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On a question of Swan

With an appendix by Kęstutis Česnavičius

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In memory of Hideyuki Matsumura

Abstract

We show that a regular local ring is a filtered inductive limit of regular local rings, essentially of finite type over $\mathbb{Z}$.

1. Introduction

Let R be a regular ring. The following is a well-known question concerning finitely generated projective modules over polynomial R-algebras.

Conjecture 1.1 (Bass–Quillen conjecture, [Bas73, Problem IX], [Qui76]). Every finitely generated projective module P over a polynomial R-algebra $R[T]$, where $T = (T_1, \dots, T_n)$, is extended from R; that is, $P \cong R[T] \otimes_R (P/(T)P)$.

The Bass–Quillen conjecture (or BQ conjecture for short) has positive answers in the following cases:

(i) if $\dim R \leqslant 1$ (Quillen and Suslin, see [Qui76, Sus76]),

(ii) if R is essentially of finite type over a field (Lindel, see [Lin81]),

(iii) if R is a local ring of unequal characteristic, essentially of finite type over $\mathbb{Z}$; let us say that $(R, \mathfrak{m}, k)$ with $p = \operatorname{char} k \notin \mathfrak{m}^2$ (Swan [Mur87]).

Swan noticed that it will be useful for the general question to have a positive answer to the following one.

Question 1.2 (Swan [Mur87]). A regular local ring is a filtered inductive limit of regular local rings, essentially of finite type over $\mathbb{Z}$.

A partial positive answer is given below.

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Theorem 1.3 ([Pop89, Theorem 3.1]). *Swan's question has a positive answer for a regular local ring* $(R, \mathfrak{m}, k)$ *in the following cases:*

(1) $p = \operatorname{char} k \notin \mathfrak{m}^2$,

(2) *R contains a field,*

(3) *R is excellent Henselian.*

Using Theorem 1.3, we showed using Lindel's and Swan's results that the BQ conjecture holds for regular local rings containing a field and for regular local rings $(R, \mathfrak{m}, k)$ with $p = \operatorname{char} k \notin \mathfrak{m}^2$ (see [Pop89, Theorem 4.1]). After almost 30 years, we noticed that complete positive answers to the above questions are missing and that there are still people interested in having them.

The purpose of the present paper is to give a complete positive answer to Swan's question (see Theorem 3.8). When $\dim R = 1$, our Theorem 2.5 uses a condition of separability which we remove in Theorem 3.6. Unfortunately, our result seems to not be very useful for the Bass–Quillen conjecture (see Remark 3.9). The appendix shows that our Corollary 3.10 could be used to reduce a well-known conjecture on *purity* to the complete case (see Proposition A.2).

2. Discrete valuation rings of unequal characteristic

Let $(A, \mathfrak{m})$ be a discrete valuation ring (a DVR for short) of unequal characteristic. Then A dominates $\mathbb{Z}_{(p)}$, where $p = \operatorname{char} k$ for $k := A/\mathfrak{m}$. We suppose $p \in \mathfrak{m}^2$ and that the field extension $k \supset \mathbb{F}_p$ is *separably generated*.

Remark 2.1. Suppose that the fraction field $K = \operatorname{Fr}(A)$ of A is a finite-type extension of $\mathbb{Q}$. Then it is possible that $k/\mathbb{F}_p$ is not a finite-type field extension [APZ90, Theorem 6.1]. Thus $k/\mathbb{F}_p$ need not be separably generated.

Lemma 2.2. *There exist a discrete valuation subring B of A and a regular parameter x of A such that*

(1) *$B \subset A$ is a ramified extension inducing an algebraic separable extension on the residue fields,*

(2) *B contains a power $b = x^e$ of x, where $1 < e \in \mathbb{N}$, and x is a regular parameter of B,*

(3) *$C = B[x]_{(x)}$ is a discrete valuation subring of A such that $C \cong \big(B[X]/\big(X^e - b\big)\big)_{(X)}$,*

(4) *the extension $C \subset A$ is unramified.*

Proof. A lifting of a separable transcendence base $\bar{y} = (\bar{y}_i)_{i \in I}$ of k over $\mathbb{F}_p$ induces a system of algebraically independent elements $y := (y_i)_{i \in I}$, where $y_i \in A$, of K over $\mathbb{Q}$. Then the ring $C_0 = \mathbb{Z}[(Y_i)_{i \in I}]_{p\mathbb{Z}[(Y_i)_{i \in I}]}$ is a DVR (see, for example, [Mat80, Theorem 83]). Consider the flat map $\psi_0 \colon C_0 \to A$ given by $Y \mapsto y$. Note that ψ_0 is ramified and induces an algebraic separable extension on the residue fields. Suppose $p = x^s t$ for some regular parameter x of A and a unit element $t \in A$, where $1 < s \in \mathbb{N}$.

We claim that t is not transcendental over $\mathbb{Q}(y)$. Indeed, choose r such that $p^r > s$. Then $B = B' \cap \mathbb{Q}\big(y, t^{p^r}\big)$ is a DVR, and the residue field extension induced by $B \subset B'$ is pure inseparable because $u^{p^r} \in B$ modulo $\mathfrak{m} \cap B$ for every $u \in B'$. As it is also algebraic separable by hypothesis, we see that this residue field extension is trivial, and so B' must be a ramified extension of B of order $e_{B'/B} = p^r > s$, which is false.

The DVR $B' = A \cap \mathbb{Q}(y,t)$ contains $x^s = p/t$. If B' is an unramified extension of C_0, then the polynomial $X^s - p/t$ is irreducible in $B'[X]$,

$$C = B'[x]_{(x)} \cong \big(B'[X]/\big(X^s - p/t\big)\big)_{(X)}$$

is a DVR subring of A and the extension $C \subset A$ is unramified and induces an algebraic separable extension on the residue fields.

Otherwise (this is possible, as Example 2.3 shows), choose a regular parameter z' of B'. Then $z' = x'^{s'} t'$ for some regular parameter x' of A, an $1 \leqslant s' < s$ and an invertible element $t' \in A$. Note that the field extensions $\mathbb{Q}(y) \subset \mathbb{Q}(y,t') \subset \mathbb{Q}(y,t)$ are finite, and the last one is of degree s' because $\mathbb{Q}(y,t') = \mathbb{Q}(y, x^{s'})$, $\mathbb{Q}(y,t) = \mathbb{Q}(y,x^s)$.

If $s' = 1$, then $B' \subset A$ is unramified. Using this procedure again, we arrive in some steps to a DVR subring B'' of B' such that the extension $B'' \subset B'$ is ramified. This is because the degree of the corresponding fraction field extensions over $\mathbb{Q}(y)$ decreases in each step; in the worst case, we have $B'' = C_0$.

Thus we may assume $s' > 1$. Repeating this procedure for $A \cap \mathbb{Q}(y,t,t')$ and so on, we arrive in some steps to a DVR subring $(B,(z))$ of A containing a power $b = x^e$, where $e > 1$, of a regular parameter x of A with $z \notin x^{e'}A$ for $e' < e$. Then $C = B[x]_{(x)} \cong \big(B[X]/\big(X^e - b\big)\big)_{(X)}$ is a DVR subring of A, and the extension $C \subset A$ is unramified and induces an algebraic separable extension on the residue fields. □

Example 2.3. Note that $B = \big(\mathbb{Z}[Y]/\big(Y^2 - 5\big)\big)_{(Y)}$ is a DVR and a ramified extension of $C = \mathbb{Z}_{(5)}$. The polynomial $f = X^4 - 5/(1+Y) \in B[X]$ is irreducible, and $D = (B[X]/(f))_{(X)}$ is a DVR and a ramified extension of B. Note that $D \cap \mathbb{Q}(Y) = B$ is a ramified extension of C. Thus in the above proof, it is possible that B' could indeed be a ramified extension of C_0.

