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On Fujita invariants of subvarieties of a uniruled variety

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Abstract

We show that if X is a smooth uniruled projective variety and L is a big and semiample $\mathbb{Q}$-divisor on X, then there exists a proper closed subset $W \subset X$ such that every subvariety Y with Fujita invariant $a(Y, L)$ greater than $a(X, L)$ is contained in W.

1. Introduction

If X is a smooth projective variety and L is a big $\mathbb{Q}$-divisor on X, then the *Fujita invariant*, or *a-constant*, is defined as follows:

$$a(X, L) = \inf\{t > 0 \mid K_X + tL \text{ is big}\}.$$

Note that $a(X, L) \in \mathbb{R}_{\geqslant 0}$ is well defined since $K_X + tL$ is big for all $t > 0$ sufficiently large and that $a(X, L) > 0$ if and only if K_X is not pseudo-effective. It is easy to see that the a-constant is a birational invariant in the sense that if $\nu : X' \to X$ is a birational morphism of smooth varieties and $L' = \nu^* L$, then $a(X, L) = a(X', L')$ (cf. [HTT15, Proposition 2.7]). Therefore, we may also define the a-constant for a big $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor L on an arbitrary projective variety X by setting

$$a(X, L) := a(X', L'),$$

where $\nu \colon X' \to X$ is a resolution of singularities and $L' = \nu^* L$. Note that if X is smooth, then the a-constant is the usual pseudo-effective threshold; however, if X is singular, these numbers may be different.

A conjecture of Batyrev and Manin relates arithmetic properties of varieties with ample anticanonical class to geometric invariants such as a-constants. Roughly speaking, this conjecture predicts that the asymptotic behavior of a point-counting function is controlled by two geometric invariants known as the a-constant and the b-constant. In view of this conjecture, it is expected that almost all subvarieties of a uniruled variety X should have a-constants not greater than that of X. See [HTT15, LTT14] for more background on the Batyrev–Manin conjecture.

In [LTT14], a-constants were intensively studied by Lehmann, Tanimoto and Tschinkel, motivated by the conjecture of Batyrev and Manin. They show that if X is a smooth uniruled

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projective variety and L is an ample $\mathbb{Q}$-divisor on X, then there exists a countable union of proper closed subsets $W \subset X$ such that every subvariety Y satisfying $a(Y,L) > a(X,L)$ is contained in W [LTT14, Theorem 1.1]. For the purpose of applications, it is expected that one may choose W to be a proper closed subset of X. The purpose of this note is to prove that this is indeed the case.

Theorem 1.1. *Let X be a smooth uniruled projective variety and L a big and semiample $\mathbb{Q}$-divisor on X. Then there exists a proper closed subset $W \subset X$ such that every subvariety Y satisfying $a(Y,L) > a(X,L)$ is contained in W.*

Note that this result is proven in [LTT14, Theorem 1.2] assuming that a weak version of the BAB conjecture (due to Borisov, Borisov and Alexeev) holds in dimension $n-1 = \dim X - 1$. We expect that Theorem 1.1 also holds if we just assume that L is big and nef (rather than big and semiample).

Our idea is to replace the WBAB conjecture assumed in [LTT14, Theorem 1.2] by constructing non-klt centers (see Definition 2.5 and Proposition 2.8) and applying the finiteness of the a-constants (see Corollary 2.15). This is an application of a recent result of Di Cerbo [DiC17] based on a boundedness result proved by Birkar [Bir16].

2. Preliminaries

In this paper, we work over the field of complex numbers $\mathbb{C}$.

2.1 Facts on a-constants

In this subsection, for the convenience of the reader, we collect several facts about a-constants that were proven in [LTT14].

