frac{n!}{m^n} h^0\big(E, \mathcal{N}(mD + j(M_1 - M_2))|_E\big) \leqslant (D + M_1)^n \cdot E + o(1) \tag{5.18}$$

by using the case $q = 0$ in (5.3). Observe that (5.3) for $q = 0$ is based only on the classical Hilbert–Samuel formula and on Proposition 3.6.2. Under the assumptions and with the notation from Lemma 5.3.1, we get

$$\frac{n!}{m^n} \ell\left(\frac{\Gamma(\mathscr{X}, \mathcal{F}_{j+1,m})}{\Gamma(\mathscr{X}, \mathcal{F}_{j,m})}\right) \geqslant \int_{X^{\mathrm{an}}} f c_1(\mathcal{O}(D + M_1))^{\wedge n} + o(1) . \tag{5.19}$$

Indeed, starting as in the proof of Lemma 5.3.1 and using (5.9), one gets that

$$\ell\left(\frac{\Gamma(\mathscr{X}, \mathcal{F}_{j,m})}{\Gamma(\mathscr{X}, \mathcal{F}_{j+1,m})}\right) \leqslant \ell\left(\Gamma(E, \mathcal{F}_{j,m})\right) = h^0\big(E, \mathcal{N}(mD + j(M_1 - M_2))|_E\big) . \tag{5.20}$$

Applying (5.18), we deduce that

$$\frac{n!}{m^n} \ell\left(\frac{\Gamma(\mathscr{X}, \mathcal{F}_{j,m})}{\Gamma(\mathscr{X}, \mathcal{F}_{j+1,m})}\right) \leqslant (D + M_1)^n \cdot E + o(1) = \int_{X^{\mathrm{an}}} (-f) c_1(\mathcal{O}(D + M_1))^{\wedge n} + o(1) ,$$

where the equality follows from Lemma 2.4.2 applied to the $\mathbb{Z}$-model function $-f$ associated with E. Multiplying by -1, we get (5.19).

Now Proposition 5.4.5 follows from the same arguments as used in the proof of Proposition 5.4.2 just by replacing the application of Lemma 5.3.1 in (5.13) by (5.19). □

6. Application to orthogonality and Monge–Ampère equation

In this section, K is a complete discretely valued field with valuation ring K° and residue field $\tilde{K}$. At the end of Subsection 6.3, we will assume $\mathrm{char}(\tilde{K}) = 0$.

6.1. A local approach to semipositivity. In this subsection, L is a line bundle on a proper variety X over K. It will be important to have a local analytic characterization of semipositive model metrics. This is done in [GK19, § 6] over an algebraically closed non-archimedean base field and can be done in a similar way over a complete discretely valued field (see [GM19] for details and generalizations). Our analytic objects will be compact strictly K-analytic domains V (see [Ber90, § 3.1, p. 48]) in the analytification X^{an} of X. We mimic the construction of algebraic metrics from Paragraph 2.2.3. We now consider *formal models* $\mathfrak{V}$ of V which are admissible formal schemes over K° (see [BL93, § 1]) with generic fiber V. As in Paragraph 2.2.3, a formal model $(\mathfrak{V}, \mathfrak{L})$ of $(V, L^{\mathrm{an}}|_V)$ induces a metric $\| \ \|_{\mathfrak{L}}$ on $L^{\mathrm{an}}|_V$ which we call *the formal metric associated with* $\mathfrak{L}$.

Following [GK19, Definition 6.2] and [GM19], we say that a model metric $\| \ \|$ on L^{an} is *semipositive in* $x \in X^{\mathrm{an}}$ if there exist $k \in \mathbb{N} \setminus \{0\}$, a compact strictly K-analytic domain V which is a neighborhood of x and a formal model $(\mathfrak{V}, \mathfrak{L})$ of $\left(V, (L^{\mathrm{an}})^{\otimes k}|_V\right)$ with $\| \ \||_V^{\otimes k} = \| \ \|_{\mathfrak{L}}$ such that for any curve Y in the special fiber of $\mathfrak{V}$ which is proper over $\tilde{K}$, we have $\deg_{\mathfrak{L}}(Y) \geqslant 0$. By [GK19, Proposition 6.5] and [GM19, Proposition 3.10], the model metric $\| \ \|$ is semipositive if and only if it is semipositive in all $x \in X^{\mathrm{an}}$.

We will need the following result from [GM19, Proposition 3.11].

