



# Wall-crossing in genus zero quasimap theory and mirror maps

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## ABSTRACT

For each positive rational number  $\varepsilon$ , the theory of  $\varepsilon$ -stable quasimaps to certain GIT quotients  $W//\mathbf{G}$  developed in [CKM14] gives rise to a Cohomological Field Theory. Furthermore, there is an asymptotic theory corresponding to  $\varepsilon \rightarrow 0$ . For  $\varepsilon > 1$  one obtains the usual Gromov–Witten theory of  $W//\mathbf{G}$ , while the other theories are new. However, they are all expected to contain the same information and, in particular, the numerical invariants should be related by wall-crossing formulas. In this paper we analyze the genus zero picture and find that the wall-crossing in this case significantly generalizes toric mirror symmetry (the toric cases correspond to abelian groups  $\mathbf{G}$ ). In particular, we give a geometric interpretation of the mirror map as a generating series of quasimap invariants. We prove our wall-crossing formulas for all targets  $W//\mathbf{G}$  which admit a torus action with isolated fixed points, as well as for zero loci of sections of homogeneous vector bundles on such  $W//\mathbf{G}$ .

## 1. Introduction

### 1.1 Overview

For a complex affine algebraic variety  $W$  acted upon by a reductive group  $\mathbf{G}$ , a choice of a character  $\theta$  of  $\mathbf{G}$  determines a linearized line bundle  $L_\theta$  on  $W$ , and hence a Geometric Invariant Theory (GIT) quotient  $W//\mathbf{G} = W//_\theta\mathbf{G}$  with a canonical (relative) polarization  $\mathcal{O}(\theta)$ . Once such a  $\theta$  is fixed, and under reasonable conditions on the triple  $(W, \mathbf{G}, \theta)$ , certain stability conditions (depending on a parameter  $\varepsilon \in \mathbb{Q}_{>0}$ ) are imposed on the moduli stacks  $\mathfrak{M}_{g,k}([W/\mathbf{G}], \beta)$  parametrizing maps of (homology) class  $\beta$  from  $k$ -pointed nodal curves of genus  $g$  to the quotient stack  $[W/\mathbf{G}]$ . This produces (relatively) proper Deligne–Mumford moduli stacks of  $\varepsilon$ -stable quasimaps, carrying virtual fundamental classes. They come equipped with evaluation maps and with tautological cotangent  $\psi$ -classes at the markings. Moreover, the same holds for an asymptotic stability condition denoted by  $\varepsilon = 0+$  and corresponding to  $\varepsilon \rightarrow 0$ .

These structures allow the introduction of numerical invariants by integrating natural cohomology classes against the virtual class. Explicitly, for  $\varepsilon \geq 0+$ , let  $\mathbb{Q}_{g,k}^\varepsilon(W//\mathbf{G}, \beta)$  denote the moduli space of  $\varepsilon$ -stable quasimaps of class  $\beta$ . When  $W//\mathbf{G}$  is projective, its (descendant)

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Received 23 July 2013, accepted in final form 22 February 2014.

2010 Mathematics Subject Classification 14D20, 14D23, 14N35

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The research of I. Ciocan-Fontanine was partially supported by the NSA grant H98230-11-1-0125 and the NSF grant DMS-1305004. The research of B. Kim was partially supported by NRF-2007-0093859.

$\varepsilon$ -quasimap invariants are defined by

$$\langle \gamma_1 \psi_1^{a_1}, \dots, \gamma_k \psi_k^{a_k} \rangle_{g,k,\beta}^\varepsilon := \int_{[\mathbb{Q}_{g,k}^\varepsilon(W//\mathbf{G},\beta)]^{\text{vir}}} \prod_{j=1}^k \text{ev}_j^*(\gamma_j) \psi_j^{a_j},$$

for integers  $a_j \geq 0$  and cohomology classes  $\gamma_j \in H^*(W//\mathbf{G}, \mathbb{Q})$ .

If  $W//\mathbf{G}$  is only quasiprojective, one may still define a good theory of *equivariant* quasimap invariants in the presence of an action with good properties by an algebraic torus, via the virtual localization formula.

For each fixed class  $\beta$ , the set  $\mathbb{Q}_{>0}$  is divided into stability chambers by finitely many walls  $1, 1/2, \dots, 1/\beta(L_\theta)$  such that the moduli spaces (and therefore the invariants) stay constant in each chamber. Here  $\beta(L_\theta)$  is essentially the degree on the curve class  $\beta$  of the polarization  $\mathcal{O}(\theta)$ . For  $\varepsilon \in (1, \infty)$  one obtains the Gromov–Witten invariants of  $W//\mathbf{G}$ ; we write  $\varepsilon = \infty$  for these stability conditions. On the other hand, the first stability chamber  $(0, 1/\beta(L_\theta)]$  depends on  $\beta$  and the asymptotic stability condition  $\varepsilon = 0+$  corresponds to being in this first chamber for *all*  $\beta$ .

The foundations of stable quasimaps were established in [CKM14], as a unified perspective on earlier constructions in [MOP11], [CK10], [MM07], [Tod11] in some special cases; a summary is included in § 2 below. The theory applies to a large class of targets, including a host of varieties whose Gromov–Witten theory has received attention over the last two decades: toric and flag varieties, as well as zero loci of sections of homogeneous bundles on them, local targets with base a GIT quotient, Nakajima quiver varieties, etc. See § 2.8 for a list and more details.

As explained in [CKM14], one has in fact a Cohomological Field Theory in the sense of Kontsevich and Manin for each value of  $\varepsilon$ , and the expectation is that these CohFTs obtained by varying a stability condition are equivalent. In particular, one should express the correspondence between the numerical invariants via wall-crossing formulas. The goal of this paper is to formulate such wall-crossing formulas for the genus zero sector of the theory. What we discover is that they turn out to be a significant generalization of Givental’s mirror theorems for semi-positive complete intersections in toric varieties. At the same time, we give a geometric interpretation to the *mirror map* as a generating series for certain quasimap invariants.

## 1.2 Wall-crossing

The genus zero wall-crossing formulas are most naturally expressed via generating functions for quasimap invariants.

Let the Novikov ring  $\Lambda = \mathbb{Q}[[q]]$  be the  $q$ -adic completion of the semigroup ring on the  $L_\theta$ -effective curve classes (see Definition 2.4.2 for the notion of  $L_\theta$ -effective classes). Let  $\{\gamma_i\}$  be a homogeneous basis of the rational cohomology  $H^*(W//\mathbf{G})$  (equivariant, in the noncompact case), and let  $\{\gamma^i\}$  be the dual basis with respect to the intersection pairing. Denote by  $\mathbf{t} = \sum_i t_i \gamma_i$  the general element and by  $\mathbb{1}$  the unit in  $H^*(W//\mathbf{G})$ .

Let  $z$  be a formal variable. For  $0+ \leq \varepsilon$  and  $\gamma \in H^*(W//\mathbf{G}, \Lambda)$ , put

$$S_{\mathbf{t}}^\varepsilon(\gamma) := \gamma + \sum_i \gamma_i \sum_{(m,\beta) \neq (0,0)} \frac{q^\beta}{m!} \left\langle \frac{\gamma^i}{z - \psi}, \gamma, \mathbf{t}, \dots, \mathbf{t} \right\rangle_{0,2+m,\beta}^\varepsilon.$$

Our wall-crossing formula (see Conjecture 6.1.1, Theorem 7.3.1, and Corollary 7.3.3) relates these series for arbitrary  $\varepsilon_1, \varepsilon_2$  in different (not necessarily adjacent) stability chambers and invertible classes  $\gamma$  of the form  $\gamma = \mathbb{1} + O(q)$ . The case of main interest, when  $\gamma = \mathbb{1}$  and we consider wall-crossings from  $\varepsilon$ -stable quasimap invariants for an arbitrary  $0+ \leq \varepsilon \leq 1$  to Gromov–Witten

invariants, is stated here for simplicity. Let

$$\tau^\varepsilon(\mathbf{t}) := \mathbf{t} + \sum_i \gamma_i \sum_{\beta \neq 0, m \geq 0} \frac{q^\beta}{m!} \langle \gamma^i, \mathbb{1}, \mathbf{t}, \dots, \mathbf{t} \rangle_{0, 2+m, \beta}^\varepsilon. \tag{1.2.1}$$

We then make the following conjecture.

CONJECTURE 1.2.1. For all targets  $W//\mathbf{G}$  and all  $\varepsilon \geq 0+$ ,

$$S_{\tau^\varepsilon(\mathbf{t})}^\infty(\mathbb{1}) = S_{\mathbf{t}}^\varepsilon(\mathbb{1}).$$

In §7 the following theorem is proved.

THEOREM 1.2.2. *Assume that  $W$  admits an action by a torus  $\mathbf{T}$ , commuting with the action of  $\mathbf{G}$  and such that the fixed points of the induced  $\mathbf{T}$ -action on  $W//\mathbf{G}$  are isolated. Then Conjecture 1.2.1 holds. Moreover, if  $E$  is a convex  $\mathbf{G}$ -representation, then the conjecture also holds for the  $E$ -twisted  $\varepsilon$ -quasimap theories of  $W//\mathbf{G}$ .*

The theorem covers the genus zero quasimap theories of almost all interesting examples of GIT targets listed in §2.8: toric varieties, flag varieties, and local targets over them all admit torus actions as above. Among Nakajima quiver varieties, interesting examples are the Hilbert schemes of points on  $\mathbb{C}^2$  and on resolutions of  $A_n$ -singularities, as well as their higher rank analogues (note that we do *not* require in Theorem 1.2.2 that the one-dimensional  $\mathbf{T}$ -orbits in  $W//\mathbf{G}$  are also isolated). Finally, zero loci of sections of globally generated homogeneous vector bundles on all the above targets are also covered by identifying their quasimap theories with twisted theories of the ambient  $W//\mathbf{G}$ ; see [CKM14] and §7.2 below for more details about this identification.

The strategy for proving Theorem 1.2.2 uses virtual localization and is very much inspired by Givental’s original approach to the proofs of toric mirror theorems in [Giv96], [Giv98a], [Giv98b]. The crucial observation that allows us to extend those arguments is that certain edge factors in localization formulas are  $\varepsilon$ -independent.

### 1.3 $J$ -functions and Birkhoff factorization

The series  $S_{\mathbf{t}}^\infty(\mathbb{1})$  in the Gromov–Witten case coincides with Givental’s big  $J$ -function of  $W//\mathbf{G}$  by the string equation. In quasimap theory, there are analogous big  $J^\varepsilon$ -functions for each stability parameter; see [CKM14] and §5. In particular, there is a *big  $I$ -function* corresponding to the  $\varepsilon = 0+$  stability condition. For  $\varepsilon \leq 1$  these  $J^\varepsilon$ -functions may contain both positive and negative powers of  $z$  and their relation with the  $S^\varepsilon$ -operators is given by a more subtle “Birkhoff factorization” formula. More precisely, we prove (see §5.4, Theorem 5.4.1) the following result.

THEOREM 1.3.1. *For all GIT targets  $W//\mathbf{G}$  with well-defined quasimap theory and for every stability parameter  $\varepsilon \geq 0+$ , there is a naturally defined series  $P^\varepsilon = P^\varepsilon(\mathbf{t}, q, z) \in H^*(W//\mathbf{G}, \Lambda)[[z]]$ , convergent in the  $q$ -adic topology and satisfying*

$$J^\varepsilon = S_{\mathbf{t}}^\varepsilon(P^\varepsilon).$$

The series  $P^\varepsilon$  (see (5.4.1)) is a generating series of equivariant virtual intersection numbers on the *quasimap graph spaces*  $QG_{0,k,\beta}^\varepsilon(W//\mathbf{G})$ . These are moduli spaces of  $\varepsilon$ -stable quasimaps whose domain curve has a component which is a parametrized  $\mathbb{P}^1$ . The standard  $\mathbb{C}^*$ -action on  $\mathbb{P}^1$  lifts to the graph spaces and the variable  $z$  appears naturally as the equivariant parameter. As the  $J^\varepsilon$ -functions themselves are defined via  $\mathbb{C}^*$ -equivariant residues on graph spaces, the Birkhoff factorization formula is obtained from virtual localization for this  $\mathbb{C}^*$ -action.

### 1.4 Semi-positive targets

Birkhoff factorization simplifies drastically for *semi-positive* targets. The triple  $(W, \mathbf{G}, \theta)$  is called semi-positive if  $\beta(\det(T_W)) \geq 0$  for all  $L_\theta$ -effective classes  $\beta$ . Here  $T_W$  is the  $\mathbf{G}$ -equivariant virtual tangent bundle of the local complete intersection (lci) variety  $W$ . Semi-positivity implies that the GIT quotient  $W//\mathbf{G}$  has numerically effective (nef) anti-canonical class.

Elementary dimension counting arguments in § 5.5 show that the  $J$ -functions of semi-positive targets have  $z$ -expansion of the form

$$J^\varepsilon(q, \mathbf{t}, z) = J_0^\varepsilon(q)\mathbb{1} + (\mathbf{t} + J_1^\varepsilon(q))\frac{1}{z} + O(1/z^2)$$

with  $J_0^\varepsilon(q) \in \Lambda$  invertible and  $J_1^\varepsilon(q) \in H^{\leq 2}(W//\mathbf{G}, \Lambda)$ , and that

$$P^\varepsilon(q, \mathbf{t}, z) = J_0^\varepsilon(q)\mathbb{1}.$$

Hence Theorem 1.3.1 has the following corollary.

COROLLARY 1.4.1. *For semi-positive  $(W, \mathbf{G}, \theta)$  and for all  $\varepsilon \geq 0+$ ,*

$$\frac{J^\varepsilon(q, \mathbf{t}, z)}{J_0^\varepsilon(q)} = S_{\mathbf{t}}^\varepsilon(\mathbb{1}).$$

A very special case of the above corollary is the main result of [CZ12]. Equating coefficients of  $1/z$  gives an expression for the transformation (1.2.1) in terms of  $J_0^\varepsilon$  and  $J_1^\varepsilon$ .

COROLLARY 1.4.2. *For semi-positive  $(W, \mathbf{G}, \theta)$  and for all  $\varepsilon \geq 0+$ ,*

$$\tau^\varepsilon(\mathbf{t}) = \frac{\mathbf{t} + J_1^\varepsilon(q)}{J_0^\varepsilon(q)}.$$

### 1.5 Mirror symmetry from wall-crossing

For semi-positive targets the wall-crossing formula translates into the following statement.

COROLLARY 1.5.1. *Assume that Conjecture 1.2.1 holds for the semi-positive triple  $(W, \mathbf{G}, \theta)$ . Then for all  $\varepsilon \geq 0+$ ,*

$$J^\infty(q, \tau^\varepsilon(\mathbf{t}), z) = \frac{J^\varepsilon(q, \mathbf{t}, z)}{J_0^\varepsilon(q)} \quad \text{with} \quad \tau^\varepsilon(\mathbf{t}) = \frac{\mathbf{t} + J_1^\varepsilon(q)}{J_0^\varepsilon(q)}.$$

*In particular, this is true in all cases covered by Theorem 1.2.2.*

It follows directly from definitions that the coefficients  $J_0^\varepsilon(q)$  and  $J_1^\varepsilon(q)$  for  $\varepsilon > 0$  depend only on the first stability chamber  $(0, 1/\beta(L_\theta)]$  and are therefore truncations of the corresponding series  $I_0$  and  $I_1$  for the  $0+$  stability condition. These are the first two terms in the  $1/z$ -expansion of the *small*  $I$ -function of  $W//\mathbf{G}$ , obtained by putting  $\mathbf{t} = 0$  in the big  $I$ -function:

$$I^{\text{small}}(q, z) := I(q, 0, z) = I_0(q) + \frac{I_1(q)}{z} + O(1/z^2). \tag{1.5.1}$$

The small  $I$ -function for a toric variety or for a complete intersection in a toric variety was introduced by Givental (it differs from (1.5.1) by an exponential factor); quasimap theory extends the notion to general GIT targets  $W//\mathbf{G}$ . Geometrically, it is obtained by summing over all classes  $\beta$  the  $\mathbb{C}^*$ -equivariant (virtual) residues of certain distinguished fixed-point components in the unpointed graph spaces  $QG_{0,0,\beta}^{0+}(W//\mathbf{G})$ ; see [CK10], [CKM14], and §§ 4–5. These residues

are explicitly computable in general and closed hypergeometric-type formulas for the small  $I$ -functions are known for almost all interesting examples of targets, such as the ones listed in § 2.8.

Givental’s Mirror Theorems for semi-positive toric complete intersections [Giv98a], and for local toric targets [Giv98b], relate the *small* (Gromov–Witten)  $J$ -function to the small  $I$ -function via a change of variables called the *mirror map* on the small parameter space  $H^{\leq 2}(W//\mathbf{G})$ .

Specializing Corollary 1.5.1 to  $\varepsilon = 0+$  and  $\mathbf{t} = 0$  and using the string and divisor equations on the Gromov–Witten side gives precisely this type of relation for all  $W//\mathbf{G}$  (after restoring the omitted exponential factors) and recovers Givental’s results in the toric cases.

Furthermore, the quantum corrections to the mirror map are given by two-pointed quasimap invariants. For example, if  $W//\mathbf{G}$  is Calabi–Yau, the classical mirror map

$$q \mapsto qe^{\int_{\beta} (I_1(q)/I_0(q))}$$

is determined by

$$\frac{I_1(q)}{I_0(q)} = \sum_i \gamma_i \sum_{\beta \neq 0} q^{\beta} \langle \gamma^i, \mathbb{1} \rangle_{0,2,\beta}^{0+}$$

with the first sum only over those cohomology classes  $\gamma_i$  which form a basis of  $H^2(W//\mathbf{G})$ .

More generally, by first taking derivatives  $d/dt_{j_1}, \dots, d/dt_{j_n}$  in the equality of Corollary 1.5.1, then specializing at  $\mathbf{t} = 0$  and applying the string and divisor equations on the Gromov–Witten side, one obtains the following result.

**COROLLARY 1.5.2.** *Assume that Conjecture 1.2.1 holds for the semi-positive triple  $(W, \mathbf{G}, \theta)$ . Then for every stability parameter  $\varepsilon \geq 0+$  and every  $n \geq 1$ , we have*

$$e^{\frac{1}{z} \int_{\beta} (J_1^{\varepsilon}(q)/J_0^{\varepsilon}(q))} \sum_i \gamma_i \sum_{\beta} q^{\beta} e^{\int_{\beta} (J_1^{\varepsilon}(q)/J_0^{\varepsilon}(q))} \left\langle \frac{\gamma^i}{z - \psi}, \gamma_{j_1}, \dots, \gamma_{j_n} \right\rangle_{0,1+n,\beta}^{\infty} = \quad (1.5.2)$$

$$(J_0^{\varepsilon}(q))^{n-1} \sum_i \gamma_i \sum_{\beta} q^{\beta} \left\langle \frac{\gamma^i}{z - \psi}, \gamma_{j_1}, \dots, \gamma_{j_n} \right\rangle_{0,1+n,\beta}^{\varepsilon} .$$

In particular, (1.5.2) is true in all cases covered by Theorem 1.2.2.

From this perspective, Conjecture 1.2.1 and Theorem 1.2.2 generalize toric mirror theorems in several different directions: they apply to general GIT quotients  $W//\mathbf{G}$  rather than just those with abelian group  $\mathbf{G}$ , there is no semi-positivity restriction, they apply to the *big* parameter space  $H^*(W//\mathbf{G})$ , and they apply to all stability conditions  $\varepsilon \geq 0+$ . In addition, the transformation (1.2.1) generalizes the mirror map.

### 1.6 Wall-crossing for $J^{\varepsilon}$ -functions

One consequence of Corollary 1.5.1 is that, in the semi-positive cases, the big  $J^{\varepsilon}$  function lies on Givental’s Lagrangian cone  $\mathcal{L}ag_{W//\mathbf{G}}$  encoding the genus zero Gromov–Witten theory of  $W//\mathbf{G}$ . Conjecture 6.4.2 asserts that this remains true for general targets and Theorem 7.3.4 proves this assertion when there is a torus action with isolated fixed points *and* isolated one-dimensional orbits. We refer the reader to § 6.4 for a detailed formulation. In this case we do not know an elegant expression similar to (1.2.1) for the required transformation of coordinates  $\tau^{\infty,\varepsilon}(\mathbf{t})$ .

### 1.7 Further directions

With appropriate modifications, the methods introduced in this paper extend to orbifold GIT quotients and will be used by the authors in a forthcoming work to obtain similar  $\varepsilon$ -wall-crossing formulas for such targets. In particular, we will obtain orbifold mirror theorems more general than the ones announced in [Iri09] for toric orbifolds and complete intersections in them.

In a different direction, we expect wall-crossing formulas to hold in higher genus quasimap theory as well and plan to investigate these in future work. In particular, we conjecture in § 6.3 that the wall-crossing is trivial for Fano triples  $(W, \mathbf{G}, \theta)$  of Fano index at least two.

## 2. Stable quasimaps

For the convenience of the reader we recall in this section the basics of the theory developed in [CKM14]. Throughout the paper we will work over the ground field  $\mathbb{C}$ .

### 2.1 A class of GIT targets

Let  $W = \text{Spec}(A)$  be an affine algebraic variety with an action by a reductive algebraic group  $\mathbf{G}$ . There are two natural quotients associated with this data: the *quotient stack*  $[W/\mathbf{G}]$  and the *affine quotient*  $W/\text{aff}\mathbf{G} = \text{Spec}(A^{\mathbf{G}})$ .

Denote by  $\text{Pic}^{\mathbf{G}}(W)$  the group of  $\mathbf{G}$ -linearized line bundles on  $W$  and by  $\chi(\mathbf{G})$  the group of characters of  $\mathbf{G}$ . Each character  $\theta$  determines a one-dimensional representation  $\mathbb{C}_{\theta}$  of  $\mathbf{G}$ , hence a linearized line bundle

$$L_{\theta} := W \times \mathbb{C}_{\theta} \in \text{Pic}^{\mathbf{G}}(W).$$

(We will often think of vector bundles in this paper as geometric objects by identifying them with their total space.)

Associated with  $\theta$  we have the GIT quotient

$$W//\mathbf{G} := W//_{\theta}\mathbf{G} := \text{Proj}(\oplus_{n \geq 0} \Gamma(W, L_{\theta}^{\otimes n})^{\mathbf{G}}),$$

which is a quasiprojective variety, with a projective morphism

$$W//\mathbf{G} \longrightarrow W/\text{aff}\mathbf{G}$$

to the affine quotient (see [Kin94, §2] for the GIT setup used in this paper).

Let

$$W^s = W^s(\theta) \quad \text{and} \quad W^{ss} = W^{ss}(\theta)$$

be the stable and semistable open subschemes, respectively, determined by the choice of linearization. The following will be assumed throughout.

- (i)  $\emptyset \neq W^s = W^{ss}$ .
- (ii) The subscheme  $W^s$  is nonsingular.
- (iii) The group  $\mathbf{G}$  acts freely on  $W^s$ .

It follows that  $W//\mathbf{G}$  is a nonsingular variety, which coincides with the stack quotient  $[W^s/\mathbf{G}]$ . In particular, it is naturally an open substack in  $[W/\mathbf{G}]$ . The relative polarization over the affine quotient is given by the descended bundle

$$\underline{L}_{\theta} := W^s \times_{\mathbf{G}} L_{\theta}.$$

We will also use the notation  $\mathcal{O}(\theta)$  for the polarization. Further, by replacing  $\theta$  by a multiple, we

may (and will from now on) assume that  $\mathcal{O}(\theta)$  is relatively very ample over the affine quotient. (Here and in the rest of the paper, the notation above is always understood as the usual *mixed construction*. Namely, given a principal  $\mathbf{G}$ -bundle  $P$  over a base scheme or algebraic stack  $B$  and a scheme  $U$  with  $\mathbf{G}$ -action, we put

$$P \times_{\mathbf{G}} U := [(P \times U)/\mathbf{G}].$$

It is an algebraic stack with a representable morphism to  $B$

$$\rho : P \times_{\mathbf{G}} U \longrightarrow B$$

which in the étale topology is a locally trivial fibration with fiber  $U$ .)

### 2.2 Maps from curves to $[W/\mathbf{G}]$

Let  $(C, x_1, \dots, x_k)$  be a prestable pointed curve, that is, a connected projective curve (of some arithmetic genus  $g$ ), with at most nodes as singularities, together with  $k$  distinct and nonsingular marked points on it. A map  $[u] : C \rightarrow [W/\mathbf{G}]$  corresponds to a pair  $(P, \tilde{u})$  with

$$P \longrightarrow C$$

a principal  $\mathbf{G}$ -bundle on  $C$  and

$$\tilde{u} : P \longrightarrow W$$

a  $\mathbf{G}$ -equivariant morphism. Equivalently (and most often), we will consider the data  $(P, u)$  with

$$u : C \longrightarrow P \times_{\mathbf{G}} W$$

a section of the fiber bundle  $\rho : P \times_{\mathbf{G}} W \rightarrow C$ . Then  $[u] : C \rightarrow [W/\mathbf{G}]$  is obtained as the composite

$$C \xrightarrow{u} P \times_{\mathbf{G}} W \longrightarrow [W/\mathbf{G}].$$

Note also that the composition

$$C \xrightarrow{[u]} [W/\mathbf{G}] \longrightarrow W/\text{aff } \mathbf{G}$$

is always a constant map, since  $C$  is projective.

### 2.3 Numerical class of a map to $[W/\mathbf{G}]$

Recall that  $L \mapsto [L/\mathbf{G}]$  gives an identification  $\text{Pic}^{\mathbf{G}}(W) = \text{Pic}([W/\mathbf{G}])$ .

Let  $(C, P, u)$  be as above. For a  $\mathbf{G}$ -equivariant line bundle  $L$  on  $W$  we get an induced line bundle  $u^*(P \times_{\mathbf{G}} L) = [u]^*([L/\mathbf{G}])$  on  $C$ .