Proposition 2.1 ([LTT14, Proposition 4.1]). *Let X be a smooth projective variety and L a big and nef $\mathbb{Q}$-divisor. Let $\mathcal{U} \to W$ be a family of subvarieties of X such that $\mathcal{U} \to X$ is dominant. Then a general member Y of the family $\mathcal{U}$ satisfies $a(Y,L) \leqslant a(X,L)$.*

Theorem 2.2 ([LTT14, Theorem 4.2]). *Let X be a smooth projective variety and L a big and nef $\mathbb{Q}$-divisor. Let $\pi\colon \mathcal{U} \to W$ be a family of subvarieties of X. There exists a proper closed subset $V \subset X$ such that if a member Y of the family $\mathcal{U}$ satisfies $a(Y,L) > a(X,L)$, then $Y \subset V$.*

Proposition 2.3 ([LTT14, Proposition 4.6]). *Let X be a smooth uniruled projective variety and L a big and nef $\mathbb{Q}$-divisor. Then either*

(i) *X is covered by proper subvarieties Y satisfying $a(Y,L) = a(X,L)$ or*

(ii) *X is birational to a $\mathbb{Q}$-factorial terminal Fano variety X' of Picard number 1.*

Lemma 2.4 ([LTT14, Lemma 4.7]). *Let X be a smooth projective variety and L a big and nef $\mathbb{Q}$-divisor on X. Fix a constant C. Then the subset of $\mathrm{Chow}(X)$ parametrizing subvarieties of X that are not contained in $\mathbf{B}_+(L)$ and are of L-degree at most C is bounded.*

2.2 Non-klt centers

We follow the standard notation and conventions of the minimal model program; see, for example, [Kol97].

Definition 2.5. Let (X, Δ) be a pair with X a normal variety and Δ an effective $\mathbb{Q}$-divisor such that $K_X + \Delta$ is $\mathbb{Q}$-Cartier. We say that a subvariety $V \subset X$ is a *non-klt center* of (X, Δ) if it is the image of a divisor of discrepancy at most -1. We will denote by $\mathrm{Nklt}(X, \Delta)$ the union of all non-klt centers of (X, Δ), which is a proper closed subset of X. A *non-klt place* is a valuation corresponding to a divisor of discrepancy at most -1. A non-klt center V is *pure* if $K_X + \Delta$ is log canonical at the generic point of V. If, moreover, there is a unique non-klt place lying over the generic point of V, we will say that V is an *exceptional* non-klt center.

The following is a weak form of Kawamata's subadjunction theorem.

Theorem 2.6 (Subadjunction; see [Jia13, Proposition 5.1]). *Let $V \subset X$ be a non-klt center of a pair (X, Δ) which is log canonical at a general point of V. Let $\nu\colon V^\nu \to V$ be the normalization. Then there is an effective $\mathbb{Q}$-divisor Δ_{V^ν} on V^ν such that*

$$\nu^*(K_X + \Delta)|_{V_\nu} \sim_{\mathbb{Q}} K_{V^\nu} + \Delta_{V^\nu} .$$

We have the following connectedness lemma of Kollár and Shokurov for the non-klt locus (cf. Shokurov [Sho93, Sho94], Kollár [Kol92, Theorem 17.4]).

Theorem 2.7 (Connectedness lemma). *Let $f\colon X \to Z$ be a proper morphism of normal varieties with connected fibers and D a $\mathbb{Q}$-divisor such that $-(K_X + D)$ is $\mathbb{Q}$-Cartier, f-nef and f-big. Write $D = D^+ - D^-$, where D^+ and D^- are effective with no common components. If D^- is f-exceptional (that is, all of its components have image of codimension at least 2), then $\mathrm{Nklt}(X, D) \cap f^{-1}(z)$ is connected for any $z \in Z$.*

We can use the following proposition to construct non-klt centers.

