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The Sarkisov program for Mori fibred Calabi–Yau pairs

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Abstract

We prove a version of the Sarkisov program for volume-preserving birational maps of Mori fibred Calabi–Yau pairs valid in all dimensions. Our theorem generalises the theorem of Usnich and Blanc on factorisations of birational maps of $(\mathbb{C}^\times)^2$ that preserve the volume form $\frac{dx}{x} \wedge \frac{dy}{y}$.

1. Introduction

Usnich [Usn06] and Blanc [Bla13] proved that the group of birational automorphisms of $\mathbb{G}_m^2$ that preserve the volume form $\frac{dx}{x} \wedge \frac{dy}{y}$ is generated by $\mathbb{G}_m^2$, $\mathrm{SL}_2(\mathbb{Z})$ and the birational map

$$P\colon\ (x,y) \dashrightarrow \left(y, \frac{1+y}{x}\right).$$

In this paper we prove a generalisation of this result valid in all dimensions. Our theorem generalises the theorem of Usnich and Blanc in the same way that the Sarkisov program [Cor95, HM13] generalises the theorem of Noether and Castelnuovo stating that Cr_2 is generated by $\mathrm{PGL}_3(\mathbb{C})$ and a standard quadratic transformation

$$C\colon\ (x_0 : x_1 : x_2) \dashrightarrow \left(\frac{1}{x_0} : \frac{1}{x_1} : \frac{1}{x_2}\right).$$

Our main result is the following.

Theorem 1.1. *A volume-preserving birational map between Mori fibred Calabi–Yau pairs is a composition of volume-preserving Sarkisov links.*

It is possible to derive the theorem of Usnich and Blanc from this statement in a similar way that in [KSC04, § 2.5] the theorem of Noether–Castelnuovo is derived from the Sarkisov program. This starts from a classification of volume-preserving Sarkisov links and proceeds by assembling batches of Sarkisov links into the map P. The resulting proof is completely elementary but long and not relevant to the theme of this paper, which is to prove Theorem 1.1, hence we omit it.

In the rest of the section, we introduce the terminology needed to make sense of the statement and, along the way, we state the more general factorisation theorem (Theorem 1.9) for volume-

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preserving birational maps of general Calabi–Yau pairs. Theorem 1.9 is used in the proof of the main result and is of independent interest. We conclude with some additional remarks.

We refer the reader to [KM98, § 2.3] for the standard definitions of terminal and canonical singularities, and of Kawamata log terminal (klt), divisorial log terminal (dlt) and log canonical (lc) singularities of pairs.

Definition 1.2. (1) Let X be a normal variety and write $F = k(X)$ for its field of fractions. A discrete valuation $\nu\colon F \to \mathbb{Z}$ is a *geometric valuation with centre on* X if

$$\nu = \operatorname{mult}_E, \quad \text{where} \quad E \subset Y \xrightarrow{f} X$$

is a prime Weil divisor on a normal variety Y, and $f\colon Y \to X$ is a morphism. We abuse language and identify E with the valuation $\nu = \operatorname{mult}_E$. The *centre* of E on X is the scheme-theoretic point $z = f(E) \in X$; we denote it by $\mathrm{z}_E\, X$. We say that E has *small centre* on X if $z \in X$ is not a divisor, that is, if it has codimension strictly greater than 1. For all $\mathbb{Q}$-Cartier divisors D on X, it makes sense to take the pull-back $f^\star(D)$ on Y; we write $\operatorname{mult}_E \overline{D}$ for the coefficient of E in $f^\star(D)$.

(2) Let (X, B) be a pair of a normal variety X and a $\mathbb{Q}$-Weil divisor $B \subset X$. The case $B = 0$ is allowed. Assume that $K_X + B$ is $\mathbb{Q}$-Cartier. Choose a meromorphic differential $\omega \in \Omega^n_{F/k}$, where $n = \dim X$. Note that, because $f\colon Y \to X$ is birational, ω is naturally a meromorphic differential on Y. In this situation, for all geometric valuations E with centre on X we have

$$\nu_E\, \omega = a + \operatorname{mult}_E \overline{\operatorname{div}_X \omega + B}$$

for some $a \in \mathbb{Q}$. This a depends only on E and the pair (X, B), not on the choice of ω. We call it the *discrepancy* of E and denote it by $a(E, K_X + B)$.

Definition 1.3. (1) A *Calabi–Yau* (*CY*) *pair* is a pair (X, D) of a normal variety X and a reduced $\mathbb{Z}$-Weil divisor $D \subset X$ such that $K_X + D \sim 0$ is a Cartier divisor linearly equivalent to 0.

(2) We say that a pair (X, D) has (*t,dlt*), respectively (*t,lc*), singularities or that it "is" (t,dlt), respectively (t,lc), if X has terminal singularities and the pair (X, D) has dlt, respectively lc, singularities.

Similarly (X, D) has (*c,dlt*), respectively (*c,lc*), singularities or "is" (c,dlt), respectively (c,lc), if X has canonical singularities and the pair (X, D) has dlt, respectively lc, singularities.

(3) We say that a pair (X, D) is $\mathbb{Q}$-*factorial* if X is $\mathbb{Q}$-factorial.

Remark 1.4. (1) We use the following observation throughout: if (X, D) is a CY pair, then, because $K_X + D$ is an integral Cartier divisor, for all geometric valuations E, we have $a(E, K_X + D) \in \mathbb{Z}$. If in addition (X, D) is lc or dlt, then $a(E, K_X + D) \leqslant 0$ implies $a(E, K + D) = -1$ or 0.

(2) If (X, D) is a dlt CY pair, then automatically it is (c,dlt). More precisely if E is a geometric valuation with small centre on X and if the centre $\mathrm{z}_X\, E$ is an element of $\operatorname{Supp} D$, then $a(E, K_X) > 0$.
Indeed, consider a valuation E with small centre on X. Then

$$a(E, K_X) = a(E, K_X + D) + \operatorname{mult}_E \overline{D}\,;$$

therefore $a(E, K_X) \leqslant 0$ implies $a(E, K_X + D) \leqslant 0$ and then, because $K_X + D$ is a Cartier divisor, *either* $a(E, K_X + D) = -1$, which is impossible because by definition of dlt, see Remark 2.4 below, $z = \mathrm{z}_X\, E \in X$ is smooth, *or* $a(E, K_X + D) = 0$ and $\operatorname{mult}_E \overline{D} = 0$.