By definition, the *numerical class*  $\beta$  of  $(P, u) \in \text{Map}(C, [W/\mathbf{G}])$  is the group homomorphism

$$\beta : \text{Pic}^{\mathbf{G}}(W) \rightarrow \mathbb{Z}, \quad \beta(L) = \deg_C(u^*(P \times_{\mathbf{G}} L)).$$

If we set

$$\mathcal{L}_{\theta} := u^*(P \times_{\mathbf{G}} L_{\theta}) = P \times_{\mathbf{G}} \mathcal{C}_{\theta}$$

and if  $[u]$  has image contained in the GIT quotient  $W//\mathbf{G}$ , then  $\mathcal{L}_{\theta} = [u]^*\mathcal{O}(\theta)$ , so  $\beta(L_{\theta})$  is the usual degree of the map with respect to the polarization  $\mathcal{O}(\theta)$ .

### 2.4 Quasimaps to $W//\mathbf{G}$ and $\varepsilon$ -stability

Fix integers  $g, k \geq 0$ . We denote by  $\mathfrak{M}_{g,k}([W/\mathbf{G}], \beta)$  the moduli stack parametrizing families of tuples

$$((C, x_1, \dots, x_k), P, u)$$

with  $(C, x_1, \dots, x_k)$  a prestable curve of genus  $g$ ,  $P \rightarrow C$  a principal  $\mathbf{G}$ -bundle on  $C$ , and  $u$  a section of  $P \times_{\mathbf{G}} W \rightarrow C$ , of degree  $\beta$ . It contains the Deligne–Mumford stack

$$M_{g,k}(W//\mathbf{G}, \beta)$$

of (stable) maps from nonsingular pointed curves to  $W//\mathbf{G}$  as an open substack. However, it is in general a nonseparated Artin stack of infinite type. More manageable moduli spaces are obtained by imposing certain *stability conditions*. These will give compactifications of  $M_{g,k}(W//\mathbf{G}, \beta)$  over  $W/_{\text{aff}}\mathbf{G}$ , which are Deligne–Mumford and carry canonical perfect obstruction theories.

For example, one such stability condition is obtained by requiring first, that the image of  $[u]$  be contained in  $[W^s/\mathbf{G}] = W//\mathbf{G}$  or, equivalently, that the section  $u$  land inside  $P \times_{\mathbf{G}} W^s$ , and second, that the map

$$[u] : (C, x_1, \dots, x_k) \longrightarrow W//\mathbf{G}$$

be Kontsevich-stable. This of course leads to the familiar moduli stack  $\overline{M}_{g,k}(W//\mathbf{G}, \beta)$  of stable maps to  $W//\mathbf{G}$ .

There are other stability conditions for which the resulting theory is that of  $\varepsilon$ -stable quasimaps from curves to  $W//\mathbf{G}$ , developed in [CKM14] (and inspired by earlier works [MOP11], [CK10], [MM07]). We recall briefly the main points, and refer the reader to *loc. cit.* for the details.

**DEFINITION 2.4.1.** A *quasimap* to  $W//\mathbf{G}$  is a map  $((C, x_1, \dots, x_k), P, u)$  to the quotient stack such that for every irreducible component  $C_i$  of  $C$ , with generic point  $\xi_i$ , we have  $u(\xi_i) \in P \times_{\mathbf{G}} W^s$ .

Hence a quasimap has at most finitely many *base points*, that is, points on  $C$  which are mapped into the unstable locus of  $W$  by the section  $u$ .

By [CKM14, Lemma 3.2.1], if  $(C, P, u)$  is a quasimap, then for any subcurve  $C'$  of  $C$ , the class  $\beta'$  of the restriction of the quasimap to  $C'$  satisfies

$$\beta'(L_\theta) \geq 0, \quad \text{with equality if and only if } \beta' = 0.$$

**DEFINITION 2.4.2.** We say that an element  $\beta \in \text{Hom}_{\mathbb{Z}}(\text{Pic}^{\mathbf{G}}(W), \mathbb{Z})$  is  $L_\theta$ -effective if it can be represented as a finite sum of classes of quasimaps.

The  $L_\theta$ -effective classes form a semigroup, denoted  $\text{Eff}(W, \mathbf{G}, \theta)$ .

*Remark 2.4.3.* The “nondegeneracy at generic points” imposed in the definition of a quasimap is clearly an open condition on the base of families of maps to the stack  $[W/\mathbf{G}]$ . It is shown in [CKM14, Theorem 3.2.5 and Remark 3.2.10] that quasimaps to  $W//\mathbf{G}$  with given underlying pointed curve and fixed  $\beta(L_\theta)$  form a bounded family. Hence we have an open Artin substack

$$\Omega\text{map}_{g,k}(W//\mathbf{G}, \beta) \subset \mathfrak{M}_{g,k}([W/\mathbf{G}], \beta)$$

which is of finite type over the stack  $\mathfrak{M}_{g,k}$  of marked prestable curves.

**DEFINITION 2.4.4.** A quasimap is *prestably* if its base points are away from the nodes and markings of the underlying curve.



DEFINITION 2.4.5. The *length*  $\ell(x)$  at a point  $x \in C$  of a prestable quasimap  $((C, x_i), P, u)$  to  $W//\mathbf{G}$  is the order of contact of  $u(C)$  with the unstable subscheme  $P \times_{\mathbf{G}} W^{us}$  at  $u(x)$ . To be more precise, consider the ideal sheaf  $\mathcal{J}$  of the closed subscheme  $P \times_{\mathbf{G}} W^{us}$  of  $P \times_{\mathbf{G}} W$ . Then

$$\ell(x) = \text{length}_x(\text{coker}(u^* \mathcal{J} \rightarrow \mathcal{O}_C)). \quad (2.4.1)$$

DEFINITION 2.4.6. Fix  $\varepsilon \in \mathbb{Q}_{>0}$ . A quasimap

$$((C, x_1, \dots, x_k), P, u)$$

is called  $\varepsilon$ -*stable* if it is prestable and the following two conditions hold:

- (i) The line bundle  $\omega_C(\sum_{i=1}^k x_i) \otimes \mathcal{L}_\theta^\varepsilon$  is ample.
- (ii) We have  $\varepsilon \ell(x) \leq 1$  for every point  $x$  in  $C$ .

Here  $\mathcal{L}_\theta = P \times_{\mathbf{G}} \mathbb{C}_\theta = u^*(P \times_{\mathbf{G}} L_\theta)$ .

It is easy to see that  $\varepsilon$ -stable quasimaps form an open moduli substack in  $\mathfrak{Q}\text{map}_{g,k}(W//\mathbf{G}, \beta)$ . We will denote it by  $\mathfrak{Q}_{g,k}^\varepsilon(W//\mathbf{G}, \beta)$ . The following is proved in [CKM14, Theorem 7.1.6].

THEOREM 2.4.7. *The stack  $\mathfrak{Q}_{g,k}^\varepsilon(W//\mathbf{G}, \beta)$  is a Deligne–Mumford stack, separated, of finite type, and with a natural proper morphism*

$$\mathfrak{Q}_{g,k}^\varepsilon(W//\mathbf{G}, \beta) \rightarrow \text{Spec}(A(W)^{\mathbf{G}}).$$

*Remark 2.4.8.* The description of the points in the moduli spaces when  $\varepsilon$  varies follows easily from the definition of  $\varepsilon$ -stability. We summarize it below.

(i) Assume  $(g, k) \neq (0, 0)$ . Then a quasimap is a *stable map* to  $W//\mathbf{G}$  in the usual sense if and only if it is  $\varepsilon$ -stable for every  $\varepsilon > 1$ . When  $(g, k) = (0, 0)$ , the same holds with  $\varepsilon > 2$ . Hence we have

$$\mathfrak{Q}_{g,k}^\infty(W//\mathbf{G}, \beta) = \overline{M}_{g,k}(W//\mathbf{G}, \beta).$$

(ii) Fix  $\beta \in \text{Eff}(W, \mathbf{G}, \theta)$ . For each  $\varepsilon \leq 1/\beta(L_\theta)$  the length inequality in the stability condition is always satisfied, while the ampleness requirement imposes first, that the underlying curve  $C$  have *no rational tails* (irreducible components which are rational and carry exactly one special point – marking or node – on them), and second, that on each *rational bridge* (irreducible component which is rational and carries exactly two special points), the line bundle  $\mathcal{L}_\theta$  have strictly positive degree.

Since we may want to deal with all possible degrees  $\beta$  at the same time, it is better to reformulate the stability condition in this case as

$$\omega_C\left(\sum_{i=1}^k x_i\right) \otimes \mathcal{L}_\theta^\varepsilon \text{ is ample for all } \varepsilon \in \mathbb{Q}_{>0}.$$

We will call these either  $(\varepsilon = 0+)$ -stable quasimaps, or, as in [CKM14], *stable quasimaps to  $W//\mathbf{G}$* ; the resulting moduli stack will be denoted by

$$\mathfrak{Q}_{g,k}^{0+}(W//\mathbf{G}, \beta),$$

or simply by  $\mathfrak{Q}_{g,k}(W//\mathbf{G}, \beta)$ . Note that these moduli spaces are empty for  $(g, k) = (0, 0), (0, 1)$ .

(iii) Assume for simplicity that  $(g, k) \neq (0, 0), (0, 1)$ . Then for every integer  $1 \leq e \leq \beta(L_\theta) - 1$ , the moduli space of  $\varepsilon$ -stable quasimaps to  $W//\mathbf{G}$  with fixed numerical data  $(g, k, \beta)$  stays constant when  $\varepsilon \in (1/(e+1), 1/e]$ . The underlying curve of a quasimap parametrized by this constant moduli space is allowed only rational tails of total degree at least  $e+1$  (with respect to  $L_\theta$ ),

while the length at each base point must be at most  $e$ . Hence, for each fixed class  $\beta$ , the set  $\mathbb{Q}_{>0}$  is divided into chambers by finitely many walls  $1, 1/2, \dots, 1/\beta(L_\theta)$ . The two extreme chambers  $(1, \infty)$  and  $(0, 1/\beta(L_\theta)]$  always correspond to stable maps and stable quasimaps, respectively.

### 2.5 Obstruction theory

The moduli stack of  $\varepsilon$ -stable quasimaps comes with natural forgetful morphisms

$$\mu : \mathbb{Q}_{g,k}^\varepsilon(W//\mathbf{G}, \beta) \longrightarrow \mathfrak{M}_{g,k}, \quad \nu : \mathbb{Q}_{g,k}^\varepsilon(W//\mathbf{G}, \beta) \longrightarrow \mathfrak{Bun}_{\mathbf{G}},$$

where  $\mathfrak{M}_{g,k}$  is the moduli stack of prestable curves and  $\mathfrak{Bun}_{\mathbf{G}}$  is the moduli stack of principal  $\mathbf{G}$ -bundles on the fibers of the universal curve  $\pi : \mathfrak{C}_{g,k} \rightarrow \mathfrak{M}_{g,k}$ .

There is a universal family

$$((\mathcal{C}^\varepsilon, x_1, \dots, x_k), \mathcal{P}, u)$$

over  $\mathbb{Q}_{g,k}^\varepsilon(W//\mathbf{G}, \beta)$ , where the universal curve

$$\pi : \mathcal{C}^\varepsilon \longrightarrow \mathbb{Q}_{g,k}^\varepsilon(W//\mathbf{G}, \beta)$$

is the pull-back of  $\mathfrak{C}_{g,k}$  via  $\mu$ ,  $\mathcal{P}$  is the pull-back of the universal principal  $\mathbf{G}$ -bundle on (the universal curve over)  $\mathfrak{Bun}_{\mathbf{G}}$  via  $\nu$ , and  $u$  is a section of the fiber bundle

$$\varrho : \mathcal{P} \times_G W \longrightarrow \mathcal{C}^\varepsilon$$

(the universal section).

The canonical obstruction theory of  $\mathbb{Q}_{g,k}^\varepsilon(W//\mathbf{G}, \beta)$ , relative to the smooth Artin stack  $\mathfrak{Bun}_{\mathbf{G}}$ , is given by the complex

$$(R^\bullet \pi_*(u^* \mathbb{R}T_\varrho))^\vee \tag{2.5.1}$$

with  $\mathbb{R}T_\varrho$  the relative tangent complex of  $\varrho$ .

The following result is part of [CKM14, Theorem 7.1.6].

**THEOREM 2.5.1.** *If  $W$  has only lci singularities, then the canonical obstruction theory (2.5.1) of  $\mathbb{Q}_{g,k}^\varepsilon(W//\mathbf{G}, \beta)$  is perfect.*

The lci condition implies that  $W$  has a (virtual) tangent bundle  $T_W$  in the  $\mathbf{G}$ -equivariant  $K$ -group. Hence  $\det(T_W)$  is a well-defined element in  $\text{Pic}^{\mathbf{G}}(W)$ . The virtual dimension of the stack  $\mathbb{Q}_{g,k}^\varepsilon(W//\mathbf{G}, \beta)$  is

$$\text{vdim}(\mathbb{Q}_{g,k}^\varepsilon(W//\mathbf{G}, \beta)) = \beta(\det(T_W)) + (1 - g)(\dim(W//\mathbf{G}) - 3) + k.$$

Note that it is independent of  $\varepsilon$ .

For classes  $\beta$  for which the moduli space  $\mathbb{Q}_{g,k}^\varepsilon(W//\mathbf{G}, \beta)$  contains a point represented by a regular map  $u : C \rightarrow W//\mathbf{G}$ , we may view  $\beta$  as the second homology class  $u_*[C] \in H_2(W//\mathbf{G}, \mathbb{Z})$ . We then have

$$\beta(\det(T_W)) = \int_\beta c_1(T_{W//\mathbf{G}}).$$

Note, however, that for some  $L_\theta$ -effective classes it is possible that every point of  $\mathbb{Q}_{g,k}^\varepsilon(W//\mathbf{G}, \beta)$  corresponds to a quasimap with base points. The simplest example is when  $W = \mathbb{C}$ ,  $\mathbf{G} = \mathbb{C}^*$  with the natural action, linearized by the identity character (so that  $W//\mathbf{G}$  is a single point), and we take  $\beta \neq 0$ . For a less trivial example, consider the conic in  $\mathbb{P}^2$  (see Example 2.8.3 below) and take the classes with  $\beta(L_\theta)$  a positive odd integer.

## 2.6 Graph spaces

We will also need moduli stacks of maps for which the domain curve contains a *parametrized*  $\mathbb{P}^1$ ; see [CKM14, § 7.2]. These are called *graph spaces*. Even though we will later only use the genus zero case of the results of this subsection, everything makes sense for arbitrary genus and we treat things in this generality.

Let  $\mathfrak{G}_{g,k,\beta}([W/\mathbf{G}])$  denote the Artin stack parametrizing tuples

$$((C, x_1, \dots, x_k), P, u, \varphi),$$

where the new data  $\varphi$  is a morphism from  $C$  to  $\mathbb{P}^1$  of degree one (that is, such that  $\varphi_*[C] = [\mathbb{P}^1]$ ). Alternatively, these are all maps

$$C \longrightarrow [W/\mathbf{G}] \times \mathbb{P}^1$$

of class  $(\beta, 1)$ . The curve  $C$  has one irreducible component  $C_0$  such that  $\varphi|_{C_0} : C_0 \xrightarrow{\sim} \mathbb{P}^1$  is an isomorphism, while the rest of the curve is contracted by  $\varphi$ .

By imposing stability conditions as in § 2, we obtain  $\varepsilon$ -stable quasimap graph spaces, denoted by

$$QG_{g,k,\beta}^\varepsilon(W//\mathbf{G}).$$

In this case, the ampleness part of the stability condition *does not* involve the parametrized component. To be more precise, we require as before that the data  $((C, x_1, \dots, x_k), P, u)$  be a prestable quasimap, that the length inequality  $\varepsilon \ell(x) \leq 1$  hold for every point  $x$  in  $C$ , and that the following modified ampleness condition hold:

$$\omega_{C'}(\sum x_i + \sum y_j) \otimes \mathcal{L}_\theta^\varepsilon \text{ is ample,}$$

where  $C'$  is the closure of  $C \setminus C_0$ , the  $x_i$  are the markings on  $C'$ , and the  $y_j$  are the nodes  $C' \cap C_0$ .

In the chamber  $1 < \varepsilon \leq \infty$  we have the usual stable map theory graph space

$$G_{g,k,\beta}(W//\mathbf{G}) = QG_{g,k,\beta}^\infty(W//\mathbf{G}) = \overline{M}_{g,k}(W//\mathbf{G} \times \mathbb{P}^1, (\beta, 1)).$$

For the other extreme chamber corresponding to  $\varepsilon = 0+$ , the quasimap graph space

$$QG_{g,k,\beta}^{0+}(W//\mathbf{G})$$

was denoted by  $\text{Qmap}_{g,k}(W//\mathbf{G}, \beta; \mathbb{P}^1)$  in [CKM14, § 7.2].

If we choose coordinates  $[\zeta_0, \zeta_1]$  on  $\mathbb{P}^1$ , then we have the standard  $\mathbb{C}^*$  action, given by

$$t[\zeta_0, \zeta_1] = [t\zeta_0, \zeta_1], \quad t \in \mathbb{C}^*.$$

It induces actions on the  $\varepsilon$ -stable quasimap graph spaces which will be of great importance for the rest of the paper.

The analogues of Theorems 2.4.7 and 2.5.1 hold for graph spaces; see [CKM14, Theorem 7.2.2]. The only difference is that the moduli stack  $\mathfrak{M}_{g,k}$  of prestable curves is replaced by the analogous moduli of prestable curves with one parametrized  $\mathbb{P}^1$  (that is, by the stack  $\mathfrak{M}_{g,k}(\mathbb{P}^1, 1)$ ) in the description of the relative obstruction theory. With this change, the obstruction theory is still given by the formula (2.5.1).

## 2.7 Quasimap invariants

The condition that base points do not appear at the markings ensures there are well-defined evaluation maps

$$\text{ev}_i : Q_{g,k}^\varepsilon(W//\mathbf{G}, \beta) \longrightarrow W//\mathbf{G}, \quad i = 1, \dots, k.$$

The moduli spaces also carry the tautological cotangent line bundles at the markings, whose first Chern classes we denote by  $\psi_i$ ,  $i = 1, \dots, k$ , as is customary in Gromov–Witten theory. These structures exist on the graph spaces as well.

Assume that  $W//\mathbf{G}$  is projective. For nonnegative integers  $a_i$  and cohomology classes  $\gamma_i \in H^*(W//\mathbf{G}, \mathbb{Q})$ , the associated descendant  $\varepsilon$ -quasimap invariant of  $W//\mathbf{G}$  is the rational number

$$\int_{[\mathcal{Q}_{g,k}^\varepsilon(W//\mathbf{G}, \beta)]^{\text{vir}}} \prod_j \text{ev}_j^*(\gamma_j) \psi_j^{a_j}.$$

If  $W//\mathbf{G}$  is only quasiprojective, one may still define a good theory of *equivariant* quasimap invariants in the presence of an action with good properties by an algebraic torus, as explained in [CKM14, § 6.3] and at the beginning of § 7.

## 2.8 Examples

We list here several classes of targets for which the theory of quasimaps applies.

*Example 2.8.1. Smooth toric varieties.* These are GIT quotients  $W//\mathbf{G}$  with  $W \cong \mathbb{C}^N$  a vector space and  $\mathbf{G} = (\mathbb{C}^*)^l$  abelian. Their quasimap theory (in the projective case) was first developed in [CK10]. There are also noncompact toric targets with very interesting theories. For example, taking  $W = \mathbb{C}^n \oplus \mathbb{C}$ ,  $\mathbf{G} = \mathbb{C}^*$  acting with weights  $(1, \dots, 1, -m)$ ,  $m \in \mathbb{Z}$ , and  $\theta$  the identity character gives the GIT quotient  $W//\mathbf{G} = |\mathcal{O}_{\mathbb{P}^{n-1}}(-m)|$ , the total space of the line bundle  $\mathcal{O}(-m)$  on  $\mathbb{P}^{n-1}$ .

*Example 2.8.2. Type A flag manifolds.* These are also quotients of a vector space, but with  $\mathbf{G}$  a product of groups  $GL_{k_i}(\mathbb{C})$  and  $\theta$  given by the determinant character on each factor (see, for example, [BCK08] for this quotient description). In particular, taking  $W = \text{Hom}(\mathbb{C}^r, \mathbb{C}^n)$  and  $\mathbf{G} = GL_r(\mathbb{C})$  with the obvious action gives the Grassmannian  $G(r, n)$  as  $W//_{\det} \mathbf{G}$ . The quasimap theory of the Grassmannian recovers the *stable quotients* theory of [MOP11]. The projective space  $\mathbb{P}^{n-1}$  may be viewed either as a special case of flag manifolds, or as a toric example.

*Example 2.8.3. Projective varieties in general.* Let  $X \subset \mathbb{P}^{n-1} = \mathbb{C}^n // \mathbb{C}^*$  be a smooth projective variety. Then  $X = W//\mathbb{C}^*$  with  $W = C(X)$ , the affine cone over  $X$ , and we have the proper moduli spaces of  $\varepsilon$ -stable quasimaps to  $X$ . However, for  $\varepsilon \leq 1$  the obstruction theory will be perfect precisely when  $X$  is a complete intersection. (For  $\varepsilon > 1$  and arbitrary  $X$  one recovers the usual stable maps to  $X$  with their canonical perfect obstruction theory.)

*Example 2.8.4. Zero loci of sections of homogeneous bundles.* The previous example of complete intersections in projective space generalizes as follows. Let  $W, \mathbf{G}, \theta$  be as in the general set-up and assume that  $W$  is smooth. Let  $E$  be a finite-dimensional  $\mathbf{G}$ -representation. Let  $W \times E$  be the corresponding  $\mathbf{G}$ -equivariant vector bundle on  $W$ , and let

$$\underline{E} := W^s \times_{\mathbf{G}} E$$

be the induced vector bundle on  $W//\mathbf{G}$ . Let  $s \in H^0(W, W \times E)^{\mathbf{G}}$  be a  $\mathbf{G}$ -invariant global section, inducing  $\underline{s} \in H^0(W//\mathbf{G}, \underline{E})$ . Assume that  $\underline{s}$  is regular and that its zero locus  $X := Z(\underline{s})$  is smooth. Then  $X = Z//_{\theta} \mathbf{G}$  with  $Z := Z(s)$ . Furthermore,  $Z$  has only lci singularities (contained in the unstable locus), hence the moduli spaces of quasimaps to  $X$  will carry perfect obstruction theories for all  $\varepsilon$ .

For toric  $W//\mathbf{G}$  this construction gives precisely the complete intersections in toric varieties (since the representation  $E$  splits into one-dimensional summands), but there are many interesting targets  $Z//\mathbf{G}$  with  $\underline{E}$  nonsplit when  $\mathbf{G}$  is nonabelian. For example, as explained in [BCK08],

all flag manifolds of classical types  $B, C$ , and  $D$  are realized in this fashion, with  $W//\mathbf{G}$  a flag manifold of type  $A$ . For another example (suggested to us by E. Scheidegger), consider on the Grassmannian  $G(2, 6)$  the vector bundle  $\underline{E} = \text{Sym}^3(S^\vee)$ , where  $S^\vee$  is the dual of the tautological rank two sub-bundle. The vector bundle  $\underline{E}$  has rank four and first Chern class equal to 6, and descends from the third symmetric power of the standard representation of  $GL_2(\mathbb{C})$ . The zero locus of a regular section defines a Calabi–Yau fourfold of the form  $Z//GL_2(\mathbb{C})$  which is not a complete intersection in  $G(2, 6)$ .

*Example 2.8.5. Local targets.* Fix again  $W, \mathbf{G}, \theta$  as in the general set-up, and assume that  $W//\mathbf{G}$  is projective. Let  $E$  be a  $\mathbf{G}$ -representation as in the previous example, and let  $\mathcal{E} := W \times E$  denote the associated  $\mathbf{G}$ -equivariant vector bundle with projection

$$p : \mathcal{E} \longrightarrow W.$$

Assume  $H^0(W, \mathcal{E})^{\mathbf{G}} = 0$ . (We call such a representation  $E$  *concave*. One may take a general  $\mathbf{G}$ -equivariant bundle  $\mathcal{E}$  on  $W$ , but most cases of interest arise from  $\mathbf{G}$ -representations and we restrict to those here for simplicity.) The line bundle  $p^*L_\theta$  gives a linearization of the  $\mathbf{G}$ -action on  $\mathcal{E}$  and the quotient  $\mathcal{E}//\mathbf{G}$  is the total space of the induced bundle  $\underline{E}$  on  $W//\mathbf{G}$ .

In Gromov–Witten theory these kind of spaces are known as *local targets*. Since they are not compact, a good theory of invariants requires the presence of a torus action. Specifically, consider the torus  $\mathbf{S} = \mathbb{C}^*$  which acts on  $\mathcal{E}$  by scalar multiplication in the fibers. Let  $\mathbb{C}_\lambda$  be the standard representation of  $\mathbf{S}$ . Let  $0 : W \longrightarrow \mathcal{E}$  be the closed embedding as the zero section. It is  $\mathbf{G}$ -equivariant and  $0^*p^*L_\theta = L_\theta$ , so it induces a closed embedding of stacks

$$i : \mathbb{Q}_{g,k}^\varepsilon(W//\mathbf{G}, \beta) \hookrightarrow \mathbb{Q}_{g,k}^\varepsilon(\mathcal{E}//\mathbf{G}, \beta).$$

(Since the pull-back via  $p$  gives an isomorphism  $\text{Pic}^{\mathbf{G}}(W) \cong \text{Pic}^{\mathbf{G}}(\mathcal{E})$ , the curve classes are the same for  $W//\mathbf{G}$  and  $\mathcal{E}//\mathbf{G}$ .) The following facts are easy to check.