Proposition 2.8 (cf. [Lai16, Lemma 3.2]). *Let X be a $\mathbb{Q}$-factorial terminal Fano variety of dimension n. Assume $(-K_X)^n > (wn)^n$ for some positive rational number w. Then for every point $p \in X$, there is an effective $\mathbb{Q}$-divisor $\Delta_p \sim_{\mathbb{Q}} -\frac{1}{w}K_X$ such that the unique minimal non-klt center $V_p \subset \mathrm{Nklt}(X, \Delta_p)$ containing p is exceptional.*

Proof. Fix a point p. Fix a positive rational number w' such that $(-K_X)^n > (w'n)^n > (wn)^n$. By [Kol97, Theorem 6.7.1], there is an effective $\mathbb{Q}$-divisor $\Delta'_p \sim_{\mathbb{Q}} -\frac{1}{w'}K_X$ such that (X, Δ'_p) is not log canonical (lc) at p. Let $0 < t \leqslant 1$ be the unique rational number such that $(X, t\Delta'_p)$ is lc but not klt at p. By [Amb98, Proposition 3.2, Lemma 3.4], we can find an effective $\mathbb{Q}$-divisor $M_p \sim_{\mathbb{Q}} -\frac{1}{w'}K_X$ and some rational number $a > 0$ such that for any rational number $0 < \epsilon \ll 1$, the pair $(X, (1-\epsilon)t\Delta'_p + \epsilon a M_p)$ has a unique minimal non-klt center V_p passing through p which is exceptional. Note that

$$(1-\epsilon)t\Delta'_p + \epsilon a M_p \sim_{\mathbb{Q}} -\frac{(1-\epsilon)t + \epsilon a}{w'}K_X$$

and $\big((1-\epsilon)t + \epsilon a\big)/w' < 1/w$ for $0 < \epsilon \ll 1$. Since $-K_X$ is ample, by adding a $\mathbb{Q}$-divisor $\mathbb{Q}$-linearly equivalent to a multiple of $-K_X$ to Δ'_p, we conclude that there exists an effective $\mathbb{Q}$-divisor Δ_p such that $\Delta_p \sim_{\mathbb{Q}} -\frac{1}{w}K_X$ and (X, Δ_p) has a unique minimal non-klt center V_p passing through p which is exceptional. □

Lemma 2.9. *We keep the notation of Proposition 2.8. If $w > 2$, then $\dim V_p > 0$ for a general point p.*

Proof. Assume to the contrary that there exist $p_1 \in X$ such that $V_{p_1} = \{p_1\}$ and $p_2 \in X \backslash \mathrm{Supp}(\Delta_{p_1})$ such that $V_{p_2} = \{p_2\}$. Then p_1 and p_2 are contained in $\mathrm{Nklt}(X, \Delta_{p_1} + \Delta_{p_2})$ and p_2

is isolated by construction. On the other hand,

$$-(K_X + \Delta_{p_1} + \Delta_{p_2}) \sim_{\mathbb{Q}} \left(1 - \frac{2}{w}\right)(-K_X)$$

is ample. By the connectedness lemma, $\mathrm{Nklt}(X, \Delta_{p_1} + \Delta_{p_2})$ is connected, which gives a contradiction. □

2.3 Finiteness of *a*-constants

We recall the main result of [DiC17] in this subsection.

Definition 2.10. Let X be a normal projective variety and H a big $\mathbb{Q}$-divisor. We define the *pseudo-effective threshold* to be

$$\tau(X, H) := \inf\{t \geqslant 0 \mid K_X + tH \text{ is big}\}\,.$$

Note that if X is smooth, the a-constant and pseudo-effective thresholds coincide.

Definition 2.11 (cf. [DiC17, Definition 3.1]). Fix a positive integer n and two positive real numbers ϵ and δ. We define $\mathcal{D}_n(\epsilon, \delta)$ to be the set of lc pairs (X, Δ) such that

(i) X is a normal projective variety of dimension n,
(ii) Δ is a big $\mathbb{Q}$-divisor with coefficients $\geqslant \delta$, and
(iii) $(X, t\Delta)$ is ϵ-lc and $K_X + t\Delta$ is pseudo-effective for some $0 \leqslant t \leqslant 1$.