Definition 1.5. A *log resolution* of a pair (X, B) is a projective morphism $f\colon Y \to X$ such that

(i) Y is smooth, the exceptional set $\operatorname{Ex} f$ is of pure codimension 1;
(ii) for B' the proper transform of B, the union of $\operatorname{Ex} f \cup \operatorname{Supp} B' \subset Y$ is a simple normal crossing (snc) divisor.

Definition 1.6. Let (X, D_X) and (Y, D_Y) be CY pairs. A birational map $\varphi\colon X \dashrightarrow Y$ is *volume preserving* if for all geometric valuations E with centre on both X and Y, we have $a(E, K_X + D_X) = a(E, K_Y + D_Y)$.

This is equivalent to saying that there exists a common log resolution

$$\begin{array}{ccc} & W & \\ {}^{p}\swarrow & & \searrow^{q} \\ X & \overset{\varphi}{\dashrightarrow} & Y \end{array}$$

such that $p^\star(K_X + D_X) = q^\star(K_Y + D_Y)$.[1]

Remark 1.7. If (X, D_X) is a CY pair and X is proper, then there is a unique (up to multiplication by a nonzero constant) rational differential $\omega_X \in \Omega^n_{k(X)/k}$ such that $D_X + \operatorname{div}_X \omega_X \geqslant 0$. Similarly, if Y is also proper, there is a distinguished rational differential ω_Y on Y. To say that φ is volume preserving is to say $\varphi_\star \omega_X = \omega_Y$.

Volume-preserving maps are called crepant birational in [Kol13].

Remark 1.8. It is obvious from the definition that the composition of two volume-preserving maps between proper varieties is volume preserving.

The first step in the proof of Theorem 1.1 is the following general factorisation theorem for volume-preserving birational maps between lc CY pairs, which is of independent interest. See [KX15, Lemma 12(4)] for a similar statement.

Theorem 1.9. *Let (X, D) and (X', D') be lc CY pairs, and let $\varphi\colon X \dashrightarrow X'$ be a volume-preserving birational map. Then there are $\mathbb{Q}$-factorial (t,dlt) CY pairs (Y, D_Y), $(Y', D_{Y'})$ and a commutative diagram of birational maps*

$$\begin{array}{ccc} Y & \overset{\chi}{\dashrightarrow} & Y' \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle g'} \\ X & \overset{\varphi}{\dashrightarrow} & X' \end{array}$$

where

(i) *the morphisms $g\colon Y \to X$, $g'\colon Y' \to X'$ are volume preserving;*
(ii) *the morphism $\chi\colon Y \dashrightarrow Y'$ is a volume-preserving isomorphism in codimension 1 which is a composition of volume-preserving Mori flips, flops and inverse flips (not necessarily in that order).*

Definition 1.10. A *Mori fibred (Mf) CY pair* is a $\mathbb{Q}$-factorial (t,lc) CY pair (X, D) together with a Mori fibration $f\colon X \to S$. Recall that this means that $f_\star \mathcal{O}_X = \mathcal{O}_S$, the divisor $-K_X$ is f-ample, and $\rho(X) - \rho(S) = 1$.

[1] By this we mean that for all meromorphic differentials $\omega \in \Omega^n_{k(X)/k}$, we have $p^\star(D_X + \operatorname{div}_X \omega) = q^\star(D_Y + \operatorname{div}_Y \varphi_* \omega)$.

Terminology 1.11. We use the following terminology throughout.

(1) A *Mori divisorial contraction* is an extremal divisorial contraction $f\colon Z \to X$ from a $\mathbb{Q}$-factorial terminal variety Z of an extremal ray R with $K_Z \cdot R < 0$. In particular, X also has $\mathbb{Q}$-factorial terminal singularities.

If (Z, D_Z) and (X, D_X) are (t,lc) CY pairs, then it makes sense to say that f is volume preserving. In this context, this is equivalent to saying that $K_Z + D_Z = f^\star(K_X + D_X)$ and, in particular, $D_X = f_\star(D_Z)$.

A birational map $t\colon Z \dashrightarrow Z'$ is a *Mori flip* if Z has $\mathbb{Q}$-factorial terminal singularities and t is the flip of an extremal ray R with $K_Z \cdot R < 0$. Note that this implies that Z' has $\mathbb{Q}$-factorial terminal singularities.

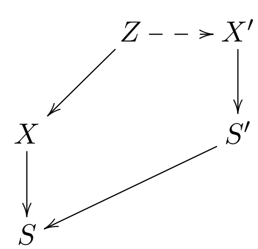
An *inverse Mori flip* is the inverse of a Mori flip.

(2) A birational map $t\colon Z \dashrightarrow Z'$ is a *Mori flop* if Z and Z' have $\mathbb{Q}$-factorial terminal singularities and t is the flop of an extremal ray R with $K_Z \cdot R = 0$.

Again, if (Z, D_Z) and $(Z', D_{Z'})$ are (t,lc) CY pairs, it makes sense to say that t is volume preserving. One can see that this just means that $D_{Z'} = t_\star D_Z$.

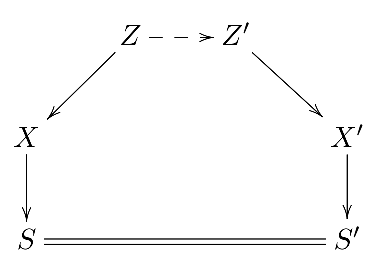
Definition 1.12. Let (X, D) and (X', D') be Mf CY pairs with Mori fibrations $X \to S$ and $X' \to S'$. A *volume-preserving Sarkisov link* is a volume-preserving birational map $\varphi\colon X \dashrightarrow X'$ that is a Sarkisov link in the sense of [Cor95]. Thus φ is of one of the following types:

(I) A *link of type I* is a commutative diagram



where $Z \to X$ is a Mori divisorial contraction and $Z \dashrightarrow X'$ is a sequence of Mori flips, flops and inverse flips.

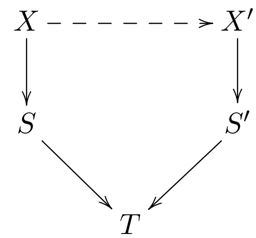
(II) A *link of type II* is a commutative diagram



where $Z \to X$ and $X' \to Z'$ are Mori divisorial contractions and $Z \dashrightarrow Z'$ is a sequence of Mori flips, flops and inverse flips.

(III) A *link of type III* is the inverse of a link of type I.

(IV) A *link of type IV* is a commutative diagram



where $X \dashrightarrow X'$ is a sequence of Mori flips, flops and inverse flips.