- (i) The  $\mathbf{S}$ -fixed substack in  $\mathbb{Q}_{g,k}^\varepsilon(\mathcal{E}//\mathbf{G}, \beta)$  is  $\mathbb{Q}_{g,k}^\varepsilon(W//\mathbf{G}, \beta)$ . In particular, it is proper.
- (ii) The fixed part of the obstruction theory coincides with the canonical obstruction theory of  $\mathbb{Q}_{g,k}^\varepsilon(W//\mathbf{G}, \beta)$ .
- (iii) The virtual normal bundle is  $R^\bullet\pi_*\mathcal{E}_{g,k,\beta}(\lambda)$ , where

$$\mathcal{E}_{g,k,\beta}(\lambda) = u^*(\mathfrak{P} \otimes_{\mathbf{G}} (\mathcal{E} \otimes \mathbb{C}_\lambda))$$

with  $\pi : \mathcal{C} \rightarrow \mathbb{Q}_{g,k}^\varepsilon(W//\mathbf{G}, \beta)$  the universal curve,  $\mathfrak{P}$  the universal principal  $\mathbf{G}$ -bundle on it, and  $u$  the universal section.

The  $\varepsilon$ -quasimap invariants of  $\mathcal{E}//\mathbf{G}$  are defined via the virtual localization formula and take values in the field  $\mathbb{Q}(\lambda)$ .

*Example 2.8.6. Nakajima quiver varieties.* These form a class of noncompact GIT targets with interesting torus-equivariant quasimap theory; see [CKM14, Example 6.3.2]. They are holomorphic symplectic varieties, a useful feature when applying to them the general results on genus zero quasimap theory we prove later in this paper; see Remark 5.5.6 (ii).

### 3. Maps between moduli spaces and universal line bundles

There are several morphisms between various moduli spaces that we will need.

### 3.1 Embeddings

The polarization  $\mathcal{O}(\theta)$  gives an embedding

$$\iota : W // \mathbf{G} \longrightarrow \mathbb{P}_{A^{\mathbf{G}}}^N = \text{Proj}(\text{Sym}_{A^{\mathbf{G}}}((\Gamma(W, L_\theta)^{\mathbf{G}})^\vee))$$

over the affine quotient  $\text{Spec}(A^{\mathbf{G}})$ . In particular, we have a basis of  $\mathbf{G}$ -equivariant global sections  $t_0, \dots, t_N \in \Gamma(W, L_\theta)^{\mathbf{G}}$  which cut out the unstable locus  $W^u$  set-theoretically:

$$W^u = \{w \in W \mid t_i(w) = 0, \quad i = 0, \dots, N\}.$$

Now  $\mathbb{P}_{A^{\mathbf{G}}}^N$  with the usual polarization  $\mathcal{O}(1)$  is itself the GIT quotient

$$(\text{Spec}(A^{\mathbf{G}}) \times \mathbb{C}^{N+1}) // \mathbf{H},$$

with  $\mathbf{H} = \mathbb{C}^*$  acting only on the factor  $\mathbb{C}^{N+1}$  by

$$\lambda \cdot (z_0, \dots, z_N) = (\lambda z_0, \dots, \lambda z_N), \quad \lambda \in \mathbf{H},$$

and linearization  $\mathbb{C}_\eta$ . Here  $\mathbb{C}_\eta$  denotes the standard one-dimensional representation of  $\mathbf{H} = \mathbb{C}^*$ , corresponding to the character  $\eta = \text{id}_{\mathbb{C}^*}$ . (We use the notation  $\mathbf{H}$  to distinguish this group from the group  $\mathbb{C}^*$  acting on graph spaces considered in § 2.6.)

Hence we have the graph space

$$\mathfrak{G}_{g,k,d}([\mathbb{C}_{A^{\mathbf{G}}}^{N+1}/\mathbf{H}]),$$

whose  $\mathbb{C}$ -points we identify as usual with the data

$$((C, x_1, \dots, x_k), \mathcal{L}, (s_i)_{i=0, \dots, N}, \varphi), \quad (3.1.1)$$

where  $\mathcal{L}$  is a line bundle on  $C$  and the  $s_i \in \Gamma(C, \mathcal{L})$  are global sections. The degree  $d$  of the map

$$[(s_0, \dots, s_N)] : C \longrightarrow [\mathbb{C}_{A^{\mathbf{G}}}^{N+1}/\mathbf{H}]$$

is in this case the unique homomorphism

$$d : \text{Pic}_{\mathbf{H}}(\mathbb{C}^{N+1}) \cong \mathbb{Z} \longrightarrow \mathbb{Z}$$

with  $d(\mathbb{C}_\eta) = \text{deg}(\mathcal{L})$ . The map (3.1.1) is a quasimap if and only if the restrictions of the sections  $s_j$  to every irreducible component of  $C$  are not all identically zero. This description agrees with the earlier one using principal  $\mathbf{H}$ -bundles  $P$  on  $C$ , via the identification  $\mathcal{L} = P \times_{\mathbf{H}} \mathbb{C}_\eta$ .

Each  $\mathbb{C}$ -point

$$((C, x_1, \dots, x_k), P, u, \varphi)$$

in  $\mathfrak{G}_{g,k,\beta}([W/\mathbf{G}])$  induces a tuple

$$((C, x_1, \dots, x_k), u^*(P \times_{\mathbf{G}} L_\theta), u^*(t_i)_{i=0, \dots, N}, \varphi)$$

which is a point in  $\mathfrak{G}_{g,k,d(\beta)}([\mathbb{C}_{A^{\mathbf{G}}}^{N+1}/\mathbf{H}])$ . Here the degree  $d(\beta)$  is the homomorphism

$$\text{Pic}_{\mathbf{H}}(\mathbb{C}^{N+1}) \longrightarrow \mathbb{Z}, \quad \mathbb{C}_\eta \longmapsto \beta(u^*(P \times_{\mathbf{G}} L_\theta)).$$

Since the construction is natural, it will work in families as well, so we obtain a morphism of stacks

$$\iota_{\mathfrak{G}} : \mathfrak{G}_{g,k,\beta}([W/\mathbf{G}]) \longrightarrow \mathfrak{G}_{g,k,d(\beta)}([\mathbb{C}_{A^{\mathbf{G}}}^{N+1}/\mathbf{H}]),$$

which is clearly equivariant for the action of  $\mathbb{C}^*$  on the graph spaces from § 2.6. It obviously depends on the choice of  $\theta$  and the sections  $t_0, \dots, t_N$ .

It is immediate to check that  $\iota_{\mathbf{G}}$  preserves the  $\varepsilon$ -stability condition for each  $0+ \leq \varepsilon \leq \infty$ . Hence there are  $\mathbb{C}^*$ -equivariant morphisms

$$\iota_{\varepsilon} : QG_{g,k,\beta}^{\varepsilon}(W//\mathbf{G}) \longrightarrow QG_{g,k,d(\beta)}^{\varepsilon}(\mathbb{P}_{A^{\mathbf{G}}}^N). \quad (3.1.2)$$

The morphism  $\iota_{\infty}$  is of course the usual embedding of moduli spaces obtained by composing a stable map with the given embedding

$$\iota \times \text{id}_{\mathbb{P}^1} : W//\mathbf{G} \times \mathbb{P}^1 \longrightarrow \mathbb{P}_{A^{\mathbf{G}}}^N \times \mathbb{P}^1.$$

Finally, note that the construction works equally well for the moduli spaces without a parametrized component, since the parametrization  $\varphi : C \rightarrow \mathbb{P}^1$  was simply carried along unmodified. We obtain morphisms

$$\iota_{\varepsilon} : Q_{g,k}^{\varepsilon}(W//\mathbf{G}, \beta) \longrightarrow Q_{g,k}^{\varepsilon}(\mathbb{P}_{A^{\mathbf{G}}}^N, d(\beta)). \quad (3.1.3)$$

### 3.2 Contractions

We'll also need various contraction morphisms between moduli spaces with target

$$\mathbb{P}_{A^{\mathbf{G}}}^N = (\text{Spec}(A^{\mathbf{G}}) \times \mathbb{C}^{N+1})//\mathbf{H}.$$

**3.2.1 Contractions to the parametrized  $\mathbb{P}^1$ .** The first kind we consider applies to graph spaces and contracts everything to base points on the parametrized  $\mathbb{P}^1$ . Hence the target of all such morphisms will be Drinfeld's original "space of quasimaps"

$$QG_{0,0,d}(\mathbb{P}_{A^{\mathbf{G}}}^N)$$

compactifying the Hom scheme  $\text{Map}_d(\mathbb{P}^1, \mathbb{P}_{A^{\mathbf{G}}}^N)$ . If  $\zeta_0, \zeta_1$  are homogeneous coordinates on  $\mathbb{P}^1$  as in §4.1 and

$$V_d \cong H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(d))$$

denotes the vector space of homogeneous polynomials of degree  $d$  in  $\zeta_0, \zeta_1$ , then  $\text{Map}_d(\mathbb{P}^1, \mathbb{P}_{A^{\mathbf{G}}}^N)$  can be identified with

$$(\mathcal{U}/\mathbf{H}) \times \text{Spec}(A^{\mathbf{G}}),$$

where

$$\mathcal{U} \subset V_d^{\oplus(N+1)}$$

is the open subset of  $(N+1)$ -tuples of polynomials with no common zero on  $\mathbb{P}^1$ . The Drinfeld compactification is simply the projectivization

$$\mathbb{P}(V_d^{\oplus(N+1)}) \times \text{Spec}(A^{\mathbf{G}}).$$

It is very easy to describe these morphisms point-wise. Let

$$((C, x_1, \dots, x_k), \mathcal{L}, (s_i)_{i=0, \dots, N}, \varphi : C \longrightarrow \mathbb{P}^1)$$

be a  $k$ -pointed  $\varepsilon$ -stable quasimap to  $\mathbb{P}_{A^{\mathbf{G}}}^N$ , of genus  $g \geq 0$  and degree  $d$ , with a parametrized  $\mathbb{P}^1$  in the domain.

Let  $C_0$  be the parametrized component of  $C$ , and let  $y_1, \dots, y_m \in C_0$  be the points where  $C_0$  meets the rest of the curve. Since  $\varphi$  contracts all other irreducible components of  $C$ ,

$$\overline{C \setminus C_0} = C_1 \amalg \cdots \amalg C_m$$

is the union of  $m$  connected components. For  $i = 0, 1, \dots, m$ , let  $d_i$  be the degree of the induced quasimap with domain curve  $C_i$ . Since the class of any quasimap is  $L_\theta$ -effective, the divisor

$$D := \sum_{i=1}^m d_i y_i$$

on  $C_0$  is effective. Consider the degree  $d$  line bundle

$$\tilde{\mathcal{L}} := (\mathcal{L}|_{C_0})(D)$$

on  $C_0$ . Composing the sections  $s_j|_{C_0}$  with the injection

$$0 \longrightarrow \mathcal{L}|_{C_0} \longrightarrow (\mathcal{L}|_{C_0})(D)$$

gives  $N + 1$  sections  $\tilde{s}_0, \dots, \tilde{s}_N$  of  $\tilde{\mathcal{L}}$ . Note that all sections  $\tilde{s}_j$  vanish at each  $y_i$  and that the cokernel of

$$\mathcal{O}_{C_0}^{\oplus(N+1)} \xrightarrow{(\tilde{s}_0, \dots, \tilde{s}_N)} \tilde{\mathcal{L}}$$

has length  $d_i$  at  $y_i$ . The data

$$(C_0, \tilde{\mathcal{L}}, \tilde{s}_0, \dots, \tilde{s}_N)$$

represents a point in  $QG_{0,0,d}(\mathbb{P}_{A\mathbb{G}}^N)$ .

Informally, each connected component  $C_i$ ,  $i = 1, \dots, N$ , is replaced by a base point at its attaching point on  $C_0$ , of length equal to the degree of the quasimap on  $C_i$ ; all remaining markings (if any) on  $C_0$  are forgotten.

A proof that the above construction can be performed in families in a functorial way can be found in [LLY01]. Strictly speaking, the proof is given there only for families of stable maps to  $\mathbb{P}^N$ , but one easily checks that the argument applies equally well for  $\varepsilon$ -stable quasimaps. We conclude that for each  $0+ \leq \varepsilon \leq \infty$ , there is a morphism of stacks

$$\Phi_\varepsilon : QG_{g,k,d}^\varepsilon(\mathbb{P}_{A\mathbb{G}}^N) \longrightarrow QG_{0,0,d}(\mathbb{P}_{A\mathbb{G}}^N). \quad (3.2.1)$$

These morphisms are clearly equivariant for the  $\mathbb{C}^*$ -action on graph spaces.

**3.2.2 Contractions of rational tails.** The second kind of contractions are morphisms

$$c : \overline{M}_{g,k}(\mathbb{P}_{A\mathbb{G}}^N, d) \longrightarrow Q_{g,k}(\mathbb{P}_{A\mathbb{G}}^N, d) \quad (3.2.2)$$

$$c : G_{g,k,d}(\mathbb{P}_{A\mathbb{G}}^N) \longrightarrow QG_{g,k,d}(\mathbb{P}_{A\mathbb{G}}^N) \quad (3.2.3)$$

from stable maps moduli to stable quasimaps moduli which replace the “rational tails” of a stable map with base points, keeping track of their degrees. They already appeared in [MM07] and are also described point-wise in [MOP11] for the case without a parametrized component in the domain curve. For example, we spell out the case with a parametrized  $\mathbb{P}^1$ .

Let

$$((C, x_1, \dots, x_k), \mathcal{L}, (s_i)_{i=0, \dots, N}, \varphi : C \longrightarrow \mathbb{P}^1)$$

be a stable map with a parametrized distinguished component. Let  $T_1, \dots, T_l$  be the maximal connected trees of rational curves in the domain curve  $C$  satisfying the following properties:

- (i) Each tree  $T_i$  contains no markings and the distinguished component  $C_0$  is not a component in  $T_i$ .
- (ii) Each tree  $T_i$  meets the rest of the curve  $C$  at a single point  $z_i$ .



Let  $d_i$  be the degree of the restriction to  $T_i$  of the stable map. Denote by  $\tilde{C}$  the closure of  $C \setminus (\cup_{i=1}^l T_i)$ ; by assumption, it contains the distinguished  $C_0$  as a component. We then have the line bundle

$$\tilde{\mathcal{L}} := \mathcal{L}|_{\tilde{C}} \otimes \mathcal{O}_{\tilde{C}} \left( \sum_{i=1}^l d_i z_i \right)$$

on  $\tilde{C}$ , with sections  $\tilde{s}_0, \dots, \tilde{s}_N$  obtained as the compositions

$$\tilde{s}_j : \mathcal{O}_{\tilde{C}} \xrightarrow{s_j} \mathcal{L}|_{\tilde{C}} \longrightarrow \mathcal{L}|_{\tilde{C}} \otimes \mathcal{O}_{\tilde{C}} \left( \sum_{i=1}^l d_i z_i \right).$$

The image of the stable map under the contraction morphism (3.2.3) is the stable quasimap

$$((\tilde{C}, x_1, \dots, x_k), \tilde{\mathcal{L}}, (\tilde{s}_i)_{i=0, \dots, N}, \varphi|_{\tilde{C}}).$$

Again, it is not difficult to see that the construction can be done functorially in families.

We have described the morphisms between the moduli spaces corresponding to the two extremal chambers for the stability parameter  $\varepsilon$ . In fact, these can be easily seen to factor into a composition of contractions

$$c_{\varepsilon_1, \varepsilon_2} : \mathbb{Q}_{g,k}^{\varepsilon_1}(\mathbb{P}_{A\mathbf{G}}^N, d) \longrightarrow \mathbb{Q}_{g,k}^{\varepsilon_2}(\mathbb{P}_{A\mathbf{G}}^N, d)$$

(and similar ones for graph spaces) for  $\varepsilon_1 > \varepsilon_2$  in adjacent chambers; see [MM07], [Tod11]. We will not really use these morphisms in the present paper, but they are important for comparing higher genus  $\varepsilon$ -stable quasimap invariants as  $\varepsilon$  varies; see § 6.3 and [MOP11], [Tod11], [CK13].

**3.2.3 Replacing markings by base points.** Another variant of the constructions above will be useful to us. Let

$$((C, x_1, \dots, x_k, y_1, \dots, y_m), \mathcal{L}, (s_i)_{i=0, \dots, N})$$

be a  $(k+m)$ -pointed  $\varepsilon$ -stable quasimap to  $\mathbb{P}_{A\mathbf{G}}^N$ , of genus  $g \geq 0$  and degree  $d$ . Further, fix positive integers  $d_1, \dots, d_m$ . The data

$$\left( (C, x_1, \dots, x_k), \mathcal{L} \otimes \mathcal{O}_C \left( \sum_{j=1}^m d_j y_j \right), (\tilde{s}_i)_{i=0, \dots, N} \right) \tag{3.2.4}$$

is a  $k$ -pointed prestable quasimap to  $\mathbb{P}_{A\mathbf{G}}^N$ , where the sections  $\tilde{s}_i$  are again obtained by composing  $s_i$  with the injection

$$\mathcal{L} \longrightarrow \mathcal{L} \otimes \mathcal{O}_C \left( \sum_{j=1}^m d_j y_j \right).$$

For each  $j$  the quasimap has a base point of length  $d_j$  at  $y_j$ . It is possible that some of the rational components of  $C$  are now rational tails, that is, they contain a single node and none of the markings  $x_i$ . Each such rational tail is part of a unique maximal tree as in the previous subsection. Let  $T_1, \dots, T_l$  be the set of all such trees and let  $e_1, \dots, e_l$  be their total degrees (that is,  $e_j$  is the degree of the restriction to the subcurve  $T_j$  of the quasimap (3.2.4)). We now contract these trees to base points of length  $e_j$ , just as in § 3.2.2, to obtain a  $k$ -pointed stable (in the sense of  $\varepsilon = 0+$ -stability) quasimap to  $\mathbb{P}_{A\mathbf{G}}^N$ . The upshot is that we obtain, for each  $0+ \leq \varepsilon \leq \infty$ , a morphism

$$b_{\varepsilon, \{d_j\}} : \mathbb{Q}_{g, k+m}^{\varepsilon}(\mathbb{P}_{A\mathbf{G}}^N, d) \longrightarrow \mathbb{Q}_{g, k}(\mathbb{P}_{A\mathbf{G}}^N, d + \sum d_j) \tag{3.2.5}$$

and similarly for graph spaces,

$$b_{\varepsilon, \{d_j\}} : QG_{g, k+m, d}^{\varepsilon}(\mathbb{P}_{A^{\mathbf{G}}}^N) \longrightarrow QG_{g, k, d+\sum d_j}(\mathbb{P}_{A^{\mathbf{G}}}^N). \quad (3.2.6)$$

### 3.3 Universal line bundles

For each  $0+ \leq \varepsilon \leq \infty$ , consider the composed  $\mathbb{C}^*$ -equivariant morphisms

$$\Phi_{\varepsilon} \circ \iota_{\varepsilon} : QG_{g, k, \beta}^{\varepsilon}(W//\mathbf{G}) \longrightarrow QG_{0, 0, d(\beta)}(\mathbb{P}_{A^{\mathbf{G}}}^N).$$

Recall that we have the identification

$$QG_{0, 0, d(\beta)}(\mathbb{P}_{A^{\mathbf{G}}}^N) = \mathbb{P}(V_{d(\beta)}^{\oplus(N+1)}) \times \text{Spec}(A^{\mathbf{G}}),$$

where  $V_{d(\beta)} \cong H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(d(\beta)))$  is the vector space of homogeneous polynomials of degree  $d(\beta)$  in two variables  $\zeta_0, \zeta_1$ . Furthermore, under this identification, the action of  $\mathbb{C}^*$  on  $V_{d(\beta)}$  is given by multiplication on the first variable  $\zeta_0$  only. It has the canonical lifting to an action on the (relative) tautological line bundle  $\mathcal{O}(1)$  on

$$\mathbb{P}(V_{d(\beta)}^{\oplus(N+1)}) \times \text{Spec}(A^{\mathbf{G}}).$$

In this paper, we always consider this canonical  $\mathbb{C}^*$ -linearization of  $\mathcal{O}(1)$ .

Following Givental, we define the *universal  $\mathbb{C}^*$ -equivariant line bundle*  $U(L_{\theta})$  on the graph space  $QG_{g, k, \beta}^{\varepsilon}(W//\mathbf{G})$  by

$$U(L_{\theta}) := (\Phi_{\varepsilon} \circ \iota_{\varepsilon})^* \mathcal{O}(1). \quad (3.3.1)$$

Similarly, given effective classes  $\beta_1, \dots, \beta_m$ , we have a universal line bundle on the graph space  $QG_{g, k+m, \beta}^{\varepsilon}(W//\mathbf{G})$ , still denoted by  $U(L_{\theta})$  and defined as the pull-back of  $\mathcal{O}(1)$  via

$$\Phi_{\varepsilon} \circ b_{\varepsilon, \{d(\beta_j)\}} \circ \iota_{\varepsilon} : QG_{g, k+m, \beta}^{\varepsilon}(W//\mathbf{G}) \longrightarrow QG_{0, 0, d(\beta)+\sum d(\beta_j)}(\mathbb{P}_{A^{\mathbf{G}}}^N).$$

As ordinary line bundles, these  $U(L_{\theta})$  on graph spaces can in fact be defined without the use of the contraction morphisms, via the universal principal  $\mathbf{G}$ -bundle and the representation  $\mathbb{C}_{\theta}$ ; however, it will be important in our arguments that they have the *canonical*  $\mathbb{C}^*$ -linearization induced from the linearization of  $\mathcal{O}(1)$  described above.

## 4. $\mathbb{C}^*$ -localization on graph spaces

One of the most important features of the graph spaces is that the natural  $\mathbb{C}^*$ -action on them has fixed-point loci described via moduli spaces of *unparametrized* quasimaps, with easy to compute virtual normal bundles. Via the virtual localization theorem of [GP99], this description leads to useful factorization properties of generating functions for the evaluations of certain cohomology classes on graph spaces against the virtual class. We recall in this section the description of the fixed point loci and of the equivariant Euler classes of their virtual normal bundles.

### 4.1 Fixed loci of $\mathbb{C}^*$ -action on graph spaces

Fix  $g, k \geq 0$  and an  $L_{\theta}$ -effective class  $\beta$ . For  $\varepsilon \in [0+, \infty]$ , we will consider moduli spaces

$$Q_{g, k+\bullet}^{\varepsilon}(W//\mathbf{G}, \beta)$$

and think of  $\bullet$  as a distinguished (last) marking. Note that for  $\beta = 0$  and  $2g + k \geq 2$ , we have

$$Q_{g, k+\bullet}^{\varepsilon}(W//\mathbf{G}, \beta) \cong \overline{M}_{g, k+\bullet} \times W//\mathbf{G},$$

and all  $k + 1$  evaluation maps at markings coincide with the projection onto the second factor.

If an  $\varepsilon$ -stable parametrized quasimap

$$((C, x_1, \dots, x_k), P, u, \varphi)$$

is  $\mathbb{C}^*$ -fixed, then all markings, all nodes, all base points, and the entire degree  $\beta$  must be supported over the fixed points  $0$  and  $\infty$  on the parametrized component  $C_0$ . We have

$$(QG_{g,k,\beta}^\varepsilon(W//\mathbf{G}))^{\mathbb{C}^*} = \coprod_{g_2, k_2, \beta_2} F_{g_2, k_2, \beta_2}^{g_1, k_1, \beta_1},$$

the union over all possible splittings

$$g = g_1 + g_2, \quad k = k_1 + k_2, \quad \beta = \beta_1 + \beta_2$$

with  $g_i, k_i \geq 0$  and  $\beta_i$  effective.

The component  $F_{g_2, k_2, \beta_2}^{g_1, k_1, \beta_1}$  is identified with the fiber product

$$Q_{g_1, k_1 + \bullet}^\varepsilon(W//\mathbf{G}, \beta_1) \times_{W//\mathbf{G}} Q_{g_2, k_2 + \bullet}^\varepsilon(W//\mathbf{G}, \beta_2) \tag{4.1.1}$$

over the evaluation maps  $\text{ev}_\bullet$  at the special markings  $\bullet$ . Under this identification, the inclusion

$$i : F_{g_2, k_2, \beta_2}^{g_1, k_1, \beta_1} \longrightarrow QG_{g,k,\beta}^\varepsilon(W//\mathbf{G})$$

is obtained as follows: given two  $\varepsilon$ -stable quasimaps to  $W//\mathbf{G}$  sending the special markings  $\bullet$  to the same point  $p$ , we glue them at  $0$  and  $\infty$  to the constant map  $\mathbb{P}^1 \rightarrow p \in W//\mathbf{G}$ .

The unstable cases  $(g_2, k_2, \beta_2) = (0, 1, 0)$  and  $(g_2, k_2, \beta_2) = (0, 0, 0)$  (and likewise for the triples  $(g_1, k_1, \beta_1)$ ) are included above by the following conventions:

$$Q_{0,0+\bullet}^\varepsilon(W//\mathbf{G}, 0) := W//\mathbf{G}, \quad Q_{0,1+\bullet}^\varepsilon(W//\mathbf{G}, 0) := W//\mathbf{G}, \quad \text{ev}_\bullet := \text{id}_{W//\mathbf{G}}.$$

This is because

$$QG_{0,1,0}(W//\mathbf{G}) \cong W//\mathbf{G} \times \mathbb{P}^1$$

with fixed point loci  $W//\mathbf{G} \times \{0\}$  and  $W//\mathbf{G} \times \{\infty\}$ , while

$$QG_{0,0,0}(W//\mathbf{G}) \cong W//\mathbf{G}$$

with trivial  $\mathbb{C}^*$ -action.