By recurrence, using Lemma 2.2, we get the following proposition.

Proposition 2.4. *In the notation of Lemma 2.2, there exist extensions of discrete valuation subrings of A*

$$C_0 \subset B_1 \subset C_1 \subset \cdots \subset B_r \subset C_r \subset B_{r+1} = A\,,$$

C_0 being defined in Lemma 2.2, such that

1. *$C_i \subset B_{i+1}$ is an unramified extension for any $0 \leqslant i \leqslant r$ inducing an algebraic separable extension on the residue fields;*
2. *$C_i \cong \big(B_i[X]/\big(X^{e_i} - b_i\big)\big)_{(X)}$ for some $b_i \in B_i$ and $1 < e_i \in \mathbb{N}$, where b_i is a regular parameter of B_i.*

Theorem 2.5. *Suppose that $K = \operatorname{Fr} A$ is a field extension of $\mathbb{Q}$ not necessarily of finite type. Then A is a filtered inductive union of regular local subrings $(R_i)_{i\in I}$ of A, essentially of finite type over $\mathbb{Z}$.*

Proof. It is enough to show that for a finite-type $\mathbb{Z}$-algebra $E \subset A$, there exists a regular local subring $R \subset A$ which contains E and is essentially of finite type over $\mathbb{Z}$. By Proposition 2.4, there exist extensions of discrete valuation subrings of A

$$C_0 \subset B_1 \subset C_1 \subset \cdots \subset B_r \subset C_r \subset B_{r+1} = A$$

such that properties (1) and (2) of Proposition 2.4 hold. Apply the classical Néron desingularization (see [Nér64], [KPP18a, Theorem 1]) for the case $C_r \subset B_{r+1} = A$. Then there exists a regular local subring $R_r \subset A$ which contains $C_r[E]$ and is essentially of finite type over C_r. If $r = 0$, then A is an unramified extension of C_0 inducing an algebraic separable extension on residue fields, and we are done.

Suppose $r > 0$. Using again the Néron desingularization for the extension $C_{r-1} \subset B_r$, we see that B_r is a filtered inductive union of regular local subrings essentially of finite type over C_{r-1}. Then there exists a regular local subring T_{r-1} of B_r essentially of finite type over C_{r-1} such that

(i) b_r belongs to a regular system of parameters z_{r-1} of T_{r-1}, and there exists a regular local subring T'_{r-1} of B_r isomorphic to $\big(T_{r-1}[X]/\big(X^{e_{r-1}} - b_{r-1}\big)\big)_{(z_{r-1},X)}$;

(ii) R_r is defined over T'_{r-1}; that is, there exists a regular local subring $R_{r-1} \supset T'_{r-1}$ of B_r such that $R_r \cong R_{r-1} \otimes_{T'_{r-1}} C_r$.

Applying this procedure recurrently, we find a regular local subring R_0 of A which is essentially of finite type over C_0 and contains $C_0[E]$. This is enough. □

Remark 2.6. If $p \notin \mathfrak{m}^2$, then the problem is easier (see [Pop89, Theorem 3.1]).

3. Regular local rings of unequal characteristic

Let $(R, \mathfrak{m}, k)$ be a regular local ring of dimension n and p =char k. We suppose $0 \neq p \in \mathfrak{m}^2$.

Lemma 3.1. *There exists a regular system of parameters $x = (x_1, \dots, x_n)$ for R such that $(p, x_2, \dots, x_n)$ is a system of parameters for R. For any such system, the map*

$$\mathbb{Z}[X_2, \dots, X_n]_{(p,X_2,\dots,X_n)} \to R$$

given by $X_i \mapsto x_i$ for $1 < i \leqslant n$ is flat and induces a ramified extension of DVRs modulo $(x_2, \dots, x_n)$.

Proof. The second statement follows from the first and the flatness criterion (see, for example, [Mat80, 20.C]). Suppose $n > 1$, and let $z = (z_1, \dots, z_n)$ be a regular system of parameters of R. Using induction on n, it is enough to choose x_n from the infinite set $z_n + \mathfrak{m}^2$ which does not divide p. □

For the next results, we need some preparations.

A ring morphism $u\colon A \to A'$ of Noetherian rings has *regular fibers* if for all prime ideals $p \in \operatorname{Spec} A$, the ring A'/pA' is a regular ring. It has *geometrically regular fibers* if for all prime ideals $p \in \operatorname{Spec} A$ and all finite field extensions K of the fraction field of A/p, the ring $K \otimes_{A/p} A'/pA'$ is regular. A flat morphism of Noetherian rings u is *regular* if its fibers are geometrically regular. If u is regular of finite type, then u is called *smooth*.

The following theorem extends Néron's desingularization [Nér64, KPP18a] and has been useful for solving different problems concerning the projective modules over regular rings or from Artin approximation theory [Pop86, Pop00, Pop89, Pop02, Swa98]. This theorem also occurs in a handwritten manuscript by M. André (Lausanne, 1991).

Theorem 3.2 (General Néron desingularization, Popescu [Pop85, Pop86, Pop90, Pop00, Pop19], Swan [Swa98]). *Let $u\colon A \to A'$ be a regular morphism of Noetherian rings and B an A-algebra of finite type. Then any A-morphism $v\colon B \to A'$ factors through a smooth A-algebra C; that is, v is a composite A-morphism $B \to C \to A'$.*

Let A be a Noetherian ring, $E = A[Y]/I$ and $Y = (Y_1, \dots, Y_q)$. If $f = (f_1, \dots, f_r)$ with $r \leqslant q$ is a system of polynomials from I, then we can define the ideal Δ_f generated by all $(r \times r)$-minors of the Jacobian matrix $(\partial f_i/\partial Y_j)$. After Elkik [Elk73], let $H_{E/A}$ be the radical of the ideal $\sum_f ((f) : I)\Delta_f B$, where the sum is taken over all systems of polynomials f from I with $r \leqslant q$. Then $H_{E/A}$ defines the non-smooth locus of E over A.

Proposition 3.3. *In the notation and hypotheses of Lemma 3.1, let $E \subset R$ be a C-subalgebra of finite type, where $C := (\mathbb{Z}[x_2, \dots, x_n])_{(p,x_2,\dots,x_n)} \cong (\mathbb{Z}[X_2, \dots, X_n])_{(p,X_2,\dots,X_n)}$. Suppose $n > 1$. Then the inclusion $v\colon E \to R$ factors through a finite-type C-algebra F; let us say that v is the composite map $E \to F_1 \xrightarrow{w} R$ such that $w(H_{F/C})R$ contains a power of p.*

Proof. Let q be a minimal prime ideal of $h_E = \sqrt{v(H_{E/C})R}$ which does not contain p. Then $C \to R_q$ is regular and, using Theorem 3.2 or [Pop19, Lemma 8], we see that v factors through a finite type C-algebra F_1; let us say that v is the composite map $E \to F \xrightarrow{w_1} R$, with w such that $h_{F_1} = \sqrt{w_1(H_{F_1/C})R}$ strictly contains q. Note that w_1 is not necessarily injective. In this way, step by step, we arrive at some new F_r and $w_r\colon F_r \to R$ such that all minimal prime ideals of $h_{F_r} := \sqrt{w_r(H_{F_r/C})R}$ contain p; that is, $p \in h_{F_r}$. We are done. □

We will need the next result, which is in fact [Pop19, Proposition 5] (see also [KPP18b, Proposition 3]) written in our special case.