Remark 1.13. It follows from the definition of Sarkisov link that all the divisorial contractions, flips, etc. that constitute it are volume preserving; in particular, all varieties in sight are naturally and automatically (t,lc) CY pairs.

In order to appreciate the statement of our main theorem, Theorem 1.1, it is important to be aware that, although all Mf CY pairs are only required to have lc singularities *as pairs*, we insist that all varieties in sight have $\mathbb{Q}$-factorial *terminal* singularities. Our factorisation theorem is at the same time a limiting case of the Sarkisov program for pairs [BM97] and a Sarkisov program for varieties [Cor95, HM13]. The Sarkisov program for pairs usually spoils the singularities of the underlying varieties, while the Sarkisov program for varieties does not preserve singularities of pairs. The proof our main result is a balancing act between singularities of pairs and of varieties.

We expect that it will be possible in some cases to classify all volume-preserving Sarkisov links and hence give useful presentations of groups of volume-preserving birational maps of interesting Mf CY pairs. We plan to return to these questions in the near future.

The paper is structured as follows. In Section 2 we develop some general results on CY pairs and volume-preserving maps between them and prove Theorem 1.9; in Section 3 we prove Theorem 1.1.

2. Birational geometry of CY pairs

Definition 2.1. Let (X, D) be a lc CY pair, and let $f\colon W \to X$ be a birational morphism. The *log transform* of D is the divisor

$$D_W = f^\flat(D) = \sum_{a(E,K_X+D)=-1} E\,,$$

where the sum is over all prime divisors $E \subset W$. (Note that D_W contains the proper transform of D.)

Lemma 2.5 is a refinement of [Kol+92, Theorem 17.10] and [Fuj11, Theorem 4.1]. In order to state it we need a definition.

Definition 2.2. Let X be a normal variety. A *geometric valuation* with centre on X is a valuation of the function field $K(X)$ of the form mult_E, where $E \subset Y$ is a divisor on a normal variety Y with a birational morphism $f\colon Y \to X$. The *centre* of E on X, denoted $\mathrm{z}_X\, E$, is the generic point of $f(E)$.

Let (X, D) be a lc pair. The *non-klt set* is the set

$$\mathrm{NKLT}(X, D) = \{z \in X \mid z = \mathrm{z}_X\, E,\ \text{where } a(E, K_X + D) = -1\}\,,$$

where E is a geometric valuation of the function field of X with centre the scheme-theoretic point $\mathrm{z}_X\, E \in X$.

Warning 2.3. Our notion of non-klt set departs from common usage. Most authors work with the *non-klt locus*—the Zariski closure of our non-klt set—which they denote $\mathrm{nklt}(X, D)$ (in lower case letters).

Remark 2.4. We use the following statement throughout: it is part of the definition of dlt pairs [KM98, Definition 2.37] that if (X, D) is dlt, where $D = \sum_{i=1}^r D_i$ with $D_i \subset X$ a prime divisor, then $\mathrm{NKLT}(X, D)$ is the set of generic points of the

$$D_I = \cap_{i\in I} D_i\,, \quad \text{where} \quad I \subset \{1, \dots, r\}$$

and X is nonsingular at all these points.

Lemma 2.5. *Let (X, D) be a lc CY pair where X is not necessarily proper, let $f\colon W \to X$ be a log resolution, and let $D_W = f^\flat(D)$.*

The minimal model program (MMP) for $K_W + D_W$ over X with scaling of a divisor ample over X exists and terminates at a minimal model (Y, D_Y) over X (that is, $K_Y + D_Y$ is nef over X). More precisely, this MMP consists of a sequence of steps

$$(W, D_W) = (W_0, D_0) \overset{t_0}{\dashrightarrow} \cdots (W_i, D_i) \overset{t_i}{\dashrightarrow} (W_{i+1}, D_{i+1}) \cdots \dashrightarrow (W_N, D_N) = (Y, D_Y)\,,$$

where $t_i\colon W_i \dashrightarrow W_{i+1}$ is the divisorial contraction or flip of an extremal ray $R_i \subset \overline{\mathrm{NE}}(W_i/X)$ with $(K_{W_i}+D_i)\cdot R_i < 0$. We denote by $g_i\colon W_i \to X$ the structure morphism and by $g\colon (Y, D_Y) \to (X, D)$ the end result.

(i) *For all i, denote by $h_i\colon W \dashrightarrow W_i$ the induced map. For all i, there are Zariski-open neighbourhoods*
$$\mathrm{NKLT}(W, D_W) \subset U \quad \textit{and} \quad \mathrm{NKLT}(W_i, D_i) \subset U_i$$
such that $h_i|U\colon U \dashrightarrow U_i$ is an isomorphism.

(ii) *We have $D_Y = g^\flat D$ and $K_Y + D_Y = g^\star(K_X + D)$ (that is, g is a dlt crepant blow-up).*

(iii) *The pair (Y, D_Y) is a (t,dlt) CY pair. In particular, Y has terminal singularities.*

(iv) *The map $h\colon W \dashrightarrow Y$ contracts precisely the prime divisors $E \subset W$ with $a(E, K_X+D) > 0$. In other words, an f-exceptional divisor $E \subset W$ is not contracted by the map $h\colon W \dashrightarrow Y$ if and only if $a(E, K_X + D) = 0$ or -1.*

Proof. The MMP exists by [Fuj11, Theorem 4.1]. In the rest of the proof we use the following well-known fact: if E is a geometric valuation with centre on W, then for all i

$$a(E, K_{W_i} + D_i) \leqslant a(E, K_{W_{i+1}} + D_{i+1})\,,$$

and the inequality is an equality if and only if $t_i\colon W_i \dashrightarrow W_{i+1}$ is an isomorphism in a neighbourhood of $z_i = \mathrm{z}_{W_i}\, E$. In particular, this at once implies $NKLT(W_i, D_i) \supset \mathrm{NKLT}(W_{i+1}, D_{i+1})$, and the two sets are equal if and only if there exist Zariski-open subsets as in assertion (i), if and only if for all E with $a(E, K + D_i) = -1$ the morphism t_i is an isomorphism in a neighbourhood of $z_{W_i} E$.