Proposition 6.1.1. *Let* $\| \ \|_1$ *and* $\| \ \|_2$ *be model metrics on* L^{an}. *Then the metric* $\| \ \| := \min(\| \ \|_1, \| \ \|_2)$ *is a model metric on* L. *If* $\| \ \|_1$ *and* $\| \ \|_2$ *are semipositive in* $x \in X^{\mathrm{an}}$, *then* $\| \ \|$ *is semipositive in* x.

6.1.2. Let $s_0 \in \Gamma(X, L) \setminus \{0\}$. We define a singular metric $\| \ \|_{s_0}$ on L^{an} by

$$\|s\|_{s_0}(x) = \begin{cases} |s/s_0(x)| & \text{if } s/s_0 \in \mathcal{O}_{X^{\mathrm{an}},x}\,, \\ \infty & \text{if } s/s_0 \notin \mathcal{O}_{X^{\mathrm{an}},x}\,. \end{cases} \tag{6.1}$$

Lemma 6.1.3. *Let* $\| \ \|$ *be a model metric on* L^{an} *and* $s_0 \in \Gamma(X, L) \setminus \{0\}$. *Let* $\| \ \|_{s_0}$ *be the singular metric defined above. Then* $\| \ \|' := \min\left(\| \ \|, \| \ \|_{s_0}\right)$ *is a model metric on* L^{an}. *If* $\| \ \|$ *is semipositive in* $x \in X^{\mathrm{an}}$, *then* $\| \ \|'$ *is also semipositive in* x.

Proof. By passing to a positive tensor power, we may assume that $\| \ \|$ is an algebraic metric. It follows from [GK17, Proposition 8.13] that algebraic metrics and formal metrics on L^{an} are the same since the argument in loc. cit. does not use that the base field is algebraically closed. Thus, to prove the first claim, it is enough to show that $\| \ \|'$ is a formal metric on L^{an}. We use the fact that being a formal metric on L^{an} is a G-local property (cf. [GK19, Proposition 5.10] and [GM19, Proposition 2.8]). By [Ber93, Lemma 1.6.2], it is enough to check that for any $y \in X^{\mathrm{an}}$, there is a neighborhood V which is a strictly affinoid domain in X^{an} such that $\| \ \|'$ restricts to a formal metric on V.

Let us first assume $s_0(y) = 0$. Since X^{an} is a good analytic space, there exist a neighborhood V of y which is a strictly affinoid domain in X^{an} and a frame s of L over V which satisfies $\|s(v)\| < \|s(v)\|_{s_0}$ for all $v \in V$. So $\| \ \|'_{|V} = \| \ \|_{|V}$ is a formal metric on $L^{\mathrm{an}}|_V$.

If $s_0(y) \neq 0$, then we can find a neighborhood V of y which is a strictly affinoid domain in X^{an} such that $s_0|_V$ is nowhere vanishing. So the restriction of $\| \ \|_{s_0}$ to V is isometric to the trivial metric on $\mathcal{O}_V$, which is formal. Hence, the restriction of $\| \ \|'$ to V is the minimum of two formal metrics on V. By [Gub98, Lemma 7.8], the restriction of $\| \ \|'$ to V is also a formal metric on L^{an}. This proves the first claim.

If $\| \ \|$ is semipositive in x, then we proceed as in the first part of the proof with $y := x$ to show that $\| \ \|'$ is semipositive in x. If $s_0(x) = 0$, then this follows from the fact that $\| \ \|'_{|V} = \| \ \|_{|V}$ is semipositive in x. If $s_0(x) \neq 0$ and V is as before, then [GK19, Corollary 5.12] and [GM19, Proposition 2.6] give the existence of an algebraic metric on L^{an} which agrees with the singular metric $\| \ \|_{s_0}$ over V. Since $\| \ \|'_{|V}$ is the restriction of the minimum of two model metrics on L^{an} which are both semipositive on V, Proposition 6.1.1 yields that $\| \ \|'$ is semipositive on V. $\square$

6.2. A useful property of the semipositive envelope of a metric. Let X be a normal projective K-variety. Let L be a line bundle on X and $\| \ \|$ a continuous metric on L^{an}. We will assume that the semipositive envelope $P(\| \ \|)$ is a continuous metric. If $\mathrm{char}(\tilde{K}) = 0$ and if L is an ample line bundle on a projective smooth variety, then the semipositive envelope $P(\| \ \|)$ of $\| \ \|$ is a continuous metric on L^{an} (see Theorem 2.5.3). Going from a continuous metric to its semipositive envelope does not change the space of small sections, as we will show next.