Proposition 3.4. *Let A and A' be Noetherian rings and $u\colon A \to A'$ a ring morphism. Suppose that A' is local, let $E = A[Y]/I$, where $Y = (Y_1, \dots, Y_s)$, let $f = (f_1, \dots, f_r)$, with $r \leqslant s$, be a system of polynomials from I, let $(M_j)_{j\in[l]}$ be some $(r \times r)$-minors of the Jacobian matrix $(\partial f_i/\partial Y_{j'})$, let $(N_j)_{j\in[l]} \in ((f) : I)$, and set $P := \sum_{j=1}^{l} N_j M_j$. Let $v\colon E \to A'$ be an A-morphism. Suppose that*

1. *there exists a non–zero divisor $d \in A$ such that $d \equiv P$ modulo I, which is also a non–zero divisor in A'; and*
2. *there exist an A-algebra D of finite type and an A-morphism $\omega\colon D \to A'$ such that d is a non–zero divisor in D, $\operatorname{Im} v \subset \operatorname{Im} \omega + d^3 A'$ and for $\bar{A} = A/(d^3)$, the map $\bar{v} = \bar{A} \otimes_A v\colon \bar{E} = E/d^3E \to \bar{A}' = A'/d^3A'$ factors through $\bar{D} = D/d^3D$.*

Then there exists a smooth D-algebra D' such that v factors through D'; let us say that v is the composite map $D \to D' \xrightarrow{w} A'$ and $h_D = \sqrt{\omega(H_{D/A})A'} \subset h_{D'} = \sqrt{w(H_{D'/A})A'}$.

Remark 3.5. Actually, the proof from [Pop19, Proposition 5] also asks for D and A' to be flat over A. In our case, it is only necessary to assume that d is a non–zero divisor in D and A'. We can even assume that d is not regular in A', but in this case we should change d^3 to d^{3c} for some c with $(0 :_{A'} d^c) = (0 :_{A'} d^{c+1})$.

The next theorem extends Theorem 2.5 in the case when k is not separably generated over $\mathbb{F}_p$.

Theorem 3.6. *Let $(A, \mathfrak{m}, k)$ be a Noetherian local ring, $s = (s_1, \dots, s_m)$ some positive integers and $\gamma = (\gamma_1, \dots, \gamma_m)$ a system of nilpotents of A. Suppose $0 \neq p \in \mathfrak{m}^2$ and that $R = A/(\gamma)$ is a DVR and A is a flat $\mathfrak{N} = \mathbb{Z}_{(p)}[\Gamma]/(\Gamma^s)$-algebra, where $\Gamma \mapsto \gamma$ with $\Gamma = (\Gamma_1, \dots, \Gamma_m)$ some variables and (Γ^s) denotes the ideal $(\Gamma_1^{s_1}, \dots, \Gamma_m^{s_m})$. Then A is a filtered inductive limit of some Noetherian local $\mathfrak{N}$-algebras $(A_i)_i$ essentially of finite type with $A_i/\Gamma A_i$ regular local rings. In particular, a DVR is a filtered inductive limit of some regular local rings, essentially of finite type over $\mathbb{Z}$.*

Proof. Let $x \in A$ be an element inducing a local parameter of R, let $E \subset A$ be a finite-type $\mathfrak{N}$-algebra, and let $v\colon E \to A$ be the inclusion. Actually, we could simply take a morphism (not necessarily injective) $v\colon E \to A$ for some finite-type $\mathfrak{N}$-algebra E. We may assume $p \equiv x^e t_0$ modulo (γ) for some $e \in \mathbb{N}$ and $t_0 \in A \setminus \mathfrak{m}$. Moreover, we may assume $p = x^e t_0 + \sum_{j=1}^m t_j \gamma_j$ for some $t_j \in A$, where $j \in [m]$. As in Proposition 3.3, we may assume that $v(H_{E/\mathfrak{N}})A$ contains a power b of p because the map $\mathfrak{N} \to A_q$ is a regular map for all $q \in \operatorname{Spec} A$ with $p \notin q$.

Following the proof of [KPP18b, Theorem 2], we assume $b = \sum_{i=1}^q v(a_i) z_i$, where $z_i \in A$ and $a_i \in H_{E/\mathfrak{N}}$. Set $E_0 = E[Z]/(f)$, where $Z = (Z_1, \dots, Z_q)$ and $f = -b + \sum_{i=1}^q a_i Z_i \in E[Z]$, and let $v_0\colon E_0 \to A$ be the map of E-algebras given by $Z \mapsto z$. Changing E to E_0, we may assume $b \in H_{E/\mathfrak{N}}$.

Let $E \cong \mathfrak{N}[Y]/I$, where $Y = (Y_1, \dots Y_m)$. Using [KPP18b, Lemma 4], we may change E again to assume that a power d of b is in $((f) : I)\Delta_f$ for some $f = (f_1, \dots, f_r)$ from I, where $r \leqslant m$ and where Δ_f denotes the ideal generated by the $(r \times r)$-minors of $(\partial f/\partial Y)$. Then d has the form $d \equiv P = \sum_{i=1}^q M_i L_i$ modulo I for some $r \times r$ minors M_i of $(\partial f/\partial Y)$ and $L_i \in ((f) : I)$.

Let $(\Lambda, (p, \Gamma), k)$ be the (unique) Cohen $\mathfrak{N}$-algebra with residue field k; that is, Λ is complete, and the map $\mathfrak{N} \to \Lambda$ is flat (even regular) (see, for example, [Mat80, Theorem 83]). Then Λ is a filtered inductive limit of some essentially smooth Noetherian local $\mathfrak{N}$-algebras $(C_j)_j$ (see Theorem 3.2). The completion of the DVR $A/(\gamma)$ is a finite free module over the DVR $\Lambda/\Gamma\Lambda$ with the base $\{1, x, \dots, x^{e-1}\}$, and so $A/(\gamma, d^3)$ is a finite free module over $\Lambda/(\Gamma, d^3)$ with the same base. Using [Mat80, (20.G)] applied to $\mathfrak{N} \to \Lambda \to A$, we get that A flat over Λ. It follows that $A/(d^3)$ is a finite free module over $\Lambda/(d^3)$ with the same base and $t_j \equiv \sum_{i=0}^{e-1} t_{ji} x^i$ modulo d^3 for some $t_{ji} \in \Lambda$. Thus there exists a smooth $\mathfrak{N}$-algebra C such that the $(t_{ji})_{ji}$ are contained in the image of the canonical map $\psi\colon C \to \Lambda$; let us say that the t_{ji} are the images of some $t'_{ji} \in C$. Let φ be the composite map

$$C[X]/(d^3) \to \Lambda[X]/(d^3) \to \tilde{A} = A/(d^3)$$

induced by ψ and $X \mapsto x$. Actually, $\tilde{A}$ is a filtered inductive limit of $\mathfrak{N}$-algebras of type $\tilde{B} = C[X]/(d^3, p - t'_0 X^e - \sum_{j=1}^m t'_j \Gamma_j)$, where $t'_j = \sum_{i=0}^{e-1} t'_{ji} X^i$ and C is smooth over $\mathfrak{N}$ and containing (t'_{ji}).

We may assume $C = (\mathfrak{N}[U]/(g))_{ag'}$, where $U = (U_1, \dots, U_l)$, for a monic polynomial g (in U_1) and $a \in \mathfrak{N}[U]$, where $g' = \partial g/\partial U_1$ (see, for example, [Swa98, Theorem 2.5]). We may take C such that the map $\tilde{v}\colon \tilde{E} := E/d^3 E \to \tilde{A}$ induced by v factors through $\tilde{B}$; let us say that $\tilde{v}$ is the composite map $\tilde{E} \to \tilde{B} \xrightarrow{\tilde{w}} \tilde{A}$, the last map being the limit map.