For stable maps ( $\varepsilon = \infty$ ), the description above covers all possibilities. However, for quasimaps there are more unstable cases which need to be treated separately, namely  $(g_2, k_2, \beta_2) = (0, 0, \beta_2)$  with  $\beta_2 \neq 0$  and  $\varepsilon \leq 1/\beta_2(L_\theta)$  (and the symmetric ones  $(0, 0, \beta_1 \neq 0)$ ). This is because in genus zero moduli of stable quasimaps in the chamber  $\varepsilon = 0+$  require at least two markings. For  $\beta \neq 0$ , we denote by

$$Q_{0,0+\bullet}(W//\mathbf{G}, \beta)_\infty$$

the moduli space parametrizing the quasimaps of class  $\beta$

$$(\mathbb{P}^1, P, u)$$

with  $P$  a principal  $\mathbf{G}$ -bundle on  $\mathbb{P}^1$  and  $u : \mathbb{P}^1 \rightarrow P \times_{\mathbf{G}} W$  a section such that  $u(x) \in W^s$  for  $x \neq \infty \in \mathbb{P}^1$  and  $\infty \in \mathbb{P}^1$  is a base point of length  $\beta(L_\theta)$ . Similarly, we have the moduli space

$$Q_{0,0+\bullet}(W//\mathbf{G}, \beta)_0$$

for which  $u$  now has a base point of length  $\beta(L_\theta)$  at  $0 \in \mathbb{P}^1$  and no other base points. (Note that the two moduli spaces are isomorphic, but they sit in  $QG_{0,0,\beta}(W//\mathbf{G})$  as different components of the  $\mathbb{C}^*$ -fixed locus, with different normal bundles.)

In either case, the regular map

$$u_{reg} : \mathbb{P}^1 \longrightarrow W//\mathbf{G}$$

induced by the quasimap is a constant map. We define the evaluation map at the special marking by setting

$$\text{ev}_\bullet((\mathbb{P}^1, P, u)) = u_{reg}(\mathbb{P}^1).$$

With these definitions, we have

$$F_{0,0,\beta_2}^{g,k,\beta_1} \cong \mathbb{Q}_{g,k+\bullet}^\varepsilon(W//\mathbf{G}, \beta_1) \times_{W//\mathbf{G}} \mathbb{Q}_{0,0+\bullet}(W//\mathbf{G}, \beta_2)_\infty$$

for  $k \geq 1$  and  $\varepsilon \leq 1/\beta_2(L_\theta)$  and similarly

$$F_{g,k,\beta_2}^{0,0,\beta_1} \cong \mathbb{Q}_{0,0+\bullet}(W//\mathbf{G}, \beta_1)_0 \times_{W//\mathbf{G}} \mathbb{Q}_{g,k+\bullet}^\varepsilon(W//\mathbf{G}, \beta_2)$$

for  $k \geq 1$  and  $\varepsilon \leq 1/\beta_1(L_\theta)$ . When  $g = k = 0$  and

$$\varepsilon \leq \min \left\{ \frac{1}{\beta_1(L_\theta)}, \frac{1}{\beta_2(L_\theta)} \right\},$$

we have

$$F_{0,0,\beta_2}^{0,0,\beta_1} \cong \mathbb{Q}_{0,0+\bullet}(W//\mathbf{G}, \beta_1)_0 \times_{W//\mathbf{G}} \mathbb{Q}_{0,0+\bullet}(W//\mathbf{G}, \beta_2)_\infty.$$

## 4.2 Euler classes of virtual normal bundles

For the rest of the section we restrict ourselves to genus zero. We drop the genus from the notation and write  $F_{k_2,\beta_2}^{k_1,\beta_1}$  for the components of the fixed point locus in  $QG_{0,k,\beta}^\varepsilon(W//\mathbf{G})$ . The Euler classes of their virtual normal bundles are essentially the products of contributions from  $(k_1, \beta_1)$  and  $(k_2, \beta_2)$ . The notation  $H^*(X)$  will mean cohomology with  $\mathbb{Q}$ -coefficients throughout the rest of the paper.

We begin with the loci  $F_0 := F_{0,0}^{k,\beta}$ , where everything is concentrated over  $0 \in \mathbb{P}^1$ . In this case only  $(k, \beta)$  contributes to the product. As explained in the previous subsection, there are two cases to consider.

Assume first that either  $k \geq 1$ , or if  $k = 0$ , that  $\varepsilon > 1/\beta(L_\theta)$  (note that this is *always* true for stable maps). Hence  $F_0 \cong \mathbb{Q}_{0,k+\bullet}^\varepsilon(W//\mathbf{G}, \beta)$ . If  $N_{F_0}$  is the *virtual* normal bundle, given by the moving part of the (absolute) obstruction theory, then its  $\mathbb{C}^*$ -equivariant Euler class is

$$e_{\mathbb{C}^*}(N_{F_0}) = \text{cont}_{(k,\beta)}(z) := \begin{cases} 1 & \text{if } (k, \beta) = (0, 0), \\ z & \text{if } (k, \beta) = (1, 0), \\ z(z - \psi_\bullet) & \text{otherwise,} \end{cases} \quad (4.2.1)$$

where  $z$  denotes the generator of  $A_{\mathbb{C}^*}^*(\text{Spec}(\mathbb{C}))$ . The formula (4.2.1) is well known; see for example [Giv98a] for stable maps and [CK10], [CKM14] for quasimaps. Note that it is the same for all targets  $W//\mathbf{G}$ .

In addition, we observe that in these cases the  $\mathbb{C}^*$ -fixed part of the obstruction theory for the graph space gives precisely the obstruction theory of  $\mathbb{Q}_{0,k+\bullet}^\varepsilon(W//\mathbf{G}, \beta)$ .

On the other hand, if  $k = 0$  and  $\varepsilon \leq 1/\beta(L_\theta)$ , then

$$F_0 \cong \mathbb{Q}_{0,0+\bullet}(W//\mathbf{G}, \beta)_0.$$

In this case.

$$e_{\mathbb{C}^*}(N_{F_0}) = \text{cont}_{(0,\beta)}(z) \in H^*(F_0, \mathbb{Q}) \otimes \mathbb{Q}(z) \quad (4.2.2)$$

will depend on the target and needs to be calculated separately. In fact, as we will see in the next section, what one is interested in calculating is the push-forward to  $H^*(W//\mathbf{G}, \mathbb{Q}) \otimes \mathbb{Q}(z)$  of  $[F_0]^{\text{vir}}/\text{cont}_{(0,\beta)}(z)$  via the evaluation map  $\text{ev}_\bullet$  (these will essentially be the coefficients of the so-called small  $I$ -function of  $W//\mathbf{G}$ ). For many of the examples of targets listed in § 2.8, closed formulas for these push-forwards are known; see Remark 5.5.5 below.

For a general component  $F_{k_2, \beta_2}^{k_1, \beta_1}$ , we have

$$e_{\mathbb{C}^*}(N_F) = \text{cont}_{(k_1, \beta_1)}(z) \boxtimes \text{cont}_{(k_2, \beta_2)}(-z) \tag{4.2.3}$$

with each factor as in (4.2.1) or (4.2.2).

### 5. Generating functions for genus zero quasimap invariants

In this section we require that quasimap invariants be well defined. Hence we will assume either that  $W//\mathbf{G}$  is projective, or that  $W//\mathbf{G}$  has an additional action by a torus  $\mathbf{T}$ , with proper fixed loci in the quasimap moduli spaces, as in [CKM14, § 6.3].

#### 5.1 The $J^\varepsilon$ and $I$ -functions

For varying  $k$  and  $\beta$ , we assemble into generating functions the residues of the virtual classes of  $QG_{0,k,\beta}^\varepsilon(W//\mathbf{G})$  at the fixed loci  $F_0 = F_{0,0}^{k,\beta}$ , paired against cohomology classes pulled back from  $W//\mathbf{G}$  via the evaluation maps at the  $k$  markings, then push-forward to  $H_*(W//\mathbf{G})$  using the evaluation  $\text{ev}_\bullet$  at the special markings.

We introduce the *Novikov ring*

$$\Lambda = \Lambda((W, \mathbf{G}, \theta)) := \left\{ \sum_{\beta \in \text{Eff}(W, \mathbf{G}, \theta)} a_\beta q^\beta \mid a_\beta \in \mathbb{Q} \right\},$$

the  $q$ -adic completion of the semigroup ring  $\mathbb{Q}[\text{Eff}(W, \mathbf{G}, \theta)]$ . In the quasi-projective case we extend the coefficients to the field

$$K := \mathbb{Q}(\{\lambda_j\}) = H_{\mathbf{T}, \text{loc}}^*(\text{Spec}(\mathbb{C}), \mathbb{Q}),$$

the *localized*  $\mathbf{T}$ -equivariant cohomology of a point. Sometimes we will also use the notation  $\mathbb{Q}[[q]]$  (or  $K[[q]]$ ) for the Novikov ring.

The generating functions will be formal functions of  $\mathbf{t} \in H^*(W//\mathbf{G})$ . (In the case of quasi-projective targets we take  $\mathbf{t}$  in the  $\mathbf{T}$ -equivariant cohomology of  $W//\mathbf{G}$ . To simplify the notation, in this section we will only write  $H^*(W//\mathbf{G})$  in either instance; the torus equivariant case is discussed in more detail in Section 7.)

DEFINITION 5.1.1. Fix  $\varepsilon \in \mathbb{Q}_{>0}$ . The (big)  $J^\varepsilon$ -function of  $W//\mathbf{G}$  is

$$\begin{aligned} J^\varepsilon(\mathbf{t}, z) &= \sum_{k \geq 0, \beta \geq 0} q^\beta (\text{ev}_\bullet)_* \left( \frac{\prod_{i=1}^k \text{ev}_i^*(\mathbf{t})}{k!} \cap \text{Res}_{F_0} [QG_{0,k,\beta}^\varepsilon(W//\mathbf{G})]^{\text{vir}} \right) \\ &= \mathbb{1} + \frac{\mathbf{t}}{z} + \sum_{k=0, \beta \neq 0, \beta(L_\theta) \leq 1/\varepsilon} q^\beta (\text{ev}_\bullet)_* \frac{[\mathbb{Q}_{0,0+\bullet}(W//\mathbf{G}, \beta)_0]^{\text{vir}}}{e_{\mathbb{C}^*}(N_{F_0})} \\ &\quad + \sum_{\substack{k \geq 1 \text{ or } \beta(L_\theta) > 1/\varepsilon, \\ (k, \beta) \neq (1, 0)}} q^\beta (\text{ev}_\bullet)_* \frac{\prod_{i=1}^k \text{ev}_i^*(\mathbf{t}) \cap [\mathbb{Q}_{0,k+\bullet}^\varepsilon(W//\mathbf{G}, \beta)]^{\text{vir}}}{z(z - \psi_\bullet)}. \end{aligned} \tag{5.1.1}$$

Here  $\mathbb{1} = [W//\mathbf{G}] \in H^0(W//\mathbf{G})$  is the unit in cohomology. The properness of the quasimap moduli spaces over the affine quotient is easily seen to imply that the evaluation maps  $\text{ev}_\bullet$  are proper, hence the push-forwards are well defined for all targets, projective or not.

For any  $\infty \geq \varepsilon > 1$ , the definition above gives the usual big  $J$ -function in Gromov–Witten theory:

$$\begin{aligned} J(\mathbf{t}, z) &= J^\infty(\mathbf{t}, z) \\ &= \mathbb{1} + \frac{\mathbf{t}}{z} + \sum_{(k,\beta) \neq (0,0), (1,0)} q^\beta (\text{ev}_\bullet)_* \left( \frac{\prod_{i=1}^k \text{ev}_i^*(\mathbf{t})}{k!} \cap \frac{[\overline{M}_{0,k+\bullet}(W//\mathbf{G}, \beta)]^{\text{vir}}}{z(z - \psi_\bullet)} \right). \end{aligned} \quad (5.1.2)$$

On the other hand, we may also consider the  $(\varepsilon = 0+)$ -stability condition to obtain a similar generating function, called *the big  $I$ -function of  $W//\mathbf{G}$*  and introduced in [CK10],[CKM14]:

$$\begin{aligned} I(\mathbf{t}, z) &= J^{0+}(\mathbf{t}, z) \\ &= \mathbb{1} + \frac{\mathbf{t}}{z} + \sum_{k=0, \beta \neq 0} q^\beta (\text{ev}_\bullet)_* \frac{[\mathbb{Q}_{0,0+\bullet}(W//\mathbf{G}, \beta)_0]^{\text{vir}}}{e_{\mathbb{C}^*}(N_{F_0})} \\ &\quad + \sum_{k \geq 1, (k,\beta) \neq (1,0)} q^\beta (\text{ev}_\bullet)_* \left( \frac{\prod_{i=1}^k \text{ev}_i^*(\mathbf{t})}{k!} \cap \frac{[\mathbb{Q}_{0,k+\bullet}(W//\mathbf{G}, \beta)]^{\text{vir}}}{z(z - \psi_\bullet)} \right). \end{aligned} \quad (5.1.3)$$

For each  $\varepsilon \geq 0+$ , the function  $J^\varepsilon$  takes values in

$$H^*(W//\mathbf{G}, \Lambda) \{ \{z, z^{-1}\} \},$$

where we have used Poincaré duality to pass to cohomology. (For quasi-projective targets, the coefficient ring is replaced with the localized  $\mathbf{T}$ -equivariant cohomology  $H_{\mathbf{T}, \text{loc}}^*(W//\mathbf{G}, \Lambda)$ .) By definition,  $H^*(W//\mathbf{G}, \Lambda) \{ \{z, z^{-1}\} \}$  is a certain completion of the ring of formal Laurent series in  $1/z$ . To be more precise, it is

$$\left\{ \sum_{j=-\infty}^{\infty} h_j z^j \mid h_j \rightarrow 0 \text{ as } j \rightarrow \infty \text{ in the } q\text{-adic topology on } \Lambda \right\}. \quad (5.1.4)$$

In fact, this completion is needed only for the  $I$ -function  $I = J^{0+}$ , as positive powers of  $z$  can appear only in the terms with  $k = 0, \beta \neq 0$ , and  $\beta(L_\theta) \leq 1/\varepsilon$ .

The first two asymptotic properties below are easily obtained from the definitions; the third follows from a dimension counting argument (in fact, we will see a more general statement in § 5.5 below).

- (i) We have  $J^\varepsilon = e^{\mathbf{t}/z} + O(q)$  for all  $\varepsilon \geq 0+$ .
- (ii)  $J^\infty = \mathbb{1} + \mathbf{t}/z + O(1/z^2)$ .
- (iii) Assume that  $W//\mathbf{G}$  is Fano of index at least two, that is, that for every  $\beta \neq 0$ , we have  $\beta(\det T_W) \geq 2$ . Then  $J^\varepsilon = \mathbb{1} + \mathbf{t}/z + O(1/z^2)$  for all  $\varepsilon \geq 0+$ .

In the rest of this section we derive other properties of these generating functions from the geometry of the  $\mathbb{C}^*$ -action on graph spaces.

## 5.2 Polynomiality

The following construction in Gromov–Witten theory is due to Givental; we just observe here that it applies to quasimap theory as well, for all  $\varepsilon \geq 0+$ .

Consider the universal line bundle  $U(L_\theta)$  on  $QG_{0,k,\beta}^\varepsilon(W//\mathbf{G})$  defined in (3.3.1). Using the identification of each  $F_{k_2,\beta_2}^{k_1,\beta_1}$  with a fibered product, we immediately see that

$$U(L_\theta)|_{F_{k_2,\beta_2}^{k_1,\beta_1}} = \text{ev}_\bullet^*(\mathcal{O}(\theta)) \boxtimes \mathbb{C}_{\beta_2(\theta)}. \tag{5.2.1}$$

Define a generating function for primary quasimap integrals on graph spaces by

$$D^\varepsilon := \sum_{k,\beta \geq 0} q^\beta \int_{[QG_{0,k,\beta}^\varepsilon(W//\mathbf{G})]^{\text{vir}}} \frac{e^{c_1^{\mathbb{C}^*}(U(L_\theta))y} \prod_{i=1}^k \text{ev}_i^*(\mathbf{t})}{k!}. \tag{5.2.2}$$

Here  $y$  is a formal variable,  $c_1^{\mathbb{C}^*}$  is the equivariant first Chern class, and the symbol  $\int$  denotes the  $\mathbb{C}^*$ -equivariant push-forward to a point.

Fix once and for all a homogeneous basis  $\{\gamma_i\}_i$  of  $H^*(W//\mathbf{G})$  and denote by  $\{\gamma^i\}_i$  the dual basis with respect to the intersection pairing on  $W//\mathbf{G}$ ,

$$\int_{W//\mathbf{G}} \gamma_i \gamma^j = \delta_i^j.$$

(As usual, in the quasi-projective  $\mathbf{T}$ -equivariant case the integral is understood via the localization theorem as an integral over the  $\mathbf{T}$ -fixed locus in  $W//\mathbf{G}$ , which is proper.)

If we write  $\mathbf{t} = \sum t_i \gamma_i$ , then  $D^\varepsilon$  is an element in the ring of formal power series  $\Lambda[[t_i, y, z]]$ . Moreover, for each  $\beta$  the coefficient of  $q^\beta$  is a polynomial in  $z$ .

We now apply the virtual localization theorem [GP99] to  $D^\varepsilon$ . Using the product expression (4.2.3) for the virtual normal bundles and the restriction formula (5.2.1), we obtain the following factorization.

**PROPOSITION 5.2.1.** *Set  $(qe^{-zyL_\theta})^\beta := q^\beta e^{-zy\beta(L_\theta)}$ . Then*

$$D^\varepsilon = \int_{W//\mathbf{G}} e^{c_1(\mathcal{O}(\theta))y} J^\varepsilon(q, \mathbf{t}, z) J^\varepsilon(qe^{-zyL_\theta}, \mathbf{t}, -z).$$

*In particular, for each  $\beta$ , the coefficient of  $q^\beta$  on the right-hand side is a polynomial in  $z$ .*

### 5.3 The $S$ -operator

We extend the customary Gromov–Witten theory bracket notation to  $\varepsilon$ -quasimap invariants: for nonnegative integers  $a_j$  and cohomology classes  $\sigma_j \in H^*(W//\mathbf{G})$ , put

$$\langle \sigma_1 \psi_1^{a_1}, \dots, \sigma_k \psi_k^{a_k} \rangle_{g,k,\beta}^\varepsilon := \int_{[Q_{g,k}^\varepsilon(W//\mathbf{G},\beta)]^{\text{vir}}} \prod_j \text{ev}_j^*(\sigma_j) \psi_j^{a_j}.$$

Extending by linearity, the cohomological insertions  $\sigma_j$  may be taken from  $H^*(W//\mathbf{G}, \Lambda)\{\{z, z^{-1}\}\}$ .

Similarly, we can consider invariants on the graph spaces

$$\langle \sigma_1, \dots, \sigma_k \rangle_{k,\beta}^{QG^\varepsilon} := \int_{[QG_{0,k,\beta}^\varepsilon(W//\mathbf{G})]^{\text{vir}}} \prod_j \tilde{\text{ev}}_j^*(\sigma_j).$$

This time the evaluation maps  $\tilde{\text{ev}}_j$  take values in  $W//\mathbf{G} \times \mathbb{P}^1$ , and the cohomology classes

$$\sigma = \gamma \otimes p \in H^*(W//\mathbf{G}) \otimes H_{\mathbb{C}^*}^*(\mathbb{P}^1) \cong H_{\mathbb{C}^*}^*(W//\mathbf{G} \times \mathbb{P}^1)$$

are pulled back from there. Again, by linearity, we may use insertions with  $\gamma \in H^*(W//\mathbf{G}, \Lambda)$ . As in the previous subsection, the virtual class is the  $\mathbb{C}^*$ -equivariant one and the integral is the equivariant push-forward to a point. We will write  $\gamma p$  for  $\gamma \otimes p$  and simply  $\gamma$  for  $\gamma \otimes 1$ .

Further, we use double brackets to denote generating series of invariants:

$$\begin{aligned} \langle\langle \sigma_1 \psi_1^{a_1}, \dots, \sigma_k \psi_k^{a_k} \rangle\rangle_{0,k}^\varepsilon &:= \sum_{m \geq 0, \beta \geq 0} \frac{q^\beta}{m!} \langle \sigma_1 \psi_1^{a_1}, \dots, \sigma_k \psi_k^{a_k}, \mathbf{t}, \dots, \mathbf{t} \rangle_{0,k+m,\beta}^\varepsilon \\ \langle\langle \sigma_1, \dots, \sigma_k \rangle\rangle_k^{QG^\varepsilon} &:= \sum_{m \geq 0, \beta \geq 0} \frac{q^\beta}{m!} \langle \sigma_1, \dots, \sigma_k, \mathbf{t}, \dots, \mathbf{t} \rangle_{k+m,\beta}^{QG^\varepsilon}, \end{aligned}$$

where, as before,  $\mathbf{t} = \sum_i t_i \gamma_i \in \mathbf{H}^*(W//\mathbf{G})$ .

Note that all terms on the right-hand side of the second formula are well defined, even when  $k = 0$ , while in the first formula we must restrict the summation to the *stable* cases (or introduce by hand terms that correspond to triples  $(k + m, \beta, \varepsilon)$  which are unstable). For example, we can write the  $J$ -functions (using the projection formula) as

$$J^\varepsilon = \sum_i \gamma_i \left\langle\left\langle \frac{\gamma^i}{z(z - \psi)} \right\rangle\right\rangle_{0,\bullet}^\varepsilon \tag{5.3.1}$$

with the understanding that the unstable terms are defined as in (5.1.1), by residues on graph spaces:

$$\begin{cases} 1 & \text{if } m = 0, \beta = 0, \\ \mathbf{t}/z & \text{if } m = 1, \beta = 0, \\ \int_{[F_0]^{\text{vir}}} \frac{\text{ev}_\bullet^* \gamma^i}{e_{\mathbf{C}^*}(N_{F_0})} & \text{if } m = 0, \beta \neq 0, \beta(L_\theta) \leq 1/\varepsilon. \end{cases}$$

The following operator on  $\mathbf{H}^*(W//\mathbf{G}, \Lambda)\{\{z, z^{-1}\}\}$ , denoted  $S^\varepsilon(z)$  and defined by

$$\begin{aligned} S^\varepsilon(z)(\gamma) &= \sum_{m \geq 0, \beta \geq 0} \frac{q^\beta}{m!} (\text{ev}_1)_* \left( \frac{[\text{Q}_{0,2+m}^\varepsilon(W//\mathbf{G}, \beta)]^{\text{vir}}}{z - \psi} \text{ev}_2^*(\gamma) \prod_{j=3}^{2+m} \text{ev}_j^*(\mathbf{t}) \right) \\ &= \sum_i \gamma_i \left\langle\left\langle \frac{\gamma^i}{z - \psi}, \gamma \right\rangle\right\rangle_{0,2}^\varepsilon, \end{aligned} \tag{5.3.2}$$

will be very important for the rest of the paper. Here the unstable term in  $\langle\langle \dots \rangle\rangle_{0,2}^\varepsilon$  corresponding to  $m = 0, \beta = 0$  is defined to be  $\int_{W//\mathbf{G}} \gamma^i \gamma$ . In fact, we have a family of linear operators depending on the parameter  $\mathbf{t} \in \mathbf{H}^*(W//\mathbf{G})$ ; we use the notation  $S_{\mathbf{t}}^\varepsilon(z)$  if the  $\mathbf{t}$ -dependence needs to be emphasized.

By the well-known calculation of  $\psi$ -integrals on  $\overline{M}_{0,m+2}$ , we have the asymptotic expansion

$$S^\varepsilon(z)(\gamma) = e^{\mathbf{t}/z} \gamma + O(q).$$

When  $\infty \geq \varepsilon > 1$ , this operator is a familiar object in Gromov–Witten theory: its components

$$S_{ij}^\infty(z) = \left\langle\left\langle \frac{\gamma^i}{z - \psi}, \gamma_j \right\rangle\right\rangle_{0,2}^\infty$$

form a fundamental solution to the quantum differential equation; see [Giv96]. It is well known that  $S^\infty(z)$  is a unitary operator. We prove here that this property holds for all  $\varepsilon \geq 0+$ .

**PROPOSITION 5.3.1.** *Consider the operator*

$$(S^\varepsilon)^*(-z)(\gamma) = \sum_i \gamma^i \left\langle\left\langle \gamma_i, \frac{\gamma}{-z - \psi} \right\rangle\right\rangle_{0,2}^\varepsilon.$$



Then

$$(S^\varepsilon)^*(-z) \circ S^\varepsilon(z) = \text{Id}. \tag{5.3.3}$$

*Proof.* Consider two equivariant cohomology classes  $p_0, p_\infty \in H_{\mathbb{C}^*}^*(\mathbb{P}^1)$ , defined by specifying their restrictions at the fixed points:

$$p_0|_0 = z, p_0|_\infty = 0 \quad \text{and} \quad p_\infty|_0 = 0, p_\infty|_\infty = -z. \tag{5.3.4}$$

By definition, for arbitrary cohomology classes  $\gamma, \delta \in H^*(W//\mathbf{G})$ , the generating series

$$\langle\langle \gamma p_0, \delta p_\infty \rangle\rangle_2^{QG^\varepsilon} = \sum_{m \geq 0, \beta \geq 0} \frac{q^\beta}{m!} \langle \gamma p_0, \mathbf{t}, \dots, \mathbf{t}, \delta p_\infty \rangle_{2+m, \beta}^{QG^\varepsilon}$$

is an element in  $\Lambda[[z]]$ . We now calculate it with the virtual localization theorem. Due to the insertions of  $p_0$  and  $p_\infty$ , only the fixed loci  $F_{k_2, \beta_2}^{k_1, \beta_1}$  for which the first marking lies over  $0 \in \mathbb{P}^1$  and the last marking lies over  $\infty \in \mathbb{P}^1$  have nonvanishing contribution to the integral  $\langle \gamma p_0, \mathbf{t}, \dots, \mathbf{t}, \delta p_\infty \rangle_{2+m, \beta}^{QG^\varepsilon}$ . In view of the description (4.1.1) of the fixed point loci,  $\mathbb{C}^*$ -localization gives

$$\langle\langle \gamma p_0, \delta p_\infty \rangle\rangle_2^{QG^\varepsilon} = \sum_i \left\langle\left\langle \frac{\gamma^i}{z - \psi}, \gamma \right\rangle\right\rangle_{0,2}^\varepsilon \left\langle\left\langle \delta, \frac{\gamma_i}{-z - \psi} \right\rangle\right\rangle_{0,2}^\varepsilon.$$

Now the right-hand side of this equality has the form

$$\int_{W//\mathbf{G}} \gamma \delta + O(1/z),$$

while the left-hand side does not have a pole at  $z = 0$ . We conclude that both sides are *constant* in  $z$  and equal to  $\int_{W//\mathbf{G}} \gamma \delta$ .