PROPOSITION 6.2.1. *For a continuous metric* $\| \ \|$ *on the line bundle* L^{an} *such that the semipositive envelope* $P(\| \ \|)$ *is a continuous metric, we have*

$$\widehat{H}^0(X, L, \| \ \|) = \widehat{H}^0(X, L, P(\| \ \|)) \,. \tag{6.2}$$

As a consequence, the non-archimedean volume satisfies

$$\mathrm{vol}(\| \ \|, P(\| \ \|)) = 0 \,. \tag{6.3}$$

Proof. Let us first prove (6.2). We have $\|s\| \leqslant P(\|s\|)$ for every section $s \in \Gamma(X, L)$ by definition of the semipositive envelope. This implies $\widehat{H}^0(X, L, P(\| \ \|)) \subseteq \widehat{H}^0(X, L, \| \ \|)$. Assume that there exists some $s_0 \in \widehat{H}^0(X, L, \| \ \|)$ which does not belong to the subset $\widehat{H}^0(X, L, P(\| \ \|))$. Then $\|s_0\| \leqslant 1$, and there is a point $x_0 \in X^{\mathrm{an}}$ with

$$P(\|s_0(x_0)\|) > 1 \,. \tag{6.4}$$

This gives $f := \log \|s_0\| \leqslant 0$, and the metric $\| \ \|_{s_0} = \| \ \| e^{-f}$ introduced in Paragraph 6.1.2 satisfies $\| \ \| \leqslant \| \ \|_{s_0}$. For a semipositive model metric $\| \ \|_1 \geqslant \| \ \|$ on L^{an}, we get

$$\| \ \| \leqslant \| \ \|' := \min(\| \ \|_{s_0}, \| \ \|_1) \leqslant \| \ \|_1 \,. \tag{6.5}$$

By Lemma 6.1.3, the metric $\| \ \|'$ is a semipositive model metric on L^{an}. Hence, $P(\| \ \|) \leqslant \| \ \|'$ by (6.5) and the construction of the semipositive envelope. However, we have $\|s_0\|_{s_0} = 1$ and get

$$\|s_0(x)\|' = \min(1, \|s_0(x)\|_1) \leqslant 1 \tag{6.6}$$

for all $x \in X^{\mathrm{an}}$. This contradicts $P(\| \ \|) \leqslant \| \ \|'$ if we compare (6.4) and (6.6).

Equation (6.3) is a direct consequence of (6.2) by the definition of the non-archimedean volume in Definition 4.1.2 and Remark 2.5.2. □

COROLLARY 6.2.2. *Let L be a line bundle on X, and let $\| \ \|_1$ and $\| \ \|_2$ be continuous metrics on L^{an} whose semipositive envelopes $P(\| \ \|_1)$ and $P(\| \ \|_2)$ are continuous metrics. Then we have $\mathrm{vol}(L, \| \ \|_1, \| \ \|_2) = E(L, P(\| \ \|_1), P(\| \ \|_2))$, and the* $\limsup$ *in the definition of the non-archimedean volume is a limit.*

Proof. For $i = 1, 2$, Proposition 6.2.1 yields

$$\widehat{H}^0(X, L, \| \ \|_i) = \widehat{H}^0(X, L, P(\| \ \|_i)) \,, \quad \mathrm{vol}(L, \| \ \|_1, \| \ \|_2) = \mathrm{vol}(L, P(\| \ \|_1), P(\| \ \|_2)) \,.$$

Hence, the result follows from Theorem 4.2.3 and Remark 2.5.2. □

6.3. The orthogonality property. Let X be a normal projective K-variety of dimension n. After the proof of Theorem 6.3.2, we will assume $\mathrm{char}(\tilde{K}) = 0$, which implies in particular that the semipositive envelope $P(\| \ \|)$ of a continuous metric $\| \ \|$ of an ample line bundle on a smooth projective variety over K is a continuous metric by a result of S. Boucksom, C. Favre and M. Jonsson (see Theorem 2.5.3).