Choose a lifting $u = (u_i)_{1 \leqslant i \leqslant l}$ of $(\tilde{w}(U_i))_{1 \leqslant i \leqslant l}$ in A. We have $g(u) = d^3 z$ for some $z \in A$. Set $D_1 = C[X, U, Z]/(g - d^3 Z)$, and let $w\colon D_1 \to A$ be given by $(X, U, Z) \mapsto (x, u, z)$. Clearly, $D_2 = (D_1/(\Gamma))_{w^{-1}((x))}$ is a regular local ring that is essentially smooth over a polynomial ring over $\mathbb{Z}$, and p and X are contained in a regular system of parameters of D_2. Then $D_2/(p - t'_0 X^e)$ is a regular local ring; denote the map

$$D = D_1 \Big/ \Big(p - t'_0 X^e - \sum_{j=1}^m t'_j \Gamma_j\Big) \to A$$

induced by w also by w.

We have $\operatorname{Im} v \subset \operatorname{Im} w + d^3 A$, and v factors modulo d^3 through w. Since d is a non-zero divisor in D and A, we may apply Proposition 3.4 (see also Remark 3.5) for $\mathfrak{N} \to A$, E, D and $\omega = w$. We get an essentially smooth D-algebra D' such that v factors through D'. Since $D/\Gamma D$ is regular local, we get that $D'/\Gamma D'$ is regular too. This is enough using, for example, [Swa98, Lemma 1.5] (see below) with $\mathcal{S}$ the set of the regular local $\mathfrak{N}$-algebras. □

Lemma 3.7. *Let $\mathcal{S}$ be a class of finitely presented $\mathfrak{N}$-algebras. Let A be a $\mathfrak{N}$-algebra. Then the following statements are equivalent:*

(1) *The $\mathfrak{N}$-algebra A is a filtered inductive limit of algebras from $\mathcal{S}$.*

(2) *If E is a finitely presented $\mathfrak{N}$-algebra and $v\colon E \to A$ is a morphism, then v factors through an algebra from $\mathcal{S}$.*

The following theorem is a positive answer to a question of Swan [Mur87].

Theorem 3.8. *Let $(A, \mathfrak{m}, k)$ be a Noetherian local ring, $s = (s_1, \ldots, s_m)$ some positive integers and $\gamma = (\gamma_1, \ldots, \gamma_m)$ a system of nilpotents of A. Suppose $0 \neq p \in \mathfrak{m}^2$ and that $R = A/(\gamma)$ is a regular local ring and A is a flat $\mathfrak{N} = \mathbb{Z}_{(p)}[\Gamma]/(\Gamma^s)$-algebra, where $\Gamma \mapsto \gamma$ with $\Gamma = (\Gamma_1, \ldots, \Gamma_m)$ some variables and (Γ^s) denotes the ideal $(\Gamma_1^{s_1} \cdots \Gamma_m^{s_m})$. Then A is a filtered inductive limit of some Noetherian local $\mathfrak{N}$-algebras $(F_i)_i$ essentially of finite type with $F_i/\Gamma F_i$ regular local rings. In particular, a regular local ring $(R, \mathfrak{m}, k)$ of unequal characteristic is a filtered inductive limit of regular local rings essentially of finite type over $\mathbb{Z}$.*

Proof. Let $x_1, \ldots, x_n \in A$, for $n = \dim A$, be a system of elements defining a regular system of parameters in R, let $E \subset A$ be a finite type $\mathfrak{N}$-algebra, and let $v\colon E \to A$ be the inclusion. Suppose $n > 1$ by Theorem 3.6, and let $p \equiv t_0 \prod_{i=1}^{l_1} z_{1i}^{\alpha_{1i}}$ modulo (γ) for some $\alpha_{1i} \in \mathbb{N}$, $t_0 \in A \setminus \mathfrak{m}$ and some elements z_{1i} of A inducing irreducible elements in R, with $(z_{1i}, \gamma) \neq (z_{1i'}, \gamma)$ for $i \neq i'$, where R is a unique factorization domain by hypothesis. Moreover, we may suppose $p = t_0 \prod_{i=1}^{l_1} z_{1i}^{\alpha_{1i}} + \sum_{j=1}^{m} t_j \gamma_j$ for some $t_j \in A$, for $j \in [m]$.

As in Proposition 3.3, we may assume that $v(H_{E/\mathfrak{N}})A$ contains a power of p and as in Theorem 3.6, we may assume $E \cong \mathfrak{N}[Y]/I$, $Y = (Y_1, \ldots, Y_m)$ and that a power d of p is in $((f) : I)\Delta_f$ for some $f = (f_1, \ldots, f_r)$ from I, where $r \leqslant m$. Then d has the form $d \equiv P = \sum_{i=1}^{q} M_i L_i$ modulo I for some $(r \times r)$-minors M_i of $(\partial f/\partial Y)$ and $L_i \in ((f) : I)$.

Consider as in Theorem 3.6 the Cohen $\mathfrak{N}$-algebra $(\Lambda_{1i}, (p, \Gamma), \mathrm{Fr}(A/(z_{1i}, \gamma)))$ with residue field $\mathrm{Fr}(A/(z_{1i}, \gamma))$; that is, Λ_{1i} is complete, and the map $\mathfrak{N} \to \Lambda_{1i}$ is flat (even regular). The $\mathfrak{N}$-algebra Λ_{1i} is a filtered inductive limit of some essentially smooth Noetherian local $\mathfrak{N}$-algebras $(C_{ij})_j$ (see Theorem 3.2). Then the completion of the DVR $R_{(z_{1i})}$ is a finite free module over the DVR $\Lambda_{1i}/\Gamma\Lambda_{1i}$ with the base $\{1, z_{1i}, \ldots, z_{1i}^{\alpha_{1i}-1}\}$, and so $\left(A/\left(\gamma, d^3\right)\right)_{(z_{1i})}$ is a finite free module over $\Lambda_{1i}/\left(\Gamma, d^3\right)$ with the same base. It follows that $\left(A/\left(d^3\right)\right)_{(z_{1i},\gamma)}$ is a finite free module over $\Lambda_{1i}/\left(d^3\right)$ with the same base.

Set $S = A \setminus \cup_{i=1}^{l_i}(z_{1i}, \gamma)A$, $\tilde{A} := A/\left(d^3\right)$ and $\tilde{\mathfrak{N}} := \mathfrak{N}/\left(d^3\right)$. Then $S^{-1}\tilde{A}$ is a product of the local Artinian rings $\tilde{A}_{(z_{1i},\gamma)}$, which are finite free modules over $\left(\tilde{\Lambda}_{1i}\right) = \Lambda_{1i}/\left(d^3\right)$; the base of $S^{-1}\tilde{A}$ over $\tilde{\Lambda}_1 := \prod_{i=1}^{l_1} \tilde{\Lambda}_{1i}$ is given by

$$\left\{z_{11}^{\varepsilon_{11}} \cdots z_{1l_1}^{\varepsilon_{1l_1}} : 0 \leqslant \varepsilon_{11} < \alpha_{11}, \ldots, 0 \leqslant \varepsilon_{1l_1} < \alpha_{1l_1}\right\}.$$

Let $t_j \equiv \sum t_{j\varepsilon_{11}\ldots\varepsilon_{1l_1}} z_{11}^{\varepsilon_{11}} \cdots z_{1l_1}^{\varepsilon_{1l_1}}$ modulo $\left(d^3\right)$ for some $t_{j\varepsilon_{11}\ldots\varepsilon_{1l_1}} \in \tilde{\Lambda}_1$.