Now write

$$K_W + D_W = f^\star(K_X + D) + F$$

with F a strictly effective f-exceptional divisor having no component in common with D_W. We are running an F-MMP, hence if $F_i \subset W_i$ denotes the image of F, then the exceptional set of the map $t_i\colon W_i \dashrightarrow W_{i+1}$ is contained in $\operatorname{Supp} F_i$; see [Kol13, § 1.35]. From this it follows that h_i is an isomorphism from $W \setminus \operatorname{Supp} F$ to its image in W_i. At the start, D_W has no components in common with F and $\operatorname{Supp}(D_W \cup F)$ is a snc divisor; thus, if $a(E, K_W + D_W) = -1$, then $z_W E \notin F$. It follows that $\mathrm{NKLT}(W, D_W) \subset \mathrm{NKLT}(W_i, D_i)$. Together with what we said, this implies assertion (i).

For assertion (ii), it is obvious that for all i, we have $D_i = g_i^\flat D$. By the negativity lemma [KM98, Lemma 3.39] $F_i \neq 0$ implies F_i not nef, so the MMP ends at $g_N = g\colon W_N = Y \to X$ when $F_N = 0$; that is, $K_Y + D_Y = g^\star(K_X + D)$.

For assertion (iii) we need to show that Y has terminal singularities. Suppose that E is a valuation with small centre $\mathrm{z}_Y\, E$ on Y. By Remark 1.4, either $a(E, K_X) > 0$ or

$$a(E, K_X) = a(E, K_X + D) = 0 \quad \text{and} \quad \mathrm{z}_Y\, E \notin \operatorname{Supp} D_Y\,,$$

and we show that this second possibility leads to a contradiction. Write $z_i = \mathrm{z}_{W_i}\, E$. Note that $z_i \in W_i$ is never a divisor, for this would imply $a(E, K_Y) > 0$. By what we said at the start of the proof, for all i, we have $a(E, K_{W_i} + D_i) \leqslant a(E, K_{W_{i+1}} + D_{i+1})$, with strict inequality if and only if $t_i\colon W_i \dashrightarrow W_{i+1}$ is not an isomorphism in a neighbourhood of $z_i \in W_i$. There must be a point where strict inequality occurs, otherwise $z_0 \notin D_0$ and $W = W_0$ is not terminal in a neighbourhood of z_0. This, however, implies $a(E, K_W + D) < 0$, that is, $a(E, K_W + D_W) = -1$ and then by assertion (i), the morphism $h\colon W \to Y$ is an isomorphism in a neighbourhood of z_0, again a contradiction.

Statement (iv) is obvious. □

Example 2.6. This example should help appreciate the statement of Theorem 1.9 and the subtleties of its proof. Let $\pi\colon W \to \mathbb{P}^1 \times \mathbb{P}^1$ be the total space of the vector bundle $\mathcal{O}(-1, -2)$ on $\mathbb{P}^1 \times \mathbb{P}^1$, let $E \subset W$ be the zero section, and let $D_W = \pi^\star p_1^\star(0)$, where $p_1\colon \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1$ is the projection to the first factor. Note that $D_W \cap E$ is a ruling in E and a -2-curve in D_W. Let $f\colon W \to Y$ be the contraction of E along the first ruling, and let $f'\colon W \to Y'$ be the contraction along the second ruling. Then (Y, D) and (Y', D') are both dlt, Y' is terminal, Y is canonical but not terminal, and the map $Y \dashrightarrow Y'$ is volume preserving.

Proof of Theorem 1.9. Let

$$\begin{array}{ccc} & W & \\ {\scriptstyle f}\swarrow & & \searrow{\scriptstyle f'} \\ X & \overset{\varphi}{\dashrightarrow} & X' \end{array}$$

be a common log resolution. Since φ is volume preserving, for all geometric valuations E we have $a(E, K_X + D) = a(E, K_{X'} + D')$ and

$$K_W + D_W = f^\star(K_X + D) + F = f'^\star(K_{X'} + D') + F\,,$$

where $D_W = f^\flat D = f'^\flat D'$ and

$$F = \sum_{a_E(K_X + D) > 0} a(E, K_X + D) E = \sum_{a_E(K_{X'} + D') > 0} a(E, K_{X'} + D') E\,.$$

Let $g\colon (Y, D_Y) \to (X, D)$ and $g'\colon (Y', D_{Y'}) \to (X', D')$ be the end products of the $(K_W + D_W)$-MMP over X and X' as in Lemma 2.5, and denote by $\chi\colon Y \dashrightarrow Y'$ the induced map. By Lemma 2.5(iv), this map χ is an isomorphism in codimension 1.

Denote by $t\colon W \dashrightarrow Y$ and $t'\colon W \dashrightarrow Y'$ the obvious maps and write $\mathrm{NKLT}(W, D_W) \subset U_W = W \setminus \operatorname{Supp} F$; by Lemma 2.5(i), the restrictions $t|U_W$ and $t'|U_W$ are isomorphisms onto their images $\operatorname{NKLT}(Y, D_Y) \subset U \subset Y$ and $\operatorname{NKLT}(Y', D_{Y'}) \subset U' \subset Y'$. It follows from this that $\chi|U$ maps U isomorphically to U'.

In the rest of the proof if N is a divisor on Y we denote by N' its transform on Y' and conversely; because χ is an isomorphism in codimension 1, it is clear what the notation means.

Let us choose, as we can by what we just said, an ample $\mathbb{Q}$-divisor L' on Y' general enough that both $(Y', D_{Y'}+L')$ and (Y, D_Y+L) are dlt. Let $0 < \varepsilon \ll 1$ be small enough that $A' = L' - \varepsilon D'$ is ample. Note that, again by what we said above, if we set $\Theta' = L' + (1-\varepsilon)D_{Y'}$, both pairs (Y', Θ') and (Y, Θ) are klt.

Since $K_{Y'} + \Theta' \sim_{\mathbb{Q}} A'$ is ample, $(Y', \Theta_{Y'})$ is the log canonical model of (Y, Θ). It follows that χ is the composition of finitely many [BCHM10, Corollary 1.4.2] flips

$$\chi\colon Y = Y_0 \overset{\chi_0}{\dashrightarrow} Y_1 \overset{\chi_1}{\dashrightarrow} \cdots \overset{\chi_{N-1}}{\dashrightarrow} Y_N = Y'$$

of the MMP for $K_Y + \Theta$. If N is a divisor on Y, denote by N_i its transform on Y_i. For all i, the map χ_i is a $(K_{Y_i} + \Theta_i)$-flip and, at the same time, a $(K_{Y_i} + D_i)$-flop, and hence all pairs (Y_i, D_i) are lc. We next argue that all (Y_i, D_i) are in fact (t,dlt).