Definition 6.3.1. Let L be a line bundle on X. Let $\| \ \|$ be a continuous metric on L^{an} whose semipositive envelope $P(\| \ \|)$ is continuous. We say that *the pair* $(L, \| \ \|)$ *satisfies the orthogonality property* if

$$\int_{X^{\mathrm{an}}} \log \frac{P(\| \ \|)}{\| \ \|} c_1(L, P(\| \ \|))^{\wedge n} = 0\,.$$

Theorem 6.3.2. *Let L be a line bundle on X and $\| \ \|$ a continuous metric on L^{an} whose semipositive envelope $P(\| \ \|)$ is a continuous metric. Then the pair $(L, \| \ \|)$ satisfies the orthogonality property.*

Proof. By assumption, the function $\varphi = \log\bigl(P(\| \ \|)/\| \ \|\bigr)$ is continuous. Fix $\varepsilon \in [0, 1]$. We have $\| \ \| \leqslant P(\| \ \|)e^{-\varepsilon\varphi} \leqslant P(\| \ \|)$. Hence, $P(P(\| \ \|)e^{-\varepsilon\varphi}) = P(\| \ \|)$. Applying Proposition 6.2.1 and then Theorem 5.4.3, we get

$$0 = \mathrm{vol}\bigl(P(\| \ \|)e^{-\varepsilon\varphi}, P(\| \ \|)\bigr) = \varepsilon \int_{X^{\mathrm{an}}} \varphi c_1(L, P(\| \ \|))^{\wedge n} + o(\varepsilon)$$

for $\varepsilon \to 0$. Dividing first by ε and then letting $\varepsilon \to 0$, we get the result. □

We now use the notation and terminology from Subsection 2.3 and, for the rest of this subsection, assume $\mathrm{char}(\tilde{K}) = 0$ and that X is a smooth projective variety over K. Let $\theta \in \mathcal{Z}^{1,1}(X)$ be a closed $(1, 1)$-form such that $\{\theta\} \in N^1(X)$ is ample. Given $f \in C^0(X^{\mathrm{an}})$, we denote by $P_\theta(f)$ the θ-psh envelope of f defined in [BFJ16, Definition 8.1] and by $\mathrm{MA}_\theta(\varphi)$ the Monge–Ampère measure on X^{an} associated with a continuous θ-psh function φ (see [BFJ15, Theorem 3.1]). The form θ is said to *satisfy the orthogonality property* if

$$\int_{X^{\mathrm{an}}} (f - P_\theta(f))\mathrm{MA}_\theta(P_\theta(f)) = 0$$

holds for all $f \in C^0(X^{\mathrm{an}})$ (see [BFJ15, Definition A.1]). In [BFJ15, Appendix A], S. Boucksom, C. Favre and M. Jonsson show that every such θ satisfies the orthogonality property if X satisfies the algebraicity condition (†) mentioned in Subsection 1.1. Using our results, we can remove (†).

Theorem 6.3.3. *Let $\theta \in \mathcal{Z}^{1,1}(X)$ be a closed form such that $\{\theta\}$ is ample. Then θ satisfies the orthogonality property.*

Proof. To deduce this from Theorem 6.3.2, we follow [BFJ15]. By [BFJ15, Lemma A.2], it is enough to show the theorem for rational classes. The homogeneity of the envelope allows us to assume that θ is an integral class. In this case, the Monge–Ampère measure $\mathrm{MA}_\theta(P_\theta(f))$ agrees with the Chambert-Loir measure $c_1(L, P(\| \ \|))^{\wedge n}$ (see [BFJ15, § 3.3]). Then the result follows from Theorem 6.3.2. □

Now we can solve the Monge–Ampère problem without the algebraicity assumption (†). For the definition of the dual complex of an SNC model, see [BFJ16, § 3].

Corollary 6.3.4. *Let $\theta \in \mathcal{Z}^{1,1}(X)$ be a closed form with $\{\theta\}$ ample and μ a positive Radon measure on X^{an} of mass $\{\theta\}^n$. If μ is supported on the dual complex of some SNC model of X, then there exists a continuous θ-psh function φ such that $\mathrm{MA}_\theta(\varphi) = \mu$.*

Proof. This follows from Theorem 6.3.3 and [BFJ15, Theorem 8.1]. □

Remark 6.3.5. By [BFJ15, Remark 7.4], the orthogonality property is equivalent to the differentiability of $E \circ P_\theta$. Note that our differentiability result in Theorem 5.4.3 is a priori different and weaker. We only proved for semipositive θ that the function $t \in \mathbb{R} \mapsto E \circ P_\theta(tf)$ is differentiable at $t = 0$ for any $f \in C^0(X^{\mathrm{an}})$. However, the orthogonality property from Theorem 6.3.3 and the proof of [BFJ15, Corollary 7.3] imply that $f \in C^0(X^{\mathrm{an}}) \mapsto E \circ P_\theta(f)$ is differentiable in the direction of any $g \in C^0(X^{\mathrm{an}})$.