Then we may choose some smooth $\tilde{\mathfrak{N}}$-algebras of type $(\tilde{C}_{1i})_i$ such that the limit maps give a map $\tilde{\psi}\colon \tilde{C}_1 := \prod_{i=1}^{l_1} \tilde{C}_{1i} \to \tilde{\Lambda}_1$ whose image contains $(t_{j\varepsilon_{11}\ldots\varepsilon_{1l_1}})$; let us say that $t_{j\varepsilon_{11}\ldots\varepsilon_{1l_1}} = \tilde{\psi}(t'_{j\varepsilon_{11}\ldots\varepsilon_{1l_1}})$ for some $t'_{j\varepsilon_{11}\ldots\varepsilon_{1l_1}} \in \tilde{C}_1$. Let $\tilde{\varphi}_1\colon \tilde{C}_1[Z_{11}, \ldots, Z_{1l_1}] \to S^{-1}\tilde{A}$ for some new variables

$Z_{11}, \dots, Z_{1l_1}$ be the map induced by $\tilde{\psi}$ and $Z_{1i} \mapsto z_{1i}$. Note that

$$t'_j = \sum t'_{j\varepsilon_{11}\dots\varepsilon_{1l_1}} Z_{11}^{\varepsilon_{11}} \cdots Z_{1l_1}^{\varepsilon_{1l_1}} \in \tilde{C}_1[Z_{11}, \dots, Z_{1l_1}]$$

is mapped to t_j modulo d^3 and $S^{-1}(\tilde{A})$ is a filtered inductive limit of localizations of $\tilde{\mathfrak{N}}$-algebras of finite type

$$\tilde{B}_1 := \tilde{C}_1[Z_{11}, \dots, Z_{1l_1}] \Big/ \Big(p - t'_0 Z_{11}^{\alpha_{11}} \cdots Z_{1l_1}^{\alpha_{1l_1}} - \sum_{j \in [m]} t'_j \Gamma_j\Big).$$

Thus the composite map $E \xrightarrow{v} A \to S^{-1}\tilde{A}$ factors through such an algebra $\tilde{B}_1$.

Clearly, $\tilde{C}_1$ is a localization of a finite-type $\tilde{\mathfrak{N}}$-algebra $\tilde{C}$ with $\tilde{\varphi}_1(\tilde{C}) \subset \tilde{A}$ and $\tilde{\varphi}_1(H_{\tilde{C}/\tilde{\mathfrak{N}}}) \not\subset \cup_{i=1}^{l_1}(z_{1i}, \gamma)\tilde{A}$. Unfortunately, we cannot assume $t'_j \in \tilde{C}[Z_{11}, \dots, Z_{1l_1}]$, but we may suppose that there exist $c \in \tilde{C}$ and $t''_j \in \tilde{C}[Z_{11}, \dots, Z_{1l_1}]$ such that $ct'_j = t''_j$ and $\tilde{\varphi}_1(c) \in \tilde{A} \setminus \cup_{i=1}^{l_i}(z_{1i}, \gamma)\tilde{A}$. It follows that $\tilde{B}_1$ is a localization of

$$\tilde{B} = \tilde{C}[T', Z_{11}, \dots, Z_{1l_1}] \Big/ \Big(cT'_0 - t''_0, \dots, cT'_m - t''_m, p - T'_0 Z_{11}^{\alpha_{11}} \cdots Z_{1l_1}^{\alpha_{1l_1}} - \sum_{j \in [m]} T'_j \Gamma_j\Big),$$

$T' = (T'_0, \dots, T'_m)$ being some new variables, and we may consider the map $\tilde{B} \xrightarrow{\tilde{\varphi}} \tilde{A}$ given by $T' \mapsto (t_0, \dots, t_m)$ and $Z_{1j} \mapsto z_{1j}$ modulo d^3.

Moreover, we may assume that the composite map $E \xrightarrow{v} A \to \tilde{A}$ factors through $\tilde{\varphi}$. The map $\tilde{G}_1 := \tilde{C}[T', Z_{11}, \dots, Z_{1l_1}]/(cT'_0 - t''_0, \dots, cT'_m - t''_m) \xrightarrow{\tilde{w}_1} \tilde{A}$ given by $T' \mapsto (t_0, \dots, t_m)$ and $Z_{1j} \mapsto z_{1j}$ modulo d^3 satisfies $\tilde{w}_1(H_{\tilde{G}_1/\tilde{\mathfrak{N}}}) \not\subset \cup_{i=1}^{l_i}(z_{1i}, \gamma)\tilde{A}$. Clearly,

$$\tilde{B} = \tilde{G}_1 \Big/ \Big(p - T'_0 Z_{11}^{\alpha_{11}} \cdots Z_{1l_1}^{\alpha_{1l_1}} - \sum_{j \in [m]} T'_j \Gamma_j)\Big).$$

We show by induction on n that there exist an essentially smooth local $\mathfrak{N}$-algebra $(D, \mathfrak{q})$ with p from a system of regular parameters modulo (Γ) of $D/\Gamma D$ and $b \in \mathfrak{q}^2$ such that v factors through a smooth $D/(p-b)$-algebra D'. This is enough because $D/(p-b, \Gamma)$ is still regular and so $D'/\Gamma D'$ is too. The case $n = 1$ is done in Theorem 3.6.

Case $n = 2$. We may suppose that x_2 is regular in $\tilde{A}$. If $\tilde{w}'_1(H_{\tilde{G}_1/\tilde{\mathfrak{N}}})\tilde{A} = \tilde{A}$, then we may change $\tilde{G}_1$ to be essentially smooth over $\tilde{\mathfrak{N}}$ (see, for example, [Pop85, Lemma 2.4]). Thus we may consider that $\tilde{G}_1/\Gamma\tilde{G}_1$ and hence also $\tilde{B}/\Gamma\tilde{B}$ are regular local rings, which ends the proof using Proposition 3.4 as in Theorem 3.6.

Otherwise, the ideal $\tilde{w}'_1(H_{\tilde{G}_1/\tilde{\mathfrak{N}}})\tilde{A}$ is $\mathfrak{m}$-primary and a power x_2^e of x_2 is in $\tilde{w}'_1(H_{\tilde{G}_1/\tilde{\mathfrak{N}}})\tilde{A}$. Moreover, as above, we may assume $x_2^e \equiv \sum_i M'_i N'_i$ modulo I' for some M'_i, N'_i, I' associated with $\tilde{G}_1$, which has the form $\tilde{\mathfrak{N}}[Y']/\tilde{I}$, where $Y' = (Y'_1, \dots, Y'_{m'})$.

Note that $\bar{A} := A/(x_2^{3e})$ is a flat $\bar{\mathfrak{N}} = \mathfrak{N}[X_2]/(X_2^{3e})$-algebra, where $X_2 \mapsto x_2$. By Theorem 3.6 applied to (Γ, X_2) and $(s, 3e)$, we see that $\bar{A} := A/(x_2^{3e})$ is a filtered inductive limit of some Noetherian local $\bar{\mathfrak{N}}$-algebras of type $G/(p-b)$, where $(G, \mathfrak{a})$ is essentially smooth over $\bar{\mathfrak{N}}$, the parameter p is from a regular system of parameters modulo (Γ, X_2) of $G/(\Gamma, X_2)G$ and $b \in \mathfrak{a}^2$. Let $\bar{\omega}\colon G/(p-b) \to \bar{A}$ be the limit map. We may suppose that the composite map $\tilde{G}_1 \xrightarrow{\tilde{w}_1} \tilde{A} \to \tilde{A}/(x_2^{3e}) = \bar{A}/(d^3)$ factors through $\tilde{\mathfrak{N}} \otimes_{\mathfrak{N}} \bar{\omega}$.