The proposition now immediately follows:

$$\begin{aligned} [(S^\varepsilon)^*(-z) \circ S^\varepsilon(z)](\gamma) &= \sum_j \gamma^j \sum_i \left\langle\left\langle \frac{\gamma^i}{z - \psi}, \gamma \right\rangle\right\rangle_{0,2}^\varepsilon \left\langle\left\langle \gamma_j, \frac{\gamma_i}{-z - \psi} \right\rangle\right\rangle_{0,2}^\varepsilon \\ &= \sum_j \gamma^j \left( \int_{W//\mathbf{G}} \gamma \gamma_j \right) = \gamma. \end{aligned} \quad \square$$

*Remark 5.3.2.* The usual proof of the unitary property in Gromov–Witten theory uses the string equation, as well as the splitting property of Gromov–Witten invariants; see [Giv98b]. While quasimap invariants satisfy splitting, the usual form of the string equation fails in general, the reason being that the universal curve over  $\mathcal{Q}_{g,k}^\varepsilon(W//\mathbf{G}, \beta)$  is *not* isomorphic to  $\mathcal{Q}_{g,k+1}^\varepsilon(W//\mathbf{G}, \beta)$ . Nevertheless, the universal curve  $\mathcal{C}_{g,k,\beta}$  is a proper Deligne–Mumford substack in  $\mathfrak{M}_{g,k+1}(W//\mathbf{G}, \beta)$  (corresponding to a different stability condition), and one may write a modified version of the string equation involving intersection numbers on  $\mathcal{C}$ . This modified equation suffices to adapt the Gromov–Witten argument to obtain Proposition 5.3.3. We prefer the proof given here as it is another illustration of the usefulness of  $\mathbb{C}^*$ -localization on graph spaces in the genus zero theories.

### 5.4 The $P$ -series and the Birkhoff factorization of $J^\varepsilon$

Next we consider yet another generating function on graph spaces:

$$P^\varepsilon(\mathbf{t}, z) := \sum_i \gamma^i \langle\langle \gamma_i p_\infty \rangle\rangle_1^{QG^\varepsilon} = \sum_i \gamma^i \sum_{m \geq 0, \beta \geq 0} \frac{q^\beta}{m!} \langle \gamma_i p_\infty, \mathbf{t}, \dots, \mathbf{t} \rangle_{1+m, \beta}^{QG^\varepsilon}. \quad (5.4.1)$$

Being  $\mathbb{C}^*$ -equivariant integrals, the graph space brackets take values in  $\mathbb{Q}[z]$ , hence  $P^\varepsilon(\mathbf{t}, z)$  is an element of  $H^*(W//\mathbf{G}, \Lambda)[[z]]$  for each  $\mathbf{t}$  (convergent in the  $q$ -adic topology in the sense of (5.1.4)).

Again, we apply virtual localization to  $P^\varepsilon$ . The calculation is similar to the one done in the proof of Proposition 5.3.1. In this case, the fixed loci that contribute are those of the form  $F_{1+k_2, \beta_2}^{k_1, \beta_1}$ , for which the first marking lies over  $\infty \in \mathbb{P}^1$ . We have

$$\begin{aligned} P^\varepsilon &= \sum_i \gamma^i \sum_j \left\langle\left\langle \frac{\gamma^j}{z(z-\psi)} \right\rangle\right\rangle_{0,1}^\varepsilon \left\langle\left\langle (-z)\gamma_i, \frac{\gamma_j}{-z(-z-\psi)} \right\rangle\right\rangle_{0,2}^\varepsilon \\ &= \sum_i \gamma^i \left\langle\left\langle \gamma_i, \frac{\sum_j \gamma_j \left\langle\left\langle \frac{\gamma^j}{z(z-\psi)} \right\rangle\right\rangle_{0,1}^\varepsilon}{-z-\psi} \right\rangle\right\rangle_{0,2}^\varepsilon \\ &= \sum_i \gamma^i \left\langle\left\langle \gamma_i, \frac{J^\varepsilon}{-z-\psi} \right\rangle\right\rangle_{0,2}^\varepsilon \\ &= (S^\varepsilon)^*(-z)(J^\varepsilon), \end{aligned}$$

where we used equality (5.3.1), with the convention explained there for the unstable terms in  $\left\langle\left\langle \frac{\gamma^j}{z(z-\psi)} \right\rangle\right\rangle_{0,1}^\varepsilon$ . Note that the relation

$$P^\varepsilon = (S^\varepsilon)^*(-z)(J^\varepsilon) \quad (5.4.2)$$

shows in particular that  $P^\varepsilon$  has the asymptotic form

$$P^\varepsilon = \mathbb{1} + O(q).$$

(This can easily be seen directly by calculating the  $\beta = 0$  integrals. Indeed, for  $\beta = 0$  and  $m \geq 0$ , there is an isomorphism

$$QG_{1+m,0}^\varepsilon(W//\mathbf{G}) \cong W//\mathbf{G} \times \mathbb{P}^1[m+1],$$

where  $\mathbb{P}^1[m+1] \cong \overline{M}_{0,m+1}(\mathbb{P}^1, 1)$  is the Fulton–MacPherson space of stable marked genus zero curves with a rigid component. Hence the  $\beta = 0$  integrals are of the form

$$\left( \int_{W//\mathbf{G}} \gamma_i \mathbf{t}^m \right) \left( \int_{\overline{M}_{0,m+1}(\mathbb{P}^1, 1)} \text{ev}_1^*(p_\infty) \right).$$

The second factor can be written as  $\int_{\mathbb{P}^1} (\text{ev}_1)_*(\mathbb{1}) \cap p_\infty$ , hence it vanishes except when  $m = 0$ , in which case  $\text{ev}_1$  is an isomorphism and the integral is equal to 1. We deduce that the only nonvanishing contribution is  $\sum_i \gamma^i \left( \int_{W//\mathbf{G}} \gamma_i \right) = \mathbb{1}$ .

Using the unitary property in equation (5.4.2), we obtain the following theorem.

**THEOREM 5.4.1.** *For every  $\varepsilon \geq 0+$ ,*

$$J^\varepsilon(z) = S^\varepsilon(z)(P^\varepsilon) = \sum_i S^\varepsilon(z)(\gamma^i) \langle \gamma_i, P^\varepsilon \rangle,$$

where  $\langle -, - \rangle$  is the intersection pairing  $\int_{W//\mathbf{G}}$ .

Observe that while the  $J$ -functions will in general depend on both positive and negative powers of  $z$  when  $\varepsilon \leq 1$ , the operator  $S^\varepsilon$  is a series of  $1/z$  and  $P^\varepsilon$  is a series of  $z$ . Hence we may view Theorem 5.4.1 as a kind of Birkhoff factorization of the  $J^\varepsilon$ -functions.

On the other hand, for  $\varepsilon > 1$ , the usual Gromov–Witten  $J$ -function is only a series of  $1/z$ , with asymptotic expansion  $\mathbb{1} + \mathbf{t}/z + O(1/z^2)$ . This implies that  $P^{\varepsilon > 1} = \mathbb{1}$  and the theorem becomes the familiar result

$$J = S(z)(\mathbb{1}),$$

which immediately follows from the string equation.

### 5.5 The case of semi-positive targets

We assume in this subsection that the triple  $(W, \mathbf{G}, \theta)$  satisfies

$$\beta(\det(T_W)) \geq 0$$

for all  $L_\theta$ -effective classes  $\beta$ . We will call such triples *semi-positive*.

In the projective case, the semi-positive triples give GIT quotients with nef anti-canonical class. Among quasiprojective targets, the main examples we have in mind are holomorphic symplectic quotients, such as Nakajima quiver varieties, and local targets  $\mathcal{E}//\mathbf{G}$  as in Example 2.8.5 with  $0 \leq \beta(\det T_W) + \beta(\det E)$  for all  $L_\theta$ -effective  $\beta$ .

First, we specialize to the case  $\mathbf{t} = 0$ . Since the virtual dimension of  $QG_{0,1,\beta}^\varepsilon(W//\mathbf{G})$  is equal to

$$\dim(W//\mathbf{G}) + 1 + \beta(\det T_W)$$

and the insertion  $\gamma_i p_\infty$  has complex degree at most  $\dim(W//\mathbf{G}) + 1$ , we conclude that

$$\langle \gamma_i, P^\varepsilon|_{\mathbf{t}=0} \rangle = 0$$

unless  $\gamma_i = [\text{pt}]$  is the point class. Furthermore,  $\langle [\text{pt}], P^\varepsilon|_{\mathbf{t}=0} \rangle$  is an element of  $\Lambda = \mathbb{Q}[[q]]$ , of the form

$$\langle [\text{pt}], P^\varepsilon|_{\mathbf{t}=0} \rangle = 1 + \sum_{\{\beta \neq 0, \beta(\det T_W) = 0\}} a_\beta q^\beta. \tag{5.5.1}$$

In particular, it is invertible, and it is equal to 1 if  $\beta(\det T_W) \geq 1$  for all effective  $\beta \neq 0$ . We will say that the triple  $(W, \mathbf{G}, \theta)$  is a *Fano triple* in this case.

Let  $S_0^\varepsilon(z)$  denote the operator  $S^\varepsilon$  at  $\mathbf{t} = 0$ . Then

$$\begin{aligned} S_0^\varepsilon(z)(\mathbb{1}) &= \mathbb{1} + \sum_i \gamma_i \left( \sum_{\beta \neq 0} q^\beta \left\langle \frac{\gamma^i}{z - \psi}, \mathbb{1} \right\rangle_{0,2,\beta}^\varepsilon \right) \\ &= \mathbb{1} + \frac{1}{z} \sum_i \gamma_i \left( \sum_{\beta \neq 0} q^\beta \langle \gamma^i, \mathbb{1} \rangle_{0,2,\beta}^\varepsilon \right) + O\left(\frac{1}{z^2}\right). \end{aligned}$$

**COROLLARY 5.5.1.** *For semi-positive triples  $(W, \mathbf{G}, \theta)$  and every  $\varepsilon \geq 0+$ ,*

$$\frac{J^\varepsilon|_{\mathbf{t}=0}}{\langle [\text{pt}], P^\varepsilon|_{\mathbf{t}=0} \rangle} = \mathbb{1} + \sum_i \gamma_i \left( \sum_{\beta \neq 0} q^\beta \left\langle \frac{\gamma^i}{z - \psi}, \mathbb{1} \right\rangle_{0,2,\beta}^\varepsilon \right). \tag{5.5.2}$$

*Proof.* Set  $\mathbf{t} = 0$  in Theorem 5.4.1. □

*Remark 5.5.2.* In the very special case when  $W//\mathbf{G}$  is a nef complete intersection in projective space  $\mathbb{P}^n$  and  $\varepsilon = 0+$ , Corollary 5.5.1 was obtained independently in [CZ12]. Their proof is different, as it uses localization with respect to the big torus action on  $\mathbb{P}^n$ .

By their definition (5.1.1), the big  $J^\varepsilon$ -functions have the form

$$\frac{\mathbf{t}}{z} + J^\varepsilon|_{\mathbf{t}=0} + O(1/z^2).$$

If we write the  $1/z$ -expansion of  $J^\varepsilon|_{\mathbf{t}=0}$  as

$$J^\varepsilon|_{\mathbf{t}=0} = J_0^\varepsilon(q)\mathbb{1} + J_1^\varepsilon(q)\frac{1}{z} + O(1/z^2),$$

then we deduce from Corollary 5.5.1 that

$$J_0^\varepsilon(q) = \langle [\text{pt}], P^\varepsilon|_{\mathbf{t}=0} \rangle = 1 + O(q) \in \Lambda \quad (5.5.3)$$

and

$$J_1^\varepsilon(q) = J_0^\varepsilon(q) \sum_i \gamma_i \left( \sum_{\beta \neq 0} q^\beta \langle \gamma^i, \mathbb{1} \rangle_{0,2,\beta}^\varepsilon \right) \in q\mathbf{H}^*(W//\mathbf{G}, \Lambda). \quad (5.5.4)$$

Hence

$$J^\varepsilon(\mathbf{t}, z) = J_0^\varepsilon(q)\mathbb{1} + \frac{\mathbf{t} + J_1^\varepsilon(q)}{z} + O(1/z^2). \quad (5.5.5)$$

This implies in particular that

$$P^\varepsilon(\mathbf{t}, z) = (S^\varepsilon(-z))^*(J^\varepsilon(\mathbf{t}, z)) = J_0^\varepsilon(q)\mathbb{1} + O(1/z).$$

Since  $P^\varepsilon$  has no pole at  $z = 0$ , it follows that  $P^\varepsilon(\mathbf{t}, z) = J_0^\varepsilon(q)\mathbb{1}$ . From this and Theorem 5.4.1, we deduce the following extension of Corollary 5.5.1.

**COROLLARY 5.5.3.** *For semi-positive  $(W, \mathbf{G}, \theta)$ , we have*

$$\frac{J^\varepsilon(\mathbf{t}, z)}{J_0^\varepsilon(q)} = S_{\mathbf{t}}^\varepsilon(z)(\mathbb{1}) = \sum_i \gamma_i \left\langle \frac{\gamma^i}{z - \psi}, \mathbb{1} \right\rangle_{0,2}^\varepsilon. \quad (5.5.6)$$

Furthermore,

$$\frac{\mathbf{t} + J_1^\varepsilon(q)}{J_0^\varepsilon(q)} = \sum_i \gamma_i \langle \gamma^i, \mathbb{1} \rangle_{0,2}^\varepsilon - \mathbb{1}. \quad (5.5.7)$$

Equation (5.5.7) allows us to easily determine, in the semi-positive case, all primary  $\varepsilon$ -quasimap invariants with a fundamental class insertion in terms of the first two coefficients  $J_0^\varepsilon(q)$  and  $J_1^\varepsilon(q)$  of the  $1/z$  expansion of  $J^\varepsilon|_{\mathbf{t}=0}$ .

**COROLLARY 5.5.4.** *Assume that  $(W, \mathbf{G}, \theta)$  is semi-positive. Then we have the following results:*

(i) *For every  $i$  and every  $m \geq 2$ ,*

$$\sum_{\beta \neq 0} q^\beta \langle \gamma^i, \mathbb{1}, \mathbf{t}, \dots, \mathbf{t} \rangle_{0,2+m,\beta}^\varepsilon = 0.$$

(ii) *For every  $i$ ,*

$$\sum_{\beta \neq 0} q^\beta \langle \gamma^i, \mathbb{1} \rangle_{0,2,\beta}^\varepsilon = \left\langle \frac{J_1^\varepsilon(q)}{J_0^\varepsilon(q)}, \gamma^i \right\rangle.$$

(iii) If  $j \neq i$ , then

$$\sum_{\beta \neq 0} q^\beta \langle \gamma^i, \mathbb{1}, \gamma_j \rangle_{0,3,\beta}^\varepsilon = 0,$$

while

$$1 + \sum_{\beta \neq 0} q^\beta \langle \gamma^j, \mathbb{1}, \gamma_j \rangle_{0,3,\beta}^\varepsilon = (J_0^\varepsilon(q))^{-1}$$

for every  $j$ .

*Proof.* Write  $\mathbf{t} = \sum_i t_i \gamma_i$  as before. Part (i) follows directly from (5.5.7), since the left-hand side is linear in  $\mathbf{t}$ . Part (ii) is (5.5.4). Using parts (i) and (ii) in (5.5.7), we get

$$\mathbf{t} = J_0^\varepsilon(q) \left( \mathbf{t} + \sum_i \gamma_i \sum_{\beta \neq 0} q^\beta \langle \gamma^i, \mathbb{1}, \mathbf{t} \rangle_{0,3,\beta}^\varepsilon \right).$$

Fixing  $j$  and making  $t_i = 0$  for  $i \neq j$  gives part (iii).  $\square$

In fact, simple degree counting gives a more precise version of part (ii). For any target  $W//\mathbf{G}$  we give a grading to  $H^*(W//\mathbf{G}, \Lambda)\{z, z^{-1}\}$  as follows:

- Elements in  $H^*(W//\mathbf{G})$  have their usual cohomological degree.
- The element  $q^\beta$  has degree  $2\beta(\det(T_W))$ .
- The element  $z$  has degree two.

It is immediate to check that the operator  $S_0^\varepsilon(z)$  preserves degree, hence  $S_0^\varepsilon(z)(\mathbb{1})$  is an element of degree zero.

Assume now that  $W//\mathbf{G}$  is projective, with  $(W, \mathbf{G}, \theta)$  semi-positive. As we have already observed,  $J_0^\varepsilon(q)$  has degree zero. The  $1/z$ -coefficient in  $S_0^\varepsilon(z)(\mathbb{1})$  is

$$\sum_i \gamma_i \left( \sum_{\beta \neq 0} q^\beta \langle \gamma^i, \mathbb{1} \rangle_{0,2,\beta}^\varepsilon \right) = \frac{J_1^\varepsilon(q)}{J_0^\varepsilon(q)},$$

which therefore has degree two. Hence the only possibly nonvanishing invariants  $\langle \gamma^i, \mathbb{1} \rangle_{0,2,\beta}^\varepsilon$  are those for which  $\gamma^i \in H^{\geq \dim_{\mathbb{R}}(W//\mathbf{G})-2}(W//\mathbf{G})$ . Write

$$J_1^\varepsilon(q) = f_0^\varepsilon(q) \mathbb{1} + \sum_{j=1}^r f_j^\varepsilon(q) D_j,$$

where  $f_0^\varepsilon(q) \in q\Lambda$  is a degree two element,

$$f_0^\varepsilon(q) = \sum_{\{\beta \neq 0, \beta(\det(T_W))=1\}} b_\beta q^\beta,$$

$\{D_1, \dots, D_r\}$  is a basis of  $H^2(W//\mathbf{G})$ , and the  $f_j^\varepsilon(q) \in q\Lambda$  for  $1 \leq j \leq r$  are degree zero elements, of the form

$$f_j^\varepsilon(q) = \sum_{\{\beta \neq 0, \beta(\det(T_W))=0\}} c_{j,\beta} q^\beta.$$

We conclude that

$$\sum_{\beta \neq 0} q^\beta \langle [\text{pt}], \mathbb{1} \rangle_{0,2,\beta}^\varepsilon = \frac{f_0^\varepsilon(q)}{J_0^\varepsilon(q)}, \quad \sum_{\beta \neq 0} q^\beta \langle D^j, \mathbb{1} \rangle_{0,2,\beta}^\varepsilon = \frac{f_j^\varepsilon(q)}{J_0^\varepsilon(q)},$$

where  $\{D^j\}$  is the dual basis in  $H^{\dim_{\mathbb{R}}(W//\mathbf{G})-2}(W//\mathbf{G})$ .

Note that if  $(W, \mathbf{G}, \theta)$  is Fano, then  $J_0^\varepsilon(q) = 1$  and  $J_1^\varepsilon(q) = f_0^\varepsilon(q)\mathbb{1}$ , and if the Fano index is at least two (that is, if  $\beta(\det T_W) \geq 2$  for all effective  $\beta \neq 0$ ), then  $J_1^\varepsilon(q) = 0$ .

*Remark 5.5.5.* The main point of the formulas in Corollary 5.5.4 is that most often  $J^\varepsilon|_{t=0}$  is explicitly computable. Indeed, for many of the examples of targets listed in § 2.8, closed formulas for the “small”  $I$ -function  $I|_{t=0} = J^{0+}|_{t=0}$  are known:

- For toric varieties and complete intersections in toric varieties the formulas are due to Givental, see [Giv98a].
- For flag varieties of classical types  $A, B, C$ , and  $D$ , as well as zero loci of regular sections of homogeneous vector bundles on such flag varieties, see [BCK08] and [CKS08].
- The  $I$ -function of a local target  $\mathcal{E}//\mathbf{G}$  over a base  $W//\mathbf{G}$  can be written explicitly in terms of the  $I$ -function of the base (this is easy when the bundle splits into a direct sum of line bundles and can be seen by “abelianization” in the general case).
- A formula in the case of the Hilbert schemes of points in  $\mathbb{C}^2$  (an example of a Nakajima quiver variety) is due to the authors together with Diaconescu and Maulik; see [CKP12] for an account. The method generalizes to other Nakajima quiver varieties.

Of course, the formulas in the case  $\varepsilon = 0+$  include the ones for any fixed  $\varepsilon > 0$ , since the corresponding  $J_0^\varepsilon(q)$  and  $J_1^\varepsilon(q)$  are just truncations of  $I_0(q)$  and  $I_1(q)$ , respectively.

*Remark 5.5.6.* The situation is even simpler in the noncompact examples.

(i) Consider a local Calabi-Yau target  $\mathcal{E}//\mathbf{G}$  as in Example 2.8.5 with  $0 = \beta(\det T_W) + \beta(\det E)$  for all effective  $\beta$ . Since the top degree of a cohomology class is strictly less than  $\dim(\mathcal{E}//\mathbf{G})$ , the dimension estimate used to conclude (5.5.1) implies that  $J_0^\varepsilon(q) = 1$ .

(ii) Consider a Nakajima quiver variety  $W//\mathbf{G}$  and let  $-\lambda$  denote the  $\mathbf{T}$ -weight of its holomorphic symplectic form. Then

$$J_0^\varepsilon(q) = 1, \quad J_1^\varepsilon(q) = \lambda f^\varepsilon(q)$$

with  $f^\varepsilon(q) \in q\mathbb{Q}[[q]]$  a (nonequivariant) scalar function of degree zero. This follows from the same dimension counting arguments and the fact that the moduli spaces of  $\varepsilon$ -stable quasimaps (and the graph spaces as well) admit *reduced* virtual classes  $[\mathbf{Q}_{g,k}^\varepsilon(W//\mathbf{G}, \beta)]_{\text{red}}^{\text{vir}}$  of dimension one larger than the usual virtual dimension and satisfying

$$[\mathbf{Q}_{g,k}^\varepsilon(W//\mathbf{G}, \beta)]^{\text{vir}} = \lambda [\mathbf{Q}_{g,k}^\varepsilon(W//\mathbf{G}, \beta)]_{\text{red}}^{\text{vir}}.$$

In terms of *reduced* quasimap invariants we have explicitly

$$f^\varepsilon(q) = \sum_{\beta \neq 0} q^\beta \int_{[\mathbf{Q}_{0,2}^\varepsilon(W//\mathbf{G}, \beta)]_{\text{red}}^{\text{vir}}} \text{ev}_1^*([\text{pt}]) \text{ev}_2^*(\mathbb{1}).$$

The reduced virtual classes for stable maps to holomorphic symplectic varieties were introduced in [MP07], [OP10]. The construction works equally well for quasimaps.

## 6. Genus zero $\varepsilon$ -wall-crossing

In this section we formulate genus zero wall-crossing formulas for quasimap invariants when varying the  $\varepsilon$ -stability parameter as equalities of generating functions after certain transformations, or changes of variables.

### 6.1 “String” transformations

Let

$$\gamma \in H^*(W//\mathbf{G}, \Lambda)$$

be an invertible element of the form  $\gamma = \mathbb{1} + O(q)$ . (In the quasi-projective cases we may also take  $\gamma = \gamma_0 + O(q)$ , with  $\gamma_0$  an invertible element in  $H_{\mathbf{T}, \text{loc}}^*(W//\mathbf{G}, \mathbb{Q})$ .) For each  $\varepsilon \geq 0+$ , consider the generating series  $\sum_i \gamma_i \langle\langle \gamma^i, \gamma \rangle\rangle_{0,2}^\varepsilon$  for the *primary*  $\varepsilon$ -quasimap invariants with a  $\gamma$ -insertion and put

$$\tau_\gamma^\varepsilon(\mathbf{t}) := \sum_i \gamma_i \langle\langle \gamma^i, \gamma \rangle\rangle_{0,2}^\varepsilon - \gamma. \quad (6.1.1)$$

We have the  $z$ -expansion

$$S_{\mathbf{t}}^\varepsilon(z)(\gamma) = \gamma + \tau_\gamma^\varepsilon(\mathbf{t}) \frac{1}{z} + O\left(\frac{1}{z^2}\right).$$

Further,

$$\mathbf{t} \mapsto \tau_\gamma^\varepsilon(\mathbf{t}) = \mathbf{t}\gamma + O(q) = \mathbf{t} + O(q),$$

hence  $\tau_\gamma^\varepsilon$  is invertible as a transformation on  $H^*(W//\mathbf{G}, \Lambda)$ . If for a pair of stability parameters  $0+ \leq \varepsilon_1 < \varepsilon_2 \leq \infty$ , we set

$$\tau_\gamma^{\varepsilon_1, \varepsilon_2}(\mathbf{t}) := (\tau_\gamma^{\varepsilon_1})^{-1} \circ \tau_\gamma^{\varepsilon_2}(\mathbf{t}), \quad (6.1.2)$$

then

$$S_{\tau_\gamma^{\varepsilon_1, \varepsilon_2}(\mathbf{t})}^{\varepsilon_1}(z)(\gamma) = S_{\mathbf{t}}^{\varepsilon_2}(z)(\gamma) \pmod{\frac{1}{z^2}}. \quad (6.1.3)$$

CONJECTURE 6.1.1. For all  $\gamma \in H^*(W//\mathbf{G}, \Lambda)$  of the form  $\mathbb{1} + O(q)$  and all  $0+ \leq \varepsilon_1 < \varepsilon_2 \leq \infty$ ,

$$S_{\tau_\gamma^{\varepsilon_1, \varepsilon_2}(\mathbf{t})}^{\varepsilon_1}(z)(\gamma) = S_{\mathbf{t}}^{\varepsilon_2}(z)(\gamma).$$

For example, let us take  $\varepsilon_1 = \varepsilon$  with  $0+ \leq \varepsilon \leq 1$ ,  $\varepsilon_2 = \infty$ , and  $\gamma = \mathbb{1}$ . Then, by the string equation in Gromov–Witten theory,  $\tau_{\mathbb{1}}^\infty$  is the identity transformation and  $S_{\mathbf{t}}^\infty(z)(\mathbb{1}) = J^\infty(\mathbf{t}, z)$ . Hence

$$\tau_{\mathbb{1}}^{\varepsilon, \infty} = (\tau_{\mathbb{1}}^\varepsilon)^{-1} \quad (6.1.4)$$

and we obtain the following special case of Conjecture 6.1.1.