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Appendix. Holomorphic Morse inequalities in arbitrary characteristic

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The holomorphic Morse inequalities give us asymptotic upper bounds for the higher cohomology of powers of line bundles. They were first proved by J.-P. Demailly [Dem85] for complex varieties. Later, F. Angelini [Ang96] gave an algebraic proof for varieties over a field of characteristic zero (see also [Kür06, Example 2.4]). In this section, we extend the holomorphic Morse inequalities to varieties over arbitrary fields.

Remark A.1. We say that a property (P) holds *at points in general position* (respectively, *at points in very general position*) of an irreducible variety T over a field k if (P) holds on the complement of a proper Zariski closed subset of T (respectively, on the complement of a countable union of proper Zariski closed subsets of T). If k is uncountable and algebraically closed and (P) holds at points in very general position, one can always pick a k-rational point where (P) holds (this is not true if k is only countable).

We have introduced the space $\mathrm{Div}(Y)_\mathbb{R}$ of real Cartier divisors on a projective scheme Y over k in Subsection 3.4. Such a divisor D is called nef if the intersection number with any closed curve in Y is non-negative. Now we come to the *holomorphic Morse inequalities.*

Theorem A.2. *Let Y be an n-dimensional projective scheme over any field k, and let $q \in \{0, \dots, n\}$. For very ample Cartier divisors D and E on Y and $F \coloneqq D - E$, we have*

$$h^q(Y, \mathcal{O}_Y(mF)) \leqslant \binom{n}{q} D^{n-q} \cdot E^q \frac{m^n}{n!} + O\big(m^{n-1}\big)\,. \tag{A.1}$$

More generally, if $D, E \in \mathrm{Div}(Y)_\mathbb{R}$ are nef, then (A.1) holds with the weaker error term $o(m^n)$ for $m \to \infty$ instead of $O\big(m^{n-1}\big)$.

Proof. Step 1: The claim holds for very ample Cartier divisors D and E on a projective variety Y over an algebraically closed field k.

The numbers h^q and the intersection numbers are invariant under base change (see [Har77, Proposition III.9.3] and [Ful98, Example 6.2.9]), and hence we may assume that the base k is uncountable. We denote by $|E|$ the space of hyperplane sections of E. According to [Kür06, Proposition 5.5], for fixed integers $m \geqslant 0$, $n \geqslant s \geqslant 0$ and $n \geqslant j \geqslant 0$,

$$h^j(s, m) \coloneqq h^j(E_1 \cap \dots \cap E_s, \mathcal{O}(mD)) \tag{A.2}$$

does not depend on the choice of divisors $E_1, \dots, E_s \in |E|$ in general position. It follows that for divisors $E_1, \dots, E_s \in |E|$ in very general position, the equality (A.2) holds simultaneously for all $m \geqslant 0$, $n \geqslant s \geqslant 0$ and $n \geqslant j \geqslant 0$. Since we assume that k is uncountable, such divisors exist. Since D is very ample, there exists an $m_0 \in \mathbb{N}$ such that

$$h^j(s,m) = 0 \text{ for all integers } m \geqslant m_0,\ n \geqslant j \geqslant 1,\ n \geqslant s \geqslant 0\,. \tag{A.3}$$

For a fixed integer s with $n \geqslant s \geqslant 0$ and varying $m \in \mathbb{N}$, we claim that

$$h^0(s,m) = D^{n-s} \cdot E^s \frac{m^{n-s}}{(n-s)!} + O\big(m^{n-s-1}\big)\,. \tag{A.4}$$

To see this, we first note that a Bertini-type argument shows that the intersection product E^s is given by the scheme-theoretic intersection $E_1 \cap \dots \cap E_s$ (see [Kür06, Lemma 5.7]). Using that D is very ample and Remark 3.3.2, we deduce (A.4).