We claim that there exists a Noetherian local $\mathfrak{N}$-algebra D such that $D/(\Gamma)$ is regular and $\bar{\omega}$

factors through $D/X_2^{3e}D$. Indeed, G is a localization of a $\bar{\mathfrak{N}}$-algebra of type $(\bar{\mathfrak{N}}[W]/(g))_{\partial g/\partial W_1}$ for some variables $W=(W_1,\dots,W_\lambda)$ and g a monic polynomial in W_1 since G is essentially smooth over $\bar{\mathfrak{N}}$. Take ε in A lifting $\bar{\omega}(W)$ and $\delta,\delta'\in A$ such that $g(\varepsilon)=x_2^{3e}\delta$ and $(p-b)(\varepsilon)=x_2^{3e}\delta'$, and let D be the corresponding localization of

$$\big(\mathfrak{N}[X_2,W,\Delta,\Delta']/\big(g-X_2^{3e}\Delta,p-b-X_2^{3e}\Delta'\big)\big)_{\partial g/\partial W_1},$$

Δ and Δ' being new variables. Let $\rho\colon D\to A$ be the map given by $W\mapsto g(W)$, $\Delta\mapsto\delta$, $\Delta'\mapsto\delta'$. Then $\bar{\omega}$ factors through $D/X_2^{3e}D\otimes\rho$. Since $\big(\bar{\mathfrak{N}}[W,\Delta]/\big(g-X_2^{3e}\Delta\big)\big)_{\partial g/\partial W_1}$ is smooth over $\bar{\mathfrak{N}}$, the corresponding localization C of $\big(\mathfrak{N}[X_2,W,\Delta]/\big(g-X_2^{3e}\Delta\big)\big)_{\partial g/\partial W_1}$ is such that $C/\Gamma C$ is regular, and p is from a regular local system of parameters of $C/\Gamma C$. It follows that $D/\Gamma D$ is regular local, which shows our claim.

Using Proposition 3.4 for $\tilde{\mathfrak{N}}[X_2]\to\tilde{A}$, $X_2\mapsto x_2$, $\tilde{\mathfrak{N}}[X_2]\otimes_{\tilde{\mathfrak{N}}}\tilde{G}_1$, $\tilde{D}=D/d^3D$ and $\tilde{\mathfrak{N}}\otimes_{\mathfrak{N}}\rho$, we see that $\tilde{w}_1$ factors through an essentially smooth $\tilde{D}$-algebra $\tilde{D}'$. Then we may find an essentially smooth D-algebra D' such that $\tilde{\mathfrak{N}}\otimes_{\mathfrak{N}}D'\cong\tilde{D}'$. Clearly, $D'/\Gamma D'$ is regular because $D/\Gamma D$ is. The map $\tilde{D}'\to\tilde{A}$ lifts by the implicit function theorem to a D-morphism $\rho'\colon D'\to A$ such that $\tilde{\mathfrak{N}}\otimes_{\mathfrak{N}}v$ factors through ρ' modulo d^3. Applying Proposition 3.4 again for $\mathfrak{N}\to A$, E, D', ρ', we see that v factors through an essentially smooth D'-algebra D'', where $D''/\Gamma D''$ is regular since $D'/\Gamma D'$ is.

Case $n>2$. As above, we may suppose $\tilde{w}_1'(H_{\tilde{G}_1/\tilde{\mathfrak{N}}})\tilde{A}\subset\mathfrak{m}\tilde{A}$. We will show that given a prime ideal $q\in\operatorname{Spec}A$ such that $q\tilde{A}$ is a minimal prime ideal of $\tilde{w}_1'(H_{\tilde{G}_1/\tilde{\mathfrak{N}}})$, the map $\tilde{w}_1'$ factors through a finite type $\tilde{\mathfrak{N}}$-algebra of the form $\tilde{P}/(p-b)$, where $b\in\tilde{P}$; let us say that $\tilde{w}_1'$ is the composite map $\tilde{G}_1\to\tilde{P}/(p-b)\xrightarrow{\tilde{\mu}}\tilde{A}'$, where $\tilde{A}'$ is a factor of $\tilde{A}$ and $\tilde{\mu}$ is induced by a map $\tilde{\mu}'\colon\tilde{P}\to\tilde{A}'$ with $\tilde{w}_1'(H_{\tilde{G}_1/\tilde{\mathfrak{N}}})\tilde{A}'\subset\sqrt{\tilde{\mu}(H_{\tilde{P}/\tilde{\mathfrak{N}}})\tilde{A}'}\not\subset q\tilde{A}'$.

For the beginning, we assume $\operatorname{height}(q)=2$. Then $qA_q=(z_1,z_2,\gamma)A_q$ for some $z_1,z_2\in q$ because R_q is a regular local ring of dimension 2. Assume that $(p,z_2,\gamma)A_q$ is a qA_q-primary ideal. Let $\mathfrak{N}_2=\mathfrak{N}[Z_2]_{(p,Z_2,\Gamma)}$, and that Λ_2 be the (unique) Cohen $\mathfrak{N}_2$-algebra with the residue field $\operatorname{Fr}(A/q)$; that is, Λ_2 is complete, and the map $\mathfrak{N}_2\to\Lambda_2$ is flat (even regular). Then the completion $\hat{R}_q$ of the regular local ring R_q is a finite free module over $\Lambda_2/(\Gamma)$. We have $\lambda_2z_2^{e_2}\in\tilde{w}_1'(H_{\tilde{G}_1/\tilde{\mathfrak{N}}})\tilde{A}$ for some $\lambda_2\in A\setminus q$ and $e_2\in\mathbb{N}$. Changing z_2 to λ_2z_2, we may suppose $\lambda_2=1$. Set $\tilde{\mathfrak{N}}_2=\mathfrak{N}_2/\big(d^3\big)$. Changing $\tilde{G}_1$ to $\tilde{\mathfrak{N}}_2\otimes_{\tilde{\mathfrak{N}}}\tilde{G}_1$ and $\tilde{w}_1'$ canonically, we may suppose that $\tilde{G}_1$ is a $\tilde{\mathfrak{N}}_2$-algebra.

As in the case $n=2$, we may assume $\tilde{G}_1\cong\tilde{\mathfrak{N}}_2[Y']/\tilde{I}$ and $z_2^{e_2}\equiv\sum_jM_jL_j$ modulo $\tilde{I}$ for a system of polynomials $f^{(2)}$ from $\tilde{I}$, some polynomials $L_j\in\big(\big(f^{(2)}\big):\tilde{I}\big)$ and some minors M_j of $\big(\partial f^{(2)}/\partial Y'\big)$. Set $\tilde{\Lambda}_2=\Lambda_2/\big(d^3\big)$. Note that $\tilde{A}_q/\big(z_2^{3e_2},\gamma\big)\cong\hat{A}_q/\big(d^3,z_2^{3e_2},\gamma\big)$ is finite free over $\tilde{\Lambda}_2/\big(Z_2^{3e_2},\Gamma\big)$ with the basis $\big\{z_1^iz_2^j\colon(i,j)\in N_2\big\}$ for some $N_2\subset[0,3e_1]\times[0,3e_2]$ and $e_1\in\mathbb{N}$ (we may suppose $p\notin\big(z_1^{e_1},z_2^{e_2}\big)$, increasing e_1 and e_2 and changing L_i if necessary). It follows that $\tilde{A}_q/\big(z_2^{3e_2}\big)$ is finite free over $\tilde{\Lambda}_2/\big(Z_2^{3e_2}\big)$ with the same basis. We have

$$p\equiv\sum_{(i,j)\in N_2'}t_{20ij}z_1^iz_2^j+\sum_{\lambda\in[m],(i,j)\in N_2''}t_{2\lambda ij}z_1^iz_2^j\gamma_\lambda\quad\operatorname{mod}\big(d^3,z_2^{3e_2}\big)$$