Definition 2.12 (cf. [DiC17, Definition 3.2]). Fix a positive integer n and two positive real numbers ϵ and δ. We define the set

$$\mathcal{T}_n(\epsilon, \delta) := \{\tau(X, \Delta) \mid (X, \Delta) \in \mathcal{D}_n(\epsilon, \delta)\}\,.$$

Theorem 2.13 ([DiC17, Corollary 3.6]). *Fix a positive integer n and three positive real numbers ϵ, δ and η. Then the set $\mathcal{T}_n(\epsilon, \delta) \cap [\eta, 1]$ is a finite set.*

Applying this theorem in our situation, we obtain Corollary 2.15. To state this, we first need to introduce the notation $\mathcal{P}_n$.

Definition 2.14. Fix a positive integer n. We define $\mathcal{P}_n$ to be the set of pairs (Y, L) such that

(i) Y is a normal projective variety of dimension n,
(ii) L is a base-point-free big Cartier divisor.

Corollary 2.15. *Fix a positive integer n and a positive real number η. Then the set*

$$\{a(Y, L) \mid (Y, L) \in \mathcal{P}_n\} \cap [\eta, \infty)$$

is a finite set.

Proof. We may assume $\eta \leqslant 1/4(n+1)$. First, we show that the set

$$\Big\{a(Y, L) \mid (Y, L) \in \mathcal{P}_n\Big\} \cap \left[\eta, \frac{1}{2}\right]$$

is a finite set. Take $(Y, L) \in \mathcal{P}_n$ and assume $a(Y, L) \in \left[\eta, \frac{1}{2}\right]$. Note that $a\big(Y, \frac{1}{2}L\big) = 2a(Y, L) \in [2\eta, 1]$. By taking a resolution, we may assume that Y is smooth. In this case, $a\big(Y, \frac{1}{2}L\big) = \tau\big(Y, \frac{1}{2}L\big)$. Replacing L by a general element in $|L|$, we may assume that L is irreducible and smooth. Moreover, $\big(Y, \frac{1}{2}L\big)$ is $\frac{1}{2}$-lc and $K_Y + \frac{1}{2}L$ is pseudo-effective, that is, $\big(Y, \frac{1}{2}L\big) \in \mathcal{D}_n\big(\frac{1}{2}, \frac{1}{2}\big)$. This implies that the set

$$\Big\{a\Big(Y, \frac{1}{2}L\Big) \;\Big|\; (Y, L) \in \mathcal{P}_n\Big\} \cap \Big[2\eta, 1\Big]$$

is finite by Theorem 2.13, and so is $\{a(Y, L) \mid (Y, L) \in \mathcal{P}_n\} \cap [\eta, \frac{1}{2}]$.

Then we show that the set

$$\Big\{a(Y, L) \mid (Y, L) \in \mathcal{P}_n\Big\} \cap \Big[\frac{1}{2}, \infty\Big)$$

is a finite set. Take $(Y, L) \in \mathcal{P}_n$ and assume $a(Y, L) \geqslant \frac{1}{2}$. By taking a resolution, we may assume that Y is smooth. By [LTT14, Proposition 2.10], we have $a(Y, L) \leqslant n + 1$. Now we consider $(Y, 2(n+1)L) \in \mathcal{P}_n$. Note that $a(Y, 2(n+1)L) = \big(1/2(n+1)\big)a(Y, L)$, hence $a(Y, 2(n+1)L) \in \big[1/4(n+1), \frac{1}{2}\big]$. By the first step, $a(Y, 2(n+1)L)$ belongs to a finite set. Hence $a(Y, L)$ belongs to a finite set. □

3. Proof of Theorem 1.1

We prove the following proposition suggested by Lehmann.

Proposition 3.1. *Fix a positive real number t. Let X be a smooth projective variety and L a big and semiample $\mathbb{Q}$-divisor. Then there is a bounded family $\mathcal{U}$ of subvarieties of X such that any subvariety Y not contained in $\mathbf{B}_+(L)$, with $a(Y, L) > t$, is dominated by some members Z of $\mathcal{U}$ such that $a(Z, L) = a(Y, L)$.*

Proof. Note that for a subvariety Y not contained in $\mathbf{B}_+(L)$, the restriction $L|_Y$ is nef and big, and so $a(Y, L)$ is well defined. Therefore, we will only consider subvarieties not contained in $\mathbf{B}_+(L)$. Replacing L by some multiple, we may assume that L is a base-point-free Cartier divisor. We construct $\mathcal{U}$ inductively by increasing induction on the dimension of Y.