Because the MMP is a MMP for $A \sim_{\mathbb{Q}} K_Y + \Theta$, the exceptional set of χ_i is contained in $\operatorname{Supp} A_i$. From this it follows that for $U_0 = U$, the restriction $\chi_0|U_0$ is an isomorphism onto its image, which we denote by U_1 and, by induction on i, the restriction $\chi_i|U_i$ is an isomorphism onto its image, which we denote by U_{i+1}. We show by induction that for all i, the set U_i is a Zariski neighbourhood of $\operatorname{NKLT}(Y_i, D_i)$, so that χ_i is a local isomorphism at the generic point of each $z \in \operatorname{NKLT}(Y_i, D_i)$ and (Y_i, D_i) is a dlt pair. Indeed, assuming the statement for $i < k$, consider $\chi_k\colon Y_k \dashrightarrow Y_{k+1}$. Let E be a valuation with discrepancy $a(E, K_{Y_{k+1}} + D_{k+1}) = -1$; then also $a(E, K_{Y_k} + D_k) = -1$, thus $z_k = \mathrm{z}_{Y_k} E \in \operatorname{NKLT}(Y_k, D_k) \subset U_k$ and then by what we just said χ_k is an isomorphism at z_k, hence $z_{k+1} = \chi_k(z_k) \in U_{k+1}$. This shows that all (Y_i, D_i) are dlt.

Finally, we prove that for all i, the variety Y_i is terminal. Assume for a contradiction that Y_j is not terminal. By Remark 1.4(2), the variety Y_j is canonical and there is a geometric valuation E with $a(E, K_{Y_j}) = a(E, K_{Y_j} + D_j) = \operatorname{mult}_E \overline{D}_j = 0$, and then also $a(E, K_Y + D_Y) = a(E, K_{Y'} + D_{Y'}) = 0$. Since Y is terminal, $a(E, K_Y) > 0$, and $\mathrm{z}_Y E \notin U$ and $\mathrm{z}_W E \in \operatorname{Supp} F$, but then $a(E, K_Y + D_Y) > a(E, K_W + D_W)$, so that we must have $a(E, K_W + D_W) = -1$, that is, $\mathrm{z}_W E \in \operatorname{NKLT}(W, D_W) \subset U_W$, which gives a contradiction. □

3. Sarkisov program under Y

3.1 Basic setup

We fix the following situation, which we keep in force throughout this section:

$$\begin{array}{ccc}
Y & \overset{\chi}{\dashrightarrow} & Y' \\
{\scriptstyle g}\downarrow & & \downarrow{\scriptstyle g'} \\
X & \overset{\varphi}{\dashrightarrow} & X' \\
{\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p'} \\
S & & S'
\end{array}$$

where

(i) Y and Y' have $\mathbb{Q}$-factorial terminal singularities and $g\colon Y \to X$ and $g'\colon Y \to X'$ are birational morphisms;

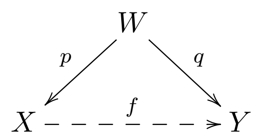
(ii) $\chi\colon Y \dashrightarrow Y'$ is the composition of Mori flips, flops and inverse flips;

(iii) $p\colon X \to S$ and $p'\colon X' \to S'$ are Mfs.

The goal of this section is to prove Theorem 3.3 below. In the final short Section 4 we show that Theorems 1.9 and 3.3 imply Theorem 1.1. The proof of Theorem 3.3 is a variation on the proof of [HM13].

Definition 3.1. A birational map $f\colon X \dashrightarrow Y$ is *contracting* if f^{-1} contracts no divisors.

Remark 3.2. If a birational map $f\colon X \dashrightarrow Y$ is contracting, then it makes sense to pull back $\mathbb{Q}$-Cartier ($\mathbb{R}$-Cartier) divisors from Y to X. Choose a normal variety W and a factorisation



with p and q proper birational morphisms. If D is a $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor or an $\mathbb{R}$-Cartier $\mathbb{R}$-divisor on Y, the pull-back $f^\star(D)$ is defined as

$$f^\star(D) = p_\star q^\star(D)$$

(this is easily seen to be independent of the factorisation).

Theorem 3.3. *The birational map $\varphi\colon X \dashrightarrow X'$ is a composition of links $\varphi_i\colon X_i/S_i \dashrightarrow X_{i+1}/S_{i+1}$ of the Sarkisov program, where all the maps $Y \dashrightarrow X_i$ are contracting.*

Terminology 3.4. We say that the link $\varphi_i\colon X_i/S_i \dashrightarrow X_{i+1}/S_{i+1}$ is *under* Y if the maps $Y \dashrightarrow X_i$ and $Y \dashrightarrow X_{i+1}$ are contracting.

3.2 Finitely-generated divisorial rings

3.2.1 *General theory*

Definition 3.5. Let $f\colon X \dashrightarrow Y$ be a contracting birational map.

Let D_X be an $\mathbb{R}$-divisor. We say that f is *D_X-nonpositive* (respectively, *D_X-negative*) if $D_Y = f_\star D_X$ is $\mathbb{R}$-Cartier and

$$D_X = f^\star(D_Y) + \sum_{\text{E} \quad f\text{-exceptional}} a_E E\,,$$

where $a_E \geqslant 0$ (respectively, $a_E > 0$) for all E.

Note the special case $D_X = K_X$ in this definition.

Definition 3.6. Let X/Z be a normal variety, proper over Z, and let D be an $\mathbb{R}$-divisor on X.

(1) A *semiample model* of D is a D-nonpositive contracting birational map $\varphi\colon X \dashrightarrow Y$ to a normal variety Y/Z, proper over Z, such that $D_Y = \varphi_\star D$ is semiample over Z.

(2) An *ample model* of D is a rational map $h\colon X \dashrightarrow W$ to a normal variety W/Z, projective over Z, together with an ample $\mathbb{R}$-Cartier divisor A, such that there is a factorisation $h = g \circ f$:

$$X \overset{f}{\dashrightarrow} Y \overset{g}{\to} W\,,$$

where $f\colon X \dashrightarrow Y$ is a semiample model of D, the map $g\colon Y \to W$ is a morphism, and $D_Y = g^\star(A)$.

Remark 3.7. Let X/Z be a normal variety, proper over Z, and let D be an $\mathbb{R}$-divisor on X.

(1) Suppose that W/Z is normal, let A be an ample $\mathbb{R}$-divisor on W, and let $h\colon X \dashrightarrow W$ be an ample model of D. If $f\colon X \dashrightarrow Y$ is a semiample model of D, then the induced rational map $g\colon Y \dashrightarrow W$ is a morphism and $D_Y = g^\star A$.

(2) All ample models of D are isomorphic over Z.

We refer to [KKL12, § 3] for basic terminology on divisorial rings.