Applying [Kür06, Lemma 5.7 and Corollary 4.2] for a fixed integer $m > n$, we deduce that for effective Cartier divisors $(E_1, \dots, E_m) \in |E|^m$ in general position, we have the following exact sequence:

$$\begin{aligned} 0 \to \mathcal{O}_Y\left(mD - \sum_{i=1}^m E_i\right) \to \mathcal{O}_Y(mD) \to \bigoplus_{1 \leqslant i \leqslant m} \mathcal{O}_{E_i}(mD) \to \bigoplus_{1 \leqslant i_1 < i_2 \leqslant m} \mathcal{O}_{E_{i_1} \cap E_{i_2}}(mD) \to \cdots \\ \to \bigoplus_{1 \leqslant i_1 < i_2 < \dots < i_n \leqslant m} \mathcal{O}_{E_{i_1} \cap E_{i_2} \cap \dots \cap E_{i_n}}(mD) \to 0\,. \end{aligned} \tag{A.5}$$

We now fix an integer $m \geqslant \max(n+1, m_0)$. There are $E_1, \dots, E_m \in |E|$ such that (A.5) is exact and such that for any integer $0 \leqslant s \leqslant n$ and for any integers $1 \leqslant i_1 < \dots < i_s \leqslant m$, the s-tuple $E_{i_1}, \dots, E_{i_s}$ is in very general position. The latter yields that $h^j(s,m) = h^j(E_{i_1} \cap \dots \cap E_{i_s}, \mathcal{O}(mD))$. We conclude from (A.3) that (A.5) gives an acyclic resolution of the sheaf $\mathcal{O}_Y(mD - \sum_{i=1}^m E_i) \simeq \mathcal{O}(mF)$. It follows that $H^q(Y, \mathcal{O}_Y(mF)) \simeq \ker(d^q)/\operatorname{im}(d^{q-1})$ for the canonical homomorphism

$$d^q \colon \bigoplus_{|I|=q} H^0(E_I, \mathcal{O}_{E_I}(mD)) \to \bigoplus_{|J|=q+1} H^0(E_J, \mathcal{O}_{E_J}(mD))\,,$$

where I and J range over subsets of $\{1, \dots, m\}$ and where $E_I := \bigcap_{i \in I} E_i$. We conclude

$$h^q(Y, \mathcal{O}_Y(mF)) \leqslant \sum_{|I|=q} h^0(E_I, \mathcal{O}_{E_I}(mD)) = \binom{m}{q} h^0(q,m)\,.$$

The first step now follows from (A.4) and the equality $\binom{m}{q} = m^q/q! + O\big(m^{q-1}\big)$ for fixed q.

Step 2: The inequalities (A.1) hold for very ample Cartier divisors D and E on a projective scheme Y over any field k.

By the same base change argument as in Step 1, we may assume that k is algebraically closed. Let $[Y] = \sum_{i \in I} b_i Y_i$ be the fundamental cycle of the projective scheme Y, where Y_i ranges over the irreducible components of Y and where b_i is the multiplicity of Y in Y_i given as the length of the local ring at the generic point of Y_i. The first step shows

$$h^q(Y_i, \mathcal{O}_{Y_i}(mF)) \leqslant \binom{n}{q} D^{n-q} \cdot E^q \cdot Y_i \frac{m^n}{n!} + O\big(m^{n-1}\big)\,,$$

and hence Lemma 3.2.3 yields Step 2 by the following computation:

$$\begin{aligned}\widehat{h}^q(Y,\mathcal{O}_Y(mF)) &\leqslant \sum_{i\in I} b_i h^q(Y_i,\mathcal{O}_{Y_i}(mF)) + O\big(m^{n-1}\big)\\ &\leqslant \sum_{i\in I} b_i \binom{n}{q} D^{n-q}\cdot E^q\cdot Y_i \frac{m^n}{n!} + O\big(m^{n-1}\big)\\ &\leqslant \binom{n}{q} D^{n-q}\cdot E^q \frac{m^n}{n!} + O\big(m^{n-1}\big)\,.\end{aligned}$$

Step 3: The claim holds for nef real divisors D and E on a projective scheme Y over any field k.

By the definition of asymptotic cohomological functions, it is equivalent to prove

$$\widehat{h}^q(Y,F) \leqslant \binom{n}{q} D^{n-q}\cdot E^q\,. \tag{A.6}$$

It is here where the error term $o(m^n)$ comes in. Since both sides are continuous (see Proposition 3.4.8) and the ample cone is dense inside the nef cone, we may assume that D and E are ample $\mathbb{Q}$-Cartier divisors. Since both sides of the equation are homogeneous of degree n (see Proposition 3.4.8), we may assume that D and E are very ample Cartier divisors on Y and hence Step 3 follows from Step 2. □

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