For a subvariety Y with $a(Y, L) > t$ and $\dim Y = 1$, it is easy to see that Y is a rational curve with

$$\deg_Y(L) = Y \cdot L = \frac{2}{a(Y, L)} < \frac{2}{t}\,.$$

By Lemma 2.4, such Y form a bounded family $\mathcal{U}_1$.

Suppose that we have constructed a bounded family $\mathcal{U}_i$ of subvarieties such that every subvariety Y with $a(Y, L) > t$ and $\dim Y \leqslant i$ is dominated by some members Z of $\mathcal{U}$ such that $a(Z, L) = a(Y, L)$. We construct $\mathcal{U}_{i+1}$ as follows. Suppose that Y is an $(i+1)$-dimensional subvariety satisfying $a(Y, L) > t$. By taking a resolution, we may assume that Y is smooth. Proposition 2.3 shows that either

(1) Y is covered by proper subvarieties Z with $a(Z, L) = a(Y, L)$ or

(2) Y is birational to a $\mathbb{Q}$-factorial terminal Fano variety Y' of Picard number 1.

In case (1), by induction, Z is dominated by some members Z' of $\mathcal{U}_i$ such that $a(Z', L) = a(Z, L)$, and so is Y.

In case (2), by taking a resolution, we may assume that $\phi\colon Y \dashrightarrow Y'$ is a morphism. By the proof of [LTT14, Proposition 4.6], we have $K_{Y'} + a(Y, L)\phi_*(L|_Y) \equiv 0$.

We define constants $c_0 < 1$ and $w > 2$ as follows: since L is base-point free, we know that the set

$$\{a(Z, L) \mid Z \text{ is a subvariety of } X\} \cap (t, \infty]$$

is finite by Corollary 2.15. Hence, we may take a rational number $c_0 < 1$ such that the set

$$\{a(Z,L) \mid Z \text{ is a subvariety of } X\} \cap [c_0 a(Z',L), a(Z',L))$$

is empty for any subvariety Z' with $a(Z',L) > t$. Take $w = 1/(1-c_0)$. We may assume $w > 2$ by taking $c_0 > \frac{1}{2}$ in the definition.

If $(-K_{Y'})^{i+1} \leqslant (w(i+1))^{i+1}$, then

$$(L|_Y)^{i+1} \leqslant (\phi^*\phi_*(L|_Y))^{i+1} = (\phi_*(L|_Y))^{i+1} \leqslant \frac{(w(i+1))^{i+1}}{a(Y,L)^{i+1}} < \frac{(w(i+1))^{i+1}}{a(X,L)^{i+1}},$$

where the first inequality holds because by the negativity lemma, $\phi^*\phi_*(L|_Y) - L|_Y = E \geqslant 0$ and hence

$$(L|_Y)^{i+1-j}(\phi^*\phi_*(L|_Y))^j = (L|_Y)^{i-j}(\phi^*\phi_*(L|_Y) - E)(\phi^*\phi_*(L|_Y))^j \leqslant (L|_Y)^{i-j}(\phi^*\phi_*(L|_Y))^{j+1}$$

for $j = 0, 1, \ldots, i$, and where we have use the fact that $\phi_*(L|_Y)$ is nef since $\rho(Y') = 1$. By Lemma 2.4, such Y form a bounded family $\mathcal{U}'_{i+1}$.