CONJECTURE 6.1.2. Let  $J^\infty(\mathbf{t}, z)$  be the Gromov–Witten theory (big)  $J$ -function of  $W//\mathbf{G}$ , let  $\varepsilon \geq 0+$  be a stability parameter, and let

$$\tau^\varepsilon(\mathbf{t}) = \sum_i \gamma_i \langle\langle \gamma^i, \mathbb{1} \rangle\rangle_{0,2}^\varepsilon - \mathbb{1} = \mathbf{t} + \sum_i \gamma_i \sum_{\beta \neq 0, m \geq 0} \frac{q^\beta}{m!} \langle\langle \gamma^i, \mathbb{1}, \mathbf{t}, \dots, \mathbf{t} \rangle\rangle_{0,2+m,\beta}^\varepsilon. \quad (6.1.5)$$

Then

$$J^\infty(\tau^\varepsilon(\mathbf{t}), z) = S_{\mathbf{t}}^\varepsilon(z)(\mathbb{1}).$$

Note that if the  $\varepsilon$ -quasimap invariants would satisfy the usual string equation, then  $\tau^\varepsilon(\mathbf{t}) = \mathbf{t}$ . For this reason we will refer to (6.1.5) as *the string transformation*. By the same token, (6.1.2) will be called a *generalized string transformation*.

### 6.2 Semi-positive targets and the mirror map

We specialize further to the semi-positive case. Using the results of § 5.5, namely Corollary 5.5.3, together with Corollary 5.5.4 and the discussion immediately following it, we restate Conjecture 6.1.2 as follows.

CONJECTURE 6.2.1. Assume that  $(W, \mathbf{G}, \theta)$  is semi-positive. Then for every  $\varepsilon \geq 0+$ ,

$$J^\infty(\tau^\varepsilon(\mathbf{t}), z) = \frac{J^\varepsilon(\mathbf{t}, z)}{J_0^\varepsilon(q)} \tag{6.2.1}$$

with

$$\tau^\varepsilon(\mathbf{t}) = \frac{\mathbf{t} + J_1^\varepsilon(q)}{J_0^\varepsilon(q)}.$$

In particular, we have the following:

(i) The function  $J^\varepsilon$  is on the overruled Lagrangian cone  $\mathcal{L}ag_{W//\mathbf{G}}$  encoding the genus zero Gromov–Witten theory of  $W//\mathbf{G}$ . (See, for example, [CG07] for Givental’s symplectic space formalism in Gromov–Witten theory and the resulting Lagrangian cone.)

(ii) If the Fano index of  $(W, \mathbf{G}, \theta)$  is at least two, then  $J^\varepsilon$  does not depend on  $\varepsilon \geq 0+$ .

*Remark 6.2.2.* We explain here why the statement above generalizes Givental’s toric Mirror Theorems in [Giv98a].

Consider the (most interesting) case when  $\varepsilon = 0+$ , so that the right-hand side of (6.2.1) is given by the  $I$ -function of  $W//\mathbf{G}$ . We then set  $\mathbf{t} = 0$  and use the string and divisor equations for Gromov–Witten invariants on the left-hand side (recall from § 5.5 that  $J_1^\varepsilon(q) \in H^{\leq 2}(W//\mathbf{G}, \Lambda)$ ). When  $W//\mathbf{G}$  is either a semi-positive toric variety, a semi-positive complete intersection in a toric variety, or a local toric target, the resulting equality coincides precisely with Givental’s toric Mirror Theorems after multiplying by an overall factor  $e^{\mathbf{t}_{\text{small}}/z}$ , where  $\mathbf{t}_{\text{small}}$  is the general element of  $H^{\leq 2}(W//\mathbf{G}, \mathbb{Q})$ , and replacing  $q^\beta$  by  $q^\beta e^{J_\beta \mathbf{t}_{\text{small}}}$ .

From this we see that the classical mirror map is

$$\mathbf{t}_{\text{small}} \longmapsto \mathbf{t}_{\text{small}} + \frac{I_1(q)}{I_0(q)} = \mathbf{t}_{\text{small}} + \sum_i \gamma_i \sum_{\beta \neq 0} q^\beta e^{J_\beta \mathbf{t}_{\text{small}}} \langle \gamma^i, \mathbb{1} \rangle_{0,2,\beta}^{0+}, \tag{6.2.2}$$

and in particular its “quantum corrections” have an interpretation as two-pointed quasimap invariants (the second equality is given by Corollary 5.5.1).

Hence Conjecture 6.1.2, and the theorems proved later in § 7, can be viewed as a significant generalization of Givental’s results

- from complete intersections in toric varieties or local toric targets to a much larger class of GIT targets;
- from semi-positive targets to targets with no restriction on the anti-canonical class;
- from  $\mathbf{t} = 0$  to general parameter  $\mathbf{t}$ , that is, from two-point invariants to invariants with any number of insertions;
- from  $(\varepsilon = 0+)$ -stability to all stability parameters  $\varepsilon$ .

Furthermore, the string transformation  $\tau^\varepsilon(\mathbf{t})$  generalizes the celebrated “mirror map”, while at the same time giving it a geometric realization as a generating series of quasimap invariants.

*Remark 6.2.3.* In [Jin10], M. Jinjenzi proposed a conjecture for writing the mirror map via two-pointed correlators in the toric cases which is similar to the second equality in (6.2.2), though it is not clear that the moduli spaces he is using coincide with the moduli of quasimaps as constructed in [CK10]. Hence Corollary 5.5.1 proves a generalization of his conjecture to all GIT targets.



### 6.3 Higher genus

We plan to study wall-crossing formulas under change of the stability parameter  $\varepsilon$  for higher genus quasimap invariants in subsequent work. Here we only formulate a conjecture in the case when the anti-canonical class is sufficiently positive.

CONJECTURE 6.3.1. If the Fano index of  $(W, \mathbf{G}, \theta)$  is least two, then the descendant quasimap invariants

$$\langle \sigma_1 \psi_1^{a_1}, \dots, \sigma_k \psi_k^{a_k} \rangle_{g,k,\beta}^\varepsilon$$

are independent of  $\varepsilon$ .

The conjecture is inspired by the following cases which are known to hold.

- When  $W//\mathbf{G}$  is a Grassmannian  $G(r, n)$ , the conjecture is proved in [MOP11] for  $\varepsilon \in \{0+, \infty\}$  and [Tod11] extends it to arbitrary  $\varepsilon$ .
- When  $W//\mathbf{G}$  is a projective Fano toric variety (with no restriction on the Fano index), the conjecture has been proved by the authors<sup>1</sup>.

### 6.4 Transformations for the big $J^\varepsilon$ -functions in the general case

In this subsection we generalize Conjecture 6.2.1 to targets with no positivity restriction on the triple  $(W, \mathbf{G}, \theta)$ .

LEMMA 6.4.1. *For every  $\varepsilon \geq 0+$ , there exist a uniquely determined element  $P^{\infty,\varepsilon}(\mathbf{t}, z)$  in  $H^*(W//\mathbf{G}, \Lambda)[[z]]$  which is convergent in the  $q$ -adic topology for each  $\mathbf{t}$ , and a uniquely determined transformation  $\mathbf{t} \mapsto \tau^{\infty,\varepsilon}(\mathbf{t})$  on  $H^*(W//\mathbf{G}, \Lambda)$  with the following properties:*

- (i)  $\tau^{\infty,\varepsilon}(\mathbf{t}) = \mathbf{t} + O(q)$ .
- (ii)  $P^{\infty,\varepsilon}(\mathbf{t}, z) = \mathbb{1} + O(q)$ .
- (iii)  $S_{\mathbf{t}}^\varepsilon(z)(P^\varepsilon(\mathbf{t}, z)) = S_{\tau^{\infty,\varepsilon}(\mathbf{t})}^\infty(z)(P^{\infty,\varepsilon}(\tau^{\infty,\varepsilon}(\mathbf{t}), z)) \pmod{1/z^2}$ .

We omit the proof of the lemma, which is an elementary but tedious check that  $P^{\infty,\varepsilon}$  and  $\tau^{\infty,\varepsilon}$  with the required properties can be constructed in a unique way and at the same time by an inductive procedure in the “degree”  $d = \beta(L_\theta)$ .

By Theorem 5.4.1, we have  $J^\varepsilon(\mathbf{t}, z) = S_{\mathbf{t}}^\varepsilon(z)(P^\varepsilon(\mathbf{t}, z))$ . On the other hand, if we write

$$P^{\infty,\varepsilon}(\mathbf{t}, z) = \sum_i C_i(q, \{t_j\}, z) \gamma_i$$

with “scalar” coefficients  $C_i(q, \{t_j\}, z) \in \Lambda[[\{t_j\}, z]]$ , then

$$S_{\mathbf{t}}^\infty(z)(P^{\infty,\varepsilon}(\mathbf{t}, z)) = \sum_i C_i z \partial_{t_i} J^\infty(\mathbf{t}, z),$$

and hence it is on the Lagrangian cone  $\text{Lag}_{W//\mathbf{G}}$ .

CONJECTURE 6.4.2. For all  $\varepsilon \geq 0+$ ,

$$J^\varepsilon(\mathbf{t}, z) = S_{\tau^{\infty,\varepsilon}(\mathbf{t})}^\infty(z)(P^{\infty,\varepsilon}(\tau^{\infty,\varepsilon}(\mathbf{t}), z)),$$

In particular, the function  $J^\varepsilon$  is on the Lagrangian cone of the Gromov–Witten theory of  $W//\mathbf{G}$ .

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<sup>1</sup>Update: Higher-genus wall-crossing for semi-positive targets is now treated in [CK13]. The result about projective toric Fanos appears in [CK13, Corollary 1.3.3].

### 7. GIT quotients with good torus actions

In this section we assume that a torus  $\mathbf{T} \cong (\mathbb{C}^*)^r$  acts on  $W$  and that this action commutes with the  $\mathbf{G}$ -action. Hence there are induced  $\mathbf{T}$ -actions on the quotients  $[W/\mathbf{G}]$ ,  $W/\text{aff } \mathbf{G}$ , and  $W//\mathbf{G}$ . Further, we assume that the  $\mathbf{T}$ -fixed locus  $(W//\mathbf{G})^{\mathbf{T}}$  is a finite set. The same is then true for the fixed locus in the affine quotient, which, in turn, implies that the  $\mathbf{T}$ -fixed loci in all quasimap moduli spaces are proper. Hence we have well-defined theories of  $\mathbf{T}$ -equivariant  $\varepsilon$ -quasimap invariants; see [CKM14, § 6.3]. Furthermore, we may (and will) consider *twisted* theories, as in [CKM14, § 6.2].

#### 7.1 Equivariant generating functions

Let

$$H_{\mathbf{T}}^*(\text{Spec}(\mathbb{C}), \mathbb{Q}) = \mathbb{Q}[\lambda_1, \dots, \lambda_r]$$

be the equivariant cohomology of a point. The equivariant quasimap invariants are defined by the virtual localization formula

$$\langle \sigma_1 \psi_1^{a_1}, \dots, \sigma_k \psi_k^{a_k} \rangle_{g,k,\beta}^\varepsilon := \int_{[(\mathbb{Q}_{g,k}^\varepsilon(W//\mathbf{G}, \beta))^{\mathbf{T}}]^{\text{vir}}} \frac{\iota^*(\prod_j \text{ev}_j^*(\sigma_j) \psi_j^{a_j})}{e(N^{\text{vir}})},$$

where

$$\iota : (\mathbb{Q}_{g,k}^\varepsilon(W//\mathbf{G}, \beta))^{\mathbf{T}} \hookrightarrow \mathbb{Q}_{g,k}^\varepsilon(W//\mathbf{G}, \beta)$$

is the inclusion of the fixed point locus and  $e(N^{\text{vir}})$  is the  $\mathbf{T}$ -equivariant Euler class of the virtual normal bundle. These are well defined by the properness of the fixed locus and take values in the localized equivariant cohomology of a point,

$$K := \mathbb{Q}(\lambda_1, \dots, \lambda_r) = H_{\mathbf{T}, \text{loc}}^*(\text{Spec}(\mathbb{C}), \mathbb{Q}).$$

Similarly, we have the  $\mathbf{T}$ -equivariant invariants on graph spaces.

The Novikov ring is now  $\Lambda = K[[q]]$ , and we have the equivariant objects  $J^\varepsilon(\mathbf{t}, z)$ ,  $S_{\mathbf{t}}^\varepsilon(z)$ ,  $P^\varepsilon(\mathbf{t}, z)$  depending on  $\mathbf{t} \in H_{\mathbf{T}, \text{loc}}^*(W//\mathbf{G}, \Lambda)$  and taking values in  $H_{\mathbf{T}, \text{loc}}^*(W//\mathbf{G}, \Lambda) \{\{z, z^{-1}\}\}$ .

If  $W//\mathbf{G}$  is *projective* and we take all insertions  $\gamma$ ,  $\mathbf{t}$ , etc. in the nonlocalized equivariant cohomology  $H_{\mathbf{T}}^*(W//\mathbf{G}, \mathbb{Q}[[q]])$ , then the  $\mathbf{T}$ -equivariant invariants may be defined without localization and take values in the ring  $\mathbb{Q}[\lambda_1, \dots, \lambda_r]$ . In this case, the generating functions specialize to their nonequivariant counterparts upon setting  $\lambda_i = 0$  for all  $i$ .

#### 7.2 Twisted theories

Twisted quasimap invariants are discussed in [CKM14, § 6.2] and we refer the reader to *loc. cit.* for the general theory. We recall here the specific situations needed for the present paper.

**7.2.1 Zero loci of sections of “positive” vector bundles and twisting.** Consider the situation described in Example 2.8.4:  $W$  is smooth,  $W//\mathbf{G}$  is projective, and we have a vector bundle  $\underline{E}$  on  $W//\mathbf{G}$  with a regular section  $\underline{s}$ , induced from a  $\mathbf{G}$ -representation  $E$  and a  $\mathbf{G}$ -invariant section  $s$  of  $W \times E$ . We are interested in the genus zero quasimap theory of  $Z//\mathbf{G}$ , where  $Z$  is the zero locus of  $s$ . Note that  $(Z, \mathbf{G}, \theta)$  is semi-positive precisely when  $\beta(\det(T_W)) \geq \beta(\det(W \times E))$  for all effective  $\beta$ .

Let  $W//\mathbf{G}$  admit an action by the torus  $\mathbf{T}$  with isolated fixed points. Except in very special cases,  $\mathbf{T}$  will not act on  $Z//\mathbf{G}$ . Nevertheless, as long as we restrict to insertions pulled back from

the cohomology of  $W//\mathbf{G}$ , one can still express its quasimap invariants as  $\mathbf{T}$ -equivariant integrals on the moduli spaces with target  $W//\mathbf{G}$  by the following construction (in Gromov–Witten theory, this construction is due to Kontsevich).

Assume that for every quasimap  $(C, P, u)$  with  $C$  a rational curve, we have  $H^1(C, P \times_G E) = 0$ . This happens for example when  $W \times E$  is generated by  $\mathbf{G}$ -invariant global sections over the stable locus (so that  $\underline{E}$  on  $W//\mathbf{G}$  is also globally generated); see [CKM14, Proposition 6.2.3 and Remark 6.2.4]. Following the terminology in Gromov–Witten theory, we will call such a representation  $E$  *convex*.

Either on the moduli space  $Q_{0,k}^\varepsilon(W//\mathbf{G}, \beta)$  or on the graph space  $QG_{0,k,\beta}^\varepsilon(W//\mathbf{G})$ , consider the universal curve  $\mathcal{C}$ , with projection  $\pi$ , the universal principal  $\mathbf{G}$ -bundle  $\mathfrak{P}$  on the universal curve, and the universal section  $u : \mathcal{C} \rightarrow \mathfrak{P} \times_{\mathbf{G}} W$ . We have an induced vector bundle

$$E_{0,k,\beta}^\varepsilon := u^*(\mathfrak{P} \times_{\mathbf{G}} (W \times E))$$

on  $\mathcal{C}$ . The  $H^1$ -vanishing assumption implies that  $R^1\pi_*E_{0,k,\beta}^\varepsilon = 0$  and  $R^0\pi_*E_{0,k,\beta}^\varepsilon$  is a vector bundle on  $Q_{0,k}^\varepsilon(W//\mathbf{G}, \beta)$  or on  $QG_{0,k,\beta}^\varepsilon(W//\mathbf{G})$  ( $\mathbf{T}$ -equivariant, if we choose a  $\mathbf{T}$ -linearization of  $W \times E$ ). Its fiber over a quasimap  $(C, P, u)$  is the space of sections  $H^0(C, P \times_G E)$ .

The  $\varepsilon$ -quasimap invariants twisted by  $E$  are defined by inserting the  $\mathbf{T}$ -equivariant Euler class of  $R^0\pi_*E_{0,k,\beta}^\varepsilon$  in the integrals:

$$\langle \sigma_1 \psi_1^{a_1}, \dots, \sigma_k \psi_k^{a_k} \rangle_{0,k,\beta}^{\varepsilon,E} = \int_{[Q_{0,k}^\varepsilon(W//\mathbf{G}, \beta)]^{\text{vir}}} e(R^0\pi_*E_{0,k,\beta}^\varepsilon) \prod_{i=1}^k \psi_i^{a_i} \text{ev}_i^*(\sigma_i). \quad (7.2.1)$$

We define the graph space twisted invariants similarly:

$$\langle \sigma_1, \dots, \sigma_k \rangle_{0,k,\beta}^{QG^\varepsilon,E} = \int_{[QG_{0,k,\beta}^\varepsilon(W//\mathbf{G})]^{\text{vir}}} e(R^0\pi_*E_{0,k,\beta}^\varepsilon) \prod_{i=1}^k \tilde{\text{ev}}_i^*(\sigma_i). \quad (7.2.2)$$

Let  $j : Z//\mathbf{G} \rightarrow W//\mathbf{G}$  be the inclusion, and let  $j_{k,\beta}$  be the induced map on quasimap moduli spaces. By [CKM14, Proposition 6.2.2], we have

$$(j_{k,\beta})_*([Q_{0,k}^\varepsilon(Z//\mathbf{G}, \beta)]^{\text{vir}}) = [Q_{0,k}^\varepsilon(W//\mathbf{G}, \beta)]^{\text{vir}} \cap e(R^0\pi_*E_{0,k,\beta}^\varepsilon).$$

Hence the twisted invariant (7.2.1) is equal to the quasimap invariant

$$\langle (j^*\sigma_1)\psi_1^{a_1}, \dots, (j^*\sigma_k)\psi_k^{a_k} \rangle_{0,k,\beta}^\varepsilon$$

of the subvariety  $Z//\mathbf{G}$ . The same argument gives the analogous equality between  $E$ -twisted graph space invariants of  $W//\mathbf{G}$  and graph space invariants of  $Z//\mathbf{G}$ . Notice that the degree  $\beta \in \text{Hom}(\text{Pic}^{\mathbf{G}}(W), \mathbb{Z})$  is measured in  $W//\mathbf{G}$ . In many (but not all) examples this will not make a difference, as we will have  $\text{Pic}^{\mathbf{G}}(Z) = \text{Pic}^{\mathbf{G}}(W)$ . In general, there is an induced map  $j_* : \text{Hom}(\text{Pic}^{\mathbf{G}}(Z), \mathbb{Z}) \rightarrow \text{Hom}(\text{Pic}^{\mathbf{G}}(W), \mathbb{Z})$  and

$$\langle (j^*\sigma_1)\psi_1^{a_1}, \dots, (j^*\sigma_k)\psi_k^{a_k} \rangle_{0,k,\beta}^\varepsilon = \sum_{j_*(\beta')=\beta} \langle (j^*\sigma_1)\psi_1^{a_1}, \dots, (j^*\sigma_k)\psi_k^{a_k} \rangle_{0,k,\beta'}^\varepsilon.$$

We now form the (equivariant) twisted generating series  $\tilde{J}^{\varepsilon,E}(\mathbf{t}, z)$ ,  $\tilde{S}_{\mathbf{t}}^{\varepsilon,E}(z)(\gamma)$ ,  $\tilde{P}^{\varepsilon,E}(\mathbf{t}, z)$  by replacing the quasimap invariants by their twisted counterparts. Equivalently, we replace in the formulas defining the series the virtual classes  $[Q_{0,k}^\varepsilon(W//\mathbf{G}, \beta)]^{\text{vir}}$  and  $[QG_{0,k,\beta}^\varepsilon(W//\mathbf{G})]^{\text{vir}}$  by their cap products with  $e(R^0\pi_*E_{0,k,\beta}^\varepsilon)$ . The intersection pairing is also modified and now reads

$$\langle \sigma_1, \sigma_2 \rangle^E = \int_{W//\mathbf{G}} \sigma_1 \sigma_2 e(\underline{E}),$$

so that it coincides with the intersection pairing of  $j^*\sigma_1$  and  $j^*\sigma_2$  on  $Z//\mathbf{G}$ .

For example,

$$\begin{aligned} \tilde{J}^{\varepsilon,E}(\mathbf{t}, z) &= \left( \mathbb{1} + \frac{\mathbf{t}}{z} \right) e(\underline{E}) + \sum_{(k,\beta) \neq (0,0),(1,0)} q^\beta \times \\ &\quad \times (\text{ev}_\bullet)_* \left( \frac{\prod_{i=1}^k \text{ev}_i^*(\mathbf{t})}{k!} \cap \text{Res}_{F_0}([QG_{0,k,\beta}^\varepsilon(W//\mathbf{G})]^{\text{vir}} \cap e(R^0\pi_*E_{0,k,\beta})) \right), \end{aligned}$$

and

$$\begin{aligned} \tilde{S}^{\varepsilon,E}(z)(\gamma) &= \gamma e(\underline{E}) + \sum_{(m,\beta) \neq (0,0)} \frac{q^\beta}{m!} \times \\ &\quad \times (\text{ev}_1)_* \left( \frac{[Q_{0,2+m}^\varepsilon(W//\mathbf{G}, \beta)]^{\text{vir}} \cap e(R^0\pi_*E_{0,k,\beta})}{z - \psi} \text{ev}_2^*(\gamma) \prod_{j=3}^{2+m} \text{ev}_j^*(\mathbf{t}) \right). \end{aligned}$$

If the insertions  $\gamma, \mathbf{t}$  are taken in  $H_{\mathbf{T}}^*(W//\mathbf{G}, \mathbb{Q}[[q]])$ , then by the discussion above the nonequivariant limits of these series will give the push-forward by the inclusion  $j$  of the corresponding objects for  $Z//\mathbf{G}$  restricted to  $j^*H^*(W//\mathbf{G})$ , and with the Novikov parameters specialized if needed:  $q^{\beta'} = q^\beta$  for  $j_*(\beta') = \beta$ .

Now define the twisted  $J^\varepsilon$ -function, the twisted  $S$ -operator, and the twisted  $P^\varepsilon$ -series by requiring

$$\langle \tilde{J}^{\varepsilon,E}(\mathbf{t}, z), \delta \rangle = \langle J^{\varepsilon,E}(\mathbf{t}, z), \delta \rangle^E, \tag{7.2.3}$$

$$\langle \tilde{S}^{\varepsilon,E}(z)(\gamma), \delta \rangle = \langle S^{\varepsilon,E}(z)(\gamma), \delta \rangle^E, \tag{7.2.4}$$

$$\langle \tilde{P}^\varepsilon, \delta \rangle = \langle P^\varepsilon, \delta \rangle^E \tag{7.2.5}$$

for every  $\delta \in H_{\mathbf{T},\text{loc}}^*(W//\mathbf{G}, \mathbb{Q})$ . Since the tilde-series are divisible by  $e(\underline{E})$ , we see that  $J^{\varepsilon,E}$ ,  $S^{\varepsilon,E}(z)(\gamma)$ , and  $P^\varepsilon$  will be elements in  $H_{\mathbf{T}}^*(W//\mathbf{G}, \mathbb{Q}[[q]])\{\{z, z^{-1}\}\}$  when the insertions are taken in the nonlocalized equivariant cohomology  $H_{\mathbf{T}}^*(W//\mathbf{G}, \mathbb{Q}[[q]])$ .

Note in addition that the same procedure gives the twisted version of the series (6.1.2) defining the generalized string transformation.

**7.2.2 Local targets and twisting.** The quasimap theory of the local targets of Example 2.8.5 can be viewed alternatively as the theory of  $W//\mathbf{G}$  twisted by the *inverse* Euler class  $e^{-1}$ . However, if  $\mathbf{T}$  acts with isolated fixed points on  $W//\mathbf{G}$ , then  $\mathbf{T} \times \mathbf{S}$  will act with the same fixed points on the total space of the vector bundle  $\underline{E}$  (and the one-dimensional orbits will also be the same). Hence we may (and will) treat this case as the untwisted theory of a target with good torus action.

### 7.3 The main theorems

Let  $W//\mathbf{G}$  be acted upon by  $\mathbf{T}$  with isolated fixed points. Let  $E$  be a convex  $\mathbf{G}$ -representation as in § 7.2.1. The statements of the results in this subsection, and the proofs we give in the rest of the section, apply to both the untwisted and the  $E$ -twisted theories of  $W//\mathbf{G}$ . To simplify the exposition, we will drop  $E$  from the notation from now on. Hence, for the rest of this section, the notation  $J^\varepsilon, S^\varepsilon, P^\varepsilon, \tau^{\varepsilon_1, \varepsilon_2} \dots$  will mean either the generating series for invariants of  $W//\mathbf{G}$ , or the series for  $E$ -twisted invariants.