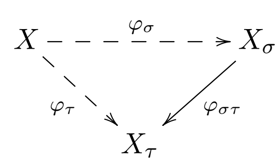
Theorem 3.8 ([KKL12, Theorem 4.2]). *Let X be a projective $\mathbb{Q}$-factorial variety, and let $\mathcal{C} \subseteq \mathrm{Div}_{\mathbb{R}}(X)$ be a rational polyhedral cone containing a big divisor*[2] *such that the ring $\mathfrak{R} = R(X, \mathcal{C})$ is finitely generated. Then there exist a finite rational polyhedral fan Σ and a decomposition*

$$\mathrm{Supp}\,\mathfrak{R} = |\Sigma| = \coprod_{\sigma \in \Sigma} \sigma$$

such that we have the following:

(i) *For all $\sigma \in \Sigma$ there exist a normal projective variety X_σ and a rational map $\varphi_\sigma\colon X \dashrightarrow X_\sigma$ such that for all $D \in \sigma$, the map φ_σ is the ample model of D. If σ contains a big divisor, then for all $D \in \overline{\sigma}$, the map φ_σ is a semiample model of D.*

(ii) *For all $\tau \subseteq \overline{\sigma}$ there exists a morphism $\varphi_{\sigma\tau}\colon X_\sigma \longrightarrow X_\tau$ such that the diagram*



commutes.

Remark 3.9. (1) Under the assumptions of Theorem 3.8, if a cone $\sigma \in \Sigma$ intersects the interior of $\mathrm{Supp}\,\mathfrak{R}$, then it consists of big divisors (this is because the big cone is the interior of the pseudoeffective cone). This holds in particular if σ is of maximal dimension.

(2) Theorem 3.8(ii) follows immediately from part (i) and Remark 3.7(1).

Definition 3.10. Let X be a projective $\mathbb{Q}$-factorial variety, and let $\mathcal{C} \subseteq \mathrm{Div}_{\mathbb{R}}(X)$ be a rational polyhedral cone containing a big divisor such that the ring $\mathfrak{R} = R(X, \mathcal{C})$ is finitely generated. We say that $\mathcal{C}$ is *generic* if

(i) for all $\sigma \in \Sigma$ of maximal dimension (that is, $\dim \sigma = \dim \mathrm{Supp}\,\mathfrak{R}$), the variety X_σ is $\mathbb{Q}$-factorial;

(ii) for all $\sigma \in \Sigma$, not necessarily of maximal dimension, and all $\tau \subset \overline{\sigma}$ of codimension 1, the morphism $X_\sigma \to X_\tau$ has relative Picard rank $\rho(X_\sigma/X_\tau) \leqslant 1$.

Notation 3.11. If V is a $\mathbb{R}$-vector space and $v_1, \ldots, v_k \in V$, then we denote by

$$\langle v_1, \ldots, v_k \rangle = \sum_{i=1}^{k} \mathbb{R}_{\geqslant 0} v_i$$

the convex cone in V spanned by the v_i.

[2]We need to assume that $\mathcal{C}$ contains a big divisor so we can say: if $D \in \mathcal{C}$ is pseudo effective, then D is effective.

Lemma 3.12. *Let X be a projective $\mathbb{Q}$-factorial variety, and let $\mathcal{C} \subseteq \operatorname{Div}_{\mathbb{R}}(X)$ be a generic rational polyhedral cone containing a big divisor.*

Let $D_1, \dots, D_k \in \mathcal{C}$ be such that the cone $\langle D_1, \dots, D_k \rangle$ contains a big divisor, and let $\varepsilon > 0$. There exist $D'_1, \dots, D'_k \in \mathcal{C}$ with $\|D_i - D'_i\| < \varepsilon$ such that the cone $\langle D'_1, \dots, D'_k \rangle$ is generic.

Proof. Make sure that all cones $\langle D'_{i_1}, \dots, D'_{i_c} \rangle$ for $i_1, \dots, i_c \in \{1, \dots, k\}$ intersect all cones $\sigma \in \Sigma$ properly. □

Theorem 3.13. *Let X be a projective $\mathbb{Q}$-factorial variety, let $\Delta_1, \dots, \Delta_r \geqslant 0$ big $\mathbb{Q}$-divisors on X such that all pairs (X, Δ_i) are klt, and let*

$$\mathcal{C} = \langle K_X + \Delta_1, \dots, K_X + \Delta_r \rangle .$$

Then $\mathfrak{R} = R(X, \mathcal{C})$ is finitely generated, and if $\operatorname{Supp} \mathfrak{R}$ spans $N^1_{\mathbb{R}}(X)$ as a vector space, then $\mathcal{C}$ is generic.

For the proof see for example [KKL12, Theorem 4.5]. Note that the assumptions readily imply that $\operatorname{Supp} \mathfrak{R}$ contains big divisors. The finite generation of $\mathfrak{R}$ is of course the big theorem of [BCHM10].

Setup 3.14. In what follows we work with a pair (X, G_X), where X is $\mathbb{Q}$-factorial and

(i) G_X is a $\mathbb{Q}$-linear combination of irreducible mobile[3] divisors;

(ii) (X, G_X) is terminal;

(iii) $K_X + G_X$ is not pseudoeffective.

Assumption (i) implies that when we run the MMP for $K_X + G_X$, no component of G_X is ever contracted, so that (X, G_X) remains terminal throughout the MMP. Assumption (iii) means that the MMP terminates with a Mf.

Corollary 3.15. *Let X be a projective $\mathbb{Q}$-factorial variety, let G_X be as in Setup 3.14, and let $\Delta_1, \dots, \Delta_r \geqslant 0$ be big $\mathbb{Q}$-divisors on X such that all pairs $(X, G_X + \Delta_i)$ are klt.*

Then for all $\varepsilon > 0$ there are ample $\mathbb{Q}$-divisors $H_1, \dots, H_r \geqslant 0$ with $\|H_i\| < \varepsilon$ such that

$$\mathcal{C}' = \langle K_X + G_X, K_X + G_X + \Delta_1 + H_1, \dots, K_X + G_X + \Delta_r + H_r \rangle$$

is generic.

Proof. Add enough ample divisors to span N^1 and then use Lemma 3.12 to perturb $\Delta_1, \dots, \Delta_r$ inside a bigger cone. Since $K_X + G_X \notin \overline{\operatorname{Eff}}\, X$, we have $K_X + G_X \notin \operatorname{Supp} \mathfrak{R}(X, \mathcal{C})$, and hence there is no need to perturb G_X. □

3.2.2 *Special case: 2-dimensional cones* Suppose that A is a big $\mathbb{Q}$-divisor on X such that

(i) $(X, G_X + A)$ is klt;

(ii) $K_X + G_X + A$ is ample on X;

(iii) $\mathcal{C} = \langle K_X + G_X, K_X + G_X + A \rangle$ is generic.