for some $t_{20ij}\in A\setminus q$ and $t_{2\lambda ij}\in A$ and some subsets $N_2',N_2''\subset[0,3e_1]\times[0,3e_2]$. The $\mathfrak{N}_2$-algebra Λ_2 is a filtered inductive limit of smooth $\mathfrak{N}_2$-algebras C_2 (see Theorem 3.2); let $\tilde{\psi}_2\colon\tilde{C}_2/\big(Z_2^{3e_2}\big)\to\tilde{\Lambda}_2/\big(Z_2^{3e_2}\big)$, where $\tilde{C}_2:=C_2/\big(d^3\big)$, be the limit map. We may choose $\tilde{C}_2$ such that the image of the map

$$\tilde{\psi}'_2\colon \tilde{C}_2[Z_1]/\big(Z_1^{3e_1}, Z_2^{3e_2}\big) \to \tilde{A}_q/\big(z_1^{3e_1}, z_2^{3e_2}\big), \quad Z_1 \mapsto z_1$$

contains $(t_{2\lambda ij})$; let us say that $t_{2\lambda ij} = \tilde{\psi}'_2(t'_{2\lambda ij})$ for some $t'_{2\lambda ij} \in \tilde{C}_2[Z_1]$, where $0 \leqslant \lambda \leqslant m$. Then $\tilde{\psi}'_2$ maps

$$t'_2 := \sum_{(i,j)\in N'_2} t'_{20ij} Z_1^i Z_2^j + \sum_{(i,j)\in N''_2, \lambda\in[m]} t'_{2\lambda ij} Z_1^i Z_2^j \Gamma_\lambda$$

to p modulo $\big(d^3, z_2^{3se_2}\big)$.

Note that $\tilde{C}_2$ is a localization of a finite type $\tilde{\mathfrak{N}}_2$-algebra $\tilde{C}'_2$ with $\tilde{\psi}'_2(\tilde{C}'_2) \subset \tilde{A}/\big(z_2^{e_2}\big)$ and $\tilde{\psi}'_2(H_{\tilde{C}'_2/\tilde{\mathfrak{N}}_2}) \not\subset q\tilde{A}/\big(z_2^{3e_2}\big)$. Usually, we have $t'_{2\lambda ij} \notin \tilde{C}'_2[Z_1]$, but there exist $c, t''_{2\lambda ij} \in \tilde{C}'_2[Z_1]$ such that $\tilde{\psi}'_2(c) \in \tilde{A}/\big(z_1^{3e_1}, z_2^{3e_2}\big) \setminus q\tilde{A}/\big(z_1^{3se_1}, z_2^{3e_2}\big)$ and $ct'_{2\lambda ij} = t''_{2\lambda ij}$. Since $\tilde{A}_q/\big(z_1^{3e_1}, z_2^{3e_2}\big)$ is a filtered inductive limit of $\mathfrak{N}_2$-algebras of type $\tilde{B}_2 = \tilde{C}_2[Z_1]/\big(Z_1^{3e_1}, Z_2^{3e_2}, p - t'_2\big)$, we see that the composite map $\tilde{G}_1 \xrightarrow{\tilde{w}'_1} \tilde{A} \to \tilde{A}_q/\big(z_1^{3e_1}, z_2^{3e_2}\big)$ factors to such a $\mathfrak{N}_2$-algebra $\tilde{B}_2$.

Set $\tilde{P}_2 := \tilde{C}'_2[Z_1, T']/\big((cT'_{2\lambda ij} - t''_{20\lambda ij})_{\lambda ij}, Z_1^{3e_1}, Z_2^{3e_2}\big)$ for some new variables $T' = (T'_{2\lambda ij})_{\lambda ij}$. Let $\tilde{\mu}'_2\colon \tilde{P}_2 \to \tilde{A}/\big(z_1^{3e_1}, z_2^{3e_2}\big)$ be given by $T' \mapsto (t_{2\lambda ij})$. Note that $\tilde{\mu}'_2(H_{\tilde{P}_2/\tilde{\mathfrak{N}}_2}) \not\subset q\tilde{A}/\big(z_1^{3e_1}, z_2^{3e_2}\big)$. We may assume that the composite map $\tilde{G}_1 \xrightarrow{\tilde{w}'_1} \tilde{A} \to \tilde{A}/\big(z_1^{3e_1}, z_2^{3e_2}\big)$ factors to a $\mathfrak{N}_2$-algebra

$$\tilde{B}'_2 = \tilde{P}_2 \Big/ \Bigg(p - \sum_{(i,j)\in N'_2} T'_{20ij} Z_1^i Z_2^j + \sum_{(i,j)\in N''_2, 1\leqslant\lambda\leqslant m} T'_{2\lambda ij} Z_1^i Z_2^j \Gamma_\lambda\Bigg).$$

Step by step, using this procedure, we may find some elements $(z_i)_{2\leqslant i\leqslant \nu}$, some numbers $(e_i)_i$, some finite-type $\tilde{\mathfrak{N}}_i = \tilde{\mathfrak{N}}_2[Z_3, \ldots, Z_i]/\big(Z_2^{3e_2}, \ldots, Z_i^{3e_i}\big)$-algebras $\tilde{P}_i$, some morphisms $\tilde{w}'_i\colon \tilde{P}_i \to A_i := \tilde{A}/\big(z_3^{3e_2}, \ldots, z_i^{3e_i}\big)$ and some elements $b_i \in \tilde{P}_i$, for $3 \leqslant i \leqslant \nu$, such that for all $3 \leqslant i \leqslant \nu$,

- (i) the composite map $\tilde{P}_{i-1} \xrightarrow{\tilde{w}'_{i-1}} A_{i-1} \to A_i$ factors through $\tilde{P}_i/(p - b_i)$; namely, $\tilde{w}'_{i-1}$ is the composite map $\tilde{P}_{i-1} \to \tilde{P}_i/(p - b_i) \xrightarrow{\tilde{w}_i} A_i$, where $\tilde{w}_i$ is induced by $\tilde{w}'_i$;
- (ii) there exists a strict inclusion $\sqrt{\tilde{w}'_{i-1}\big(H_{\tilde{P}_{i-1}/\tilde{\mathfrak{N}}_{i-1}}\big) A_i} \subset \sqrt{\tilde{w}'_i\big(H_{\tilde{P}_i/\tilde{\mathfrak{N}}_i} A_i\big)}$;
- (iii) $\big(0 :_{A_{i-1}} z_i^{e_i}\big) = \big(0 :_{A_{i-1}} z_i^{e_i+1}\big)$ and $z_i^{e_i} \in \tilde{w}'_{i-1}\big(H_{\tilde{P}_{i-1}/\tilde{\mathfrak{N}}_{i-1}}\big) A_i$.

By Noetherian induction, we may assume that $\tilde{w}'_\nu(H_{\tilde{P}_\nu/\tilde{\mathfrak{N}}_\nu})\tilde{A}_\nu$ is $\mathfrak{m}\tilde{A}$-primary. We may choose x_n such that it is regular in $(R_i)_{i\in[\nu]}$ and a power $x_n^{e'}$ of x_n is in $\tilde{w}'_\nu\big(H_{\tilde{P}_\nu/\tilde{\mathfrak{N}}_\nu}\big)\tilde{A}_\nu$ (we can assume $x_n^{e'} \in \Delta_f((f) : J)$ for some f and J, defining P_ν as usual).