**THEOREM 7.3.1.** *Let  $0+ \leq \varepsilon_1 < \varepsilon_2 \leq \infty$  be stability parameters. Let  $\gamma \in \mathbf{H}_{\mathbf{T},\text{loc}}^*(W//\mathbf{G}, \Lambda)$  be invertible, of the form  $\mathbb{1} + O(q)$ . Let  $\tau_\gamma^{\varepsilon_1, \varepsilon_2}(\mathbf{t})$  be the ( $\mathbf{T}$ -equivariant) generalized string transformation (6.1.2). Then*

$$S_{\tau_\gamma^{\varepsilon_1, \varepsilon_2}(\mathbf{t})}^{\varepsilon_1}(z)(\gamma) = S_{\mathbf{t}}^{\varepsilon_2}(z)(\gamma).$$

**COROLLARY 7.3.2.** *The equivariant version of Conjecture 6.1.2 holds for both the  $E$ -twisted and the untwisted theories of  $W//\mathbf{G}$ . The equivariant version of Conjecture 6.2.1 holds for semi-positive  $(W, \mathbf{G}, \theta)$ , and when  $\beta(\det(T_W)) - \beta(\det(W \times E))$  is nonnegative for all  $L_\theta$ -effective classes  $\beta$ , it holds for the  $E$ -twisted theory as well.*

Assuming that  $W//\mathbf{G}$  is projective and  $\gamma \in \mathbf{H}_{\mathbf{T}}^*(W//\mathbf{G}, \mathbb{Q}[[q]])$  is a nonlocalized equivariant class, we may pass to the nonequivariant limit in Theorem 7.3.1.

**COROLLARY 7.3.3.** *Let  $W//\mathbf{G}$  be projective, admitting an action by a torus with isolated fixed points.*

- (i) *Conjecture 6.1.1 holds for  $W//\mathbf{G}$ ; in particular, Conjecture 6.1.2 holds as well. Further, Conjecture 6.2.1 holds for  $W//\mathbf{G}$  with  $(W, \mathbf{G}, \theta)$  semi-positive.*
- (ii) *If  $j : Z//\mathbf{G} \hookrightarrow W//\mathbf{G}$  is the zero locus of a section of a vector bundle  $E$  induced by a convex representation, then Conjectures 6.1.1 and 6.1.2 hold for the quasimap theory of  $Z//\mathbf{G}$  restricted to  $j^*\mathbf{H}^*(W//\mathbf{G})$ , and with the Novikov parameters specialized by setting  $q^{\beta'} = q^\beta$  for  $j_*(\beta') = \beta$ . If  $(Z, \mathbf{G}, \theta)$  is semi-positive, then Conjecture 6.1.2 holds with the same restriction for insertions and the same Novikov specialization.*

For the proof of the next result, we will need that the one-dimensional  $\mathbf{T}$ -orbits in  $W//\mathbf{G}$  are also well behaved.

Let  $P^{\infty, \varepsilon}(\mathbf{t}, z)$  and  $\tau^{\infty, \varepsilon}(\mathbf{t})$  be the  $\mathbf{T}$ -equivariant series from Lemma 6.4.1. As mentioned already, we will use the same notation for the corresponding objects in the  $E$ -twisted theory and the statement in the following theorem includes both the untwisted and twisted cases.

**THEOREM 7.3.4.** *If the  $\mathbf{T}$ -action on  $W//\mathbf{G}$  has isolated fixed points and isolated one-dimensional orbits, then for all  $\varepsilon \geq 0+$ ,*

$$S_{\mathbf{t}}^\varepsilon(z)(P^\varepsilon(\mathbf{t}, z)) = S_{\tau^{\infty, \varepsilon}(\mathbf{t})}^\infty(z)(P^{\infty, \varepsilon}(\tau^{\infty, \varepsilon}(\mathbf{t}), z)).$$

*In particular, the function  $J^\varepsilon(\mathbf{t}, z) = S_{\mathbf{t}}^\varepsilon(z)(P^\varepsilon(\mathbf{t}, z))$  is on the Lagrangian cone of the ( $E$ -twisted) Gromov–Witten theory of  $W//\mathbf{G}$ .*

The construction of  $P^{\infty, \varepsilon}(\mathbf{t}, z)$  and  $\tau^{\infty, \varepsilon}(\mathbf{t})$  in Lemma 6.4.1 does not involve any inversion, hence for  $\mathbf{t} \in \mathbf{H}_{\mathbf{T}}^*(W//\mathbf{G}, \mathbb{Q})$  we may pass to the nonequivariant limit in Theorem 7.3.4 to obtain the following corollary.

**COROLLARY 7.3.5.** *Let  $W//\mathbf{G}$  be projective, admitting an action by a torus with isolated fixed points and isolated one-dimensional orbits.*

- (i) *Conjecture 6.4.2 holds for  $W//\mathbf{G}$ .*
- (ii) *If  $j : Z//\mathbf{G} \hookrightarrow W//\mathbf{G}$  is the zero locus of a section of a vector bundle  $E$  induced by a convex representation, then Conjecture 6.4.2 holds for the quasimap theory of  $Z//\mathbf{G}$  restricted to  $j^*\mathbf{H}^*(W//\mathbf{G})$ , with the Novikov parameters specialized by setting  $q^{\beta'} = q^\beta$  for  $j_*(\beta') = \beta$ .*

The proofs of Theorems 7.3.1 and 7.3.4 use virtual localization for the  $\mathbf{T}$ -action on moduli spaces of quasimaps and will occupy the rest of this section.

### 7.4 Fixed loci

7.4.1  *$\mathbf{T}$ -fixed quasimaps.* Fix  $\varepsilon > 0$ , an effective class  $\beta \neq 0$ , and an integer  $k \geq 0$ . A  $\mathbf{T}$ -fixed quasimap

$$((C, x_1, x_2 \dots, x_{m+2}), P, u)$$

in  $\mathcal{Q}_{0,2+m}^\varepsilon(W//\mathbf{G}, \beta)$  must have all its nodes, markings, ramifications, and base points lying over the fixed locus  $(W//\mathbf{G})^\mathbf{T}$ . Explicitly, this means the following. For any irreducible component  $C' \cong \mathbb{P}^1$  of  $C$ , let  $(P', u')$  be the restriction of the pair  $(P, u)$  to  $C'$ . We have a rational map

$$[u'] : C' \dashrightarrow W//\mathbf{G}$$

inducing a regular map

$$[u']_{reg} : C' \longrightarrow W//\mathbf{G}$$

(possibly of smaller degree than the degree of the quasimap  $(C', P', u')$ ). Then one of the following must hold:

- (i) The map  $[u']_{reg}$  is constant with image a fixed point  $\mu \in (W//\mathbf{G})^\mathbf{T}$ . We will say in this case that  $C'$  is a *contracted component* of the quasimap.
- (ii) We have  $[u']_{reg} = [u']$  (that is, there are no base points on  $C'$ ) and the map is a cover of a one-dimensional  $\mathbf{T}$ -orbit in  $W//\mathbf{G}$ , totally ramified over the two fixed points in the orbit. The nodes and markings on  $C'$ , if any, must be mapped to the fixed points as well.
- (iii) We have  $[u']_{reg} \neq [u']$  and the map  $[u']_{reg}$  is a ramified cover of an orbit as in case (ii), while  $[u']$  has a base point at one of the ramification points and a node at the other ramification point. This possibility may occur only when  $\beta(L_\theta) < \varepsilon \leq 1$ .

7.4.2 *Components of the fixed loci.* When the  $\mathbf{T}$ -action on  $W//\mathbf{G}$  satisfies the additional condition that the one-dimensional  $\mathbf{T}$ -orbits are isolated, the description of the connected components of the  $\mathbf{T}$ -fixed loci in the stable map moduli spaces  $\mathcal{Q}_{0,2+m}^{\varepsilon > 1}(W//\mathbf{G}, \beta)$  goes back to Kontsevich and is well known. They are indexed by certain decorated trees and each such component is isomorphic to (a finite quotient of) a product of moduli spaces of stable pointed curves  $\overline{M}_{0,n}$ , with the factors corresponding to the vertices of the tree. For general  $\varepsilon$ , a similar description holds, except that each vertex factor now is a  $\mathbf{T}$ -fixed component in a  $\varepsilon$ -quasimap space with target a fixed point  $\mu \in (W//\mathbf{G})^\mathbf{T}$ , viewed as a GIT quotient by  $\mathbf{G}$ . In special cases, for example when  $W//\mathbf{G}$  is a Grassmannian or a toric variety, they turn out to be certain Hassett moduli spaces  $\overline{M}_{0,k|d}^\varepsilon$  of stable marked curves with weights; see [MOP11], [Tod11], [CK13].

When one-dimensional orbits vary in families, the fixed loci can still be indexed by decorated trees, but their precise structure is more complicated, since in this case there will be factors of positive dimension corresponding not just to the vertices of the trees, but also to the edges. These edge moduli spaces will depend on the target and may be difficult to identify explicitly. For an example in the case of stable maps to the Hilbert scheme of points in  $\mathbb{C}^2$ , see [OP10, §§3.8.2–3.8.3]. However, we will only need the coarser (and obvious) information that certain edge factors are *independent of  $\varepsilon$*  for a given target  $W//\mathbf{G}$ , rather than a full description of either the edge or the vertex moduli spaces.

7.4.3 *Unbroken quasimaps and unbroken components.* Each marking  $x$  of  $C$  has a fractional cotangent weight  $\alpha_x$  given by the  $\mathbf{T}$ -representation induced by the quasimap on the cotangent space  $T_x^*C$ . If  $x$  is on a contracted component, then the weight is zero. Otherwise,  $-\nu\alpha_x = w$  for

some integer  $n \geq 1$  and a weight  $w$  of the representation  $T_\mu(W//\mathbf{G})$ , where  $\mu \in (W//\mathbf{G})^{\mathbf{T}}$  is the image of  $x$ . Note that the weights of  $T_\mu(W//\mathbf{G})$  are all nonzero since the fixed points are isolated.

Similarly, if  $x$  is a node of  $C$  and  $C', C''$  are the components of  $C$  incident to  $x$ , there are fractional cotangent weights  $\alpha_{C',x}$  and  $\alpha_{C'',x}$  from the  $\mathbf{T}$ -representations  $T_x^*C'$  and  $T_x^*C''$ , respectively. The node  $x$  is called a *breaking node* if

$$\alpha_{C',x} + \alpha_{C'',x} \neq 0.$$

As in [OP10], we call a  $\mathbf{T}$ -fixed  $\varepsilon$ -stable quasimap of class  $\beta \neq 0$  to  $W//\mathbf{G}$  *unbroken* if all nodes of its domain curve are nonbreaking. This implies that either the entire domain curve is contracted by the quasimap, or that no components are contracted. We discuss the latter situation. Since we consider only quasimaps with at least two markings, the number of markings must be exactly equal to two and there can be no base points. Hence a (noncontracting) unbroken quasimap must be a two-pointed stable map  $f : (C, x_1, x_2) \rightarrow W//\mathbf{G}$ . The domain curve

$$C = C_1 \cup \dots \cup C_l$$

is a chain of projective lines  $\mathbb{P}^1$  with  $x_1 \in C_1, x_2 \in C_l$ , and nodes

$$y_j = C_j \cap C_{j+1}, \quad j = 1, \dots, l-1.$$

The weights at markings and nodes are all nonzero and satisfy

$$\alpha_{x_1} = -\alpha_{C_1, y_1}, \quad \alpha_{C_j, y_j} = -\alpha_{C_{j+1}, y_j}, \quad \alpha_{C_l, y_{l-1}} = -\alpha_{x_2},$$

and in particular  $\alpha_{x_1} = -\alpha_{x_2}$ . We say that the stable map connects the fixed points  $\mu = f(x_1)$  and  $\nu = f(x_2)$ .

If  $\mathcal{O}(\theta)$  is given a nontrivial  $\mathbf{T}$ -linearization, let  $w_\mu$  and  $w_\nu$  be the weights of the  $\mathbf{T}$ -action on the fibers over  $\mu$  and  $\nu$ , respectively. Then a localization computation as in [OP10, Lemma 4] shows that

$$\alpha_{x_1} = \frac{-w_\mu + w_\nu}{\beta(L_\theta)}.$$

By the nonvanishing of  $\alpha_{x_1}$ , we also conclude that there are no noncontracting unbroken quasimaps connecting a  $\mathbf{T}$ -fixed point to itself.

Let  $M$  be a connected component of the  $\mathbf{T}$ -fixed locus in the moduli space  $\mathbf{Q}_{0,2}^\varepsilon(W//\mathbf{G}, \beta)$ . If  $M$  contains a noncontracting unbroken quasimap with starting weight  $\alpha_{x_1}$  and joining two fixed points  $\mu$  and  $\nu$ , then all quasimaps in  $M$  are unbroken, with the same starting weight, and join the points  $\mu$  and  $\nu$ . (Note, however, that the number of irreducible components of the domain curve may vary within  $M$ .) The important fact for us will be that these unbroken components  $M$  are the same for all values of  $\varepsilon$ , since they always parametrize stable *maps* of class  $\beta$ .

When there are more than two markings, the unbroken components will appear as the  $\varepsilon$ -independent edge moduli spaces that were mentioned earlier. (There may also be edge factors corresponding to unbroken quasimaps with only one marking, but we will not be concerned with these explicitly.)

### 7.5 Recursion

We divide connected components  $M$  of the fixed locus  $\mathbf{Q}_{0,2+m}^\varepsilon(W//\mathbf{G}, \beta)^{\mathbf{T}}$  into two types. We say that  $M$  is of *initial type* if it parametrizes quasimaps for which the *first* marking  $x_1$  is on a contracted irreducible component of the domain curve. We say  $M$  is of *recursion type* if it parametrizes quasimaps with the first marking on a noncontracted irreducible component. Every  $M$  is either of initial type or of recursion type.

Every recursion component is of the form

$$M \cong M' \times_{(W//\mathbf{G})^{\mathbf{T}}} M'', \tag{7.5.1}$$

where

- (i) the factor  $M' \subset \overline{M}_{0,2}(W//\mathbf{G}, \beta_{M'})^{\mathbf{T}}$  is an unbroken component for some  $\beta_{M'} \neq 0$ ;
- (ii) the factor  $M''$  is a connected component of  $Q_{0,2+m}^{\varepsilon}(W//\mathbf{G}, \beta - \beta_{M'})^{\mathbf{T}}$ ;
- (iii) we have  $\alpha_M = \alpha_{M'} \neq \alpha_{M''}$ , where  $\alpha_M$  denotes the cotangent weight at the first marking in any element of  $M$  (and similarly for  $M', M''$ ).

The fiber product is taken using the evaluation map at the second marking in  $M'$  and the evaluation map at the first marking in  $M''$ . The component  $M''$  may be of either initial or recursion type. In the unstable case  $m = 0$  and  $\beta_{M'} = \beta$ , we take  $M''$  to be a fixed point in  $W//\mathbf{G}$  and  $\alpha_{M''} = 0$ .

We conclude that

$$Q_{0,2+m}^{\varepsilon}(W//\mathbf{G}, \beta)^{\mathbf{T}} = \coprod_{M \text{ initial}} M \coprod_{M' \text{ unbroken}} \left( M' \times_{(W//\mathbf{G})^{\mathbf{T}}} \coprod_{\alpha_{M'} \neq \alpha_{M''}} M'' \right).$$

Let

$$\{\delta_{\mu} \mid \mu \in (W//\mathbf{G})^{\mathbf{T}}\}$$

be the basis of the localized cohomology  $H_{\mathbf{T}, \text{loc}}^*(W//\mathbf{G}, \mathbb{Q})$  consisting of the equivariant fundamental classes of the fixed points, that is,  $\delta_{\mu}$  is equal to (the dual of)  $(i_{\mu})_*[\mu]$ , where  $i_{\mu} : \{\mu\} \hookrightarrow W//\mathbf{G}$  is the embedding.

For a cohomology class  $\gamma \in H_{\mathbf{T}}^*(W//\mathbf{G}, \Lambda)$  of the form  $\gamma = \gamma_0 + O(q)$ , we consider the components

$$S_{\mu}^{\varepsilon}(\gamma) := \langle S_{\mathbf{t}}^{\varepsilon}(z)(\gamma), \delta_{\mu} \rangle = i_{\mu}^*(\gamma) + \sum_{(\beta, m) \neq (0, 0)} \frac{q^{\beta}}{m!} \left\langle \frac{\delta_{\mu}}{z - \psi}, \gamma, \mathbf{t}, \dots, \mathbf{t} \right\rangle_{0, 2+m, \beta}^{\varepsilon}$$

of  $S_{\mathbf{t}}^{\varepsilon}(z)(\gamma)$  in the fixed point basis. Alternatively, these are the restrictions of  $S_{\mathbf{t}}^{\varepsilon}(z)(\gamma)$  at the fixed points,

$$S_{\mu}^{\varepsilon}(\gamma) = i_{\mu}^*(S_{\mathbf{t}}^{\varepsilon}(z)(\gamma)).$$

If, as in § 6,  $\{t_i\}$  are the coordinates of  $\mathbf{t}$  in a basis of  $H_{\mathbf{T}, \text{loc}}^*(W//\mathbf{G})$  over  $K = \mathbb{Q}(\lambda_j)$ , then  $S_{\mu}^{\varepsilon}(\gamma)$  is an element of  $\Lambda[[t_i, 1/z]] = K[[q, t_i, 1/z]]$ .

Given a  $\mathbf{T}$ -fixed point  $\mu$ , a class  $\beta$ , and an integer  $m \geq 0$ , we partition the set of connected components of  $Q_{0,2+m}^{\varepsilon}(W//\mathbf{G}, \beta)^{\mathbf{T}}$  into three disjoint subsets

$$V(\mu, \beta, m) \coprod \text{In}(\mu, \beta, m) \coprod \text{Rec}(\mu, \beta, m) \tag{7.5.2}$$

as follows:

- The subset  $V(\mu, \beta, m)$  consists of all components for which the first marking does *not* lie over  $\mu$ .
- The subset  $\text{In}(\mu, \beta, m)$  consists of all components of initial type for which the first marking lies over  $\mu$ .
- The subset  $\text{Rec}(\mu, \beta, m)$  consists of all components of recursion type for which the first marking lies over  $\mu$ .



LEMMA 7.5.1. *For each fixed  $\beta$  and each tuple  $\underline{k} := (k_i)_i$  of nonnegative integers, the coefficient of the monomial  $q^\beta \prod_i t_i^{k_i}$  in  $S_\mu^\varepsilon(\gamma)$  is a rational function of  $z$  with coefficients in  $K$ . This rational function decomposes as a sum of simple partial fractions with denominators either powers of  $z$  or powers of linear factors  $z - \alpha$ , where  $-\alpha$  is one of weights of the  $\mathbf{T}$ -representation  $T_\mu(W//\mathbf{G})$  for some  $n \in \mathbb{Z}_{>0}$*

*Proof.* It suffices to assume  $\gamma = \gamma_0 \in H_{\mathbf{T}}^*(W//\mathbf{G}, \mathbb{Q})$ . Let  $a_{\beta, \underline{k}}$  be the coefficient in the lemma and let  $m = \sum_i k_i$  be the total degree in the  $t_i$ . Then

$$\begin{aligned} a_{\beta, \underline{k}} &= \int_{[\mathbb{Q}_{0, 2+m}^\varepsilon(W//\mathbf{G}, \beta)]^{\text{vir}}} \frac{\text{ev}_1^*(\delta_\mu) c_{\beta, \underline{k}}}{z - \psi} \\ &= \sum_{j=0}^{\infty} \int_{[\mathbb{Q}_{0, 2+m}^\varepsilon(W//\mathbf{G}, \beta)]^{\text{vir}}} \frac{\psi^j \text{ev}_1^*(\delta_\mu) c_{\beta, \underline{k}}}{z^{j+1}} \end{aligned}$$

for some cohomology class  $c_{\beta, \underline{k}}$  pulled back from  $(W//\mathbf{G})^{m+1}$  by the product of the remaining evaluation maps. In the case of  $E$ -twisted theory, the Euler class of the twisting bundle is also incorporated in  $c_{\beta, \underline{k}}$ . By the formula above,  $a_{\beta, \underline{k}}$  is a priori an element in  $K[[1/z]]$ .

We apply virtual localization to the integrals in the expression above. Each integral is a sum of contributions from the components of the fixed point locus. We use the partition (7.5.2) to analyze these contributions.

First, the contribution from components in  $V(\mu, \beta, m)$  vanishes, due to the factor  $\text{ev}_1^*(\delta_\mu)$  in the integrand.

Next, each initial component  $M$  in  $\text{In}(\mu, \beta, m)$  contributes only fractions with denominator a power of  $z$ . This is because  $\psi$  restricts to a *nonequivariant* class to  $M$ , which is therefore nilpotent.

Finally, if  $M$  is a recursion component in  $\text{Rec}(\mu, \beta, m)$ , then the restriction of  $\psi$  to  $M$  is  $\psi_0 + \alpha_M$ , with  $\psi_0$  a nonequivariant class. If  $M$  is positive-dimensional, the contribution can be written as

$$\sum_{j=0}^{\infty} \int_{[M]^{\text{vir}}} \frac{\psi_0^j (\text{ev}_1^*(\delta_\mu) c_{\beta, \underline{k}})|_M}{(z - \alpha_M)^{j+1} e(N^{\text{vir}})}.$$

Again, by the nilpotency of  $\psi_0$ , it follows that  $M$  contributes finitely many fractions with denominator powers of  $z - \alpha_M$ . If  $M$  has dimension zero, then  $\psi$  restricts to the pure weight  $\alpha_M$ , so the series

$$\sum_{j=0}^{\infty} \int_{[M]^{\text{vir}}} \frac{\alpha_M^j (\text{ev}_1^*(\delta_\mu) c_{\beta, \underline{k}})|_M}{z^{j+1} e(N^{\text{vir}})}$$

is a geometric series and therefore the contribution is a fraction with a simple pole at  $z = \alpha_M$ . The lemma is proved.  $\square$

With a more careful analysis of the contributions from the recursion-type fixed loci, the virtual localization theorem shows, in fact, that the restrictions to fixed points  $S_\mu^\varepsilon$  satisfy a certain recursion relation.

For a given  $\mu \in (W//\mathbf{G})^{\mathbf{T}}$ , let  $U(\mu)$  denote the set of all components  $M' \subset \overline{M}_{0,2}(W//\mathbf{G}, \beta_{M'})^{\mathbf{T}}$  (for varying  $\beta_{M'} \neq 0$ ) parametrizing *unbroken* stable maps  $(C, x_1, x_2, f)$  with  $f(x_1) = \mu$ . For such an  $M'$  we set  $\nu_{M'} = f(x_2)$ .

LEMMA 7.5.2. *The restriction  $S_\mu^\varepsilon$  satisfies the recursion relation*

$$S_\mu^\varepsilon(z) = R_\mu^\varepsilon(z) + \sum_{M' \in U(\mu)} q^{\beta_{M'}} \left\langle \frac{\delta_\mu}{z - \alpha_{M'} - \psi_0}, S_{\nu_{M'}, \alpha_{M'}}^\varepsilon \Big|_{z=\alpha_{M'} - \psi_\infty} \right\rangle_{M'}, \quad (7.5.3)$$

where

- the classes  $\psi_0, \psi_\infty$  are nonequivariant cohomology classes;
- each  $(q, \{t_i\})$ -coefficient of  $R_\mu^\varepsilon(z)$  is an element in  $K[1/z]$ ;
- for  $\nu \in (W//\mathbf{G})^\mathbf{T}$ , the notation  $S_{\nu, \alpha}^\varepsilon(z)$  means the part of  $S_\nu^\varepsilon(z)$  that remains after removing the partial fraction terms with poles at  $z = \alpha$  (this is well defined by Lemma 7.5.1);
- the subscript  $M'$  for the bracket means the  $\mathbf{T}$ -virtual localization contribution of the component  $M'$  to the intersection number on  $\overline{M}_{0,2}(W//\mathbf{G}, \beta_{M'})$ .

*Proof.* We apply virtual  $\mathbf{T}$ -localization to  $S_\mu^\varepsilon$ . The terms with  $\beta = 0$  and varying  $m$  sum to  $i_\mu^*(e^{t/z}\gamma)$ . For each  $\beta \neq 0$  and each  $m \geq 0$ , the components of the fixed locus in  $V(\mu, \beta, m)$  contribute zero, while the components of initial type in  $\text{In}(\mu, \beta, m)$  contribute polynomials in  $1/z$  with *polynomial* coefficients (homogeneous, of degree  $m$ ) in  $K[\{t_i\}]$ . Summing over all  $m, \beta$  gives the initial terms  $R_\mu^\varepsilon(z)$  as described in the lemma. The remaining part is obtained by summing over all  $m$  and  $\beta$  the contributions of components of recursion type in  $\text{Rec}(\mu, \beta, m)$ .

The contribution of an  $M \in \text{Rec}(\mu, \beta, m)$  is

$$q^\beta \int_{[M]^{\text{vir}}} \frac{1}{e^{\mathbf{T}}(N_M^{\text{vir}})} \left( \frac{\text{ev}_1^*(\delta_\mu) \text{ev}_2^*(\gamma) \prod_{i=1}^m \text{ev}_{2+i}^*(\mathbf{t})}{(z - \psi)m!} \Big|_M \right). \quad (7.5.4)$$

By (7.5.1),

$$M \cong M' \times_{(W//\mathbf{G})^\mathbf{T}} M'',$$

with  $M' \in U(\mu)$  parametrizing two-pointed maps of some class  $\beta_{M'} \neq 0$  and

$$M'' \in \text{In}(\nu_{M'}, \beta - \beta_{M'}, m) \coprod \text{Rec}(\nu_{M'}, \beta - \beta_{M'}, m).$$

Moreover,  $\alpha_{M'} = \alpha_M$  and  $\alpha_{M''} \neq \alpha_M$ .