[3] A $\mathbb{Q}$-divisor M is mobile if for some integer $n > 0$ such that nM is integral, the linear system $|nM|$ has no fixed (divisorial) part.

(1) The decomposition of $\operatorname{Supp}\mathfrak{R}(X,\mathcal{C})$ given by Theorem 3.8 corresponds to running a MMP for K_X+G_X with scaling by A. This MMP exists by [BCHM10, Corollary 1.4.2]. In more detail, let

$$1=t_0>t_1>\cdots>t_{N+1}>0$$

be rational numbers such that $\operatorname{Supp}\mathfrak{R}(X,\mathcal{C})=\langle K_X+G_X+A, K_X+G_X+t_{N+1}A\rangle$ and the maximal cones of the decomposition correspond to the intervals (t_i,t_{i+1}). For all $t\in(t_i,t_{i+1})$, the divisor K_X+G_X+tA is ample on $X_i=\operatorname{Proj}R(X,K_X+G_X+tA)$. Then

$$X=X_0 \dashrightarrow X_1 \dashrightarrow \cdots \dashrightarrow X_i \dashrightarrow X_{i+1} \dashrightarrow \cdots \dashrightarrow X_N$$

is a minimal model program for K_X+G_X with scaling by A; that is,

(i)
$$t_{i+1}=\inf\{\tau\in\mathbb{R}\mid K_{X_i}+G_{X_i}+\tau A_i \text{ is nef}\},$$
where A_i denotes the push-forward of A and $X_i \dashrightarrow X_{i+1}$ is the divisorial contraction or flip of an extremal ray $R_i\subset\overline{\mathrm{NE}}(X_i)$ with
$$(K_{X_i}+G_{X_i}+t_{i+1}A_i)\cdot R_i=0 \quad\text{and}\quad (K_{X_i}+G_{X_i})\cdot R_i<0;$$

(ii)
$$t_{N+1}=\inf\{\lambda\mid K_X+G_X+\lambda A \text{ is effective}\},$$
where $K_X+G_X+\lambda A$ effective means that it is $\mathbb{Q}$-linearly equivalent to an effective divisor, and
$$X_N\to\operatorname{Proj}R(X,K_X+G_X+t_{N+1}A)$$
is a Mf.

(2) Genericity means that at each step there is a unique extremal ray $R_i\subset\overline{\mathrm{NE}}(X_i)$ with $(K_{X_i}+G_{X_i}+t_{i+1}A_i)\cdot R_i=0$ and $(K_{X_i}+G_{X_i})\cdot R_i<0$.

(3) The genericity immediately implies the following: If $0<\varepsilon\ll 1$ is small enough, then for all ample $\mathbb{Q}$-divisors H with $\|H\|<\varepsilon$, the divisor K_X+G_X+A+H is ample on X, the cone $\mathcal{C}'=\langle K_X+G_X, K_X+G_X+A'\rangle$ is still generic, and the MMP for K_X+G_X with scaling by $A+H$ is identical to the MMP for K_X+G_X with scaling by A, in the sense that the sequence of steps and end product are identical.

3.2.3 *Special case: 3-dimensional cones* In this subsection we prove the following special case of Theorem 3.3.

Lemma 3.16. *Suppose that (Y,G_Y) is as in Setup 3.14 and that A and A' are big $\mathbb{Q}$-divisors on Y such that*

(i) *(Y,G_Y+A) and (Y,G_Y+A') are klt;*
(ii) *K_Y+G_Y+A and K_Y+G_Y+A' are both ample on Y;*
(iii) *$\mathcal{C}=\langle K_Y+G_Y, K_Y+G_Y+A\rangle$ and $\mathcal{C}'=\langle K_Y+G_Y, K_Y+G_Y+A'\rangle$ are generic;*
(iv) *the MMP for K_Y+G_Y with scaling by A, respectively A', ends in a Mf X/S, respectively X'/S'.*

Then the birational map $\varphi\colon X\dashrightarrow X'$ is a composition of links $\varphi_i\colon X_i/S_i\dashrightarrow X_{i+1}/S_{i+1}$ of the Sarkisov program, where each map $Y\dashrightarrow X_i$ is contracting.

Proof. The proof is the argument of [HM13], which we sketch here for the reader's convenience. After a small perturbation of A and A' as in Corollary 3.15 that, as stated in Section 3.2.2(3), does not change the two MMPs or their end products, the cone $\widetilde{\mathcal{C}} = \langle K_Y + G_Y, K_Y + G_Y + A, K_Y + G_Y + A' \rangle$ is generic. The argument of [HM13] then shows how walking along the boundary of $\operatorname{Supp} \widetilde{\mathcal{C}}$ corresponds to a chain of Sarkisov links from X/S to X'/S'. By construction, all maps from Y are contracting. □

3.3 Proof of Theorem 3.3

Write $\chi = \chi_{N-1} \circ \cdots \circ \chi_0$, where each

$$\chi_i \colon Y_i \dashrightarrow Y_{i+1}$$

is a Mori flip, flop or inverse flip, and $Y = Y_0$, $Y' = Y_N$.

For all $\mathbb{Q}$-divisors G_Y on Y denote by G_{Y_i} the strict transform on Y_i. Choose G_Y such that for all $i \in \{0, \ldots, N\}$

(i) G_{Y_i} satisfies the Setup 3.14;

(ii) χ_i is either a $(K_{Y_i} + G_{Y_i})$-flip or antiflip.[4]

One way to choose G_Y is as follows: if χ_i is a flop, choose G_{Y_i} ample, general and very small on Y_i. If G_{Y_i} is small enough, then for all j if χ_j was a flip or antiflip, then it still is a flip or antiflip. On the other hand, now χ_i is a $(K_{Y_i} + G_{Y_i})$-flip. Some other flops may have become flips or antiflips. If there are still flops, repeat the process by adding, on Y_k such that χ_k is a $(K_{Y_k} + G_{Y_k})$-flop, a very small ample divisor to G_{Y_k}, and so on until there are no flops left.