Now, we assume $(-K_{Y'})^{i+1} > (w(i+1))^{i+1}$. By Proposition 2.8, for a general point $p \in Y'$, there exists an effective $\mathbb{Q}$-divisor $\Delta'_p \sim_{\mathbb{Q}} -\frac{1}{w}K_{Y'}$ such that $V'_p \subset \mathrm{Nklt}(Y', \Delta'_p)$ is the minimal exceptional non-klt center containing p. Note that by Lemma 2.9 and the inequality $w > 2$, we have $\dim V'_p > 0$. Let $\nu\colon \tilde{V}^\nu_p \to V'_p$ be the normalization. For any $\mathbb{Q}$-Cartier divisor G on V'_p, we write $G|_{\tilde{V}^\nu_p} = \nu^* G$. By Theorem 2.6, there is an effective $\mathbb{Q}$-divisor $\Delta_{\tilde{V}^\nu_p}$ such that

$$\big(K_{Y'} + \Delta'_p\big)|_{\tilde{V}^\nu_p} \sim_{\mathbb{Q}} K_{\tilde{V}^\nu_p} + \Delta_{\tilde{V}^\nu_p}.$$

Note that since $K_{Y'} + a(Y,L)\phi_* L \equiv 0$, we have

$$K_{\tilde{V}^\nu_p} + \Delta_{\tilde{V}^\nu_p} + \left(1 - \frac{1}{w}\right) a(Y,L)\phi_* L|_{\tilde{V}^\nu_p} \sim_{\mathbb{Q}} 0.$$

Let V_p be the strict transform of V'_p on Y. Let $\tilde{V}_p$ be a common resolution of $\tilde{V}^\nu_p$ and V_p with morphisms $f\colon \tilde{V}_p \to V_p$ and $g\colon \tilde{V}_p \to \tilde{V}^\nu_p$. Then

$$\begin{aligned} &K_{\tilde{V}_p} + \left(1 - \frac{1}{w}\right) a(Y,L) f^*(L|_{V_p}) \\ &\quad = g^*\left(K_{\tilde{V}^\nu_p} + \Delta_{\tilde{V}^\nu_p} + \left(1 - \frac{1}{w}\right) a(Y,L)\phi_* L|_{\tilde{V}^\nu_p}\right) - g^{-1}_* \Delta_{\tilde{V}^\nu_p} + E \\ &\quad \sim_{\mathbb{Q}} -g^{-1}_* \Delta_{\tilde{V}^\nu_p} + E, \end{aligned}$$

where E is a g-exceptional $\mathbb{Q}$-divisor. Note that the $\mathbb{Q}$-divisor $-g^{-1}_*\Delta_{\tilde{V}^\nu_p} + E$ is not big. Hence $K_{\tilde{V}_p} + \big(1 - (1/w)\big)a(Y,L)f^*(L|_{V_p})$ is not big and therefore

$$a(V_p, L) \geqslant \left(1 - \frac{1}{w}\right) a(Y,L) = c_0 a(Y,L).$$

By the definition of c_0, this implies that $a(V_p, L) \geqslant a(Y,L)$. Since p is a general point, Y is dominated by such V_p. By induction, V_p is dominated by some members Z of $\mathcal{U}_i$ such that $a(Z,L) = a(V_p,L) \geqslant a(Y,L)$. Hence Y is also dominated by some members Z of $\mathcal{U}_i$ such that $a(Z,L) \geqslant a(Y,L)$. By Proposition 2.1, by taking general members, Y is dominated by some members Z of $\mathcal{U}_i$ such that $a(Z,L) = a(Y,L)$.

Hence we may take $\mathcal{U}_{i+1} = \mathcal{U}_i \cup \mathcal{U}'_{i+1}$, and the proof is completed. □

Proof of Theorem 1.1. Take $t = a(X, L)$ in Proposition 3.1. There is a bounded family $\mathcal{U}$ of subvarieties of X such that any subvariety Y not contained in $\mathbf{B}_+(L)$ with $a(Y, L) > a(X, L)$ is dominated by some members Z of $\mathcal{U}$ such that $a(Z, L) = a(Y, L) > a(X, L)$. By Theorem 2.2, there exists a proper closed subset $W \subset X$ such that any member Z of the family $\mathcal{U}$ satisfying $a(Z, L) > a(X, L)$ is contained in W. Hence, any subvariety Y with $a(Y, L) > a(X, L)$ is contained in W. □

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