Recall that the virtual class of the fixed locus  $M$  and the virtual normal bundle are obtained from the fixed part and the moving part of the *absolute* obstruction theory of  $\mathbb{Q}_{0,2+m}^\varepsilon(W//\mathbf{G}, \beta)$ , respectively. In turn, the absolute obstruction theory is obtained via distinguished triangles from the complex (2.5.1)

$$(R^\bullet \pi_*(u^* \mathbb{R}T_\varrho))^\vee,$$

the relative cotangent complex  $\mathbb{L}_{\mathfrak{Bun}_\mathbf{G}/\mathfrak{M}_{0,m+2}}^\bullet$ , and the cotangent complex  $\mathbb{L}_{\mathfrak{M}_{0,m+2}}^\bullet$ ; see [CKM14, Remark 4.5.3]. The relative obstruction theory  $E^\bullet$  over  $\mathfrak{M}_{0,m+2}$  is obtained from the distinguished triangle

$$E^\bullet \longrightarrow (R^\bullet \pi_*(u^* \mathbb{R}T_\varrho))^\vee \longrightarrow g^* \mathbb{L}_{\mathfrak{Bun}_\mathbf{G}/\mathfrak{M}_{0,m+2}}^\bullet[1],$$

where  $g$  is the projection to  $\mathfrak{Bun}_\mathbf{G}$ .

Consider the exact sequence

$$0 \longrightarrow \mathcal{O}_C \longrightarrow \mathcal{O}_{C'} \oplus \mathcal{O}_{C''} \longrightarrow \mathcal{O}_{C' \cap C''} \longrightarrow 0 \quad (7.5.5)$$

coming from splitting the domain curve at the node where the unbroken map  $f : C' \rightarrow W//\mathbf{G}$  meets the rest of the quasimap. The complex  $E^\bullet$  on  $M$  differs from the sum of the corresponding complexes on  $M'$  and  $M''$  only by a complex supported at the node, whose cohomology is the

representation  $T_{\nu_{M'}}(W//\mathbf{G})$  in degree zero. Since all its  $\mathbf{T}$ -weights are nontrivial, this term will only appear in the difference between the virtual normal bundles. The complex  $\mathbb{L}_{\mathfrak{y}_{0,m+2}^\bullet}$  differs from the sum of the complexes on the factors only by the deformation space at the node which is the product

$$T_{x_2}C' \boxtimes T_{x_1}C''$$

of tangent spaces at the indicated markings. It has weight  $\alpha_M - \alpha_{M''} \neq 0$ , so will contribute again to the normal bundle difference.

We conclude that

$$[M]^{\text{vir}} = [M']^{\text{vir}} \times [M'']^{\text{vir}}$$

and that

$$\frac{1}{e^{\mathbf{T}}(N_M^{\text{vir}})} = \frac{e^{\mathbf{T}}([T_{\nu_{M'}}(W//\mathbf{G})])}{e^{\mathbf{T}}(N_{M'}^{\text{vir}}) e^{\mathbf{T}}(N_{M''}^{\text{vir}})(-\psi_2^{M'} - \psi_1^{M''})}.$$

Furthermore, the cotangent weight at the second marking on  $M'$  is  $-\alpha_M = -\alpha_{M'}$ , hence  $-\psi_2^{M'} = \alpha_M - \psi_\infty$  with  $\psi_\infty$  nonequivariant. Hence the contribution (7.5.4) can be written as a product of two virtual localization integrals on the factors:

$$\left( q^{\beta_{M'}} \int_{[M']^{\text{vir}}} \frac{\text{ev}_1^*(\delta_\mu)}{(z - \alpha_{M'} - \psi_0) e^{\mathbf{T}}(N_{M'}^{\text{vir}})} \right) \times \left( q^{\beta - \beta_{M'}} \int_{[M'']^{\text{vir}}} \frac{\text{ev}_1^*(\delta_{\nu_{M'}}) \text{ev}_2^*(\gamma) \prod_{i=1}^m \text{ev}_{2+i}^*(\mathbf{t})}{(\alpha_{M''} - \psi_\infty - \psi) e^{\mathbf{T}}(N_{M''}^{\text{vir}})} \right).$$

The sum over all recursion components is then easily rearranged to obtain the recursion part of (7.5.3).

Finally, we note that the argument also works in the twisted case, since the exact sequence (7.5.5) shows that

$$\mathbb{H}^0(C, P \times_{\mathbf{G}} E) \oplus \underline{E}_\mu = \mathbb{H}^0(C', P|_{C'} \times_{\mathbf{G}} E) \oplus \mathbb{H}^0(C'', P|_{C''} \times_{\mathbf{G}} E)$$

as  $\mathbf{T}$ -representations, and hence that the Euler class of the twisting bundle in the integrand also factors.  $\square$

*Remark 7.5.3.* (i) The main point of Lemma 7.5.2 is the following: up to initial terms which are polynomial in  $1/z$ , the  $q^\beta$ -term in  $S_\mu^\varepsilon$  is calculated from various  $q^{\beta-\beta'}$ -terms with  $\beta' \neq 0$  in  $S_\nu^\varepsilon$  for other fixed points  $\nu$  by a universal formula whose coefficients are *independent of  $\varepsilon$*  (but may depend on  $\gamma$ ). It is in this sense that we think of (7.5.3) as a recursion relation. Given the initial terms  $R_\mu^\varepsilon(z)$ , the set of all  $S_\mu^\varepsilon$ ,  $\mu \in (W//\mathbf{G})^{\mathbf{T}}$  is uniquely recovered from (7.5.3).

(ii) The statement of the lemma remains valid if we replace  $\mathbf{t}$  by  $\tau(\mathbf{t}, q) \in \mathbb{H}_{\mathbf{T}, \text{loc}}^*(W//\mathbf{G}, \Lambda)$ .

(iii) If the one-dimensional  $\mathbf{T}$ -orbits in  $W//\mathbf{G}$  are isolated, then each unbroken component is just a point and the classes  $\psi_0$  and  $\psi_\infty$  vanish. The proof of Lemma 7.5.1 shows that in this case the coefficients of  $S_\mu^\varepsilon(\gamma)$  have only *simple* poles at  $w/-n$ , where  $n$  runs over the positive integers and  $w$  runs over the tangent weights at  $\mu$  (and high order poles at  $z = 0$ ). Furthermore, since the tangent weights at each fixed point are independent, formula (7.5.3) for  $\gamma = \mathbb{1}$  turns into the usual Givental recursion

$$S_\mu^\varepsilon(z) = R_\mu^\varepsilon(z) + \sum_{\nu \in o(\mu)} \sum_{n=1}^{\infty} q^{n\beta(\mu, \nu)} \frac{C_{\mu, \nu, n}}{z + w(\mu, \nu)/n} S_\nu^\varepsilon\left(-\frac{w(\mu, \nu)}{n}\right) \quad (7.5.6)$$

obtained in [Giv96], [Giv98a] for Gromov–Witten invariants. Here the first sum is over the set  $o(\mu)$  of all fixed points  $\nu$  connected to  $\mu$  by a one-dimensional  $\mathbf{T}$ -orbit,  $\beta(\mu, \nu)$  is the homology class of

the orbit, and  $w(\mu, \nu)$  is the tangent weight at  $\mu$  corresponding to the orbit. The coefficients  $C_{\mu, \nu, n}$  are explicit rational functions in  $K = \mathbb{Q}(\lambda_j)$  and are *independent* of  $\varepsilon$ .

Hence the lemma generalizes Givental's result in two main directions: to all  $\varepsilon$ -stable quasimap invariants, and to torus actions with not necessarily isolated one-dimensional orbits.

## 7.6 Polynomiality revisited

All the generating functions  $S_{\mathbf{t}}^\varepsilon(z)(\gamma)$  satisfy a polynomiality property analogous to the one given in § 5.2 for the  $J^\varepsilon$ -functions, and proved in a similar manner. However, we will need a slightly more refined localized version for each of the fixed-point components  $S_\mu^\varepsilon$ .

LEMMA 7.6.1. *For each  $\mu \in (W//\mathbf{G})^{\mathbf{T}}$ , the series*

$$D(S_\mu^\varepsilon) := S_\mu^\varepsilon(q, \mathbf{t}, z) S_\mu^\varepsilon(qe^{-zyL_\theta}, \mathbf{t}, -z)$$

*has no pole at  $z = 0$ . In other words, it is naturally an element in the formal power series ring  $K[[q, t_i, y, z]]$ .*

*Proof.* For effective classes  $\beta, \beta_1, \beta_2$ , consider the morphism

$$\Phi_\varepsilon \circ b_{\varepsilon, \{d(\beta_1), d(\beta_2)\}} \circ \iota_\varepsilon : QG_{0, 2+m, \beta}^\varepsilon(W//\mathbf{G}) \longrightarrow QG_{0, 0, d(\beta)+d(\beta_1)+d(\beta_2)}^\varepsilon(\mathbb{P}_{A\mathbf{G}}^N)$$

from § 3.3 and denote by  $U_{\beta_1, \beta_2}(L_\theta)$  the universal  $\mathbb{C}^*$ -equivariant line bundle obtained by pulling back  $\mathcal{O}(1)$  with the canonical linearization.

Next, for a given  $\mu \in (W//\mathbf{G})^{\mathbf{T}}$ , let

$$QG_{0, 2+m, \beta}^\varepsilon(W//\mathbf{G})_\mu$$

be the  $\mathbf{T}$ -fixed locus in the graph space  $QG_{0, 2+m, \beta}^\varepsilon(W//\mathbf{G})$  consisting of quasimaps for which the parametrized  $\mathbb{P}^1$  is contracted to the point  $\mu$ . It is a  $\mathbb{C}^*$ -invariant substack. Denote by  $\kappa$  the inclusion map into the graph space. We then put

$$\text{Res}_\mu[QG_{0, 2+m, \beta}^\varepsilon(W//\mathbf{G})]^\text{vir} := \kappa_* \left( \frac{[QG_{0, 2+m, \beta}^\varepsilon(W//\mathbf{G})_\mu]^\text{vir}}{e^{\mathbf{T}}(N^\text{vir})} \right),$$

the  $\mathbf{T}$ -localization residue of the virtual class  $[QG_{0, 2+m, \beta}^\varepsilon(W//\mathbf{G})]^\text{vir}$  of the entire graph space at this locus. (In the case of twisted theory, we replace, as usual,  $[QG_{0, 2+m, \beta}^\varepsilon(W//\mathbf{G})_\mu]^\text{vir}$  by  $\kappa^* e(R^0 \pi_* E_{0, 2+m, \beta}) \cap [QG_{0, 2+m, \beta}^\varepsilon(W//\mathbf{G})_\mu]^\text{vir}$ .)

Finally, we write  $\gamma$  as a  $q$ -series

$$\gamma = \gamma_0 + \sum_{\beta \neq 0} q^\beta \gamma_\beta$$

with  $\gamma_\beta \in H_{\mathbf{T}, \text{loc}}^*(W//\mathbf{G})$ .

Consider the generating series

$$\begin{aligned} \langle\langle \gamma p_0, \gamma p_\infty; e^{c_1(U(L_\theta))} \rangle\rangle_{2; \mu}^{QG^\varepsilon} &:= \sum_{m, \beta \geq 0} \frac{q^\beta}{m!} \sum_{\beta_1, \beta_2 \geq 0} q^{\beta_1} q^{\beta_2} \times \\ &\times \int_{\text{Res}_\mu[QG_{0, 2+m, \beta}^\varepsilon(W//\mathbf{G})]^\text{vir}} e^{c_1(U_{\beta_1, \beta_2}(L_\theta))y} \text{ev}_1^*(\gamma_{\beta_1} p_0) \text{ev}_2^*(\gamma_{\beta_2} p_\infty) \prod_{i=1}^m \text{ev}_{2+i}^*(\mathbf{t}). \end{aligned}$$

It is defined *without*  $\mathbb{C}^*$ -localization and hence it is naturally a power series in  $z$ .

On the other hand, we can calculate it by applying virtual localization for the  $\mathbb{C}^*$ -action on the graph space, as in the proofs of Lemma 5.2.1 and of Proposition 5.3.3. The result is

$$e^{w_\mu y} D(S_\mu^\varepsilon),$$

where  $w_\mu = w(\mathcal{O}(\theta); \mu)$  is the  $\mathbf{T}$ -weight on the fiber of  $\mathcal{O}(\theta)$  at  $\mu$ . (In the twisted case, there is an additional factor given by the character  $e(\underline{E}_\mu)$  of the  $\mathbf{T}$  representation of the fiber of  $\underline{E}$  at  $\mu$ .) The lemma is proved.  $\square$

*Remark 7.6.2.* Since the series  $P^\varepsilon$  and  $P^{\infty, \varepsilon}$  appearing in Theorem 7.3.4 contain only nonnegative powers of  $z$ , it is clear that (after making appropriate adjustments to the universal bundles  $U(L_\theta)$ ) the proof also applies to the components of  $S_{\mathbf{t}}^\varepsilon(z)(P^\varepsilon(\mathbf{t}, z))$  and  $S_\tau^\infty(z)(P^{\infty, \varepsilon}(\tau, z))$ , where  $\tau = \tau^{\infty, \varepsilon}(q, \mathbf{t})$ .

### 7.7 Uniqueness Lemma

The following statement is an appropriately generalized version of Givental's Uniqueness Lemma; see [Giv98a, Proposition 4.5] and [Kim99, Lemma 3].

LEMMA 7.7.1. *For  $j = 1, 2$ , let*

$$\{S_{j, \mu} \mid \mu \in (W//\mathbf{G})^{\mathbf{T}}\}$$

*be two systems of power series in  $\Lambda[[t_i]]\{\{z, z^{-1}\}\} = K[[q, t_i]]\{\{z, z^{-1}\}\}$  satisfying the following properties:*

(1) *If we write*

$$S_{j, \mu} = \sum_{\beta} q^{\beta} \sum_{\underline{k}} a_{j, \mu, \beta, \underline{k}}(z) \prod_i t_i^{k_i}$$

*for  $j = 1, 2$ , then the  $a_{j, \mu, \beta, \underline{k}}(z)$  are in  $K(z)$  and decompose into sums of partial fractions with specified poles, as given by Lemma 7.5.1.*

(2) *The systems*

$$\{S_{j, \mu} \mid \mu \in (W//\mathbf{G})^{\mathbf{T}}\}, \quad j = 1, 2$$

*satisfy the recursion relation (7.5.3).*

(3) *For  $j = 1, 2$  and all  $\mu$ , the convolution*

$$D(S_{j, \mu}) := S_{j, \mu}(q, \mathbf{t}, z) S_{j, \mu}(q e^{-zy L_\theta}, \mathbf{t}, -z)$$

*has no pole at  $z = 0$ .*

(4) *For all  $\mu$ ,*

$$S_{1, \mu} = S_{2, \mu} \pmod{1/z^2}.$$

(5) *For  $j = 1, 2$  and all  $\mu$ ,*

$$S_{j, \mu} = i_\mu^*(e^{\mathbf{t}/z}) + O(q).$$

*Then  $S_{1, \mu} = S_{2, \mu}$  for all  $\mu \in (W//\mathbf{G})^{\mathbf{T}}$ .*

*Proof.* For each tuple  $\underline{k} := (k_i)_i$  of nonnegative integers and each effective  $\beta$  define the *bi-degree* of the monomial  $q^\beta \prod_i t_i^{k_i}$  to be

$$\left( \sum_i k_i, \beta(L_\theta) \right) \in \mathbb{N} \times \mathbb{N}.$$

(Note the reversal of ordering in the pair: the total degree in the  $t$ -variables appears first!)

For a pair  $(m, d)$  of nonnegative integers, denote by  $S_{j,\mu}^{(m,d)}$  the part of bi-degree  $(m, d)$  of  $S_{j,\mu}$ ,

$$S_{j,\mu}^{(m,d)} := \sum_{\{\beta: \beta(L_\theta)=d\}} q^\beta \sum_{\{k: \sum k_i=m\}} a_{j,\mu,\beta,\underline{k}}(z) \prod_i t_i^{k_i}.$$

To prove the lemma, it suffices to show that for all  $\mu \in (W//\mathbf{G})^{\mathbf{T}}$  and all  $(m, d) \in \mathbb{N} \times \mathbb{N}$ , we have

$$S_{1,\mu}^{(m,d)} = S_{2,\mu}^{(m,d)}. \quad (7.7.1)$$

We do so by induction on  $(m, d)$ , using the lexicographic order

$$(m', d') < (m, d) \iff m' < m, \text{ or } m' = m \text{ and } d' < d.$$

The base case  $d = 0$  and all  $m \in \mathbb{N}$  holds by property (5) above. Let  $(m, d)$  be fixed with  $d \geq 1$ . We assume that (7.7.1) is true for all fixed points  $\mu$  and all  $(m', d') < (m, d)$ . Consider the difference

$$\Delta := D(S_{1,\mu}) - D(S_{2,\mu})$$

and let  $\Delta^{(m,d)}$  be its part of bi-degree  $(m, d)$ . By the induction assumption,

$$\Delta^{(m,d)} = (S_{1,\mu}^{(m,d)} - S_{2,\mu}^{(m,d)}) + e^{-zyd}(S_{1,\mu}^{(m,d)}(-z) - S_{2,\mu}^{(m,d)}(-z)). \quad (7.7.2)$$

By the recursion property (2) and the induction assumption again, the difference

$$S_{1,\mu}^{(m,d)} - S_{2,\mu}^{(m,d)}$$

is a polynomial in  $1/z$ . We write it as

$$z^{-2a} \left( \frac{A}{z} + B + O(z) \right),$$

with  $a \geq 0$  and  $A, B \in \Lambda[\{t_i\}]$  homogeneous polynomials of bi-degree  $(m, d)$ . Then (7.7.2) can be rewritten as

$$\Delta^{(m,d)} = z^{-2a} (2B + ydA + O(z)).$$

By the polynomiality property (3), we know that  $\Delta^{(m,d)}$  has no pole at  $z = 0$ . Hence if  $a \geq 1$ , we must have  $2B + ydA = 0$ , which in turn implies  $A = B = 0$ . By descending induction we obtain  $a = 0$ . But then  $S_{1,\mu}^{(m,d)} = S_{2,\mu}^{(m,d)}$ , since they agree modulo  $1/z^2$  by property (4).  $\square$

### 7.8 Proof of Theorem 7.3.1

For every fixed point  $\mu \in (W//\mathbf{G})^{\mathbf{T}}$ , put

$$S_{1,\mu} := \langle S_{\tau_\gamma^{\varepsilon_1, \varepsilon_2}(\mathbf{t})}^{\varepsilon_1}(z)(\gamma), \delta_\mu \rangle,$$

$$S_{2,\mu} := \langle S_{\mathbf{t}}^{\varepsilon_2}(z)(\gamma), \delta_\mu \rangle.$$

Theorem 7.3.1 will be proved if we show that

$$S_{1,\mu} = S_{2,\mu}, \quad \forall \mu \in (W//\mathbf{G})^{\mathbf{T}}.$$

But this follows from the Uniqueness Lemma 7.7.1, since the  $S_{j,\mu}$  are power series in  $\Lambda[[t_i, 1/z]]$  satisfying the five properties listed in that lemma.

- Property (1) holds by Lemma 7.5.1.
- Property (2) holds by Lemma 7.5.2.
- Property (3) holds by Lemma 7.6.1.

- Property (4) holds by (6.1.3).
- Property (5) holds by the definitions of the operators  $S^\varepsilon(z)$  and of the transformation  $\tau_\gamma^{\varepsilon_1, \varepsilon_2}$ .

### 7.9 Proof of Theorem 7.3.4

For every fixed point  $\mu \in (W//\mathbf{G})^{\mathbf{T}}$ , put

$$\begin{aligned} S_{1,\mu} &:= \langle S_{\mathbf{t}}^\varepsilon(z)(P^\varepsilon(\mathbf{t}, z)), \delta_\mu \rangle, \\ S_{2,\mu} &:= \langle S_{\tau^\infty, \varepsilon(\mathbf{t})}^\infty(z)(P^{\infty, \varepsilon}(\tau^{\infty, \varepsilon}(\mathbf{t}), z)), \delta_\mu \rangle. \end{aligned}$$

Then the  $S_{j,\mu}$  are power series in  $\Lambda[[t_i]]\{\{z, z^{-1}\}\}$ . By their definition, they satisfy properties (4) and (5) in the Uniqueness Lemma. By Remark 7.6.2, they satisfy property (3) as well.

Recall that in Theorem 7.3.4, we make the additional assumption that the one-dimensional  $\mathbf{T}$ -orbits in  $W//\mathbf{G}$  are isolated. This implies that the unbroken components of the  $\mathbf{T}$ -fixed loci are of dimension zero.

Let  $o(\mu)$  denote the set of all fixed points  $\nu$  connected to  $\mu$  by a one-dimensional  $\mathbf{T}$ -orbit. The proof of Lemma 7.5.1 shows that each  $(q, \{t_i\})$ -coefficient of  $S_{j,\mu}$  is a rational function in  $z$  which decomposes as the sum of a Laurent polynomial in  $K[z, z^{-1}]$  and a sum of partial fractions with *simple* poles at  $w(\mu, \nu)/-n$ , where  $n$  runs over the positive integers and for all  $\nu \in o(\mu)$ , the numerator  $w(\mu, \nu)$  is the tangent weight at  $\mu$  of the corresponding orbit. Furthermore,  $S_{j,\nu}$  is regular at  $z = w(\mu, \nu)/-n$ .

Finally, we claim that for  $j = 1, 2$ , the system  $\{S_{j,\mu} \mid \mu \in (W//\mathbf{G})^{\mathbf{T}}\}$  satisfies Givental's recursion relation (7.5.6). To be more precise,

$$S_{j,\mu}(z) = R_{j,\mu}(z) + \sum_{\nu \in o(\mu)} \sum_{n=1}^{\infty} q^{n\beta(\mu, \nu)} \frac{C_{\mu, \nu, n}}{z + w(\mu, \nu)/n} S_{j,\nu} \left( -\frac{w(\mu, \nu)}{n} \right),$$

where the  $(q, \{t_i\})$ -coefficients of  $R_{j,\mu}(z)$  are in  $K[z, z^{-1}]$ , while the recursion coefficients  $C_{\mu, \nu, n}$  are in  $K$  and do not depend on  $j$ .

This is very easy to see. For example, take  $j = 1$ . For each  $\nu \in o(\mu)$  and each  $n \geq 1$ , write

$$P^\varepsilon(z) = P^\varepsilon \left( -\frac{w(\mu, \nu)}{n} \right) + \left( z + \frac{w(\mu, \nu)}{n} \right) A_{\nu, n}^\varepsilon(z). \quad (7.9.1)$$

For each  $(m, d)$ , the part of bi-degree  $(m, d)$  of  $A_{\nu, n}^\varepsilon(z)$  is a polynomial in  $z$ .

Calculating  $S_{1,\mu}(z) = i_\mu^*(S^\varepsilon(z)(P^\varepsilon(z)))$  via virtual localization as in the proof of Lemma 7.5.2 gives

$$S_{1,\mu}(z) = \hat{R}_\mu^\varepsilon(z) + \sum_{\nu \in o(\mu)} \sum_{n=1}^{\infty} q^{n\beta(\mu, \nu)} \frac{C_{\mu, \nu, n}}{z + w(\mu, \nu)/n} i_\nu^* S^\varepsilon \left( -\frac{w(\mu, \nu)}{n} \right) (P^\varepsilon(z)),$$

where  $\hat{R}_\mu^\varepsilon(z)$  is obtained by summing the contributions from the fixed-point components of initial type. As a power series of  $q$  and  $t_i$ ,  $\hat{R}_\mu^\varepsilon(z)$  has coefficients in  $K[z, z^{-1}]$ . The recursion coefficients  $C_{\mu, \nu, n}$  do not depend on  $\varepsilon$  or on the insertion  $\gamma$  in  $S(z)(\gamma)$ . Now use (7.9.1) to get

$$\begin{aligned} S_{1,\mu}(z) &= \hat{R}_\mu^\varepsilon(z) + \sum_{\nu \in o(\mu)} \sum_{n=1}^{\infty} q^{n\beta(\mu, \nu)} C_{\mu, \nu, n} i_\nu^* S^\varepsilon \left( -\frac{w(\mu, \nu)}{n} \right) (A_{\nu, n}^\varepsilon(z)) \\ &\quad + \sum_{\nu \in o(\mu)} \sum_{n=1}^{\infty} q^{n\beta(\mu, \nu)} \frac{C_{\mu, \nu, n}}{z + w(\mu, \nu)/n} i_\nu^* S^\varepsilon \left( -\frac{w(\mu, \nu)}{n} \right) \left( P^\varepsilon \left( -\frac{w(\mu, \nu)}{n} \right) \right). \end{aligned}$$

Setting

$$R_{1,\mu}(z) = \hat{R}_\mu^\varepsilon(z) + \sum_{\nu \in o(\mu)} \sum_{n=1}^{\infty} q^{n\beta(\mu,\nu)} C_{\mu,\nu,n} i_\nu^* S^\varepsilon \left( -\frac{w(\mu,\nu)}{n} \right) (A_{\nu,n}^\varepsilon(z))$$

gives the claimed recursion relation for  $S_{1,\mu}(z)$ . The same argument obviously works for  $S_{2,\mu}(z)$ .

Note that the proof of the Uniqueness Lemma goes through if we allow the initial terms in the recursion relation to have coefficients in  $K[z, z^{-1}]$  rather than in  $K[z^{-1}]$ . Hence we may apply it to the systems  $\{S_{j,\mu} \mid \mu \in (W//\mathbf{G})^{\mathbf{T}}\}$  to conclude the theorem.

#### ACKNOWLEDGEMENTS

Some of the ideas appearing in this work originate from early discussions with Rahul Pandharipande on big  $I$ -functions of toric varieties and from a joint project of the authors with Duiliu Emanuel Diaconescu and Davesh Maulik on the relation between the small  $I$  and  $J$ -functions of the Hilbert scheme of points in  $\mathbb{C}^2$ . It is a pleasure to thank all three of them here. In addition, special thanks are due to Duiliu Emanuel Diaconescu for many helpful conversations during the preparation of the paper. We thank Jeongseok Oh for pointing out some typos in an earlier version.

The majority of the writing has been done while the first named author visited KIAS in 2011-2012. He thanks KIAS for financial support, excellent working conditions, and an inspiring research environment.

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