For all $i \in \{0, \ldots, N\}$, we choose by induction on i a big divisor A_i on Y_i such that $K_{Y_i} + G_{Y_i} + A_i$ is ample, $\langle K_{Y_i} + G_{Y_i}, K_{Y_i} + G_{Y_i} + A_i \rangle$ is generic, and the MMP for $K_{Y_i} + G_{Y_i}$ with scaling by A_i terminates with a Mf $p_i \colon X_i \to S_i$. (Note that A_j is not the transform of A_i on Y_j: it is just another divisor.) At the start $p_0 = p \colon X_0 = X \to S_0 = S$, but it will not necessarily be the case that $p_N = p'$. We prove, also by induction on i, that for all i the induced map $\varphi_i \colon X_i \dashrightarrow X_{i+1}$ is the composition of Sarkisov links under Y. Finally, we prove that the induced map $X_N \dashrightarrow X'$ is the composition of Sarkisov links under Y.

Suppose that for all $j < i$, we have constructed A_j. We consider two cases.

(1) If χ_{i-1} is a $(K_{Y_{i-1}} + G_{Y_{i-1}})$-flip, choose an ample $\mathbb{Q}$-divisor A'_{i-1} on Y_{i-1} such that $\langle K_{Y_{i-1}} + G_{Y_{i-1}}, K_{Y_{i-1}} + G_{Y_{i-1}} + A'_{i-1} \rangle$ is generic and the MMP for $K_{Y_{i-1}} + G_{Y_{i-1}}$ with scaling by A'_{i-1} begins with the flip χ_{i-1}. This can be accomplished as follows: If χ_{i-1} is the flip of the extremal contraction $\gamma_{i-1} \colon Y_{i-1} \to Z_{i-1}$, then $A'_{i-1} = L_{i-1} + \gamma^\star_{i-1}(N_{i-1})$, where L_{i-1} is ample on Y_{i-1} and N_{i-1} is ample enough on Z_{i-1}. Now set

$$A_i = \chi_{i-1\star}\big((t_1 - \varepsilon) A'_{i-1}\big) ,$$

where $K_{Y_{i-1}} + G_{Y_{i-1}} + t_1 A'_{i-1}$ is γ_{i-1}-trivial and $0 < \varepsilon \ll 1$. Note that $\langle K_{Y_{i-1}} + G_{Y_{i-1}}, K_{Y_{i-1}} + G_{Y_{i-1}} + A'_{i-1} \rangle$ generic implies $\langle K_{Y_i} + G_{Y_i}, K_{Y_i} + G_{Y_i} + A_i \rangle$ generic.[5] We take $p_i \colon X_i \to S_i$ to be the end product of the MMP for $K_{Y_i} + G_{Y_i}$ with scaling by A_i. It follows from Lemma 3.16, applied to Y_{i-1} and the divisors A_{i-1} and A'_{i-1}, that the induced map $\varphi_i \colon X_{i-1} \dashrightarrow X_i$ is a composition of Sarkisov links under Y_{i-1} and hence, since $Y \dashrightarrow Y_{i-1}$ is an isomorphism in codimension 1, under Y.

(2) If χ_{i-1} is a $(K_{Y_{i-1}} + G_{Y_{i-1}})$-antiflip, choose A_i ample on Y_i such that $\langle K_{Y_i} + G_{Y_i}, K_{Y_i} + G_{Y_i} + A_i \rangle$ is generic and the MMP for $K_{Y_i} + G_{Y_i}$ with scaling by A_i begins with the flip χ^{-1}_{i-1}.

[4] The purpose of G is to make sure that there are no flops.

[5] The divisor A'_{i-1} is ample, hence (moving in the linear equivalence class) $(Y_{i-1}, G_{Y_{i-1}} + A'_{i-1})$ is klt—in fact even terminal if we want. So since $t_1 < 1$, we even have that $(Y_i, G_{Y_i} + A_i)$ is klt.

We take $p_i\colon X_i \to S_i$ to be the end product of the MMP for $K_{Y_i} + G_{Y_i}$ with scaling by A_i. It follows from Lemma 3.16, applied to Y_{i-1} and the divisors A_{i-1} and $A'_{i-1} = \chi^{-1}_{i-1\star}((t_1 - \varepsilon)A_i)$, where $K_{Y_i} + G_{Y_i} + t_1A_i$ is χ^{-1}_{i-1}-trivial and $0 < \varepsilon \ll 1$, that the induced map $\varphi_{i-1}\colon X_{i-1} \dashrightarrow X_i$ is a composition of Sarkisov links under Y_{i-1} and hence, since $Y \dashrightarrow Y_{i-1}$ is an isomorphism in codimension 1, under Y.

Finally, choose A' ample on Y' such that $\langle K_{Y'}+G_{Y'}, K_{Y'}+G_{Y'}+A'\rangle$ is generic and the MMP for $K_{Y'}+G_{Y'}$ with scaling by A' terminates with the Mf $p'\colon X' \to S'$. It follows from Lemma 3.16, applied to $Y_N = Y'$ and the divisors A_N and A', that the induced map $\varphi_N\colon X_r \dashrightarrow X'$ is a composition of Sarkisov links under Y' and hence, since $Y \dashrightarrow Y'$ is an isomorphism in codimension 1, under Y.

4. Proof of Theorem 1.1

Let (X, D) and (X', D') with $p\colon X \to S$ and $p'\colon X' \to S'$ be (t,lc) Mf CY pairs, and let $\varphi\colon X \dashrightarrow X'$ be a volume-preserving birational map. Theorem 1.9 gives a diagram

$$\begin{array}{ccc} (Y, D_Y) & \overset{\chi}{\dashrightarrow} & (Y', D_{Y'}) \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle g'} \\ (X, D) & \overset{\varphi}{\dashrightarrow} & (X', D') \end{array}$$

where (Y, D_Y) and $(Y', D_{Y'})$ are (t,dlt) $\mathbb{Q}$-factorial CY pairs, g and g' are volume preserving, and χ is a volume-preserving composition of Mori flips, flops and inverse flips. In particular, if we forget the divisors D, we are in the situation of Section 3.1, so that by Theorem 3.3, the map $\varphi\colon X \dashrightarrow X'$ is the composition of Sarkisov links $\varphi_i\colon X_i/S_i \dashrightarrow X_{i+1}/S_{i+1}$ such that all induced maps $g_i\colon Y \dashrightarrow X_i$ are contracting. It is clear that for all i, the map g_i is volume preserving, hence for $D_i = g_{i\star}D_Y$, the map $\varphi_i\colon (X_i, D_i) \dashrightarrow (X_{i+1}, D_{i+1})$ also is volume preserving, and (X_i, D_i) is a (t,lc) CY pair.

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