Induct on $n > 1$. Using the induction hypothesis, as in the case $n = 2$, the quotient $A/\big(x_n^{3e'}\big)$ is a filtered inductive limit of some Noetherian local $\tilde{\mathfrak{N}}' = \tilde{\mathfrak{N}}[X_n]/\big(X_n^{3e'}\big)$-algebras of type $G/(p - b)$, where $(G, \mathfrak{a})$ is essentially smooth over $\tilde{\mathfrak{N}}'$, the parameter p is from a system of regular parameters of $G/(\Gamma, X_n)$ and $b \in \mathfrak{a}^2$. Let $\bar{\omega}\colon G/(p - b) \to A/\big(x_n^{3e'}\big)$ be the limit map. Then as in the case $n = 2$, we find a Noetherian local $\mathfrak{N}$-algebra D such that $D/\big(X_n^{3e'}\big) \cong G/(p - b)$, $D/(\Gamma)$ is regular and the composite map $\tilde{P}_\nu \xrightarrow{\tilde{w}'_\nu} A_\nu \to A_\nu/\big(x_n^{3e'}\big)$ factors through $\tilde{\mathfrak{N}}_\nu \otimes_{\tilde{\mathfrak{N}}} \bar{\omega}$. As in the case $n = 2$, using Proposition 3.4 again, we find a Noetherian $\mathfrak{N}$-algebra D' and a map $\rho'\colon D' \to A$ such that $D'/(\Gamma)D'$ is regular and $\tilde{w}'_\nu$ factors through $\tilde{\mathfrak{N}}_\nu \otimes_{\mathfrak{N}} D'$.

In particular, the composite map $\tilde{P}_{\nu-1} \xrightarrow{\tilde{w}'_{\nu-1}} A_{\nu-1} \to A_\nu$ factors through $\tilde{\mathfrak{N}}_\nu \otimes_\varphi D'$, and the procedure continues. By recurrence, using Proposition 3.4 and Remark 3.5, we find a Noetherian local $\mathfrak{N}$-algebra D'' essentially smooth over D and a map $\rho''\colon D'' \to A$ such that $D''/\Gamma D''$ is regular and $\tilde{w}_1'$ factors through $\tilde{\mathfrak{N}} \otimes_{\mathfrak{N}} D''$. Finally the same procedure shows that v factors through a $\mathfrak{N}$-algebra F that is essentially smooth over D, with $F/\Gamma F$ regular. □

Remark 3.9. To show the BQ conjecture for a regular local ring of unequal characteristic by applying Theorem 3.8 we need to prove the following facts:

(i) If the BQ conjecture holds for Λ'_{i-1}, then it holds for Λ_i with $1 \leqslant i \leqslant r$.

(ii) If the BQ conjecture holds for Λ_i, then it holds for Λ'_i with $1 \leqslant i < r$.

The first fact could be proved using the ideas of Lindel and Swan. For the second fact, we have to show the following result:

(iii) If the BQ conjecture holds for a regular local ring A and $b \in A$ belongs to a regular system of parameters of A, then it holds for a localization of $A[W]/\big(W^e - b\big)$, where $e \in \mathbb{N}$ (in fact, for $A[W]/\big(W^e - b\big)$ using [Roi79]).

This might be wrong, as an example from [Lam78] suggests. If $A = \mathbb{R}[X_1]_{(X_1)}$ and $B = \big(A[X_2]/\big(X_2^3 - X_1^2\big)\big)_{(X_1,X_2)}$, then there exist projective modules over $B[T]$ of rank 1 which are not free. On the other hand, Vorst shows in [Vor83] that if the BQ conjecture holds for a regular local ring A, then it holds for a factor of a polynomial A-algebra by a monomial ideal.

Corollary 3.10. *All regular local rings are filtered inductive limits of excellent regular local rings.*

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Appendix. A purity conjecture for flat cohomology

Kęstutis Česnavičius

The following purity conjecture seems to be a part of the folklore.

Conjecture A.1. For a regular local ring $(R, \mathfrak{m})$ of dimension d and a commutative, finite, flat R-group scheme G, the flat, locally of finite presentation (fppf) cohomology with supports vanishes in low degrees as follows:

$$H^i_{\mathfrak{m}}(R, G) = 0 \quad \text{for} \quad i < d\,;$$

equivalently, setting $U_R := \operatorname{Spec}(R) \setminus \{\mathfrak{m}\}$, the pullback map

$$H^i(R, G) \to H^i(U_R, G) \quad \text{is an isomorphism for} \quad i < d - 1 \quad \text{and injective for} \quad i = d - 1\,.$$

As usual, we form the fppf cohomology of a scheme S in the site whose objects are all the S-schemes and coverings are jointly surjective morphisms that are flat and locally of finite presentation.

Examples of known cases

(i) In its equivalent form that involves U_R, the conjecture is known for cohomological degrees $i = 0$ and $i = 1$. More precisely, the map $H^0(R, G) \to H^0(U_R, G)$ is bijective when $d > 0$ by a well-known lemma on sections of finite schemes over normal bases (for the lack of a better reference, see [Čes17, Lemma 3.1.9]). This lemma and the representability of G-torsors also show that the map $H^1(R, G) \to H^1(U_R, G)$ is injective when $d > 0$; by a result of Moret-Bailly [Mor85, Lemme 2] (whose detailed proof may be found, for instance, in [Mar16, Section 3.1]), this map is bijective when $d > 1$. Of course, there is no need to assume that G is commutative for these results in low degrees.

(ii) The case when the order of G is invertible in R is also known; in fact, in this case, by the absolute cohomological purity conjecture proved by Gabber, $H^i_{\mathfrak{m}}(R, G) = 0$ for $i < 2d$; see [Fuj02].

(iii) We have $H^i_{\mathfrak{m}}(R, \mathbb{G}_a) = 0$ for $i < d$ because the depth of R is d (see [Gro05, III.3.4]), so, in the case when R is an $\mathbb{F}_p$-algebra, the conjecture holds for $G = \alpha_p$ (where α_p is the kernel of the absolute Frobenius morphism of $\mathbb{G}_a$).

(iv) By the purity for the Brauer group, whose proof was finished in [Čes19], the groups $H^i_{\mathfrak{m}}(R, \mathbb{G}_m)$ vanish when $d \geqslant 2$ and $i \leqslant 3$; thus, $H^i_{\mathfrak{m}}(R, \mu_p) = 0$ when $i \leqslant 3$ with $i < d$ (where μ_p is the kernel of the absolute Frobenius morphism of $\mathbb{G}_m$).

The goal of this appendix is to illustrate the main results of the paper with the following reduction.

Proposition A.2. *For every regular local ring $(R, \mathfrak{m})$ that is a filtered direct limit of* excellent *regular local rings, the conjecture above reduces to the case when R is complete.*

Proof. By descending the elements that comprise an $(R/\mathfrak{m})$-basis of $\mathfrak{m}/\mathfrak{m}^2$ to a finite level in the direct limit, we may assume that $(R, \mathfrak{m})$ is a filtered direct limit of excellent regular local rings each of whose dimension is at least that of R. Thus, the limit formalism reduces us to the case when R is excellent. Excision (see [Mil80, proof of Proposition III, 1.27]) and [GD67, 18.7.6] then reduce us further to the case when R is both excellent and Henselian. We then use the Néron–Popescu desingularization (see Theorem 3.2) to express the $\mathfrak{m}$-adic completion $(\widehat{R}, \widehat{\mathfrak{m}})$ of $(R, \mathfrak{m})$ as a filtered direct limit of essentially smooth, local R-algebras $(R_j, \mathfrak{m}_j)$. Since R is Henselian local and shares the residue field with $\widehat{R}$, Hensel's lemma [GD67, 18.5.17] ensures that the structure maps $R \to R_j$ have sections $R_j \to R$. In particular, they induce the injections $H^i_{\mathfrak{m}}(R, G) \hookrightarrow H^i_{\mathfrak{m}_j}(R_j, G)$ which give the injection $H^i_{\mathfrak{m}}(R, G) \hookrightarrow H^i_{\widehat{\mathfrak{m}}}(\widehat{R}, G)$. This achieves the promised reduction to the case when R is complete. □

Remark A.3. By Corollary 3.10, we see that Conjecture A.1 is reduced to the complete case.

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